

ON THE UNSYMMETRICAL TOP.

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1. This note is a continuation of the paper published in Vol. 56, pp. 355—362, of this Journal and the notation is the same as was used at that time. The moments of inertia for the principal axes of the body at the fixed point are I_1 , I_2 , and I_3 and the components of the angular velocity along these axes are ω_1 , ω_2 , ω_3 . The centroid is at $(0, 0, h)$, Euler's angles are denoted by (θ, φ, ψ) , and the weight of the body by w .

2. If the usual notation for a derivative with respect to time is used, Euler's equations for this case are

$$I_1 \dot{\omega}_1 + (I_3 - I_2) \omega_2 \omega_3 = hw \sin \theta \cos \varphi \quad (1)$$

$$I_2 \dot{\omega}_2 + (I_1 - I_3) \omega_3 \omega_1 = -hw \sin \theta \sin \varphi \quad (2)$$

$$I_3 \dot{\omega}_3 + (I_2 - I_1) \omega_1 \omega_2 = 0. \quad (3)$$

The values of ω_1 , ω_2 , ω_3 are

$$\omega_1 = \dot{\theta} \cos \varphi + \dot{\psi} \sin \theta \sin \varphi \quad (4)$$

$$\omega_2 = -\dot{\theta} \sin \varphi + \dot{\psi} \sin \theta \cos \varphi \quad (5)$$

$$\omega_3 = \dot{\varphi} + \dot{\psi} \cos \theta. \quad (6)$$

In addition there are the two well known integrals

$$I_1 \omega_1^2 + I_2 \omega_2^2 + I_3 \omega_3^2 - T = -2wh \cos \theta \quad (7)$$

$$I_1 \omega_1 \sin \theta \sin \varphi + I_2 \omega_2 \sin \theta \cos \varphi + I_3 \omega_3 \cos \theta = k \quad (8)$$

k and T being constants of integration.

3. The case which it is desired to study is that in which I_2 approaches I_1 and I_3 becomes small in such a way that $\frac{I_1 - I_2}{I_3}$ approaches the value p . If $I_{x_1 y_1}$, $I_{x_1 z_1}$, and $I_{y_1 z_1}$ are the moments of inertia of the body for its three principal planes, the value of p in terms of them is

$$\frac{I_{x_1 z_1} - I_{y_1 z_1}}{I_{x_1 z_1} + I_{y_1 z_1}}$$

Hence if we choose the axes so that $I_{x_1 z_1}$ is greater than $I_{y_1 z_1}$, it may be assumed that p is a positive constant whose value lies between zero and one. Such a situation can arise in case the body with a fixed point has the shape of a rod.

4. With these restrictions on the values of the moments of inertia, equations 1, 2, 3, 7, and 8 become

$$\dot{\omega}_1 - \omega_2 \omega_3 = \frac{h w}{I_1} \sin \theta \cos \varphi \quad (1')$$

$$\dot{\omega}_2 + \omega_3 \omega_1 = -\frac{h w}{I_1} \sin \theta \sin \varphi \quad (2')$$

$$\dot{\omega}_3 = p \omega_1 \omega_2 \quad (3')$$

$$\omega_1^2 + \omega_2^2 = \frac{T - 2 w h \cos \theta}{I_1} \quad (7')$$

$$\omega_1 \sin \theta \sin \varphi + \omega_2 \sin \theta \cos \varphi = \frac{k}{I_1}. \quad (8')$$

The values of ω_1 and ω_2 given in (4) and (5) substituted in (7') and (8') give

$$\dot{\theta}^2 + \dot{\psi}^2 \sin^2 \theta = \frac{T - 2 w h \cos \theta}{I_1} \quad \text{and} \quad \dot{\psi} \sin^2 \theta = \frac{k}{I_1}.$$

Therefore

$$\dot{\psi} = \frac{k}{I_1 \sin^2 \theta} \quad (9)$$

and

$$I_1^2 \sin^2 \theta \dot{\theta}^2 = I_1 \sin^2 \theta (T - 2 w h \cos \theta) - k^2. \quad (10)$$

These equations determine ψ and θ in terms of the time and show how ψ and θ vary in the course of the motion. They need not be discussed as they are the well known equations of motion for a spherical pendulum.

5. If $\omega_1, \omega_2, \omega_3,$ and ψ are replaced by their values in terms of φ and $\theta,$ equation (3') becomes

$$\begin{aligned} \dot{\omega}_3 &= \frac{d}{dt} \left(\dot{\varphi} + \frac{k \cos \theta}{I_1 \sin^2 \theta} \right) = \frac{p}{2} \left(\frac{k^2}{I_1^2 \sin^2 \theta} - \dot{\theta}^2 \right) \sin 2\varphi + \frac{p k \dot{\theta}}{I_1 \sin \theta} \cos 2\varphi \\ &= \frac{p}{2} \left(\frac{k^2}{I_1^2 \sin^2 \theta} + \dot{\theta}^2 \right) \sin 2(\varphi + \varepsilon) \end{aligned} \quad (11)$$

where

$$\varepsilon = \arctan \frac{-k}{I_1 \dot{\theta} \sin \theta}.$$

6. The integration of equation (11) is a simple matter if $\dot{\theta} = 0.$ So far as the variation in θ and ψ is concerned, this corresponds to the case of a conical pendulum. Equation (11) then becomes

$$\ddot{\varphi} = \frac{p k^2}{2 I_1^2 \sin^2 \theta} \sin 2\varphi = \frac{p k^2}{I_1^2 \sin^2 \theta} \sin \varphi \cos \varphi.$$

From this it follows that

$$\dot{\varphi}^2 = \frac{p k^2}{I_1^2 \sin^2 \theta} \sin^2 \varphi + \dot{\varphi}_0^2,$$

φ_0 being the value of φ when $\dot{\varphi} = 0.$

Finally

$$t = \pm \int \frac{d\varphi}{\sqrt{\dot{\varphi}_0^2 + \frac{p k^2}{I_1^2 \sin^2 \theta} \sin^2 \varphi}}.$$

7. The case $h = 0.$ Geometrically, this is a particular case of Poinsot motion. Analytically, it is a particular case of the solution given by Euler. It seems of interest to note the simplifications introduced by the special values of the moments of inertia. In this case it is no restriction to take the angular momentum vector along the vertical $OZ.$ This makes $\theta = \frac{\pi}{2},$ and from (10) $k = \sqrt{I_1 T}.$ From (9)

$$\dot{\psi} = \frac{\dot{k}}{I_1 \sin^2 \theta} = \frac{\dot{k}}{I_1}, \quad \psi = \frac{k}{I_1} t + \psi_0.$$

Equation (11) becomes

$$\ddot{\varphi} = \frac{p}{2} \frac{k^2}{I_1^2 \sin^2 \theta} \sin 2\varphi = \frac{p T}{2 I_1} \sin 2\varphi.$$

This equation differs from the one treated in paragraph six only in that $\sin^2 \theta$ has been replaced by unity.

