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A parametrized index theorem for the algebraic K-theory Euler class

by

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0. Introduction

The classical setting. Suppose that $p: E \to B$ is a bundle of smooth compact manifolds. By this we mean that p comes with an atlas of local trivializations $\varphi_i: p^{-1}(U_i) \to U_i \times M_i$, where each M_i is a smooth compact manifold, the changes of charts

$$\varphi_j \varphi_i^{-1}: (U_i \cap U_j) \times M_i \to (U_i \cap U_j) \times M_j$$

are fiberwise smooth and the induced maps $(U_i \cap U_j) \times T^r M_i \to (U_i \cap U_j) \times T^r M_j$ are continuous for each $r > 0$.

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Let V be a complex vector bundle on E with discrete structure group (*flat* complex vector bundle for short). Let V_i be the complex vector bundle on B whose fiber over $b \in B$ is the local-coefficient homology group $H_i(p^{-1}(b); V)$. Denote by $K(\mathbb{C}_t)$ the K-theory of the topological ring $\mathbb{C}$, so that $K(\mathbb{C}_t) \simeq \mathbb{Z} \times \mathrm{BU}$ and for a space X the group of homotopy classes $[X, K(\mathbb{C}_t)]$ is the topological complex K-theory of X. The vector bundles V and V_i for $i \geqslant 0$ give classes $[V] \in [E, K(\mathbb{C}_t)]$, $[V_i] \in [B, K(\mathbb{C}_t)]$. The Atiyah–Singer index theorem for families leads to the equation [BeS]

$$\mathrm{tr}^*[V] = \sum (-1)^i [V_i] \in [B, K(\mathbb{C}_t)]. \tag{0.1}$$

Here tr is the homotopy transfer of Becker–Gottlieb and Dold, a stable map from B_+ to E_+ determined by p, and $\mathrm{tr}^*[V]$ is the image of $[V]$ under the map on topological K-theory induced by tr. (The subscript $+$ indicates an added disjoint base point.)

The Bismut–Lott theorem. Let $K(\mathbb{C})$ be the K-theory space of the discrete ring $\mathbb{C}$, in other words, the space whose homotopy groups are the algebraic K-groups of $\mathbb{C}$. Like $K(\mathbb{C}_t)$, the space $K(\mathbb{C})$ represents a cohomology theory. The natural ring homomorphism from $\mathbb{C}$ with the discrete topology to $\mathbb{C}$ with the ordinary topology induces a map $K(\mathbb{C}) \to K(\mathbb{C}_t)$. Since the vector bundles V and V_i above are flat, the elements $[V]$ and $[V_i]$ lift back to elements in $[E, K(\mathbb{C})]$ and $[B, K(\mathbb{C})]$, respectively, which we still denote $[V]$ and $[V_i]$. It is natural to ask whether equation (0.1) holds in $[B, K(\mathbb{C})]$.

Bismut and Lott have given strong evidence that this is the case. There are certain characteristic classes for flat bundles, constructed by Kamber and Tondeur, which for odd k yield homomorphisms $c_k\colon [B, K(\mathbb{C})] \to H^k(B; \mathbb{R})$. The Riemann–Roch theorem of Bismut and Lott [BL] then states that equation (0.1) holds in $[B, K(\mathbb{C})]$ after the homomorphisms c_k are applied:

$$c_k(\mathrm{tr}^*[V]) = \sum (-1)^i c_k[V_i] \in H^k(B; \mathbb{R}). \tag{0.2}$$

To prove this Bismut and Lott use Bismut's local version of the Atiyah–Singer index theorem for families. (They assume that the fibers of p are *closed* manifolds, but we have been told that this was mostly to simplify the presentation.) See also [Lo].

An improvement. In this paper we show that the equation $\mathrm{tr}^*[V] = \sum (-1)^i [V_i]$ holds in $[B, K(\mathbb{C})]$, but we go a little beyond that, answering a question also raised by John Lott.

Suppose that $p\colon E \to B$ is a smooth fiber bundle with compact fiber F, as above. Let R be *any discrete ring*. Let V be a local coefficient system of f.g. projective left R-modules on E, in other words: a bundle of f.g. projective left R-modules on E. As above, V determines an element $[V] \in [E, K(R)]$. Each fiber $E_b = p^{-1}(b)$ of p has local-coefficient homology groups $H_i(E_b;V)$. For fixed i these homology groups form a bundle V_i of left R-modules on B. We assume that the fibers of V_i are again projective, in which case they are also finitely generated. Each V_i determines an element $[V_i]\in[B,K(R)]$. In §8, especially Corollary 8.12, we show:

$$\text{Riemann–Roch Theorem:}\quad \mathrm{tr}^*[V]=\sum(-1)^i[V_i]\in[B,K(R)]. \tag{0.3}$$

Smoothness is an essential hypothesis in (0.3) and (0.2). In Part III of this paper we show, again in answer to a question raised by John Lott, that (0.2) and hence (0.3) can fail for a fiber bundle whose fibers are closed *topological* manifolds.

We emphasize that both sides of equation (0.3) are fiber homotopy invariants of $p: E\to B$. This is obvious for the right-hand side. It is true for the left-hand side because there exists a fiber homotopy invariant definition of the transfer tr. See [BeG2]; compare the older definition in [BeG1]. Thus, (0.3) expresses a fiber homotopy theoretic property of bundles of smooth compact manifolds.

Like the Bismut–Lott proof of (0.2), our proof of (0.3) uses a family index theorem. It is an index theorem of Hopf–Pontryagin type, that is, an index theorem stating that Poincaré duals of certain generalized Euler classes of tangent bundles of compact manifolds agree with certain generalized Euler characteristics of those manifolds.

Generalized Euler characteristics. The notion of generalized Euler characteristic that we use relies on the work of Waldhausen, especially [W2]. Waldhausen associates with any space Y an infinite loop space $A(Y)$. This is designed to be the universal receptacle for Euler characteristics of retractive spaces over Y subject to certain relative finiteness conditions. (A retractive space over Y is a space X together with maps $r: X\to Y$ and $i: Y\to X$ such that $ri=\mathrm{id}_Y$.)

We write $\mathbb{S}^n\subset\mathbb{R}^{n+1}$ for the standard n-sphere, with base point $(1,0,\ldots)$. If the space Y itself has suitable finiteness properties, then $\mathbb{S}^0\times Y$ viewed as a retractive space over Y satisfies those relative finiteness conditions, so that the relative Euler characteristic of $\mathbb{S}^0\times Y$ is defined. We denote it by $\chi(Y)\in A(Y)$ and we think of it as the "absolute" Euler characteristic of Y. Note that it lives in a space $A(Y)$ which depends on Y.

The finiteness property that we require of Y is that it be *homotopy finitely dominated*; this means that there exist a compact CW-space Z and maps

$$f: Z\to Y,\quad g: Y\to Z$$

such that fg is homotopic to id_Y. The relative finiteness conditions that we impose on retractive spaces over Y are of a similar nature, with the result that $\pi_0 A(Y)$ is isomorphic to $K_0(\mathbb{Z}\pi_1(Y))$ in the case where Y is path-connected and based. In that case

$$K_0(\mathbb{Z}\pi_1(Y))\cong\mathbb{Z}\oplus\widetilde{K}_0(\mathbb{Z}\pi_1(Y))$$

and the component of $\chi(Y)$ is the sum of the ordinary Euler characteristic of Y, in $\mathbb{Z}$, and the Wall finiteness obstruction [Wa] of Y, in the reduced K_0-group of the ring $\mathbb{Z}\pi_1(Y)$.

The characteristic $\chi(Y)$ depends "continuously" on Y in the following sense. If $p\colon E \to B$ is any fibration with homotopy finitely dominated fibers, then we can evaluate χ fiberwise. We obtain a section $\chi(p)$ of another fibration, $A_B(E) \to B$, whose fiber over $b \in B$ is $A(p^{-1}(b))$.

Generalized Euler classes. The generalized Euler class in our index theorem is a class $[e_n]$ in the nth cohomology of the pair $(B\,\mathrm{TOP}(n), B\,\mathrm{TOP}(n-1))$ with twisted coefficients in the spectrum $\mathbf{A}(*)$ associated with the infinite loop space $A(*)$. Here $B\,\mathrm{TOP}(n)$ is the classifying space for euclidean n-bundles, fiber bundles with fibers homeomorphic to $\mathbb{R}^n$. We work at this level of generality because our index theorem is for topological manifolds: tangent bundles of topological manifolds are euclidean bundles. Compare [Ki].

Assembly. The final ingredient in our index theorem is the assembly transformation [WW1]. Let $\mathbf{A}(Y)$ be the spectrum associated with the infinite loop space $A(Y)$. Assembly is essentially a natural transformation $\alpha\colon Y_+ \wedge \mathbf{A}(*) \to \mathbf{A}(Y)$. It agrees with the obvious identification $*_+ \wedge \mathbf{A}(*) \cong \mathbf{A}(*)$ when $Y = *$ and is essentially characterized by that property. (Every functor from spaces to spectra which respects homotopy equivalences has such an assembly transformation.) We like to think of α as a map of infinite loop spaces, from the space $A^{\%}(Y) := \Omega^\infty(Y_+ \wedge \mathbf{A}(*))$ to $A(Y)$.

(Most index theorems involve some map from the home of the symbols to the home of the indices. The map is usually some form of assembly, or assembly following on Poincaré duality, depending on whether those symbols are viewed as representatives of generalized homology or cohomology. However, it is not always easily identified as such. See for example [BD1], [BD2] and the reformulation of the Mishchenko–Fomenko index theorem [MF] in [Ro1, 3.1], [Ro2, 3.3], [Ro3, 2.2].)

The index theorem. Assume now that $p\colon E \to B$ is a bundle of compact *topological* n-manifolds, possibly with boundary. Let (τ, τ_∂) be the vertical tangent bundle pair of p. (This is short for a certain euclidean n-bundle τ on E, a certain euclidean $(n-1)$-bundle τ_∂ on $\partial E = \bigcup_b \partial E_b$ and an identification of $\tau|\partial E$ with $\varepsilon \oplus \tau_\partial$ where ε is a trivial line bundle on ∂E.) The fiberwise Poincaré dual of $[e_n](\tau, \tau_\partial)$ turns out to be a vertical homotopy class of sections of a fibration

$$A^{\%}_B(E) \to B$$

with fiber $A^{\%}(p^{-1}(b))$ over $b \in B$. The content of the index theorem is that fiberwise assembly $\alpha\colon A^{\%}_B(E) \to A_B(E)$ takes the fiberwise Poincaré dual of $[e_n](\tau, \tau_\partial)$ to the fiberwise characteristic, up to a vertical homotopy:

$$\alpha\wp[e_n](\tau, \tau_\partial) \simeq \chi(p). \tag{0.4}$$

A vanishing theorem. Becker [Be] has defined a *stable cohomotopy Euler class*, living in nth cohomology of the pair $(BO(n), BO(n-1))$ with twisted coefficients in the sphere spectrum $\mathbf{S}^0$. Using the unit map $\mathbf{S}^0 \to \mathbf{A}(*)$ determined by the point $\chi(*) \in A(*) = \Omega^\infty \mathbf{A}(*)$, we can view $[b_n]$ as a class with twisted coefficients in the spectrum $\mathbf{A}(*)$. It turns out that

$$[b_n] = j^*[e_n] \tag{0.5}$$

where $j\colon (BO(n), BO(n-1)) \to (B\mathrm{TOP}(n), B\mathrm{TOP}(n-1))$ is the inclusion. The unit map $\mathbf{S}^0 \to \mathbf{A}(*)$ has a canonical splitting up to homotopy, so that (0.5) identifies the stable cohomotopy component of $[e_n]$ applied to any vector bundle (pair) and implies the vanishing of the other component. The reasoning behind (0.5) is that the forgetful passage from vector bundles to euclidean bundles factors through disk bundles, bundles with fibers homeomorphic to the n-disk $\mathbb{D}^n$ for some n; and we can learn something about those disk bundles by applying (0.4) with p equal to the disk bundle projection. Briefly, (0.5) is a distant corollary of (0.4).

Outline of proof of (0.3). For a pointed space X, let $Q(X) := \Omega^\infty \Sigma^\infty(X)$. One needs to know that the fiberwise Poincaré dual of $[b_n](\tau)$, viewed as a vertical homotopy class of sections of a fibration on B with fiber $Q(p^{-1}(b)_+)$ over $b \in B$, refines the Becker–Gottlieb–Dold transfer, viewed as a homotopy class of maps $B \to Q(E_+)$. (This refinement is still a fiber homotopy invariant of $p\colon E \to B$.) Combining (0.4) with (0.5) gives

$$\alpha \wp [b_n](\tau) \simeq \chi(p). \tag{0.6}$$

Applying an appropriate map of infinite loop spaces $A_B(E) \to K(R)$ determined by the module bundle V to the two sides of equation (0.6), we obtain (0.3). In [Wi] the transition from (0.6) to (0.3) is shown to be analogous to how Fulton and MacPherson [FM] establish a Riemann–Roch theorem in algebraic geometry by first proving a *bivariant* version.

This outline of proof also suggests that (0.6) should be viewed as essentially a special case of (0.3), the universal case—with suitably generalized notions of ring and flat vector bundle. We therefore sometimes refer to (0.6) as the *universal Riemann-Roch formula* for bundles of smooth compact manifolds.

A converse Riemann–Roch theorem. The two sides of equation (0.6) are defined for *any* fibration with homotopy finitely dominated fibers, $p\colon E \to B$. It is therefore tempting to look for a geometric characterization of those fibrations $p\colon E \to B$ for which the universal Riemann-Roch equation (0.6) is a *true* equation. We show in Part III of this work that (0.6) holds for a fibration $p\colon E \to B$ with homotopy finitely dominated fibers if and only if p is fiber homotopy equivalent to a bundle of smooth compact manifolds. This converse depends very strongly on Waldhausen's work relating stabilized h-cobordism theory (alias concordance theory, alias pseudoisotopy theory) to algebraic K-theory.

Further remarks. (1) Our construction of the characteristic class $[e_n]$ is designed to give the shortest possible proof of the index theorem (0.4). There is an alternative description in which $[e_n]$ appears as the stable part of the obstruction to splitting off trivial line bundles from euclidean n-bundles. For motivation we recall that Becker designed his characteristic class $[b_n]$ to have the following property:

Let ξ be an n-dimensional vector bundle on a CW-space X of dimension $\leqslant 2n-4$. Then $[b_n](\xi)=0$ if and only if ξ has a vector bundle splitting $\xi\cong\xi'\oplus\varepsilon$, where ε is a trivial line bundle.

Becker found that the correct coefficient spectrum for a characteristic class $[b_n]$ with this property had to be a spectrum with ith term $\Sigma(\mathrm{O}(i)/\mathrm{O}(i-1))$, in other words the sphere spectrum. By analogy, in the alternative description of $[e_n]$ the coefficient spectrum has ith term $\Sigma(\mathrm{TOP}(i)/\mathrm{TOP}(i-1))$. Again Waldhausen's theory is needed to identify the coefficient spectrum with $\mathbf{A}(*)$. Details may appear elsewhere.

(2) After reading a preliminary version of the present paper, Waldhausen found another proof of (0.6) (for bundles of smooth compact manifolds) and the "converse" mentioned above. His proof is based on his manifold approach to (his) algebraic K-theory of spaces. An advantage of this proof is that it is very direct, for a reader who knows his way through Waldhausen's work on h-cobordism theory and the algebraic K-theory of spaces. A disadvantage is that it does not make the connection with index theory. As a result it does not suggest the generalizations that our proof suggests. (The generalizations that we have in mind are index theorems involving families of *closed* manifolds and their Euler characteristics, where the Euler characteristics are promoted to homotopy fixed points of actions of $\mathbb{Z}/2$ by duality on the appropriate algebraic K-theory spaces.) Waldhausen's alternative proof will be sketched at the end of Part III.

Guide. In §1 we introduce the notion of a characteristic for a functor F from a small category $\mathcal{C}$ to spaces. We use it to explain how certain generalized Euler characteristics can be evaluated on "families" of spaces, such as the fibers of a fibration. What one should have in mind here is the A-theory Euler characteristic described above.

In §§2–5 we have a much narrower framework, as follows. Let $\mathcal{E}^\bullet$ be the category of euclidean neighborhood retracts (ENR's), where a morphism from X to Y is a *partial* proper map from X to Y; that is, a proper map $V\to Y$, where V is an open subset of X. Such a morphism, denoted $X\rightsquigarrow Y$, is a *localization* if the underlying proper map $V\to Y$ is cell-like [L1], [L2], [L3]; the cases where $V\to Y$ is an identity map are particularly important to us.

The localization morphisms form a subcategory $l\mathcal{E}^\bullet$ of $\mathcal{E}^\bullet$. We now fix a functor F from $\mathcal{E}^\bullet$ to pointed spaces which is pro-excisive; this means that the abelian group-valued functor $\pi_* F$ is a locally finite generalized homology theory on $\mathcal{E}^\bullet$. We then fix a characteristic χ for $F|l\mathcal{E}^\bullet$.

The F that we have in mind in §§2–5 is the pro-excisive functor corresponding to $\mathbf{A}(*)$, the connective spectrum determined by the infinite loop space $A(*)$. The χ that we have in mind in §§2–5 is a refinement of the A-theory Euler characteristic outlined above—where that makes sense, i.e., for compact ENR's and cell-like maps between such. Our strategy is to prove the index theorem (0.4) by stating and proving a pre-assembly version of (0.4), a much sharper index theorem, in the abstract framework of pro-excisive functors F on $\mathcal{E}^\bullet$ and characteristics for $F|l\mathcal{E}^\bullet$.

The characteristic classes needed in the (sharper) index theorem are developed in §2. The index theorem itself occupies §3. In §4 we prove the vanishing theorem (0.5), in the abstract framework. §5 is about a pre-assembly version of (0.6), a consequence of §3 and §4.

In §6 and §7 we finally present those examples of characteristics that we had in mind in §1 and §§2–5, respectively. It is shown in §8 that the §7 example does indeed refine the main example of §6. This completes the proofs of (0.4) and (0.6). In §8 we show how (0.3) follows from (0.6) and discuss how (0.3) needs to be corrected to be valid for bundles of compact topological manifolds. In the last subsection of §8 we explain how our results give rise to a family Reidemeister torsion; compare [BL] and [IK], [I3].

In §9 we state the Waldhausen theorems on the homotopy types of stabilized spaces of h-cobordisms, with all the background that we need to use them. The last part of §9 is a guided tour around Waldhausen's writings on the subject. The main result of §10 is a characterization of bundles of compact topological manifolds among fibrations $p\colon E \to B$ with finitely dominated fibers. This characterization is in terms of the fiberwise index $\chi(p)$. The behavior of the basic index theorem of §3 under stabilization, i.e., product with $[0,1]$, is the subject of §11. Using §10 and §11, we finally obtain in §12 a characterization of bundles of compact smooth manifolds among fibrations $p\colon E \to B$ with finitely dominated fibers. This characterization is again in terms of the fiberwise index $\chi(p)$.

Conventions. Unless otherwise stated, the sets that we use are *small sets*, i.e., they are elements of a fixed universe [Mac, Chapter I, §6].

We use the word *space* to mean a compactly generated weak Hausdorff space (whose underlying set is small), unless otherwise stated. Products and mapping spaces are formed in the category of these spaces in the usual way.

All *cofibrations* are closed maps having the homotopy extension property.

A *homotopy cartesian square* is a commutative square of spaces in which the canonical map from the initial term to the homotopy pullback of the other three terms is a weak homotopy equivalence.

We tend to use boldface notation for spectra. Specifically, if X is an infinite loop space, then $\mathbf{X}$ is the default notation for the corresponding (-1)-connected spectrum.

Part I. An index theorem

1. Characteristics

Overview. A finitely dominated space X determines an Euler characteristic in the group $K_0(\mathbb{Z}\pi_1(X))$ whose image in the reduced K_0-group is the Wall finiteness obstruction. With a view to extracting Wall finiteness obstructions fiberwise, i.e., from a fibration with finitely dominated fibers, we propose an abstraction of the notion *Euler characteristic* which is applicable to *parametrized families* (of finitely dominated spaces, or other objects).

To keep notation simple, we work in an abstract setting which does not even mention algebraic K-theory. For illustrations with algebraic K-theory, see §6, which can be read right after §1.

Let $\mathcal{C}$ be a small category and let F be a functor from $\mathcal{C}$ to the category of spaces. For each C in $\mathcal{C}$ let $\mathcal{C}/C$ be the *over category* whose objects are the morphisms in $\mathcal{C}$ with codomain C. Later we will also need the *under category* $C/\mathcal{C}$ whose objects are the morphisms in $\mathcal{C}$ with domain C.

Definition 1.1. A *characteristic* for F is a natural transformation χ from the functor $C \mapsto |\mathcal{C}/C|$ to F.

Remarks. (i) The *space* of characteristics for F is exactly $\operatorname{holim} F$, the homotopy limit of F.

(ii) For C in $\mathcal{C}$ let $\chi(C) := \chi(\mathrm{id}_C) \in F(C)$, the image under χ of the identity vertex in $|\mathcal{C}/C|$. In our favorite examples $\chi(C)$ is something like an Euler characteristic of C. It is a point in a space $F(C)$ which may depend on C.

Example 1.2. Let F be the constant functor with constant value equal to the classifying space $|\mathcal{C}|$. For each C in $\mathcal{C}$ we have the forgetful functor $\mathcal{C}/C \to \mathcal{C}$ which, on passage to classifying spaces, yields $\chi\colon |\mathcal{C}/C| \to F(C)$.

Example 1.3. Let $\mathcal{F}$ be a functor from $\mathcal{C}$ to the category of small categories. Suppose that we have a rule selecting, for each object C in $\mathcal{C}$, an object $C^!$ in $\mathcal{F}(C)$ and, for each morphism $e\colon C \to D$ in $\mathcal{C}$, a morphism $e^!\colon e_*(C^!) \to D^!$ in $\mathcal{F}(D)$. We assume that $e^!$ is an identity morphism whenever e is an identity morphism and that the chain rule alias 1-cocycle condition is satisfied: $(ef)^! = e^! \cdot e_*(f^!)$ whenever e and f are composable.

In these circumstances we can define a functor F from $\mathcal{C}$ to spaces by $F(C) := |\mathcal{F}(C)|$. We can also define a characteristic for F, as follows. For a fixed D in $\mathcal{C}$ the rule taking $f: C \to D$ to $f_*(C^!)$ is a functor from $\mathcal{C}/D$ to $\mathcal{F}(D)$, inducing $\chi: |\mathcal{C}/D| \to F(D)$.

Suppose again that F is a functor from $\mathcal{C}$ to spaces. The *homotopy colimit* of F, denoted $\operatorname{hocolim} F$, is the geometric realization of the simplicial space

$$[n] \mapsto \coprod_{u:[n]\to\mathcal{C}} Fu(0).$$

See [BK]. When it is convenient we write $\operatorname{hocolim}_C F(C)$ or $\operatorname{hocolim} F(?)$ instead of $\operatorname{hocolim} F$. The homotopy colimit of F comes with a canonical map to $|\mathcal{C}|$, because $|\mathcal{C}|$ is the homotopy colimit of the terminal functor from $\mathcal{C}$ to spaces. A characteristic for F determines, *up to homotopy*, a section of the canonical map $\operatorname{hocolim} F \to |\mathcal{C}|$. More precisely:

OBSERVATION 1.4. *A characteristic χ for F induces a map from* $\operatorname{hocolim} |\mathcal{C}/?|$ *to* $\operatorname{hocolim} F$. *The composition* $\operatorname{hocolim} |\mathcal{C}/?| \to \operatorname{hocolim} F \to |\mathcal{C}|$ *is a homotopy equivalence.*

Remark 1.5. Let $u: B' \to B$ be a homotopy equivalence and let $p: X \to B$ be a fibration. Then

$$\Gamma(p) = \operatorname{map}_B(B, X) \xrightarrow{u^*} \operatorname{map}_B(B', X)$$

is a homotopy equivalence. Hence every map $s: B' \to X$ over B determines, up to contractible choice, a section of p.

In particular, in the situation of Observation 1.4, let $u: B' \to B$ be the projection from $\operatorname{hocolim} |\mathcal{C}/?|$ to $|\mathcal{C}|$ and let $p: X \to B$ be the fibration associated with the projection $\operatorname{hocolim} F \to |\mathcal{C}|$. Now Observation 1.4 gives us a canonical map from $\operatorname{holim} F$ to

$$\operatorname{map}_B(B', X) \simeq \Gamma(p).$$

It follows from [Dw, 3.12] that this map is a weak homotopy equivalence if F takes all morphisms to weak homotopy equivalences. Under these circumstances, therefore, we like to think of elements of $\operatorname{holim} F$ as sections of the fibration associated with the quasifibration $\operatorname{hocolim} F \to |\mathcal{C}|$.

1.6. *Main application.* Let $p: E \to B$ be a fibration, where B is the geometric realization of a simplicial set $\mathfrak{B}$. Suppose that $\mathcal{C}$ is a full subcategory of the category consisting of all spaces and the homotopy equivalences between them. Let F be a functor from $\mathcal{C}$ to spaces taking all morphisms to homotopy equivalences. Suppose finally that, for any simplex x in $\mathfrak{B}$, the pullback E_x of E under the characteristic map $\Delta^{|x|} \to B$ belongs to $\mathcal{C}$. We will make a fibration $F_B(E) \to B$ essentially by applying F to the fibers of p. If F comes with a characteristic χ, we can evaluate it on the fibers of p to obtain a section $\chi(p)$ of $F_B(E) \to B$.

Details. Let simp($\mathfrak{B}$) be the category of simplices of $\mathfrak{B}$. The objects are the simplices of $\mathfrak{B}$; a morphism $x \to y$ is a monotone map f from $\{0, 1, ..., |x|\}$ to $\{0, 1, ..., |y|\}$ for which $f^*y = x$.

Let B' be the homotopy colimit of the functor $x \mapsto |\mathrm{simp}(\mathfrak{B})/x|$ on simp($\mathfrak{B}$). This comes with a canonical projection to $|\mathrm{simp}(\mathfrak{B})|$ which is a homotopy equivalence. For cosmetic purposes we like to compose it with another homotopy equivalence from $|\mathrm{simp}(\mathfrak{B})|$ to B. This is induced by Kan's 'last vertex map' [Kan], a simplicial map from the nerve of simp($\mathfrak{B}$) to $\mathfrak{B}$. The last vertex map takes an n-simplex

$$x_0 \xrightarrow{g_0} x_1 \xrightarrow{g_1} \dots \xrightarrow{g_{n-1}} x_n$$

in the nerve to the n-simplex g^*x_n in $\mathfrak{B}$, where $g: \{0, ..., n\} \to \{0, ..., |x_n|\}$ is defined by $g(i) = g_{n-1} \dots g_{i+1} g_i(|x_i|)$ for $0 \leqslant i \leqslant n$.

Now define $F_B(E) \to B$ as the fibration associated with the composite projection (which is already a quasifibration) $\mathrm{hocolim}_x F(E_x) \to |\mathrm{simp}(\mathfrak{B})| \to B$, where x runs through the simplices of $\mathfrak{B}$. The characteristic χ gives us an element

$$\chi(p) \in \operatorname*{holim}_x F(E_x). \tag{1.7}$$

This in turn gives us a map $B' \to F_B(E)$ over B, which we may regard as a section of $F_B(E) \to B$, determined up to contractible choice, as explained in Remark 1.5. We emphasize that (1.7) is our rigorous definition of $\chi(p)$. But we sometimes find it suggestive to write $\chi(p): B \to F_B(E)$ for (1.7).

2. Excisive characteristics and characteristic classes

Overview. In this section we introduce *excisive* characteristics $\chi(Y) \in F(Y)$ defined for euclidean neighborhood retracts Y. We assume therefore that F is defined on the category $\mathcal{E}^\bullet$ whose morphisms are pointed maps between one-point compactifications of euclidean neighborhood retracts, and that π_*F is a locally finite generalized homology theory (see Remark 2.2). However, we do not assume that the characteristics $\chi(Y)$ behave naturally for *all* morphisms in $\mathcal{E}^\bullet$; we only assume that they behave naturally for what we call *localization* morphisms.

Again we work in an abstract setting to keep the notation simple. For illustrations with algebraic K-theory, see §7, which can be read right after §2.

Locally finite homology theories and excisive characteristics

A map $X \to Y$ between locally compact spaces is *proper* if it extends to a (continuous) pointed map $X^{\bullet} \to Y^{\bullet}$ of the one-point compactifications. Euclidean neighborhood retracts (ENR's) are locally compact, so we can make a category $\mathcal{E}$ whose objects are the ENR's and whose morphisms are the proper maps. There is a larger category $\mathcal{E}^{\bullet}$ with the same objects, where a morphism from X to Y is a (continuous) pointed map $f: X^{\bullet} \to Y^{\bullet}$.

Remark–Notation. It is appropriate to think of a morphism in $\mathcal{E}^{\bullet}$ from X to Y as a *partial* proper map, i.e., a proper map to Y defined on an open subset of X. Namely, the morphism is a pointed map $f: X^{\bullet} \to Y^{\bullet}$ and so determines an open subset $V := f^{-1}(Y) = X \setminus f^{-1}(\infty)$ of X and a proper map $g: V \to Y$ given by $g(x) := f(x)$ for $x \in V$. (We have $V = X$ if and only if f is a morphism in $\mathcal{E} \subset \mathcal{E}^{\bullet}$.) Conversely, given X and Y, an arbitrary open set V in X and an arbitrary proper map $g: V \to Y$, we obtain a pointed map $f: X^{\bullet} \to Y^{\bullet}$ alias morphism f from X to Y in $\mathcal{E}^{\bullet}$, by $f(x) := g(x)$ for $x \in V$ and $f(x) := \infty$ for $x \in X^{\bullet} \setminus V$.

To remind the reader that morphisms from X to Y in $\mathcal{E}^{\bullet}$ are, in general, not maps from X to Y, but only partial (proper) maps, we will preferably write such morphisms in the form

$$f: X \rightsquigarrow Y.$$

2.1. *Terminology* (compare [WW2, 1.1]). A commutative square of locally compact spaces and proper maps

$$\begin{tikzcd} X_1 \arrow[r] \arrow[d] & X_2 \arrow[d] \\ X_3 \arrow[r] & X_4 \end{tikzcd}$$

is a *proper homotopy pushout square* if the resulting map from the homotopy pushout of $X_3 \leftarrow X_1 \to X_2$ to X_4 is a proper homotopy equivalence. A covariant functor F from $\mathcal{E}$ to pointed spaces is *homotopy invariant* if it takes proper homotopy equivalences to weak homotopy equivalences. A homotopy invariant F is *excisive* if it takes proper homotopy pushout squares in $\mathcal{E}$ to weak homotopy pullback squares (also called *homotopy cartesian* squares) of spaces and $F(X \times [0, \infty[\,)$ is weakly contractible (weakly homotopy equivalent to a point) for all X in $\mathcal{E}$.

Finally we call F *pro-excisive* if it is homotopy invariant, excisive and satisfies an appropriate wedge axiom as follows. Suppose that X is the coproduct in $\mathcal{E}$ of objects X_i for $i \in \mathbb{N} = \{0, 1, 2, \ldots\}$. For each $j \in \mathbb{N}$ let X'_j be the coproduct of X_j and all $X_i \times [0, \infty[$ for $i \neq j$. The obvious inclusions induce

$$\pi_* F(X) \to \pi_* F(X'_j) \leftarrow \pi_* F(X_j)$$

where the right-hand arrow is an isomorphism by excision. We use this as an identification, take the product over all j, and obtain $u: \pi_* F(X) \to \prod_{j\in\mathbb{N}} \pi_* F(X_j)$. The wedge axiom we want stipulates that u be an isomorphism.

