MEAN MOTIONS AND VALUES OF THE RIEMANN ZETA FUNCTION.

By

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Contents.

Introduction.

1. In the theory of almost periodic functions the study of mean motions and of problems of distribution forms an interesting chapter.

Historically, the subject begins with Lagrange's treatment of the perturbations of the large planets, which leads to a study of the variation of the argument of a trigonometric polynomial $F(t) = a_0 e^{i\lambda_0 t} + \cdots + a_N e^{i\lambda_N t}$. Apart from some cases considered by Lagrange, this problem was first treated rigorously by Bohl [1] and Weyl [1], who by means of the theory of equidistribution proved the existence of a mean motion whenever the numbers $\lambda_1 - \lambda_0, \ldots, \lambda_N - \lambda_0$ are linearly independent.

Closely related to their method is the treatment by Bohr [1] of the distribution of the values of the Riemann zeta function $\zeta(s) = \zeta(\sigma + it)$, or rather of the function $\log \zeta(s)$, in the half-plane $\sigma > \frac{1}{2}$. It depends on a certain mean convergence of the Euler product, first applied by Bohr and Landau [1], and on the linear independence of the logarithms of the primes. The method has been further developed in Bohr and Jessen [1], [2]. Two main results have been obtained. One concerns the distribution of the values of $\log \zeta(s)$ on vertical lines and states (in the terminology now used) the existence of an asymptotic distribution function of the function $\log \zeta(\sigma + it)$ for every fixed $\sigma > \frac{1}{2}$, possessing a continuous density $F_\sigma(x)$ which is positive in the whole x-plane when $\sigma \leqq 1$. The other result concerns the distribution of the values in vertical strips and states for every strip $(\frac{1}{2} <) \sigma_1 < \sigma < \sigma_2$ and every x the existence of a relative frequency of the zeros of $\log \zeta(s) - x$ in the strip, which depends continuously on σ_1 and σ_2 and is positive for all x when $\sigma_1 \leqq 1$.

The function $\zeta(s)$ is almost periodic in $[1, +\infty]$ and so is, too, the function $\log \zeta(s)$, whereas a certain generalized almost periodicity is present for $\frac{1}{2} < \sigma \leqq 1$. This almost periodicity makes the results less surprising, but was not used in the proofs.

2. A new treatment of the distribution of the values of $\log \zeta(s)$ on vertical lines was given in Jessen and Wintner [1] in connection with a general treatment by means of Fourier transforms of distribution functions of functions of a real variable which are almost periodic in the ordinary or in a generalized sense. The existence of the density $F_\sigma(x)$ is here obtained by an estimate of the Fourier transform of the distribution function, and from its expression as a Fourier integral it followed, among other things, that it possesses continuous partial derivatives of arbitrarily high order. In the case $\frac{1}{2} < \sigma < 1$ it was even shown that it is a regular analytic function of the coordinates. Similar results were obtained regarding the distribution of the values of $\zeta(s)$ itself on vertical lines.

The distribution of the zeros of an arbitrary analytic almost periodic function $f(s)$ in vertical strips was studied by Jessen [1] by means of the so-called Jensen function defined as the mean value $\varphi_f(\sigma) = \underset{t}{M}\{\log |f(\sigma + it)|\}$; it was shown that this function is a continuous convex function and that the relative frequency of zeros of $f(s)$ in a strip $\sigma_1 < \sigma < \sigma_2$ inside the strip of almost periodicity exists and is equal to $(\varphi_f'(\sigma_2) - \varphi_f'(\sigma_1))/2\pi$ whenever $\varphi_f(\sigma)$ is differentiable at the points σ_1 and σ_2. An addition on the variation of the argument of $f(s)$

on vertical lines has been given by Hartman [1], who proved that the mean motion of $f(\sigma + it)$ exists and is equal to $\varphi_f'(\sigma)$ whenever $\varphi_f(\sigma)$ is differentiable at the point σ. A systematic exposition of this subject, including a complete treatment of Lagrange's problem, has been given in Jessen and Tornehave [1].

These investigations of the Jensen function concern functions which are almost periodic in the ordinary sense. In the case of the zeta function they are therefore only applicable in the half-plane $\sigma > 1$. Together with the above mentioned result on the existence and continuity in σ_1 and σ_2 of the frequency of zeros of $\log \zeta(s) - x$ in $\sigma_1 < \sigma < \sigma_2$ they show that the Jensen function $\varphi_{\log \zeta - x}(\sigma)$ of $\log \zeta(s) - x$ is differentiable in $\sigma > 1$ for all x. In the closely related case of an almost periodic function $f(s)$ with linearly independent exponents in the Dirichlet series an even preciser result is known. It has been shown in Jessen [2] by a combination and extension of the method from the zeta function and the Fourier transform method, that in this case the relative frequency of zeros of $f(s) - x$ exists for any strip $\sigma_1 < \sigma < \sigma_2$ and any x and is the integral over the interval $\sigma_1 < \sigma < \sigma_2$ of a certain continuous function. This means that the Jensen function $\varphi_{f-x}(\sigma)$ of $f(s) - x$ is twice differentiable with a continuous second derivative.

3. The object of the present paper is to round off the previous work by a treatment of mean motions on vertical lines and of zeros in vertical strips of the functions $\log \zeta(s) - x$ and $\zeta(s) - x$ in the half-plane $\sigma > \frac{1}{2}$.

Since the zeta function is almost periodic only in a generalized sense in the strip $\frac{1}{2} < \sigma \leqq 1$ we must first extend the results connected with the Jensen function to certain cases of generalized almost periodic functions general enough to include the functions $\log \zeta(s) - x$ and $\zeta(s) - x$. This extension, which may be of some interest in itself, is given in Chapter I.

In Chapter II the functions $\log \zeta(s)$ and $\zeta(s)$ are dealt with. We prove that the Jensen function $\varphi_{\log \zeta - x}(\sigma) = \underset{t}{M}\{\log |\log \zeta(\sigma + it) - x|\}$ of $\log \zeta(s) - x$ exists and is a twice differentiable convex function in the interval $\frac{1}{2} < \sigma < +\infty$. For $\sigma \to \frac{1}{2}$ we have $\varphi_{\log \zeta - x}(\sigma) \to \infty$ for any x. For every $\sigma > \frac{1}{2}$ the function $\log \zeta(\sigma + it) - x$ possesses a mean motion which is equal to $\varphi_{\log \zeta - x}'(\sigma)$, and for any strip $(\frac{1}{2} <)\, \sigma_1 < \sigma < \sigma_2$ there exists a relative frequency of the zeros of $\log \zeta(s) - x$ in the strip, which is equal to $(\varphi_{\log \zeta - x}'(\sigma_2) - \varphi_{\log \zeta - x}'(\sigma_1))/2\pi$. The second derivative $\varphi_{\log \zeta - x}''(\sigma)$ is obtained in the form $\varphi_{\log \zeta - x}''(\sigma) = 2\pi G_\sigma(x)$, where $G_\sigma(x)$ is a continuous function of σ and x, which for every $\sigma > \frac{1}{2}$ represents the density of a certain distribution function analogous to the distribution function of $\log \zeta(\sigma + it)$. It has similar properties as the density $F_\sigma(x)$ mentioned above; thus it possesses continuous partial derivatives of arbitrarily high order and is in the case $\frac{1}{2} < \sigma < 1$ even a regular analytic function of the coordinates; if $\frac{1}{2} < \sigma \leqq 1$ it is positive for all x. Similar results are proved for the functions $\zeta(s) - x$.

For the convenience of the reader we have included proofs of most of the known results which we need. In particular we have included a treatment of the asymptotic distribution functions of the functions $\log \zeta(\sigma + it)$ and $\zeta(\sigma + it)$ for every $\sigma > \frac{1}{2}$.

Of earlier results in our subject we have mentioned above only those which are of direct importance for the present paper. A detailed account of the development of the subject has been given in the introduction to Jessen and Tornehave [1].

CHAPTER I.

Mean Motions and Zeros of Generalized Analytic Almost Periodic Functions.

Ordinary Analytic Almost Periodic Functions.

4. We shall begin by stating the above mentioned results of Jessen and Hartman as they appear in Jessen and Tornehave [1]. First we must mention certain definitions which will be used throughout.

Let $f(s)$ denote an arbitrary function of the complex variable $s = \sigma + it$, which is regular in an open domain G and is not identically zero. The function $\arg f(s)$ is then defined mod. 2π, by the condition $f(s) = |f(s)| e^{i \arg f(s)}$, for all s in G, with the exception of the zeros of $f(s)$.

Let L denote an orientated straight line (or segment) belonging to G. We then define the left argument $\arg^- f(s)$ of $f(s)$ on L as an arbitrary branch of the argument, which is continuous except at the zeros of $f(s)$ on L, whereas it is discontinuous with a jump of $-p\pi$, when s passes, in the positive direction of L, a zero of $f(s)$ of the order p. Similarly we define the right argument $\arg^+ f(s)$ of $f(s)$ on L as an arbitrary branch of the argument, which is continuous except at the zeros of $f(s)$ on L, whereas it is discontinuous with a jump of $+p\pi$, when s passes, in the positive direction of L, a zero of $f(s)$ of the order p. In a discontinuity point we use as value the mean value of the limits from the two sides; the two functions $\arg^- f(s)$ and $\arg^+ f(s)$ are hereby defined for all s on L.

If s_1 and s_2 are points of L, so that the direction from s_1 to s_2 coincides with the positive direction of L, the differences $\arg^- f(s_2) - \arg^- f(s_1)$ and $\arg^+ f(s_2) - \arg^+ f(s_1)$ are independent of the choice of the branches of the arguments and are called the variation of the argument of $f(s)$ from s_1 to s_2 along the left or right side of L, or simply the left or right variation of the argument of $f(s)$ along the segment from s_1 to s_2.

When speaking of the left and right argument of a function on a vertical or horizontal line (or segment) we suppose the line orientated after increasing values of t or σ respectively.

5. The results referred to are now as follows.

Let $f(s) = f(\sigma + it)$ be almost periodic in the strip $[\alpha, \beta]$ and not identically zero. Then the mean value[1]

$$\varphi_f(\sigma) = \underset{t}{M}\{\log|f(\sigma+it)|\} = \lim_{(\delta-\gamma)\to\infty} \frac{1}{\delta-\gamma}\int_\gamma^\delta \log|f(\sigma+it)|\,dt$$

exists uniformly in the interval $[\alpha, \beta]$ and is a convex function of σ. It is called the *Jensen function* of $f(s)$.

Moreover, if $\arg^- f(\sigma+it)$ and $\arg^+ f(\sigma+it)$ denote the left and right argument of $f(s)$ on the line $s = \sigma + it$, $-\infty < t < +\infty$, then the lower and upper, left and right mean motions of $f(s)$ on this line, defined by

$$\left.\begin{matrix} c_f^-(\sigma) \\ \bar{c}_f^-(\sigma) \end{matrix}\right\} = \lim_{(\delta-\gamma)\to\infty} \begin{matrix}\inf \\ \sup\end{matrix} \frac{\arg^- f(\sigma+i\delta) - \arg^- f(\sigma+i\gamma)}{\delta-\gamma}$$

and

$$\left.\begin{matrix} c_f^+(\sigma) \\ \bar{c}_f^+(\sigma) \end{matrix}\right\} = \lim_{(\delta-\gamma)\to\infty} \begin{matrix}\inf \\ \sup\end{matrix} \frac{\arg^+ f(\sigma+i\delta) - \arg^+ f(\sigma+i\gamma)}{\delta-\gamma}$$

[1] For an arbitrary real function $\varrho(\gamma, \delta)$ defined when $-\infty < \gamma < \delta < +\infty$ we denote by $\liminf\limits_{(\delta-\gamma)\to\infty} \varrho(\gamma,\delta)$ the least upper bound of those numbers r for which there exists a number $T = T(r)$ such that $\varrho(\gamma,\delta) > r$ for $(\delta-\gamma) > T$, and, similarly, by $\limsup\limits_{(\delta-\gamma)\to\infty} \varrho(\gamma,\delta)$ the greatest lower bound of those numbers r for which there exists a number $T = T(r)$ such that $\varrho(\gamma,\delta) < r$ for $(\delta-\gamma) > T$. If these limits are equal, we denote their common value by $\lim\limits_{(\delta-\gamma)\to\infty} \varrho(\gamma,\delta)$. When $\varrho(\gamma,\delta)$ is complex-valued, we write $\lim\limits_{(\delta-\gamma)\to\infty} \varrho(\gamma,\delta) = a$ if there exists to every $\varepsilon > 0$ a $T = T(\varepsilon)$ such that $|\varrho(\gamma,\delta) - a| < \varepsilon$ for $(\delta-\gamma) > T$. For a set of functions $\varrho(\gamma,\delta)$ the limits are said to exist uniformly, if, for an arbitrary ε, the same $T = T(\varepsilon)$ may be used for all functions of the set.

satisfy the inequalities

$$\varphi_f'(\sigma - 0) \leqq \underline{c}_f^-(\sigma) \leqq \left\{ \begin{matrix} \underline{c}_f^+(\sigma) \\ \bar{c}_f^-(\sigma) \end{matrix} \right\} \leqq \bar{c}_f^+(\sigma) \leqq \varphi_f'(\sigma + 0).$$

Finally, if $N_f(\sigma_1, \sigma_2; \gamma, \delta)$ denotes the number of zeros[1] of $f(s)$ in the rectangle $(\alpha <) \sigma_1 < \sigma < \sigma_2 (< \beta)$, $\gamma < t < \delta$, then the lower and upper relative frequencies of zeros of $f(s)$ in the strip (σ_1, σ_2), defined by

$$\left. \begin{matrix} \underline{H}_f(\sigma_1, \sigma_2) \\ \bar{H}_f(\sigma_1, \sigma_2) \end{matrix} \right\} = \lim_{(\delta-\gamma) \to \infty} \begin{matrix} \inf \\ \sup \end{matrix} \frac{N_f(\sigma_1, \sigma_2; \gamma, \delta)}{\delta - \gamma}$$

satisfy the inequalities

$$\frac{1}{2\pi}(\varphi_f'(\sigma_2 - 0) - \varphi_f'(\sigma_1 + 0)) \leqq \underline{H}_f(\sigma_1, \sigma_2) \leqq \bar{H}_f(\sigma_1, \sigma_2) \leqq \frac{1}{2\pi}(\varphi_f'(\sigma_2 + 0) - \varphi_f'(\sigma_1 - 0)).$$

As a corollary we have, that if $\varphi_f(\sigma)$ is differentiable at the point σ, then the left and right mean motions

$$c_f^-(\sigma) = \lim_{(\delta-\gamma) \to \infty} \frac{\arg^- f(\sigma + i\delta) - \arg^- f(\sigma + i\gamma)}{\delta - \gamma}$$

and

$$c_f^+(\sigma) = \lim_{(\delta-\gamma) \to \infty} \frac{\arg^+ f(\sigma + i\delta) - \arg^+ f(\sigma + i\gamma)}{\delta - \gamma}$$

both exist and are determined by

$$c_f^-(\sigma) = c_f^+(\sigma) = \varphi_f'(\sigma).$$

Similarly, if $\varphi_f(\sigma)$ is differentiable at σ_1 and σ_2, then the relative frequency of zeros

$$H_f(\sigma_1, \sigma_2) = \lim_{(\delta-\gamma) \to \infty} \frac{N_f(\sigma_1, \sigma_2; \gamma, \delta)}{\delta - \gamma}$$

exists and is determined by

$$H_f(\sigma_1, \sigma_2) = \frac{1}{2\pi}(\varphi_f'(\sigma_2) - \varphi_f'(\sigma_1)).$$

The latter formula is called the *Jensen formula* for almost periodic functions.

If for $m > 0$ we put

$$|f(s)|_m = \max \{|f(s)|, m\}$$

the Jensen function $\varphi_f(\sigma)$ is also determined as the limit of the mean value

$$M_t \{\log |f(\sigma + it)|_m\}$$

as $m \to 0$, the convergence being again uniform in $[\alpha, \beta]$.

[1] Throughout, multiple zeros are counted according to their order of multiplicity.

Among the further properties of the Jensen function we mention that, if a sequence of functions $f_1(s), f_2(s), \ldots$ almost periodic in $[\alpha, \beta]$, none of which is identically zero, converges uniformly towards $f(s)$ in $[\alpha, \beta]$, then the Jensen function $\varphi_{f_n}(\sigma)$ of $f_n(s)$ converges uniformly in $[\alpha, \beta]$ towards $\varphi_f(\sigma)$.

6. We shall now establish connections between the Jensen function and certain distribution functions.

Let R_x be the complex plane with $x = \xi_1 + i\xi_2$ as variable point. A completely additive, non-negative set-function $\mu(E)$, defined for all Borel sets E in R_x, for which $\mu(R_x)$ is finite, will be called a *distribution function* in R_x. We shall not suppose that $\mu(R_x) = 1$.

For distribution functions in this general sense we have a theory similar to that of the case $\mu(R_x) = 1$. We briefly recall the parts of the theory which will be applied, referring for details e. g. to the monograph by Cramér [1] and to the summary in Jessen and Wintner [1].

Our notation for an integral with respect to a distribution function μ will be

$$\int_E h(x)\,\mu(d\,R_x).$$

A similar notation will be used for ordinary Lebesgue integrals. The Lebesgue measure (in any number of dimensions) will be denoted throughout by m. Thus the notation for an ordinary Lebesgue integral in R_x will be

$$\int_E h(x)\,m(d\,R_x).$$

A set E is called a continuity set of μ if $\mu(E') = \mu(E'')$, where E' denotes the set formed by all interior points of E, and E'' the closure of E. If $\mu(E) = \nu(E)$ or $\mu(E) \leqq \nu(E)$ for the common continuity sets of μ and ν, then $\mu(E) = \nu(E)$ or $\mu(E) \leqq \nu(E)$ for all Borel sets E.

A sequence of distribution functions μ_n is said to be convergent if there exists a distribution function μ such that $\mu_n(E) \to \mu(E)$ for all continuity sets of the limit function μ, which is then unique. The symbol $\mu_n \to \mu$ will be used only in this sense.

We have $\mu_n \to \mu$, if and only if the relation

$$\int_{R_x} h(x)\,\mu_n(d\,R_x) \to \int_{R_x} h(x)\,\mu(d\,R_x)$$

holds for all bounded continuous functions $h(x)$ in R_x. If $\mu_n \to \mu$, then $\liminf \mu_n(E) \geqq \geqq \mu(E)$ for any open set E, and $\limsup \mu_n(E) \leqq \mu(E)$ for any closed set E.

A distribution function μ_σ depending on a parameter σ, which runs in an interval (α, β), is said to depend continuously on σ, if $\mu_{\sigma_n} \to \mu_{\sigma_0}$ when $\sigma_n \to \sigma_0$. Then $\mu_\sigma(R_x)$ is continuous and therefore bounded in any closed interval $(\alpha <)\sigma_1 \leqq \leqq \sigma \leqq \sigma_2 (< \beta)$. Moreover, $\mu_\sigma(E)$, considered as a function of σ, is semi-continuous from below for any open set E and semi-continuous from above for any closed set E. In particular, $\mu_\sigma(E)$ is a Baire function for any open or closed set E and hence for any Borel set E.[1] The integral

$$\mu(E) = \int_{\sigma_1}^{\sigma_2} \mu_\sigma(E)\, d\sigma$$

is again a distribution function.

Suppose now that μ_σ for every σ is the limit of a distribution function $\mu_{n,\sigma}$, and suppose that $\mu_{n,\sigma}$ for every n depends continuously on σ; consider the distribution function

$$\mu_n(E) = \int_{\sigma_1}^{\sigma_2} \mu_{n,\sigma}(E)\, d\sigma.$$

Then μ will be the limit of μ_n, if $\mu_{n,\sigma}(R_x)$ is uniformly bounded for all n and all σ in $\sigma_1 \leqq \sigma \leqq \sigma_2$. For if E is a Borel set, and E' denotes the set formed by all interior points of E, and E'' the closure of E, then, by Fatou's theorem, we have

$$\liminf \mu_n(E) \geqq \liminf \mu_n(E') \geqq \int_{\sigma_1}^{\sigma_2} \liminf \mu_{n,\sigma}(E')\, d\sigma \geqq \int_{\sigma_1}^{\sigma_2} \mu_\sigma(E')\, d\sigma = \mu(E')$$

and

$$\limsup \mu_n(E) \leqq \limsup \mu_n(E'') \leqq \int_{\sigma_1}^{\sigma_2} \limsup \mu_{n,\sigma}(E'')\, d\sigma \leqq \int_{\sigma_1}^{\sigma_2} \mu_\sigma(E'')\, d\sigma = \mu(E''),$$

so that $\mu_n(E) \to \mu(E)$ if E is a continuity set of μ.

A distribution function μ is called absolutely continuous if $\mu(E) = 0$ for every Borel set E of measure 0; this is the case if and only if there exists in R_x a Lebesgue integrable point function $F(x)$ such that

$$\mu(E) = \int_E F(x)\, m(d R_x)$$

for any Borel set E; we call $F(x)$ the density of μ.

Let R_y be the complex plane with $y = \eta_1 + i\eta_2$ as variable point, and let throughout xy denote not the usual product of the two complex numbers, but

[1] Since the system of sets E for which $\mu_\sigma(E)$ is a Baire function contains the limit of any decreasing or increasing sequence of sets from the system.

the inner product $\xi_1 \eta_1 + \xi_2 \eta_2$ of the vectors $x = (\xi_1, \xi_2)$ and $y = (\eta_1, \eta_2)$. If μ is a distribution function in R_x then the integral

$$\Lambda(y; \mu) = \int_{R_x} e^{ixy} \mu(d R_x)$$

defines in R_y a function $\Lambda(y; \mu)$ which is uniformly continuous and bounded, the maximum of its absolute value being $\Lambda(0; \mu) = \mu(R_x)$. We call $\Lambda(y; \mu)$ the *Fourier transform* of μ. If $\Lambda(y; \mu) \equiv \Lambda(y; \nu)$, then $\mu = \nu$.

If $\mu_n \to \mu$, then $\Lambda(y; \mu_n) \to \Lambda(y; \mu)$ holds uniformly in every circle $|y| \leqq a$; conversely, if a sequence of Fourier transforms $\Lambda(y; \mu_n)$ is uniformly convergent in every circle $|y| \leqq a$, then the limit function also is the Fourier transform $\Lambda(y; \mu)$ of a distribution function μ, and $\mu_n \to \mu$.

If the integral

$$\int_{R_y} |y|^p \, |\Lambda(y; \mu)| \, m(d R_y)$$

is finite for an integer $p \geqq 0$, then μ is absolutely continuous and its density $F(x) = F(\xi_1, \xi_2)$, determined by the inversion formula

$$F(x) = (2\pi)^{-2} \int_{R_y} e^{-ixy} \Lambda(y; \mu) \, m(d R_y)$$

is continuous and possesses in the case $p > 0$ continuous partial derivatives of order $\leqq p$, which may be obtained by differentiation under the integral sign. This is in particular the case if for some $\varepsilon > 0$

$$\Lambda(y; \mu) = O(|y|^{-(2+p+\varepsilon)}) \quad \text{as} \quad |y| \to \infty.$$

If the estimate

$$\Lambda(y; \mu) = O(e^{-c|y|}) \quad \text{as} \quad |y| \to \infty$$

holds for some $c > 0$, then $F(x) = F(\xi_1, \xi_2)$ is a regular analytic function of the two real variables ξ_1, ξ_2. If c may be taken arbitrarily large, then $F(x)$ is an entire function of the two variables ξ_1, ξ_2.

7. Let again $f(s)$ be an analytic almost periodic function in the strip $[\alpha, \beta]$. We shall prove a theorem on the distribution of the values of $f(s)$ on vertical lines which is a special case of a general theorem on asymptotic distribution functions, to be found in Jessen and Wintner [1].

For an arbitrary σ and an arbitrary interval $(-\infty <) \gamma < t < \delta (< +\infty)$ let $\mu_{\sigma; \gamma, \delta}$ and $\nu_{\sigma; \gamma, \delta}$ denote the distribution functions of $f(\sigma + it)$ and of $f(\sigma + it)$ with respect to $|f'(\sigma + it)|^2$ over the interval $\gamma < t < \delta$, defined by

$$\mu_{\sigma;\gamma,\delta}(E) = \frac{m(A_{\sigma;\gamma,\delta}(E))}{\delta - \gamma} \quad \text{and} \quad \nu_{\sigma;\gamma,\delta}(E) = \frac{1}{\delta-\gamma}\int\limits_{A_{\sigma;\gamma,\delta}(E)} |f'(\sigma+it)|^2\,dt,$$

where $A_{\sigma;\gamma,\delta}(E)$ denotes the set of points in $\gamma < t < \delta$ for which $f(\sigma+it)$ belongs to E.

Then $\mu_{\sigma;\gamma,\delta}$ and $\nu_{\sigma;\gamma,\delta}$ converge for $(\delta-\gamma)\to\infty$[1] towards certain distribution functions μ_σ and ν_σ. We call these distribution functions the *asymptotic distribution functions* of $f(\sigma+it)$ and of $f(\sigma+it)$ with respect to $|f'(\sigma+it)|^2$.

The proof is immediate. By the definition of the integral we have

(1) $$\Lambda(y;\mu_{\sigma;\gamma,\delta}) = \frac{1}{\delta-\gamma}\int_\gamma^\delta e^{if(\sigma+it)y}\,dt \quad \text{and}$$

$$\Lambda(y;\nu_{\sigma;\gamma,\delta}) = \frac{1}{\delta-\gamma}\int_\gamma^\delta e^{if(\sigma+it)y}|f'(\sigma+it)|^2\,dt,$$

where $f(\sigma+it)y$ denotes the inner product. Since the functions $e^{if(\sigma+it)y}$ and $e^{if(\sigma+it)y}|f'(\sigma+it)|^2$ are almost periodic for every y and form a uniformity set of almost periodic functions for $|y|\leqq a$ for any a, the mean values

$$\underset{t}{M}\{e^{if(\sigma+it)y}\} = \lim_{(\delta-\gamma)\to\infty}\frac{1}{\delta-\gamma}\int_\gamma^\delta e^{if(\sigma+it)y}\,dt \quad \text{and}$$

$$\underset{t}{M}\{e^{if(\sigma+it)y}|f'(\sigma+it)|^2\} = \lim_{(\delta-\gamma)\to\infty}\frac{1}{\delta-\gamma}\int_\gamma^\delta e^{if(\sigma+it)y}|f'(\sigma+it)|^2\,dt$$

exist uniformly in every circle $|y|\leqq a$. This implies the theorem, and we obtain

(2) $$\Lambda(y;\mu_\sigma) = \underset{t}{M}\{e^{if(\sigma+it)y}\} \quad \text{and} \quad \Lambda(y;\nu_\sigma) = \underset{t}{M}\{e^{if(\sigma+it)y}|f'(\sigma+it)|^2\}.$$

From the expressions (1) and (2) it follows that the Fourier transforms are continuous functions of y and σ together. This implies that the distribution functions μ_σ and ν_σ, and $\mu_{\sigma;\gamma,\delta}$ and $\nu_{\sigma;\gamma,\delta}$ for fixed γ and δ, depend continuously on σ.

8. By means of the distribution functions μ_σ we obtain for the Jensen function $\varphi_f(\sigma)$ of $f(s)$ the expression

(3) $$\varphi_f(\sigma) = \int\limits_{R_x} \log|x|\,\mu_\sigma(dR_x).$$

[1] I. e. for any sequence of intervals $\gamma_n < t < \delta_n$, where $(\delta_n - \gamma_n)\to\infty$.

For by the definition of the integral we obtain for every $m > 0$

$$\frac{1}{\delta-\gamma}\int_{\gamma}^{\delta} \log |f(\sigma+it)|_m \, dt = \int_{R_x} \log |x|_m \, \mu_{\sigma;\gamma,\delta}(dR_x),$$

whence

$$M_t\{\log |f(\sigma+it)|_m\} = \int_{R_x} \log |x|_m \, \mu_\sigma(dR_x), \tag{4}$$

since we may replace $\log |x|_m$ by $\log M_m$ for $|x| > M$, where M denotes the upper bound of $|f(\sigma+it)|$, and thus obtain a bounded continuous function under the integral sign. By § 5 the left-hand side of (4) converges for $m \to 0$ towards $\varphi_f(\sigma)$. This implies the existence of the integral on the right in (3) and the relation (3).

Similarly, the Jensen function

$$\varphi_{f-x}(\sigma) = M_t\{\log |f(\sigma+it) - x|\}$$

of the function $f(s) - x$ is for any complex number x determined by the expression

$$\varphi_{f-x}(\sigma) = \int_{R_u} \log |u - x| \, \mu_\sigma(dR_u).$$

9. We will now establish a connection between the Jensen functions $\varphi_{f-x}(\sigma)$ and the distribution functions ν_σ.

For a fixed strip $(\alpha <) \sigma_1 < \sigma < \sigma_2 (< \beta)$ let $N_{f-x}(\gamma, \delta)$ denote the number of zeros of $f(s) - x$ in the rectangle $\sigma_1 < \sigma < \sigma_2$, $\gamma < t < \delta$, and let us consider the distribution function

$$\nu_{\gamma,\delta}(E) = \int_E \frac{N_{f-x}(\gamma,\delta)}{\delta-\gamma} m(dR_x).$$

By the area theorem we have

$$\nu_{\gamma,\delta}(E) = \frac{1}{\delta-\gamma} \int_{A_{\gamma,\delta}(E)} |f'(\sigma+it)|^2 \, d\sigma \, dt,$$

where $A_{\gamma,\delta}(E)$ denotes the set of points in the rectangle for which $f(s)$ belongs to E. Hence, by Fubini's theorem,

$$\nu_{\gamma,\delta}(E) = \int_{\sigma_1}^{\sigma_2} \nu_{\sigma;\gamma,\delta}(E) \, d\sigma.$$

Also, $\nu_{\sigma;\gamma,\delta}(R_x)$ is uniformly bounded for all γ and δ and all σ in $\sigma_1 \leqq \sigma \leqq \sigma_2$ (since $f'(s)$ is bounded in the strip $\sigma_1 \leqq \sigma \leqq \sigma_2$). Hence by § 6 $\nu_{\gamma,\delta}$ converges for $(\delta-\gamma) \to \infty$ towards the distribution function ν determined by

$$(5) \qquad \nu(E) = \int_{\sigma_1}^{\sigma_2} \nu_\sigma(E)\, d\sigma.^1$$

By § 5 we have for every x the inequalities

$$(6) \qquad \frac{1}{2\pi}(\varphi'_{f-x}(\sigma_2 - 0) - \varphi'_{f-x}(\sigma_1 + 0)) \leqq \liminf_{(\delta-\gamma)\to\infty} \frac{N_{f-x}(\gamma,\delta)}{\delta-\gamma}$$
$$\leqq \limsup_{(\delta-\gamma)\to\infty} \frac{N_{f-x}(\gamma,\delta)}{\delta-\gamma} \leqq \frac{1}{2\pi}(\varphi'_{f-x}(\sigma_2 + 0) - \varphi'_{f-x}(\sigma_1 - 0)).$$

By § 5 the convex functions $\varphi_{f-x}(\sigma)$ depend continuously on x. This implies that the left and right derivatives $\varphi'_{f-x}(\sigma-0)$ and $\varphi'_{f-x}(\sigma+0)$ considered as functions of x for every fixed σ will be lower and upper semi-continuous, respectively. Hence, the functions to the left and right in (6) are lower and upper semi-continuous, respectively. By Fatou's theorem we conclude that if E is a continuity set of $\nu(E)$, then

$$(7) \quad \frac{1}{2\pi}\int_E (\varphi'_{f-x}(\sigma_2-0)-\varphi'_{f-x}(\sigma_1+0))\, m(dR_x) \leqq \nu(E) \leqq \frac{1}{2\pi}\int_E (\varphi'_{f-x}(\sigma_2+0)-\varphi'_{f-x}(\sigma_1-0))\, m(dR_x).$$

These inequalities must then hold for all Borel sets E.

If we put $E = R_x$, $\sigma_1 = \sigma - \varepsilon$, and $\sigma_2 = \sigma + \varepsilon$, then $\nu(E)$ will by (5) approach zero for $\varepsilon \to 0$, whereas the first term in (7) will converge towards the integral over R_x of $(\varphi'_{f-x}(\sigma+0) - \varphi'_{f-x}(\sigma-0))/2\pi$. This integral must therefore be zero. Hence the two functions $\varphi'_{f-x}(\sigma-0)$ and $\varphi'_{f-x}(\sigma+0)$ will differ only in a null-set. This implies that the first and last terms in (7) are equal. Thus we have proved that for an arbitrary Borel set E

$$(8) \quad \frac{1}{2\pi}\int_E (\varphi'_{f-x}(\sigma_2-0)-\varphi'_{f-x}(\sigma_1+0))\, m(dR_x) = \nu(E) = \frac{1}{2\pi}\int_E (\varphi'_{f-x}(\sigma_2+0)-\varphi'_{f-x}(\sigma_1-0))\, m(dR_x).$$

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[1] Actually, the relation $\nu_{\gamma,\delta}(E) \to \nu(E)$ holds not only for the continuity sets of ν, but for all Borel sets E. This property depends on the fact that the densities $N_{f-x}(\gamma,\delta)/(\delta-\gamma)$ of the distribution functions $\nu_{\gamma,\delta}$ are uniformly bounded for $(\delta-\gamma) > 1$, say.

If in particular ν_σ for every σ is absolutely continuous with a density $G_\sigma(x)$ which is a continuous function of x and σ together, the relations (8) show (on account of the semi-continuity of the integrands) that for every x

$$\varphi'_{f-x}(\sigma_2 - 0) - \varphi'_{f-x}(\sigma_1 + 0) \leqq 2\pi \int_{\sigma_1}^{\sigma_2} G_\sigma(x)\, d\sigma \leqq \varphi'_{f-x}(\sigma_2 + 0) - \varphi'_{f-x}(\sigma_1 - 0).$$

The continuity in σ_1 and σ_2 of the term in the middle then implies that $\varphi_{f-x}(\sigma)$ is differentiable, and we obtain

$$\varphi'_{f-x}(\sigma_2) - \varphi'_{f-x}(\sigma_1) = 2\pi \int_{\sigma_1}^{\sigma_2} G_\sigma(x)\, d\sigma,$$

which shows that $\varphi_{f-x}(\sigma)$ is twice differentiable with the second derivative

$$\varphi''_{f-x}(\sigma) = 2\pi\, G_\sigma(x).$$

The Jensen Function of a Type of Generalized Analytic Almost Periodic Functions.

10. We shall now give an extension of some of the preceding results to a type of generalized analytic almost periodic functions. The functions which we will consider will be supposed to be almost periodic in a strip $[\alpha_0, \beta_0]$ and continuable not in a strip (α, β), but in a half-strip, say $\alpha < \sigma < \beta$, $t > \gamma_0$. The type of generalized almost periodicity with which we shall be concerned will be an extension to analytic functions of almost periodicity in Besicovitch's sense with index p; but while Besicovitch takes $p \geqq 1$, it is sufficient for our purposes to take $p > 0$. No theory of this type of generalized almost periodicity will be needed, since all results follow directly from the definition.

It will not be possible to maintain the above definition of the Jensen function. As might be expected, since we are dealing with generalized almost periodicity of the Besicovitch type, the limit has to be replaced by a limit, in which $\delta \to \infty$ for fixed γ, but not necessarily uniformly in γ.

The notion of a mean value will be taken throughout in this sense, i. e. a function $H(t)$ defined on a half-line $t > \gamma_0$ will be said to possess the *mean value*

$$\underset{t}{M}\{H(t)\} = \lim_{\delta \to \infty} \frac{1}{\delta - \gamma} \int_{\gamma}^{\delta} H(t)\, dt,$$

if the limit on the right exists for a fixed $\gamma > \gamma_0$ (the integral need not exist for $\gamma = \gamma_0$). Evidently the existence of the limit for one $\gamma > \gamma_0$ implies its existence for all $\gamma > \gamma_0$, and the value is independent of γ. If $H(t)$ is a real function defined on a half-line $t > \gamma_0$, the *upper mean value* is defined by

$$\overline{\underset{t}{M}}\{H(t)\} = \limsup_{\delta \to \infty} \frac{1}{\delta - \gamma} \int_{\gamma}^{\delta} H(t)\, dt,$$

where $\gamma > \gamma_0$ is again arbitrary, but fixed.

A similar change will have to be made in the definitions of the mean motions, the frequencies of zeros, and the asymptotic distribution functions.

The usual notation for strips will be maintained for half-strips without change. Thus, if we are dealing with functions defined in a half-strip $\alpha < \sigma < \beta$, $t > \gamma_0$, a statement is said to hold in $[\alpha, \beta]$, if it holds in the part of the half-strip belonging to an arbitrary reduced strip $(\alpha <)\, \alpha_1 < \sigma < \beta_1 \,(< \beta)$.

Suppose that $p > 0$, and that $f(s)$, $f_1(s)$, $f_2(s)$, ... are functions defined in the half-strip $\alpha < \sigma < \beta$, $t > \gamma_0$. Then we shall say that $f_n(s)$ converges in the mean, with index p, towards $f(s)$ in $[\alpha, \beta]$ if

$$\left[\overline{\underset{t}{M}}\left\{\int_{\alpha_1}^{\beta_1} |f(\sigma + it) - f_n(\sigma + it)|^p d\sigma\right\}\right]^{\frac{1}{p}} \to 0 \quad \text{as} \quad n \to \infty$$

for any reduced strip $(\alpha <)\, \alpha_1 < \sigma < \beta_1 \,(< \beta)$. Since the left side decreases as p decreases it is plain that convergence in mean with index p implies convergence in mean with index p_1, if $0 < p_1 < p$.

11. We shall prove the following theorems.

Theorem 1. *Let* $-\infty \leqq \alpha < \alpha_0 < \beta_0 < \beta \leqq +\infty$ *and* $-\infty < \gamma_0 < +\infty$, *and let* $f_1(s)$, $f_2(s)$, ... *be a sequence of functions almost periodic in* $[\alpha, \beta]$ *converging uniformly in* $[\alpha_0, \beta_0]$ *towards a function* $f(s)$, *which is then almost periodic in* $[\alpha_0, \beta_0]$. *Suppose, that none of the functions is identically zero. Suppose further, that* $f(s)$ *may be continued as a regular function in the half-strip* $\alpha < \sigma < \beta$, $t > \gamma_0$, *and that* $f_n(s)$ *converges in mean with an index* $p > 0$ *towards* $f(s)$ *in* $[\alpha, \beta]$.

Then the Jensen function

$$\varphi_f(\sigma) = \underset{t}{M}\{\log |f(\sigma + it)|\}$$

exists uniformly in $[\alpha, \beta]$, *i. e. the function*

$$\varphi_f(\sigma; \gamma, \delta) = \frac{1}{\delta - \gamma} \int_{\gamma}^{\delta} \log |f(\sigma + it)|\, dt$$

converges for $\delta \to \infty$ *for any fixed* $\gamma > \gamma_0$ *uniformly in* $[\alpha, \beta]$ *towards a limit function* $\varphi_f(\sigma)$. *The Jensen function* $\varphi_{f_n}(\sigma)$ *of* $f_n(s)$ *converges for* $n \to \infty$ *uniformly in* $[\alpha, \beta]$ *towards* $\varphi_f(\sigma)$.

The function $\varphi_f(\sigma)$ *is convex in* (α, β), *and, for every* σ *in* (α, β), *the four mean motions defined by*

$$\left.\begin{matrix} \underline{c}_f^-(\sigma) \\ \bar{c}_f^-(\sigma) \end{matrix}\right\} = \lim_{\delta\to\infty} \begin{matrix}\inf\\ \sup\end{matrix} \frac{\arg^- f(\sigma + i\delta) - \arg^- f(\sigma + i\gamma)}{\delta - \gamma}$$

and

$$\left.\begin{matrix} \underline{c}_f^+(\sigma) \\ \bar{c}_f^+(\sigma) \end{matrix}\right\} = \lim_{\delta\to\infty} \begin{matrix}\inf\\ \sup\end{matrix} \frac{\arg^+ f(\sigma + i\delta) - \arg^+ f(\sigma + i\gamma)}{\delta - \gamma}$$

satisfy the inequalities

$$(9) \qquad \varphi_f'(\sigma - 0) \leqq \underline{c}_f^-(\sigma) \leqq \left\{\begin{matrix} \underline{c}_f^+(\sigma) \\ \bar{c}_f^-(\sigma) \end{matrix}\right\} \leqq \bar{c}_f^+(\sigma) \leqq \varphi_f'(\sigma + 0).$$

Further, for every strip (σ_1, σ_2) *where* $\alpha < \sigma_1 < \sigma_2 < \beta$, *the two relative frequencies of zeros defined by*

$$\left.\begin{matrix} \underline{H}_f(\sigma_1, \sigma_2) \\ \bar{H}_f(\sigma_1, \sigma_2) \end{matrix}\right\} = \lim_{\delta\to\infty} \begin{matrix}\inf\\ \sup\end{matrix} \frac{N_f(\sigma_1, \sigma_2; \gamma, \delta)}{\delta - \gamma},$$

where $N_f(\sigma_1, \sigma_2; \gamma, \delta)$ *denotes the number of zeros of* $f(s)$ *in the rectangle* $\sigma_1 < \sigma < \sigma_2$, $\gamma < t < \delta$, *satisfy the inequalities*

$$(10) \quad \frac{1}{2\pi}(\varphi_f'(\sigma_2 - 0) - \varphi_f'(\sigma_1 + 0)) \leqq \underline{H}_f(\sigma_1, \sigma_2) \leqq \bar{H}_f(\sigma_1, \sigma_2) \leqq \frac{1}{2\pi}(\varphi_f'(\sigma_2 + 0) - \varphi_f'(\sigma_1 - 0)).$$

As a corollary we have that if $\varphi_f(\sigma)$ is differentiable at the point σ, then the left and right mean motions

$$c_f^-(\sigma) = \lim_{\delta\to\infty} \frac{\arg^- f(\sigma + i\delta) - \arg^- f(\sigma + i\gamma)}{\delta - \gamma}$$

and

$$c_f^+(\sigma) = \lim_{\delta\to\infty} \frac{\arg^+ f(\sigma + i\delta) - \arg^+ f(\sigma + i\gamma)}{\delta - \gamma}$$

both exist and are determined by

$$c_f^-(\sigma) = c_f^+(\sigma) = \varphi_f'(\sigma).^1$$

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[1] This means that $\arg^- f(\sigma + it)$ and $\arg^+ f(\sigma + it)$ are both $= ct + o(t)$, where $c = \varphi_f'(\sigma)$.

Similarly, if $\varphi_f(\sigma)$ is differentiable at σ_1 and σ_2, then the relative frequency of zeros

$$H_f(\sigma_1, \sigma_2) = \lim_{\delta \to \infty} \frac{N_f(\sigma_1, \sigma_2; \gamma, \delta)}{\delta - \gamma}$$

exists and is determined by the Jensen formula

$$H_f(\sigma_1, \sigma_2) = \frac{1}{2\pi}(\varphi_f'(\sigma_2) - \varphi_f'(\sigma_1)).$$

Theorem 2. *The function $f(\sigma + it)$ possesses for every σ in (α, β) an asymptotic distribution function $\mu_{f,\sigma}$, i. e. the distribution function $\mu_{f,\sigma;\gamma,\delta}$ of $f(\sigma + it)$ in the interval $\gamma < t < \delta$, defined by*

$$\mu_{f,\sigma;\gamma,\delta}(E) = \frac{m(A_{f,\sigma;\gamma,\delta}(E))}{\delta - \gamma},$$

where $A_{f,\sigma;\gamma,\delta}(E)$ for an arbitrary Borel set E denotes the set of points of $\gamma < t < \delta$ for which $f(\sigma + it)$ belongs to E, converges for $\delta \to \infty$ for any fixed $\gamma > \gamma_0$ towards a distribution function $\mu_{f,\sigma}$. The asymptotic distribution function $\mu_{f_n,\sigma}$ of $f_n(\sigma + it)$ converges for $n \to \infty$ towards $\mu_{f,\sigma}$.

There are of course similar theorems for functions $f(s)$ which may be continued in a half-strip $\alpha < \sigma < \beta$, $t < \delta_0$. The limits must then be taken for $\gamma \to -\infty$ and fixed $\delta < \delta_0$. If both pairs of theorems are applicable for the same sequence $f_1(s), f_2(s), \ldots$, the Jensen function $\varphi_f(\sigma)$ and the asymptotic distribution function $\mu_{f,\sigma}$ will be the same in both cases, since in both cases they are the limits of $\varphi_{f_n}(\sigma)$ and $\mu_{f_n,\sigma}$ respectively.

We shall not go into the extension of the results of §§ 8—9, since their extension is not needed for the treatment of the zeta function.

12. Let $\alpha_1, \beta_1, \alpha_2, \beta_2$ be chosen such that $\alpha < \alpha_1 < \alpha_0 < \alpha_2 < \beta_2 < \beta_0 < \beta_1 < \beta$, and let $\delta > 0$ be chosen so small that $\alpha < \alpha_1 - 8\delta$, $\beta_1 + 8\delta < \beta$, $\alpha_0 < \alpha_2 - 3\delta$, and $\beta_2 + 3\delta < \beta_0$; we may suppose $\delta \leqq \frac{1}{2}$. Define the rectangles $R_\nu(t_0)$ for $0 \leqq \nu \leqq 8$ by

$$R_\nu(t_0): \alpha_1 - \nu\delta \leqq \sigma \leqq \beta_1 + \nu\delta, \qquad |t - t_0| \leqq \tfrac{1}{2}(1 + \nu),$$

and the rectangles $S_\nu(t_0)$ for $0 \leqq \nu \leqq 3$ by

$$S_\nu(t_0): \alpha_2 - \nu\delta \leqq \sigma \leqq \beta_2 + \nu\delta, \qquad |t - t_0| \leqq \tfrac{1}{2}(1 + \nu).[1]$$

[1] Not all of these rectangles will be used in the proofs of Theorems 1 and 2. The remainder are being kept in reserve for the proofs of Theorems 3 and 4.

Since $\delta \leqq \frac{1}{2}$, the distance between the frontiers of two successive rectangles $R_\nu(t_0)$ or $S_\nu(t_0)$ is δ. The rectangles $R_\nu(t_0)$ are contained in the half-strip $\alpha < \sigma < \beta$, $t > \gamma_0$ for $t_0 > \gamma_0 + \frac{9}{2}$.

We shall begin with some lemmas, which may be proved without difficulty from the assumptions of Theorem 1.

Lemma 1. If for $t_0 > \gamma_0 + \frac{9}{2}$ we put

$$K(t_0) = \max_{R_7(t_0)} |f(s)|, \qquad K_n(t_0) = \max_{R_7(t_0)} |f_n(s)|, \qquad L_n(t_0) = \max_{R_7(t_0)} |f(s) - f_n(s)|,$$

the functions $K(t_0)^p$, $K_n(t_0)^p$, $L_n(t_0)^p$ possess mean values, and

$$\underset{t_0}{M}\{L_n(t_0)^p\} \to 0 \quad \text{and} \quad \underset{t_0}{M}\{K_n(t_0)^p\} \to \underset{t_0}{M}\{K(t_0)^p\} \quad \text{as} \quad n \to \infty.$$

Proof. If $g(s)$ is regular in $|s - s_0| \leqq \delta$, the mean value

$$M(\varrho) = \frac{1}{2\pi}\int_0^{2\pi} |g(s_0 + \varrho e^{i\theta})|^p \, d\theta$$

is, according to Hardy [1], increasing for $0 \leqq \varrho \leqq \delta$. Hence

$$|g(s_0)|^p = M(0) \leqq \frac{1}{\frac{1}{2}\delta^2}\int_0^{\delta} M(\varrho)\,\varrho\, d\varrho = \frac{1}{\pi\delta^2}\iint\limits_{|s-s_0|\leqq\delta} |g(s)|^p \, d\sigma\, dt.$$

Consequently

$$L_n(t_0)^p \leqq \frac{1}{\pi\delta^2}\iint\limits_{R_8(t_0)} |f(s) - f_n(s)|^p \, d\sigma\, dt.$$

The mean convergence therefore implies that

(11) $$\underset{t_0}{\overline{M}}\{L_n(t_0)^p\} \to 0 \quad \text{as} \quad n \to \infty.$$

Now,

$$K(t_0) \leqq K_n(t_0) + L_n(t_0) \quad \text{and} \quad K_n(t_0) \leqq K(t_0) + L_n(t_0).$$

Hence[1]

(12) $$\left|\left[\frac{1}{\delta - \gamma}\int_\gamma^\delta K(t_0)^p\, dt_0\right]^P - \left[\frac{1}{\delta - \gamma}\int_\gamma^\delta K_n(t_0)^p\, dt_0\right]^P\right| \leqq \left[\frac{1}{\delta - \gamma}\int_\gamma^\delta L_n(t_0)^p\, dt_0\right]^P,$$

where $P = 1$ if $p \leqq 1$ and $P = 1/p$ if $p > 1$. From the almost periodicity of $f_n(s)$ in $[\alpha, \beta]$ it follows that $K_n(t_0)$ is almost periodic. Hence $K_n(t_0)^p$ possesses a mean value. The inequality (12) together with (11) then shows, that $K(t_0)^p$ possesses

[1] See e. g. Hardy, Littlewood and Pólya [1], Theorems 28, 198, and 199.

a mean value, and that $\underset{t_0}{M}\{K_n(t_0)^p\} \to \underset{t_0}{M}\{K(t_0)^p\}$ as $n \to \infty$. Finally, the mean value of $L_n(t_0)^p$ will exist, since $f_m(s) - f_n(s)$ converges in the mean towards $f(s) - f_n(s)$ for $m \to \infty$.

Lemma 2. $K(t_0)^p = o(t_0)$ as $t_0 \to \infty$.

Proof. Since $|f(s)|$ takes the value $K(t_0)$ in some point of $R_7(t_0)$, there will (for $t_0 > \gamma_0 + \frac{11}{2}$) exist an interval of length $\geqq 1$ containing t_0 in which $K(t) \geqq K(t_0)$. Hence

$$K(t_0)^p \leqq \int_{t_0-1}^{t_0+1} K(t)^p \, dt,$$

and the right hand side is $o(t_0)$, since $\underset{t}{M}\{K(t)^p\}$ exists.

Lemma 3. There exists a constant $k > 0$ such that for all t_0

$$\max_{S_0(t_0)} |f(s)| \geqq k \quad \text{and} \quad \max_{S_0(t_0)} |f_n(s)| \geqq k \quad \text{for all } n.$$

Proof. Since the functions are almost periodic in $[\alpha_0, \beta_0]$ and not identically zero, and since $f_n(s)$ converges uniformly in $[\alpha_0, \beta_0]$ towards $f(s)$, there exists a constant $h > 0$ and a bounded closed sub-set S of the strip (α_0, β_0) such that for every function $f(s + i t_0)$ or $f_n(s + i t_0)$ the absolute value is $\geqq h$ in some point of S.

If the lemma were false, we could extract from the system of functions $f(s + i t_0)$ and $f_n(s + i t_0)$ a sequence converging uniformly towards zero in $S_0(0)$. Since the functions are uniformly bounded in $[\alpha_0, \beta_0]$, this sequence would converge uniformly to zero in any bounded closed sub-set of (α_0, β_0), in particular in S, and this is impossible.

13. We shall also use certain general function theoretic lemmas.[1] Let $F(s)$ be a regular function in $R_3(0)$ for which

$$\max_{S_0(0)} |F(s)| \geqq k,$$

where k is a given positive number.[2] The lemmas will give estimates depending on the number

$$K = \max_{R_3(0)} |F(s)| \qquad (\geqq k).$$

[1] Some of these lemmas are well known, but for the convenience of the reader the proofs are given.

[2] When the lemmas are applied k will be the number of Lemma 3.

By A we denote constants (not necessarily the same at each occurrence) depending on the rectangles involved and on k (but not on $F(s)$). In one of the lemmas the constant depends on a parameter m and is therefore denoted by $A(m)$. Besides $R_3(0)$, the rectangles $R_2(0)$, $R_1(0)$, and $S_0(0)$ occur. Any sequence of four rectangles, each of which contains the next in its interior, would do. Later on we shall sometimes use the lemmas for other sets of four rectangles.

Lemma 4. The number N of zeros of $F(s)$ in $R_2(0)$ satisfies an inequality

$$N \leqq A \log (K + 1).$$

Proof. Let s_0 be a point of $S_0(0)$ in which $|F(s)| \geqq k$. Let $z = z(s)$ be a regular function in $R_3(0)$ which maps $R_3(0)$ on the circle $|z| \leqq 1$ so that $z(s_0) = 0$, and let $s = s(z)$ be its inverse function.[1] The image of $R_2(0)$ will depend on s_0, but will for all possible s_0 be contained in a circle $|z| \leqq \varrho < 1$, where ϱ is independent of s_0. Applying Jensen's inequality to the function $H(z) = F(s(z))$ we obtain

$$\frac{k}{\varrho^N} \leqq K \quad \text{or} \quad N \leqq \frac{1}{\log \frac{1}{\varrho}} \log \frac{K}{k},$$

whence the desired result.[2]

The next two lemmas will be proved together.

Lemma 5. If $s_1, \ldots, s_N$ are the zeros of $F(s)$ in $R_2(0)$, and we put

$$F(s) = F_1(s) \prod_{n=1}^{N} (s - s_n),$$

then in $R_1(0)$

$$\log |F_1(s)| \geqq - A \log (K + 1), \quad \text{i. e.} \quad |F_1(s)| \geqq \frac{1}{(K+1)^A}.$$

Lemma 6. The left or right variation V of the argument of $F(s)$ along an arbitrary straight segment in $R_1(0)$ satisfies an inequality

$$|V| \leqq A \log (K + 1).$$

Proof. The function $F_1(s)$ is regular in $R_3(0)$ and $\neq 0$ in $R_2(0)$. If d denotes the diameter of $R_2(0)$, we have $|F_1(s_0)| \geqq k/d^N$, where s_0 is the point introduced in the proof of Lemma 4. On the frontier of $R_3(0)$, and hence in $R_3(0)$, we have $|F_1(s)| \leqq K/\delta^N$.

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[1] Thus $s = s(z)$ is continuous in $|z| \leqq 1$ and regular except in four points on $|z| = 1$ corresponding to the vertices of $R_3(0)$.

[2] Since $K/k \leqq K + 1$ when $k \geqq 1$, and $K/k \leqq (K+1)^{1/k}$ when $k < 1$. The expression $\log (K+1)$ is introduced for the sake of uniformity throughout the lemmas.