A functor F from $\mathcal{E}^\bullet$ to pointed spaces is *pro-excisive* if its restriction to $\mathcal{E}$ is pro-excisive. In this situation the wedge axiom simplifies to the statement that, for $X = \coprod_{i\in\mathbb{N}} X_i$ in $\mathcal{E}^\bullet$, the morphisms $p_i: X \rightsquigarrow X_i$ given by $p_i(x) = x$ for $x \in X_i$ and $p_i(x) = \infty$ for x not in X_i induce a weak homotopy equivalence $F(X) \to \prod_{i\in\mathbb{N}} F(X_i)$.

Remark 2.2. Suppose that F is pro-excisive (defined on $\mathcal{E}$, to begin with). Then each $F(X)$ is an infinite loop space. Indeed, excision implies that the commutative diagram of pointed spaces

$$\begin{array}{ccc} F(X) & \longrightarrow & F(X\times[0,\infty[) \\ \downarrow & & \downarrow \\ F(X\times]-\infty,0]) & \longrightarrow & F(X\times\mathbb{R}) \end{array}$$

is a homotopy cartesian square with contractible lower left and upper right term. Let $\mathbf{F}(X)$ be the CW-spectrum made from the spaces $F(X\times\mathbb{R}^i)$, using their singular simplicial sets and the method of [Go1, 0.1]. Then $\mathbf{F}$ is a functor from $\mathcal{E}$ to CW-spectra; it is pro-excisive in the sense of [WW2, 1.1]. Conversely, if $\mathbf{J}$ from $\mathcal{E}$ to CW-spectra is pro-excisive as in [WW2, 1.1], then $\Omega^\infty\mathbf{J}$ is pro-excisive in the sense of §2.1 above. (Use [WW2, 1.2].) Here Ω^∞ of a spectrum can be defined as the geometric realization of the simplicial set of maps to that spectrum from the sphere spectrum.

Consequently, the classification theorem of [WW2, 1.2] is also a classification theorem for pro-excisive functors on $\mathcal{E}$ as in §2.1 above. The result of the classification is that every pro-excisive F from $\mathcal{E}$ to pointed spaces is related by a chain of natural weak equivalences to the functor $X \mapsto \Omega^\infty(X^\bullet \wedge \mathbf{F}(*))$. (A natural transformation between functors F_1, F_2 from $\mathcal{E}$ to pointed spaces is a *weak equivalence* if the map $F_1(X) \to F_2(X)$ is a weak homotopy equivalence for every X in $\mathcal{E}$.)

There is a similar classification theorem for pro-excisive functors from $\mathcal{E}^\bullet$ to spectra as in §2.1: every pro-excisive F from $\mathcal{E}^\bullet$ to pointed spaces is related by a chain of natural weak equivalences to the functor $X \mapsto \Omega^\infty(X^\bullet \wedge \mathbf{F}(*))$. See [WW2, 2.1]. If F from $\mathcal{E}^\bullet$ to pointed spaces is pro-excisive, then the functor $X \mapsto \pi_* F(X)$ on $\mathcal{E}^\bullet$ has all the properties that one expects from a *locally finite generalized homology theory.*

2.3. *More terminology.* A compact subset C of $\mathbb{R}^n$ is *cellular* [Br] if each neighborhood of C in $\mathbb{R}^n$ contains another neighborhood of C which is an n-cell, i.e., homeomorphic to $\mathbb{R}^n$. Then

$$\check{H}^*(C;\mathbb{Z}) \cong \check{H}^*(\text{point};\mathbb{Z})$$

by the continuity property of Čech cohomology. A map $g: X \to Y$ between ENR's is *cell-like* if $g^{-1}(y)$ is homeomorphic to a cellular subset of some euclidean space, for all $y \in Y$. Lacher, who introduced the notion, also gave in [L1, 1.2] the following beautiful characterization of proper cell-like maps among the proper maps from X to Y. A proper map $g: X \to Y$ (between ENR's) is cell-like if and only if, for every open subset U of Y, the appropriate restriction of g is a proper homotopy equivalence from $g^{-1}(U)$ to U.

The only 'explicit' cell-like maps that we are going to see in this paper are homeomorphisms and disk bundle projections.

We call a morphism $f: X \rightsquigarrow Y$ in $\mathcal{E}^\bullet$ a *localization* if it is cell-like as a partial map; in other words, if the proper map from $f^{-1}(Y) = X \setminus f^{-1}(\infty)$ to Y obtained by restricting f is cell-like. We denote the subcategory of $\mathcal{E}^\bullet$ consisting of all objects and the localization morphisms by $l\mathcal{E}^\bullet$. Then $\mathcal{E} \cap l\mathcal{E}^\bullet$ (intersection in $\mathcal{E}^\bullet$) is the category of all ENR's, with (proper) cell-like maps as morphisms.

Examples. Let X be an ENR. For an arbitrary open subset $V \subset X$, the pointed map $X^\bullet \to V^\bullet$ given by $x \mapsto x$ for $x \in V \subset X^\bullet$ and $x \mapsto \infty$ for $x \in X^\bullet \setminus V$ is a morphism $X \rightsquigarrow V$ in $l\mathcal{E}^\bullet \subset \mathcal{E}^\bullet$. Indeed, the identity $V \to V$ is cell-like.

For any X in $\mathcal{E}^\bullet$, the projection $X \times [0,1] \to X$ is proper and cell-like, and so belongs to the intersection $\mathcal{E} \cap l\mathcal{E}^\bullet \subset \mathcal{E}^\bullet$.

Assumption 2.4. For the remainder of this section (and for use in §§3–6) we fix a pro-excisive functor F from $\mathcal{E}^\bullet$ to pointed spaces and a characteristic χ for $F|l\mathcal{E}^\bullet$. We call such a characteristic an *excisive* characteristic.

Excisive characteristics for bundles

Let $p: E \to B$ be a fiber bundle whose fibers are compact topological manifolds. Assume that $B = |\mathfrak{B}|$ for a simplicial set $\mathfrak{B}$. Our goal is to associate with this setup a section $\chi(p)$ of $F_B(E) \to B$.

This is of course the informal notation of §1.6. The true home of $\chi(p)$ should be the homotopy limit $\mathrm{holim}_x\, F(E_x)$, where x runs over the objects of $\mathrm{simp}(\mathfrak{B})$. The space $\mathrm{holim}_x\, F(E_x)$ can be viewed as a section space; see Observation 1.4 and Remark 1.5.

Unfortunately the characteristic χ does not determine a point in $\mathrm{holim}_x\, F(E_x)$. The difficulty stems from the fact that χ is a characteristic for $F|l\mathcal{E}^\bullet$, while $x \mapsto E_x$ is not a functor from $\mathrm{simp}(\mathfrak{B})$ to $l\mathcal{E}^\bullet$ except in the trivial case $B = \varnothing$. (It is a functor from $\mathrm{simp}(\mathfrak{B})$ to $\mathcal{E} \subset \mathcal{E}^\bullet$, but not to $\mathcal{E} \cap l\mathcal{E}^\bullet$, which is the category of ENR's and cell-like maps. The maps $E_x \to E_y$ induced by morphisms $x \to y$ in $\mathrm{simp}(\mathfrak{B})$ are not always surjective, therefore not always cell-like.) Heavy guns are needed to overcome this difficulty.

Let $t\mathfrak{B}$ be the simplicial set whose k-simplices are pairs (x,θ), where x is a k-simplex in $\mathfrak{B}$ and θ is an equivalence relation on E_x, with quotient space E_x^θ, such that the two projections make up a homeomorphism $E_x \to \Delta^k \times E_x^\theta$.

THEOREM 2.5. *The forgetful map $t\mathfrak{B}\to\mathfrak{B}$ is a Kan fibration. Its fiber over any vertex x is isomorphic to $\mathrm{TOP}(E_x,\partial E_x)/\mathrm{TOP}_0(E_x,\partial E_x)$, where $\mathrm{TOP}(E_x,\partial E_x)$ is the simplicial group of automorphisms of the pair $(E_x,\partial E_x)$ and $\mathrm{TOP}_0(E_x,\partial E_x)$ is the corresponding discrete group. Hence by* [Mc2] *the fibers of $t\mathfrak{B}\to\mathfrak{B}$ are acyclic.*

Remark. Suppose that B is connected, with base vertex z. Let c from B to $B\mathrm{TOP}(E_z,\partial E_z)$ be the classifying map for p. Then we think of $tB:=|t\mathfrak{B}|$ as the homotopy pullback of

$$B \xrightarrow{c} B\mathrm{TOP}(E_z,\partial E_z) \hookleftarrow B\mathrm{TOP}_0(E_z,\partial E_z).$$

Proof of Theorem 2.5. The Kan fibration statement is true by inspection. Now fix a vertex x in B and write $L=E_x=p^{-1}(x)$. There is a map from $\mathrm{TOP}(L,\partial L)$ to the fiber of $t\mathfrak{B}\to\mathfrak{B}$ over x which takes a k-simplex $f\colon L\times\Delta^k\to L\times\Delta^k$ to the equivalence relation θ on $L\times\Delta^k$ given by

$$z_1\theta z_2 \iff f_1(z_1)=f_1(z_2)\in L,$$

where f_1 is the first component of f. The map factors through the space of left cosets $\mathrm{TOP}(L,\partial L)/\mathrm{TOP}_0(L,\partial L)$, giving the isomorphism claimed in Theorem 2.5. The main result of [Mc2], building on work of Mather [Mat], Thurston [Thu] and Segal [S2], is that $\mathrm{TOP}(L,\partial L)/\mathrm{TOP}_0(L,\partial L)$ is acyclic. □

Let $tB=|t\mathfrak{B}|$ and let $\mathrm{simp}(t\mathfrak{B})$ be the category of simplices of $t\mathfrak{B}$. The projections $E_x\to E_x^\theta$ for (x,θ) in $\mathrm{simp}(t\mathfrak{B})$ constitute a natural transformation which induces a map of homotopy limits:

$$\operatorname*{holim}_{x} F(E_x) \to \operatorname*{holim}_{(x,\theta)} F(E_x^\theta). \tag{2.6}$$

COROLLARY 2.7. *This map is a weak homotopy equivalence.*

Proof. Since we are comparing two functors—on two different but related categories, $\mathrm{simp}(\mathfrak{B})$ and $\mathrm{simp}(t\mathfrak{B})$—taking all morphisms to weak homotopy equivalences, we can translate the statement into one about section spaces. That statement is a special case of the following: Let

$$\begin{tikzcd} E_1 \arrow[r] \arrow[d, "q_1"] & E_2 \arrow[d, "q_2"] \\ B_1 \arrow[r, "g"] & B_2 \end{tikzcd}$$

be a commutative pullback square, where q_2 is a fibration whose fibers are componentwise nilpotent [HMR] (we have in mind the infinite loop spaces $F(E_x^\theta)$) and g is a map whose homotopy fiber over any point in B_2 is acyclic. Suppose also that B_1 and B_2 are homotopy equivalent to CW-spaces. Then the pullback map from $\Gamma(q_2)$ to $\Gamma(q_1)$ is a homotopy equivalence.

The proof of the general statement is as follows. We can assume that g is a fibration. Then $\Gamma(q_1)$ can be identified with a space of sections $\Gamma(q_2')$ where $q_2'\colon E_2'\to B_2$ is the fibration whose fiber over $x\in B_2$ is the space of maps from $g^{-1}(x)$ to $q_2^{-1}(x)$. Our map $\Gamma(q_2)\to\Gamma(q_1)$ becomes a map $\Gamma(q_2)\to\Gamma(q_2')$ induced by an evident map $E_2\to E_2'$ over B_2. This is given on the fibers over $x\in B_2$ by the inclusion of the space of constant maps from $g^{-1}(x)$ to $q_2^{-1}(x)$ in the space of all maps from $g^{-1}(x)$ to $q_2^{-1}(x)$. By our hypotheses, $E_2\to E_2'$ is a fiberwise weak homotopy equivalence. □

2.8. *Definition-Summary.* Let $p\colon E\to B$ be a fiber bundle with compact topological manifold fibers. Up to contractible choice, p determines a section $\chi(p)$ of $F_B(E)\to B$, in the informal notation of §1.6, but with Assumption 2.4.

Details. We assume $B=|\mathfrak{B}|$ as in Theorem 2.5 and observe that $(x,\theta)\mapsto E_x^\theta$ is a functor taking all morphisms to *homeomorphisms*, hence a functor from $\operatorname{simp}(t\mathfrak{B})$ to $l\mathcal{E}^\bullet$. Therefore the characteristic χ for $F|l\mathcal{E}^\bullet$ determines a point

$$\chi(p)\in \operatorname*{holim}_{(x,\theta)} F(E_x^\theta). \tag{2.9}$$

The homotopy fiber C of (2.6) over $\chi(p)$ is a weakly contractible space. Each point $c\in C$ determines a point in $\operatorname{holim}_x F(E_x)$, which in turn determines, as in Remark 1.5 and §1.6, a section of $F_B(E)\to B$. We emphasize once again that (2.9) is our rigorous definition of $\chi(p)$; but we find the informal notation $\chi(p)\colon B\to F_B(E)$ suggestive. □

Characteristic classes

We will show that χ in Assumption 2.4 determines a characteristic cohomology class $[\bar{e}_n]$ for euclidean n-bundles, with twisted coefficients in the spectrum $\mathbf{F}(*)$. For more precision, write $\mathrm{TOP}(n)=\mathrm{TOP}(\mathbb{R}^n)$ and let $\gamma(n)$ be the tautological euclidean n-bundle on $B\mathrm{TOP}(n)$, with fiberwise one-point compactification $\gamma(n)^\bullet$. We write

$$[\bar{e}_n]\in H^{\gamma(n)}(B\mathrm{TOP}(n);\mathbf{F}(*))$$

to indicate that $[\bar{e}_n]$ is a vertical homotopy class of sections of a bundle with base $B\mathrm{TOP}(n)$ and fiber $\Omega^\infty(\gamma(n)_x^\bullet\wedge\mathbf{F}(*))$ over $x\in B\mathrm{TOP}(n)$.

Remark 2.10. We take the following view of euclidean n-bundles. Let $\mathcal{B}(n)$ be the category of n-*balls*, i.e., the category of all spaces homeomorphic to $\mathbb{R}^n$, with *reverse* embeddings as morphisms. That is, a morphism from V to W is an embedding of W in V. By our conventions, $\mathcal{B}(n)$ is a small category. Then

$$|\mathcal{B}(n)| \simeq B\,\mathrm{TOP}(n) \tag{2.11}$$

by [Mc2, 2.15]. A map from a paracompact space X to $|\mathcal{B}(n)|$ is therefore as good as a euclidean n-bundle on X.

Remark 2.12. We take the following view of cohomology. Let $\mathcal{C}$ be a small category and let $\mathbf{E}$ be a functor from $\mathcal{C}$ to the category of Ω-spectra; the morphisms in the category of Ω-spectra are the *functions* in the sense of [A1, p. 140]. Suppose that $\mathbf{E}$ takes all morphisms in $\mathcal{C}$ to weak homotopy equivalences.

Then for fixed $n\in\mathbb{Z}$ we have a functor $C\mapsto E_n(C)$ from $\mathcal{C}$ to spaces, picking out the nth term of $\mathbf{E}(C)$. Now we regard $\mathbf{E}$ as a (possibly twisted) coefficient system on $|\mathcal{C}|$ for generalized cohomology and write

$$\pi_k \operatorname{holim} E_n = H^{n-k}(|\mathcal{C}|; \mathbf{E}(-)) \tag{2.13}$$

for $k\geqslant 0$. We justify this by interpreting $\operatorname{holim} E_n$ as the section space of a suitable fibration on $|\mathcal{C}|$, with fiber $\simeq E_n(C)$ over a vertex C of $|\mathcal{C}|$. The details are as in Remark 1.5.

We identify $\mathcal{B}(n)$ with a subcategory of $l\mathcal{E}^{\bullet}$, consisting of the objects $V, W, \ldots$ which are homeomorphic to $\mathbb{R}^n$ and the morphisms $f\colon V \rightsquigarrow W$ which restrict to homeomorphisms from $f^{-1}(W)\subset V$ to W. Let

$$\bar{e}_n := \chi\,|\,\mathcal{B}(n) \in \operatorname{holim}(F\,|\,\mathcal{B}(n)). \tag{2.14}$$

Now (2.13) says that $\pi_0 \operatorname{holim}(F\,|\,\mathcal{B}(n))$ is the 0th cohomology of $|\mathcal{B}(n)|$ with coefficients in the spectrum-valued functor $\mathbf{F}\,|\,\mathcal{B}(n)$. Combining this observation with Remarks 2.10 and 2.2, we can say that $[\bar{e}_n]\in\pi_0 \operatorname{holim}(F\,|\,\mathcal{B}(n))$ is a characteristic cohomology class for euclidean n-bundles,

$$[\bar{e}_n] \in H^{\gamma(n)}(B\,\mathrm{TOP}(n); \mathbf{F}(*)).$$

Relative characteristic classes

Remark 2.15. We take the following view of relative cohomology. Let $\mathcal{A}$ be a small category, $\partial\mathcal{A}\subset\mathcal{A}$ a subcategory. Suppose that any morphism in $\mathcal{A}$ whose codomain is in $\partial\mathcal{A}$ belongs to $\partial\mathcal{A}$. Let $\mathbf{E}$ be a functor from $\mathcal{A}$ to the category of Ω-spectra (defined as in Remark 2.12) taking all morphisms to weak homotopy equivalences. Let $\partial\mathbf{E}$ be a functor from $\partial\mathcal{A}$ to *weakly contractible* Ω-spectra. Let $\psi\colon\partial\mathbf{E}\to\mathbf{E}|\partial\mathcal{A}$ be a natural transformation. Define another functor $\mathbf{E}^{\mathrm{rel}}$ from $\mathcal{A}$ to Ω-spectra by

$$\mathbf{E}^{\mathrm{rel}}(a) := \begin{cases} \partial\mathbf{E}(a) & \text{if } a \text{ is in } \partial\mathcal{A}, \\ \mathbf{E}(a) & \text{otherwise,} \end{cases}$$

$$\mathbf{E}^{\mathrm{rel}}(f) := \begin{cases} \partial\mathbf{E}(f) & \text{if } f \text{ is in } \partial\mathcal{A}, \\ \mathbf{E}(f) & \text{if the domain of } f \text{ is not in } \partial\mathcal{A}, \\ \mathbf{E}(f)\cdot\psi_d & \text{otherwise (where } d \text{ is the domain of } f). \end{cases}$$

Then we allow ourselves to write

$$\pi_k \operatorname{holim} E_n^{\mathrm{rel}} = H^{n-k}(|\mathcal{A}|, |\partial\mathcal{A}|; \mathbf{E})$$

for $k\geqslant 0$. To justify this, we refer to Remark 2.12 and observe in addition that $\operatorname{holim} E_n^{\mathrm{rel}}$ fits into a homotopy cartesian square with contractible lower left-hand term,

$$\begin{tikzcd} \operatorname{holim} E_n^{\mathrm{rel}} \arrow[r] \arrow[d] & \operatorname{holim} E_n \arrow[d] \\ \operatorname{holim} \partial E_n \arrow[r] & \operatorname{holim}(E_n|\partial\mathcal{A}). \end{tikzcd}$$ □

For $n\geqslant 0$ let $\mathcal{GB}(n)$ be the category of *generalized balls* whose objects are the spaces homeomorphic to $\mathbb{R}^n$ or $[0,\infty[\times\mathbb{R}^{n-1}$, with reverse embeddings taking boundary to boundary as morphisms. That is, a morphism from V to W is an embedding of $(W,\partial W)$ in $(V,\partial V)$. Note that the inclusion $\mathcal{B}(n)\to\mathcal{GB}(n)$ induces a homotopy equivalence of the classifying spaces—because it has a left adjoint, the *delete boundary* functor. Let $\partial\mathcal{GB}(n)\subset\mathcal{GB}(n)$ be the full subcategory consisting of the objects homeomorphic to $[0,\infty[\times\mathbb{R}^{n-1}$. Then we have

$$\bar{e}_n = \chi|\mathcal{GB}(n) \in \operatorname{holim}(F|\mathcal{GB}(n))^{\mathrm{rel}}, \tag{2.16}$$

improving on (2.14). It is shown in Lemma 2.17 below that the pair $(|\mathcal{GB}(n)|, |\partial\mathcal{GB}(n)|)$ is homotopy equivalent to $(B\mathrm{TOP}(n), B\mathrm{TOP}(n-1)^{\urcorner})$ for $n\geqslant 0$, where $\mathrm{TOP}(n-1)^{\urcorner}$ is the group of homeomorphisms $]-\infty,+\infty]\times\mathbb{R}^{n-1}\to]-\infty,+\infty]\times\mathbb{R}^{n-1}$, viewed as a subgroup of $\mathrm{TOP}(n)$ by restriction. (For $n=0$ read $B\mathrm{TOP}(n-1)^{\urcorner}=\varnothing$.) Therefore, using Remark 2.15, we may write

$$[\bar{e}_n] \in H^{\gamma(n)}(B\mathrm{TOP}(n), B\mathrm{TOP}(n-1)^{\urcorner}; \mathbf{F}(*)).$$

LEMMA 2.17. $(|\mathcal{G}\mathcal{B}(n)|, |\partial\mathcal{G}\mathcal{B}(n)|) \simeq (B\mathrm{TOP}(n), B\mathrm{TOP}(n-1)^{\urcorner})$.

Proof. Let $\mathcal{B}_t(n)$ and $\partial\mathcal{G}\mathcal{B}_t(n)$ be the topological categories corresponding to $\mathcal{B}(n)$ and $\mathcal{G}\mathcal{B}(n)$. It is enough to show that the vertical arrows in the diagram of classifying spaces and inclusion maps

$$\begin{tikzcd} |\partial\mathcal{G}\mathcal{B}(n)| \arrow[r] \arrow[d] & |\mathcal{B}(n)| \arrow[d] \\ |\partial\mathcal{G}\mathcal{B}_t(n)| \arrow[r] & |\mathcal{B}_t(n)| \end{tikzcd}$$

are homotopy equivalences (the horizontal arrows are induced by the *delete boundary* functors). This is done in [Mc2, 2.18] for the right-hand vertical arrow. An entirely analogous argument works for the left-hand arrow. □

Remark 2.18. Pairs of categories $(\mathcal{A}, \partial\mathcal{A})$ as in Remark 2.15 tend to arise in the following way. Let $v: \mathcal{B} \to \mathcal{C}$ be any functor between small categories. The *right cylinder* of v is the category obtained from $\mathcal{B} \amalg \mathcal{C}$ by adjoining, for each object b in $\mathcal{B}$ and each object c in $\mathcal{C}$, a morphism set $\mathrm{mor}(b,c)$ equal to $\mathrm{mor}(v(b),c)$. The composition maps from $\mathrm{mor}(c_1,c_2) \times \mathrm{mor}(b,c_1)$ to $\mathrm{mor}(b,c_2)$ and from $\mathrm{mor}(b_2,c) \times \mathrm{mor}(b_1,b_2)$ to $\mathrm{mor}(b_1,c)$ are the obvious ones. See [Tho] for a generalization of this construction.

Now let $\mathcal{A}$ be the right cylinder of v and let $\partial\mathcal{A} := \mathcal{B}$, viewed as a subcategory of $\mathcal{A}$. Then the pair $(\mathcal{A}, \partial\mathcal{A})$ has the property required in Remark 2.15.

For example, the pair $(\mathcal{G}\mathcal{B}(n), \partial\mathcal{G}\mathcal{B}(n))$ can be obtained in this way: $\mathcal{G}\mathcal{B}(n)$ is isomorphic to the right cylinder of the *delete boundary* functor, a functor from $\partial\mathcal{G}\mathcal{B}(n)$ to $\mathcal{B}(n)$.

3. The index theorem

Overview. We keep the notation and hypotheses of §2, specifically Assumption 2.4. The goal is to show that, for a closed topological manifold M^n, the element $\chi(M) \in F(M)$ is Poincaré dual to e_n of the tangent bundle τ of M. (However, e_n is only defined informally at the very end of the section, in Remark 3.19; it is the image of $\bar{e}_n$ under a canonical involution on the space(s) in which $\bar{e}_n$ lives. See Definition 3.9.) This formulation of the goal indicates that we want to think of Poincaré duality for M as a map between certain infinite loop spaces. The strategy of our proof is to look for a description of the classifying map for τ adapted to the view of euclidean bundles developed in Remark 2.10. In the process we will be led to a description of Poincaré duality which is adapted to the view of cohomology developed in Remark 2.12.

The classifying map for the tangent bundle

Assume for now that M^n is a manifold without boundary, not necessarily compact. Let $\mathcal{O}=\mathcal{O}(M)$ be the set of open subsets of M which are homeomorphic to $\mathbb{R}^n$. This is a poset, ordered by *reverse* inclusion. We will see that

$$|\mathcal{O}| \simeq M.$$

For a more precise statement, we introduce an open subset $W \subset |\mathcal{O}| \times M$. Note first that $|\mathcal{O}|$ is the geometric realization of a simplicial set whose k-simplices are of the form $(U_0, U_1, ..., U_k)$ with $U_i \in \mathcal{O}$ and $U_i \supset U_{i+1}$. We decree that $(x,y) \in |\mathcal{O}| \times M$ belongs to W if the (open) cell containing x corresponds to a nondegenerate simplex $(U_0, ..., U_k)$ which has $y \in U_0$.

PROPOSITION 3.1. *The projections $W \to |\mathcal{O}|$ and $W \to M$ are homotopy equivalences.*

Proof. Since W is open in the product $|\mathcal{O}| \times M$, the projections in question are *almost locally trivial* in the sense of [S2, A.1]. By [S2, A.2] it is enough to verify that both have contractible fibers. Each fiber of $W \to |\mathcal{O}|$ is homeomorphic to euclidean space $\mathbb{R}^n$.

Let W_y be the fiber of $W \to M$ over $y \in M$. Under the projection this embeds in $|\mathcal{O}|$, and we can describe it as the union of all open cells corresponding to nondegenerate simplices $(U_0, ..., U_k)$ where U_0 contains y. There is a subspace $V_y \subset W_y$ defined as the union of all open cells corresponding to nondegenerate simplices $(U_0, ..., U_k)$ where U_k contains y. Note the following:

(1) V_y is a deformation retract of W_y. Namely, suppose that x in W_y belongs to a cell corresponding to a simplex $(U_0, ..., U_k)$ with $y \in U_0$. Let $(x_0, x_1, ..., x_k)$ be the barycentric coordinates of x in that simplex, all $x_i > 0$, and let $j \leqslant k$ be the largest integer such that $y \in U_j$. Define a deformation retraction by

$$h_{1-t}(x) := (tx_{\mathrm{no}} + x_{\mathrm{yes}})^{-1}(x_0, ..., x_j, tx_{j+1}, ..., tx_k),$$
$$x_{\mathrm{no}} := \sum_{i>j} x_i, \quad x_{\mathrm{yes}} := \sum_{i \leqslant j} x_i,$$

for $t \in [0,1]$, using the barycentric coordinates in the same simplex.

(2) V_y is a CW-subspace of $|\mathcal{O}|$ which can also be described as the classifying space of a subposet $\mathcal{O}_y \subset \mathcal{O}$, consisting of the $U \in \mathcal{O}$ containing y.

Finally we observe that the category $\mathcal{O}_y$ is a directed poset. Therefore $|\mathcal{O}_y|$ is contractible. $\square$

THEOREM 3.2. *With the identifications of Proposition* 3.1 *and Remark* 2.10, *the inclusion of* $|\mathcal{O}(M)|$ *in* $|\mathcal{B}(n)|$ *is the classifying map for the tangent bundle* τ *of* M^n.

Proof. We cannot prove anything without explaining to some extent why Remark 2.10 holds. Write $\mathcal{B}:=\mathcal{B}(n)$ and let $|\mathcal{B}|_{\text{big}}$ be the geometric realization of the Δ-set [RS1], [RS2] alias incomplete simplicial set whose k-simplices are the functors from the poset $\{0,1,...,k\}$ (with $0\leqslant 1\leqslant ...\leqslant k$) to $\mathcal{B}$. (In other words, in making $|\mathcal{B}|_{\text{big}}$ we choose to ignore degeneracy operators.) It is well known that the identification map $|\mathcal{B}|_{\text{big}}\to|\mathcal{B}|$ is a homotopy equivalence. $|\mathcal{B}|_{\text{big}}$ is the codomain of a map with fibers homeomorphic to $\mathbb{R}^n$, say $V\to|\mathcal{B}|_{\text{big}}$. The description of V is similar to that of W in Proposition 3.1. Namely, V is the coend (diagonal colimit, see [Mac]) of

$$(s,t)\mapsto s(\min)\times \operatorname{star}(t)$$

where the variables s and t are simplices of $|\mathcal{B}|_{\text{big}}$, or equivalently, functors from one of the posets $\{0,...,k\}$ to $\mathcal{B}$; we write $s(\min)$ for the value of s on the minimal element, which is a space homeomorphic to $\mathbb{R}^n$. We project V to the coend of

$$(s,t)\mapsto \operatorname{star}(t)$$

which is $|\mathcal{B}|_{\text{big}}$. The projection is almost locally trivial in the sense of [S2, A.1]. If $f\colon X\to|\mathcal{B}|_{\text{big}}$ is any map where X is a CW-space, then $f^*V\to X$ is another almost locally trivial map with fibers homeomorphic to $\mathbb{R}^n$. It has no preferred section, but if we are willing to replace X by the homotopy equivalent f^*V, then we have

$$f^*V\times_X f^*V\to f^*V,\quad (z_1,z_2)\mapsto z_1, \tag{3.3}$$

which has a preferred section (the diagonal) and qualifies therefore as a microbundle. This is how maps to $|\mathcal{B}|_{\text{big}}$ give rise to microbundles on the domain. It is not a very serious objection that we had to modify the domain in order to see a microbundle on it.

Returning to our business, let f be the inclusion map from $|\mathcal{O}(M)|$ to $|\mathcal{B}|_{\text{big}}$. (We get an automatic factorization through $|\mathcal{B}|_{\text{big}}$ because the nondegenerate simplices in the nerve of $\mathcal{O}(M)$ form a Δ-set.) Then f^*V is homeomorphic to what we previously (in the proof of Proposition 3.1) called W. We now have to find an isomorphism from the microbundle (3.3) on $f^*V=W$ to the pullback of the tangent microbundle τ under the projection $W\to M$. This is easy. $\square$

Inverse Poincaré duality by scanning

Our next order of business is to come up with a description of (inverse) Poincaré duality adapted to Remark 2.12. The method we use is inspired by [Mcl], [Bö], [S3]. The characteristic χ on $F|l\mathcal{E}^{\bullet}$ is not relevant at this stage (but it will be later on, in Theorem 3.11).

Let M^n be a closed manifold. We note that $\mathcal{O}(M)\subset\mathcal{B}(n)\subset l\mathcal{E}^{\bullet}$ and that each U in $\mathcal{O}(M)$ determines a morphism $M\rightsquigarrow U$ in $l\mathcal{E}^{\bullet}$, the formal extension of id_U to a partial map from M to U. The morphisms $M\rightsquigarrow U$ in turn induce maps $F(M)\to F(U)$. These maps for the various U constitute a natural transformation from the constant functor on $\mathcal{O}(M)$ with value $F(M)$ to the functor $F|\mathcal{O}(M)$. The natural transformation induces

$$F(M)\to \operatorname{holim} F|\mathcal{O}(M). \tag{3.4}$$

We have already observed in Remark 2.12 that the homotopy groups of the codomain of (3.4) are the cohomology groups of M with twisted coefficients in $\mathbf{F}(*)$. We will show that (3.4) is a variant of (inverse) Poincaré duality.