Let $z_1 = z_1(s)$ be a regular function in $R_2(0)$ which maps $R_2(0)$ on the circle $|z_1| \leqq 1$ such that $z_1(s_0) = 0$, and let $s = s(z_1)$ be the inverse function. The image of $R_1(0)$ will be contained in a circle $|z_1| \leqq \varrho_1 < 1$, where ϱ_1 is independent of s_0. Applying Carathéodory's inequalities[1] to a branch of the function $\log H_1(z_1) = \log |H_1(z_1)| + i \arg H_1(z_1)$, where $H_1(z_1) = F_1(s(z_1))$, we obtain for $|z_1| \leqq \varrho_1$ the inequalities

$$\log |H_1(z_1)| \geqq \frac{1+\varrho_1}{1-\varrho_1} \log \frac{k}{d^N} - \frac{2\varrho_1}{1-\varrho_1} \log \frac{K}{\delta^N}$$

and

$$|\arg H_1(z_1) - \arg H_1(0)| \leqq \frac{2\varrho_1}{1-\varrho_1^2}\left(\log \frac{K}{\delta^N} - \log \frac{k}{d^N}\right).$$

Hence in $R_1(0)$

$$(13) \qquad \log |F_1(s)| \geqq -\left(\frac{2\varrho_1}{1-\varrho_1} \log \frac{K}{\delta^N} - \frac{1+\varrho_1}{1-\varrho_1} \log \frac{k}{d^N}\right)$$

and

$$|\arg F_1(s) - \arg F_1(s_0)| \leqq \frac{2\varrho_1}{1-\varrho_1^2} \log \frac{K d^N}{k \delta^N}.$$

The latter inequality gives the estimate

$$(14) \qquad |V| \leqq \frac{4\varrho_1}{1-\varrho_1^2} \log \frac{K d^N}{k \delta^N} + N\pi.$$

In (13) and (14) we may by Lemma 4 replace N by $A \log (K+1)$, since $d > 1$ and $\delta < 1$. We then obtain the desired results.[2]

Lemma 7. There exists a horizontal segment $\alpha_1 \leqq \sigma \leqq \beta_1$, $t = t^*$ in $R_0(0)$ on which $F(s) \neq 0$ and

$$\left|\frac{d}{d\sigma} \arg F(\sigma + it)\right| \leqq A \log^2 (K+1).$$

Proof. Let $s_n = \sigma_n + i t_n$ be the zeros introduced in Lemma 5 and let t^* be chosen in the interval $|t^*| \leqq \frac{1}{2}$ such that $\min_n \{|t^* - t_n|\}$ is as large as possible. By Lemma 4 the distance of the segment $\alpha_1 \leqq \sigma \leqq \beta_1$, $t = t^*$ from the zeros s_n and the frontier of $R_1(0)$ is $\geqq 1/A \log (K+1)$, and we may therefore find a rectangle $\alpha_1 - r \leqq \sigma \leqq \beta_1 + r$, $|t - t^*| \leqq r$ belonging to $R_1(0)$, where $r \geqq 1/A \log (K+1)$, in which $F(s) \neq 0$. It follows from Lemma 6 that, if s^* lies on the segment, then

[1] See e. g. Carathéodory [1], § 74. The inequalities as there given must be applied to the function $f(z_1) = a \log H_1(z_1) + b$ for suitable values of a and b.

[2] For the first estimate we have to apply that, since $K \geqq k$, a constant (depending on the rectangles and on k) will be $\leqq$ an expression $A \log (K+1)$.

a branch of $\arg F(s)$ satisfies in this rectangle, and a fortiori in the circle $|s - s^*| \leqq r$, an inequality

$$|\arg F(s) - \arg F(s^*)| \leqq A \log (K + 1).$$

By the familiar inequality[1]

$$|u'_x(0, 0)| \leqq \frac{4M}{\pi r}$$

for a harmonic function $u(x, y)$ in the circle $x^2 + y^2 \leqq r^2$, for which $|u(x, y)| \leqq M$, we obtain the desired result.

Lemma 8. If we put

$$|F(s)|_m = \max \{|F(s)|, m\} \quad \text{for} \quad 0 < m \leqq 1,$$

the integral

$$I = \int_\gamma^\delta (\log |F(\sigma + it)|_m - \log |F(\sigma + it)|)\, dt \qquad (\geqq 0)$$

satisfies for $-\frac{1}{2} \leqq \gamma < \delta \leqq \frac{1}{2}$ and $\alpha_1 \leqq \sigma \leqq \beta_1$ an inequality

(15) $$I \leqq A(m) \log^2 (K + 1),$$

where $A(m) \to 0$ as $m \to 0$.[2]

Proof. Since $m \leqq 1$

$$\log |F(\sigma + it)|_m - \log |F(\sigma + it)| \leqq -\log^- |F(\sigma + it)|.^3$$

Hence by Lemma 5 for $\alpha_1 \leqq \sigma \leqq \beta_1$ and $|t| \leqq \frac{1}{2}$

$$\log |F(\sigma + it)|_m - \log |F(\sigma + it)| \leqq -\log^- |F_1(s)| - \sum_{n=1}^{N} \log^- |s - s_n|$$

$$\leqq A \log (K + 1) - \sum_{n=1}^{N} \log^- |t - t_n|.$$

Since

$$-\int_\gamma^\delta \log^- |t - t_n|\, dt < -\int_{-1}^{1} \log^- |u|\, du = 2$$

for any t_n, we find that

$$I \leqq A \log (K + 1) + 2N,$$

and hence, on using Lemma 4, that

(16) $$I \leqq A \log (K + 1).$$

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[1] Cf. Schwarz [1], § 6.

[2] Instead of $\log^2 (K + 1)$ we might use any positive function which does not take arbitrarily small values and which tends to infinity more rapidly than $\log (K + 1)$.

[3] In analogy to the notation $\log^+ x$ for the function $\max \{\log x, 0\}$, $x > 0$, we denote by $\log^- x$ the function $\min \{\log x, 0\}$, $x > 0$. The function $-\log^- x$ is non-negative and decreasing; moreover, if $x = x_1 \ldots x_N$, we have $-\log^- x \leqq -\log^- x_1 - \cdots - \log^- x_N$.

This estimate does not depend on m and implies (15) for every m. It remains to prove that if $\varepsilon > 0$ is given, then $I \leqq \varepsilon \log^2 (K+1)$ for $-\frac{1}{2} \leqq \gamma < \delta \leqq \frac{1}{2}$ and $\alpha_1 \leqq \sigma \leqq \beta_1$, provided that m is sufficiently small. From (16) it follows that $I \leqq \varepsilon \log^2 (K+1)$ for all m, provided that $K \geqq$ (some) K_0 depending on ε. Thus, to complete the proof we must show that $I \leqq \varepsilon \log^2 (K+1)$ for $K \leqq K_0$, when m is sufficiently small. It will be sufficient to prove that $I \leqq \varepsilon \log^2 (k+1)$.

When $K \leqq K_0$, Lemma 4 shows that $N \leqq N_0 = A \log (K_0 + 1)$, and Lemma 5 then shows that when s belongs to $R_1(0)$ and all $|s - s_n| \geqq r$, where $0 < r < 1$, then

$$|F(s)| \geqq \frac{r^N}{(K+1)^A} \geqq \frac{r^{N_0}}{(K_0+1)^A}.$$

Thus, if we put

$$m = \frac{r^{N_0}}{(K_0+1)^A},$$

the total length of those sub-intervals of $|t| \leqq \frac{1}{2}$ in which the integrand in I is positive, is at most $N 2 r$. Consequently

$$I \leqq A \log (K+1) N 2 r - N \int_{-Nr}^{Nr} \log^- |u| \, du$$

$$\leqq A \log (K_0 + 1) N_0 2 r - N_0 \int_{-N_0 r}^{N_0 r} \log^- |u| \, du.$$

The last expression tends to zero as $r \to 0$. Hence $I \leqq \varepsilon \log^2 (k+1)$ when r is sufficiently small, i. e. when m is sufficiently small.

Connected with Lemma 8 is the following lemma, which will be used later on in the proof of Theorem 3.

Lemma 9. The integral

$$J = \int_\gamma^\delta \log |F(\sigma + it)| \, dt$$

satisfies for $-\frac{1}{2} \leqq \gamma < \delta \leqq \frac{1}{2}$ and $\alpha_1 \leqq \sigma \leqq \beta_1$ an inequality

$$|J| \leqq A \log (K+1).$$

Proof. We have

$$J = \int_\gamma^\delta \log^+ |F(\sigma + it)| \, dt - \int_\gamma^\delta - \log^- |F(\sigma + it)| \, dt.$$

The first integral on the right is $\leqq \log^+ K < \log (K+1)$. The second integral is the integral I of Lemma 8, for $m = 1$, which by (16) is $\leqq A \log (K+1)$.

Remark. All the preceding lemmas remain valid if in the estimates on the right we replace $\log(K+1)$ or $\log^2(K+1)$ by K^q, where q is a given positive number.

14. We now turn to the proof of Theorem 1. On account of Lemma 3 we may apply the lemmas of § 13 to the functions $f(s+it_0)$ and $f_n(s+it_0)$ and hereby obtain estimates on the functions $f(s)$ and $f_n(s)$ in $R_3(t_0)$. Instead of the number K we may use the numbers $K(t_0)$ and $K_n(t_0)$ introduced in Lemma 1.

From Lemma 8[1] it thus follows that for $\alpha_1 \leqq \sigma \leqq \beta_1$

$$\int_{t_0-\frac{1}{2}}^{t_0+\frac{1}{2}} (\log|f(\sigma+it)|_m - \log|f(\sigma+it)|)\,dt \leqq A(m)\,K(t_0)^p$$

and

$$\int_{t_0-\frac{1}{2}}^{t_0+\frac{1}{2}} (\log|f_n(\sigma+it)|_m - \log|f_n(\sigma+it)|)\,dt \leqq A(m)\,K_n(t_0)^p.$$

Let us now suppose, as we may according to § 10, that $p \leqq 1$. Then if u_1 and u_2 are both $\geqq m$

$$|\log u_2 - \log u_1| \leqq \log(m + |u_2 - u_1|) - \log m \leqq a\,|u_2 - u_1|^p,$$

where a is the (finite) upper bound of $(\log(m+x) - \log m)/x^p$ for $x > 0$. Hence

$$\Big|\log|f(s)|_m - \log|f_n(s)|_m\Big| \leqq a\,\Big||f(s)|_m - |f_n(s)|_m\Big|^p \leqq a\,|f(s) - f_n(s)|^p$$

and consequently for $\alpha_1 \leqq \sigma \leqq \beta_1$

$$\int_{t_0-\frac{1}{2}}^{t_0+\frac{1}{2}} \Big|\log|f(\sigma+it)|_m - \log|f_n(\sigma+it)|_m\Big|\,dt \leqq a\,L_n(t_0)^p.$$

It follows that for $\alpha_1 \leqq \sigma \leqq \beta_1$

$$\left|\frac{1}{\delta-\gamma}\int_{\gamma}^{\delta} \log|f(\sigma+it)|\,dt - \frac{1}{\delta-\gamma}\int_{\gamma}^{\delta} \log|f_n(\sigma+it)|\,dt\right|$$

$$\leqq A(m)\frac{1}{\delta-\gamma}\int_{\gamma-\frac{1}{2}}^{\delta+\frac{1}{2}} K(t_0)^p\,dt_0 + A(m)\frac{1}{\delta-\gamma}\int_{\gamma-\frac{1}{2}}^{\delta+\frac{1}{2}} K_n(t_0)^p\,dt_0 + a\frac{1}{\delta-\gamma}\int_{\gamma-\frac{1}{2}}^{\delta+\frac{1}{2}} L_n(t_0)^p\,dt_0.$$

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[1] With $\gamma = -\frac{1}{2}$, $\delta = \frac{1}{2}$ and with $A(m)K^p$ on the right instead of $A(m)\log^2(K+1)$.

For fixed γ, and $\delta \to \infty$, the expression on the right converges by Lemma 1 towards

$$A(m) \underset{t_0}{M} \{K(t_0)^p\} + A(m) \underset{t_0}{M} \{K_n(t_0)^p\} + a \underset{t_0}{M} \{L_n(t_0)^p\}, \tag{17}$$

and this expression converges for $n \to \infty$, again by Lemma 1, towards

$$2\,A(m) \underset{t_0}{M} \{K(t_0)^p\}. \tag{18}$$

Let $\varepsilon > 0$ be given, and let m be chosen so small that the expression (18) is $< \varepsilon$. Then the expression (17) is $< \varepsilon$ for $n \geqq$ (some) n_0, and consequently

$$\left| \frac{1}{\delta - \gamma} \int_\gamma^\delta \log |f(\sigma + it)|\, dt - \frac{1}{\delta - \gamma} \int_\gamma^\delta \log |f_n(\sigma + it)|\, dt \right| < \varepsilon$$

for $\alpha_1 \leqq \sigma \leqq \beta_1$ if $n \geqq n_0$ and $\delta \geqq$ (some) $\delta_0 = \delta_0(n)$. For every fixed n we know (cf. § 5) that the limit

$$\varphi_{f_n}(\sigma) = \lim_{\delta \to \infty} \frac{1}{\delta - \gamma} \int_\gamma^\delta \log |f_n(\sigma + it)|\, dt$$

exists uniformly for $\alpha_1 \leqq \sigma \leqq \beta_1$. It follows that

$$\varphi_f(\sigma) = \lim_{\delta \to \infty} \frac{1}{\delta - \gamma} \int_\gamma^\delta \log |f(\sigma + it)|\, dt$$

also exists uniformly for $\alpha_1 \leqq \sigma \leqq \beta_1$ and that

$$|\varphi_f(\sigma) - \varphi_{f_n}(\sigma)| \leqq \varepsilon$$

for $\alpha_1 \leqq \sigma \leqq \beta_1$ if $n \geqq n_0$. This establishes the first part of the theorem.

15. The convexity of $\varphi_f(\sigma)$ follows immediately from the convexity of the functions $\varphi_{f_n}(\sigma)$. It will be sufficient to prove (9) for $\alpha_1 < \sigma < \beta_1$ and (10) for $\alpha_1 < \sigma_1 < \sigma_2 < \beta_1$.

Since γ may be chosen arbitrarily, we may suppose that $f(s)$ has no zeros on the segment $\alpha_1 \leqq \sigma \leqq \beta_1$, $t = \gamma$. By Lemmas 2, 4, and 6 it makes no difference if in the definition of the mean motions and the frequencies of zeros we restrict δ to a set of values, so that any interval $|t - t_0| \leqq \frac{1}{2}$, where $t_0 > \gamma + \frac{1}{2}$, contains at least one value from the set.

Let us first merely suppose that δ is restricted to values for which $f(s)$ has no zeros on the segment $\alpha_1 \leqq \sigma \leqq \beta_1$, $t = \delta$. Then by Cauchy's theorem, applied to the rectangle $(\alpha_1 <)\,\sigma_1 < \sigma < \sigma_2\,(< \beta_1)$, $\gamma < t < \delta$,

(19) $$N_f(\sigma_1, \sigma_2; \gamma, \delta) =$$

$$\frac{1}{2\pi}\left[(\arg^- f(\sigma_2 + i\delta) - \arg^- f(\sigma_2 + i\gamma)) - (\arg^+ f(\sigma_1 + i\delta) - \arg^+ f(\sigma_1 + i\gamma)) + R(\sigma_1, \sigma_2; \gamma, \delta)\right],$$

where $R(\sigma_1, \sigma_2; \gamma, \delta)$ denotes the contribution to the variation of the argument from the horizontal sides of the rectangle. By Lemmas 2 and 6

$$R(\sigma_1, \sigma_2; \gamma, \delta) = o(\delta).$$

Hence

$$\frac{1}{2\pi}(\underline{c}_f^-(\sigma_2) - \bar{c}_f^+(\sigma_1)) \leqq \underline{H}_f(\sigma_1, \sigma_2) \leqq \bar{H}_f(\sigma_1, \sigma_2) \leqq \frac{1}{2\pi}(\bar{c}_f^-(\sigma_2) - \underline{c}_f^+(\sigma_1)),$$

so that (10) is a consequence of (9). Of the inequalities (9) it is sufficient to prove

(20) $$\varphi_f'(\sigma - 0) \leqq \underline{c}_f^-(\sigma) \quad \text{and} \quad \bar{c}_f^+(\sigma) \leqq \varphi_f'(\sigma + 0),$$

the others being trivial.

For any t_0 for which $f(s)$ has no zeros on the segment $\alpha_1 \leqq \sigma \leqq \beta_1$, $t = t_0$ put

$$\max_{\alpha_1 \leqq \sigma \leqq \beta_1} \left| \frac{d}{d\sigma} \arg f(\sigma + i t_0) \right| = C(t_0).$$

Then

$$R(\sigma_1, \sigma_2; \gamma, \delta) \leqq (C(\gamma) + C(\delta))(\sigma_2 - \sigma_1).$$

By Lemmas 2 and 7, together with the above remark, we may suppose that δ is restricted to values for which $C(\delta) = o(\delta)$.

The remainder of the proof now follows that of ordinary almost periodic functions, and a brief indication of how it runs will suffice.[1] The function

$$\varphi_f(\sigma; \gamma, \delta) = \frac{1}{\delta - \gamma} \int_\gamma^\delta \log |f(\sigma + i t)|\, dt$$

is continuous and stretchwise differentiable and has the left and right derivatives

(21) $$\varphi_f'(\sigma - 0; \gamma, \delta) = \frac{\arg^- f(\sigma + i\delta) - \arg^- f(\sigma + i\gamma)}{\delta - \gamma} \quad \text{and}$$

$$\varphi_f'(\sigma + 0; \gamma, \delta) = \frac{\arg^+ f(\sigma + i\delta) - \arg^+ f(\sigma + i\gamma)}{\delta - \gamma}.$$

[1] For details see Jessen and Tornehave [1], pp. 186—187.

The relation (19) therefore takes the form

$$\frac{N_f(\sigma_1, \sigma_2; \gamma, \delta)}{\delta - \gamma} = \frac{1}{2\pi}(\varphi_f'(\sigma_2 - 0; \gamma, \delta) - \varphi_f'(\sigma_1 + 0; \gamma, \delta) + r(\sigma_1, \sigma_2; \gamma, \delta)),$$

where

$$|r(\sigma_1, \sigma_2; \gamma, \delta)| \leqq \frac{C(\gamma) + C(\delta)}{\delta - \gamma}(\sigma_2 - \sigma_1).$$

Since $N_f(\sigma_1, \sigma_2; \gamma, \delta) \geqq 0$ the function

$$\varphi_*(\sigma; \gamma, \delta) = \varphi_f(\sigma; \gamma, \delta) + \frac{C(\gamma) + C(\delta)}{2(\delta - \gamma)}\sigma^2$$

is convex, and since $C(\delta) = o(\delta)$ we have uniformly in (α_1, β_1)

$$\varphi_f(\sigma) = \lim_{\delta \to \infty} \varphi_*(\sigma; \gamma, \delta).$$

This shows, once more, that $\varphi_f(\sigma)$ is convex, and also that for every σ in (α_1, β_1)

$$\varphi_f'(\sigma - 0) \leqq \liminf_{\delta \to \infty} \varphi_*'(\sigma - 0; \gamma, \delta) = \liminf_{\delta \to \infty} \varphi_f'(\sigma - 0; \gamma, \delta)$$

and

$$\limsup_{\delta \to \infty} \varphi_f'(\sigma + 0; \gamma, \delta) = \limsup_{\delta \to \infty} \varphi_*'(\sigma + 0; \gamma, \delta) \leqq \varphi_f'(\sigma + 0).$$

Combining this with (21) we find the inequalities (20).

16. Next we turn to the proof of Theorem 2.

By the definition of the integral we have

$$\Lambda(y; \mu_{f,\sigma;\gamma,\delta}) = \frac{1}{\delta - \gamma}\int_\gamma^\delta e^{if(\sigma+it)y}\,dt.$$

Let us suppose, as we may, that $p \leqq 1$. Then, if u_1, u_2, and y are arbitrary complex numbers,

$$|e^{iu_2y} - e^{iu_1y}| \leqq \min\{2, |(u_2 - u_1)y|\} \leqq \min\{2, |u_2 - u_1|\,|y|\} \leqq c\,|u_2 - u_1|^p\,|y|^p,$$

where c is the (finite) upper bound of $\min\{2, x\}/x^p$ for $x > 0$. Hence, if $\alpha_1 \leqq \sigma \leqq \beta_1$ and $|y| \leqq a$

$$\left|\frac{1}{\delta - \gamma}\int_\gamma^\delta e^{if(\sigma+it)y}\,dt - \frac{1}{\delta - \gamma}\int_\gamma^\delta e^{if_n(\sigma+it)y}\,dt\right| \leqq c\,a^p\frac{1}{\delta - \gamma}\int_\gamma^\delta L_n(t)^p\,dt.$$

For fixed γ, and $\delta \to \infty$, the expression on the right converges towards $c\,a^p \underset{t}{M}\{L_n(t)^p\}$, which converges towards zero as $n \to \infty$. For every fixed n we know (cf. § 7) that the limit

$$\Lambda(y;\mu_{f_n,\sigma}) = \lim_{\delta\to\infty} \frac{1}{\delta-\gamma}\int_{\gamma}^{\delta} e^{i f_n(\sigma+it) y}\, dt$$

exists uniformly in $|y| \leqq a$. Consequently

$$\lim_{\delta\to\infty} \frac{1}{\delta-\gamma}\int_{\gamma}^{\delta} e^{i f(\sigma+it) y}\, dt$$

also exists uniformly in $|y| \leqq a$ and is the limit of $\Lambda(y;\mu_{f_n,\sigma})$ as $n\to\infty$ uniformly in $|y| \leqq a$.

This establishes the theorem.

Extension of the Results to the Logarithm of a Generalized Analytic Almost Periodic Function.

17. On certain additional assumptions the previous theorems may be extended to the logarithm of $f(s)$, or rather a branch of $\log f(s)$. Since this branch will be discontinuous on certain cuts it will be necessary to make some additions to the definitions in § 4 of the left or right variation of the argument along a segment.

Let $f(s)$ be regular in a vertical half-strip (or a rectangle with sides parallel to the axes) and suppose that $f(s)$ has no zeros on the vertical line $\sigma=\sigma^*$. Let $g(s)$ denote a branch of $\log f(s)$ in the domain Δ obtained from the half-strip (or rectangle) by omitting all points on the half-lines $-\infty<\sigma\leqq\sigma_0$, $t=t_0$, where σ_0+it_0 denote the zeros of $f(s)$ with $\sigma_0<\sigma^*$, and on the half-lines $\sigma_0\leqq\sigma<+\infty$, $t=t_0$, where σ_0+it_0 denote the zeros of $f(s)$ with $\sigma_0>\sigma^*$. On the cuts we may define border values of $g(s)$ from each side except in the zeros of $f(s)$.[1] When s approaches a zero, $g(s)$ will vary in a horizontal strip of the complex plane, and the real part of $g(s)$ will approach $-\infty$.

We shall now define what we will mean by the left or right variation of the argument of $g(s)$ along an arbitrary vertical segment $s=\sigma+it$, $t_1\leqq t\leqq t_2$, or horizontal segment $s=\sigma+it$, $\sigma_1\leqq\sigma\leqq\sigma_2$, which contains points of the cuts.

For a vertical segment, which belongs to Δ with the exception of one end-point which is no zero of $f(s)$, we define the variations as those which we should

[1] Naturally the half-line corresponding to a zero may contain other zeros, so that a cut may contain zeros of $f(s)$ besides the end-point.

obtain, if $g(s)$ were continued across the cut in this point. If the end-point is a zero of $f(s)$ we define the variations as the limits of the variations along a smaller segment obtained by replacing the end-point with a point of the segment, which converges towards the end-point. The limits will exist in virtue of the above remark on the variation of $g(s)$. An arbitrary vertical segment, which contains points of the cuts, may be divided into segments of the above types,[1] and we define the variations for the segment as the sums of the variations for the parts.

For a horizontal segment, which lies on a cut and contains no zero of $f(s)$, we define the left and right variation as the left or right variation along the segment of the function obtained by continuing $g(s)$ across the cut from the left or right side respectively. For a horizontal segment, one end-point of which is a zero of $f(s)$, but which otherwise contains no zero of $f(s)$, we define the variations as the limits of the variations along a smaller segment obtained by replacing the end-point with a point of the segment, which converges towards the end-point. An arbitrary horizontal segment, which contains points of the cuts, may be divided into segments of the above types, and we define the variations for the segment as the sums of the variations for the parts.

It is easily seen that the left and right variations along the vertical segment $s=\sigma+it$, $t_1 \leqq t \leqq t_2$, considered as functions of σ for fixed t_1 and t_2, are continuous from the left and right respectively. Similarly, the left and right variations along the horizontal segment $s=\sigma+it$, $\sigma_1 \leqq \sigma \leqq \sigma_2$, considered as functions of t for fixed σ_1 and σ_2, are continuous from the right and left respectively.

18. We shall prove the following theorems.

Theorem 3. *Let $f(s)$ and $f_n(s)$ be as in Theorem* I, *and suppose that $f_n(s)=e^{g_n(s)}$, where $g_n(s)$ is a sequence of functions almost periodic in $[\alpha, \beta]$ converging uniformly in $[\alpha_0, \beta_0]$ towards a function $g(s)$, which is then almost periodic in $[\alpha_0, \beta_0]$ and satisfies $f(s)=e^{g(s)}$. Suppose also that none of the functions $f(s)$ or $f_n(s)$ is constant. Let the branch $g(s)=\log f(s)$ be continued in the domain Δ obtained by omitting from the half-strip $\alpha<\sigma<\beta$, $t>\gamma_0$ all segments $\alpha<\sigma\leqq\sigma_0$, $t=t_0$, where σ_0+it_0 denote the zeros of $f(s)$ with $\sigma_0\leqq\alpha_0$, and all segments $\sigma_0\leqq\sigma<\beta$, $t=t_0$, where σ_0+it_0 denote the zeros of $f(s)$ with $\sigma_0\geqq\beta_0$.*

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[1] As we are considering closed segments a point of division must be counted to both of the adjoining segments.

Then the Jensen function

$$\varphi_g(\sigma) = \underset{t}{M}\{\log|g(\sigma + it)|\}$$

exists uniformly in $[\alpha, \beta]$, *i. e. the function*

$$\varphi_g(\sigma; \gamma, \delta) = \frac{1}{\delta - \gamma}\int_\gamma^\delta \log|g(\sigma + it)|\, dt$$

converges for $\delta \to \infty$ *for any fixed* $\gamma > \gamma_0$ *uniformly in* $[\alpha, \beta]$ *towards a limit function* $\varphi_g(\sigma)$. *The Jensen function* $\varphi_{g_n}(\sigma)$ *of* $g_n(\sigma)$ *converges for* $n \to \infty$ *uniformly in* $[\alpha, \beta]$ *towards* $\varphi_g(\sigma)$.

The function $\varphi_g(\sigma)$ *is convex in* (α, β), *and, for every* σ *in* (α, β), *the four mean motions defined by*

$$\left.\begin{matrix}\underline{c}_g^-(\sigma)\\ \bar{c}_g^-(\sigma)\end{matrix}\right\} = \lim_{\delta\to\infty} \begin{matrix}\inf\\ \sup\end{matrix} \frac{V_g^-(\sigma; \gamma, \delta)}{\delta - \gamma} \quad \textit{and} \quad \left.\begin{matrix}\underline{c}_g^+(\sigma)\\ \bar{c}_g^+(\sigma)\end{matrix}\right\} = \lim_{\delta\to\infty} \begin{matrix}\inf\\ \sup\end{matrix} \frac{V_g^+(\sigma; \gamma, \delta)}{\delta - \gamma}, \tag{22}$$

where $V_g^-(\sigma; \gamma, \delta)$ *and* $V_g^+(\sigma; \gamma, \delta)$ *denote the left and right variation of the argument of* $g(s)$ *along the segment* $s = \sigma + it$, $\gamma \leqq t \leqq \delta$, *satisfy the inequalities*

$$\varphi_g'(\sigma - 0) \leqq \underline{c}_g^-(\sigma) \leqq \left\{\begin{matrix}\underline{c}_g^+(\sigma)\\ \bar{c}_g^-(\sigma)\end{matrix}\right\} \leqq \bar{c}_g^+(\sigma) \leqq \varphi_g'(\sigma + 0). \tag{23}$$

Further, for every strip (σ_1, σ_2) *where* $\alpha < \sigma_1 < \sigma_2 < \beta$, *the two relative frequencies of zeros defined by*

$$\left.\begin{matrix}\underline{H}_g(\sigma_1, \sigma_2)\\ \bar{H}_g(\sigma_1, \sigma_2)\end{matrix}\right\} = \lim_{\delta\to\infty} \begin{matrix}\inf\\ \sup\end{matrix} \frac{N_g(\sigma_1, \sigma_2; \gamma, \delta)}{\delta - \gamma}, \tag{24}$$

where $N_g(\sigma_1, \sigma_2; \gamma, \delta)$ *denotes the number of zeros of* $g(s)$ *in the part of the rectangle* $\sigma_1 < \sigma < \sigma_2$, $\gamma < t < \delta$ *which belongs to* Δ, *satisfy the inequalities*

$$\frac{1}{2\pi}(\varphi_g'(\sigma_2 - 0) - \varphi_g'(\sigma_1 + 0)) \leqq \underline{H}_g(\sigma_1, \sigma_2) \leqq \bar{H}_g(\sigma_1, \sigma_2) \leqq \frac{1}{2\pi}(\varphi_g'(\sigma_2 + 0) - \varphi_g'(\sigma_1 - 0)). \tag{25}$$

As a corollary we have, that if $\varphi_g(\sigma)$ is differentiable at the point σ, then the left and right mean motions

$$c_g^-(\sigma) = \lim_{\delta\to\infty} \frac{V_g^-(\sigma; \gamma, \delta)}{\delta - \gamma} \quad \text{and} \quad c_g^+(\sigma) = \lim_{\delta\to\infty} \frac{V_g^+(\sigma; \gamma, \delta)}{\delta - \gamma}$$

both exist and are determined by

$$c_g^-(\sigma) = c_g^+(\sigma) = \varphi_g'(\sigma).$$

Similarly, if $\varphi_g(\sigma)$ is differentiable at σ_1 and σ_2, then the relative frequency of zeros

$$H_g(\sigma_1, \sigma_2) = \lim_{\delta \to \infty} \frac{N_g(\sigma_1, \sigma_2; \gamma, \delta)}{\delta - \gamma}$$

exists and is determined by the Jensen formula

$$H_g(\sigma_1, \sigma_2) = \frac{1}{2\pi}(\varphi_g'(\sigma_2) - \varphi_g'(\sigma_1)).$$

Theorem 4. *The function $g(\sigma + it)$ possesses for every σ in (α, β) an asymptotic distribution function $\mu_{g,\sigma}$, i. e. the distribution function $\mu_{g,\sigma;\gamma,\delta}$ of $g(\sigma + it)$ in the interval $\gamma < t < \delta$, defined by*

$$\mu_{g,\sigma;\gamma,\delta}(E) = \frac{m(A_{g,\sigma;\gamma,\delta}(E))}{\delta - \gamma},$$

where $A_{g,\sigma;\gamma,\delta}(E)$ for an arbitrary Borel set E denotes the set of points of $\gamma < t < \delta$ for which $g(\sigma + it)$ belongs to E, converges for $\delta \to \infty$ for any fixed $\gamma > \gamma_0$ towards a distribution function $\mu_{g,\sigma}$. The asymptotic distribution function $\mu_{g_n,\sigma}$ of $g_n(\sigma + it)$ converges for $n \to \infty$ towards $\mu_{g,\sigma}$.

There are of course similar theorems for functions $g(s)$ which may be continued in a half-strip $\alpha < \sigma < \beta$, $t < \delta_0$. The limits must then be taken for $\gamma \to -\infty$ and fixed $\delta < \delta_0$. If both pairs of theorems are applicable, the Jensen function $\varphi_g(\sigma)$ and the asymptotic distribution function $\mu_{g,\sigma}$ will be the same in both cases.

We shall not go into the extension of the results of §§ 8—9, which will not be needed.

19. Since $f_n(s) = e^{g_n(s)}$ its Jensen function $\varphi_{f_n}(\sigma)$ is the real part of the mean value $\underset{t}{M}\{g_n(\sigma + it)\}$, which is constant in (α, β). Hence, by Theorem 1, the Jensen function $\varphi_f(\sigma)$ of $f(s)$ is also constant in (α, β), and, consequently, the relative frequency $H_f(\sigma_1, \sigma_2)$ of zeros of $f(s)$ exists and is equal to zero for any strip (σ_1, σ_2).

20. We shall need some more lemmas.

Lemma 10. On placing $\theta(t_0) = 0$ when $R_5(t_0)$ belongs to Δ, and $\theta(t_0) = 1$ otherwise, the mean value $\underset{t_0}{M}\{\theta(t_0)\}$ exists and is equal to zero.

Proof. This is an immediate consequence of § 19. For an arbitrary zero $\sigma^* + it^*$ of $f(s)$ with $\alpha_1 - 5\delta \leqq \sigma^* \leqq \beta_1 + 5\delta$ we put $\theta^*(t_0) = 1$ for $|t_0 - t^*| \leqq 3$, and $\theta^*(t_0) = 0$ otherwise. Then

$$\theta(t_0) \leqq \Sigma\, \theta^*(t_0),$$

where the sum is extended over all the zeros $\sigma^* + i t^*$ in question. Hence

$$\int_\gamma^\delta \theta(t_0)\, d t_0 \leqq \Sigma \int_\gamma^\delta \theta^*(t_0)\, d t_0 \leqq 6 N_f(\alpha_1 - 6\delta, \beta_1 + 6\delta; \gamma - 3, \delta + 3),$$

whence

$$\underset{t_0}{\overline{M}} \{\theta(t_0)\} \leqq 6 H_f(\alpha_1 - 6\delta, \beta_1 + 6\delta) = 0.^1$$

Lemma 11. For $0 < \varrho < 1$ put $\psi_n(\varrho, t_0) = 0$ when $R_5(t_0)$ belongs to Δ and

$$\max_{R_5(t_0)} | g(s) - g_n(s) | \leqq \varrho,$$

and $\psi_n(\varrho, t_0) = 1$ otherwise. Then, for ϱ fixed,

$$\Psi_n(\varrho) = \underset{t_0}{\overline{M}} \{\psi_n(\varrho, t_0)\} \to 0 \quad \text{as} \quad n \to \infty.$$

Proof. For an $\eta > 0$ put $\lambda_n(t_0) = 0$ when $L_n(t_0) \leqq \eta$, and $\lambda_n(t_0) = 1$ otherwise. Then by Lemma 1

(26) $$\Lambda_n = \underset{t_0}{\overline{M}} \{\lambda_n(t_0)\} \to 0 \quad \text{as} \quad n \to \infty.$$

Also, for a $K > 0$ put $\varkappa_n(t_0) = 0$ when $K_n(t_0) \leqq K$, and $\varkappa_n(t_0) = 1$ otherwise. Then by Lemma 1

(27) $$\mathrm{K}_n = \underset{t_0}{\overline{M}} \{\varkappa_n(t_0)\} \leqq \frac{M}{K^p},$$

where M denotes a constant exceeding all the mean values $\underset{t_0}{M} \{K_n(t_0)^p\}$.

Since $f_n(s)$ has no zeros it follows from Lemma 5 (with $R_{\nu+4}(0)$ instead of $R_\nu(0)$) that when $\varkappa_n(t_0) = 0$

$$\min_{R_5(t_0)} | f_n(s) | \geqq \frac{1}{(K + 1)^A}.$$

Hence, when $\lambda_n(t_0) = 0$ and $\varkappa_n(t_0) = 0$

(28) $$\max_{R_5(t_0)} \left| \frac{f(s)}{f_n(s)} - 1 \right| \leqq \eta (K + 1)^A.$$

Let η and K be chosen such that $\eta (K + 1)^A = 1 - e^{-\varrho}$. Then (28) implies (since $1 - e^{-\varrho} < 1$) that $f(s)$ has no zeros in $R_5(t_0)$, i. e. $R_5(t_0)$ belongs to Δ, and (28) may be written

(29) $$\max_{R_5(t_0)} | e^{g(s) - g_n(s)} - 1 | \leqq 1 - e^{-\varrho}.$$

[1] The lemma may also be proved without any appeal to Theorem 1, by means of the following lemma.

From (29) it follows that

$$\max_{R_5(t_0)} |g(s) - g_n(s) - \nu 2\pi i| \leqq \varrho$$

for some integer $\nu = \nu(n, t_0)$ which must be zero when $n \geqq$ (some) n_0, since $g_n(s)$ converges uniformly towards $g(s)$ in $[\alpha_0, \beta_0]$, and $\varrho < 1$.

Hence $\psi_n(\varrho, t_0) \leqq \lambda_n(t_0) + \varkappa_n(t_0)$, and, consequently, $\Psi_n(\varrho) \leqq \Lambda_n + K_n$ for $n \geqq n_0$. By (26) and (27) we obtain

$$\limsup_{n\to\infty} \Psi_n(\varrho) \leqq \frac{M}{K^p}.$$

Since K may be chosen arbitrarily large the lemma is proved.

Lemma 12. On placing $\chi(Q, t_0) = 0$ when $R_5(t_0)$ belongs to Δ and

$$\max_{R_5(t_0)} |g(s)| \leqq Q,$$

and $\chi(Q, t_0) = 1$ otherwise, we have

$$\mathrm{X}(Q) = \overline{M}_{t_0} \{\chi(Q, t_0)\} \to 0 \quad \text{as} \quad Q \to \infty.$$

Proof. Let Q_n denote the upper bound of $|g_n(s)|$ in the strip $\alpha_1 - 5\delta \leqq \sigma \leqq \beta_1 + 5\delta$. Then $\chi(Q, t_0) \leqq \psi_n(1, t_0)$, and hence $\mathrm{X}(Q) \leqq \Psi_n(1)$, if $Q \geqq Q_n + 1$. The lemma is therefore a consequence of Lemma 11.

Lemma 13. There exists a constant $k_1 > 0$ such that for all t_0

$$\max_{S_0(t_0)} |f(s) - 1| \geqq k_1 \quad \text{and} \quad \max_{S_0(t_0)} |f_n(s) - 1| \geqq k_1 \quad \text{for all } n,$$

$$\max_{S_0(t_0)} |f(s) + 1| \geqq k_1 \quad \text{and} \quad \max_{S_0(t_0)} |f_n(s) + 1| \geqq k_1 \quad \text{for all } n,$$

$$\max_{S_0(t_0)} |g(s)| \geqq k_1 \quad \text{and} \quad \max_{S_0(t_0)} |g_n(s)| \geqq k_1 \quad \text{for all } n.$$

Proof. Since none of the functions is constant, the proof runs as the proof of Lemma 3.

Lemma 14. There exists a constant $k_2 > 0$ such that for all t_0 either

$$\max_{S_0(t_0)} |f(s) - \overline{f(\bar{s} + 2it_0)}| \geqq k_2 \quad \text{or} \quad \max_{S_0(t_0)} |f(s)\overline{f(\bar{s} + 2it_0)} - 1| \geqq k_2,$$

and similarly, for every n, either

$$\max_{S_0(t_0)} |f_n(s) - \overline{f_n(\bar{s} + 2it_0)}| \geqq k_2 \quad \text{or} \quad \max_{S_0(t_0)} |f_n(s)\overline{f_n(\bar{s} + 2it_0)} - 1| \geqq k_2.$$ [1]

[1] Note that the point $\bar{s} + 2it_0$ is the symmetric point of s with respect to the line $t = t_0$.

Proof. If the lemma were false it would be possible to extract from the system of functions $f(s+it_0)$ and $f_n(s+it_0)$ a sequence of functions $h_\nu(s)$ for which $h_\nu(s) - \overline{h_\nu(\bar{s})} \to 0$ and $h_\nu(s)\overline{h_\nu(\bar{s})} \to 1$ uniformly in $S_0(0)$. Since the functions are uniformly bounded in $[\alpha_0, \beta_0]$ we may suppose without loss of generality that the sequence $h_\nu(s)$ converges uniformly in $S_0(0)$ (otherwise we consider a subsequence). The limit function $h(s)$ then satisfies the conditions $h(s) = \overline{h(\bar{s})}$ and $h(s)\overline{h(\bar{s})} = 1$, which show that $h(s)$ is either identically 1 or identically -1, and this is impossible by Lemma 13.

Lemma 15. There exists a constant C such that for all t_0

$$\max_{S_3(t_0)} |g(s)| \leqq C \quad \text{and} \quad \max_{S_3(t_0)} |g_n(s)| \leqq C \quad \text{for all } n.$$

Proof. This is an immediate consequence of the almost periodicity and the uniform convergence of $g_n(s)$ towards $g(s)$ in $[\alpha_0, \beta_0]$.[1]

21. We shall need some more general function theoretic lemmas. Let $F(s)$ be a regular function in $R_5(0)$ which has no zeros in the part of $R_5(0)$ which belongs to the strip (α_0, β_0), and let $G(s)$ be a branch of $\log F(s)$ in the domain obtained from $R_5(0)$ by omitting all segments $\alpha_1 - 5\delta \leqq \sigma \leqq \sigma_0$, $t = t_0$, where $\sigma_0 + it_0$ denote the zeros of $F(s)$ with $\sigma_0 \leqq \alpha_0$, and all segments $\sigma_0 \leqq \sigma \leqq \beta_1 + 5\delta$, $t = t_0$, where $\sigma_0 + it_0$ denote the zeros of $F(s)$ with $\sigma_0 \geqq \beta_0$. Suppose, as in § 13, that

$$\max_{S_0(0)} |F(s)| \geqq k,$$

and, further, that

$$\max_{S_0(0)} |F(s) - 1| \geqq k_1. \tag{30}$$

$$\max_{S_0(0)} |G(s)| \geqq k_1, \tag{31}$$

$$\max_{S_3(0)} |G(s)| \leqq C, \tag{32}$$

and that for every t_1 in $|t_1| \leqq 1$ either

$$\max_{S_0(t_1)} |F(s) - \overline{F(\bar{s} + 2it_1)}| \geqq k_2 \quad \text{or} \quad \max_{S_0(t_1)} |F(s)\overline{F(\bar{s} + 2it_1)} - 1| \geqq k_2. \tag{33}$$ [2]

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[1] This trivial lemma is formulated explicitly merely to introduce the constant C.

[2] We notice that since $F(s)$ is regular in $R_5(0)$ the functions $F(s)$ and $\overline{F(\bar{s} + 2it_1)}$ are regular in $R_3(t_1)$ (and a fortiori in $S_0(t_1)$).

The lemmas will then give estimates depending on the number

$$K = \max_{R_5(0)} |F(s)|.$$[1]

By A we shall now denote constants (not necessarily the same at each occurrence) depending only on the rectangles and the constants k, k_1, k_2, and C.

Lemma 16. The number N of zeros of $G(s)$ in $R_2(0)$ satisfies an inequality

$$N \leqq A \log(K+1).$$

Proof. The number is $\leqq$ the number of zeros of $F(s) - 1$, which, by (30) and Lemma 4, is $\leqq A \log(K+2)$, and this again is $\leqq A \log(K+1)$.

Lemma 17. The left and right variations $V^-(\sigma_1, \sigma_2; t_1)$ and $V^+(\sigma_1, \sigma_2; t_1)$ of $G(s)$ along an arbitrary horizontal segment $s = \sigma + i t_1$, $\sigma_1 \leqq \sigma \leqq \sigma_2$, in $R_1(0)$ satisfy inequalities

$$|V^-(\sigma_1, \sigma_2; t_1)| \leqq A \log(K+1) \quad \text{and} \quad |V^+(\sigma_1, \sigma_2; t_1)| \leqq A \log(K+1).$$

Proof. The proof is an adaptation of a well-known argument due to Backlund.

Since $V^-(\sigma_1, \sigma_2; t_1)$ and $V^+(\sigma_1, \sigma_2; t_1)$, considered as functions of t_1 for fixed σ_1 and σ_2, are continuous functions from the right and left respectively, we may suppose that the segment contains no point of the cuts and no zero of $G(s)$. The two variations are then equal, and may be denoted briefly by V.

If $G(s)$ is either real or purely imaginary on the segment, we have $V = 0$. Otherwise

$$|V| \leqq (\nu + 1)\pi,$$

where ν may denote either the number of points on the segment in which $G(s)$ is real, or the number of points on the segment in which $G(s)$ is purely imaginary. In the first case, $F(s)$ is also real in the said points, which are therefore zeros of the function $F(s) - \overline{F(\bar{s} + 2 i t_1)}$, the absolute value of which is $\leqq 2K$ in $R_3(t_1)$. In the second case, $|F(s)| = 1$ in the said points, which are therefore zeros of the function $F(s)\overline{F(\bar{s} + 2 i t_1)} - 1$, the absolute value of which is $\leqq K^2 + 1$ in $R_3(t_1)$. By (33) and Lemma 4 it follows that either

$$\nu \leqq A \log(2K+1) \quad \text{or} \quad \nu \leqq A \log(K^2+2),$$

whence the desired result.

[1] Since this K is $\geqq$ the K of § 13 the estimates of § 13 remain valid with the new K.

Lemma 18. The left and right variations $V^-(\sigma; \gamma, \delta)$ and $V^+(\sigma; \gamma, \delta)$ of $G(s)$ along an arbitrary vertical segment $s = \sigma + it$, $\gamma \leqq t \leqq \delta$, in $R_1(0)$ satisfy inequalities

$$|V^-(\sigma; \gamma, \delta)| \leqq A \log^2 (K+1) \quad \text{and} \quad |V^+(\sigma; \gamma, \delta)| \leqq A \log^2 (K+1).$$

If the segment is not divided by the cuts we have

$$|V^-(\sigma; \gamma, \delta)| \leqq A \log (K+1) \quad \text{and} \quad |V^+(\sigma; \gamma, \delta)| \leqq A \log (K+1).$$

Proof. Since $V^-(\sigma; \gamma, \delta)$ and $V^+(\sigma; \gamma, \delta)$, considered as functions of σ for fixed γ and δ, are continuous from the left and right respectively, we may suppose that the segment contains no zero of $G(s)$, not even on the borders of the cuts. The two variations are then equal and may be denoted briefly by V.

By Lemma 4 the number of cuts going into $R_2(0)$ is $\leqq A \log (K+1)$. It is therefore sufficient to prove that

$$|V| \leqq A \log (K+1) \tag{34}$$

when the segment is not divided by the cuts.

From (31) and (32) it follows by Lemma 6 (with $S_\nu(0)$ instead of $R_\nu(0)$) that V is bounded for segments in $S_1(0)$. In the general case we join the end-points of the segment by means of horizontal segments with the end-points of a vertical segment in $S_1(0)$ which does not contain zeros of $G(s)$. The estimate (34) then follows from Lemmas 16 and 17.

Lemma 19. There exists a horizontal segment $\alpha_1 \leqq \sigma \leqq \beta_1$, $t = t^*$ in $R_0(0)$ on which $F(s) \neq 0$ and $G(s) \neq 0$ and

$$\left| \frac{d}{d\sigma} \arg G(\sigma + it) \right| \leqq A \log^2 (K+1).$$

Proof. By Lemmas 4 and 16 the number of zeros in $R_2(0)$ of $F(s)$ and $G(s)$ together is $\leqq A \log (K+1)$. As in the proof of Lemma 7 the segment may therefore be chosen such that neither $F(s)$ nor $G(s)$ have zeros in a rectangle $\alpha_1 - r \leqq \sigma \leqq \beta_1 + r$, $|t - t^*| \leqq r$ belonging to $R_1(0)$, where $r \geqq 1/A \log (K+1)$. It follows from Lemmas 17 and 18 that if s^* lies on the segment, then a branch of $\arg G(s)$ satisfies in this rectangle, and a fortiori in $|s - s^*| \leqq r$, an inequality

$$|\arg G(s) - \arg G(s^*)| \leqq A \log (K+1),$$

whence the desired result.

Lemma 20. The integral

$$I(\sigma) = \int_{\gamma}^{\delta} \log |G(\sigma + it)| \, dt$$

satisfies for $-\frac{1}{2} \leqq \gamma < \delta \leqq \frac{1}{2}$ and $\alpha_1 \leqq \sigma \leqq \beta_1$ an inequality

$$|I(\sigma)| \leqq A \log^2 (K + 1).$$

Proof. The function $I(\sigma)$ is a continuous and stretchwise differentiable function of σ with $V^-(\sigma; \gamma, \delta)$ and $V^+(\sigma; \gamma, \delta)$ as left and right derivatives. Hence

$$I(\sigma) = I(\sigma_0) + \int_{\sigma_0}^{\sigma} V^-(\sigma; \gamma, \delta) \, d\sigma.$$

Let σ_0 be chosen in the interval (α_0, β_0). From (31) and (32) it follows by Lemma 9 (with $S_\nu(0)$ instead of $R_\nu(0)$) that $I(\sigma_0)$ is bounded. Lemma 18 therefore gives the desired result.

22. We now turn to the proof of Theorem 3.

For $0 < \varrho < 1$ and $Q > 1$ let us consider the function $\theta_n(\varrho, Q, t_0)$, which is zero, when the functions $\psi_n(\varrho, t_0)$ and $\chi(Q, t_0)$ introduced in Lemmas 11 and 12 are both zero, and 1 otherwise. Then

$$\Theta_n(\varrho, Q) = \underset{t_0}{\overline{M}} \{\theta_n(\varrho, Q, t_0)\} \leqq \Psi_n(\varrho) + \mathrm{X}(Q). \tag{35}$$

If $\theta_n(\varrho, Q, t_0) = 0$, i. e. if $R_5(t_0)$ belongs to Δ and

$$\max_{R_5(t_0)} |g(s) - g_n(s)| \leqq \varrho \quad \text{and} \quad \max_{R_5(t_0)} |g(s)| \leqq Q,$$

we have by Lemma 8[1], which is applicable on account of Lemma 13, for $\alpha_1 \leqq \sigma \leqq \beta_1$

$$\int_{t_0 - \frac{1}{2}}^{t_0 + \frac{1}{2}} (\log |g(\sigma + it)|_m - \log |g(\sigma + it)|) \, dt \leqq A(m) \, Q$$

and (since $|g_n(s)| \leqq Q + \varrho < 2Q$ in $R_5(t_0)$)

$$\int_{t_0 - \frac{1}{2}}^{t_0 + \frac{1}{2}} (\log |g_n(\sigma + it)|_m - \log |g_n(\sigma + it)|) \, dt \leqq A(m) \, 2 \, Q.$$

Also (since $|\log u_2 - \log u_1| \leqq |u_2 - u_1|/m$, when u_1 and u_2 are both $\geqq m$)

$$\int_{t_0 - \frac{1}{2}}^{t_0 + \frac{1}{2}} \Big| \log |g(\sigma + it)|_m - \log |g_n(\sigma + it)|_m \Big| \, dt \leqq \frac{\varrho}{m}.$$

[1] With $\gamma = -\frac{1}{2}$, $\delta = \frac{1}{2}$, and with $A(m)K$ on the right instead of $A(m) \log^2 (K + 1)$.

Hence

$$\int_{t_0-\frac{1}{2}}^{t_0+\frac{1}{2}} \Big| \log|g(\sigma+it)| - \log|g_n(\sigma+it)| \Big| \, dt \leqq A(m)\, 3\, Q + \frac{\varrho}{m}. \tag{36}$$

For all t_0 we have by Lemma 20 for $\alpha_1 \leqq \sigma \leqq \beta_1$ and $t_0 - \frac{1}{2} \leqq \gamma_1 < \delta_1 \leqq t_0 + \frac{1}{2}$

$$\Big| \int_{\gamma_1}^{\delta_1} \log|g(\sigma+it)|\, dt \Big| \leqq A\, K(t_0)^{\frac{1}{2}p} \tag{37}$$

and

$$\Big| \int_{\gamma_1}^{\delta_1} \log|g_n(\sigma+it)|\, dt \Big| \leqq A\, K_n(t_0)^{\frac{1}{2}p}. \tag{38}$$

Now

$$\frac{1}{\delta-\gamma}\int_\gamma^\delta \log|g(\sigma+it)|\,dt - \frac{1}{\delta-\gamma}\int_\gamma^\delta \log|g_n(\sigma+it)|\,dt = \frac{1}{\delta-\gamma}\int_{\gamma-\frac{1}{2}}^{\delta+\frac{1}{2}} F(t_0)\,dt_0,$$

where

$$F(t_0) = \int_{\max\{\gamma,\, t_0-\frac{1}{2}\}}^{\min\{\delta,\, t_0+\frac{1}{2}\}} (\log|g(\sigma+it)| - \log|g_n(\sigma+it)|)\,dt.$$

If $\theta_n(\varrho, Q, t_0) = 0$, it follows from (36) that

$$|F(t_0)| \leqq A(m)\, 3\, Q + \frac{\varrho}{m},$$

whereas for all t_0, on account of (37) and (38),

$$|F(t_0)| \leqq A\, K(t_0)^{\frac{1}{2}p} + A\, K_n(t_0)^{\frac{1}{2}p}.$$

Hence

$$\begin{aligned}
&\Big| \frac{1}{\delta-\gamma}\int_\gamma^\delta \log|g(\sigma+it)|\,dt - \frac{1}{\delta-\gamma}\int_\gamma^\delta \log|g_n(\sigma+it)|\,dt \Big| \\
&\leqq \frac{1}{\delta-\gamma}\int_{\gamma-\frac{1}{2}}^{\delta+\frac{1}{2}} (1-\theta_n(\varrho, Q, t_0)) \left(A(m)\, 3\, Q + \frac{\varrho}{m}\right) dt_0 + \\
&+ \frac{1}{\delta-\gamma}\int_{\gamma-\frac{1}{2}}^{\delta+\frac{1}{2}} \theta_n(\varrho, Q, t_0)\, A\, K(t_0)^{\frac{1}{2}p}\,dt_0 + \frac{1}{\delta-\gamma}\int_{\gamma-\frac{1}{2}}^{\delta+\frac{1}{2}} \theta_n(\varrho, Q, t_0)\, A\, K_n(t_0)^{\frac{1}{2}p}\,dt_0 \\
&\leqq \frac{\delta-\gamma+1}{\delta-\gamma}\left(A(m)\, 3\, Q + \frac{\varrho}{m}\right) + \\
&+ A\left[\frac{1}{\delta-\gamma}\int_{\gamma-\frac{1}{2}}^{\delta+\frac{1}{2}} \theta_n(\varrho, Q, t_0)\,dt_0\right]^{\frac{1}{2}} \left(\left[\frac{1}{\delta-\gamma}\int_{\gamma-\frac{1}{2}}^{\delta+\frac{1}{2}} K(t_0)^p\,dt_0\right]^{\frac{1}{2}} + \left[\frac{1}{\delta-\gamma}\int_{\gamma-\frac{1}{2}}^{\delta+\frac{1}{2}} K_n(t_0)^p\,dt_0\right]^{\frac{1}{2}}\right)
\end{aligned}$$

For fixed γ, and $\delta \to \infty$, the expression on the right has by Lemma 1 the upper limit

$$A(m)\, 3\, Q + \frac{\varrho}{m} + A\, \Theta_n(\varrho, Q)^{\frac{1}{2}} \left(\underset{t_0}{M}\{K(t_0)^p\}^{\frac{1}{2}} + \underset{t_0}{M}\{K_n(t_0)^p\}^{\frac{1}{2}}\right),$$

which by (35) is $\leqq$

$$A(m)\, 3\, Q + \frac{\varrho}{m} + A\left(\Psi_n(\varrho) + \mathrm{X}(Q)\right)^{\frac{1}{2}} \left(\underset{t_0}{M}\{K(t_0)^p\}^{\frac{1}{2}} + \underset{t_0}{M}\{K_n(t_0)^p\}^{\frac{1}{2}}\right). \tag{39}$$

By Lemmas 1 and 11 this expression converges for $n \to \infty$ towards

$$A(m)\, 3\, Q + \frac{\varrho}{m} + 2\, A\, \mathrm{X}(Q)^{\frac{1}{2}}\, \underset{t_0}{M}\{K(t_0)^p\}^{\frac{1}{2}} = T_1 + T_2 + T_3 \text{ (say)}. \tag{40}$$

Let $\varepsilon > 0$ be given, and let first, by Lemma 12, Q be chosen so large that $T_3 < \frac{1}{3}\varepsilon$, then m so small that $T_1 < \frac{1}{3}\varepsilon$, and finally ϱ so small that $T_2 < \frac{1}{3}\varepsilon$. Then the expression (40) is $< \varepsilon$. Hence the expression (39) is $< \varepsilon$ for $n \geqq$ (some) n_0 and, consequently,

$$\left| \frac{1}{\delta - \gamma} \int_{\gamma}^{\delta} \log|g(\sigma + it)|\, dt - \frac{1}{\delta - \gamma} \int_{\gamma}^{\delta} \log|g_n(\sigma + it)|\, dt \right| < \varepsilon$$

for $\alpha_1 \leqq \sigma \leqq \beta_1$ if $n \geqq n_0$ and $\delta \geqq$ (some) $\delta_0(n)$. Arguing as in § 14 we obtain from this the first part of the theorem.