To begin, we simplify (3.4). By the classification theorem for pro-excisive functors, we may assume that $F(X)=\Omega^\infty(X^{\bullet}\wedge\mathbf{J})$ for some Ω-CW-spectrum $\mathbf{J}$. Then our map takes the form

$$\Omega^\infty(M^{\bullet}\wedge\mathbf{J})\to \operatorname*{holim}_{U\in\mathcal{O}(M)}\Omega^\infty(U^{\bullet}\wedge\mathbf{J}) \tag{3.5}$$

and is induced by the collapse maps from $M^{\bullet}=M_+$ to the various $U^{\bullet}$. We will compare it with another map for which we give two versions:

$$\Omega^\infty(M^{\bullet}\wedge\mathbf{J})\to s\Gamma(\tilde\tau^{\bullet};\mathbf{J}), \tag{3.6i}$$

$$\Omega^\infty(M^{\bullet}\wedge\mathbf{J})\to s\Gamma(\tau^{\bullet};\mathbf{J}). \tag{3.6ii}$$

Explanations. Here $\tau^{\bullet}$ is the fiberwise one-point compactification of the tangent bundle τ of M; see [Ki]. It is fiber homotopy equivalent to $\tilde\tau^{\bullet}$, the bundle on M with fiber $M_z\times z$ over $z\in M$, where M_z is the reduced mapping cone of the inclusion $M^{\bullet}\setminus z\to M^{\bullet}$, or equivalently, the unreduced mapping cone of the inclusion $M\setminus z\to M$. An explicit chain of fiber homotopy equivalences will be given below.

Let q be any fibration on a space Y with a distinguished 'trivial' section, which we assume is a fiberwise cofibration; compare [Ja]. We write $s\Gamma(q;\mathbf{J})$ for the space of *stable sections of q with coefficient spectrum* $\mathbf{J}$, that is, the space of sections of a fibration on Y with fiber $\Omega^\infty(q^{-1}(y)\wedge\mathbf{J})$ over $y\in Y$.

The map (3.6i) is induced by the inclusions $M^{\bullet}\to M_z$. For (3.6ii), we need an exponential map for τ, which is a continuous family of embeddings $\exp_x\colon\tau_x\to M$ for $x\in M$ such that $x\in\operatorname{im}(\exp_x)$ for all $x\in M$. Then we have collapse maps from $M^{\bullet}$ to $(\operatorname{im}(\exp_x))^{\bullet}\cong\tau_x^{\bullet}$ which lead to (3.6ii).

To compare the two versions of (3.6) we need an explicit chain of fiber homotopy equivalences relating $\tau^{\bullet}$ to $\tilde\tau^{\bullet}$. For this purpose we introduce an intermediate bundle q on M whose fiber over $x\in M$ is the reduced mapping cone of the inclusion $\tau_x^{\bullet}\setminus x_\tau\to\tau_x^{\bullet}$, where $x_\tau=\exp_x^{-1}(x)$. Both τ^k and $\tilde\tau^{\bullet}$ admit fairly obvious fiber homotopy equivalences to the new bundle q, respecting the trivial sections. Now we can compare the versions (3.6i) and (3.6ii) by viewing them as maps from $\Omega^\infty(M^\bullet \wedge \mathbf{J})$ to $s\Gamma(q;\mathbf{J})$. We find that as such they are equal.

To compare (3.5) with (3.6i) we note that compactness of M has not been used in the definition of (3.6i). Thus we have similar maps

$$\Omega^\infty(U^\bullet \wedge \mathbf{J}) \to s\Gamma(\tilde{\tau}(U)^\bullet;\mathbf{J})$$

for a noncompact manifold U with tangent bundle $\tau(U)$. Here it is however important not to confuse $U^\bullet$ with U_+. It is also important to define U_z for $z \in U$ as the reduced mapping cone of $U^\bullet \setminus z \hookrightarrow U^\bullet$, not as the unreduced mapping cone of $U \setminus z \hookrightarrow U$. We take $U \in \mathcal{O}(M)$ and obtain a commutative diagram

$$\begin{tikzcd} \Omega^\infty(M^\bullet \wedge \mathbf{J}) \arrow[r, "(3.5)"] \arrow[d, "(3.6)"] & \operatorname{holim}_U \Omega^\infty(U^\bullet \wedge \mathbf{J}) \arrow[d, "(3.6)"] \\ s\Gamma(\tau^\bullet;\mathbf{J}) \arrow[r] & \operatorname{holim}_U s\Gamma(\tau(U)^\bullet;\mathbf{J}) \end{tikzcd}$$

where the lower vertical arrow is induced by restriction and certain quotient maps $M_z \to U_z$ which are pointed homotopy equivalences.

OBSERVATION 3.7. *The lower horizontal and right-hand vertical arrows in the diagram are homotopy equivalences.*

Proof. For the right-hand vertical arrow, this follows from the homotopy invariance property of homotopy limits, since $\Omega^\infty(U^\bullet \wedge \mathbf{J}) \to s\Gamma(\tilde{\tau}(U)^\bullet;\mathbf{J})$ is a homotopy equivalence for every U in $\mathcal{O}(M)$. The lower horizontal arrow can be rewritten as a composition

$$s\Gamma(\tilde{\tau}^\bullet;\mathbf{J}) \to \operatorname*{holim}_U s\Gamma(\tilde{\tau}^\bullet|U;\mathbf{J}) \to \operatorname*{holim}_U s\Gamma(\tilde{\tau}(U)^\bullet;\mathbf{J})$$

where the second arrow is induced by fiberwise weak homotopy equivalences from $\tilde{\tau}^\bullet|U$ to $\tilde{\tau}(U)^\bullet$, one for each U. The first can be rewritten in the form

$$s\Gamma(\tilde{\tau}^\bullet;\mathbf{J}) \to s\Gamma(p^*\tilde{\tau}^\bullet;\mathbf{J})$$

where $p\colon \operatorname{hocolim}_U U \to M$ is the obvious map. (Here $\operatorname{hocolim}_U U$ is the homotopy colimit of the functor on $\mathcal{O}(U)^{\mathrm{op}}$ which to each object U in $\mathcal{O}(M)^{\mathrm{op}}$ associates the space U itself.) Thus we need to know that p is a homotopy equivalence. This follows from Proposition 3.1, since p factors through W in Proposition 3.1 and the factorizing map $\operatorname{hocolim}_U U \to W$ is a homotopy equivalence. $\square$

We conclude that (3.5) and (3.6) can be identified by means of the diagram preceding Observation 3.7. It remains to understand (3.6). For this purpose we need "standard" Poincaré duality, a weak homotopy equivalence

$$\wp\colon s\Gamma(\tau^\bullet;\mathbf{J}) \to \Omega^\infty(M^\bullet \wedge \mathbf{J}) \tag{3.8}$$

which is commonly defined as the composition of a 'Thom' weak homotopy equivalence (inducing the Thom isomorphism on homotopy groups) with an Atiyah–Milnor–Spanier–Whitehead duality map.

Details. Let $\nu=\nu(M)$ be a normal bundle of an embedding of M in $\mathbb{R}^w$, with Thom space $\mathrm{th}(\nu)$, collapse map $c\colon \mathbb{S}^w \to \mathrm{th}(\nu)$ and 'diagonal' $d\colon \mathrm{th}(\nu)\to M^\bullet\wedge\mathrm{th}(\nu)$. The Thom weak homotopy equivalence

$$s\Gamma(\tau^\bullet;\mathbf{J}) \to \mathrm{map}(\mathrm{th}(\nu)\wedge\mathbf{S}^0, \mathbb{S}^w\wedge\mathbf{J})$$

associates to $t\in s\Gamma(\tau^\bullet;\mathbf{J})$ the map whose restriction to $\nu_x^\bullet\wedge\mathbf{S}^0$ for $x\in M$ is $\mathrm{id}\wedge t(x)$ from $\nu_x^\bullet\wedge\mathbf{S}^0$ to $\nu_x^\bullet\wedge\tau_x^\bullet\wedge\mathbf{J}\cong\mathbb{S}^w\wedge\mathbf{J}$. The duality map

$$\mathrm{map}(\mathrm{th}(\nu)\wedge\mathbf{S}^0, \mathbb{S}^w\wedge\mathbf{J}) \to \Omega^\infty(M^\bullet\wedge\mathbf{J})$$

associates to $\mathbf{g}\colon \mathrm{th}(\nu)\wedge\mathbf{S}^0\to\mathbb{S}^w\wedge\mathbf{J}$ the composition

$$\begin{tikzcd}
\mathbb{S}^w\wedge\mathbf{S}^0 \arrow[r, "dc\wedge\mathbf{id}"] & (M^\bullet\wedge\mathrm{th}(\nu))\wedge\mathbf{S}^0 \arrow[d, "\cong"] & \\
& M^\bullet\wedge(\mathrm{th}(\nu)\wedge\mathbf{S}^0) \arrow[r, "\mathrm{id}\wedge\mathbf{g}"] & M^\bullet\wedge(\mathbb{S}^w\wedge\mathbf{J}).
\end{tikzcd}$$

Strictly speaking, that composition is an element of $\Omega^w\Omega^\infty(M_+\wedge\mathbb{S}^w\wedge\mathbf{J})$. However, we have an inclusion $\Omega^\infty(M^\bullet\wedge\mathbf{J})\to\Omega^w\Omega^\infty(M^\bullet\wedge\mathbb{S}^w\wedge\mathbf{J})$ which is a homotopy equivalence.

Definition 3.9. We define a map called *reflection*, $t\mapsto\bar{t}$, from $s\Gamma(\tau^\bullet;\mathbf{J})$ to $s\Gamma(\tau^\bullet;\mathbf{J})$. It is the map induced by the sphere bundle automorphism

$$\tau^\bullet\wedge_M\nu^\bullet\wedge_M\tau^\bullet \to \tau^\bullet\wedge_M\nu^\bullet\wedge_M\tau^\bullet,$$

which interchanges the first and third factors. We regard it here as a stable fiberwise automorphism of $\tau^\bullet$ via identifications $\tau_x^\bullet\wedge(\nu_x^\bullet\wedge\tau_x^\bullet)\cong\tau_x^\bullet\wedge\mathbb{S}^w$. (Strictly speaking, reflection is an involution on $\Omega^w s\Gamma(\tau^\bullet;\mathbb{S}^w\wedge\mathbf{J})$.)

Remark. If τ is a vector bundle, the involution $t\mapsto\bar{t}$ has a simpler description. Up to homotopy it is the map induced by the automorphism of τ which is scalar multiplication by -1 on each fiber. For the proof, let μ_s for nonzero $s\in\mathbb{C}$ be the real vector bundle automorphism of $\tau\oplus\tau\cong\tau\otimes_{\mathbb{R}}\mathbb{C}$ given by complex multiplication with s. Then conjugation with μ_s for s on the unit circle, between 1 and $e^{i\pi/4}$, deforms the automorphism of $\tau\oplus\tau$ given by permutation of the summands into that given by $(x,y)\mapsto(-x,y)$.

PROPOSITION–DEFINITION–SUMMARY 3.10. *Up to a canonical vertical homotopy, the composition* (3.6)$\cdot$(3.8) *is given by reflection,* $t \mapsto \bar{t}$. *Hence* (3.6) *is a weak homotopy inverse for* $\bar{\wp}$, *the composition of* $\wp$ *in* (3.8) *with reflection* $t \mapsto \bar{t}$. *Therefore the maps* (3.6) *and* (3.4) *can be labelled*

$$\bar{\wp}^{-1}\colon \Omega^\infty(M_+ \wedge \mathbf{J}) \to s\Gamma(\tilde{\tau}^\bullet; \mathbf{J}),$$
$$\bar{\wp}^{-1}\colon F(M) \to \operatorname{holim} F\,|\,\mathcal{O}(M).$$

Proof. The second sentence follows from the first because $\wp$ is a weak homotopy equivalence. (A pointed map $f\colon X \to Y$ is *weakly homotopy inverse* to another, $g\colon Y \to X$, if the singular simplicial set functor turns f and g into reciprocal homotopy inverses.) The third follows from the second; we use the 'identification' of (3.6) with (3.5) alias (3.4) resulting from Observation 3.7.

We come to the proof of the first sentence. In the construction of (3.8), compactness of M was not really used; so if U is a noncompact manifold, embedded in $\mathbb{R}^w$ with a specified normal bundle $\nu(U)$, we obtain a similar map

$$\wp\colon s\Gamma(\tau(U)^\bullet; \mathbf{J}) \to \Omega^w\Omega^\infty(U^\bullet \wedge \mathbb{S}^w \wedge \mathbf{J}).$$

(There is no obvious collapse map c from $\mathbb{S}^w$ to $\operatorname{th}(\nu(U))$, but there is still an obvious map $\mathbb{S}^w \to U^\bullet \wedge \operatorname{th}(\nu(U))$ replacing what was called dc earlier.) Moreover, the construction is natural; that is, for open $U \subset M$, we get a commutative square

$$\begin{tikzcd}
s\Gamma(\tau^\bullet; \mathbf{J}) \arrow[r, "\wp"] \arrow[d, "\text{restriction}"] & \Omega^\infty(M^\bullet \wedge \mathbf{J}) \arrow[d, "\text{collapse}"] \\
s\Gamma(\tau(U)^\bullet; \mathbf{J}) \arrow[r, "\wp"] & \Omega^\infty(U^\bullet \wedge \mathbf{J}).
\end{tikzcd}$$

(To achieve strict commutativity, choose $\tau(U)$, $\nu(U)$ and the trivialization of $\tau(U)^\bullet \wedge \nu(U)^\bullet$ by restricting τ, ν and the trivialization of $\tau^\bullet \wedge \nu^\bullet$ on M.) We apply this insight with $U = \operatorname{im}(\exp_x) \cong \tau_x$, for variable $x \in M$. The result is a commutative square

$$\begin{tikzcd}
s\Gamma(\tau^\bullet; \mathbf{J}) \arrow[r] \arrow[d, "(3.8)"] & s\Gamma(\exp^*(\tau^\bullet); \mathbf{J}) \arrow[d, "(3.8)"] \\
\Omega^w\Omega^\infty(M^\bullet \wedge \mathbb{S}^w \wedge \mathbf{J}) \arrow[r, "(3.6)"] & \Omega^w s\Gamma(\tau^\bullet; \mathbb{S}^w \wedge \mathbf{J})
\end{tikzcd}$$

where the upper horizontal arrow is induced by the exponential map, from the total space of τ to M. The right-hand vertical arrow is induced by maps of type (3.8),

$$s\Gamma(\tau(\tau_x)^\bullet; \mathbf{J}) \to \Omega^w\Omega^\infty(\tau_x^\bullet; \mathbb{S}^w \wedge \mathbf{J})$$

for $x \in M$. We use the identification $\tau(\tau_x) \cong \exp_x^*(\tau)$. (Thus we view the upper right-hand term in the square as the space of sections of a bundle on M whose fiber over $x \in M$ is $s\Gamma(\tau(\tau_x)^\bullet; \mathbf{J})$.)

Going through the definition of the Milnor duality map $\wp$ once again (but now with $\tau_x \cong \mathrm{im}(\exp_x)$ in place of M), one finds that the composite map in the above square, from the upper left-hand term via the upper right-hand term to the lower right-hand term, is up to a canonical vertical homotopy the map induced by the compositions

$$\begin{tikzcd}
 & \Omega^\infty(\tau_x^\bullet \wedge \mathbf{J}) \arrow[r, "="] & \mathrm{map}(\mathbf{S}^0, \tau_x^\bullet \wedge \mathbf{J}) \arrow[d, "\text{stabilization}"] \\
 & \mathrm{map}(\nu_x^\bullet \wedge \mathbf{S}^0, \mathbb{S}^w \wedge \mathbf{J}) \arrow[d, "\text{stabilization}"] & \mathrm{map}(\nu_x^\bullet \wedge \mathbf{S}^0, \nu_x^\bullet \wedge \tau_x^\bullet \wedge \mathbf{J}) \arrow[l, "\cong"'] \\
 & \mathrm{map}(\tau_x^\bullet \wedge \nu_x^\bullet \wedge \mathbf{S}^0, \tau_x^\bullet \wedge \mathbb{S}^w \wedge \mathbf{J}) \arrow[r, "\cong"] & \Omega^w \Omega^\infty(\tau_x^\bullet \wedge \mathbb{S}^w \wedge \mathbf{J}),
\end{tikzcd}$$

one such for each $x \in M$. Hence, up to that canonical vertical homotopy, it is the map obtained by composing the stabilization $s\Gamma(\tau^\bullet; \mathbf{J}) \to \Omega^w s\Gamma(\tau^\bullet; \mathbb{S}^w \wedge \mathbf{J})$ with reflection on $\Omega^w s\Gamma(\tau^\bullet; \mathbb{S}^w \wedge \mathbf{J})$. □

The index theorem for a single closed manifold

Returning to (3.4), we ask how the image of $\chi(M) \in F(M)$ under

$$\wp^{-1}: F(M) \to \mathrm{holim}\, F \,|\, \mathcal{O}(M)$$

is related to the image of $\bar{e}_n \in \mathrm{holim}\, F \,|\, \mathcal{B}(n)$ under the restriction map

$$\mathrm{holim}\, F \,|\, \mathcal{B}(n) \to \mathrm{holim}\, F \,|\, \mathcal{O}(M).$$

THEOREM 3.11 (the index theorem for a single closed manifold). *The two image points are connected by a canonical path $\omega(M)$ in* $\mathrm{holim}(F \,|\, \mathcal{O}(M))$.

Proof. Enlarge $\mathcal{O} = \mathcal{O}(M)$ by adding M as an initial object. We can still write $(\mathcal{O} \cup \{M\}) \subset l\mathcal{E}^\bullet$, by identifying the unique morphism in $\mathcal{O} \cup \{M\}$ from M to an arbitrary U in $\mathcal{O}$ with the canonical morphism $M \rightsquigarrow U$ in $l\mathcal{E}^\bullet$ (formal extension of id_U to a partial map from M to U). So the characteristic χ gives us an element z in the homotopy limit of $F \,|\, (\mathcal{O} \cup \{M\})$, which is the same as the homotopy limit of the diagram

$$F(M) \xrightarrow{\wp^{-1}} \mathrm{holim}\, F \,|\, \mathcal{O}.$$

Therefore z consists of a point z_0 in $F(M)$ and a path ω in $\mathrm{holim}\, F \,|\, \mathcal{O}$ starting at $\wp^{-1}(z_0)$. Inspection shows that $z_0 = \chi(M)$ and that the endpoint of ω is the image of $\bar{e}_n$ in $\mathrm{holim}\, F \,|\, \mathcal{O}$. □

The case of a manifold with boundary

Remark 3.12. We indicate how Theorem 3.2 can be generalized to manifolds with boundary. Let M^n be a manifold with boundary ∂M and interior M_0, where $n>0$. Let $\mathcal{O}(M)$ be the poset whose objects are the open subsets of M homeomorphic to either $\mathbb{R}^n$ or $]-\infty,\infty]\times\mathbb{R}^{n-1}$. The partial order on $\mathcal{O}(M)$ is reverse inclusion as usual. Let $\partial\mathcal{O}(M)\subset\mathcal{O}(M)$ consist of the open subsets of M which are homeomorphic to $]-\infty,\infty]\times\mathbb{R}^{n-1}$.

The *delete interior* functor from $\partial\mathcal{O}(M)$ to $\mathcal{O}(\partial M)$ induces a homotopy equivalence of the classifying spaces. (Namely, by Quillen's Theorem A [Ql] it is enough to show that for every $U\in\mathcal{O}(\partial M)$ the poset of $U'\in\partial\mathcal{O}(M)$ with $U'\supset U$ has a contractible classifying space. This is clearly a directed poset.) Also, the inclusion $\mathcal{O}(M_0)\to\mathcal{O}(M)$ induces a homotopy equivalence of the classifying spaces, since it has a left adjoint. It follows that the pair $(|\mathcal{O}(M)|,|\partial\mathcal{O}(M)|)$ is homotopy equivalent to $(M,\partial M)$. The following inclusion can be thought of as the classifying map for the tangent bundle pair $(\tau(M),\tau(\partial M))$:

$$(|\mathcal{O}(M)|,|\partial\mathcal{O}(M)|)\to(|\mathcal{GB}(n)|,|\partial\mathcal{GB}(n)|). \tag{3.13}$$

Remark 3.14. We indicate how (3.6) and Theorem 3.11 can be generalized to compact manifolds M with boundary. The analog of (3.5) has the form

$$\bar{\wp}^{-1}\colon F(M)\to\operatorname{holim}(F|\mathcal{O}(M)). \tag{3.15}$$

It is a weak homotopy equivalence and can be analyzed as in Proposition 3.10. Note that $F|\mathcal{O}(M)$ takes objects in $\partial\mathcal{O}(M)\subset\mathcal{O}(M)$ to contractible spaces; therefore the codomain of (3.15) can be interpreted as *relative cohomology* of the pair $(M,\partial M)$. Compare Remark 2.15. The domain of (3.15) should be interpreted as absolute homology of M.

The analog of the Index Theorem 3.11 states that the images of $\chi(M)\in F(M)$ and $\bar{e}_n\in\operatorname{holim}(F|\mathcal{GB}(n))$ in $\operatorname{holim}(F|\mathcal{O}(M))$ are connected by a canonical path $\omega(M)$.

The index theorem for families

Let M^n be a compact manifold. As we have just observed, stating and proving the index theorem for M amounts to specifying a point $\mathcal{I}_M=(\bar{e}_n,\omega(M),\chi(M))$ in the homotopy pullback of the diagram

$$\operatorname{holim}(F|\mathcal{GB}(n))\xrightarrow{\text{restriction}}\operatorname{holim}(F|\mathcal{O}(M))\xleftarrow{\bar{\wp}^{-1}}F(M). \tag{3.16}$$

To make a family index theorem, we must understand how $\mathcal{I}_M$ depends on M as a variable. We therefore return to the assumptions and notation of §2.8. In particular, $p\colon E\to B$ is a bundle with compact manifold fibers (fiber dimension n).

PROPOSITION 3.17. *The restriction map in* (3.16) *has a refinement, or parametrized version,*

$$\operatorname{holim}(F\,|\,\mathcal{GB}(n)) \to \operatorname*{holim}_{(x,\theta)} \operatorname{holim}(F\,|\,\mathcal{O}(E_x^\theta)).$$

Proof. Denote (somewhat informally) by $\mathcal{O}(p)$ the category whose objects are triples (x,θ,U) with (x,θ) in $\operatorname{simp}(t\mathfrak{B})$ and $U\in\mathcal{O}(E_x^\theta)$. A morphism from (x,θ,U) to (y,σ,V) is a morphism g from (x,θ) to (y,σ) in $\operatorname{simp}(t\mathfrak{B})$ such that the image of U under the induced homeomorphism $g_*\colon E_x^\theta \to E_y^\sigma$ contains V. Let $\varphi_1\colon \mathcal{O}(p)\to\mathcal{GB}(n)$ be the forgetful functor taking (x,θ,U) to U. The right-hand side in Proposition 3.17 simplifies to $\operatorname{holim}(F\,|\,\mathcal{GB}(n))\cdot\varphi_1$. This makes the map from $\operatorname{holim}(F\,|\,\mathcal{GB}(n))$ to it obvious. □

Again we ask how the image of $\chi(p)\in\operatorname{holim}_{(x,\theta)} F(E_x^\theta)$, compare (2.9), under the fiberwise version of $\bar{\wp}^{-1}$ in (3.4),

$$\operatorname*{holim}_{(x,\theta)} F(E_x^\theta) \xrightarrow{\simeq} \operatorname*{holim}_{(x,\theta)} \operatorname{holim}(F\,|\,\mathcal{O}(E_x^\theta)),$$

is related to the image of $\bar{e}_n$ under the map in Proposition 3.17,

$$\operatorname{holim}(F\,|\,\mathcal{GB}(n)) \to \operatorname*{holim}_{(x,\theta)} \operatorname{holim}(F\,|\,\mathcal{O}(E_x^\theta)).$$

THEOREM 3.18 (the family index theorem). *The two image points are related by a canonical path $\omega(p)$ in* $\operatorname{holim}_{(x,\theta)} \operatorname{holim}(F\,|\,\mathcal{O}(E_x^\theta))$.

Proof. We use $\mathcal{O}(p)$ from the proof of Proposition 3.17. Denote (somewhat informally) by $\mathcal{O}(p)\cup\operatorname{simp}(t\mathfrak{B})$ the category obtained from $\mathcal{O}(p)\amalg\operatorname{simp}(t\mathfrak{B})$ by adjoining, for each (x,θ) in $\operatorname{simp}(t\mathfrak{B})$ and (y,σ,V) in $\mathcal{O}(p)$, a set of morphisms from (x,θ) to (y,σ,V) which is equal to the set of morphisms from (x,θ) to (y,σ) in $\operatorname{simp}(t\mathfrak{B})$. Let $\varphi_2\colon \mathcal{O}(p)\cup\operatorname{simp}(t\mathfrak{B})\to l\mathcal{E}^\bullet$ be the functor given by

$$\begin{aligned}(y,\sigma,V)&\mapsto V,\\ (x,\theta)&\mapsto E_x^\theta\end{aligned}$$

on objects; the morphism $\varphi_2(f)$ induced by a morphism f in $\mathcal{O}(p)\cup\operatorname{simp}(t\mathfrak{B})$ is an appropriate collapse map (a formal extension of a homeomorphism to a partial proper map). The path $\omega(p)$ is the image of $\chi\in\operatorname{holim} F$ under the evident map from $\operatorname{holim} F$ to $\operatorname{holim} F\varphi_2$.

As in the proof of Theorem 3.11, this involves a reinterpretation of $\operatorname{holim} F\varphi_2$. Think of $\operatorname{holim} F\varphi_2$ as the space of paths in $\operatorname{holim}(F\cdot(\varphi_2|\mathcal{O}(p)))=\operatorname{holim} F\varphi_1$ together with a lift of the starting point to $\operatorname{holim}(F\cdot(\varphi_2|\operatorname{simp}(t\mathfrak{B}))$. □

Remark 3.19. We emphasize that Theorem 3.18 is the rigorous form of the family index theorem. A more portable (but less precise) form of Theorem 3.18 is $\bar{\wp}(\bar{e}_n(\tau)) \simeq \chi(p)$ or $\wp(e_n(\tau)) \simeq \chi(p)$, where $\bar{e}_n(\tau)$ and $e_n(\tau)$ have the following meaning.

(1) Formally, $\bar{e}_n(\tau)$ is the image of $\bar{e}_n$ under the map in Proposition 3.17.

(2) Informally, $\bar{e}_n(\tau)$ is the pullback of a certain section $\bar{e}_n$ under the classifying map $(E, \partial E) \to (B\,\mathrm{TOP}(n), B\,\mathrm{TOP}(n-1)^{\urcorner})$ for the vertical tangent bundle τ of p. Furthermore (and still informally) $\bar{e}_n$ is a section of a certain fibration on $B\,\mathrm{TOP}(n)$ whose fibers are infinite loop spaces homotopy equivalent to $F(\mathbb{R}^n)$. It vanishes on $B\,\mathrm{TOP}(n-1)^{\urcorner})$ and so *represents* a class in the nth cohomology of $(B\,\mathrm{TOP}(n), B\,\mathrm{TOP}(n-1)^{\urcorner})$ with twisted coefficients in the spectrum $\mathbf{F}(*)$. Hence $\bar{e}_n(\tau)$ is a section of a fibration on E with fibers $\simeq F(\mathbb{R}^n)$. It vanishes on ∂E and so *represents* a class in the cohomology of $(E, \partial E)$ with twisted coefficients in the spectrum $\mathbf{F}(*)$.

(3) Informally, $e_n(\tau)$ is the image of $\bar{e}_n(\tau)$ under reflection, Definition 3.9, with the informal definition of $\bar{e}_n(\tau)$. We do not give a formal definition of $e_n(\tau)$.

4. Disk bundles and the unit transformation

The unit transformation

Definition 4.1. We make a category of pairs (F, z) where F is a pro-excisive functor from $\mathcal{E}^{\bullet}$ to pointed spaces, see §2.1 and Remark 2.2, and z is a point in $F(*)$. A *morphism* from (F, z) to (F', z') is a natural transformation $F \to F'$ taking z to z'. Such a morphism is a *weak equivalence* if $F(X) \to F'(X)$ is a weak homotopy equivalence for all X. We say that two objects (F, z) and (F', z') are *weakly equivalent* if they can be related by a chain of weak equivalences.

LEMMA 4.2. *For every (F, z), there exists a morphism $\eta\colon (C, y) \to (F, z)$ where (C, y) is weakly equivalent to the pair*

$$(X \mapsto Q(X^{\bullet}), 1).$$

Proof. By the classification theorem for excisive functors, mentioned in Remark 2.2, there exists a diagram of natural transformations

$$F(X) \xrightarrow{g} F'(X) \xleftarrow{h} F''(X) \tag{4.3}$$

where $F''(X) = \Omega^{\infty}(X^{\bullet} \wedge \mathbf{F}(*))$ and g, h are weak homotopy equivalences for every X. (The classification theorem speaks of a *chain* of natural transformations, but it is easy to reduce to a chain of length 2.) Specializing (4.3) to the case $X = *$, choose a point $\bar{z}$ in the homotopy fiber of h over $g(z)$. An argument involving homotopy pullbacks shows that it is enough to prove Lemma 4.2 for the excisive functor F'' and the point $z''\in F''(*)$ which is the image of $\bar{z}$ under the projection. However, that case is easy because z'' determines a map of spectra $\mathbf{S}^0\to\mathbf{F}(*)$ and then a natural transformation from $Q(X^{\bullet})=\Omega^{\infty}(X^{\bullet}\wedge\mathbf{S}^0)$ to $F''(X)$. This takes the unit element in $Q\mathbb{S}^0=Q(*^{\bullet})$ to z''. $\square$

Remark. The existence statement, Lemma 4.2, implies a uniqueness statement, as follows. Suppose that $(C_1,y_1)\to(F,z)$ and $(C_2,y_2)\to(F,z)$ are two morphisms as in Lemma 4.2. Assume or arrange that $C_1(X)\to F(X)$ and $C_2(X)\to F(X)$ are Serre fibrations for every X. Make another object (P,x) where $P(X)$ is the pullback of

$$C_1(X)\to F(X)\leftarrow C_2(X)$$

and $x\in P(*)$ is the point determined by y_1 and y_2. Finally choose a morphism $(C_3,y_3)\to(P,x)$ as in Lemma 4.2. Then the following compositions are weak equivalences over (F,z):

$$(C_3,y_3)\to(P,x)\to(C_1,y_1),$$
$$(C_3,y_3)\to(P,x)\to(C_2,y_2).$$

Definition 4.4. If F is any pro-excisive functor from $\mathcal{E}^{\bullet}$ to pointed spaces and χ is a characteristic for $F|l\mathcal{E}^{\bullet}$, then we can take $z=\chi(*)\in F(*)$ in Lemma 4.2. The resulting natural transformation $\eta\colon C\to F$ constructed as in Lemma 4.2 is the *unit transformation* for F and χ. We shall also use informal notation such as $\eta\colon Q(X^{\bullet})\to F(X)$ or even $\eta\colon Q^{\bullet}(X)\to F(X)$. Further, we write $\boldsymbol{\eta}\colon\mathbf{S}^0\to\mathbf{F}(*)$ for the map of spectra corresponding to $\eta\colon Q(X^{\bullet})\to F(X)$ via the classification theorem; see Remark 2.2.