23. The convexity of $\varphi_g(\sigma)$ follows at once from the convexity of the functions $\varphi_{g_n}(\sigma)$. But as to the rest of the proof we cannot proceed exactly as in § 15, but must first introduce some modifications of the functions $\varphi_g(\sigma; \gamma, \delta)$, $V_g^-(\sigma; \gamma, \delta)$, and $V_g^+(\sigma; \gamma, \delta)$, which we obtain by adding certain terms corresponding to the cuts.

For an arbitrary cut C defined by $\alpha < \sigma \leqq \sigma_0$, $t = t_0$ or $\sigma_0 \leqq \sigma < \beta$, $t = t_0$[1] let $v_C(\sigma)$ for $\sigma < \sigma_0$ or $\sigma > \sigma_0$ respectively denote the variation of the argument of $g(s)$ along the lower border of the cut from $\sigma + it_0$ to $\sigma_0 + it_0$ and back to $\sigma + it_0$ along the upper border of the cut.[2] For $\sigma \geqq \sigma_0$ or $\sigma \leqq \sigma_0$ respectively let us put $v_C(\sigma) = 0$.

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[1] It will be understood that the cut does not go beyond $\sigma_0 + it_0$; naturally there may be more zeros of $f(s)$ on the cut than the end-point $\sigma_0 + it_0$.

[2] More precisely, if $V^-(\sigma_1, \sigma_2; t_0)$ and $V^+(\sigma_1, \sigma_2; t_0)$ denote the left and right variations of the argument of $g(s)$ along the segment $\sigma_1 \leqq \sigma \leqq \sigma_2$, $t = t_0$, we put $v_C(\sigma) = V^+(\sigma, \sigma_0; t_0) - V^-(\sigma, \sigma_0; t_0)$ or $v_C(\sigma) = -V^+(\sigma_0, \sigma; t_0) + V^-(\sigma_0, \sigma; t_0)$ respectively.

The expressions

$$W_g^-(\sigma;\gamma,\delta) = V_g^-(\sigma;\gamma,\delta) + \sum_{\gamma}^{\delta} v_C(\sigma - 0)$$

and

$$W_g^+(\sigma;\gamma,\delta) = V_g^+(\sigma;\gamma,\delta) + \sum_{\gamma}^{\delta} v_C(\sigma + 0),$$

in which the sums are extended over all cuts between the lines $t=\gamma$ and $t=\delta$,[1] will represent the variation of the argument of $g(s)$ from $\sigma + i\gamma$ to $\sigma + i\delta$ along a path composed of the left or right sides of the parts into which the segment $s = \sigma + it$, $\gamma \leqq t \leqq \delta$, is divided by the cuts, and joining loops around the cuts.

The function

$$\psi_g(\sigma;\gamma,\delta) = \varphi_g(\sigma;\gamma,\delta) + \frac{1}{\delta-\gamma}\sum_{\gamma}^{\delta}\int_{\sigma_0}^{\sigma} v_C(\sigma)\,d\sigma$$

will be continuous and stretchwise differentiable with the left and right derivatives

$$(41) \qquad \psi_g'(\sigma - 0;\gamma,\delta) = \frac{W_g^-(\sigma;\gamma,\delta)}{\delta-\gamma} \quad \text{and} \quad \psi_g'(\sigma + 0;\gamma,\delta) = \frac{W_g^+(\sigma;\gamma,\delta)}{\delta-\gamma}.$$

We shall now prove that $\psi_g(\sigma;\gamma,\delta)$ for γ fixed and $\delta \to \infty$ converges uniformly in $[\alpha,\beta]$ towards $\varphi_g(\sigma)$, and also that the four mean motions remain unchanged if in their definition (22) we replace $V_g^-(\sigma;\gamma,\delta)$ and $V_g^+(\sigma;\gamma,\delta)$ by $W_g^-(\sigma;\gamma,\delta)$ and $W_g^+(\sigma;\gamma,\delta)$. This is proved by proving that

$$(42) \qquad \frac{1}{\delta-\gamma}\sum_{\gamma}^{\delta} |v_C(\sigma)| \to 0$$

uniformly in $[\alpha,\beta]$.

By Lemma 4 the number of cuts which go into $R_0(t_0)$ is $\leqq A \log(K(t_0)+1)$, and by Lemma 17 $|v_C(\sigma)| \leqq A\log(K(t_0)+1)$ for $\alpha_1 \leqq \sigma \leqq \beta_1$ for each such cut. Thus for $\alpha_1 \leqq \sigma \leqq \beta_1$

$$\sum_{t_0-\frac{1}{2}}^{t_0+\frac{1}{2}} |v_C(\sigma)| \leqq A\,\theta(t_0)\,K(t_0)^{\frac{1}{2}\nu},$$

where $\theta(t_0)$ is the function introduced in Lemma 10. Hence for $\alpha_1 \leqq \sigma \leqq \beta_1$

[1] There may be an infinite number of such cuts, but for every σ in (α,β) only the finite number which intersect the segment $s = \sigma + it$, $\gamma < t < \delta$, contribute to the sum.

$$\frac{1}{\delta-\gamma}\sum_{\gamma}^{\delta}|v_C(\sigma)|\leqq\frac{1}{\delta-\gamma}\int_{\gamma-\frac{1}{2}}^{\delta+\frac{1}{2}}\Big(\sum_{t_0-\frac{1}{2}}^{t_0+\frac{1}{2}}|v_C(\sigma)|\Big)\,d\,t_0$$

$$\leqq A\left[\frac{1}{\delta-\gamma}\int_{\gamma-\frac{1}{2}}^{\delta+\frac{1}{2}}\theta(t_0)\,d\,t_0\right]^{\frac{1}{q}}\left[\frac{1}{\delta-\gamma}\int_{\gamma-\frac{1}{2}}^{\delta+\frac{1}{2}}K(t_0)^p\,d\,t_0\right]^{\frac{1}{p}}$$

By Lemmas 1 and 10 the expression on the right tends to zero, whence the desired result.

24. We may now proceed essentially as in § 15. It will be sufficient to prove (23) for $\alpha_1 < \sigma < \beta_1$ and (25) for $\alpha_1 < \sigma_1 < \sigma_2 < \beta_1$.

Since γ may be chosen arbitrarily we may suppose that $f(s) \neq 0$ and $g(s) \neq 0$ on the segment $\alpha_1 \leqq \sigma \leqq \beta_1$, $t=\gamma$. By Lemmas 2, 16, and 18 it makes no difference if in the definitions (22) and (24) of the mean motions and of the frequencies of zeros we restrict δ to a set of values so that any interval $|t-t_0| \leqq \frac{1}{2}$ contains at least one value from the set. By (42) the expressions

$$\left.\begin{matrix} \underline{c}_g^-(\sigma) \\ \bar{c}_g^-(\sigma)\end{matrix}\right\} = \lim_{\delta\to\infty}\begin{matrix}\inf\\ \sup\end{matrix}\ \frac{W_g^-(\sigma;\gamma,\delta)}{\delta-\gamma} \quad\text{and}\quad \left.\begin{matrix} \underline{c}_g^+(\sigma) \\ \bar{c}_g^+(\sigma)\end{matrix}\right\} = \lim_{\delta\to\infty}\begin{matrix}\inf\\ \sup\end{matrix}\ \frac{W_g^+(\sigma;\gamma,\delta)}{\delta-\gamma}$$

for the mean motions are also valid when δ is restricted in this manner.

Let us first merely suppose that δ is restricted to values for which $f(s) \neq 0$ and $g(s) \neq 0$ on the segment $\alpha_1 \leqq \sigma \leqq \beta_1$, $t=\delta$. Then by Cauchy's theorem, applied to the part of the rectangle $(\alpha_1 <)\,\sigma_1 < \sigma < \sigma_2\,(< \beta_1)$, $\gamma < t < \delta$ which belongs to Δ,

$$(43)\qquad N_g(\sigma_1,\sigma_2;\gamma,\delta)=\frac{1}{2\pi}\left[W_g^-(\sigma_2;\gamma,\delta)-W_g^+(\sigma_1;\gamma,\delta)+R(\sigma_1,\sigma_2;\gamma,\delta)\right],$$

where $R(\sigma_1,\sigma_2;\gamma,\delta)$ denotes the contribution to the variation of the argument from the horizontal sides of the rectangle. By Lemmas 2 and 17

$$R(\sigma_1,\sigma_2;\gamma,\delta)=o(\delta).$$

Hence

$$\frac{1}{2\pi}(\underline{c}_g^-(\sigma_2)-\bar{c}_g^+(\sigma_1))\leqq \underline{H}_g(\sigma_1,\sigma_2)\leqq \bar{H}_g(\sigma_1,\sigma_2)\leqq\frac{1}{2\pi}(\bar{c}_g^-(\sigma_2)-\underline{c}_g^+(\sigma_1)),$$

so that (25) is a consequence of (23). Of (23) it is sufficient to prove

$$\varphi_g'(\sigma-0)\leqq \underline{c}_g^-(\sigma)\quad\text{and}\quad \bar{c}_g^+(\sigma)\leqq\varphi_g'(\sigma+0).$$

For any t_0 for which $f(s) \neq 0$ and $g(s) \neq 0$ on the segment $\alpha_1 \leqq \sigma \leqq \beta_1$, $t = t_0$ put

$$\max_{\alpha_1 \leqq \sigma \leqq \beta_1} \left| \frac{d}{d\sigma} \arg g(\sigma + i t_0) \right| = C(t_0).$$

Then

$$| R(\sigma_1, \sigma_2; \gamma, \delta) | \leqq (C(\gamma) + C(\delta)) (\sigma_2 - \sigma_1).$$

By Lemmas 2 and 19 we may suppose that δ is restricted to values for which $C(\delta) = o(\delta)$. By means of (41) the relation (43) assumes the form

$$\frac{N_g(\sigma_1, \sigma_2; \gamma, \delta)}{\delta - \gamma} = \frac{1}{2\pi} (\psi'_g(\sigma_2 - 0; \gamma, \delta) - \psi'_g(\sigma_1 + 0; \gamma, \delta) + r(\sigma_1, \sigma_2; \gamma, \delta)),$$

where

$$| r(\sigma_1, \sigma_2; \gamma, \delta) | \leqq \frac{C(\gamma) + C(\delta)}{\delta - \gamma} (\sigma_2 - \sigma_1),$$

and the proof is completed by the argument used in § 15.

25. Next we turn to the proof of Theorem 4.

By the definition of the integral we have

$$\Lambda(y; \mu_{g, \sigma; \gamma, \delta}) = \frac{1}{\delta - \gamma} \int_\gamma^\delta e^{i g(\sigma + it) y} \, dt.$$

Now, if $\alpha_1 \leqq \sigma \leqq \beta_1$, and the function $\psi_n(\varrho, t)$ introduced in Lemma 11 is zero, then

$$| e^{i g(\sigma + it) y} - e^{i g_n (\sigma + it) y} | \leqq \varrho \, | y |,$$

whereas the expression on the left is $\leqq 2$ for all t.

Hence, if $|y| \leqq a$,

$$\left| \frac{1}{\delta - \gamma} \int_\gamma^\delta e^{i g(\sigma + it) y} \, dt - \frac{1}{\delta - \gamma} \int_\gamma^\delta e^{i g_n (\sigma + it) y} \, dt \right| \leqq \varrho a + \frac{1}{\delta - \gamma} \int_\gamma^\delta \psi_n(\varrho, t) \, 2 \, dt.$$

For fixed γ, and $\delta \to \infty$, the expression on the right converges towards $\varrho a + 2 \Psi_n(\varrho)$, which by Lemma 11 converges towards ϱa when $n \to \infty$ for any ϱ. For every fixed n we know (cf. § 7) that the limit

$$\Lambda(y; \mu_{g_n, \sigma}) = \lim_{\delta \to \infty} \frac{1}{\delta - \gamma} \int_\gamma^\delta e^{i g_n (\sigma + it) y} \, dt$$

exists uniformly in $|y| \leqq a$. It follows that

$$\lim_{\delta \to \infty} \frac{1}{\delta - \gamma} \int_\gamma^\delta e^{i g (\sigma + it) y} \, dt$$

also exists uniformly in $|y| \leqq a$ and is the limit of $\Lambda(y; \mu_{g_n, \sigma})$ when $n \to \infty$ uniformly in $|y| \leqq a$.

Thus the theorem is established.

CHAPTER II.

The Riemann Zeta Function.

Application of the Previous Results to the Zeta Function and its Logarithm.

26. Let us now consider the Riemann zeta function $\zeta(s)$. It is regular in the whole plane with the exception of the point $s = 1$, where it has a pole of the first order. In the half-plane $\sigma > 1$ the function is determined by the Euler product

$$\zeta(s) = \prod_{k=1}^{\infty} (1 - p_k^{-s})^{-1},$$

in which $p_1, p_2, \ldots$ denote the primes $2, 3, \ldots$; in consequence of this expression we have $\zeta(s) \neq 0$ for $\sigma > 1$. We shall also consider the partial product

$$\zeta_n(s) = \prod_{k=1}^{n} (1 - p_k^{-s})^{-1}$$

of the Euler product. The function $\zeta_n(s)$ is regular and $\neq 0$ for $\sigma > 0$.

By $\log \zeta(s)$ and $\log \zeta_n(s)$ we shall denote the functions

$$\log \zeta(s) = \sum_{k=1}^{\infty} - \log(1 - p_k^{-s})$$

and

$$\log \zeta_n(s) = \sum_{k=1}^{n} - \log(1 - p_k^{-s}),$$

where in each term on the right $-\log(1 - z) = z + \frac{1}{2}z^2 + \cdots$. The function $\log \zeta(s)$ is regular for $\sigma > 1$ and $\log \zeta_n(s)$ for $\sigma > 0$. By the function $\log \zeta(s)$ in the half-plane $\sigma > \frac{1}{2}$ we shall mean the analytic continuation of $\log \zeta(s)$ in the domain Δ obtained from $\sigma > \frac{1}{2}$ by omitting the segment $\frac{1}{2} < \sigma \leqq 1$, $t = 0$ and all segments $\frac{1}{2} < \sigma \leqq \sigma_0$, $t = t_0$, where $\sigma_0 + i t_0$ denote the zeros (if any) of $\zeta(s)$ in $\sigma > \frac{1}{2}$.

27. The functions $\zeta_n(s)$ and $\log \zeta_n(s)$ are almost periodic in $[0, +\infty]$ and converge for $n \to \infty$ uniformly in $[1, +\infty]$ towards $\zeta(s)$ and $\log \zeta(s)$.

Let us consider the functions $\zeta_n(s)$ and $\zeta(s)$ in the half-strip $\frac{1}{2} < \sigma < +\infty$, $t > 0$. It is known that $\zeta_n(s)$ converges in the mean with the index $p = 2$ towards

$\zeta(s)$ in $[\frac{1}{2}, +\infty]$.[1] Hence, if we take $\alpha = \frac{1}{2}$, $\alpha_0 = 1$, $1 < \beta_0 < +\infty$, $\beta = +\infty$, and $\gamma_0 = 0$, the assumptions of Theorem 1 are satisfied, if for $f(s)$ and $f_n(s)$ we take the functions

$$f(s) = \zeta(s) - x \text{ and } f_n(s) = \zeta_n(s) - x,$$

where x is an arbitrary complex number, and the assumptions of Theorem 3 are satisfied, if for $f(s)$, $f_n(s)$, $g(s)$, and $g_n(s)$ we take

$$f(s) = \zeta(s)\, e^{-x}, \quad f_n(s) = \zeta_n(s)\, e^{-x}, \quad g(s) = \log \zeta(s) - x, \quad g_n(s) = \log \zeta_n(s) - x.$$

By means of Theorems 1 and 3 the study of the mean motions and zeros of the functions $\zeta(s) - x$ and $\log \zeta(s) - x$ is therefore reduced to a study of the functions $\zeta_n(s) - x$ and $\log \zeta_n(s) - x$, and a passage to the limit. Similarly, the study of the asymptotic distribution functions of $\zeta(s)$ and $\log \zeta(s)$ on vertical lines is by Theorems 2 and 4 reduced to a study of the functions $\zeta_n(s)$ and $\log \zeta_n(s)$ and a passage to the limit.

Two Types of Distribution Functions.

28. The investigation depends on the discussion of two types of distribution functions which are closely related to each other. The first type, leading to the asymptotic distribution functions of $\zeta(s)$ and $\log \zeta(s)$ on vertical lines, has already been considered in Jessen and Wintner [1].[2]

First we shall prove the following theorem.

Theorem 5. *Let $l(z) = l_1 z + l_2 z^2 + \cdots$ and $m(z) = m_1 z + m_2 z^2 + \cdots$ be power series convergent in a circle $|z| < \varrho\, (\leqq \infty)$, and such that $l_1 \neq 0$ and $m_1 \neq 0$. Let $r_1, r_2, \ldots$ be a sequence of real numbers > 0, such that $r_n < \varrho$ for all n, and let $\lambda_1, \lambda_2, \ldots$ be a sequence of real numbers differing from each other and from zero.*

Consider for every n the functions

$$f_n(\theta_1, \ldots, \theta_n) = \sum_{k=1}^{n} l(r_k e^{2\pi i \theta_k}) \quad \text{and} \quad g_n(\theta_1, \ldots, \theta_n) = \sum_{k=1}^{n} \lambda_k\, m(r_k e^{2\pi i \theta_k}),$$

---

[1] This follows e. g. from a result of Besicovitch [1], pp. 163—169, with an addition on uniformity in σ which readily follows from his proof. It is essentially this property which forms the basis for the investigations by Bohr and Landau [1] and by Bohr [1] on the distribution of the values of the zeta function.

[2] Our treatment of this type has been given a different form to match the treatment of the second type. Also, the results regarding the first type have been given with certain additions which are necessary for the treatment of the second type.

where each θ_k describes the real axis considered mod. 1 as a circle c_k, so that $(\theta_1, \ldots, \theta_n)$ describes the corresponding n-dimensional torus-space $Q_n = (c_1, \ldots, c_n)$.

Let μ_n and ν_n denote the distribution functions of $f_n(\theta_1, \ldots, \theta_n)$ and of $f_n(\theta_1, \ldots, \theta_n)$ with respect to $|g_n(\theta_1, \ldots, \theta_n)|^2$, defined by

$$\mu_n(E) = m(\Omega(E)) \quad \text{and} \quad \nu_n(E) = \int_{\Omega(E)} |g_n(\theta_1, \ldots, \theta_n)|^2 \, m(d\,Q_n),$$

respectively, where $\Omega(E)$ for an arbitrary Borel set E in R_x denotes the set of points in Q_n for which $f_n(\theta_1, \ldots, \theta_n)$ belongs to E.

Then, if $r_n \to 0$ for $n \to \infty$, the distribution functions μ_n and ν_n are absolutely continuous with continuous densities $F_n(x) = F_n(\xi_1, \xi_2)$ and $G_n(x) = G_n(\xi_1, \xi_2)$ for $n \geqq$ (some) n_0, and $F_n(\xi_1, \xi_2)$ and $G_n(\xi_1, \xi_2)$ possess continuous partial derivatives of order $\leqq p$ for $n \geqq$ (some) n_p.

If, moreover, the three series

$$S_0 = \sum_{k=1}^{\infty} r_k^2, \quad S_1 = \sum_{k=1}^{\infty} |\lambda_k| \, r_k^2, \quad S_2 = \sum_{k=1}^{\infty} \lambda_k^2 r_k^2$$

are convergent[1]*, then μ_n and ν_n converge for $n \to \infty$ towards distribution functions μ and ν which are absolutely continuous with continuous densities $F(x) = F(\xi_1, \xi_2)$ and $G(x) = G(\xi_1, \xi_2)$ possessing continuous partial derivatives of arbitrarily high order. The functions $F_n(x)$ and $G_n(x)$ and their partial derivatives converge uniformly towards $F(x)$ and $G(x)$ and their partial derivatives as $n \to \infty$.*

29. To prove the first part of the theorem it is by § 6 sufficient to prove that for every $p \geqq 0$ the Fourier transforms $\Lambda(y; \mu_n)$ and $\Lambda(y; \nu_n)$ for $n \geqq$ (some) n_p and some $\varepsilon > 0$ are $O(|y|^{-(2+p+\varepsilon)})$ as $|y| \to \infty$. To prove the second part of the theorem it is sufficient to prove that for $n \geqq n_p$ the functions $\Lambda(y; \mu_n)$ and $\Lambda(y; \nu_n)$ have a bounded majorant which for some $\varepsilon > 0$ is $O(|y|^{-(2+p+\varepsilon)})$ as $|y| \to \infty$, and that $\Lambda(y; \mu_n)$ and $\Lambda(y; \nu_n)$ converge uniformly in every circle $|y| \leqq a$ towards functions, which are then $\Lambda(y; \mu)$ and $\Lambda(y; \nu)$.

By the definition of the integral we get

$$\Lambda(y; \mu_n) = \int_{Q_n} e^{i f_n(\theta_1, \ldots, \theta_n) y} \, m(d\,Q_n) \quad \text{and} \tag{44}$$

$$\Lambda(y; \nu_n) = \int_{Q_n} e^{i f_n(\theta_1, \ldots, \theta_n) y} \, |g_n(\theta_1, \ldots, \theta_n)|^2 \, m(d\,Q_n),$$

[1] We shall use S_0, S_1, S_2 not only as notations for the sums of the series, but also as notations for the series themselves. We notice that by Cauchy's inequality the convergence of S_1 follows from that of S_0 and S_2.

where $f_n(\theta_1, \ldots, \theta_n)\,y$ denotes the inner product. Here

$$e^{i f_n(\theta_1, \ldots, \theta_n)\,y} = \prod_{k=1}^{n} e^{i\,l(r_k e^{2\pi i \theta_k})\,y}$$

and

$$|g_n(\theta_1, \ldots, \theta_n)|^2 = \sum_{l=1}^{n} \lambda_l^2 \,|m(r_l e^{2\pi i \theta_l})|^2 + \sum_{\substack{l, m=1\\ l \neq m}}^{n} \lambda_l \lambda_m\, m(r_l e^{2\pi i \theta_l})\, \overline{m(r_m e^{2\pi i \theta_m})}.$$

Hence, on placing for $0 < r < \varrho$

(45) $$K_0(y, r) = \int_c e^{i l(r e^{2\pi i \theta})\,y}\, d\theta, \quad K_1(y, r) = \int_c e^{i l(r e^{2\pi i \theta})\,y}\, m(r e^{2\pi i \theta})\, d\theta,$$

$$\text{and } K_2(y, r) = \int_c e^{i l(r e^{2\pi i \theta})\,y}\, |m(r e^{2\pi i \theta})|^2\, d\theta,$$

where c is the real axis considered mod. 1, we obtain

(46) $$\Lambda(y; \mu_n) = \prod_{k=1}^{n} K_0(y, r_k)$$

and

(47) $$\Lambda(y; \nu_n) = \sum_{l=1}^{n} \lambda_l^2\, K_2(y, r_l) \prod_{\substack{k=1\\ k \neq l}}^{n} K_0(y, r_k) +$$

$$+ \sum_{\substack{l, m=1\\ l \neq m}}^{n} \lambda_l \lambda_m\, K_1(y, r_l)\, \overline{K_1(-y, r_m)} \prod_{\substack{k=1\\ k \neq l, m}}^{n} K_0(y, r_k).$$

30. We shall need some estimates of the functions (45).

For all $r < \varrho$

(48) $$|K_0(y, r)| \leqq K_0(0, r) = 1$$

and

(49) $$K_1(0, r) = 0.$$

For an arbitrary $\varrho_0 < \varrho$ there exists a constant A such that $|l(z)| \leqq A|z|$ and $|m(z)| \leqq A|z|$ for $|z| \leqq \varrho_0$. Suppose now that $r \leqq \varrho_0$. Since $l(0) = 0$, the integrals over c of the real and imaginary parts of $l(r e^{2\pi i \theta})$ and hence of the inner product $l(r e^{2\pi i \theta})\,y$ are zero. Moreover, the inner product is numerically $\leqq A r |y|$. Hence, since $|e^{it} - (1 + it)| \leqq \frac{1}{2} t^2$,

(50) $$|K_0(y, r) - 1| \leqq \tfrac{1}{2} A^2 r^2 |y|^2.$$

Also, since $|e^{it}-(1+it-\frac{1}{2}t^2)| \leqq \frac{1}{6}|t|^3$, and since (according to Parseval's formula) the integral over c of $[l(re^{2\pi i\theta})y]^2$ is $=\frac{1}{2}(|l_1|^2r^2+|l_2|^2r^4+\cdots)|y|^2 \geqq \frac{1}{2}|l_1|^2r^2|y|^2$,

$$|\mathrm{K}_0(y,r)| \leqq 1-\tfrac{1}{4}|l_1|^2r^2|y|^2+\tfrac{1}{6}A^3r^3|y|^3$$

and, consequently, for certain constants B_1 and B_2,

$$|\mathrm{K}_0(y,r)| \leqq 1-B_1r^2|y|^2 \text{ when } r|y| \leqq B_2. \tag{51}$$

Similarly, since $m(0)=0$ and $|e^{it}-1| \leqq |t|$,

$$|\mathrm{K}_1(y,r)| \leqq A^2r^2|y|. \tag{52}$$

Also

$$|\mathrm{K}_2(y,r)| \leqq A^2r^2. \tag{53}$$

Finally, it is known[1] that there exist constants $\varrho_1<\varrho$ and B such that for $r \leqq \varrho_1$

$$|\mathrm{K}_0(y,r)| \leqq Br^{-\frac{1}{2}}|y|^{-\frac{1}{2}}. \tag{54}$$

31. Suppose first that $r_n \to 0$ for $n \to \infty$. Then all $r_n \leqq$ (some) $\varrho_0<\varrho$. Let this ϱ_0 be used in § 30. Then the estimates (50), (52), and (53) are valid for $r=r_n$ and all n. Moreover, for the ϱ_1 of § 30 we have $r_n \leqq \varrho_1$ for all $n>$ $>$ (some) $h \geqq 0$. Hence (54) is valid for $r=r_n$, $n>h$. Consequently, if $n \geqq h+$ $+11+2p$, each of the products in (46) and (47) is $O(|y|^{-(\frac{5}{2}+p)})$, and each term in (47) is therefore $O(|y|^{-(\frac{5}{2}+p)})$. This establishes the first part of the theorem with $n_p=h+11+2p$.

Suppose now that the series S_0, S_1, S_2 are convergent.

For all n

$$|\Lambda(y;\mu_n)| \leqq \Lambda(0;\mu_n)=1. \tag{55}$$

If $n \geqq n_p$ let us apply (54) to the factors in (46) with $h<k \leqq h+5+2p$, and (48) to the rest. We obtain

$$|\Lambda(y;\mu_n)| \leqq B^{5+2p}\left(\prod_{k=h+1}^{h+5+2p} r_k^{-\frac{1}{2}}\right)|y|^{-(\frac{5}{2}+p)}. \tag{56}$$

From (55) and (56) it is seen that the functions $\Lambda(y;\mu_n)$ for $n \geqq n_p$ have a bounded majorant which is $O(|y|^{-(\frac{5}{2}+p)})$ as $|y| \to \infty$.

[1] See Jessen and Wintner [1], Theorem 13.

From (46) and (50) it follows that

(57) $$|\Lambda(y;\mu_{n+1})-\Lambda(y;\mu_n)|=|\Lambda(y;\mu_n)||K_0(y,r_{n+1})-1|\leqq \tfrac{1}{2}A^2 r_{n+1}^2|y|^2,$$

whence by the convergence of the series S_0 the uniform convergence of $\Lambda(y;\mu_n)$ for $n\to\infty$ in any circle $|y|\leqq a$.

From (47), (48), (49), and (53) it follows that for all n

$$\Lambda(0;\nu_n)=\sum_{l=1}^{n}\lambda_l^2 K_2(0,r_l)\leqq A^2\sum_{l=1}^{n}\lambda_l^2 r_l^2\leqq A^2 S_2.$$

Hence

(58) $$|\Lambda(y;\nu_n)|\leqq A^2 S_2.$$

If $n\geqq n_p$ each of the products in (47) is numerically $\leqq B^{9+2p}P|y|^{-(\frac{9}{2}+p)}$, where P is the product of the $9+2p$ largest of the numbers $r_k^{-\frac{1}{2}}$, $h<k\leqq n_p$. Hence, by (52) and (53)

(59) $$|\Lambda(y;\nu_n)|\leqq (A^2\sum_{l=1}^{n}\lambda_l^2 r_l^2+A^4\sum_{\substack{l,m=1\\ l\neq m}}^{n}|\lambda_l||\lambda_m|r_l^2 r_m^2|y|^2)B^{9+2p}P|y|^{-(\frac{9}{2}+p)}$$

$$\leqq A^2 S_2 B^{9+2p}P|y|^{-(\frac{9}{2}+p)}+A^4 S_1^2 B^{9+2p}P|y|^{-(\frac{5}{2}+p)}.$$

From (58) and (59) it will be seen that the functions $\Lambda(y;\nu_n)$ for $n\geqq n_p$ have a bounded majorant which is $O(|y|^{-(\frac{5}{2}+p)})$ as $|y|\to\infty$.

From (47) it follows that

$$\Lambda(y;\nu_{n+1})-\Lambda(y;\nu_n)=\Lambda(y;\nu_n)(K_0(y,r_{n+1})-1)+\lambda_{n+1}^2K_2(y,r_{n+1})\prod_{k=1}^{n}K_0(y,r_k)+$$

$$+\lambda_{n+1}K_1(y,r_{n+1})\sum_{m=1}^{n}\lambda_m\overline{K_1(-y,r_m)}\prod_{\substack{k=1\\ k\neq m}}^{n}K_0(y,r_k)+$$

$$+\lambda_{n+1}\overline{K_1(-y,r_{n+1})}\sum_{l=1}^{n}\lambda_l K_1(y,r_l)\prod_{\substack{k=1\\ k\neq l}}^{n}K_0(y,r_k).$$

Hence, by (48), (50), (52), (53), and (58),

(60) $$|\Lambda(y;\nu_{n+1})-\Lambda(y;\nu_n)|\leqq\tfrac{1}{2}A^4S_2r_{n+1}^2|y|^2+A^2\lambda_{n+1}^2r_{n+1}^2+$$

$$+2A^4|\lambda_{n+1}|r_{n+1}^2|y|^2\sum_{l=1}^{n}|\lambda_l|r_l^2$$

$$\leqq\tfrac{1}{2}A^4S_2r_{n+1}^2|y|^2+A^2\lambda_{n+1}^2r_{n+1}^2+2A^4S_1|\lambda_{n+1}|r_{n+1}^2|y|^2,$$

whence the uniform convergence of $\Lambda(y;\nu_n)$ for $n\to\infty$ in any circle $|y|\leqq a$.

This completes the proof of the theorem.

32. Since $\Lambda(y; \mu_n)$ converges towards $\Lambda(y; \mu)$ when $n \to \infty$ we obtain from (46) the expression

$$(61) \qquad \Lambda(y; \mu) = \prod_{k=1}^{\infty} \mathrm{K}_0(y, r_k),$$

where the product is absolutely convergent in consequence of (50) and the convergence of the series S_0.

Similarly, from (47) we shall deduce the expression

$$(62) \quad \Lambda(y; \nu) = \sum_{l=1}^{\infty} \lambda_l^2 \mathrm{K}_2(y, r_l) \prod_{\substack{k=1 \\ k \neq l}}^{\infty} \mathrm{K}_0(y, r_k) + \\ + \sum_{\substack{l, m=1 \\ l \neq m}}^{\infty} \lambda_l \lambda_m \mathrm{K}_1(y, r_l) \overline{\mathrm{K}_1(-y, r_m)} \prod_{\substack{k=1 \\ k \neq l, m}}^{\infty} \mathrm{K}_0(y, r_k).$$

Here the infinite products are absolutely convergent, and since by (48) the products are numerically $\leqq 1$, the series are absolutely convergent in consequence of (52), (53), and the convergence of the series S_1 and S_2.

We know that $\Lambda(y; \nu_n)$ converges towards $\Lambda(y; \nu)$ when $n \to \infty$. Now $\Lambda(y; \nu_n)$ differs from the expression

$$(63) \quad \sum_{l=1}^{n} \lambda_l^2 \mathrm{K}_2(y, r_l) \prod_{\substack{k=1 \\ k \neq l}}^{\infty} \mathrm{K}_0(y, r_k) + \sum_{\substack{l, m=1 \\ l \neq m}}^{n} \lambda_l \lambda_m \mathrm{K}_1(y, r_l) \overline{\mathrm{K}_1(-y, r_m)} \prod_{\substack{k=1 \\ k \neq l, m}}^{\infty} \mathrm{K}_0(y, r_k)$$

by the factor $\prod_{k=n+1}^{\infty} \mathrm{K}_0(y, r_k)$, which converges towards 1 when $n \to \infty$. Hence (63) converges towards $\Lambda(y; \nu)$, and this establishes (62).

33. For every n the densities $F_n(x)$ and $G_n(x)$ vanish outside the closed bounded set of values assumed by $f_n(\theta_1, \ldots, \theta_n)$. Since $F_n(x)$ and $G_n(x)$ and their partial derivatives converge uniformly towards $F(x)$ and $G(x)$ and their partial derivatives when $n \to \infty$ it is plain that all the latter functions will approach zero when $|x| \to \infty$.[1] We shall now prove a much preciser result.

Theorem 6. *For any $\lambda > 0$ the densities $F(x)$, $G(x)$ and $F_n(x)$, $G_n(x)$, $n \geqq n_0$, have a majorant of the form $K_0 e^{-\lambda |x|^2}$, and the partial derivatives of $F(x)$, $G(x)$ and $F_n(x)$, $G_n(x)$, $n \geqq n_p$, of order $\leqq p$, have a majorant of the form $K_p e^{-\lambda |x|^2}$.*

[1] This is also an easy consequence of the explicit expression of the functions by means of the Fourier transforms.

34. For $n > q > 0$ let us write

$$f_{q,n}(\theta_{q+1}, \ldots, \theta_n) = \sum_{k=q+1}^{n} l(r_k e^{2\pi i \theta_k}) \text{ and } g_{q,n}(\theta_{q+1}, \ldots, \theta_n) = \sum_{k=q+1}^{n} \lambda_k m(r_k e^{2\pi i \theta_k})$$

and let $Q_{q,n}$ denote the torus-space with $(\theta_{q+1}, \ldots, \theta_n)$ as variable point. Let $\mu_{q,n}$ denote the distribution function of $f_{q,n}(\theta_{q+1}, \ldots, \theta_n)$ and $\nu_{q,n}$ the distribution function of $f_{q,n}(\theta_{q+1}, \ldots, \theta_n)$ with respect to $|g_{q,n}(\theta_{q+1}, \ldots, \theta_n)|^2$.

From the definition of μ_n we obtain by Fubini's theorem for an arbitrary Borel set E

$$\mu_n(E) = \int_{Q_{q,n}} m(\Omega(\theta_{q+1}, \ldots, \theta_n))\, m(d\, Q_{q,n}),$$

where $\Omega(\theta_{q+1}, \ldots, \theta_n)$ denotes the set of points in Q_q for which $f_q(\theta_1, \ldots, \theta_q)$ belongs to $E - f_{q,n}(\theta_{q+1}, \ldots, \theta_n)$.[1] Hence

$$\mu_n(E) = \int_{R_u} \mu_q(E - u)\, \mu_{q,n}(d\, R_u) \tag{64}$$

and consequently if $q \geqq n_0$

$$F_n(x) = \int_{R_u} F_q(x - u)\, \mu_{q,n}(d\, R_u). \tag{65}$$

Since $g_q(\theta_1, \ldots, \theta_q)$ is bounded, say $|g_q(\theta_1, \ldots, \theta_q)| \leqq C_q$, and $|a|^2 \leqq 2|a-b|^2 + 2|b|^2$ for arbitrary complex numbers, we have

$$|g_n(\theta_1, \ldots, \theta_n)|^2 \leqq 2\, C_q^2 + 2\, |g_{q,n}(\theta_{q+1}, \ldots, \theta_n)|^2.$$

Hence we obtain by Fubini's theorem from the definition of ν_n

$$\begin{aligned} \nu_n(E) &\leqq 2\, C_q^2 \mu_n(E) + 2 \int_{Q_{q,n}} m(\Omega(\theta_{q+1}, \ldots, \theta_n))\, |g_{q,n}(\theta_{q+1}, \ldots, \theta_n)|^2\, m(d\, Q_{q,n}) \\ &= 2\, C_q^2 \mu_n(E) + 2 \int_{R_u} \mu_q(E - u)\, \nu_{q,n}(d\, R_u) \end{aligned}$$

and consequently if $q \geqq n_0$

$$G_n(x) \leqq 2\, C_q^2 F_n(x) + 2 \int_{R_u} F_q(x - u)\, \nu_{q,n}(d\, R_u). \tag{66}$$

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[1] By $E - x$ we denote the set of all points $y - x$, where y belongs to E. Similarly, we denote by $x - E$ the set of all points $x - y$, where y belongs to E.

35. Let us write

$$s_{q,n}(\theta_{q+1},\ldots,\theta_n)=\sum_{k=q+1}^{n} l_1 r_k e^{2\pi i\theta_k} \quad\text{and}\quad t_{q,n}(\theta_{q+1},\ldots,\theta_n)=\sum_{k=q+1}^{n}\lambda_k m_1 r_k e^{2\pi i\theta_k}.$$

There exists a constant A_1 such that $|l(z)-l_1 z|\leqq A_1|z|^2$ and $|m(z)-m_1 z|\leqq \leqq A_1|z|^2$ for $|z|\leqq$ the number ϱ_0 introduced in § 31. Hence

$$(67)\qquad |f_{q,n}(\theta_{q+1},\ldots,\theta_n)-s_{q,n}(\theta_{q+1},\ldots,\theta_n)|\leqq\sum_{k=q+1}^{n}A_1 r_k^2\leqq A_1 S_0$$

and

$$(68)\qquad |g_{q,n}(\theta_{q+1},\ldots,\theta_n)-t_{q,n}(\theta_{q+1},\ldots,\theta_n)|\leqq\sum_{k=q+1}^{n}|\lambda_k|A_1 r_k^2\leqq A_1 S_1.$$

From (67) it follows that

$$(69)\quad \int_{Q_{q,n}} e^{8\lambda|f_{q,n}(\theta_{q+1},\ldots,\theta_n)|^2} m(d\,Q_{q,n})\leqq e^{16\lambda A_1^2 S_0^2}\int_{Q_{q,n}} e^{16\lambda|s_{q,n}(\theta_{q+1},\ldots,\theta_n)|^2} m(d\,Q_{q,n}).$$

If we apply Parseval's equation to the function $(s_{q,n}(\theta_{q+1},\ldots,\theta_n))^p$, where p is any positive integer, we obtain

$$\int_{Q_{q,n}} |s_{q,n}(\theta_{q+1},\ldots,\theta_n)|^{2p} m(d\,Q_{q,n})=$$

$$=\sum_{p_{q+1}+\cdots+p_n=p}\left|\frac{p!}{p_{q+1}!\ldots p_n!}(l_1 r_{q+1})^{p_{q+1}}\ldots(l_1 r_n)^{p_n}\right|^2\leqq$$

$$\leqq p!\sum_{p_{q+1}+\cdots+p_n=p}\frac{p!}{p_{q+1}!\ldots p_n!}|l_1 r_{q+1}|^{2p_{q+1}}\ldots|l_1 r_n|^{2p_n}=p!\left(\sum_{k=q+1}^{n}|l_1 r_k|^2\right)^p.$$

Hence, if q is chosen so large that

$$d=1-16\lambda|l_1|^2\sum_{k=q+1}^{\infty}r_k^2$$

is positive, the integral on the right in (69) is $\leqq d^{-1}$. The integral on the left is therefore $\leqq e^{16\lambda A_1^2 S_0^2}d^{-1}=C$ (say).

Let S be a fixed bounded set contained e. g. in the circle $|x|\leqq a$. Then if $|x_0|>2a$ the set x_0-S is contained in $|x|>\frac{1}{2}|x_0|$. Hence $e^{8\lambda(\frac{1}{2}|x_0|)^2}\mu_{q,n}(x_0-S)\leqq C$. For all x_0 we have $\mu_{q,n}(x_0-S)\leqq 1$. Thus we have proved that the functions $\mu_{q,n}(x-S)$ possess a majorant of the form $Ke^{-2\lambda|x|^2}$, and hence (for the same K) the majorant $Ke^{-\lambda|x|^2}$.

Since $|a|^4 \leqq 8|a-b|^4 + 8|b|^4$ for arbitrary complex numbers we obtain from (68)

$$\int_{Q_{q,n}} |g_{q,n}(\theta_{q+1}, \ldots, \theta_n)|^4 m(d\,Q_{q,n}) \leqq 8\,A_1^4 S_1^4 + 8\int_{Q_{q,n}} |t_{q,n}(\theta_{q+1}, \ldots, \theta_n)|^4 m(d\,Q_{q,n}).$$

By Parseval's formula applied to $(t_{q,n}(\theta_{q+1}, \ldots, \theta_n))^2$ the integral on the right is

$$\leqq 2\left(\sum_{k=q+1}^{n} |\lambda_k m_1 r_k|^2\right)^2 < 2|m_1|^4 S_2^2.$$

Hence the integral on the left is $< 8\,A_1^4 S_1^4 + 16|m_1|^4 S_2^2 = D^2$ (say). From the definitions of $\mu_{q,n}$ and $\nu_{q,n}$ we therefore obtain for an arbitrary Borel set E by Schwarz's inequality

$$\nu_{q,n}(E) \leqq D\,\mu_{q,n}(E)^{\frac{1}{2}}.$$

Hence the functions $\nu_{q,n}(x-S)$ also possess a majorant of the form $K e^{-\lambda|x|^2}$.

36. From (65) it follows that $F_n(x) \leqq M_q\,\mu_{q,n}(x-S_q)$, where S_q denotes the set of values of $f_q(\theta_1, \ldots, \theta_q)$, and M_q denotes the maximum of $F_q(x)$. This shows that the functions $F_n(x)$ for $n > q$, and hence for $n \geqq n_0$, have a majorant of the form $K_0 e^{-\lambda|x|^2}$. Since $F_n(x)$ converges towards $F(x)$ as $n \to \infty$, this function also majorizes $F(x)$.

If $q \geqq n_p$ the densities occurring in (65) will possess continuous partial derivatives of order $\leqq p$, and we may differentiate under the integral sign in (65). The same argument then shows that the partial derivatives of order $\leqq p$ of $F_n(x)$ for $n > q$, and hence for $n \geqq n_p$, have a majorant of the form $K_p e^{-\lambda|x|^2}$. Since the partial derivatives converge towards the partial derivatives of $F(x)$ when $n \to \infty$, this function also majorizes the partial derivatives of order $\leqq p$ of $F(x)$.

The corresponding results on the functions $G_n(x)$ and $G(x)$ follow in the same manner from (66).

37. If the series $\sum_{k=1}^{\infty} r_k$ converges, it is plain, since $|l(z)| \leqq A|z|$ for $|z| \leqq \varrho_0$, that all $f_n(\theta_1, \ldots, \theta_n)$ are uniformly bounded, say $|f_n(\theta_1, \ldots, \theta_n)| \leqq K$. This implies that all $F_n(x)$ and $G_n(x)$ and hence $F(x)$ and $G(x)$ vanish for $|x| > K$. We shall now prove the following theorem.

Theorem 7. *If the series* $\sum_{k=1}^{\infty} r_k$ *diverges, then the densities* $F(x)$ *and* $G(x)$ *are* > 0 *for all* x.

38. Let us first consider the function $F(x)$.

For an arbitrary $\varepsilon > 0$ let C_ε denote the circle $|x| < \varepsilon$. Then if x_0 is an arbitrary point of R_x we obtain from (64)

$$\mu_n(x_0 + C_{2\varepsilon}) \geqq \mu_q(x_0 + C_\varepsilon)\,\mu_{q,n}(C_\varepsilon);$$

for when u belongs to C_ε the set $x_0 + C_{2\varepsilon} - u$ will contain $x_0 + C_\varepsilon$. Now

$$\int\limits_{Q_{q,n}} |f_{q,n}(\theta_{q+1}, \ldots, \theta_n)|^2 m(d\,Q_{q,n}) = \sum_{k=q+1}^{n} \int\limits_{c_k} |l(r_k e^{2\pi i\theta_k})|^2 d\theta_k +$$

$$+ \sum_{\substack{k,l=q+1\\ k\neq l}}^{n} \int\limits_{c_k} l(r_k e^{2\pi i\theta_k})\,d\theta_k \int\limits_{c_l} \overline{l(r_l e^{2\pi i\theta_l})}\,d\theta_l.$$

Here the last term vanishes, and since $|l(z)| \leqq A|z|$ for $|z| \leqq \varrho_0$ the first term is $\leqq A^2(r_{q+1}^2 + \cdots + r_n^2)$. Hence

$$\varepsilon^2(1 - \mu_{q,n}(C_\varepsilon)) \leqq A^2(r_{q+1}^2 + r_{q+2}^2 + \cdots).$$

If q is large enough the right-hand side is $\leqq \frac{1}{2}\varepsilon^2$. Then $\mu_{q,n}(C_\varepsilon) \geqq \frac{1}{2}$ and consequently $\mu_n(x_0 + C_{2\varepsilon}) \geqq \frac{1}{2}\mu_q(x_0 + C_\varepsilon)$ for all n. For $n \to \infty$ this yields

$$\mu(x_0 + C_{2\varepsilon}) \geqq \tfrac{1}{2}\mu_q(x_0 + C_\varepsilon). \tag{70}$$

Since $|l(z) - l_1 z| \leqq A_1|z|^2$ for $|z| \leqq \varrho_0$ we have for $q > p > 0$

$$|f_{p,q}(\theta_{p+1}, \ldots, \theta_q) - s_{p,q}(\theta_{p+1}, \ldots, \theta_q)| \leqq A_1(r_{p+1}^2 + r_{p+2}^2 + \cdots).$$

Let p be chosen so large that the right-hand side is $< \varepsilon$. Let $\theta_1, \ldots, \theta_p$ be arbitrarily chosen and put $x_1 = f_p(\theta_1, \ldots, \theta_p)$. Then if q is large enough we have $|l_1|r_{p+1} + \cdots + |l_1|r_q > |x_0 - x_1|$, and none of the numbers $|l_1|r_{p+1}, \ldots, |l_1|r_q$ is larger than the sum of the $q - p - 1$ others. As is easily seen this implies that we may choose $\theta_{p+1}, \ldots, \theta_q$ such that $s_{p,q}(\theta_{p+1}, \ldots, \theta_q) = x_0 - x_1$. This implies that $f_q(\theta_1, \ldots, \theta_q)$ belongs to $x_0 + C_\varepsilon$, so that $\mu_q(x_0 + C_\varepsilon) > 0$. On account of (70) this shows that $\mu(x_0 + C_{2\varepsilon}) > 0$. Thus we have proved that $\mu(E)$ is > 0 for any set E which contains interior points.

For $q \geqq n_0$ we obtain from (65) for $n \to \infty$

$$F(x) = \int\limits_{R_u} F_q(x - u)\,\varrho_q(d\,R_u), \tag{71}$$

where ϱ_q denotes the distribution function towards which $\mu_{q,n}$ converges for $n \to \infty$. Evidently this distribution function also has the property that $\varrho_q(E) > 0$

for any set E containing interior points. The relation (71) therefore implies that $F(x) > 0$ for all x.

39. Next we shall consider the function $G(x)$.

Since $r_n \to 0$ when $n \to \infty$, and since $G(x)$ is not altered if for an arbitrary N we make a permutation of the numbers $r_1, \ldots, r_N$ and the same permutation of the numbers $\lambda_1, \ldots, \lambda_N$, we may suppose that the numbers r_1 and r_2 are as small as we please and that $r_1 > r_2$.

The proof depends on an elementary proposition, viz. that if r_1 and r_2 are sufficiently small, and $r_1 > r_2$, then there exist two pairs of values (θ_1', θ_2') and (θ_1'', θ_2''), such that

$$f_2(\theta_1', \theta_2') = f_2(\theta_1'', \theta_2'') \quad \text{whereas} \quad g_2(\theta_1', \theta_2') \neq g_2(\theta_1'', \theta_2''), \tag{72}$$

and such that if we write $f_2(\theta_1, \theta_2) = u_1(\theta_1, \theta_2) + i u_2(\theta_1, \theta_2)$, the Jacobian

$$\frac{\partial(u_1, u_2)}{\partial(\theta_1, \theta_2)} \tag{73}$$

is $\neq 0$ in both of the points (θ_1', θ_2') and (θ_1'', θ_2''). We prove this as follows.

It is known that the curve S_r with the parametric representation $x = x(\theta) = l(re^{2\pi i\theta})$ is convex if r is sufficiently small, say for $r \leqq r_0$. Since $x'(\theta) = 2\pi i r e^{2\pi i\theta} l'(re^{2\pi i\theta})$, the outer normal of S_r at a point z is determined by $l^*(z) = z l'(z) = l_1 z + 2 l_2 z^2 + \cdots$ provided that $l^*(z) \neq 0$. We may suppose that $l^*(z) \neq 0$ for $|z| \leqq r_0$. For an arbitrary x the points (θ_1, θ_2) with $f_2(\theta_1, \theta_2) = x$ are determined by the common points of the curves S_{r_1} and $x - S_{r_2}$. For an arbitrary point (θ_1, θ_2) the Jacobian (73) is equal to the area of the parallelogram determined by the vectors $2\pi l^*(r_1 e^{2\pi i\theta_1})$ and $2\pi l^*(r_2 e^{2\pi i\theta_2})$. If $r_1 \leqq r_0$ and $r_2 \leqq r_0$ there exists to every θ_1 a unique θ_2 such that these vectors have the same direction. Let (θ_1^0, θ_2^0) be a pair of such values. Then, if we place $x^0 = f_2(\theta_1^0, \theta_2^0)$, the curves S_{r_1} and $x^0 - S_{r_2}$ are externally tangent to each other. Hence, if x^0 is moved slightly in the opposite direction of $l^*(r_1 e^{2\pi i\theta_1^0})$ to a point x^*, the curves S_{r_1} and $x^* - S_{r_2}$ will have two points of intersection near the former point of contact. This shows that in any neighbourhood of (θ_1^0, θ_2^0) there exist points (θ_1', θ_2') and (θ_1'', θ_2'') for which the Jacobian (73) is $\neq 0$, and for which the first of the conditions (72) is satisfied.

If $l(z)$ and $m(z)$ are proportional, i. e. if $m(z) = m_1 l_1^{-1} l(z)$, we have $g_2(\theta_1, \theta_2) = \lambda_1 m_1 l_1^{-1} f_2(\theta_1, \theta_2) + (\lambda_2 - \lambda_1) m_1 l_1^{-1} l(r_2 e^{2\pi i\theta_2})$. The second of the conditions (72) is therefore satisfied. It remains to consider the case where $l(z)$ and $m(z)$ are not proportional.

If we place $g_2(\theta_1, \theta_2) = v_1(\theta_1, \theta_2) + i v_2(\theta_1, \theta_2)$ and $m^*(z) = z m'(z) = m_1 z + 2 m_2 z^2 + \cdots$, the Jacobian

$$\frac{\partial(v_1, v_2)}{\partial(\theta_1, \theta_2)}$$

is equal to the area of the parallelogram determined by $2\pi\lambda_1 m^*(r_1 e^{2\pi i\theta_1})$ and $2\pi\lambda_2 m^*(r_2 e^{2\pi i\theta_2})$. If the Jacobian is $\neq 0$ at (θ_1^0, θ_2^0), the function $g_2(\theta_1, \theta_2)$ will take different values in different points of a neighbourhood. It is therefore sufficient to prove that, when r_1 and r_2 are sufficiently small and $r_1 > r_2$, there exist such points z_1 and z_2 on the circles $|z| = r_1$ and $|z| = r_2$, that $l^*(z_1)$ and $l^*(z_2)$ have the same direction, whereas $m^*(z_1)$ and $m^*(z_2)$ are not parallel.

Suppose that r_0 has been chosen so small that the function $w = l^*(z)$ for $|z| \leqq r_0$ has a regular inverse function $z = z(w) = l_1^{-1} w + \cdots$, and put $m^*(z(w)) = m_1 l_1^{-1} w + c_1 w^2 + c_2 w^3 + \cdots = h(w)$. Since $l(z)$ and $m(z)$ are not proportional, the functions $l^*(z)$ and $m^*(z)$ are not proportional either, i. e. the coefficients $c_1, c_2, \ldots$ do not all vanish; let c_ν be the first which is $\neq 0$. The images of the circles $|z| = r_1$ and $|z| = r_2$ in the w-plane are two curves C_1 and C_2 each of which intersects an arbitrary half-line with origin 0 in one point. Since $r_1 > r_2$ the curve C_1 surrounds C_2. Our object is to choose the half-line in such a manner that for the corresponding points w_1 and w_2 on C_1 and C_2 the vectors $h(w_1)$ and $h(w_2)$ are not parallel, i. e., on placing $k(w) = h(w)/m_1 l_1^{-1} w = 1 + d_\nu w^\nu + \cdots$, in such a manner that the vectors $k(w_1)$ and $k(w_2)$ are not parallel.

Suppose that r_0 has been chosen so small that $|k(w) - 1| \leqq (\text{some})\ a < 1$ in the domain of the w-plane which corresponds to $|z| \leqq r_0$, and that in addition $y = k(w)$ for this domain has an inverse function $w = w((y-1)^{1/\nu})$, which is regular on the Riemann surface of $(y-1)^{1/\nu}$. Then the images of C_1 and C_2 in the y-plane are two curves D_1 and D_2 on this surface, such that D_1 surrounds D_2, and these curves belong to $|y - 1| \leqq a$. Let y_1 be a point on D_1 with maximal argument; then there is no point y_2 on D_2 with the same argument. Hence, if the half-line is chosen such that $k(w_1) = y_1$, the vectors $k(w_1)$ and $k(w_2)$ will not be parallel. This completes the proof of our elementary proposition.