The Becker–Euler classes

Definition 4.5. Let $\mathrm{G}(n)$ be the simplicial monoid of homotopy equivalences $\mathbb{S}^{n-1}\to\mathbb{S}^{n-1}$. The classifying space $B\mathrm{G}(n)$ carries a tautological spherical quasifibration $\zeta(n)$ with fibers $\simeq\mathbb{S}^{n-1}$. We include $\mathrm{G}(n-1)$ in $\mathrm{G}(n)$ by

$$(f\colon\mathbb{S}^{n-2}\to\mathbb{S}^{n-2})\mapsto(1*f\colon\mathbb{S}^0*\mathbb{S}^{n-2}\to\mathbb{S}^0*\mathbb{S}^{n-2})$$

using an identification of the join $\mathbb{S}^0*\mathbb{S}^{n-2}$ with $\mathbb{S}^{n-1}$.

Definition 4.6. Let θ be a spherical quasifibration on a space X and let $\mathbf{J}$ be any spectrum. For $x\in X$, let θ_x be the fiber of θ over x. Let ε be the trivial fibration $\mathbb{S}^0\times X\to X$ and let $\varepsilon\oplus\theta$ be the fiberwise join of ε and θ. We denote by $\Gamma^{\sim}(\varepsilon\oplus\theta;\mathbf{J})$

the space of sections of the fibration associated with the quasifibration on X with fiber $\Omega^\infty((\varepsilon\oplus\theta)_x\wedge\mathbf{J})$ over x. Here the base point of $\mathbb{S}^0\cong\varepsilon_x$ serves as base point for $(\varepsilon\oplus\theta)_x$, for each $x\in X$. Let Y be a subspace of X. We write

$$H^\theta(X;\mathbf{J}),$$
$$H^\theta(X,Y;\mathbf{J})$$

for π_0 of $\Gamma^{\tilde{}}(\varepsilon\oplus\theta;\mathbf{J})$ and π_0 of the subspace consisting of the sections vanishing on Y, respectively. (A section *vanishes* on Y if its restriction to Y is the zero-section.)

Suppose now that X has the homotopy type of a compact CW-space. Choose a spherical quasifibration ξ on X such that the fiberwise join $\theta\oplus\xi$ admits a trivialization, i.e., a fiber homotopy equivalence $\theta\oplus\xi\to\varepsilon^w$ where ε^w is the trivial fibration $\mathbb{S}^{w-1}\times X\to X$. We can use this to define an involution $s\mapsto\bar{s}$ called *reflection* (compare Definition 3.9) on $\Gamma^{\tilde{}}(\varepsilon\oplus\theta;\mathbf{J})$, strictly speaking on the homotopy equivalent space $\Omega^w\Gamma^{\tilde{}}(\varepsilon\oplus\theta\oplus\theta\oplus\xi;\mathbf{J})$. This is induced by the automorphism of $\theta\oplus\theta$ which permutes the factors. Reflection induces involutions on $H^\theta(X;\mathbf{J})$ and $H^\theta(X,Y;\mathbf{J})$.

Definition 4.7. Following [Be], we now describe characteristic classes for spherical quasifibrations: the Becker–Euler classes

$$[b_n],[\bar{b}_n]\in H^{\zeta(n)}(B\mathrm{G}(n),B\mathrm{G}(n-1);\mathbf{S}^0). \tag{4.8}$$

Becker designed $[b_n]$ to have the following property. Let (X,Y) be a CW-pair of relative dimension $\leqslant 2n-4$. A map of pairs $f\colon(X,Y)\to(B\mathrm{G}(n),B\mathrm{G}(n-1))$ factors up to homotopy through $(B\mathrm{G}(n-1)^\urcorner,B\mathrm{G}(n-1))$ if and only if $f^*[b_n]=0$ in $H^{\zeta(n)}(X,Y;\mathbf{S}^0)$. We will see that $[\bar{b}_n]$ has exactly the same property.

We begin by making a section of ε, the trivial bundle $\mathbb{S}^0\times B\mathrm{G}(n)\to B\mathrm{G}(n)$: the section picking the nonbase point of $\mathbb{S}^0$ in each fiber. Then we regard it as a section s of $\varepsilon\oplus\zeta(n)$. The section s is vertically nullhomotopic over $B\mathrm{G}(n-1)$. Namely, $\varepsilon\oplus\zeta(n)\,|\,B\mathrm{G}(n-1)$ is canonically identified with $\varepsilon\oplus\varepsilon\oplus\zeta(n-1)$, and s is already vertically nullhomotopic when viewed as a section of the subquasifibration $\varepsilon\oplus\varepsilon\subset\varepsilon\oplus\varepsilon\oplus\zeta(n-1)$. However, $s\,|\,B\mathrm{G}(n-1)$ has *two* canonical vertical nullhomotopies, h and $\bar{h}$, as a section of $\varepsilon\oplus\varepsilon$. We may identify $\varepsilon\oplus\varepsilon$ with a trivial bundle with fibers $\cong\mathbb{S}^1\subset\mathbb{C}$; then s picks out -1 in each fiber and h uses the canonical contraction of $\mathbb{S}^1(+)$, the intersection of $\mathbb{S}^1$ with the upper half-plane, while $\bar{h}$ uses the canonical contraction of $\mathbb{S}^1(-)$, the intersection of $\mathbb{S}^1$ with the lower half-plane. These two vertical nullhomotopies h and $\bar{h}$ of $s\,|\,B\mathrm{G}(n-1)$ allow us, by dint of a vertical homotopy extension property, to deform s into sections trivial on $B\mathrm{G}(n-1)$ of the fibration on $B\mathrm{G}(n)$ associated with $\varepsilon\oplus\zeta(n)$. We view these sections as sections b_n and $\bar{b}_n$ of a larger fibration on $B\mathrm{G}(n)$, namely, the one associated with the quasifibration on $B\mathrm{G}(n)$ obtained from $\varepsilon\oplus\zeta(n)$ by applying $Q=\Omega^\infty\Sigma^\infty$ to each fiber of $\varepsilon\oplus\zeta(n)$. The relative vertical homotopy classes are denoted $[b_n]$ and $[\bar{b}_n]$, respectively.

We need to understand the homotopy theoretic relationship between b_n and $\bar{b}_n$. As the notation suggests, they are related by *reflection*, see Definition 4.6. However, this is only meaningful when we restrict to a compact CW-subpair (X,Y) of the CW-pair $(B\mathrm{G}(n), B\mathrm{G}(n-1))$. To show that $\bar{b}_n|X$ is indeed (up to a specified vertical homotopy) the reflection of $b_n|X$, recall how b_n and $\bar{b}_n$ were obtained from the pairs (s,h) and $(s,\bar{h})$, respectively, by an application of the vertical homotopy extension property. Thus it suffices to show that reflection takes $(s|X, h|Y)$ to $(s|X, \bar{h}|Y)$, perhaps up to a specified deformation. Reflection does leave $s|X$ invariant, on the nose. The reflection of $h|Y$ can be deformed relative to endpoints into $\bar{h}|Y$ by multiplying with e^{it}, where $0\leqslant t\leqslant\pi$; here we are using an evident fiber-preserving action of the unit circle in $\mathbb{C}$ on $\varepsilon\oplus\varepsilon|Y$, hence on $\varepsilon\oplus\zeta(n-1)\oplus\varepsilon\oplus\zeta(n-1)|Y$, which is identified with $\zeta(n)\oplus\zeta(n)|Y$. (The fibers of $\varepsilon\oplus\varepsilon|Y$ are circles, since the fibers of ε are 0-spheres and $\oplus$ stands for fiberwise join in the present context.)

In the next subsection we will also think of $[\bar{b}_n]$ as an element of the cohomology group $H^{\gamma(n)}(B\mathrm{TOP}(n), B\mathrm{TOP}(n-1)^\urcorner; \mathbf{S}^0)$. We can justify this for example by invoking a certain map of pairs from $((B\mathrm{TOP}(n), B\mathrm{TOP}(n-1)^\urcorner)$ to $(B\mathrm{G}(n), B\mathrm{G}(n-1))$. Alternatively, we could redefine $\bar{b}_n$ as a certain section of the bundle $\gamma(n)^\bullet$ on $B\mathrm{TOP}(n)$, vanishing on $B\mathrm{TOP}(n-1)^\urcorner$. The details are left to the reader.

Characteristic classes associated with disk bundles

Notation. For a manifold V, let $\mathrm{TOP}(V,\partial V)$ be the topological group of homeomorphisms $V\to V$; note that such a homeomorphism will automatically map ∂V to ∂V. Let $\mathrm{TOP}(V)\subset\mathrm{TOP}(V,\partial V)$ be the subgroup consisting of the homeomorphisms $V\to V$ which restrict to the identity on ∂V.

Let $W=]-\infty,\infty]\times\mathbb{R}^{n-1}$. Then $W^\bullet$ is an n-disk. There is a commutative diagram of topological groups and homomorphisms:

$$\begin{tikzcd}
\mathrm{TOP}(W,\partial W) \arrow[r, "\subset"] \arrow[d, "="] & \mathrm{TOP}(W^\bullet, \partial(W^\bullet)) \arrow[d, "\text{restriction}"] \\
\mathrm{TOP}(W,\partial W) \arrow[r, "\text{restriction}"] & \mathrm{TOP}(W\setminus\partial W).
\end{tikzcd}$$

It leads to a map ι of pairs:

$$\begin{array}{c}(B\,\mathrm{TOP}(W^{\bullet},\partial(W^{\bullet})),B\,\mathrm{TOP}(W,\partial W))\\ \downarrow{\scriptstyle\iota}\\ (B\,\mathrm{TOP}(W\backslash\partial W),B\,\mathrm{TOP}(W,\partial W))\\ \|\\ (B\,\mathrm{TOP}(n),B\,\mathrm{TOP}(n-1)^{\urcorner}).\end{array}\tag{4.9}$$

THEOREM 4.10. *The equation* $\iota^*[\bar{e}_n]=\boldsymbol{\eta}_*\iota^*[\bar{b}_n]$ *holds in*

$$H^{\gamma(n)}(B\,\mathrm{TOP}(W^{\bullet},\partial(W^{\bullet})),B\,\mathrm{TOP}(W,\partial W);\mathbf{F}(*)).$$

Informally, Theorem 4.10 says that $[\bar{e}_n]$ applied to a euclidean n-bundle ξ equals $\boldsymbol{\eta}_*[\bar{b}_n]$ applied to ξ, provided ξ has been obtained from an n-disk bundle by deleting the boundary sphere bundle. In particular, $[\bar{e}_n]$ applied to an n-dimensional vector bundle ξ is $\boldsymbol{\eta}_*[\bar{b}_n]$ of ξ. (Note however that the formal statement is not only more formal but also more relative and therefore stronger.)

Outline of proof. Let V be a space homeomorphic to $\mathbb{R}^n$. We need to know that any embedding of V into a manifold D homeomorphic to the n-disk determines a "lift" of $\chi(V)\in F(V)$ to $Q^{\bullet}(V)$, across the unit map $\eta\colon Q^{\bullet}(V)\to F(V)$. By a "lift" we mean a point in the homotopy fiber of $\eta\colon Q^{\bullet}(V)\to F(V)$ projecting to $\chi(V)\in F(V)$. The embedding $V\to D$ determines a wrong-way morphism $D\rightsquigarrow V$ in $l\mathcal{E}^{\bullet}$ (the inverse of that embedding, formally extended to a partial map from D to V). The naturality property of χ applied to this morphism $D\rightsquigarrow V$ shows that it is enough to specify a lift of $\chi(D)\in F(D)$ to $Q^{\bullet}(D)$. Now the unique map $D\to *$ is proper and cell-like! It is therefore a morphism in $\mathcal{E}\cap l\mathcal{E}^{\bullet}\subset\mathcal{E}^{\bullet}$. The naturality property of χ (which is a characteristic for $F|l\mathcal{E}^{\bullet}$ only, not for F) applied to that morphism, which induces homotopy equivalences $Q^{\bullet}(D)\to Q^{\bullet}(*)$ and $F(D)\to F(*)$, shows that it is enough to specify a lift of $\chi(*)\in F(*)$ to $Q^{\bullet}(*)$, across $\eta\colon Q^{\bullet}(*)\to F(*)$. But such a specified lift is part of the definition of η.

For the honest proof of Theorem 4.10 we replace the classifying spaces of topological groups involved by classifying spaces of appropriate discrete categories. We describe those discrete categories first.

Our model for the pair $(B\,\mathrm{TOP}(W\backslash\partial W),B\,\mathrm{TOP}(W,\partial W))$ is that of Lemma 2.17, the classifying space pair of the category pair $(\mathcal{GB}(n),\partial\mathcal{GB}(n))$. Our model for the pair $(B\,\mathrm{TOP}(W^{\bullet},\partial(W^{\bullet})),B\,\mathrm{TOP}(W,\partial W))$ will be a homology approximation only, again a classifying space pair determined by a category pair $(\mathcal{OD}(n),\partial\mathcal{OD}(n))$ which we now describe. It comes with a forgetful functor to $(\mathcal{GB}(n),\partial\mathcal{GB}(n))$ which we again call ι because it is our categorical model for the map ι mentioned in Theorem 4.10.

The objects of $\mathcal{OD}(n)$ are pairs (D,H) where D is homeomorphic to the n-disk and $H\in\mathcal{O}(D)$. A morphism from (D,H) to (D',H') is a homeomorphism $f\colon D\to D'$ for which $H'\subset f(H)$. The forgetful functor ι from $\mathcal{OD}(n)$ to $\mathcal{GB}(n)$ is given by $(D,H)\mapsto H$. Let $\partial\mathcal{OD}(n)$ be the inverse image of $\partial\mathcal{GB}(n)$ under the forgetful functor.

By construction, the category pair $(\mathcal{OD}(n),\partial\mathcal{OD}(n))$ is equivalent to a semidirect product of the discrete group $\mathrm{TOP}_0(\mathbb{D}^n,\partial\mathbb{D}^n)$ with the pair $(\mathcal{O}(\mathbb{D}^n),\partial\mathcal{O}(\mathbb{D}^n))$. So the classifying space pair determined by $(\mathcal{OD}(n),\partial\mathcal{OD}(n))$ is homotopy equivalent to the homotopy orbit pair of the canonical action of $\mathrm{TOP}_0(\mathbb{D}^n,\partial\mathbb{D}^n)$ on

$$(|\mathcal{O}(\mathbb{D}^n)|,|\partial\mathcal{O}(\mathbb{D}^n)|)\simeq(\mathbb{D}^n,\partial\mathbb{D}^n).$$

It admits therefore a *homology* equivalence to the homotopy orbit pair of the canonical action of $\mathrm{TOP}(\mathbb{D}^n,\partial\mathbb{D}^n)$ on $(\mathbb{D}^n,\partial\mathbb{D}^n)$, which in turn is homotopy equivalent to $(B\mathrm{TOP}(W^\bullet,\partial(W^\bullet)),B\mathrm{TOP}(W,\partial W))$ of Theorem 4.10. (This homotopy equivalence is a consequence of the fact that the map $\mathrm{TOP}(W^\bullet,\partial(W^\bullet))\to\partial(W^\bullet)\cong\partial\mathbb{D}^n$ given by $g\mapsto g(\infty)$ is a fibration, with fiber over the base point $\infty\in\partial(W^\bullet)$ equal to the subgroup $\mathrm{TOP}(W,\partial W)$.) More precisely, we can make a commutative square of pairs

$$\begin{tikzcd}
(|\mathcal{OD}(n)|,|\partial\mathcal{OD}(n)|) \arrow[r] \arrow[d, "|\iota|"] & (B\mathrm{TOP}(W^\bullet,\partial(W^\bullet)),B\mathrm{TOP}(W,\partial W)) \arrow[d, "\iota"] \\
(|\mathcal{GB}(n)|,|\partial\mathcal{GB}(n)|) \arrow[r, "\simeq"] & (B\mathrm{TOP}(W\backslash\partial W),B\mathrm{TOP}(W,\partial W))
\end{tikzcd} \tag{4.11}$$

where the upper horizontal arrow is a homology equivalence. When we apply $H^{\gamma(n)}(\ldots;\mathbf{F}(*))$ or $H^{\gamma(n)}(\ldots;\mathbf{S}^0)$ to (4.11), we obtain a commutative square of abelian groups and homomorphisms in which the two horizontal arrows are isomorphisms. Here $\gamma(n)$ denotes the usual bundle over $B\mathrm{TOP}(n)=B\mathrm{TOP}(W\backslash\partial W)$, but also its pullback to the various other spaces appearing in (4.11).

The proof of Theorem 4.10 and also a sharper formulation of Theorem 4.10 will come out of the commutative diagram

$$\begin{tikzcd}
\operatorname{holim}_D Q^\bullet(*) \arrow[r, "\eta"] & \operatorname{holim}_D F(*) \\
\operatorname{holim}_D Q^\bullet(D) \arrow[r, "\eta"] \arrow[u, "\simeq"] \arrow[d, "\simeq"] & \operatorname{holim}_D F(D) \arrow[u, "\simeq"] \arrow[d, "\simeq"] \\
\operatorname{holim}_{(D,H)} Q^\bullet(H) \arrow[r, "\eta"] & \operatorname{holim}_{(D,H)} F(H).
\end{tikzcd} \tag{4.12}$$

Here D runs through the category $\mathcal{D}(n)$ of all n-disks, with homeomorphisms as morphisms, and (D,H) runs through $\mathcal{OD}(n)$. (The vertical arrows between rows 2 and 3 are weak homotopy equivalences because (3.15) is a weak homotopy equivalence, in particular when M is a disk.)

We begin by making a combinatorial model for $\iota^*\bar{b}_n$. Choose a point z_1 in the homotopy inverse limit of the left-hand column of (4.12) which projects to $z \in Q^\bullet(*) \subset \operatorname{holim}_D Q^\bullet(*)$. This is a contractible choice. The image of z_1 under the projection to $\operatorname{holim}_{(D,H)} Q^\bullet(H)$ is by definition

$$\iota^*\bar{b}_n \in \operatorname*{holim}_{(D,H)} Q^\bullet(H).$$

Of course we also have a combinatorial model for $\iota^*\bar{e}_n$. This is simply the pullback of χ under the forgetful functor $(D,H) \mapsto H$ from $\mathcal{OD}(n)$ to $l\mathcal{E}^\bullet$:

$$\iota^*\bar{e}_n \in \operatorname*{holim}_{(D,H)} F(H).$$

THEOREM 4.13 (sharp version of Theorem 4.10). *We construct a path v_n in $\operatorname{holim}_{(D,H)} F(H)$ ending at $\iota^*\bar{e}_n$ and starting at the image of $\iota^*\bar{b}_n$ under*

$$\operatorname*{holim}_{(D,H)} Q^\bullet(H) \xrightarrow{\eta} \operatorname*{holim}_{(D,H)} F(H).$$

Proof (construction). The constructions and hypotheses so far determine a point z_2 in the homotopy limit of the following subdiagram of (4.12):

$$\begin{array}{ccc} \operatorname{holim}_D Q^\bullet(*) & \xrightarrow{\quad\eta\quad} & \operatorname{holim}_D F(*) \\ \uparrow{\scriptstyle\simeq} & & \uparrow{\scriptstyle\simeq} \\ \operatorname{holim}_D Q^\bullet(D) & & \operatorname{holim}_D F(D) \\ \downarrow{\scriptstyle\simeq} & & \downarrow{\scriptstyle\simeq} \\ \operatorname{holim}_{(D,H)} Q^\bullet(H) & & \operatorname{holim}_{(D,H)} F(H). \end{array} \tag{4.14}$$

Namely, to specify z_2, we have to specify, for each space in the diagram (4.14), a point in that space; and for each arrow in (4.14), a path in the codomain of that arrow connecting the image under that arrow of the specified point in the domain with the specified point in the codomain; and for each pair of composable arrows in (4.14), a map from a standard 2-simplex, and so on. For the spaces and arrows in the left-hand column, such choices were made when we selected z_1. For the spaces and arrows in the right-hand column, such choices are made for us by the characteristic χ; see also the remark just below. For the horizontal arrow, we take the appropriate constant path; this works because of our hypothesis

$$\eta(z) = \chi(*) \in F(*) \subset \operatorname*{holim}_D F(*).$$

There is just one instance of (two) composable arrows in the diagram; for that we have to specify a map from a 2-simplex to $\operatorname{holim}_D F(*)$. The map is prescribed on two edges of the 2-simplex and constant on one of these; we can define it as the appropriate degeneracy of its (prescribed) restriction to the appropriate edge.

Now that z_2 has been specified, we note that the projection from the holim of (4.12) to the holim of (4.14) is a fibration and a homotopy equivalence. Therefore its fiber over z_2 is contractible. Choose a point z_3 in there. The image of z_3 in the holim of the lower row of (4.12) is a triple of the form

$$(\iota^* \bar{b}_n, v_n, \iota^* \bar{e}_n). \qquad \square$$

Remark 4.15. We have used the following general principle, which we shall have occasion to use again. Let U be a functor from a small category $\mathcal{K}$ to the category of small categories. With U one can associate a new category, the (opposite) Grothendieck construction

$$\mathcal{K}\textstyle\int^{\mathrm{op}} U$$

as follows. The objects are pairs (k, x) where k is an object of $\mathcal{K}$ and x is an object of $U(k)$. A morphism from (k_0, x_0) to (k_1, x_1) is a pair (f, g) where $f: k_1 \to k_0$ is a morphism in $\mathcal{K}$ and $g: x_0 \to f(x_1)$ is a morphism in $U(k)$. Compare [Tho].

Let E be a functor from $\mathcal{K}\int^{\mathrm{op}} U$ to spaces. For each object k in $\mathcal{K}$ we have an evident inclusion $U(k) \to \mathcal{K}\int^{\mathrm{op}} U$ and hence a projection

$$\operatorname{holim} E \to \operatorname{holim}(E \mid U(k)).$$

The general principle that we are after states that these projections, taken together, have a canonical refinement to a map

$$\operatorname{holim} E \to \operatorname*{holim}_k \operatorname{holim}(E \mid U(k)).$$

The proof is an exercise. — The special case that we have used in Theorem 4.13 is this. Let $\mathcal{K}$ have three objects $1, 2, 3$ and just two nonidentity morphisms $1 \to 2$ and $3 \to 2$. Put $U(1) = U(2) = \mathcal{D}(n)$ and $U(3) = \mathcal{OD}(n)$. The functor induced by $1 \to 2$ is the identity of $\mathcal{D}(n)$; the functor induced by $3 \to 2$ is the forgetful one, $(D, H) \mapsto D$. We define E from $\mathcal{K}\int^{\mathrm{op}} U$ to spaces as a composition, $E = F \circ E'$, where

$$E' : \mathcal{K}\textstyle\int^{\mathrm{op}} U \to l\mathcal{E}^{\bullet}$$

is given by $(1, D) \mapsto *$, $(2, D) \mapsto D$, $(3, (D, H)) \mapsto H$ on objects. (We leave it to the reader to define E' on morphisms. Note that E' does not land in the subcategory $l\mathcal{E}$ of $l\mathcal{E}^{\bullet}$.) The characteristic χ is a point in $\operatorname{holim} F$ and determines a point in $\operatorname{holim} F \circ E' = \operatorname{holim} E$. That in turn determines, by the above reasoning, a point in $\operatorname{holim}_{k=1,2,3} \operatorname{holim}(E \mid U(k))$, the homotopy limit of the right-hand column in (4.14).

5. The transfer of Becker–Gottlieb and Dold

Our goal here is to combine the Family Index Theorem 3.18 with Theorem 4.10 stating equality of $[\bar{e}_n]$ and $\boldsymbol{\eta}_*[\bar{b}_n]$ for n-disk bundles. Therefore we need to understand $\bar{\wp}[\bar{b}_n]$, the reflected Poincaré dual of $[\bar{b}_n]$ applied to the vertical tangent bundle pair of a bundle $p: E \to B$ with compact n-manifold fibers. This is a vertical homotopy class of sections of $Q_B^\bullet(E) \to B$ which we identify with a well-known fiber homotopy invariant of the bundle $p: E \to B$ due to Becker–Gottlieb [BeG2] and Dold [D2]. See also [Cl] and [DP].

The Poincaré dual of the Becker–Euler class

Let M^n be compact, with tangent bundle τ. We write b_M and $\bar{b}_M$ for the Becker–Euler sections $b_n(\tau)$ and $\bar{b}_n(\tau)$, understood as sections of $\tau^\bullet$ which are trivial over ∂M. The goal here is to describe explicitly $\wp b_M$, the Poincaré dual of b_M. Our interest in $\wp b_M$ comes from the fact that it agrees with $\bar{\wp}\bar{b}_M$ (since $\bar{\wp} = \wp r$ and $\bar{b}_M = r(b_M)$ where r is the reflection involution, Definition 3.9).

Choose a locally flat embedding $(M, \partial M) \to (\mathbb{R}^{w-1} \times [-\infty, \infty[, \mathbb{R}^{w-1} \times -\infty)$. Choose a normal bundle ν, increasing w if necessary. Let $\partial\nu = \nu|\partial M$. Let $c: \mathbb{S}^w \to \mathrm{th}(\nu)/\mathrm{th}(\partial\nu)$ be the collapse map. Now the Poincaré dual $\wp b_M$ is the composition

$$\mathbb{S}^w \xrightarrow{c} \mathrm{th}(\nu)/\mathrm{th}(\partial\nu) \xrightarrow{(\mathrm{id}, b_M)} \mathrm{th}(\nu \oplus \tau) \tag{5.1}$$

where the map labelled (id, b_M) takes $z \in \nu_x$ to $(z, b_M(x))$ in $\nu_x \times \tau_x$. We think of (5.1) as an element in $\Omega^w(\mathbb{S}^w \wedge M^\bullet) \subset \Omega^w\Omega^\infty\Sigma^\infty(\mathbb{S}^w \wedge M^\bullet)$, to be consistent.

The above description of $\wp b_M$ involves a contractible choice of embedding of M in a high-dimensional euclidean half-space and a choice of normal bundle $\nu(M)$. Also, b_M is only determined up to contractible choice.

Becker–Euler class and Spanier–Whitehead duality

We recall the homotopy transfer of Becker–Gottlieb and Dold. This is defined for any fibration $p: E \to B$ where each fiber $p^{-1}(b)$ is homotopy equivalent to a homotopy finitely dominated CW-space. It is a section tr_p of a fibration

$$(Q_+)_B(E) \to B$$

with fiber $Q_+(p^{-1}(x)) = Q(p^{-1}(x)_+)$ over $x \in B$, where the $+$ denotes an added base point. (Here we cannot write $Q^\bullet$ because the fibers $p^{-1}(x)$ might not be compact.)

The composition of tr_p with the inclusion of $(Q_+)_B(E)$ in $Q_+(E)$ is a map $B \to Q_+(E)$, in other words a stable map from B_+ *to* E_+; hence the word *transfer*.

First we assume $B=*$. Let E_+^* be a Spanier–Whitehead 0-dual of E_+. In other words, E_+^* is a CW-spectrum with finitely many cells, equipped with a map

$$\varkappa\colon \mathbf{S}^0 \to E_+^* \wedge E_+$$

which is nondegenerate, i.e., slant product with $[\varkappa]$ is an isomorphism from the cohomology of E_+ to the homology of E_+^* (with integer coefficients). In this case tr_p is a point in $Q_+(E)$, namely, the composition

$$\begin{array}{c} \mathbf{S}^0 \\ \Big\downarrow{\scriptstyle \varkappa} \\ E_+^* \wedge E_+ \\ \Big\downarrow{\scriptstyle \mathrm{id}\wedge \mathrm{diag}} \\ E_+^* \wedge (E_+ \wedge E_+) \\ \Big\downarrow{\scriptstyle \cong} \\ (E_+^* \wedge E_+) \wedge E_+) \\ \Big\downarrow{\scriptstyle \varkappa^* \wedge \mathrm{id}} \\ \mathbf{S}^0 \wedge E_+ . \end{array} \tag{5.2}$$

Here $\varkappa^*\colon E_+^* \wedge E_+ \to \mathbf{S}^0$ is what one might call the adjunct of $\varkappa$. It must be chosen together with a homotopy h connecting the maps of bispectra

$$\mathbf{S}^0 \wedge \mathbf{S}^0 \xrightarrow{\mathrm{id}\wedge\varkappa} \mathbf{S}^0 \wedge (E_+^* \wedge E_+),$$

$$\mathbf{S}^0 \wedge \mathbf{S}^0 \xrightarrow{\sigma_{(24)}(\varkappa\wedge\varkappa)} (E_+^* \wedge E_+) \wedge (E_+^* \wedge E_+) \xrightarrow{\varkappa^* \wedge \mathrm{id}} \mathbf{S}^0 \wedge (E_+^* \wedge E_+)$$

where $\sigma_{(24)}$ is the automorphism of $(E_+^* \wedge E_+) \wedge (E_+^* \wedge E_+)$ interchanging the two factors E_+. The choice of a pair $(\varkappa^*, h)$ is a contractible choice. The point is that $E_+^* \wedge E_+$ is self-dual, the self-duality being given by $\sigma_{(24)}(\varkappa \wedge \varkappa)$.

In the case of an arbitrary base B, the choices of dual, etc., must be made fiberwise, for each fiber of $p\colon E \to B$. We omit the details.

Example 5.3. Returning to the case $B=*$, assume that $E=M$ is a compact n-manifold, embedded in $\mathbb{R}^w$ with normal bundle ν, and so on as in (5.1). Then we have geometric choices for E^*_+, $\varkappa$ and $\varkappa^*$. For E^*_+ we take the formal w-fold desuspension of $\mathrm{th}(\nu)/\mathrm{th}(\partial\nu)$. For $\varkappa$ we take the formal w-fold desuspension of the composition

$$\mathbb{S}^w \xrightarrow{c} \mathrm{th}(\nu)/\mathrm{th}(\partial\nu) \xrightarrow{\mathrm{diag}} \mathrm{th}(\nu)/\mathrm{th}(\partial\nu)\wedge M_+$$

where "diag" denotes the map taking $z\in\nu_x$ to (z,x) in $\nu_x\times M$. To make $\varkappa^*$ we choose an embedding $u\colon M\to\mathrm{int}(M)$, isotopic to id_M. Then we choose a (continuous) function $\psi\colon [0,\infty[\,\to[0,\infty[$ with $\psi^{-1}(0)=[d,\infty[$, where d is the distance from $u(M)$ to the complement in $\mathbb{R}^w$ of the total space of ν. Let $\varkappa^*$ be the formal w-fold desuspension of the map

$$\mathrm{th}(\nu)/\mathrm{th}(\partial\nu)\wedge M_+ \to \mathbb{R}^w\cup\infty\cong\mathbb{S}^w,$$
$$(x,y)\mapsto \frac{x-u(y)}{\psi(|x-u(y)|)}.$$

With these choices, (5.2) is the composition

$$\mathbb{S}^w \xrightarrow{c} \mathrm{th}(\nu)/\mathrm{th}(\partial\nu) \xrightarrow{j} \mathbb{S}^w\wedge M_+$$

where j is induced by a certain map over M from the total space of ν to $\mathbb{R}^w\times M$. By inspection, j agrees with (id, b_M) in (5.1), more precisely, the two agree after composition with $\mathbb{S}^w\wedge M_+\to\mathbb{S}^w/K\wedge M_+$ where $K\subset\mathbb{S}^w\cong\mathbb{R}^w\cup\infty$ is a suitable disk containing the base point ∞. Hence (5.2) and (5.1) agree in the case at hand. We spell out the fiberwise version of this insight (returning to the $Q^\bullet$-notation since the fibers are compact):

THEOREM 5.4. *Let $p\colon E\to B$ be a bundle with compact n-manifold fibers. Then $\mathrm{tr}_p\colon B\to Q^\bullet_B(E)$ agrees with $\wp b_n$, the fiberwise Poincaré dual of b_n of the vertical tangent bundle pair of p.*

Remark 5.5. Let Γ be the space of sections of $Q^\bullet_B(E)\to B$. Then tr_p is, strictly speaking, not a point in Γ, but a map $C_1\to\Gamma$ where C_1 is a contractible space (the space of choices needed in the construction). Similarly, $\wp b_n$ is a map $C_2\to\Gamma$ with contractible C_2. Our proof of Theorem 5.4 produces another map $C_3\to\Gamma$ with contractible C_3 and maps $C_3\to C_1$ as well as $C_3\to C_2$ over Γ. In this sense tr_p and $\wp b_n$ *agree*.