40. By means of this proposition the theorem may now be proved as follows.

If we denote by M a sufficiently small neighbourhood of the point $x^* = f_2(\theta_1', \theta_2') = f_2(\theta_1'', \theta_2'')$, the functions $\xi_1 = u_1(\theta_1, \theta_2)$ and $\xi_2 = u_2(\theta_1, \theta_2)$ will determine a mapping of certain neighbourhoods A' and A'' of (θ_1', θ_2') and (θ_1'', θ_2'') on M, and the inverse transformations will be determined by functions

(74) $$\begin{aligned} \theta_1 &= \gamma_1'(\xi_1, \xi_2) \\ \theta_2 &= \gamma_2'(\xi_1, \xi_2) \end{aligned} \quad \text{and} \quad \begin{aligned} \theta_1 &= \gamma_1''(\xi_1, \xi_2) \\ \theta_2 &= \gamma_2''(\xi_1, \xi_2) \end{aligned}$$

with continuous partial derivatives and with Jacobians

$$\frac{\partial(\gamma_1', \gamma_2')}{\partial(\xi_1, \xi_2)} \quad \text{and} \quad \frac{\partial(\gamma_1'', \gamma_2'')}{\partial(\xi_1, \xi_2)}$$

which are numerically $\geqq$ (some) $k_1 > 0$. Introducing the functions (74) in $g_2(\theta_1, \theta_2)$ we obtain two functions

$$\Gamma'(x) = g_2(\gamma_1'(\xi_1, \xi_2), \gamma_2'(\xi_1, \xi_2)) \quad \text{and} \quad \Gamma''(x) = g_2(\gamma_1''(\xi_1, \xi_2), \gamma_2''(\xi_1, \xi_2))$$

for which $\Gamma'(x^*) \neq \Gamma''(x^*)$. We may, therefore, suppose that M has been chosen so small that in M

(75) $$|\Gamma'(x) - \Gamma''(x)| \geqq \text{(some) } k_2 > 0.$$

From the definition of ν_n we obtain by Fubini's theorem for an arbitrary Borel set E

$$\nu_n(E) = \int_{Q_2} m(d\,Q_2) \int_{\Omega(\theta_1, \theta_2)} |g_n(\theta_1, \ldots, \theta_n)|^2 \, m(d\,Q_{2,n}),$$

where $\Omega(\theta_1, \theta_2)$ denotes the set of points in $Q_{2,n}$ for which $f_{2,n}(\theta_3, \ldots, \theta_n)$ belongs to $E - f_2(\theta_1, \theta_2)$. Hence

$$\nu_n(E) \geqq \int_{A'} m(d\,Q_2) \int_{\Omega(\theta_1, \theta_2)} |g_n(\theta_1, \ldots, \theta_n)|^2 \, m(d\,Q_{2,n}) + \\ + \int_{A''} m(d\,Q_2) \int_{\Omega(\theta_1, \theta_2)} |g_n(\theta_1, \ldots, \theta_n)|^2 \, m(d\,Q_{2,n}).$$

In these integrals we apply the substitutions (74) and thus obtain

(76) $$\nu_n(E) \geqq \int_M k_1 \, m(d\,R_x) \int_{\Omega(x)} (|\Gamma'(x) + g_{2,n}(\theta_3, \ldots, \theta_n)|^2 + \\ + |\Gamma''(x) + g_{2,n}(\theta_3, \ldots, \theta_n)|^2) \, m(d\,Q_{2,n}),$$

where $\Omega(x)$ denotes the set of points in $Q_{2,n}$ in which $f_{2,n}(\theta_3, \ldots, \theta_n)$ belongs to $E - x$. Now, $|a + c|^2 + |b + c|^2 \geqq \frac{1}{2}|a - b|^2$ for arbitrary complex numbers.

Hence by (75) the integrand in the inner integral in (76) is $\geqq \frac{1}{2} k_2^2$ for all x in M. Consequently

$$\nu_n(E) \geqq \int\limits_M k_1 m(dR_x) \int\limits_{\Omega(x)} \tfrac{1}{2} k_2^2 m(dQ_{2,n}) = \tfrac{1}{2} k_1 k_2^2 \int\limits_M \mu_{2,n}(E-x)\, m(dR_x),$$

whence for $n \to \infty$

$$\nu(E) \geqq \tfrac{1}{2} k_1 k_2^2 \int\limits_M \varrho_2(E-x)\, m(dR_x),$$

so that for an arbitrary x_0

$$G(x_0) \geqq \tfrac{1}{2} k_1 k_2^2 \int\limits_M R_2(x_0 - x)\, m(dR_x),$$

where $R_2(x)$ denotes the density of ϱ_2. Since, by the first part of the theorem, $R_2(x) > 0$ for all x, this shows that $G(x) > 0$ for all x.

41. Next we shall prove the following theorem.

Theorem 8. *If $r_n^{-1} = O(n)$, then the densities $F(x) = F(\xi_1, \xi_2)$ and $G(x) = G(\xi_1, \xi_2)$ are regular analytic in every point of the real plane R_x. If $r_n^{-1} = o(n)$, then $F(x)$ and $G(x)$ are entire functions of the two variables ξ_1, ξ_2.*

Consider the products

$$\prod_{k=1}^{\infty} \mathrm{K}_0(y, r_k), \quad \prod_{\substack{k=1\\k \neq l}}^{\infty} \mathrm{K}_0(y, r_k), \quad \text{and} \quad \prod_{\substack{k=1\\k \neq l,m}}^{\infty} \mathrm{K}_0(y, r_k)$$

occurring in the expressions (61) and (62) for $\Lambda(y;\mu)$ and $\Lambda(y;\nu)$. Let $b = \limsup_{n\to\infty} r_n^{-1}/n$. Then if $a > b$ there exists a p_0 such that $r_n \leqq$ the number ϱ_1 introduced in § 30, and $r_n^{-1} \leqq a n$, for $n > p_0$. We have then $|\mathrm{K}_0(y, r_n)| \leqq$ $\leqq B r_n^{-\frac{1}{2}} |y|^{-\frac{1}{2}} \leqq B a^{\frac{1}{2}} n^{\frac{1}{2}} |y|^{-\frac{1}{2}}$ for every $n > p_0$. The p^{th} factor in each of the products corresponds to a value k such that $p \leqq k \leqq p+2$. Consequently, $|\mathrm{K}_0(y, r_k)| \leqq B a^{\frac{1}{2}} (p+2)^{\frac{1}{2}} |y|^{-\frac{1}{2}} = (p+2)^{\frac{1}{2}} t^{-\frac{1}{2}}$, where $t = B^{-2} a^{-1} |y|$, if $p > p_0$. Since $|\mathrm{K}_0(y, r_k)| \leqq 1$ for all k it follows that for $t \geqq p_0 + 3$ each product is numerically

$$\leqq \prod_{p=p_0+1}^{\infty} \min\{1, (p+2)^{\frac{1}{2}} t^{-\frac{1}{2}}\} = \prod_{p=p_0+1}^{t-2} (p+2)^{\frac{1}{2}} t^{-\frac{1}{2}} = \prod_{q \leqq t} q^{\frac{1}{2}} t^{-\frac{1}{2}} \Big/ \prod_{q \leqq p_0+2} q^{\frac{1}{2}} t^{-\frac{1}{2}},$$

which by Stirling's formula is $O(t^{\frac{5}{4} + \frac{1}{2} p_0} e^{-\frac{1}{2} t})$. Thus each product is $\leqq C e^{-c|y|}$ for every $c < \frac{1}{2} B^{-2} a^{-1}$, the constant C (depending on c) being the same for all products. Hence

$$|\Lambda(y;\mu)| \leqq C e^{-c|y|}$$

and

$$|\Lambda(y;\nu)| \leqq \left(\sum_{l=1}^{\infty} \lambda_l^2 A^2 r_l^2 + \sum_{\substack{l,m=1\\ l\neq m}}^{\infty} |\lambda_l|\,|\lambda_m|\, A^4 r_l^2 r_m^2 \,|y|^2\right) C e^{-c|y|}$$

$$\leqq (A^2 S_2 + A^4 S_1^2 |y|^2)\, C e^{-c|y|}$$

Consequently, $\Lambda(y;\mu)$ and $\Lambda(y;\nu)$ are $O(e^{-c|y|})$ for every $c < \frac{1}{2} B^{-2} a^{-1}$, which proves the first part of the theorem (cf. § 6). If $b = 0$ we may take a arbitrarily small; hence $\Lambda(y;\mu)$ and $\Lambda(y;\nu)$ are $O(e^{-c|y|})$ for arbitrarily large c which proves the second part of the theorem.

42. In the applications the numbers $r_1, r_2, \ldots$ will depend on a parameter σ (whereas $\lambda_1, \lambda_2, \ldots$ remain constants).

Theorem 9. *If $r_1, r_2, \ldots$ are continuous functions of a parameter σ in a closed interval $\sigma_1 \leqq \sigma \leqq \sigma_2$, and $r_n \to 0$ uniformly in σ, then the distribution functions μ_n and ν_n will for every n depend continuously on σ, the numbers n_p, $p \geqq 0$, may be chosen independent of σ, and the densities $F_n(x)$ and $G_n(x)$ and their partial derivatives will be continuous functions of x and σ together.*

If, moreover, the series S_0, S_1, S_2 have convergent majorants, then the distribution functions μ and ν will depend continuously on σ, and the densities $F(x)$ and $G(x)$ and their partial derivatives will be continuous functions of x and σ together. Further, the densities $F_n(x)$ and $G_n(x)$ and their partial derivatives will converge uniformly in x and σ together towards $F(x)$ and $G(x)$ and their partial derivatives. Finally, the majorants of Theorem 6 may for every $\lambda > 0$ be chosen independent of σ.

From the expressions (44) it will be seen that $\Lambda(y;\mu_n)$ and $\Lambda(y;\nu_n)$ for every n depend continuously on y and σ together. On examination of § 31 we see that ϱ_0, h, and n_p successively may be chosen independent of σ. Also, since each $r_n^{-\frac{1}{2}}$ is a bounded function of σ, there will for every $n \geqq n_p$ exist bounded majorants of $\Lambda(y;\mu_n)$ and $\Lambda(y;\nu_n)$ which are $O(|y|^{-(\frac{5}{2}+p)})$ and are independent of σ. This establishes the first part of the theorem.

The estimates (57) and (60) show that the uniform convergence of $\Lambda(y;\mu_n)$ and $\Lambda(y;\nu_n)$ towards $\Lambda(y;\mu)$ and $\Lambda(y;\nu)$ in any circle $|y| \leqq a$ is also uniform in σ. Hence, $\Lambda(y;\mu)$ and $\Lambda(y;\nu)$ depend continuously on y and σ together. For every $p \geqq 0$ the estimates (55), (56), (58), and (59) show that $\Lambda(y;\mu_n)$ and $\Lambda(y;\nu_n)$ for $n \geqq n_p$ possess bounded majorants which are $O(|y|^{-(\frac{5}{2}+p)})$ and are independent of σ. This establishes the second part of the theorem except the last statement, which follows on examination of the proof of Theorem 6, where again all constants may be chosen independent of σ.

Theorem 10. *Let $r_1, r_2, \ldots$ be continuous functions of a parameter σ in a closed interval $\sigma_1 \leqq \sigma \leqq \sigma_2$, such that $r_n \to 0$ uniformly in σ. Let the series S_0, S_1, S_2 be convergent for $\sigma_1 < \sigma \leqq \sigma_2$, but let S_0 be divergent for $\sigma = \sigma_1$. Then the density $F(x)$ of the distribution function μ and each of its partial derivatives will converge uniformly in x towards zero when $\sigma \to \sigma_1$.*

By the expression of $F(x)$ and its partial derivatives it is sufficient to prove that

$$\int_{R_y} |y|^p \, |\Lambda(y; \mu)| \, m(dR_y) \to 0$$

as $\sigma \to \sigma_1$ for every $p \geqq 0$.

Since $|\Lambda(y; \mu)| \leqq |\Lambda(y; \mu_n)|$ for every n there exists according to the proof of Theorem 9 a bounded majorant of $\Lambda(y; \mu)$ which is $O(|y|^{-(\frac{5}{2}+p)})$ and is independent of σ. We therefore only need to prove that $\Lambda(y; \mu) \to 0$ uniformly in every domain $0 < c \leqq |y| \leqq C$ when $\sigma \to \sigma_1$. Let q be chosen so large that $r_k C \leqq B_2$ for $k \geqq q$ and all σ, where B_2 is the constant occurring in the estimate (51). Then, if $c \leqq |y| \leqq C$, we have

$$|\Lambda(y; \mu)| = \prod_{k=1}^{\infty} |\mathrm{K}_0(y, r_k)| \leqq \prod_{k=q}^{\infty} (1 - B_1 r_k^2 |y|^2) \leqq \prod_{k=q}^{\infty} (1 - B_1 r_k^2 c^2).$$

Since the series S_0 diverges for $\sigma = \sigma_1$, the last product converges towards zero when $\sigma \to \sigma_1$, and this establishes the theorem.

Distribution Functions Connected with the Zeta Function and its Logarithm.

43. In § 27 we have reduced the study of the functions $\zeta(s)$ and $\log \zeta(s)$ to a study of the functions $\zeta_n(s)$ and $\log \zeta_n(s)$. Together with $\zeta_n(s)$ we shall consider the whole class of functions

$$\zeta_n(s; \theta_1, \ldots, \theta_n) = \prod_{k=1}^{n} (1 - p_k^{-s} e^{2\pi i \theta_k})^{-1}.$$

These functions are all regular and $\neq 0$ for $\sigma > 0$.

Let us now consider the functions

$$\log \zeta_n(s; \theta_1, \ldots, \theta_n) = \sum_{k=1}^{n} - \log (1 - p_k^{-s} e^{2\pi i \theta_k}),$$

where in each term on the right $-\log(1 - z) = z + \frac{1}{2} z^2 + \cdots$, and their derivatives with respect to s

$$\zeta_n'/\zeta_n(s;\theta_1,\ldots,\theta_n)=\sum_{k=1}^{n}-\frac{(\log p_k)\,p_k^{-s}\,e^{2\pi i\theta_k}}{1-p_k^{-s}\,e^{2\pi i\theta_k}}.$$

For $s=\sigma>0$ these are the functions $f_n(\theta_1,\ldots,\theta_n)$ and $g_n(\theta_1,\ldots,\theta_n)$ of Theorem 5, if we take $l(z)=-\log(1-z)$, $m(z)=z\,l'(z)=z/(1-z)$, where $|z|<1$, $r_n=p_n^{-\sigma}$, and $\lambda_n=-\log p_n$. Then $r_n\to 0$ when $n\to\infty$ for any $\sigma>0$, and the three series S_0, S_1, S_2 are convergent for $\sigma>\frac{1}{2}$, so that Theorems 5 and 6 are applicable.

The estimate (54) in this case holds for any $\varrho_1<1$.[1] The proof of Theorem 5 therefore shows that the theorem is valid with $n_0=11$ and $n_p=11+2p$. Theorem 7 is applicable for $\frac{1}{2}<\sigma\leqq 1$, and Theorem 8 for $\frac{1}{2}<\sigma<1$, in which case $r_n^{-1}=o(n)$. Finally, $r_n\to 0$ as $n\to\infty$ uniformly for $\sigma>$ (any) $\alpha>0$, and the three series S_0, S_1, S_2 have convergent majorants for $\sigma>$ (any) $\alpha>\frac{1}{2}$, so that the first part of Theorem 9 is applicable for any interval $(0<)\,\sigma_1\leqq\sigma\leqq\sigma_2\,(<+\infty)$, while the second part of the theorem is applicable for any interval $(\frac{1}{2}<)\,\sigma_1\leqq\sigma\leqq\sigma_2\,(<+\infty)$. Finally, the series S_0 is divergent for $\sigma=\frac{1}{2}$, so that Theorem 10 is applicable for $\sigma_1=\frac{1}{2}$.

Thus we obtain the following theorem.

Theorem 11. *For an arbitrary $\sigma>0$ the distribution functions $\mu_{n,\sigma}$ and $\nu_{n,\sigma}$ of $\log\zeta_n(\sigma;\theta_1,\ldots,\theta_n)$ and of $\log\zeta_n(\sigma;\theta_1,\ldots,\theta_n)$ with respect to $|\zeta_n'/\zeta_n(\sigma;\theta_1,\ldots,\theta_n)|^2$ are for $n\geqq 11$ absolutely continuous with continuous densities $F_{n,\sigma}(x)$ and $G_{n,\sigma}(x)$ which for $n\geqq 11+2p$ possess continuous partial derivatives of order $\leqq p$.*

If $\sigma>\frac{1}{2}$, the distribution functions $\mu_{n,\sigma}$ and $\nu_{n,\sigma}$ converge for $n\to\infty$ towards distribution functions μ_σ and ν_σ which are absolutely continuous with continuous densities $F_\sigma(x)$ and $G_\sigma(x)$ possessing continuous partial derivatives of arbitrarily high order. The functions $F_{n,\sigma}(x)$ and $G_{n,\sigma}(x)$ and their partial derivatives converge uniformly towards $F_\sigma(x)$ and $G_\sigma(x)$ and their partial derivatives for $n\to\infty$. If $\frac{1}{2}<\sigma\leqq 1$, then $F_\sigma(x)>0$ and $G_\sigma(x)>0$ for all x. If $\frac{1}{2}<\sigma<1$, then $F_\sigma(x)$ and $G_\sigma(x)$ are entire functions of the two variables ξ_1, ξ_2.

The distribution functions all depend continuously on σ, and their densities and the partial derivatives of the densities are continuous functions of x and σ together. Further, if $\frac{1}{2}<\alpha<\beta<+\infty$, the convergence of $F_{n,\sigma}(x)$ and $G_{n,\sigma}(x)$ and their partial derivatives towards $F_\sigma(x)$ and $G_\sigma(x)$ and their partial derivatives is uniform in x and σ together for all x and $\alpha\leqq\sigma\leqq\beta$. If $\lambda>0$ is arbitrary and $\frac{1}{2}<\alpha<\beta<+\infty$, the functions $F_\sigma(x)$, $G_\sigma(x)$ and $F_{n,\sigma}(x)$, $G_{n,\sigma}(x)$, $n\geqq 11$, have for $\alpha\leqq\sigma\leqq\beta$ a majorant

[1] See Jessen and Wintner [1], p. 70.

of the form $K_0 e^{-\lambda |x|^2}$, *and for every* p *the partial derivatives of* $F_\sigma(x)$, $G_\sigma(x)$ *and* $F_{n,\sigma}(x)$, $G_{n,\sigma}(x)$, $n \geqq 11 + 2p$, *of order* $\leqq p$, *have a majorant of the form* $K_p e^{-\lambda |x|^2}$.

The density $F_\sigma(x)$ *and each of its partial derivatives converge uniformly towards zero as* $\sigma \to \frac{1}{2}$.

We remark that since $\log \zeta_n(\sigma; \theta_1, \ldots, \theta_n)$ and $\zeta_n'/\zeta_n(\sigma; \theta_1, \ldots, \theta_n)$ take conjugate values in the points $(\theta_1, \ldots, \theta_n)$ and $(-\theta_1, \ldots, -\theta_n)$, all the distribution functions and hence also their densities are symmetric with respect to the line $\xi_2 = 0$.

44. Let R_x be mapped on itself by the transformation e^x; every point $x = \xi_1 + i\xi_2 \neq 0$ is then the image of the enumerable set of points $\log x = \log|x| + i \arg x$. In the neighbourhood of each of these points the Jacobian of the transformation is equal to $|x|^2$. If E is an arbitrary set in R_x we denote by $\log E$ the set of all points x such that e^x belongs to E. We shall now prove the following theorem.

Theorem 12. *For an arbitrary* $\sigma > 0$ *the distribution functions* $\bar{\mu}_{n,\sigma}$ *and* $\bar{\nu}_{n,\sigma}$ *of* $\zeta_n(\sigma; \theta_1, \ldots, \theta_n)$ *and of* $\zeta_n(\sigma; \theta_1, \ldots, \theta_n)$ *with respect to* $|\zeta_n'(\sigma; \theta_1, \ldots, \theta_n)|^2$ *are determined by*

$$(77) \qquad \bar{\mu}_{n,\sigma}(E) = \mu_{n,\sigma}(\log E) \quad \textit{and} \quad \bar{\nu}_{n,\sigma}(E) = \int_{\log E} e^{2\xi_1} \nu_{n,\sigma}(dR_x).$$

For $n \geqq 11$ *they are absolutely continuous with continuous densities* $\bar{F}_{n,\sigma}(x)$ *and* $\bar{G}_{n,\sigma}(x)$ *which are zero for* $x = 0$ *and for* $x \neq 0$ *are determined by*

$$(78) \qquad \bar{F}_{n,\sigma}(x) = |x|^{-2} \sum_{\log x} F_{n,\sigma}(\log x) \quad \textit{and} \quad \bar{G}_{n,\sigma}(x) = \sum_{\log x} G_{n,\sigma}(\log x),$$

where the summations are with respect to all values of $\log x$. *For* $n \geqq 11 + 2p$ *the densities possess continuous partial derivatives of order* $\leqq p$.

If $\sigma > \frac{1}{2}$, *the distribution functions* $\bar{\mu}_{n,\sigma}$ *and* $\bar{\nu}_{n,\sigma}$ *converge for* $n \to \infty$ *towards distribution functions* $\bar{\mu}_\sigma$ *and* $\bar{\nu}_\sigma$ *which are determined by*

$$(79) \qquad \bar{\mu}_\sigma(E) = \mu_\sigma(\log E) \quad \textit{and} \quad \bar{\nu}_\sigma(E) = \int_{\log E} e^{2\xi_1} \nu_\sigma(dR_x)$$

and are absolutely continuous with continuous densities $\bar{F}_\sigma(x)$ *and* $\bar{G}_\sigma(x)$ *which are zero for* $x = 0$ *and for* $x \neq 0$ *are determined by*

$$(80) \qquad \bar{F}_\sigma(x) = |x|^{-2} \sum_{\log x} F_\sigma(\log x) \quad \textit{and} \quad \bar{G}_\sigma(x) = \sum_{\log x} G_\sigma(\log x).$$

The densities possess continuous partial derivatives of arbitrarily high order which all vanish for $x = 0$. *The functions* $\bar{F}_{n,\sigma}(x)$ *and* $\bar{G}_{n,\sigma}(x)$ *and their partial derivatives* *converge uniformly towards* $\bar{F}_\sigma(x)$ *and* $\bar{G}_\sigma(x)$ *and their partial derivatives when* $n \to \infty$. *If* $\frac{1}{2} < \sigma \leqq 1$, *then* $\bar{F}_\sigma(x) > 0$ *and* $\bar{G}_\sigma(x) > 0$ *for all* $x \neq 0$. *If* $\frac{1}{2} < \sigma < 1$, *then* $\bar{F}_\sigma(e^x)$ *and* $\bar{G}_\sigma(e^x)$ *are entire functions of the two variables* ξ_1, ξ_2.

The distribution functions all depend continuously on σ, *and their densities and the partial derivatives of the densities are continuous functions of* x *and* σ *together. Further, if* $\frac{1}{2} < \alpha < \beta < +\infty$, *then the convergence of* $\bar{F}_{n,\sigma}(x)$ *and* $\bar{G}_{n,\sigma}(x)$ *and their partial derivatives towards* $\bar{F}_\sigma(x)$ *and* $\bar{G}_\sigma(x)$ *and their partial derivatives is uniform in* x *and* σ *together for all* x *and* $\alpha \leqq \sigma \leqq \beta$. *If* $\lambda > 0$ *is arbitrary, and* $\frac{1}{2} < \alpha < \beta < +\infty$, *then the functions* $\bar{F}_\sigma(x)$, $\bar{G}_\sigma(x)$ *and* $\bar{F}_{n,\sigma}(x)$, $\bar{G}_{n,\sigma}(x)$, $n \geqq 11$, *have for* $\alpha \leqq \sigma \leqq \beta$ *for* $x \neq 0$ *a majorant of the form* $K_0 e^{-\lambda(\log|x|)^2}$, *and for every* p *the partial derivatives of* $\bar{F}_\sigma(x)$, $\bar{G}_\sigma(x)$ *and* $\bar{F}_{n,\sigma}(x)$, $\bar{G}_{n,\sigma}(x)$, $n \geqq 11 + 2p$, *of order* $\leqq p$, *have for* $x \neq 0$ *a majorant of the form* $K_p e^{-\lambda(\log|x|)^2}$.

The density $\bar{F}_\sigma(x)$ *multiplied by* $|x|^2$ *and each of its partial derivatives of order* p *multiplied by* $|x|^{2+p}$ *tend uniformly to zero when* $\sigma \to \frac{1}{2}$.

We observe that the distribution functions and hence also their densities are symmetric with respect to the line $\xi_2 = 0$.

Most of the statements are immediate consequences of Theorem 11. By definition we have.

$$\bar{\mu}_{n,\sigma}(E) = m(\Omega(E)) \quad \text{and} \quad \bar{\nu}_{n,\sigma}(E) = \int\limits_{\Omega(E)} |\zeta_n'(\sigma; \theta_1, \ldots, \theta_n)|^2 m(d\,Q_n),$$

where $\Omega(E)$ denotes the set of points in Q_n for which $\zeta_n(\sigma; \theta_1, \ldots, \theta_n)$ belongs to E. Since $\Omega(E)$ is also the set of points in Q_n for which $\log \zeta_n(\sigma; \theta_1, \ldots, \theta_n)$ belongs to $\log E$, the expressions (77) follow immediately. The remainder of the first part of the theorem follows from (77), since for every n the functions $F_{n,\sigma}(x)$ and $G_{n,\sigma}(x)$ are zero outside the bounded set of values of $\log \zeta_n(\sigma; \theta_1, \ldots, \theta_n)$. The sums (78) therefore contain only a finite number of terms different from zero.

The expressions (77) may for $n \geqq 11$ be written

$$\bar{\mu}_{n,\sigma}(E) = \int\limits_{\log E} F_{n,\sigma}(x)\, m(d\,R_x) \quad \text{and} \quad \bar{\nu}_{n,\sigma}(E) = \int\limits_{\log E} e^{2\xi_1} G_{n,\sigma}(x)\, m(d\,R_x).$$

Since the integrands for $n \to \infty$ converge towards $F_\sigma(x)$ and $e^{2\xi_1} G_\sigma(x)$, and since the convergence for every $\lambda > 0$ is majorized by integrable functions $K_0 e^{-\lambda|x|^2}$ and $K_0 e^{2\xi_1} e^{-\lambda|x|^2}$, it is plain that $\bar{\mu}_{n,\sigma}$ and $\bar{\nu}_{n,\sigma}$ converge towards the distribution functions (79), which may be written

$$\bar{\mu}_\sigma(E) = \int\limits_{\log E} F_\sigma(x)\, m(d\,R_x) \quad \text{and} \quad \bar{\nu}_\sigma(E) = \int\limits_{\log E} e^{2\xi_1} G_\sigma(x)\, m(d\,R_x).$$

Since $\log E$ is a null-set when E is a null-set, it is obvious that $\bar{\mu}_\sigma$ and $\bar{\nu}_\sigma$ are absolutely continuous, and also that their densities are given by (80).

On placing

(81) $$\Phi_{n,\sigma}(x) = \sum_{h=-\infty}^{\infty} F_{n,\sigma}(x + 2\pi i h), \qquad \Phi_\sigma(x) = \sum_{h=-\infty}^{\infty} F_\sigma(x + 2\pi i h)$$

and

(82) $$\Gamma_{n,\sigma}(x) = \sum_{h=-\infty}^{\infty} G_{n,\sigma}(x + 2\pi i h), \qquad \Gamma_\sigma(x) = \sum_{h=-\infty}^{\infty} G_\sigma(x + 2\pi i h)$$

we have

$$\bar{F}_{n,\sigma}(x) = |x|^{-2}\, \Phi_{n,\sigma}(\log x), \qquad \bar{F}_\sigma(x) = |x|^{-2}\, \Phi_\sigma(\log x)$$

and

$$\bar{G}_{n,\sigma}(x) = \Gamma_{n,\sigma}(\log x) \qquad \bar{G}_\sigma(x) = \Gamma_\sigma(\log x).$$

By Theorem 11 the series (81) and (82) are in any interval $(\frac{1}{2} <)\, \alpha \leqq \sigma \leqq \beta\, (< +\infty)$ and for any $\lambda > 0$ majorized by a series

$$\sum_{h=-\infty}^{\infty} K e^{-\lambda |x + 2\pi i h|^2} = K e^{-\lambda \xi_1^2} \sum_{h=-\infty}^{\infty} e^{-\lambda (\xi_2 + 2\pi h)^2} \leqq K' e^{-\lambda \xi_1^2},$$

and for every $p > 0$ the series obtained by partial derivation of order $\leqq p$ have similar majorants. As is easily seen, this implies all the remaining statements of the theorem except the last statements of the second part and the last part. The first of these statements, viz. that $\bar{F}_\sigma(x) > 0$ and $\bar{G}_\sigma(x) > 0$ for all $x \neq 0$ if $\frac{1}{2} < \sigma \leqq 1$ follows immediately from Theorem 7. We proceed to prove that $\bar{F}_\sigma(e^x)$ and $\bar{G}_\sigma(e^x)$ are entire functions of ξ_1 and ξ_2 when $\frac{1}{2} < \sigma < 1$, i. e. that $\Phi_\sigma(x)$ and $\Gamma_\sigma(x)$ are entire functions of ξ_1 and ξ_2.