The index theorem for regular manifolds

Definition 5.6. A *regular* manifold M^n is a topological manifold (with boundary) together with

(1) an n-disk bundle $q\colon L\to M$;

(2) an open embedding $j\colon U\to L$ over M, where U is an open neighborhood of the diagonal in $M\times M$ (viewed as a space over M by means of the first projection).

Remark. In practice we are only interested in the *germ* of (q,j). Two structures (q_1,j_1) and (q_2,j_2) of regular manifold on M determine the same germ if $q_1=q_2$ and there exists an open neighborhood V of the diagonal in $M\times M$ such that both j_1 and j_2 are defined on V and agree there.

The homotopy theoretic content of Definition 5.6 is that the classifying map for the tangent bundle pair of $(M,\partial M)$ comes with a specified lift across (4.9) to a map of pairs

$$(M,\partial M)\to(B\,\mathrm{TOP}(W^{\bullet},\partial(W^{\bullet})),B\,\mathrm{TOP}(W,\partial W))$$

where $W=]-\infty,+\infty]\times\mathbb{R}^{n-1}$.

By a *bundle of regular n-manifolds*, we mean a bundle $p\colon E\to B$ with n-manifold fibers, together with a disk bundle $D\to E$, an open neighborhood U of the diagonal in $E\times_B E$ and an embedding $U\to D$ over E which has the properties listed above fiberwise; that is, the embedding $U|E_b\to D|E_b$ has those properties for each $b\in B$.

Example. A bundle $p\colon E\to B$ of smooth n-manifolds has a canonical structure of bundle of regular n-manifolds. More precisely, the "smooth manifold bundle" structure on p determines up to contractible choice a germ of "regular manifold bundle" structures on p.

THEOREM 5.7. *Let $p\colon E\to B$ be a bundle of compact regular n-manifolds. Then $\chi(p)\colon B\to F_B(E)$ is vertically homotopic to the composition*

$$B\xrightarrow{\mathrm{tr}_p}Q^{\bullet}_B(E)\xrightarrow{\boldsymbol{\eta}_*}F_B(E).$$

Remark. Our proof gives a contractible choice of preferred vertical homotopies.

Proof. Let τ stand for the vertical tangent bundle pair of $p\colon E\to B$. We showed $\chi(p)\simeq\bar{\wp}\bar{e}_n(\tau)$ in Theorem 3.18, see also Remark 3.19; and $\bar{e}_n(\tau)=\boldsymbol{\eta}_*\bar{b}_n(\tau)$ in Theorem 4.10. Therefore $\chi(p)\simeq\boldsymbol{\eta}_*(\bar{\wp}\bar{b}_n(\tau))=\boldsymbol{\eta}_*(\wp b_n(\tau))\simeq\boldsymbol{\eta}_*\mathrm{tr}_p$ by Theorem 5.4. □

Part II. Applications

In Part II we present the functors F and characteristics χ that we had in mind in Part I, beginning with Definition 1.1. Specifically, §6 builds and explores characteristics (for certain spaces) which are natural under homotopy equivalences, whereas §7 builds an excisive characteristic (see Assumption 2.4).

6. Homotopy invariant characteristics

The A-theory characteristic

Notation. A space Y is *homotopy finite* if it is homotopy equivalent to a compact CW-space. It is *homotopy finitely dominated* if there exists a compact CW-space Z and maps $r: Z \to Y$, $i: Y \to Z$ such that $ri \simeq \mathrm{id}_Y$.

Example 6.1. *The A-theory characteristic.* We want to construct a characteristic as in Definition 1.1 for Waldhausen's A-theory functor, restricted to the category $\mathcal{C}$ of homotopy finitely dominated spaces with homotopy equivalences as morphisms. First we need to recall the definition of $A(Y)$ for a space Y.

K-theory of Waldhausen categories. For any category $\mathcal{D}$ with cofibrations $\operatorname{cof}\mathcal{D}$ and weak equivalences $w\mathcal{D}$, Waldhausen [W2, §1.3] has constructed an infinite loop space $\Omega|w\mathcal{S}_\bullet(\mathcal{D})|$ which we shall denote by $K(\mathcal{D})$. Furthermore, Waldhausen constructs a natural transformation $|w\mathcal{D}| \to K(\mathcal{D})$ which he observes is reminiscent of Segal's "group completion" process. Following Thomason we call a category with cofibrations and weak equivalences a *Waldhausen category.*

Retractive spaces over a space Y. Waldhausen defines $A(Y)$ as the K-theory of a certain Waldhausen category of retractive spaces over Y which we now describe; see [W2, §2.1]. A retractive space over Y consists of a space X and a diagram

$$X \underset{s}{\overset{r}{\rightleftarrows}} Y$$

such that $rs = \mathrm{id}_Y$ and s is a closed embedding having the homotopy extension property. The retractive spaces over Y form a category where the morphisms are maps over and relative to Y. A morphism is a *cofibration* if the underlying map of spaces is a closed embedding having the homotopy extension property. It is a *weak equivalence* if the underlying map of spaces is a homotopy equivalence. With these notions of cofibration and weak equivalence, the category of retractive spaces over Y is a Waldhausen category. This follows from [Str]. A retractive space X over Y is *homotopy finite* if it is the codomain of a weak equivalence from another retractive space over Y which is a CW-space relative to Y, with finitely many cells. A retractive space X over Y is *homotopy finitely dominated* if it fits into a diagram $X' \to W \to X$ of retractive spaces over Y such that W is homotopy finite and the composite morphism from X' to X is a weak equivalence.

Let $\mathcal{R}^{fd}(Y)$ be the Waldhausen category of homotopy finitely dominated retractive spaces over Y. Define $A(Y)$ as the K-theory of $\mathcal{R}^{fd}(Y)$.

A continuous map $f: Y_1 \to Y_2$ induces a functor $f_*: \mathcal{R}^{fd}(Y_1) \to \mathcal{R}^{fd}(Y_2)$, pushout with f alias cobase change along f, and then a map $A(Y_1) \to A(Y_2)$. This suggests that $Y \mapsto A(Y)$

is a functor. Unfortunately, to establish that we have to say exactly what we mean by pushout or cobase change. The procedure must be strictly associative and unital. We therefore make (belatedly) the following rather pedantic additional assumptions on our retractive spaces $X \rightleftarrows Y$:

(1) X contains $\{1\}\times Y$, and $s: Y\to X$ is given by $y\mapsto(1,y)$.

(2) No element of $X\setminus s(Y)$ is an ordered pair of the form $(1,\ldots)$.

(Think of 1 as an element of $\mathbb{R}$, using your favorite description of $\mathbb{R}$ within the universe.) For a retractive space X over Y_1 satisfying these conditions (with retraction r and section s) and $f: Y_1\to Y_2$ as above, let f_*X be the (automatically disjoint) union of $X\setminus s(Y_1)$ and $\{1\}\times Y_2$. Then f_*X has an evident topology making it into a pushout (colimit) of

$$X \xleftarrow{s} Y_1 \xrightarrow{f} Y_2,$$

and it is also a retractive space over Y_2 satisfying the additional pedantic conditions. Most important: the rule $f\mapsto f_*$ respects composition, so that $g_*f_*=(gf)_*$.

We now construct a characteristic χ for the A-theory functor defined on the category $\mathcal{C}$ (whose objects are the finitely dominated spaces and whose morphisms are the homotopy equivalences).

Since we have a natural map $|w\mathcal{R}^{fd}(Y)|\to A(Y)$ (the one that is reminiscent of group completion) it is enough to construct a characteristic χ for the functor $Y\mapsto|w\mathcal{R}^{fd}(Y)|$ on $\mathcal{C}$. We will do this using Example 1.3, with $\mathcal{F}=w\mathcal{R}^{fd}$. Accordingly, we must associate with every Y in $\mathcal{C}$ an object $Y^!$ in $w\mathcal{R}^{fd}(Y)$. Let

$$Y^! = \mathbb{S}^0\times Y \tag{6.2}$$

with retraction $r: Y^!\to Y$ equal to the projection and section $s: Y\to Y^!$ given by $y\mapsto(1,y)$. For a morphism $e: X\to Y$ in $\mathcal{C}$, we have

$$e_*(X^!) = (\{-1\}\times X)\cup(\{1\}\times Y).$$

Let $e^!$ from $e_*(X^!)$ to $Y^!$ be given by $(-1,x)\mapsto(-1,e(x))$, $(1,y)\mapsto(1,y)$. Then $e^!$ is a morphism in $w\mathcal{R}^{fd}(Y)$. The 1-cocycle condition is satisfied: $(ef)^!=e^!\cdot e_*(f^!)$ whenever e and f are composable. By Example 1.3 we get the desired characteristic χ for the functor $Y\mapsto|w\mathcal{R}^{fd}(Y)|$.

6.3. *Definition-Summary.* Let $p: E\to B$ be a fibration with homotopy finitely dominated fibers, where B is the geometric realization of a simplicial set $\mathfrak{B}$. We apply §1.6 with $\mathcal{C}$ equal to the category of homotopy finitely dominated spaces, with homotopy equivalences as morphisms, and $F=A|\mathcal{C}$. The characteristic χ from (6.1) and (6.2) then yields a section $\chi(p): B \to A_B(E)$, in the informal notation of §1.6. We call it the *parametrized A-theory Euler characteristic* of p. We will also sometimes write $\chi_h(p): B \to A_B(E)$, to stress the homotopy invariant nature of the construction.

Remark. Let $B = B\mathrm{G}(Y)$, where $\mathrm{G}(Y)$ is the simplicial monoid of homotopy automorphisms of Y. Let $p: E \to B\mathrm{G}(Y)$ be the universal fibration with fibers homotopy equivalent to Y. We obtain $\chi(p): B \to A_B(E)$ as above. In homotopy theoretic terms, the fibration $A_B(E) \to B$ amounts to an A_∞-action of $\mathrm{G}(Y)$ on $A(Y)$, and the section $\chi(p)$ means that $\chi(Y) \in A(Y)$ has been promoted to a homotopy fixed point of the action.

In the case where Y is connected and based, this refines the rather trivial observation that the sum of the ordinary Euler characteristic of Y, in $\mathbb{Z}$, and the Wall finiteness obstruction of Y, in the reduced K_0-group of $\mathbb{Z}\pi_1(Y)$, is an element of $K_0(\mathbb{Z}\pi_1(Y))$ which is invariant under the action of $\pi_0(G(Y))$.

Linearized characteristics

Let R be a (discrete) ring, with unit. Let Y be a space, homotopy equivalent to a CW-space. The bundles of f.g. projective left R-modules on Y, in other words covering spaces of Y where the fibers are continuously equipped with a structure of f.g. projective left R-module, are the objects of an exact category in the sense of Quillen [Ql]. See also [Kar]. Let $K_0(Y, R)$ be the Grothendieck group of that exact category. There is a canonical homomorphism

$$\beta_Y: K_0(Y, R) \to [Y, K(R)]$$

where the square brackets denote a set of homotopy classes of maps. It takes the class of a module bundle V to the homotopy class of ιc_V, where c_V is a classifying map for the bundle V and ι is the inclusion of the classifying space for such bundles into its group completion, which is $K(R)$. (The target of c_V is $\coprod B\mathrm{GL}(P)$ where P runs through a set of representatives of the isomorphism classes of f.g. projective left R-modules.)

Now let $p: E \to B$ be a fibration where the base is homotopy equivalent to a CW-space and the fibers are homotopy finitely dominated. The Riemann–Roch problem raised by Bismut and Lott, as we see it, is to factorize a certain homomorphism

$$\beta_{E\downarrow B}: K_0(E, R) \to [B, K(R)]$$

through $\beta_E: K_0(E, R) \to [E, K(R)]$. In §8 we obtain such factorizations under special hypotheses on $p: E \to B$. In this subsection, we merely define $\beta_{E\downarrow B}$, relate it to the parametrized characteristics of Example 6.1 and make it more explicit in special cases.

Definition of $\beta_{E\downarrow B}$. Because of [W2, §1.7, §1.9], we may redefine $K(R)$ as the K-theory of a Waldhausen category $\mathcal{C}(R)$ of certain chain complexes. The objects of $\mathcal{C}(R)$ are the chain complexes of left projective R-modules which are chain homotopy equivalent to finitely generated ones (bounded below and above). The morphisms are the R-module chain maps. A morphism is a weak equivalence if it is a chain homotopy equivalence. It is a cofibration if it is split injective in each dimension. (We will usually shorten 'chain homotopic' and 'chain homotopy equivalence' to 'homotopic' and 'homotopy equivalence', respectively.)

As for $p: E \to B$, we adopt the notation of §1.6, assuming that B is the geometric realization of a simplicial set $\mathfrak{B}$. Let V be a bundle of f.g. projective R-modules on E. For each simplex z in $\mathfrak{B}$ let $J_V(z)$ be the singular chain complex of E_z with twisted coefficients in V. Then J_V is a functor from simp($\mathfrak{B}$) to $w\mathcal{C}(R)$ and so induces a map from $|\text{simp}(\mathfrak{B})| \simeq B$ to $K(R)$, via the subspace $|w\mathcal{C}(R)|$ of $K(R)$. The homotopy class of this map depends only on the class $[V] \in K_0(E, R)$. We call it $\beta_{E\downarrow B}([V])$.

Relation with the A-theory characteristic. Fix a bundle V of f.g. left projective R-modules on E. For a homotopy finitely dominated retractive space

$$X \underset{s}{\overset{r}{\rightleftarrows}} E$$

let $\lambda_V(X \leftrightarrows E)$ be the relative singular chain complex of the pair (X, E) with (twisted) coefficients in the bundle of modules $r^*(V)$. The functor λ_V induces a map of K-theory spaces, $A(E) \to K(R)$; see the remark just below.

Remark. John Klein pointed out that the functor λ_V is not an *exact* functor. It respects cofibrations and weak equivalences, but it does not respect cobase change. For example, suppose that $X \to Y$ is a cofibration (and also an inclusion) in the domain category of λ_V. We let 'Y/X' denote the pushout $Y \cup_X E$ in $\mathcal{R}^{fd}(E)$. Then there is a canonical map from $\lambda_V(Y)/\lambda_V(X)$ to $\lambda_V(Y/X)$, but the map is rarely an isomorphism. Fortunately however, it is a homotopy equivalence. This suggests the remedy: Instead of using Waldhausen's $S_\bullet$-construction to define the K-theory of a category with cofibrations and weak equivalences, use Thomason's variation, given at the end of [W2, §1.3]. This is natural with respect to functors which are slightly less than exact. In particular, λ qualifies.

OBSERVATION 6.4. *$\beta_{E\downarrow B}$ applied to the class of a bundle V is the homotopy class of the composition (explanation follows)*

$$B \xrightarrow{\chi(p)} A_B(E) \to A(E) \xrightarrow{\lambda_V} K(R).$$

Explanation of terms. We still assume $B=|\mathfrak{B}|$. The canonical maps $E_z \to E$ induce a map from $\operatorname{hocolim}_z A(E_z)$ to $A(E)$, which extends to a map from $A_B(E)$ to $A(E)$. Compare §1.6. The characteristic determines as in Observation 1.4, Remark 1.5 and §1.6 a section $\chi(p)$ of the projection $A_B(E)\to B$. □

Assuming that for each $b\in B$ and $i\geqslant 0$ the homology module $H_i(p^{-1}(b);V)$ has a finite length resolution by f.g. projective left R-modules, we shall give a homology theoretic description of $\beta_{E\downarrow B}$ applied to the class of V. Here is some K-theory background.

Recall that an exact category $\mathcal{M}$ is determined by an embedding of an additive category as a full subcategory of an abelian category $\mathcal{A}$ where $\mathcal{M}$ is closed under extensions in $\mathcal{A}$. See [Ql, p. 16] and [ThoT, Appendix A]. A map f in $\mathcal{M}$ is an admissible monomorphism if it is a monomorphism in $\mathcal{A}$ and the cokernel is isomorphic to an object in $\mathcal{M}$. Dually, a map f in $\mathcal{M}$ is an admissible epimorphism if it is an epimorphism in $\mathcal{A}$ and the kernel is isomorphic to an object in $\mathcal{M}$. The two main examples for us are

$$\mathcal{M}=\mathcal{P}\mathcal{A},$$

the category of projective objects in the abelian category $\mathcal{A}$; and

$$\mathcal{M}=\mathcal{N}\mathcal{P}\mathcal{A},$$

consisting of those objects in the abelian category $\mathcal{A}$ which have finite length resolutions by objects in $\mathcal{P}\mathcal{A}$. The letter $\mathcal{N}$ can be read as *nearly.* If $\mathcal{A}$ is the category of finitely generated left modules over a ring R, then we write $\mathcal{P}_R$ and $\mathcal{N}\mathcal{P}_R$ instead of $\mathcal{P}\mathcal{A}$ and $\mathcal{N}\mathcal{P}\mathcal{A}$.

Notice that any morphism in $\mathcal{P}\mathcal{A}$ which is epic in $\mathcal{A}$ is admissible. A morphism in $\mathcal{P}\mathcal{A}$ which is monic in $\mathcal{A}$ is admissible if and only if it splits. However, all morphisms in $\mathcal{N}\mathcal{P}\mathcal{A}$ which are epic/monic in $\mathcal{A}$ are admissible. See [Ba, I.6.2]. Quillen's Q-construction associates to an exact category $\mathcal{M}$ an infinite loop space $K(\mathcal{M})$. See [Ql]. Quillen's resolution theorem [Ql] implies that the inclusion of $K(\mathcal{P}\mathcal{A})$ in $K(\mathcal{N}\mathcal{P}\mathcal{A})$ is a homotopy equivalence.

If $\mathcal{M}$ is an exact category, we can make $\mathcal{M}$ into a category with cofibrations and weak equivalences, by letting $\operatorname{cof}\mathcal{M}$ be the admissible monomorphisms in $\mathcal{M}$ and by letting $w\mathcal{M}$ be the isomorphisms in $\mathcal{M}$. Then there is a natural equivalence from $K(\mathcal{M})$ in the sense of Quillen to $K(\mathcal{M})$ in the sense of Waldhausen [W2, §1.9], [Gi, 9.3].

For geometric applications we want to have a chain complex theoretic description of $K(\mathcal{M})$. Let $\operatorname{ch}(\mathcal{M})$ be the category of chain complexes in $\mathcal{M}$ which are graded over $\mathbb{Z}$ and bounded above and below. We make $\operatorname{ch}(\mathcal{M})$ into a category with cofibrations and weak equivalences by letting $\operatorname{cof}\operatorname{ch}(\mathcal{M})$ be the chain maps which are degreewise admissible monomorphisms and by letting $w\operatorname{ch}(\mathcal{M})$ be the quasi-isomorphisms, i.e., chain maps which induce isomorphisms in homology. (The homology groups of a chain complex in $\mathcal{M}$ are objects of $\mathcal{A}$.) The chain complexes concentrated in degree zero form a full subcategory which we identify with $\mathcal{M}$. In many cases the inclusion of Waldhausen K-theories, $K(\mathcal{M}) \to K(\mathrm{ch}(\mathcal{M}))$, is a homotopy equivalence.

EXAMPLE 6.5. *The commutative square of inclusion maps*

$$\begin{tikzcd} K(\mathcal{P}\mathcal{A}) \arrow[r] \arrow[d] & K(\mathrm{ch}(\mathcal{P}\mathcal{A})) \arrow[d] \\ K(\mathcal{N}\mathcal{P}\mathcal{A}) \arrow[r] & K(\mathrm{ch}(\mathcal{N}\mathcal{P}\mathcal{A})) \end{tikzcd}$$

consists entirely of homotopy equivalences.

Proof. The upper horizontal arrow is a homotopy equivalence by [W2, 1.7.1] and the approximation theorem [W2, 1.6.7]. See [ThoT]. The right-hand vertical arrow is a homotopy equivalence by the approximation theorem, and the left-hand vertical arrow is a homotopy equivalence by Quillen's resolution theorem, mentioned earlier. □

Remark. Let $\mathcal{A}$ be the category of finitely generated left R-modules, so that $\mathcal{P}\mathcal{A} = \mathcal{P}_R$. Replacing $\mathrm{ch}(\mathcal{P}_R)$ by the larger category $\mathcal{C}(R)$ of chain complexes of left projective R-modules which are *homotopy equivalent* to objects in $\mathrm{ch}(\mathcal{P}\mathcal{A})$ does not change the homotopy type of the K-theory space. This follows directly from the approximation theorem.

Returning to an arbitrary exact category $\mathcal{M} \subset \mathcal{A}$, we introduce certain full subcategories of $\mathrm{ch}(\mathcal{M})$. Let $\mathrm{tch}(\mathcal{M})$ consist of the *trivial* chain complexes (with trivial differential) and let $\mathrm{sch}(\mathcal{M})$ consist of the very *special* chain complexes C whose homology groups H_iC belong to $\mathcal{M}$ for all i. Thus

$$\mathrm{tch}(\mathcal{M}) \subset \mathrm{sch}(\mathcal{M}) \subset \mathrm{ch}(\mathcal{M}),$$

and we define $w\,\mathrm{tch}(\mathcal{M}) := \mathrm{tch}(\mathcal{M}) \cap w\mathcal{M}$, $w\,\mathrm{sch}(\mathcal{M}) := \mathrm{sch}(\mathcal{M}) \cap w\mathcal{M}$. Note: it is irrelevant to us whether $\mathrm{sch}(\mathcal{M})$ is an exact subcategory of $\mathrm{ch}(\mathcal{M})$ or not.

In the proposition just below, we regard the homology functor H_* as a functor from $\mathrm{sch}(\mathcal{M})$ to $\mathrm{tch}(\mathcal{M})$. We also use a *restricted* product $\prod'$ of pointed spaces. It consists of those points (x_i) in the honest product for which $x_i \neq *$ for only finitely many i.

PROPOSITION 6.6. *The following diagram commutes up to homotopy*:

$$\begin{tikzcd} {|w\,\mathrm{sch}(\mathcal{M})|} \arrow[r, "H_*"] \arrow[d, "\subset"] & {|w\,\mathrm{tch}(\mathcal{M})|} \arrow[r, "\cong"] \arrow[d, "\subset"] & \prod'_{i\in\mathbb{Z}} |w\mathcal{M}| \arrow[d, "\subset"] \\ K(\mathrm{ch}(\mathcal{M})) \arrow[r, "="] & K(\mathrm{ch}(\mathcal{M})) & \prod'_{i\in\mathbb{Z}} K(\mathrm{ch}(\mathcal{M})). \arrow[l, "\text{altern. sum}"'] \end{tikzcd}$$

Proof. Given a chain complex C in $\mathrm{sch}(\mathcal{M})$, let P_kC be the kth Postnikov approximation to C. Thus $(P_kC)_i = C_i$ for $i \leqslant k$, $(P_kC)_{k+1} = \mathrm{im}(\partial : C_{k+1} \to C_k)$ and $(P_kC)_i = 0$ for $i > k+1$. Let Q_kC be the kernel of the canonical projection from P_kC to $P_{k-1}C$. Then

$$Q_kC \to P_kC \to P_{k-1}C$$

is a functorial cofibration sequence in $\mathrm{sch}(\mathcal{M})$. The projection

$$Q_kC \to H_*(Q_kC)$$

is a weak equivalence in $\mathrm{sch}(\mathcal{M})$. Of course $H_*(Q_kC)$ is concentrated in degree k and equal to H_kC there. Using this and an observation [W2, 1.3.3] related to the additivity theorem, one finds that the left-hand square in Proposition 6.6 commutes up to homotopy. Commutativity of the right-hand square follows from [W2, 1.6.2]. □

We return to $\beta_{E\downarrow B}$. Recall that $\mathcal{NP}_R$ is the exact category of those left R-modules which admit finite resolutions by f.g. projective ones.

PROPOSITION 6.7. *Suppose that B is connected and $H_i(p^{-1}(b); V)$ is in $\mathcal{NP}_R$ for each $b \in B$ and $i \geqslant 0$. Then $\beta_{E\downarrow B}([V])$ is the alternating sum $\sum (-1)^i \ldots$ of the homotopy classes of the composite maps*

$$B \xrightarrow{k(i)} |\mathrm{iso}(\mathcal{NP}_R)| \hookrightarrow K(\mathcal{NP}_R) \simeq K(R) \quad (i \geqslant 0)$$

where $k(i)$ classifies the bundle on B with fiber $H_i(p^{-1}(b); V)$ over $b \in B$.

Proof. Let j be the largest integer for which $H_j(p^{-1}(b); V)$ is nonzero for some (hence all) $b \in B$. Let $\mathcal{U} \subset w\mathcal{C}(R)$ be the full subcategory consisting of the chain complexes having all homology groups in $\mathcal{NP}_R$ and those in dimensions <0 and $>j$ equal to 0. In view of our definition of $\beta_{E\downarrow B}$, it suffices to show that the composite inclusion $|\mathcal{U}| \hookrightarrow |w\mathcal{C}(R)| \hookrightarrow K(\mathcal{C}(R)) = K(R)$ is homotopic to the alternating sum, over i with $0 \leqslant i \leqslant j$, of the composite maps $|U| \to K(\mathcal{NP}_R) \simeq K(R)$ determined by the functors $C \mapsto H_i(C; V)$ for C in $\mathcal{U}$. We may replace $|\mathcal{U}|$ by the homotopy equivalent $|\mathcal{U} \cap w\mathrm{ch}(\mathcal{P}_R)|$. This is contained in $|w\mathrm{sch}(\mathcal{M})|$ with $\mathcal{M} = \mathcal{NP}_R$, so we can use Proposition 6.6 to complete the proof. □

7. An excisive characteristic

Controlled A-theory

Following [ACFP], [CP] and [CPV], we introduce control. For us, a *control space* is a pair $(\bar{Q}, Q)$ of locally compact Hausdorff spaces, with Q open in $\bar{Q}$.

Let X_1 and X_2 be retractive spaces over Q (with retractions $r_i: X_i \to Q$ for $i=1,2$). A retractive map $X_1 \to X_2$ is, as usual, a map over Q and relative to Q. A *retractive map germ* from X_1 to X_2 is an equivalence class of pairs (U, f), where $U \subset \overline{Q}$ is an open neighborhood of $\overline{Q} \backslash Q$ and $f: r_1^{-1}(U \cap Q) \to X_2$ is a retractive map (that is, a map relative to $U \cap Q$ and over Q). Two such pairs (U, f) and (V, g) are equivalent if f and g agree on $r_1^{-1}(W \cap Q)$ for some open neighborhood W of $\overline{Q} \backslash Q$ in $\overline{Q}$, with $W \subset U \cap V$.

A *controlled map* from X_1 to X_2 is a map $f: X_1 \to X_2$ (not necessarily over Q) which is relative to Q and satisfies the following *control condition*:

For every $z \in (\overline{Q} \backslash Q)^{\bullet}$ and every neighborhood L of z in $\overline{Q}^{\bullet}$, there exists a smaller neighborhood $L' \subset L$ of z in $\overline{Q}^{\bullet}$ such that $r_2 f(x) \in L$ whenever $x \in X_1$ and $r_1(x) \in L'$.

A *controlled map germ* from X_1 to X_2 is an equivalence class of pairs (U, f), where U is an open neighborhood of $\overline{Q} \backslash Q$ in P and $f: r_1^{-1}(U \cap Q) \to X_2$ is a map relative to $U \cap Q$ which satisfies the control condition above, for all $x \in r_1^{-1}(U \cap Q)$. Again, two such pairs (U, f) and (V, g) define the same controlled map germ if f and g agree on $r_1^{-1}(W \cap Q)$ for a sufficiently small open $W \subset \overline{Q}$ with $\overline{Q} \backslash Q \subset W$. Note that a retractive map germ is also a controlled map germ.

Let $\mathcal{R}(\overline{Q}, Q)$, $\mathcal{R}\mathcal{G}(\overline{Q}, Q)$, $\mathcal{C}(\overline{Q}, Q)$, $\mathcal{C}\mathcal{G}(\overline{Q}, Q)$ be the categories whose objects are the retractive spaces over Q and whose morphisms are the retractive maps, retractive map germs, controlled maps and controlled map germs, respectively. (To ensure that these categories depend functorially on $(\overline{Q}, Q)$, we impose certain conditions on the underlying sets of the retractive spaces involved, as in Example 6.1.) Then

$$\mathcal{R}(\overline{Q}, Q) \subset \mathcal{C}(\overline{Q}, Q),$$
$$\mathcal{R}\mathcal{G}(\overline{Q}, Q) \subset \mathcal{C}\mathcal{G}(\overline{Q}, Q).$$

We will now define notions of *homotopy* in $\mathcal{C}(\overline{Q}, Q)$ and $\mathcal{C}\mathcal{G}(\overline{Q}, Q)$. This will lead very naturally to notions of *cofibration* and *weak equivalence* and *homotopy locally finitely dominated object* in $\mathcal{C}(\overline{Q}, Q)$ and $\mathcal{C}\mathcal{G}(\overline{Q}, Q)$. We will then use these notions in $\mathcal{R}(\overline{Q}, Q)$ and $\mathcal{R}\mathcal{G}(\overline{Q}, Q)$.

Homotopy. Let X be retractive over Q. We write $X \curlywedge I$ for the colimit of

$$X \times I \hookleftarrow Q \times I \to Q,$$

which is again a retractive space over Q. Two morphisms $\varphi, \psi: X \to Y$ in $\mathcal{C}(\overline{Q}, Q)$ are homotopic if there exists a morphism $\gamma: X \curlywedge I \to Y$ in $\mathcal{C}(\overline{Q}, Q)$ such that $\gamma i_0 = \varphi$ and $\gamma i_1 = \psi$, where $i_0: X \to X \curlywedge I$ and $i_1: X \to X \curlywedge I$ are the maps (morphisms) given by $x \mapsto (x, 0)$ and $x \mapsto (x, 1)$, respectively. Likewise, two morphisms $\varphi, \psi: X \to Y$ in $\mathcal{C}\mathcal{G}(\overline{Q}, Q)$ are homotopic if there exists a morphism $\gamma: X \curlywedge I \to Y$ in $\mathcal{C}\mathcal{G}(\overline{Q}, Q)$ such that $\gamma i_0 = \varphi$ and $\gamma i_1 = \psi$.

Weak equivalences. We obtain homotopy categories $\mathcal{HC}(\overline{Q},Q)$ and $\mathcal{HCG}(\overline{Q},Q)$ by identifying homotopic morphisms in $\mathcal{C}(\overline{Q},Q)$ and in $\mathcal{CG}(\overline{Q},Q)$, respectively. A morphism in $\mathcal{C}(\overline{Q},Q)$ is a *weak equivalence* if it becomes an isomorphism in $\mathcal{HC}(\overline{Q},Q)$. A morphism in $\mathcal{CG}(\overline{Q},Q)$ is a *weak equivalence* if it becomes an isomorphism in $\mathcal{HCG}(\overline{Q},Q)$.

Cofibrations. Let $\varphi: X_1 \to X_2$ be a morphism in $\mathcal{C}(\overline{Q},Q)$. Let $r_i: X_i \to Q$ be the retractions. We say that φ is a *cofibration* if it is a closed embedding and has the homotopy extension property (for homotopies in $\mathcal{C}(\overline{Q},Q)$, as defined above).