It is plain from the expressions (44) that $\Lambda(y; \mu_{n,\sigma})$ and $\Lambda(y; \nu_{n,\sigma})$ possess continuous partial derivatives of the first order with respect to η_2. According to (46) and (47) they are for $n \geqq 4$ sums of terms each of which contains at least $n - 3$ factors $K_0(y, r_k)$, while the other factors are bounded. Hence, if $n \geqq 8$

$$\frac{\partial}{\partial \eta_2} \Lambda(y; \mu_{n,\sigma}) = O(|y|^{-\frac{5}{2}}) \quad \text{and} \quad \frac{\partial}{\partial \eta_2} \Lambda(y; \nu_{n,\sigma}) = O(|y|^{-\frac{5}{2}}).$$

This implies[1] for $\Phi_{n,\sigma}(x)$ and $\Gamma_{n,\sigma}(x)$ the representations

$$\Phi_{n,\sigma}(x) = (2\pi)^{-2} \int_{-\infty}^{\infty} \sum_{j=-\infty}^{\infty} e^{-i x (\eta_1 + i j)}\, \Lambda(\eta_1 + i j;\, \mu_{n,\sigma})\, d\eta_1$$

and

[1] See Jessen and Wintner [1], p. 73.

$$\Gamma_{n,\sigma}(x) = (2\pi)^{-2} \int_{-\infty}^{\infty} \sum_{j=-\infty}^{\infty} e^{-ix(\eta_1+ij)} \Lambda(\eta_1 + ij;\, \nu_{n,\sigma})\, d\eta_1$$

as combined Fourier series and Fourier integrals. Since $\Lambda(y;\mu_{n,\sigma})$ and $\Lambda(y;\nu_{n,\sigma})$ for $n \geqq 11$ have bounded majorants which are $O(|y|^{-\frac{3}{2}})$ we obtain for $n \to \infty$ similar expressions for $\Phi_\sigma(x)$ and $\Gamma_\sigma(x)$ with μ_σ and ν_σ instead of $\mu_{n,\sigma}$ and $\nu_{n,\sigma}$. These expressions show[1] that $\Phi_\sigma(x)$ and $\Gamma_\sigma(x)$ are entire functions of the two variables ξ_1, ξ_2 if $\frac{1}{2} < \sigma < 1$.

The last part of the theorem is equivalent to the statement that $\Phi_\sigma(x)$ and each of its partial derivatives tend uniformly to zero when $\sigma \to \frac{1}{2}$, which by the argument used in the proof of Theorem 10 follows from the above mentioned expression for $\Phi_\sigma(x)$.[2]

Main Results.

45. We are now in a position to prove our main theorems.

Let us first consider the functions $\log \zeta_n(s)$ for $\sigma > 0$. According to §§ 7 and 27 there exist for every σ asymptotic distribution functions of $\log \zeta_n(\sigma + it)$ and of $\log \zeta_n(\sigma + it)$ with respect to $|\zeta_n'/\zeta_n(\sigma + it)|^2$. For $\zeta_n(\sigma + it)$ we have the expression

$$\zeta_n(\sigma + it) = \prod_{k=1}^{n} (1 - p_k^{-\sigma} e^{-(\log p_k) it})^{-1} = \zeta_n(\sigma;\, \lambda_1 t, \ldots, \lambda_n t),$$

---

[1] See Jessen and Wintner [1], p. 73.

[2] We notice that the last statement of Theorem 12 is not true if the factors $|x|^2$ and $|x|^{2+p}$ are omitted. It is not even true that $\bar{\mu}_\sigma(E) \to 0$ for any bounded set E as $\sigma \to \frac{1}{2}$. This may be proved as follows.

For every n and every $\sigma > 0$ we have

$$\log \zeta_n(\sigma;\, \theta_1, \ldots, \theta_n) + \log \zeta_n(\sigma;\, \theta_1 + \tfrac{1}{2}, \ldots, \theta_n + \tfrac{1}{2}) = \log \zeta_n(2\sigma;\, 2\theta_1, \ldots, 2\theta_n).$$

The right-hand side has the distribution function $\mu_{n,2\sigma}$. From Theorem 11' follows therefore for any $\varepsilon > 0$ the existence of a constant K such that for all $\sigma > \frac{1}{2}$ and all n the measure of the set in Q_n in which

$$|\log \zeta_n(\sigma;\, \theta_1, \ldots, \theta_n) + \log \zeta_n(\sigma;\, \theta_1 + \tfrac{1}{2}, \ldots, \theta_n + \tfrac{1}{2})| \leqq K$$

is $\geqq 1 - \varepsilon$. For any point of this set we have either

$$\log|\zeta_n(\sigma;\, \theta_1, \ldots, \theta_n)| \leqq \tfrac{1}{2}K \text{ or } \log|\zeta_n(\sigma;\, \theta_1 + \tfrac{1}{2}, \ldots, \theta_n + \tfrac{1}{2})| \leqq \tfrac{1}{2}K.$$

Since the two sets in Q_n determined by these inequalities are congruent, it is plain that their measures must be $\geqq \frac{1}{2}(1 - \varepsilon)$. Hence, if we denote by E the circle $|x| \leqq e^{\frac{1}{2}K}$, we have $\bar{\mu}_{n,\sigma}(E) \geqq \frac{1}{2}(1 - \varepsilon)$ for all $\sigma > \frac{1}{2}$ and all n and, consequently, $\mu_\sigma(E) \geqq \frac{1}{2}(1 - \varepsilon)$ for all $\sigma > \frac{1}{2}$, so that $\liminf_{\sigma \to \frac{1}{2}} \bar{\mu}_\sigma(E) \geqq \frac{1}{2}(1 - \varepsilon)$.

This remark provides an answer to a desideratum mentioned in Jessen and Wintner [1], p. 74.

where, by way of abbreviation, we have put $-(\log p_k)/2\pi = \lambda_k$. These numbers $\lambda_1, \ldots, \lambda_n$ are linearly independent. Similarly,

$$\zeta_n'(\sigma + it) = \zeta_n'(\sigma; \lambda_1 t, \ldots, \lambda_n t),$$

$$\log \zeta_n(\sigma + it) = \log \zeta_n(\sigma; \lambda_1 t, \ldots, \lambda_n t),$$

and

$$\zeta_n'/\zeta_n(\sigma + it) = \zeta_n'/\zeta_n(\sigma; \lambda_1 t, \ldots, \lambda_n t).$$

For the Fourier transforms of the distribution functions $\mu_{n,\sigma}$ and $\nu_{n,\sigma}$ we have by (44) the expressions

$$\Lambda(y; \mu_{n,\sigma}) = \int_{Q_n} e^{i \log \zeta_n(\sigma; \theta_1, \ldots, \theta_n) y}\, m(d\,Q_n) \text{ and}$$

$$\Lambda(y; \nu_{n,\sigma}) = \int_{Q_n} e^{i \log \zeta_n(\sigma; \theta_1, \ldots, \theta_n) y}\, |\zeta_n'/\zeta_n(\sigma; \theta_1, \ldots, \theta_n)|^2\, m(d\,Q_n).$$

Now, if $H(\theta_1, \ldots, \theta_n)$ is any continuous function in Q_n and if $\lambda_1, \ldots, \lambda_n$ are linearly independent, we have

$$\underset{t}{M}\{H(\lambda_1 t, \ldots, \lambda_n t)\} = \lim_{(\delta-\gamma)\to\infty} \frac{1}{\delta-\gamma}\int_\gamma^\delta H(\lambda_1 t, \ldots, \lambda_n t)\, dt =$$

$$= \int_{Q_n} H(\theta_1, \ldots, \theta_n)\, m(d\,Q_n).$$[1]

Hence

$$\Lambda(y; \mu_{n,\sigma}) = \underset{t}{M}\{e^{i \log \zeta_n(\sigma+it) y}\} \quad \text{and}$$

$$\Lambda(y; \nu_{n,\sigma}) = \underset{t}{M}\{e^{i \log \zeta_n(\sigma+it) y}\, |\zeta_n'/\zeta_n(\sigma+it)|^2\}.$$

Together with § 7 this shows that the distribution functions $\mu_{n,\sigma}$ and $\nu_{n,\sigma}$ of Theorem 11 are also the asymptotic distribution functions of $\log \zeta_n(\sigma + it)$ and of $\log \zeta_n(\sigma + it)$ with respect to $|\zeta_n'/\zeta_n(\sigma + it)|^2$.

By § 8 this gives for an arbitrary x for the Jensen function $\varphi_{\log \zeta_n - x}(\sigma)$ of $\log \zeta_n(s) - x$ the expressions

$$(83) \quad \varphi_{\log \zeta_n - x}(\sigma) = \int_{R_u} \log|u - x|\, \mu_{n,\sigma}(d\,R_u) = \int_{R_u} \log|u - x|\, F_{n,\sigma}(u)\, m(d\,R_u),$$

where the last expression is valid for $n \geqq 11$.

[1] This classical result, due in principle to Bohl, which is an easy consequence of Weierstrass' approximation theorem, was used by Weyl as basis for his theorem on equidistribution mod. 1 of the points $(\lambda_1 t, \ldots, \lambda_N t)$. Weyl's theorem was a main tool in Bohr's study of the distribution of the values of the zeta function. In the present exposition we use only the above statement. As to this way of avoiding the explicit use of Weyl's theorem, cf. Jessen and Wintner [1], p. 79.

From Theorem 11 and § 9 it follows that $\varphi_{\log \zeta_n - x}(\sigma)$ for any $n \geqq 11$ and any x is twice differentiable with the second derivative

(84) $$\varphi''_{\log \zeta_n - x}(\sigma) = 2\pi G_{n,\sigma}(x).$$

46. By means of these results we shall now deduce the following theorem connecting the function $\log \zeta(s)$ with the distribution functions described in Theorem 11.

Theorem 13. *For every $\sigma > \frac{1}{2}$ the function $\log \zeta(\sigma + it)$ possesses the asymptotic distribution function μ_σ, i. e. the distribution function*

$$\mu_{\sigma;\gamma,\delta}(E) = \frac{m(A_{\sigma;\gamma,\delta}(E))}{\delta - \gamma},$$

where $A_{\sigma;\gamma,\delta}(E)$ for an arbitrary Borel set E denotes the set of points in $\gamma < t < \delta$ for which $\log \zeta(\sigma + it)$ belongs to E, converges for $\delta \to \infty$ and any fixed $\gamma > 0$ towards μ_σ.

The Jensen function

$$\varphi_{\log \zeta - x}(\sigma) = \lim_{\delta \to \infty} \frac{1}{\delta - \gamma} \int_\gamma^\delta \log |\log \zeta(\sigma + it) - x| \, dt$$

exists for every x uniformly in $[\frac{1}{2}, +\infty]$ and is a twice differentiable convex function with the second derivative

(85) $$\varphi''_{\log \zeta - x}(\sigma) = 2\pi G_\sigma(x).$$

It is expressible as

(86) $$\varphi_{\log \zeta - x}(\sigma) = \int_{R_u} \log |u - x| \, \mu_\sigma(dR_u) = \int_{R_u} \log |u - x| \, F_\sigma(u) \, m(dR_u).$$

For $\sigma >$ (some) $\sigma_0(x)$ we have $\varphi_{\log \zeta - x}(\sigma) = \log |x|$, if $x \neq 0$, and $\varphi_{\log \zeta - x}(\sigma) = -(\log 2)\sigma$, if $x = 0$. We have $\varphi_{\log \zeta - x}(\sigma) \to \infty$ as $\sigma \to \frac{1}{2}$.

For every $\sigma > \frac{1}{2}$ the two mean motions

$$c^-_{\log \zeta - x}(\sigma) = \lim_{\delta \to \infty} \frac{V^-(\sigma; \gamma, \delta)}{\delta - \gamma} \quad \text{and} \quad c^+_{\log \zeta - x}(\sigma) = \lim_{\delta \to \infty} \frac{V^+(\sigma; \gamma, \delta)}{\delta - \gamma},$$

where $V^-(\sigma; \gamma, \delta)$ and $V^+(\sigma; \gamma, \delta)$ denote the left and right variations of the argument of $\log \zeta(s) - x$ along the segment $s = \sigma + it$, $\gamma \leqq t \leqq \delta$, exist and are determined by

$$c^-_{\log \zeta - x}(\sigma) = c^+_{\log \zeta - x}(\sigma) = \varphi'_{\log \zeta - x}(\sigma).$$

Further, for every strip (σ_1, σ_2), *where* $\frac{1}{2} < \sigma_1 < \sigma_2 < +\infty$, *the relative frequency*

$$H_{\log \zeta - x}(\sigma_1, \sigma_2) = \lim_{\delta \to \infty} \frac{N(\sigma_1, \sigma_2; \gamma, \delta)}{\delta - \gamma},$$

where $N(\sigma_1, \sigma_2; \gamma, \delta)$ *denotes the number of zeros of* $\log \zeta(s) - x$ *in the part of the rectangle* $\sigma_1 < \sigma < \sigma_2$, $\gamma < t < \delta$ *which belongs to* Δ, *exists and is determined by*

$$H_{\log \zeta - x}(\sigma_1, \sigma_2) = \frac{1}{2\pi}(\varphi'_{\log \zeta - x}(\sigma_2) - \varphi'_{\log \zeta - x}(\sigma_1)) = \int_{\sigma_1}^{\sigma_2} G_\sigma(x)\, d\sigma.$$

47. According to § 27 we may apply Theorems 3 and 4.

The first part of the theorem follows from § 45 by means of Theorems 4 and 11.

From Theorem 3 it follows that $\varphi_{\log \zeta - x}(\sigma)$ exists uniformly in $[\frac{1}{2}, +\infty]$ and is a convex function, and that $\varphi_{\log \zeta_n - x}(\sigma)$ converges uniformly towards $\varphi_{\log \zeta - x}(\sigma)$ in $[\frac{1}{2}, +\infty]$. From (84) and Theorem 11 it follows that $\varphi_{\log \zeta - x}(\sigma)$ is twice differentiable with the second derivative $2\pi G_\sigma(x)$. From (83) follows (86), since by Theorem 11 the function $\log |u - x| F_{n,\sigma}(u)$ converges, for a fixed σ and $n \to \infty$, towards $\log |u - x| F_\sigma(u)$, and the convergence is majorized by a function of the form $K_0 |\log |u-x|| e^{-\lambda |u|^2}$, which is integrable over R_u. The statements concerning $\varphi_{\log \zeta - x}(\sigma)$ for large σ are obvious consequences of the behaviour of $\log \zeta(s) - x$ for large σ.[1] That $\varphi_{\log \zeta - x}(\sigma) \to \infty$ for $\sigma \to \frac{1}{2}$ follows from (86) together with Theorem 11, since the integral of $F_\sigma(u)$ over R_u is 1.

The remainder of the theorem is now implied by Theorem 3.

48. From the remark at the end of § 18 it follows that Theorem 13 remains valid if the limits are taken for $\gamma \to -\infty$ and a fixed $\delta < 0$. This follows also from the remark at the end of § 43, since $\log \zeta(s)$ takes conjugate values for conjugate values of s.

49. We shall now prove the following analogous theorem, connecting the function $\zeta(s)$ itself with the distribution functions introduced in Theorem 12.

Theorem 14. *For every* $\sigma > \frac{1}{2}$ *the function* $\zeta(\sigma + it)$ *possesses the asymptotic distribution function* $\bar{\mu}_\sigma$, *i. e. the distribution function*

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[1] Cf. Jessen and Tornehave [1], Theorem 9.

$$\bar{\mu}_{\sigma;\gamma,\delta}(E) = \frac{m(A_{\sigma;\gamma,\delta}(E))}{\delta - \gamma},$$

where $A_{\sigma;\gamma,\delta}(E)$ *for an arbitrary Borel set* E *denotes the set of points in* $\gamma < t < \delta$ *for which* $\zeta(\sigma + it)$ *belongs to* E, *converges for* $\delta \to \infty$ *and any fixed* $\gamma > 0$ *towards* $\bar{\mu}_\sigma$.

The Jensen function

$$\varphi_{\zeta - x}(\sigma) = \lim_{\delta \to \infty} \frac{1}{\delta - \gamma} \int_\gamma^\delta \log |\zeta(\sigma + it) - x| \, dt$$

exists for every σ *uniformly in* $[\frac{1}{2}, +\infty]$ *and is a twice differentiable function with the second derivative*

$$\varphi''_{\zeta - x}(\sigma) = 2\pi \bar{G}_\sigma(x). \tag{87}$$

It is expressible as

$$\varphi_{\zeta - x}(\sigma) = \int_{R_u} \log |u - x| \, \bar{\mu}_\sigma(dR_u) = \int_{R_u} \log |u - x| \, \bar{F}_\sigma(u) \, m(dR_u). \tag{88}$$

For $\sigma >$ *(some)* $\sigma_0(x)$ *we have* $\varphi_{\zeta - x}(\sigma) = \log |1 - x|$, *if* $x \neq 1$, *and* $\varphi_{\zeta - x}(\sigma) = -(\log 2)\,\sigma$, *if* $x = 1$. *For* $x = 0$ *we have* $\varphi_{\zeta - x}(\sigma) = 0$ *for all* $\sigma > \frac{1}{2}$. *For* $x \neq 0$ *we have* $\varphi_{\zeta - x}(\sigma) \to \infty$ *when* $\sigma \to \frac{1}{2}$.

For every $\sigma > \frac{1}{2}$ *the two mean motions*

$$c^-_{\zeta - x}(\sigma) = \lim_{\delta \to \infty} \frac{\arg^-(\zeta(\sigma + i\delta) - x) - \arg^-(\zeta(\sigma + i\gamma) - x)}{\delta - \gamma}$$

and

$$c^+_{\zeta - x}(\sigma) = \lim_{\delta \to \infty} \frac{\arg^+(\zeta(\sigma + i\delta) - x) - \arg^+(\zeta(\sigma + i\gamma) - x)}{\delta - \gamma}$$

exist and are determined by

$$c^-_{\zeta - x}(\sigma) = c^+_{\zeta - x}(\sigma) = \varphi'_{\zeta - x}(\sigma).$$

Further, for every strip (σ_1, σ_2), *where* $\frac{1}{2} < \sigma_1 < \sigma_2 < +\infty$, *the relative frequency*

$$H_{\zeta - x}(\sigma_1, \sigma_2) = \lim_{\delta \to \infty} \frac{N(\sigma_1, \sigma_2; \gamma, \delta)}{\delta - \gamma},$$

where $N(\sigma_1, \sigma_2; \gamma, \delta)$ *denotes the number of zeros of* $\zeta(s) - x$ *in the rectangle* $\sigma_1 < \sigma < \sigma_2$, $\gamma < t < \delta$, *exists and is determined by*

$$H_{\zeta-x}(\sigma_1, \sigma_2) = \frac{1}{2\pi}(\varphi'_{\zeta-x}(\sigma_2) - \varphi'_{\zeta-x}(\sigma_1)) = \int_{\sigma_1}^{\sigma_2} \bar{G}_\sigma(x)\, d\sigma.$$

50. To prove this theorem let us first consider the functions $\zeta_n(s)$. By considerations exactly like those of § 45 we see that for every $\sigma > 0$ the distribution functions $\bar{\mu}_{n,\sigma}$ and $\bar{\nu}_{n,\sigma}$ of Theorem 12 are also the asymptotic distribution functions of $\zeta_n(\sigma + it)$ and of $\zeta_n(\sigma + it)$ with respect to $|\zeta'_n(\sigma + it)|^2$. Consequently, the Jensen function $\varphi_{\zeta_n - x}(\sigma)$ of $\zeta_n(s) - x$ is for an arbitrary x determined by

$$(89) \qquad \varphi_{\zeta_n - x}(\sigma) = \int_{R_u} \log|u - x|\,\bar{\mu}_{n,\sigma}(dR_u) = \int_{R_u} \log|u - x|\,\bar{F}_{n,\sigma}(u)\, m(dR_u),$$

where the last expression is valid for $n \geqq 11$.

From Theorem 12 and § 9 it follows that $\varphi_{\zeta_n - x}(\sigma)$ for any $n \geqq 11$ and any x is twice differentiable with the second derivative

$$(90) \qquad \varphi''_{\zeta_n - x}(\sigma) = 2\pi\,\bar{G}_{n,\sigma}(x).$$

51. According to § 27 we may apply Theorems 1 and 2.

The first part of Theorem 14 then follows from § 50 by means of Theorems 2 and 12.

From Theorem 1 it follows that $\varphi_{\zeta-x}(\sigma)$ exists uniformly in $[\frac{1}{2}, +\infty]$ and is a convex function, and that $\varphi_{\zeta_n - x}(\sigma)$ converges uniformly towards $\varphi_{\zeta-x}(\sigma)$ in $[\frac{1}{2}, +\infty]$. From (90) and Theorem 12 it follows that $\varphi_{\zeta-x}(\sigma)$ is twice differentiable with the second derivative $2\pi\,\bar{G}_\sigma(x)$. From (89) follows (88), since by Theorem 12 the function $\log|u - x|\,\bar{F}_{n,\sigma}(u)$ for a fixed σ and $n \to \infty$ converges towards $\log|u - x|\,\bar{F}_\sigma(u)$, and the convergence is majorized by a function of the form $K_0|\log|u - x||\,e^{-\lambda(\log|u|)^2}$, which is integrable over R_u. The statements concerning $\varphi_{\zeta-x}(\sigma)$ for large σ are obvious consequences of the behaviour of $\zeta(s) - x$ for large σ.[1] In particular, $\varphi_\zeta(\sigma) = 0$ for $\sigma > \sigma_0(0)$; that we may take $\sigma_0(0) = \frac{1}{2}$ is a consequence of § 19.

To prove that $\varphi_{\zeta-x}(\sigma) \to \infty$ for $\sigma \to \frac{1}{2}$ when $x \neq 0$ we use the relation

$$\varphi''_{\zeta-x}(\sigma) = \sum_{\log x} \varphi''_{\log\zeta - \log x}(\sigma),$$

[1] Cf. Jessen and Tornehave [1], Theorem 9.

which follows from (80), (85), and (87). On account of Theorem 11 the series possesses in every interval $(\frac{1}{2}<)\,\alpha \leqq \sigma \leqq \beta\,(<+\infty)$ a convergent majorant. By integration we therefore obtain for an arbitrary σ_1

$$(91) \qquad \varphi'_{\zeta-x}(\sigma)-\varphi'_{\zeta-x}(\sigma_1)=\sum_{\log x}\left(\varphi'_{\log\zeta-\log x}(\sigma)-\varphi'_{\log\zeta-\log x}(\sigma_1)\right).$$

For $\sigma>\sigma_0(x)$ we have $\zeta(s) \neq x$ and hence $\log\zeta(s) \neq \log x$ for all values of $\log x$. Hence, if $\sigma_1>\sigma_0(x)$, all the terms $\varphi'_{\zeta-x}(\sigma_1)$ and $\varphi'_{\log\zeta-\log x}(\sigma_1)$ vanish if $x \neq 1$, whereas, if $x=1$, the term $\varphi'_{\zeta-x}(\sigma_1)$ is $=-\log 2$, and of the terms $\varphi'_{\log\zeta-\log x}(\sigma_1)$ one is $=-\log 2$ and the others vanish. Hence the relation (91) takes the form

$$\varphi'_{\zeta-x}(\sigma)=\sum_{\log x}\varphi'_{\log\zeta-\log x}(\sigma).$$

By another integration we obtain

$$\varphi_{\zeta-x}(\sigma)-\varphi_{\zeta-x}(\sigma_1)=\sum_{\log x}\left(\varphi_{\log\zeta-\log x}(\sigma)-\varphi_{\log\zeta-\log x}(\sigma_1)\right).$$

For $\frac{1}{2}<\sigma<\sigma_1$ all the differences are $\geqq 0$. Moreover, by Theorem 13 each of the differences on the right will $\to\infty$ when $\sigma\to\frac{1}{2}$. This shows that $\varphi_{\zeta-x}(\sigma)\to\infty$ when $\sigma\to\frac{1}{2}$.

The remainder of the theorem is implied by Theorem 1.

52. From the remark at the end of § 11 it follows that Theorem 14 remains valid if the limits are taken for $\gamma\to-\infty$ and a fixed $\delta<0$. This follows also from the remark after Theorem 12, since $\zeta(s)$ takes conjugate values for conjugate values of s.

53. As a corollary of Theorems 13 and 14 we have the following theorem.

Theorem 15. *If $N(T)$ denotes either the number of zeros of $\log\zeta(s)-x$ in the part of the domain $\sigma>\frac{1}{2}$, $0<t<T$ which belongs to Δ, for an arbitrary x, or the number of zeros of $\zeta(s)-x$ in the domain $\sigma>\frac{1}{2}$, $0<t<T$, for an arbitrary $x\neq 0$, then*

$$\frac{N(T)}{T}\to\infty \quad \text{when} \quad T\to\infty.$$

Bibliography.

A. S. BESICOVITCH. [1] Almost periodic functions. Cambridge 1932.

P. BOHL. [1] Über ein in der Theorie der säkularen Störungen vorkommendes Problem. J. reine angew. Math. *135*, 189—283 (1909).

H. BOHR. [1] Zur Theorie der Riemannschen Zetafunktion im kritischen Streifen. Acta math. *40*, 67—100 (1915).

H. BOHR and B. JESSEN. [1], [2] Über die Werteverteilung der Riemannschen Zetafunktion I, II. Acta math. *54*, 1—35 (1930), *58*, 1—55 (1932).

H. BOHR and E. LANDAU. [1] Sur les zéros de la fonction $\zeta(s)$ de Riemann. C. R. Acad. Sci., Paris *158*, 106—110 (1914).

C. CARATHÉODORY. [1] Conformal representation. Cambridge 1932.

H. CRAMÉR. [1] Mathematical methods of statistics. Princeton 1946.

G. H. HARDY. [1] The mean value of the modulus of an analytic function. Proc. London Math. Soc. (2) *14*, 269—277 (1915).

G. H. HARDY, J. E. LITTLEWOOD, and G. PÓLYA. [1] Inequalities. Cambridge 1934.

PH. HARTMAN. [1] Mean motions and almost periodic functions. Trans. Amer. Math. Soc. *46*, 64—81 (1939).

B. JESSEN. [1] Über die Nullstellen einer analytischen fastperiodischen Funktion. Eine Verallgemeinerung der Jensenschen Formel. Math. Ann. *108*, 485—516 (1933).

[2] The theory of integration in a space of an infinite number of dimensions. Acta math. *63*, 249—323 (1934).

B. JESSEN and H. TORNEHAVE. [1] Mean motions and zeros of almost periodic functions. Acta math. *77*, 137—279 (1945).

B. JESSEN and A. WINTNER. [1] Distribution functions and the Riemann zeta function. Trans. Amer. Math. Soc. *38*, 48—88 (1935).

H. A. SCHWARZ. [1] Zur Integration der partiellen Differentialgleichung $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0$. J. reine angew. Math. *74*, 218—253 (1872). Mathematische Abhandlungen II, 175—210.

H. WEYL. [1] Über die Gleichverteilung von Zahlen mod. Eins. Math. Ann. *77*, 313—352 (1916).