Let $\varphi: X_1 \to X_2$ be a morphism in $\mathcal{CG}(\overline{Q},Q)$. Let $r_i: X_i \to Q$ be the retractions. Say that φ is a *closed embedding* if, for some representative (U,f) of φ, there exists an open neighborhood V of $\overline{Q}\backslash Q$ in $\overline{Q}$ such that $(r_2 f)^{-1}(V\cap Q)$ is contained in $r_1^{-1}(U\cap Q)$ and f restricts to a closed embedding from $(r_2 f)^{-1}(V\cap Q)$ to $r_2^{-1}(V\cap Q)$. Say that φ is a *cofibration* if it is a closed embedding and has the homotopy extension property (for homotopies in $\mathcal{CG}(\overline{Q},Q)$, as defined above).

Homotopy locally finite domination. Let X be a retractive space over Q with retraction $r: X \to Q$. A *controlled CW-structure* on X is a structure of relative CW-space on X (relative to Q) with the following property:

For every $z \in (\overline{Q}\backslash Q)^{\bullet}$ and neighborhood L of z in $\overline{Q}^{\bullet}$, there exists a smaller neighborhood L' of z in $\overline{Q}^{\bullet}$ such that any cell of X which has nonempty intersection with $r^{-1}(L'\cap Q)$ is contained in $r^{-1}(L\cap Q)$.

An object Y in $\mathcal{C}(\overline{Q},Q)$ or $\mathcal{CG}(\overline{Q},Q)$ is *homotopy locally finitely dominated* if, in $\mathcal{HC}(\overline{Q},Q)$ or in $\mathcal{HCG}(\overline{Q},Q)$, as appropriate, it is a retract of some X with a controlled CW-structure which is

(1) locally finite (i.e., $r: X \to Q$ is proper);

(2) finite-dimensional (i.e., X equals its relative n-skeleton for some n).

Definition 7.1. A morphism in $\mathcal{R}(\overline{Q},Q)$ is a *cofibration* if it becomes a cofibration in $\mathcal{C}(\overline{Q},Q)$; it is a *weak equivalence* if it becomes a weak equivalence in $\mathcal{C}(\overline{Q},Q)$. A morphism in $\mathcal{RG}(\overline{Q},Q)$ is a *cofibration* if it becomes a cofibration in $\mathcal{CG}(\overline{Q},Q)$; it is a *weak equivalence* if it becomes a weak equivalence in $\mathcal{CG}(\overline{Q},Q)$.

Let $\mathcal{R}^{ld}(\overline{Q},Q)$ be the full subcategory of $\mathcal{R}(\overline{Q},Q)$ consisting of the objects which are homotopy locally finitely dominated, as objects of $\mathcal{C}(\overline{Q},Q)$. Let $\mathcal{RG}^{ld}(\overline{Q},Q)$ be the full subcategory of $\mathcal{RG}(\overline{Q},Q)$ consisting of the objects which are homotopy locally finitely dominated, as objects of $\mathcal{CG}(\overline{Q},Q)$.

We leave it to the reader to verify that $\mathcal{R}^{ld}(\overline{Q},Q)$ and $\mathcal{RG}^{ld}(\overline{Q},Q)$ are Waldhausen categories, with the notions of cofibration and weak equivalence inherited from $\mathcal{R}(\overline{Q},Q)$ and $\mathcal{RG}(\overline{Q},Q)$, respectively.

OBSERVATION 7.2. *Let $(\bar{Q},Q)$ and $(\bar{Q}',Q')$ be control spaces. A proper map u from $\bar{Q}$ to $\bar{Q}'$ with $u^{-1}(Q')=Q$ induces an exact functor from $\mathcal{R}^{ld}(\bar{Q},Q)$ to $\mathcal{R}^{ld}(\bar{Q}',Q')$ and another from $\mathcal{RG}^{ld}(\bar{Q},Q)$ to $\mathcal{RG}^{ld}(\bar{Q}',Q')$, by pushout along u (compare Example 6.1).*

Proof. We begin with the following fact. Suppose given a closed $C'\subset\bar{Q}'$. For any neighborhood V of $u^{-1}(C')$, there exists a neighborhood V' of C' such that $u^{-1}(V')\subset V$.

Now take $C'=\bar{Q}'\setminus Q'$ and deduce that, for any neighborhood V of $\bar{Q}\setminus Q$ in $\bar{Q}$, there exists a neighborhood V' of $\bar{Q}'\setminus Q'$ in $\bar{Q}'$ such that $u^{-1}(V')\subset V$. Also, try letting C' consist of a single point z' in $\bar{Q}'\setminus\bar{Q}$ and deduce that for any neighborhood V of $u^{-1}(z')$ there exists a neighborhood V' of z' such that $u^{-1}(V')\subset V$. Together, these observations show that pushout along u takes controlled maps to controlled maps and controlled map germs to controlled map germs. They also show that pushout along u takes objects with controlled CW-structures to objects with controlled CW-structures. It follows easily that $u_*:\mathcal{R}(\bar{Q},Q)\to\mathcal{R}(\bar{Q}',Q')$ and $u_*:\mathcal{RG}(\bar{Q},Q)\to\mathcal{RG}(\bar{Q}',Q')$ respect cofibrations, pushouts, weak equivalences and homotopy locally finite dominations. □

PROPOSITION 7.3. *Let $(\bar{Q},Q)$ be a control space. Let $\bar{U}\subset\bar{Q}$ be open and let $U=\bar{U}\cap Q$. For a retractive space X over Q, with retraction $r:X\to Q$, let $X|U:=r^{-1}(U)$, a retractive space over U.*

The rule $X\mapsto X|U$ is an exact functor from $\mathcal{RG}^{ld}(\bar{Q},Q)$ to $\mathcal{RG}^{ld}(\bar{U},U)$. If U contains $\bar{Q}\setminus Q$, it is also an equivalence of categories.

Details. The rule $X\mapsto X|U$ can first of all be viewed as a functor from $\mathcal{CG}(\bar{Q},Q)$ to $\mathcal{CG}(\bar{U},U)$. Namely, suppose that a morphism $X_1\to X_2$ in $\mathcal{CG}(\bar{Q},Q)$ is represented by (W_Q,f_Q) where W_Q is an open neighborhood of $\bar{Q}\setminus Q$ in $\bar{Q}$ and f_Q is a map from the portion of X_1 lying over W_Q to X_2, subject to the appropriate conditions. Choose an open neighborhood W_U of $\bar{U}\setminus U$ in $\bar{U}$, small enough so that the restriction f_U of f_Q to the portion of X_1 lying over $W_U\cap U$ is defined and maps that portion of X_1 to $X_2|U$. Then the pair (W_U,f_U) represents a morphism $X_1|U\to X_2|U$ in $\mathcal{CG}(\bar{U},U)$, which does not depend on the choice of W_U.

A homotopy between morphisms $\varphi,\psi:X_1\to X_2$ in $\mathcal{CG}(\bar{Q},Q)$ induces a homotopy between the induced morphisms $X_1|U\to X_2|U$ in $\mathcal{CG}(\bar{U},U)$. It follows easily that the restriction functor $X\mapsto X|U$, now viewed as a functor from $\mathcal{RG}(\bar{Q},Q)$ to $\mathcal{RG}(\bar{U},U)$, is exact.

Moreover, if X in $\mathcal{CG}(\bar{Q},Q)$ is isomorphic in $\mathcal{CG}(\bar{Q},Q)$ to an object with a controlled CW-structure which is locally finite and finite-dimensional, then the object $X|U$ in $\mathcal{CG}(\bar{U},U)$ is isomorphic to one with a controlled CW-structure which is locally finite and finite-dimensional. It follows that the restriction functor $X\mapsto X|U$, again viewed as a functor from $\mathcal{RG}(\bar{Q},Q)$ to $\mathcal{RG}(\bar{U},U)$, takes homotopy locally finitely dominated objects to homotopy locally finitely dominated objects.

For the last sentence of Proposition 7.3, suppose that $U \supset \overline{Q} \setminus Q$. Then we have a functor $\mathcal{R}\mathcal{G}^{ld}(\overline{U}, U) \to \mathcal{R}\mathcal{G}^{ld}(\overline{Q}, Q)$ taking X in $\mathcal{R}\mathcal{G}^{ld}(\overline{U}, U)$ to the pushout of the diagram $X \hookleftarrow U \hookrightarrow Q$. Up to natural isomorphisms, this is inverse to the restriction functor $\mathcal{R}\mathcal{G}^{ld}(\overline{Q}, Q) \to \mathcal{R}\mathcal{G}^{ld}(\overline{U}, U)$. $\square$

COROLLARY 7.4. *Let* $\mathbb{J}Y = (Y \times [0,\infty], Y \times [0,\infty[)$. *The rule* $Y \mapsto K(\mathcal{R}\mathcal{G}^{ld}(\mathbb{J}Y))$ *is a functor from* $\mathcal{E}^{\bullet}$ *to pointed spaces.*

Proof. Let $f: Y_1 \rightsquigarrow Y_2$ be a morphism in $\mathcal{E}^{\bullet}$. Let $V = f^{-1}(Y_2)$. The restriction of f to V is a proper map $V \to Y_2$ which induces an exact functor

$$\mathcal{R}\mathcal{G}^{ld}(\mathbb{J}V) \to \mathcal{R}\mathcal{G}^{ld}(\mathbb{J}Y_2)$$

as in Observation 7.2. There is another exact functor

$$\mathcal{R}\mathcal{G}^{ld}(\mathbb{J}Y_1) \to \mathcal{R}\mathcal{G}^{ld}(\mathbb{J}V)$$

given by restriction as in Proposition 7.3. The map of K-theories induced by f that we need to define is the map induced by the composite exact functor

$$\mathcal{R}\mathcal{G}^{ld}(\mathbb{J}Y_1) \to \mathcal{R}\mathcal{G}^{ld}(\mathbb{J}V) \to \mathcal{R}\mathcal{G}^{ld}(\mathbb{J}Y_2). \qquad \square$$

THEOREM 7.5. *The functor* $Y \mapsto K(\mathcal{R}\mathcal{G}^{ld}(\mathbb{J}Y))$ *on* $\mathcal{E}^{\bullet}$ *is pro-excisive.*

This, together with Proposition 7.6 below, is a mild variation on [CPV, 2.21]; precursors are the main theorems of [PW] and [Vo]. See [We] for a direct proof. $\square$

PROPOSITION 7.6. *The commutative square of* K*-theory spectra determined by the commutative square of Waldhausen categories and exact inclusion functors*

$$\begin{tikzcd}
\mathcal{R}^{fd}(Y \times 0) \arrow[r, "\subset"] \arrow[d] & \mathcal{R}^{ld}(\mathbb{J}Y) \arrow[d] \\
* \arrow[r, "\subset"] & \mathcal{R}\mathcal{G}^{ld}(\mathbb{J}Y)
\end{tikzcd}$$

is homotopy cartesian. In particular, this holds for $Y = *$; *hence the pro-excisive functor* $Y \mapsto K(\mathcal{R}\mathcal{G}^{ld}(\mathbb{J}Y))$ *on* $\mathcal{E}^{\bullet}$ *has coefficient spectrum* $\simeq \mathbb{S}^1 \wedge \mathbf{A}(*)$, *where* $\mathbf{A}(*)$ *is the spectrum determined by the infinite loop space* $A(*)$.

Proof. For the first sentence, see [We]. For the second sentence, note that the K-theory spectrum of $\mathcal{R}^{ld}(\mathbb{J}*)$ is contractible by a well-known Eilenberg swindle; see also [We].

The excisive A-theory characteristic

The goal is to define an excisive characteristic as in Assumption 2.4. The pro-excisive functor F on $\mathcal{E}^{\bullet}$ which we have in mind is essentially Ω of the one in Theorem 7.5. The characteristic is only defined for $F|l\mathcal{E}^{\bullet}$, as in §2.

Outline. For Y in $\mathcal{E}^{\bullet}$, we define a full subcategory $\mathcal{V}(Y)$ of $\mathcal{R}^{ld}(\mathbb{J}Y)$ and make it into a Waldhausen category in its own right. (The symbol $\mathcal{V}$ should suggest something in the direction of *vanishing.*) The important properties of $\mathcal{V}(Y)$ are these:

(1) the inclusion functor $\mathcal{V}(Y)\to\mathcal{R}^{ld}(\mathbb{J}Y)$ is exact;

(2) $K(\mathcal{V}(Y))$ is contractible;

(3) the space $(Y\times 0)\amalg(Y\times[0,\infty[)$, as a retractive space over $Y\times[0,\infty[$, belongs to $\mathcal{V}(Y)$.

The object $(Y\times 0)\amalg(Y\times[0,\infty[)$ of $\mathcal{V}(Y)$ becomes isomorphic to the zero-object in $\mathcal{R}\mathcal{G}^{ld}(\mathbb{J}Y)$, and so determines a point $\chi(Y)$ in

$$F(Y):=\mathrm{hofiber}[K(\mathcal{V}(Y))\to K(\mathcal{R}\mathcal{G}^{ld}(\mathbb{J}Y))]\simeq\Omega K(\mathcal{R}\mathcal{G}^{ld}(\mathbb{J}Y)).$$

Convention. In the remainder of this section, we want to make sure that all Waldhausen categories in sight have a unique zero-object (an object which is initial and terminal). This means that our earlier definition of $\mathcal{R}\mathcal{G}^{ld}(\mathbb{J}Y)$ needs to be modified slightly: identify all zero-objects.

7.8. *Details. Definition of $\mathcal{V}(Y)$ and exactness of the inclusion functor.* The objects of $\mathcal{V}(Y)$ are, briefly, the proper retractive ENR's over $Y\times[0,\infty[$. A *proper retractive* ENR over $Y\times[0,\infty[$ is a retractive space of the form

$$X\underset{s}{\overset{r}{\rightleftarrows}}Y\times[0,\infty[$$

where X is an ENR and r is a proper map. The morphisms in $\mathcal{V}(Y)$ are the retractive maps. A morphism in $\mathcal{V}(Y)$ is a *cofibration* if it is injective.

For real $s\geqslant 0$ let $\varphi^s\colon Y\times[0,\infty[\to Y\times[0,\infty[$ be the shift, $(y,t)\mapsto(y,t+s)$. A morphism $f\colon X_1\to X_2$ is a *weak equivalence* in $\mathcal{V}(Y)$ if, for every sequence of positive integers $c_0,c_1,c_2,\ldots$, the morphism

$$\coprod_i c_i\varphi^i_*X_1\to\coprod_i c_i\varphi^i_*X_2$$

induced by f is a weak equivalence in $\mathcal{R}^{ld}(\mathbb{J}Y)$. Here $c_i\varphi^i_*X_1$ is short for a coproduct of c_i copies of $\varphi^i_*X_1$; similarly with X_2 instead of X_1. With these notions of cofibration and weak equivalence, $\mathcal{V}(Y)$ is a Waldhausen category. In particular, (i) the pushout of a morphism $f\colon X_1\to X_2$ in $\mathcal{V}(Y)$ and a cofibration $g\colon X_1\to X_3$ in $\mathcal{V}(Y)$ exists in $\mathcal{V}(Y)$; and

(ii) if f is a weak equivalence, then the canonical morphism from X_3 to the pushout is also a weak equivalence. Statement (i) is a consequence of (i′): The pushout of a diagram $A \leftarrow B \rightarrow C$ of ENR's, where $A \leftarrow B$ is proper and $B \rightarrow C$ is a closed embedding, is an ENR. Statement (i′) follows easily from [Hu, VI.1.2]. Statement (ii) is a consequence of the fact that a closed embedding of ENR's has the homotopy extension property. See [Hu, IV.3.2], noting that ENR's are ANR's (absolute neighborhood retracts); compare [D1, IV.8.13.1].

Since closed embeddings of ENR's have the homotopy extension property, cofibrations in $\mathcal{V}(Y)$ are still cofibrations in the larger category $\mathcal{R}^{ld}(\mathbb{J}Y)$. (The homotopy extension property yields at first 'uncontrolled' extensions of controlled homotopies. An uncontrolled extension $\{h_t\}$ can always be improved to a suitably controlled one, $\{h_t^\varkappa\}$, by restriction and reparametrization:

$$h_t^\varkappa(x) := h_{\varkappa(x)t}(x)$$

for $0 \leqslant t \leqslant 1$, where $\varkappa$ is a function satisfying $0 \leqslant \varkappa(x) \leqslant 1$ for all x, among other things. We omit the details.) Hence the inclusion functor from $\mathcal{V}(Y)$ to $\mathcal{R}^{ld}(\mathbb{J}Y)$ is exact.

Contractibility of $K(\mathcal{V}(Y))$. This is based on an Eilenberg swindle. We will show that there exist self-maps

$$a\colon K(\mathcal{V}(Y)) \to K(\mathcal{V}(Y)), \quad b\colon K(\mathcal{V}(Y)) \to K(\mathcal{V}(Y))$$

such that $a \simeq \mathrm{id}$ and $b \simeq \mathrm{id} + ab$ (where we use the infinite loop space structure on $K(\mathcal{V}(Y))$ to make sense of the addition). Hence the identity map of $K(\mathcal{V}(Y))$ is nullhomotopic.

We write $\varphi := \varphi^1$. The map a is induced by the exact functor φ_* from $\mathcal{V}(Y)$ to $\mathcal{V}(Y)$. The map b is induced by the exact functor

$$X \mapsto \coprod_{i \in \mathbb{N}} \varphi_*^i X$$

($\mathbb{N} = \{0, 1, 2, \ldots\}$) from $\mathcal{V}(Y)$ to $\mathcal{V}(Y)$. The additivity theorem implies immediately that $b \simeq \mathrm{id} + ab$. To show that $a \simeq \mathrm{id}$, we introduce another exact functor

$$\psi\colon \mathcal{V}(Y) \to \mathcal{V}(Y)$$

and natural weak equivalences $X \to \psi(X) \leftarrow \varphi_*(X)$. Namely, $\psi(X)$ is the quotient of $X \times I$ by the relations $(x, s) \simeq (\varphi^s(x), 0)$ for $x \in Y \times [0, \infty[$. (Remember that X contains $Y \times [0, \infty[$ as a retract.) Then $\psi(X)$ maps to $Y \times [0, \infty[$ by means of $(x, s) \mapsto \varphi^s(r(x))$, where r is the retraction for X, and $Y \times [0, \infty[$ maps to $\psi(X)$ by $z \mapsto (j(z), 0)$ where $j\colon Y \times [0, \infty[\to X$ is the structural section for X. Therefore $\psi(X)$ is retractive over

$Y\times[0,\infty[$. The natural weak equivalences $X\to\psi(X)$ and $\varphi_*(X)\to\psi(X)$ are given by $x\mapsto(x,0)$ and $\varphi(x)\mapsto(x,1)$. (We have written $\varphi(x)$ for the image of $x\in X$ under an obvious embedding $X\to\varphi_*(X)$ which covers the map $\varphi\colon Y\times[0,\infty[\to Y\times[0,\infty[$.)

The characteristic object. $(Y\times 0)\amalg(Y\times[0,\infty[)$, with the evident retraction to $Y\times[0,\infty[$, is an object of $\mathcal{V}(Y)$ which we denote by $Y^!$. It maps to the zero-object in $\mathcal{R}\mathcal{G}^{ld}(\mathbb{J}Y)$, because the identity endomorphism and the zero-endomorphism of $Y^!$ determine the same controlled map germ from $Y^!$ to $Y^!$. (Remember the convention preceding §7.8.) In this way, $Y^!$ determines a point $\chi(Y)\in F(Y)$.

(The notation $(Y\times 0)\amalg(Y\times[0,\infty[)$ only describes the object $Y^!$ of $\mathcal{V}(Y)$ up to unique isomorphism. What we really mean is $(\{-1\}\times Y\times\{0\})\cup(\{1\}\times Y\times[0,\infty[)$. Compare Example 6.1.)

Naturality. Let $f\colon Y_1^\bullet\to Y_2^\bullet$ be a pointed map, where Y_1 and Y_2 are ENR's. We decompose f as in the proof of Corollary 7.4:

$$Y_1^\bullet\to W^\bullet\to Y_2^\bullet$$

where $W=f^{-1}(Y_2)$. Using that decomposition, we found that f induces an exact functor

$$\mathcal{R}\mathcal{G}^{ld}(\mathbb{J}Y_1)\to\mathcal{R}\mathcal{G}^{ld}(\mathbb{J}Y_2).$$

This led to Corollary 7.4. The same recipe gives an exact functor from $\mathcal{V}(Y_1)$ to $\mathcal{V}(Y_2)$, and we conclude that $Y\mapsto K(\mathcal{V}(Y))$ is a functor on $\mathcal{E}^\bullet$. Hence F is a functor on $\mathcal{E}^\bullet$.

Moreover, it is clear that the pullback alias restriction functor $\mathcal{V}(Y_1)\to\mathcal{V}(W)$ takes the characteristic object $Y_1^!$ to $W^!$. If $f|W$ from W to Y_2 is cell-like, then the retractive map

$$(W\times 0)\amalg(Y_2\times[0,\infty[)\to(Y_2\times 0)\amalg(Y_2\times[0,\infty[)$$

taking $(w,0)$ in the first summand $W\times 0$ to $(f(w),0)$ in the first summand $Y_2\times 0$ is a weak equivalence in $\mathcal{V}(Y_2)$. Hence we obtain by the recipe of Example 1.3 a characteristic χ for the functor on $l\mathcal{E}^\bullet$ given by

$$Y\mapsto\operatorname{hofiber}(|w\mathcal{V}(Y)|\to|w\mathcal{R}\mathcal{G}^{ld}(\mathbb{J}Y)|).$$

This functor is a subfunctor of $F|l\mathcal{E}^\bullet$, so that, finally, we have a characteristic χ for the functor $F|l\mathcal{E}^\bullet$.

Remark 7.9. The functor F on $\mathcal{E}^\bullet$ is pro-excisive, with coefficient spectrum

$$\mathbf{F}(*)\simeq\mathbf{A}(*).$$

Proof. This follows from Proposition 7.6 and the contractibility of $K(\mathcal{V}(Y))$ proved in §7.8. □

Notation 7.10. Because of Remark 7.9, we write $F(Y) =: {}_{lf}A^{\%}(Y)$ or, if Y is compact, $F(Y) = A^{\%}(Y)$. See §8 for more justification.

7.11. *Definition–Summary.* Let $p: E \to B$ be a fiber bundle with compact topological manifold fibers, where B is the geometric realization of a simplicial set $\mathfrak{B}$. Applying §2.8 with $F = {}_{lf}A^{\%}$ and χ as in §7.8 yields a section

$$\chi(p): B \to A_B^{\%}(E)$$

of $A_B^{\%}(E) \to B$. This is in the informal notation of §1.6. We call $\chi(p)$ the *parametrized $A^{\%}$-theory Euler characteristic* of p.

8. Riemann–Roch theorems

Assembly

Definition 8.1. Let $\mathcal{C}$ be the category of all spaces which are homotopy equivalent to compact CW-spaces. A functor $\mathbf{F}$ from $\mathcal{C}$ to CW-spectra is *homotopy invariant* if it takes homotopy equivalences to homotopy equivalences. A homotopy invariant $\mathbf{F}$ is *excisive* if $\mathbf{F}(\varnothing)$ is contractible and if $\mathbf{F}$ preserves homotopy pushout squares.

PROPOSITION–DEFINITION 8.2. *For any homotopy invariant functor $\mathbf{F}$ from $\mathcal{C}$ to CW-spectra, there exist an excisive (and homotopy invariant) functor $\mathbf{F}^{\%}$ from $\mathcal{C}$ to CW-spectra and a natural transformation*

$$\alpha = \alpha_{\mathbf{F}}: \mathbf{F}^{\%} \to \mathbf{F}$$

such that $\alpha: \mathbf{F}^{\%}(*) \to \mathbf{F}(*)$ is a homotopy equivalence. There is a construction of $\mathbf{F}^{\%}$ and $\alpha_{\mathbf{F}}$ which is natural in $\mathbf{F}$.

Proposition 8.2 is a mild variation on [WW1, Theorem 1.1]. The proof goes through. We call α the *assembly*. The concept and the name originated in Frank Quinn's thesis [Qn1], [Qn2].

Remark. There is a chain of natural homotopy equivalences relating $\mathbf{F}^{\%}(X)$ to $X_+ \wedge \mathbf{F}(*)$. See [WW1, 1.2].

Remark. Natural dependence of $\alpha_{\mathbf{F}}$ on $\mathbf{F}$ implies "universality" of $\alpha_{\mathbf{F}}$. That is, $\alpha_{\mathbf{F}}: \mathbf{F}^{\%} \to \mathbf{F}$ is the best approximation (from the left) of $\mathbf{F}$ by an excisive functor. Namely, suppose that $\beta: \mathbf{E} \to \mathbf{F}$ is another natural transformation where $\mathbf{E}$ is excisive on $\mathcal{C}$. We have a commutative square

$$\begin{array}{ccc} \mathbf{E}^{\%}(X) & \xrightarrow{\alpha_{\mathbf{E}}} & \mathbf{E}(X) \\ \downarrow{\scriptstyle \beta^{\%}} & & \downarrow{\scriptstyle \beta} \\ \mathbf{F}^{\%}(X) & \xrightarrow{\alpha_{\mathbf{F}}} & \mathbf{F}(X) \end{array}$$

which is natural in X. The arrow $\alpha_{\mathbf{E}}$ is a homotopy equivalence, showing that β essentially factors through $\alpha_{\mathbf{F}}$. If $\beta: \mathbf{E}(*) \to \mathbf{F}(*)$ happens to be a homotopy equivalence, then $\beta^{\%}: \mathbf{E}^{\%}(X) \to \mathbf{F}^{\%}(X)$ is also a homotopy equivalence for all X. See [WW1, §1]. For this reason, we will say that $\beta: \mathbf{E} \to \mathbf{F}$ is *an assembly transformation* if $\mathbf{E}$ is excisive on $\mathcal{C}$ and $\beta: \mathbf{E}(*) \to \mathbf{F}(*)$ is a homotopy equivalence.

8.3. *Notation–Example.* The functor $Y \mapsto A(Y)$ has an Ω-spectrum-valued version, $Y \mapsto \mathbf{A}(Y)$, so that the zeroth term of $\mathbf{A}(Y)$ is $A(Y)$. We abbreviate $\Omega^\infty \mathbf{A}^{\%}(Y)$ to $A^{\%}(Y)$, for Y in $\mathcal{C}$.

Comments. The space $A(Y)$ is the underlying space of a Γ-space [S1] determined by the coproduct of retractive spaces over Y, and the abelian monoid $\pi_0 A(Y)$ is an abelian group. Therefore Segal's machine delivers an Ω-spectrum $\mathbf{A}(Y)$ with zeroth term $A(Y)$.

We already have a definition of $A^{\%}(Y)$ when Y is a compact ENR, from Notation 7.10; but there is no serious clash, since both versions of $A^{\%}(Y)$ are related to the infinite loop space $\Omega^\infty(Y_+ \wedge \mathbf{A}(*))$ by a chain of natural homotopy equivalences.

Universal Riemann–Roch

As in §1.6, let $p: E \to B$ be a fibration, where B is the geometric realization of a simplicial set. Assume also that the fibers of p belong to $\mathcal{C}$, the category of Definition 8.1; in other words, they are homotopy finitely dominated. We apply §1.6 with this $\mathcal{C}$ and, in place of F, the functors A (as defined in Example 6.1) and $A^{\%}$ (constructed from the functor A following Proposition 8.2). The result is a fiberwise assembly map

$$A_B^{\%}(E) \to A_B(E),$$

in the informal notation of §1.6. It is a map over B. Recall from §6.3 that the parametrized Euler characteristic $\chi_h(p)$ is a section $B \to A_B(E)$ of the projection $A_B(E) \to B$.

Note that the fiberwise assembly $A_B^{\%}(E) \to A_B(E)$ and the parametrized Euler characteristic $\chi_h(p): B \to A_B(E)$ have been defined in fiber homotopy theoretic terms. That is, if $p': E' \to B$ is another fibration and $g: E' \to E$ is a fiberwise homotopy equivalence over B, then

$$\begin{array}{ccc} A_B^{\%}(E') & \xrightarrow{\alpha} & A_B(E') \\ {\scriptstyle g_*}\downarrow{\scriptstyle \simeq} & & {\scriptstyle g_*}\downarrow{\scriptstyle \simeq} \\ A_B^{\%}(E) & \xrightarrow{\alpha} & A_B(E) \end{array}$$

commutes and the composition of $\chi_h(p'): B \to A_B(E')$ with $g_*: A_B(E') \to A_B(E)$ agrees with $\chi_h(p)$ up to a preferred vertical homotopy.

THEOREM 8.4. *Let $p: E \to B$ be a bundle of compact topological manifolds (possibly with boundary). Then the section $\chi_h(p): B \to A_B(E)$ from §6.3 is vertically homotopic, by a preferred vertical homotopy, to the composition of $\chi(p)$ in §7.11 with fiberwise assembly:*

$$B \xrightarrow{\chi(p)} A_B^{\%}(E) \xrightarrow{\alpha} A_B(E).$$

The proof will be given in the next subsection.

Remark. The section $\chi(p)$ and the preferred vertical homotopy from $\alpha\chi(p)$ to $\chi_h(p)$ in Theorem 8.4 are not fiber homotopy invariants. They depend very strongly on the structure of p as a bundle of compact manifolds. The following example (without proofs) may serve as an illustration. Let $g: M' \to M$ be a homotopy equivalence between compact topological manifolds. We view it as a fiber homotopy equivalence between bundles $p': M' \to *$ and $p: M \to *$. Applying Theorem 8.4 to both of these and using the homotopy invariance properties of χ_h, we get an element δ in the homotopy pullback of

$$A^{\%}(M') \xrightarrow{g_*\alpha} A(M) \xleftarrow{\alpha} A^{\%}(M).$$

More precisely, δ is the element determined by $\chi(p')$ alias $\chi(M')$ in $A^{\%}(M')$, $\chi(p)$ alias $\chi(M)$ in $A^{\%}(M)$ and a three-segment path in $A(M)$ obtained by concatenating

(1) the image under $g_*: A(M') \to A(M)$ of the preferred path in $A(M')$ connecting $\alpha(\chi(M'))$ with $\chi_h(M')$, which we get from Theorem 8.4;

(2) the preferred path from $g_*(\chi_h(M'))$ to $\chi_h(M)$ which we get from the homotopy invariance properties of χ_h;

(3) the preferred path in $A(M)$ from $\chi_h(M)$ to $\alpha(\chi(M))$, which we again get from Theorem 8.4.

It is easy to identify π_0 of the above homotopy pullback with the Whitehead group of $\pi_1 M$, so that δ represents an element in the Whitehead group. That element is of course the Whitehead torsion of g; in particular, it can be nonzero. For more on this example and its long history, see [Cha1] and [RaY].

THEOREM 8.5. *Let $p: E \to B$ be a bundle of compact regular manifolds (see Definition* 5.6*), possibly with boundary. Then the section $\chi_h(p): B \to A_B(E)$ from* §6.3 *is vertically homotopic to the composition of Becker–Gottlieb–Dold transfer $\mathrm{tr}_p: B \to Q^{\bullet}_B(E)$ with $\alpha\eta: Q^{\bullet}_B(E) \to A_B(E)$.*

This follows directly from Theorems 8.4 and 5.7. Smooth manifolds are regular manifolds, so that Theorem 8.5 can be seen as a statement about smooth manifold bundles.

Proof of Theorem 8.4

Here we must unravel the relationship between the excisive characteristic of §7.11 and the homotopy invariant characteristic of §6.3, in the case when Y is a compact ENR. At this point we clearly need different symbols for the two characteristics, so we write $\chi(Y) \in F(Y)$ for the excisive one, §7.11, and $\chi_h(Y) \in F_h(Y)$ for the homotopy invariant one, §6.3. Thus, F is the pro-excisive functor of §7.8 and Remark 7.9, with coefficient spectrum $\simeq \mathbf{A}(*)$, and F_h is the algebraic K-theory of spaces functor $Y \mapsto A(Y)$. We shall find that, modulo formal inversion of a natural homotopy equivalence, there is an *assembly* transformation $F(Y) \to F_h(Y)$ which is defined when Y is a compact ENR and takes $\chi(Y)$ to $\chi_h(Y)$. The word *assembly* is justified because α is a natural map between infinite loop spaces.

For a more precise statement we need the commutative diagram of Waldhausen categories and exact functors

$$\begin{tikzcd} \mathcal{V}(Y) \arrow[r, "\subset"] \arrow[d] & \mathcal{R}^{ld}(\mathbb{J}Y) \arrow[d] & \mathcal{R}^{fd}(Y) \arrow[l, "\supset"'] \arrow[d] \\ \mathcal{V}(Y) \arrow[r, "\subset"] & \mathcal{R}\mathcal{G}^{ld}(\mathbb{J}Y) & * \arrow[l, "\supset"'] \end{tikzcd} \tag{8.6}$$

where Y in the right-hand column has been identified with $Y \times 0$. We apply the K-theory functor and take homotopy pullbacks of the resulting rows to obtain a forgetful map

$$\begin{tikzcd} F'(Y) \arrow[d] \\ F(Y). \end{tikzcd}$$

LEMMA 8.7. *That map is a homotopy equivalence.*

Proof. The right-hand square in (8.6) is homotopy cartesian. See [CPV, 2.12]. □

OBSERVATION 8.8. *There is a canonical lift* $\chi'(Y)\in F'(Y)$ *of* $\chi(Y)\in F(Y)$. *Under the projection* $F'(Y)\to F_h(Y)$, *this lift* $\chi'(Y)$ *maps to* $\chi_h(Y)$.

Explanation. Note that the object $Y^!$ in $\mathcal{V}(Y)$ also belongs to the subcategory $\mathcal{R}^{fd}(Y)\subset\mathcal{R}^{ld}(\mathbb{J}Y)$. □

OBSERVATION 8.9. *The projection* $F'\to F_h$ *is an assembly transformation.*

Explanation and proof. We view $F'\to F_h$ as a natural transformation between infinite loop space-valued functors and show that the corresponding natural transformation $\mathbf{F}'\to\mathbf{F}_h$ between spectrum-valued functors is an assembly transformation.

We know already that $\mathbf{F}'$ is excisive. It remains to show that $\mathbf{F}'(*)\to\mathbf{F}_h(*)$ is a homotopy equivalence, or equivalently, that $F'(*)\to F_h(*)$ is a homotopy equivalence. But this is obvious from the definition of F' and the contractibility of $K(\mathcal{R}^{ld}(\mathbb{J}*))$. □

Remark. Why did we bother with χ when χ' is apparently so much more convenient? The answer is, of course, that χ has better naturality properties: it is a characteristic on $l\mathcal{E}^{\bullet}$.

Conclusion of proof of Theorem 8.4. The best policy here is to stick with homotopy limits and to avoid mentioning section spaces explicitly. Therefore we use the rigorous definitions of (2.9) and (1.7):

$$\chi'(p)\in\operatorname*{holim}_{(x,\theta)}F'(E_x^\theta),$$

$$\chi_h(p)\in\operatorname*{holim}_{x}F_h(E_x).$$

To find out how $\chi'(p)$ and $\chi_h(p)$ might be related, we set up a commutative square with forgetful vertical arrows

$$\begin{array}{ccc} \operatorname{holim}_{(x,\theta)}F'(E_x^\theta) & \xleftarrow{\simeq} & \operatorname{holim}_x F'(E_x) \\ \downarrow & & \downarrow \\ \operatorname{holim}_{(x,\theta)}F_h(E_x^\theta) & \xleftarrow{\simeq} & \operatorname{holim}_x F_h(E_x). \end{array} \tag{8.10}$$

We are searching for a path connecting the images of $\chi'(p)$ and $\chi_h(p)$ in the lower left-hand term of (8.10).

Now recall the natural transformation $E_x\to E_x^\theta$ for (x,θ) in $\operatorname{simp}(t\mathfrak{B})$, which induces the horizontal arrows in (8.10). View it as a functor T on $\operatorname{simp}(t\mathfrak{B})\times\{0,1\}$, where $\{0,1\}$ is to be understood as a poset, with $0<1$, hence as a category (with a single nonidentity morphism, from 0 to 1). Using χ_h once more, we obtain a point $z\in\operatorname{holim}F_hT$. But $\operatorname{holim}F_hT$ is exactly the space of natural transformations from the diagram $1\hookleftarrow 0$ to the lower row of (8.10). In particular, z determines a path of the kind we are searching for. That path does what we want. □

Linear Riemann–Roch: Generalities

In the remaining subsections we work mostly with a bundle $p: E \to B$ of compact topological manifolds and a ring R. We produce a transfer homomorphism t from $[E, K(R)]$ to $[B, K(R)]$ such that

$$\beta_{E\downarrow B} = t\beta_E : K_0(E, R) \to [B, K(R)]$$

(see §6, the subsection *Linearized characteristics*). The case where p is a bundle of regular compact manifolds is easier to understand, so we deal with it first. In this case t is induced by the Becker–Gottlieb–Dold transfer tr_p, and so depends only on the fiber homotopy type of p.

Linear Riemann–Roch for regular manifolds

THEOREM 8.11. *Let $p: E \to B$ be a bundle of regular compact manifolds and let R be any ring. Then $\beta_{E\downarrow B}: K_0(E, R) \to [B, K(R)]$ equals $\mathrm{tr}_p^* \cdot \beta_E$, where $\mathrm{tr}_p^*: [E, K(R)] \to [B, K(R)]$ is induced by the Becker–Gottlieb–Dold transfer, a stable map from B_+ to E_+.*

COROLLARY 8.12. *Let V be a bundle of f.g. projective left R-modules on E. Suppose that the homology groups $H_i(E_b; V)$ all admit finite length resolutions by f.g. projective left R-modules; let V_i be the R-module bundle on B with fiber $H_i(E_b; V)$ over $b \in B$. Then*

$$\mathrm{tr}_p^*[V] = \sum_i (-1)^i [V_i] \in [B, K(R)].$$

Proof of Corollary 8.12 *modulo Theorem* 8.11. Use Proposition 6.7. □

Proof of Theorem 8.11. Assume first that B is the geometric realization of a simplicial set with finitely many nondegenerate simplices only. Combining Observation 6.4 and Theorem 8.5, we find that $\beta_{E\downarrow B}$ applied to the class of some module bundle V on E is the homotopy class of the composition

$$B \xrightarrow{\mathrm{tr}_p} Q^{\bullet}_B(E) \xrightarrow{\alpha\eta} A_B(E) \to A(E) \xrightarrow{\lambda_V} K(R),$$

which is also the homotopy class of the composition

$$B \xrightarrow{\mathrm{tr}_p} Q^{\bullet}_B(E) \to Q^{\bullet}(E) \xrightarrow{\alpha\eta} A(E) \xrightarrow{\lambda_V} K(R).$$

Note that $\lambda_V \alpha\eta: Q^{\bullet}(E) \to K(R)$ is a map of infinite loop spaces. It is therefore determined by its restriction to E. We now examine that restriction.

To this end we observe first that $\eta: Q^\bullet(E) \to A^\%(E)$ is homotopic to the composition of $\chi(\mathrm{id}_E): E \to A^\%_E(E)$ with $A^\%_E(E) \to A^\%(E)$. This follows from Theorem 5.7, because $\mathrm{id}: E \to E$ is a bundle of regular compact 0-manifolds. Then we conclude that $\alpha\eta: Q^\bullet(E) \to A(E)$ is homotopic to the composition of $\chi_h(\mathrm{id}_E): E \to A_E(E)$ with the map $A_E(E) \to A(E)$. This makes it obvious that $\lambda_V \alpha\eta | E$ is the map which "classifies" the module bundle V; so it is in the homotopy class $\beta_E(V)$.

This completes the proof in the case where B is the geometric realization of a finitely generated simplicial set. In the case where B is the geometric realization of a larger simplicial set, we reduce to the previous case using naturality arguments. □

Products

For now we assume only that $p: E \to B$ is a fibration with finitely dominated fibers. We shall reformulate $\beta_{E\downarrow B}: K_0(E, R) \to [B, K(R)]$ in terms of a product

$$\mu: \mathbf{K}(E, R) \wedge \mathbf{A}(E) \to \mathbf{K}(R)$$

where $\mathbf{K}(E, R)$ is the K-theory spectrum of the Waldhausen category of bundles of f.g. projective left R-modules on E.

Ingredients. We need three Waldhausen categories. The first is $\mathcal{P}(E, R)$, the category of bundles of f.g. projective left R-modules on E. We make it into a Waldhausen category by decreeing that the split monomorphisms are the cofibrations and the isomorphisms are the weak equivalences. The second Waldhausen category that we need is $\mathcal{R}^{fd}(E)$, defined in Example 6.1. The third is $\mathrm{ch}(R)$, the category of chain complexes C of projective left R-modules which, up to (chain) homotopy equivalence, are bounded and f.g. projective in each degree. The cofibrations in $\mathrm{ch}(R)$ are the chain maps which are split monomorphisms of R-modules in each degree. The weak equivalences are the homotopy equivalences. The $\mathrm{ch}(R)$-notation is not entirely consistent with §6, but it is short. The K-theory spaces of these categories are then $K(E, R)$ (by definition) and $A(E)$ and $K(R)$, respectively. To be more precise, however, we decree

$$\begin{aligned} K(E, R) &:= \Omega |w\mathcal{T}_\bullet \mathcal{P}(E, R)|, \\ A(E) &:= \Omega |w\mathcal{T}_\bullet \mathcal{R}^{fd}(E)|, \\ K(R) &:= \Omega^2 |ww\mathcal{T}_\bullet \mathcal{T}_\bullet \mathrm{ch}(R)|, \end{aligned}$$

where $\mathcal{T}_\bullet$ is the Thomason construction, a mild variation on Waldhausen's $\mathcal{S}_\bullet$-construction which we have used earlier (just before Observation 6.4). It is described at the end of §1.3 in [W2]. Our unusual definition of $K(R)$ is blessed by the digression following 1.5.3 in [W2].

The product at the space level. Here we describe a functor from $\mathcal{P}(E,R)\times\mathcal{R}^{fd}(E)$ to $\mathrm{ch}(R)$. The functor is given by

$$(V,(X\rightleftarrows E))\mapsto\lambda_V(X\rightleftarrows E)$$

where λ_V is the relative singular chain complex of (X,E) with coefficients in V, as in Observation 6.4. The functor is not quite biexact because λ_V is not quite exact for fixed V, but it does induce (with the definitions just above) a map

$$\mu\colon K(E,R)\wedge A(E)\to K(R).$$

The product at the spectrum level. We make the spectra we need out of Γ-spaces, using Segal's machine [S1]. A Γ-space is a contravariant functor from a certain category Γ to spaces, subject to certain conditions. The category Γ^{op} has as objects the finite sets S and as morphisms from S to T the pointed maps $S_+\to T_+$. The *underlying space* of a Γ-space $S\to Z(S)$ is the space $Z(\{1\})$. In particular, $K(E,R)$ and $A(E)$ are underlying spaces of Γ-spaces given by

$$S\mapsto K(S\times E,R),$$
$$S\mapsto A(S\times E),$$

respectively, with structure maps whose definition we leave to the reader. We like to think of $K(R)$ as the underlying space of a *bi*-Γ-space (contravariant functor from $\Gamma\times\Gamma$ to spaces, subject to some conditions), as follows:

$$(S,T)\mapsto K(R^{S\times T}).$$

(Note in this connection that the category of left modules over $R^{S\times T}$ is equivalent to the product of $S\times T$ copies of the category of left R-modules.) The Segal machine makes this bi-Γ-space into a bispectrum, which we denote (here) by $\mathbf{K}(R)$. In conclusion, we see that it is easy to promote $\mu\colon K(E,R)\wedge A(E)\to K(R)$ to a binatural map

$$K(S\times E,R)\wedge A(T\times E)\to K(R^{S\times T})$$

for S,T in Γ. The Segal machine translates this into a map of bispectra,

$$\mu\colon\mathbf{K}(E,R)\wedge\mathbf{A}(E)\to\mathbf{K}(R).\qquad\square$$

Description of $\beta_{E\downarrow B}$ in terms of μ. We remember $K_0(E,R)=\pi_0K(E,R)$ and recover $\beta_{E\downarrow B}$ as a composition

$$K_0(E,R)\to[A(E),K(R)]\to[B,K(R)],$$

where the first arrow is obtained from μ by adjunction and the second is composition with $\chi(p)$. Compare Observation 6.4.

Assembly (second encounter) and coassembly

Here we need an explicit description of the assembly transformation $\alpha_{\mathbf{J}}\colon \mathbf{J}^{\%}\to\mathbf{J}$ for a *covariant* functor $\mathbf{J}$ from compact ENR's to spectra. The standard description, taken from [WW1], is

$$\mathbf{J}^{\%}(Y)=\operatorname*{hocolim}_{x\in Y^s}\mathbf{J}(\Delta^{|x|})$$

where Y^s denotes the singular simplicial set of Y and the simplices of Y^s are viewed as objects of a category in the usual way. The characteristic maps $\Delta^{|x|}\to Y$ for the various x in Y^s give rise to a natural transformation $\mathbf{J}(\Delta^{|x|})\to\mathbf{J}(Y)$, which in turn induces the required map α from $\mathbf{J}^{\%}(Y)$, as defined just above, to $\mathbf{J}(Y)$.

Coassembly is the contravariant analog of assembly. Therefore let $\mathbf{J}$ be a *contravariant* homotopy functor from compact ENR's to spectra. Then there exists an essentially unique natural transformation $\omega\colon\mathbf{J}\to\mathbf{J}^{\mho}$ with the following properties:

(1) $\omega\colon\mathbf{J}(*)\to\mathbf{J}^{\mho}(*)$ is a weak equivalence;

(2) $\mathbf{J}^{\mho}(*)$ respects homotopy pushout squares and $\mathbf{J}(\varnothing)\simeq *$.

One possible definition of $\mathbf{J}^{\mho}$: First make sure that the values of $\mathbf{J}$ are Ω-spectra, made up of (geometric realizations of) Kan simplicial sets. Then let

$$\mathbf{J}^{\mho}(Y):=\operatorname*{holim}_{x\in Y^s}\mathbf{J}(\Delta^{|s|})$$

where the homotopy limit is again taken over the category of singular simplices of Y. Proofs are very much like those in the covariant case.

Now let $\mathbf{J}_i$ for $i=1,2$ be homotopy functors from the category of compact ENR's to the category of spectra. Suppose that $\mathbf{J}_1$ is contravariant and $\mathbf{J}_2$ is covariant. Let $\mathbf{J}_3$ be a single bispectrum. Let

$$\mu\colon\mathbf{J}_1(X)\wedge\mathbf{J}_2(X)\to\mathbf{J}_3$$

be a *dinatural* transformation. This is an informal extension of, or variation on, the definition of *dinatural* given in [Mac]; what it means here is that for any map $f\colon X\to Y$ of ENR's, the following diagram of bispectra is commutative:

$$\begin{tikzcd}
\mathbf{J}_1(Y)\wedge\mathbf{J}_2(X) \arrow[r] \arrow[d] & \mathbf{J}_1(X)\wedge\mathbf{J}_2(X) \arrow[d, "\mu"] \\
\mathbf{J}_1(Y)\wedge\mathbf{J}_2(Y) \arrow[r, "\mu"] & \mathbf{J}_3.
\end{tikzcd}$$

The example to have in mind is: $\mathbf{J}_1(X)=\mathbf{K}(X,R)$ and $\mathbf{J}_2(X)=\mathbf{A}(X)$ and $\mathbf{J}_3=\mathbf{K}(R)$ (bispectrum version) and μ as in the previous subsection.

OBSERVATION 8.13. *In this situation there exists a dinatural transformation*

$$\mathbf{J}_1^{\mho}(X)\wedge\mathbf{J}_2^{\%}(X)\to\mathbf{J}_3$$

making the following diagram of dinatural transformations commutative:

$$\begin{tikzcd}
\mathbf{J}_1\wedge\mathbf{J}_2^{\%} \arrow[r, "\omega\wedge\mathrm{id}"] \arrow[d, "\mathrm{id}\wedge\alpha"] & \mathbf{J}_1^{\mho}\wedge\mathbf{J}_2^{\%} \arrow[d] \\
\mathbf{J}_1\wedge\mathbf{J}_2 \arrow[r, "\mu"] & \mathbf{J}_3.
\end{tikzcd}$$

Proof. We use the above definitions of α and ω. Fix Y, a compact ENR. Write simp for the category of singular simplices of Y. For x in simp let $E(x)$ be the classifying space of the category of simp-objects under x, or equivalently, the classifying space of the category of $\mathrm{simp}^{\mathrm{op}}$-objects over x. Then

$$\mathbf{J}_1^{\mho}(Y)=\underset{x\in\mathrm{simp}^{\mathrm{op}}}{\mathrm{holim}}\,\mathbf{J}_1(\Delta^{|x|})=\mathrm{end}[(x,y)\mapsto\mathbf{J}_1(\Delta^{|x|})^{E(y)}],$$
$$\mathbf{J}_2^{\%}(Y)=\underset{x\in\mathrm{simp}}{\mathrm{hocolim}}\,\mathbf{J}_2(\Delta^{|x|})=\mathrm{coend}[(x,y)\mapsto E(x)_+\wedge\mathbf{J}_2(\Delta^{|y|})].$$

Here $\mathbf{J}_1(\Delta^{|x|})^{E(y)}$ denotes a mapping spectrum, obtained from $\mathbf{J}_1(\Delta^{|x|})$ by applying $\mathrm{map}(E(y),-)$ termwise.

Now define the missing map $\mathbf{J}_1^{\mho}(Y)\wedge\mathbf{J}_2^{\%}(Y)\to\mathbf{J}_3$ in such a way that, for each x in simp, the following commutes:

$$\begin{tikzcd}
\mathbf{J}_1^{\mho}(Y)\wedge(E(x)_+\wedge\mathbf{J}_2(\Delta^{|x|})) \arrow[r] \arrow[d] & \mathbf{J}_1^{\mho}(Y)\wedge\mathbf{J}_2^{\%}(Y) \arrow[d] \\
\mathbf{J}_1(\Delta^{|x|})^{E(x)}\wedge(E(x)_+\wedge\mathbf{J}_2(\Delta^{|x|})) \arrow[r] & \mathbf{J}_3
\end{tikzcd}$$

where the arrow in the lower row is the composition of an evaluation map to $\mathbf{J}_1(\Delta^{|x|})\wedge\mathbf{J}_2(\Delta^{|x|})$ with $\mu\colon\mathbf{J}_1(\Delta^{|x|})\wedge\mathbf{J}_2(\Delta^{|x|})\to\mathbf{J}_3$. □

Linear Riemann–Roch for topological manifolds

Example 8.14. With μ and $\mathbf{K}(E,R)$ as in the previous subsection, we easily get $\mathbf{K}(E,R)^{\mho}\simeq\mathbf{K}(R)^{E}$ and then, from Observation 8.13, a homotopy commutative square of bispectra

$$\begin{tikzcd}
\mathbf{K}(E,R)\wedge\mathbf{A}^{\%}(E) \arrow[r, "\omega\wedge\mathrm{id}"] \arrow[d, "\mathrm{id}\wedge\alpha"] & \mathbf{K}(R)^{E}\wedge\mathbf{A}^{\%}(E) \arrow[d] \\
\mathbf{K}(E,R)\wedge\mathbf{A}(E) \arrow[r, "\mu"] & \mathbf{K}(R)
\end{tikzcd}$$

and from that, a square of spaces and maps which commutes up to homotopy,

$$\begin{tikzcd} K_0(E,R)\times A^{\%}(E) \arrow[r] \arrow[d] & {[E,K(R)]}\times A^{\%}(E) \arrow[d] \\ K_0(E,R)\times A(E) \arrow[r] & K(R). \end{tikzcd} \tag{8.15}$$

If now $p\colon E\to B$ is a bundle with compact manifold fibers, then $\chi(p)\colon B\to A_B(E)$ has a canonical lift $B\to A_B^{\%}(E)$ by Theorem 8.4, and we obtain from (8.15) another square which commutes up to homotopy,

$$\begin{tikzcd} K_0(E,R)\times B \arrow[r, "\beta_E\times\mathrm{id}"] \arrow[d, "\mathrm{id}\times\chi(p)"] & {[E,K(R)]}\times B \arrow[d] \\ K_0(E,R)\times A(E) \arrow[r, "\mu"] & K(R). \end{tikzcd} \tag{8.16}$$

Definition 8.17. Let $t\colon[E,K(R)]\to[B,K(R)]$ be the adjoint of the right-hand vertical arrow in the commutative square (8.16).

THEOREM 8.18. $t\beta_E=\beta_{E\downarrow B}\colon K_0(E,R)\to[B,K(R)]$.

Proof. Clearly $t\beta_E$ is adjoint to the diagonal map in (8.16), from upper left-hand term to lower right-hand term. Our earlier description of $\beta_{E\downarrow B}$ in terms of μ shows that $\beta_{E\downarrow B}$ is also adjoint to the diagonal map in (8.16). □

8.19. *Summary.* Let $p\colon E\to B$ be a fibration with homotopy finitely dominated fibers. In and around Observation 6.4, we defined an 'algebraic' transfer map $\beta_{E\downarrow B}$ from $K_0(E,R)$ to $[B,K(R)]$ and showed that $\beta_{E\downarrow B}$ applied to the class of a bundle V of f.g. projective left R-modules on E can be thought of as a linearized fiberwise Euler characteristic of p, with coefficients in V.

Theorem 8.18 states that, in the case where p is a bundle of compact topological manifolds, the algebraic transfer $\beta_{E\downarrow B}$ is the composition of an obvious 'forgetful' homomorphism $\beta_E\colon K_0(E,R)\to[E,K(R)]$ with a geometric transfer homomorphism $t\colon[E,K(R)]\to[B,K(R)]$. In this sense, Theorem 8.18 is another Riemann–Roch theorem, formally quite similar to Theorem 8.11. However, t in Theorem 8.18 is not defined in fiber homotopy invariant terms, unlike tr_p^* in Theorem 8.11.

Reidemeister torsion

Let $p\colon E\to B$ be a fibration with homotopy finitely dominated fibers E_b, for $b\in B$. Let R be a ring and let V be a bundle of f.g. projective left R-modules on E. For a space X

over E with reference map $r: X \to E$ let

$$\Phi^h(X) := \text{hofiber}[\lambda_V r_*: A(X) \to K(R)],$$
$$\Phi^t(X) := \text{hofiber}[\lambda_V r_* \alpha: A^{\%}(X) \to K(R)],$$
$$\Phi^d(X) := \text{hofiber}[\lambda_V r_* \alpha\eta: Q(X_+) \to K(R)],$$

with $\lambda_V: A(E) \to K(R)$ as in Observation 6.4 and α, η the assembly and unit maps (defined under mild conditions on X which we suppress). Now suppose that $H_i(E_b; V)=0$ for all $b \in B$; equivalently, the singular chain complexes with V-coefficients of the fibers E_b are all contractible (=exact). Then the composition

$$B \xrightarrow{\chi_h(p)} A_B(E) \to K(R)$$

is canonically nullhomotopic. In other words, we have lifted $\chi_h(p)$ to a section of a fibration

$$\Phi^h_B(E) \to B$$

with fiber $\Phi^h(E_b)$ over $b \in B$. We call this lift the *homotopy Reidemeister torsion* of p. If $p: E \to B$ happens to be a bundle of compact topological manifolds, then using Theorem 8.4 we have a lift of the excisive family characteristic $\chi(p): B \to A^{\%}_B(E)$ to a section of

$$\Phi^t_B(E) \to B.$$

This is the *topological* Reidemeister torsion of p. And finally, if p happens to be a bundle of compact regular manifolds, we have from Theorem 8.5 a lift of the Becker–Gottlieb–Dold section $B \to (Q_+)_B(E)$ to a section of

$$\Phi^d_B(E) \to B,$$

which we call the *regular* Reidemeister torsion of p. In particular, if p is a bundle of compact *smooth* manifolds, its regular Reidemeister torsion is defined—in that case we also call it the smooth Reidemeister torsion.

Remark. Earlier, Igusa and Klein [IK], [I3] used parametrized generalized Morse functions to define the (parametrized) Reidemeister torsion of a bundle of smooth compact manifolds, $p: E \to B$. Their Reidemeister torsion is a map from B to the homotopy fiber of $\lambda_V \alpha\eta: Q(E_+) \to K(R)$. Also Bismut and Lott have used a differential form version of their Riemann–Roch theorem to construct an analytic version of parametrized Reidemeister torsion in the special case $R=\mathbb{C}$. It is not clear at this stage whether the parametrized Reidemeister torsions produced by [BL] are in agreement with ours. In the unparametrized setting, $B=*$, they are; this is the theorem of Cheeger and Müller, [Che], [Mü]. Put differently again, our Theorem 8.5 and [BL] together make it possible to *state* a family version of the Cheeger–Müller theorem; we would like to know whether this is in fact true.

Part III. Converse Riemann–Roch theory

9. Waldhausen's theorems in h-cobordism theory

The theorems in question state, briefly, the following. Firstly, the space of stabilized topological h-cobordisms on a compact topological manifold L is homotopy equivalent to the homotopy fiber of the assembly map $\alpha: A^{\%}(L) \to A(L)$. Secondly, the space of stabilized smooth h-cobordisms on a compact smooth manifold L is homotopy equivalent to the homotopy fiber of the map $\alpha\eta: Q(L_+) \to A(L)$, where $\eta: Q(L_+) \to A^{\%}(L)$ is the unit transformation. More details are given below; the theorems are stated again in Corollaries 9.7 and 9.9.

These theorems have a long history. Modulo some stability theory they subsume for example the h-cobordism theorem of Smale [Sm], the classification of h-cobordisms due to Barden–Mazur–Stallings [Maz], the pseudoisotopy implies isotopy theorem of Cerf [Ce] and the smooth pseudoisotopy classification of Hatcher and Wagoner [HaW], with corrections in [I1]. Most of the required stability theory can be found in Igusa's big work [I2].

However that may be, to use the Waldhausen theorems about h-cobordism spaces, we need to have a user-friendly description of the homotopy equivalences involved. We do not wish to prove again that they *are* homotopy equivalences, but we offer a guide to Waldhausen's proofs.

Retractive manifolds

Definition 9.1. A retractive manifold over a compact topological manifold L is a compact subset N of $L \times I$ which

(1) contains a neighborhood of $L \times 0$ in $L \times I$,

(2) is a manifold (with boundary) in its own right.

A retractive manifold N over L becomes a retractive space $N \rightleftarrows L$ with retraction equal to the projection and zero-section equal to the inclusion of $L \cong L \times 0$ in N.

A *family* of retractive manifolds over L, parametrized by Δ^j, is a retractive manifold over $\Delta^j \times L$ for which the composite projection $N \to \Delta^j \times L \to \Delta^j$ is a bundle. Such families are the j-simplices of a simplicial set $\mathfrak{P}(L)$. (The letter $\mathfrak{P}$ indicates *partitions*; see the remark just below.) We abbreviate

$$\mathcal{P}(L) := |\mathrm{simp}(\mathfrak{P}(L))|.$$

This will be slightly more important to us than the realization $|\mathfrak{P}(L)|$, although it is $|\mathfrak{P}(L)|$ which carries a tautological bundle of retractive manifolds over L. The two are of course homotopy equivalent; see §1.6.

For a retractive manifold N over L, let

$$\sigma(N) := w(N \times I) \subset (L \times I) \times I$$

where $w(z,s,t)=(z,t,s)$ for $(z,s,t) \in L \times I \times I$. Then $\sigma(N)$ is a retractive manifold over $L \times I$. This procedure extends in a straightforward way to families and gives the stabilization map $\sigma: \mathcal{P}(L) \to \mathcal{P}(L \times I)$. We let

$$\mathcal{P}_{\$}(L) := \operatorname*{hocolim}_{r} \mathcal{P}(L \times I^r).$$

We get a map $\varkappa_h: \mathcal{P}(L) \to A(L)$ by viewing retractive manifolds over L as retractive spaces over L. More precisely, we have an obvious functor from $\mathrm{simp}(\mathfrak{P}(L))$ to $w\mathcal{R}^{fd}(L)$ which induces a map from $\mathcal{P}(L)=|\mathrm{simp}(\mathfrak{P}(L))|$ to $|w\mathcal{R}^{fd}(L)| \subset A(L)$. We can extend $\varkappa_h: \mathcal{P}(L) \to A(L)$ to a map

$$\varkappa_h: \mathcal{P}_{\$}(L) \to A(L)$$

in the following way. We identify $\mathcal{P}_{\$}(L)$ with a subspace of the classifying space of the category whose objects are pairs (r,N) with $r \geqslant 0$ and N in $\mathrm{simp}(\mathfrak{P}(L \times I^r))$ and where a morphism from (q,M) to (r,N) is a morphism from M to $\sigma^{q-r}N$ in $\mathrm{simp}(\mathfrak{P}(L \times I^q))$ if $q \geqslant r$. (There is no morphism if $q<r$.) Then we make a functor from that category to $w\mathcal{R}^{fd}(L)$ by associating to an object (r,N) the pushout of $N \leftarrow (L \times I^r) \times 0 \to L$, viewed as a retractive space over L.

Remark. What we call a retractive manifold N over L is viewed in [W1] as the lower half of a partition (N,M) of $L \times I$, with M equal to the closure of the complement of N in $L \times I$. Waldhausen imposes a few more conditions to ensure that both parts N, M of a partition (N,M) are submanifolds of codimension zero in $L \times I$.

Naturality and addition laws

It is clear from the above definitions that a homeomorphism $L \to L'$ of compact manifolds induces homeomorphisms $\mathcal{P}(L) \to \mathcal{P}(L')$ and $\mathcal{P}_{\$}(L) \to \mathcal{P}_{\$}(L')$. This is not much in the way of naturality; we need rather more and we can get it by imposing some boundary conditions on retractive manifolds over L.

Definition 9.2. Let $(L; \partial_0 L, \partial_1 L)$ be a compact manifold triad; i.e., the boundary ∂L of L is the union of two compact submanifolds $\partial_0 L$ and $\partial_1 L$ whose intersection is $\partial(\partial_0 L)=\partial(\partial_1 L)$. By a *retractive manifold over L rel $\partial_0 L$* we mean a compact subset N of $L \times I$ which

(1) contains a neighborhood of $(L \times 0) \cup (\partial_0 L \times I)$ in $L \times I$,
(2) is a manifold (with boundary) in its own right.

A retractive manifold N over L rel $\partial_0 L$ is also a retractive manifold over L. Thus the retractive manifolds over L rel $\partial_0 L$ determine a simplicial subset $\mathfrak{P}(L;\partial_0 L)$ of $\mathfrak{P}(L)$. We let $\mathcal{P}(L;\partial_0 L):=|\mathrm{simp}(\mathfrak{P}(L;\partial_0 L))|$ and

$$\mathcal{P}_{\$}(L;\partial_0 L):=\operatorname*{hocolim}_r \mathcal{P}(L\times I^r;\partial_0 L\times I^r).$$

A retractive manifold N over L rel $\partial_0 L$ determines a retractive space over L: the pushout of

$$N \hookleftarrow (L\times 0)\cup(\partial_0 L\times I) \xrightarrow{\text{proj}} L.$$

Extending this to families, we obtain by analogy with Definition 9.1 a map

$$\varkappa_h\colon \mathcal{P}_{\$}(L;\partial_0 L)\to A(L).$$

Now assume given two compact manifold triads $(L;\partial_0 L,\partial_1 L)$ and $(L';\partial_0 L',\partial_1 L')$ and an embedding (read: injective continuous map) $f\colon L'\to L$ such that $f(\partial_1 L')$ is contained in $\partial_1 L$. With a retractive manifold N over L' rel $\partial_0 L'$ we can then associate a retractive manifold f_*N over L rel $\partial_0 L$: the union of $(f\times \mathrm{id}_I)(N)$ and $(L\setminus f(L'))\times I$ in $L\times I$. In this way we obtain induced maps

$$f_*\colon \mathcal{P}(L';\partial_0 L')\to \mathcal{P}(L;\partial_0 L),$$
$$f_*\colon \mathcal{P}_{\$}(L';\partial_0 L')\to \mathcal{P}_{\$}(L;\partial_0 L).$$

We make a category $\mathcal{W}$ having the compact manifold triads $(L;\partial_0 L,\partial_1 L)$ as its objects. A morphism from $(L';\partial_0 L',\partial_1 L')$ to $(L;\partial_0 L,\partial_1 L)$ is an embedding $f\colon L'\to L\times I^k$, with $k=\dim(L')-\dim(L)$, such that $f(\partial_1 L')$ is contained in $\partial_1(L\times I^k)$, the closure of the complement in $\partial(L\times I^k)$ of $\partial_0 L\times I^k$. There are no morphisms from $(L';\partial_0 L',\partial_1 L')$ to $(L;\partial_0 L,\partial_1 L)$ if $\dim(L')<\dim(L)$.

A morphism in $\mathcal{W}$ is a *homotopy equivalence* if the underlying map (from L' to $L\times I^k$ in the above notation) is a homotopy equivalence. A functor Z from $\mathcal{W}$ to spaces is *homotopy invariant* if it takes homotopy equivalences in $\mathcal{W}$ to homotopy equivalences of spaces.

A functor Z from $\mathcal{W}$ to spaces is *separating* if it comes with a natural and associative homeomorphism

$$Z(L\amalg L')\cong Z(L)\times Z(L'),$$

where L and L' are variable objects of $\mathcal{W}$ (in shorthand). *Associativity* means that the two resulting ways of identifying $Z(L\amalg L'\amalg L'')$ with $Z(L)\times Z(L')\times Z(L'')$ agree.

Example. We note that $(L;\partial_0 L,\partial_1 L)\mapsto\mathcal{P}_\$(L;\partial_0 L)$ is a homotopy invariant and separating functor on $\mathcal{W}$. Namely, an embedding $f\colon L'\to L\times I^k$ as in Definition 9.2, definition of $\mathcal{W}$, induces maps

$$\mathcal{P}_\$(L';\partial_0 L')\to\mathcal{P}_\$(L\times I^k;\partial_0 L\times I^k)\hookrightarrow\mathcal{P}_\$(L;\partial_0 L).$$

Example. There is a functor of the form $(L;\partial_0 L,\partial_1 L)\mapsto A(L)$ on $\mathcal{W}$. An embedding $f\colon L'\to L\times I^k$ as in Definition 9.2, definition of $\mathcal{W}$, induces a map $A(L')\to A(L)$ which is the composition of $A(L')\to A(L\times I^k)$ induced by f and $A(L\times I^k)\to A(L)$ induced by the projection $L\times I^k\to L$. This functor is also homotopy invariant and separating.

Example. The map $\varkappa_h$ from $\mathcal{P}_\$(L;\partial_0 L)$ to $A(L)$ constructed in Definition 9.2 is unfortunately not a natural transformation between functors on $\mathcal{W}$. In other words, for a morphism $f\colon(L';\partial_0 L',\partial_1 L')\to(L;\partial_0 L,\partial_1 L)$ in $\mathcal{W}$, the diagram

$$\begin{tikzcd}
\mathcal{P}(L';\partial_0 L') \arrow[r, "\varkappa_h"] \arrow[d, "f_*"] & A(L') \arrow[d, "f_*"] \\
\mathcal{P}(L;\partial_0 L) \arrow[r, "\varkappa_h"] & A(L)
\end{tikzcd}$$

is usually not commutative. It is however commutative up to a preferred homotopy, essentially because of a preferred weak equivalence $\varkappa_h(f_*N)\to\varkappa_h(N)$ between retractive spaces over L, for every retractive manifold N over L' rel $\partial_0 L'$. In the same spirit, there are preferred higher homotopies corresponding to strings of composable morphisms in $\mathcal{W}$, leading after all to a natural transformation

$$\operatorname*{hocolim}_{(L';\partial_0 L',\partial_1 L')\to(L;\partial_0 L,\partial_1 L)}\mathcal{P}(L';\partial_0 L',\partial_1 L')\to A(L).$$

Here the hocolim is taken over the category of $\mathcal{W}$-objects over $(L;\partial_0 L,\partial_1 L)$. Note that it contains $\mathcal{P}_\$(L;\partial_0 L)$ as a deformation retract. We omit the details.

LEMMA 9.3. *Let Z be a homotopy invariant and separating functor on $\mathcal{W}$. Then Z can be refined to a functor from $\mathcal{W}$ to Γ-spaces. More precisely, there exist a functor $Z^\sharp$ from $\mathcal{W}$ to Γ-spaces and a natural homotopy equivalence from $Z^\sharp(L)(1)$ to $Z(L)$ where $Z^\sharp(L)(1)$ is the underlying space of the Γ-space $Z^\sharp(L)$, for $L=(L;\partial_0 L,\partial_1 L)$ in $\mathcal{W}$.*

We will prove Lemma 9.3 at the end of this chapter. Here we only describe the resulting addition law on $Z(L)$, up to homotopy. This is given by

$$Z(L)\times Z(L)\simeq Z(L\times I')\to Z(L\times I)\simeq Z(L)$$

where $I'=[0,\frac{1}{3}]\cup[\frac{2}{3},1]\subset I$ and the arrow is induced by inclusion. We assume of course that L is an arbitrary object of $\mathcal{W}$ and make $L\times I'$ and $L\times I$ into objects of $\mathcal{W}$ by letting $\partial_0(L\times I')=\partial_0 L\times I'$ and $\partial_0(L\times I)=\partial_0 L\times I$.

Remark. Because of Lemma 9.3 and the preceding examples, $\varkappa_h$ in Definition 9.1 has now been promoted to a map respecting certain homotopy everything addition laws. Note that $A(L)$ with the addition law just sketched is group-like, i.e., the abelian monoid $\pi_0 A(L)$ is an abelian group; but $\mathcal{P}_\$(L;\partial_0 L)$ is not group-like unless $L=\varnothing$. Hence $A(L)$ is an infinite loop space, but $\mathcal{P}_\$(L)$ is not.

The theorems

We now state the Waldhausen theorems in h-cobordism theory, in the relatively user-friendly language of Waldhausen's "manifold approach" [W1], [W3]. It is important to realize that the manifold approach is not a self-contained approach to the theory—it relies heavily on the combinatorial approach of [W2, §3] and [WV1], [WV2] for proofs. We give detailed references in the next subsection.

9.4. *Notation.* Define $\mathcal{H}(L)$ like $\mathcal{P}(L)$, allowing however only retractive manifolds N in which N is an h-cobordism on $L\times 0$ (and families of such, parametrized by Δ^k for some k). Define $\mathcal{P}^j(L)$ like $\mathcal{P}(L)$, allowing only retractive manifolds N in which N is obtained from an h-cobordism by attaching a finite number of handles of index j inside $L\times I$ and families of such. (Because of the retraction, the attaching maps for such handles will always be nullhomotopic; and if $\dim(L)>2j+1$ they will be trivial.) There are stabilized versions

$$\mathcal{H}_\$(L),\quad \mathcal{P}^j_\$(L).$$

They come with homotopy everything addition laws, induced/restricted from the one on $\mathcal{P}_\$(L)$ described in and around Lemma 9.3.

The composition of the inclusion $\mathcal{H}_\$(L)\to\mathcal{P}_\$(L)$ with $\varkappa_h\colon\mathcal{P}_\$(L)\to A(L)$ is canonically nullhomotopic, in other words it has a canonical (nameless) extension

$$\operatorname{cone}(\mathcal{H}_\$(L))\to A(L).$$

Namely, $\varkappa_h$ restricted to $\mathcal{H}_\$(L)$ factors through the classifying space of the full subcategory of $w\mathcal{R}^{fd}(L)$ consisting of all objects for which the morphism to the zero-object is a weak equivalence. The subcategory has a terminal object, so its classifying space has a canonical contraction.

THEOREM 9.5 (fibration theorem). *For any compact topological manifold L, the commutative square*

$$\begin{tikzcd} \mathcal{H}_\$(L) \arrow[r, "\subset"] \arrow[d, "\subset"] & \mathcal{P}^j_\$(L) \arrow[d, "\varkappa_h"] \\ \operatorname{cone}(\mathcal{H}_\$(L)) \arrow[r] & A(L) \end{tikzcd}$$

becomes $(j-\varepsilon)$-cartesian upon applying group completion ΩB. Here ε is some integer independent of j and L.

Remark. Note that $\mathcal{H}_\$(L)$ is a connected component of $\mathcal{P}^j_\$(L)$; see §9.4. Group completion changes the homotopy type of only one of the four spaces in the diagram, $\mathcal{P}^j_\$(L)$, since the other three are group-like. Group completion is possible because we can promote all the maps in the diagram to maps of Γ-spaces; the details are as in Lemma 9.3.

We denote by $B^\infty Z$ the spectrum obtained by Segal's method from a Γ-space with underlying space Z.

THEOREM 9.6 (excision theorem). *Let L_a, L_b and L_c be compact manifold triads. Let homeomorphisms $f\colon \partial_1 L_a \cong \partial_0 L_b$ and $g\colon \partial_1 L_b \cong \partial_0 L_c$ be given. The following square of maps of spectra, induced by evident morphisms in $\mathcal{W}$, is $(j-\varepsilon)$-cartesian:*

$$\begin{tikzcd} B^\infty\mathcal{P}^j_\$(L_b;\partial L_b) \arrow[r] \arrow[d] & B^\infty\mathcal{P}^j_\$(L_b\amalg_g L_c;\partial_0 L_b) \arrow[d] \\ B^\infty\mathcal{P}^j_\$(L_a\amalg_f L_b;\partial_1 L_b) \arrow[r] & B^\infty\mathcal{P}^j_\$(L_a\amalg_f L_b\amalg_g L_c). \end{tikzcd}$$

COROLLARY 9.7. *For a compact topological manifold we have*

$$\mathcal{H}_\$(L) \simeq \operatorname{hofiber}[\alpha\colon A^\%(L)\to A(L)].$$

Let $d\mathcal{H}_\$(L)$ and $d\mathcal{P}^j_\$(L)$ be the smooth analogs of $\mathcal{H}_\$(L)$ and $\mathcal{P}^j_\$(L)$, respectively. They are defined for smooth compact L. (We give more details in the next subsection.) Here we only need the case $L=*$. Note that $\pi_0 B^\infty d\mathcal{P}^j_\$(*)\cong\mathbb{Z}$. The generator can be represented by some map of spectra

$$u_j\colon \mathbf{S}^0 \to B^\infty d\mathcal{P}^j_\$(*).$$

THEOREM 9.8 (vanishing theorem). *The connectivity of u_j tends to infinity as $j\to\infty$.*

COROLLARY 9.9. *For a compact smooth manifold L we have*

$$d\mathcal{H}_{\$}(L) \simeq \text{hofiber}[\alpha\eta\colon Q(L_+) \to A(L)].$$

Remark. We present Corollary 9.7 as a corollary to Theorems 9.5 and 9.6 because this is reasonable from a user perspective. Namely, Theorems 9.5 and 9.6 are matter-of-fact statements and they lead in a highly intelligible way to a homotopy fiber sequence

$$\mathcal{H}_{\$}(L) \to A^{\%}(L) \to A(L).$$

The hierarchy of proofs is different, though. Waldhausen proves Corollary 9.7 first and foremost. He offers Theorems 9.5 and 9.6 as afterthoughts.

Guide to the literature

The fibration theorem. This is simply the combination of parts (1) and (2) of Theorem 1 of [W1]. Both are proved in §5 of [W1]; in the proof, however, Waldhausen uses a nonstandard definition of $A(L)$. In Theorem 1.7.1 of [W2], this is shown to be equivalent to the standard definition. □

The excision theorem. This is the hardest of the three. The proof is outlined in §5 of [W1] and to some extent in [W4]. The details of the proof can be found in §3 of [W2] and the preprints [WV1], [WV2]. The papers [Ste] and [Cha2] are closely related to [WV1] and [WV2], respectively, although unlike [WV1] and [WV2] they were not written to complement §3 of [W2] exactly. We feel compelled to present yet another outline, as follows.

A disk bundle transfer argument [BuL2] allows us to reduce to the case where L is a piecewise linear manifold. In that case, the piecewise linear version of $\mathcal{P}_{\$}(L)$ is defined. Triangulation theory shows that it is homotopy equivalent to the original. Hence it suffices to prove the piecewise linear version of Theorem 9.6. The proof is in two major steps.

The first step is to construct a map t from the square in Theorem 9.5 (in the piecewise linear setting) to the homotopy cartesian square

$$\begin{tikzcd} A_{\%}(L) \arrow[r] \arrow[d] & A^{\%}(L) \arrow[d, "\alpha"] \\ * \simeq \text{hofiber}(\text{id}_{A(L)}) \arrow[r] & A(L) \end{tikzcd}$$

where α is the assembly map, compare §8, and $A_{\%}(L)$ is the homotopy fiber of the assembly map. (Think of the two commutative squares as two functors defined on a certain category with four objects; and think of t as a natural transformation between these two functors.) The construction of t is achieved in §3 of [W2], modulo some fine points treated in [WV1]. More specifically, §3 of [W2] supplies convenient combinatorial models for $A_{\%}(L)$ and $A^{\%}(L)$, in the setting of simplicial sets; [WV1] makes the transition to the piecewise linear setting and interprets those models, or suitable subspaces, as classifying spaces for piecewise linear fibrations, subject to suitable conditions. The piecewise linear fibrations of interest here are the 'tautological bundles' on $|\mathfrak{P}(L\times I^r)|\simeq \mathcal{P}(L\times I^r)$, for all $r\geqslant 0$; see also the remark just below. The maps which classify them (suitably restricted and followed by appropriate inclusions) make up $t^{\urcorner}$, the upper right-hand component of t. The lower right-hand coordinate $t_{\lrcorner}$ of t is the identity $A(L)\to A(L)$. The remaining two components ${}^{\ulcorner}t$ and ${}_{\llcorner}t$ of t are essentially restrictions of $t^{\urcorner}$ and $t_{\lrcorner}$, respectively.

The second step is to show that ${}^{\ulcorner}t\colon \mathcal{H}_{\$}(L)\to A_{\%}(L)$ is a homotopy equivalence. This is done in [WV2]. The proof uses an identification of $A_{\%}(L)$ with the loop space of the so-called piecewise linear Whitehead space of L; this is Theorem 3.3.1 of [W2].

It follows from these two steps, in conjunction with the Fibration Theorem 9.5, that the map

$$t^{\urcorner}\colon \mathcal{P}^j_{\$}(L)\to A^{\%}(L)$$

is $(j-\varepsilon)$-connected for some ε independent of j. A slight modification gives a $(j-\varepsilon)$-connected map

$$\mathcal{P}^j_{\$}(L;\partial_0 L)\to A^{\%}(L;\partial_0 L)$$

defined for any compact manifold triad $(L;\partial_0 L,\partial_1 L)$. This is sufficiently natural in the variable $(L;\partial_0 L,\partial_1 L)$, viewed as an object in $\mathcal{W}$; compare Definition 9.2. Now Theorem 9.6 follows since $A^{\%}$ is excisive. □

The vanishing theorem. This has another formulation in functor calculus language. We begin with the other formulation. For a space X, let $\alpha\eta\colon Q(X_+)\to A(X)$ be the composition of the unit transformation $Q(X_+)\to A^{\%}(X)$ with the assembly map $A^{\%}(X)\to A(X)$.

There exists an integer c such that the following commutative square is $(2r-c)$-cartesian for any r-connected X:

$$\begin{array}{ccc} Q(X_+) & \xrightarrow{\alpha\eta} & A(X) \\ \downarrow & & \downarrow \\ Q(*_+) & \xrightarrow{\alpha\eta} & A(*). \end{array}$$

Modulo the fact that the functor A is analytic, see [Go1], [Go2], this is equivalent to saying that *stabilization* turns $\alpha\eta$ into a homotopy equivalence,

$$Q(X_+) \simeq Q^S(X_+) \xrightarrow{(\alpha\eta)^S} A^S(X).$$

In this form the statement is proved in [W3]. The proof uses the manifold approach. An alternative proof which does not use manifolds has recently been given by [Du]. See also the earlier work by [DuM].

Now we translate the above into manifold approach language. Smoothing theory [Mo], [KS], [BuL1] applied to the maps

$$d\mathcal{H}(L \times I^r) \to \mathcal{H}(L \times I^r),$$
$$d\mathcal{P}^j(L \times I^r) \to \mathcal{P}^j(L \times I^r)$$

and the group completion theorem [A2] show that for any compact smooth L, the commutative square

$$\begin{array}{ccc} d\mathcal{H}_\$(L) & \longrightarrow & d\mathcal{P}^j_\$(L) \\ \downarrow & & \downarrow \\ \mathcal{H}_\$(L) & \longrightarrow & \mathcal{P}^j_\$(L) \end{array}$$

becomes $(j-\varepsilon)$-cartesian after group completion, for some ε independent of j and L. It follows immediately that the Fibration and Excision Theorems 9.5 and 9.6 have smooth versions involving $d\mathcal{H}_\$(L)$ and $d\mathcal{P}^j_\$(L)$ instead of $\mathcal{H}_\$(L)$ and $\mathcal{P}^j_\$(L)$. Now suppose that $L = \mathbb{S}^n$ for some large n and let $D \subset \mathbb{S}^n$ be an n-disk. Choose $j \gg 2n$. Then we have a commutative diagram

$$\begin{array}{ccccc} d\mathcal{H}_\$(D) & \xrightarrow{\subset} & (\Omega B)d\mathcal{P}^j_\$(D) & \xrightarrow{\varkappa_h} & A(D) \\ \downarrow{\scriptstyle\subset} & & \downarrow{\scriptstyle\subset} & & \downarrow{\scriptstyle\subset} \\ d\mathcal{H}_\$(\mathbb{S}^n) & \xrightarrow{\subset} & (\Omega B)d\mathcal{P}^j_\$(\mathbb{S}^n) & \xrightarrow{\varkappa_h} & A(\mathbb{S}^n) \end{array}$$

with vertical arrows induced by inclusion, where the rows approximate homotopy fiber sequences by the smooth version of Theorem 9.5. It is a consequence of Morlet's disjunction lemma [BuLR] that the left-hand vertical homotopy fiber is approximately $2n$-connected. It follows that the map from middle vertical homotopy fiber to right-hand vertical homotopy fiber is $(2n-\varepsilon)$-connected for some ε independent of n. But the middle homotopy fiber has the $(2n-\varepsilon)$-type of an $(n-1)$-fold delooping of

$$(\Omega B)d\mathcal{P}^j_\$(*)$$

by the smooth version of Theorem 9.6. The right-hand vertical homotopy fiber has the $(2n-\varepsilon)$-type of an $(n-1)$-sphere by the result about A^S. Summarizing, for sufficiently large j the connectivity of u_j in Theorem 9.8 is at least $n-\varepsilon$. But n was arbitrary. $\square$

The corollaries. We remind the reader that the deduction of Corollary 9.7 from Theorems 9.5 and 9.6 is intended as a formal exercise; see the remark after Corollary 9.9. The purpose of the exercise is to clarify what that homotopy equivalence in Corollary 9.7 is.

The key ingredient here is *upper stabilization*, a natural map $\bar{\sigma}: \mathcal{P}(L) \to \mathcal{P}(L \times I)$ closely related to (lower) stabilization σ. It is given by $\bar{\sigma}(N) := \sigma(N) \cup (L \times J) \times I$ where $J := [0, \frac{1}{3}] \cup [\frac{2}{3}, 1]$ for a retractive manifold N over L. Upper stabilization commutes with lower stabilization, in other words,

$$\begin{tikzcd}
\mathcal{P}(L) \arrow[r, "\sigma"] \arrow[d, "\bar{\sigma}"] & \mathcal{P}(L \times I^{\{a\}}) \arrow[d, "\bar{\sigma}"] \\
\mathcal{P}(L \times I^{\{b\}}) \arrow[r, "\sigma"] & \mathcal{P}(L \times I^{\{a,b\}})
\end{tikzcd} \tag{9.10}$$

commutes. Hence $\bar{\sigma}$ stabilizes to a map $\bar{\sigma}: \mathcal{P}_\$(L) \to \mathcal{P}_\$(L \times I)$. Mostly for notational convenience we compose with $\iota: \mathcal{P}_\$(L \times I) \to \mathcal{P}_\$(L)$, the inclusion of a subtelescope. This gives $\iota\bar{\sigma}: \mathcal{P}_\$(L) \to \mathcal{P}_\$(L)$. Restricting $\iota\bar{\sigma}$ we obtain maps

$$\begin{aligned}
&\mathcal{H}_\$(L) \to \mathcal{H}_\$(L), \\
&\mathcal{P}^j_\$(L) \to \mathcal{P}^{j+1}_\$(L),
\end{aligned}$$

of which the first is a homotopy equivalence. — On the K-theoretic side, we have a closely related map $A(L) \to A(L)$ induced by the fiberwise suspension Σ_L. It is a homotopy equivalence. The square

$$\begin{tikzcd}
\mathcal{P}^j_\$(L) \arrow[r, "\iota\bar{\sigma}"] \arrow[d, "\varkappa_h"] & \mathcal{P}^{j+1}_\$(L) \arrow[d, "\varkappa_h"] \\
A(L) \arrow[r, "\Sigma_L"] & A(L)
\end{tikzcd}$$

commutes up to a preferred homotopy. Summarizing the reasoning so far: It is possible to stabilize Theorems 9.5, 9.6 and 9.8 with respect to j. This procedure does not alter the homotopy types of the terms $\mathcal{H}_\$(L)$ and $A(L)$ in the diagram in Theorem 9.5.

Now we can do our deduction: By Theorem 9.5, stabilized with respect to j and promoted to a statement about functors on $\mathcal{W}$, there is a homotopy fiber sequence

$$\mathcal{H}_\$(L; \partial_0 L) \to \operatorname*{hocolim}_j (\Omega B)\mathcal{P}^j_\$(L; \partial_0 L) \to A(L).$$

By Theorem 9.6, the term in the middle is an excisive functor of the variable $(L;\partial_0 L,\partial_1 L)$, in $\mathcal{W}$. By Theorem 9.5 again, the map on the right is an assembly map, since it has a contractible homotopy fiber when L is contractible. (To make sense of this statement, replace the infinite loop spaces involved by the associated spectra. Note that $\mathcal{H}(\mathbb{D}^i)$ is contractible for all $i\geqslant 0$ by the Alexander trick.) Hence there is a homotopy fiber sequence of infinite loop spaces

$$\mathcal{H}_{\$}(L)\to A^{\%}(L)\to A(L).$$

In the case where L is smooth, we need the smooth versions of Theorems 9.5 and 9.6, which we deduce from the topological versions using smoothing theory. We find using the smooth Theorems 9.6 and 9.8 that the stabilization with respect to j of

$$(\Omega B)d\mathcal{P}^j_{\$}(L;\partial_0 L)$$

is an excisive functor of $(L;\partial_0 L,\partial_1 L)$ and is homotopy equivalent (as an infinite loop space) to $Q(L_+)$ when L is contractible; hence it is always homotopy equivalent to $Q(L_+)$ as an infinite loop space. The smooth version of Theorem 9.5, stabilized with respect to j, therefore implies a homotopy fiber sequence

$$d\mathcal{H}_{\$}(L)\to Q(L_+)\to A(L). \qquad \square$$

Smooth and regular retractive manifolds. Here we explain exactly what is meant by $d\mathcal{P}(L)$. There is a need to explain because, without special precautions, the stabilization σ will introduce "corners". Actually we find it more convenient to work with regular manifolds, for this reason.

Suppose therefore that L^n is regular; see Definition 5.6. Then $L\times I$ has a canonical regular structure (q,U,j) in which $q\colon K\to L\times I$ is an $(n+1)$-disk bundle, U is an open neighborhood of the diagonal in a certain subspace of $(L\times I)\times(L\times I)$ containing the diagonal, and $j\colon U\to K$ is an embedding over $L\times I$, subject to certain conditions. A *regular retractive manifold* over L is a retractive manifold N over L together with

(1) a regular structure (q',U',j') on N,

(2) a one-parameter family, parametrized by $[0,1]$, of regular structures on $\operatorname{int}(N)\cup(\operatorname{int}(L)\times 0)$, specializing to the restrictions of (q',U',j') and (q,U,j) for the parameter values 0 and 1, respectively.

It is straightforward to define families of regular retractive manifolds over L, parametrized by Δ^j for some j. These are the j-simplices of a simplicial set $d\mathfrak{P}(L)$. Let $d\mathcal{P}(L)=|\operatorname{simp}(d\mathfrak{P}(L))|$. We also define $d\mathcal{H}(L)\subset d\mathcal{P}(L)$ as the space of regular (invertible) h-cobordisms on L.

The letter d is meant to indicate something like *differentiable relative to L*. Indeed smoothing theory shows (see the remark just below) that, in the case where L is smooth, $d\mathcal{P}(L)$ does have the same homotopy type as the space of smooth retractive manifolds over L according to your favorite definition, provided $\dim(L)\neq 3,4$. Also $d\mathcal{H}(L)$ in that case has the same homotopy type as the space of smooth (invertible) h-cobordisms on L, provided $\dim(L)\neq 3,4$.

The stabilization $\sigma(N)$ of a regular retractive manifold N over L is in a canonical way a regular retractive manifold over $L\times I$. In this way we get a stabilization map $\sigma\colon d\mathcal{P}(L)\to d\mathcal{P}(L\times I)$ and

$$d\mathcal{P}_{\$}(L) := \operatorname*{hocolim}_{s} d\mathcal{P}(L\times I^{s}),$$
$$d\mathcal{H}_{\$}(L) := \operatorname*{hocolim}_{s} d\mathcal{H}(L\times I^{s}).$$

We leave it to the reader to define an upper stabilization map $\bar{\sigma}$ from $d\mathcal{P}(L)$ to $d\mathcal{P}(L\times I)$, compatible with the original $\bar{\sigma}$ from $\mathcal{P}_{\$}(L)$ to $\mathcal{P}_{\$}(L\times I)$. This appears to be canonical up to contractible choice only, but it is easy to define it in such a way that the regular analog of (9.10) commutes. Hence it stabilizes (by means of lower stabilization σ) to a map

$$\bar{\sigma}\colon d\mathcal{P}_{\$}(L)\to d\mathcal{P}_{\$}(L\times I).$$

Remark. Suppose that N is a retractive manifold over L, where L happens to be equipped with a smooth structure. The smooth structure on L determines, up to contractible choice, a regular structure on L. Assuming $\dim(L)\neq 3,4$ we will show that the space X_d of smooth structures on N (as a retractive manifold over L) maps by a homotopy equivalence to the space X_r of regular structures on N (as a retractive manifold over L). In the course of this discussion, which we will keep informal, we will say what we mean by a smooth structure on N *as a retractive manifold over L*.

Let W be the union of $L\times I$ and an open collar attached (outside) to $\partial(L\times I)$. We think of N as a subspace of the smooth manifold W. The boundary ∂N is a union $\partial_- N\cup\partial_+ N$ where $\partial_- N=L\times 0\subset\partial N$ and $\partial_+ N$ is the closure of the complement of $\partial_- N$ in ∂N.

Let $C\subset N$ be an open collar, $C\cong\partial N\times[0,1[$. Write $C=C_-\cup C_+$ where $C_\pm\cong\partial_\pm N\times[0,1[$. Observe that C has a structure of regular manifold which is unique up to contractible choice. This induces a regular structure on $C\setminus\partial N$ which we call ϱ_1. On the other hand, $C\setminus\partial N$ also inherits a structure ϱ_2 of regular (even smooth) manifold from W. The structures ϱ_1 and ϱ_2 agree on $C_-\setminus\partial_- N$.

On inspection, our definition of *regular retractive manifolds* over L boils down to the following. The space X_r of regular structures on N, as a retractive manifold over L, is (homotopy equivalent to) the space of isotopies from ϱ_1 to ϱ_2, relative to $C_-\setminus\partial_- N$.

We now produce a similar description of the space X_d of smooth structures on N, as a retractive manifold over L. By Morlet's smoothing theory [KS], we don't need to mention charts and atlases; it is enough to equip certain tangent bundles with certain structures. To be more precise, the classifying maps for various tangent bundles lead to a map $u: C \to B$ where B is the homotopy pullback of

$$\begin{tikzcd} & B\mathrm{TOP}(n-1) \arrow[d] \\ B\mathrm{O}(n) \arrow[r] & B\mathrm{TOP}(n) \end{tikzcd}$$

and $n = \dim(N) = \dim(C) = \dim(L) + 1$. The restriction $u|C_-$ has canonical factorization $C_- \to B\mathrm{O}(n-1) \to B$. We can define X_d as the space of factorizations of $u: C \to B$ through $B\mathrm{O}(n-1)$, up to a specified homotopy and relative to C_-.

Finally we have to check that a certain forgetful map from X_d to X_r is a homotopy equivalence. Up to homotopy equivalences, X_d and X_r can be viewed as spaces of sections, subject to boundary conditions, of certain fibrations over C_+. The fibers of these fibrations are total homotopy fibers of the commutative squares

$$\begin{tikzcd} B\mathrm{O}(n-1) \arrow[r] \arrow[d, "\subset"] & B\mathrm{TOP}(n-1) \arrow[d, "\subset"] \\ B\mathrm{O}(n) \arrow[r] & B\mathrm{TOP}(n), \end{tikzcd} \qquad \begin{tikzcd} B\mathrm{TOP}(\mathbb{S}^{n-1}) \arrow[r, "\mathrm{id}"] \arrow[d, "\mathrm{id}"] & B\mathrm{TOP}(\mathbb{S}^{n-1}) \arrow[d] \\ B\mathrm{TOP}(\mathbb{S}^{n-1}) \arrow[r] & B\mathrm{TOP}(n), \end{tikzcd}$$

respectively. (A total homotopy fiber of a commutative square of spaces is any homotopy fiber of the associated map from initial term to the homotopy pullback of the other three terms.) Hence it suffices to show that the diagram in the shape of a 3-cube with these two squares as top and bottom faces, respectively, is homotopy cartesian. But this is easy. $\square$

More on addition laws

Here we prove Lemma 9.3. Our point of view is that a Γ-space is a *covariant* functor from a certain category Γ^{op} to spaces, subject to certain conditions. The objects of Γ^{op} are the finite sets. The morphisms from S to T are the pointed maps $S_+ \to T_+$. Hence we have to say what $Z^\sharp(L)(S)$ is for any $L = (L; \partial_0 L, \partial_1 L)$ in $\mathcal{W}$ and finite set S. We decree

$$Z^\sharp(L)(S) = \operatorname*{hocolim}_{S \leftarrow L' \to L} Z(q, V)$$

where the arrow $L' \to L$ signifies a morphism in $\mathcal{W}$ with codomain $(L; \partial_0 L, \partial_1 L)$ and $S \leftarrow L'$ signifies an honest continuous map from the compact manifold L' to S. The diagrams of the form $S \leftarrow L' \rightarrow L$ then form a category $\mathcal{I}(L,S)$ in an obvious way. The rule associating to such a diagram the space $Z(L')$ is a functor on $\mathcal{I}(L,S)$. It is clear that $Z^\sharp(L)(S)$ is natural in L and S. It is also clear that the natural projection from $Z^\sharp(L)(1)$ to $Z(L)$ is a homotopy equivalence, since $\mathcal{I}(L,1)$ has a terminal object; here the number 1 denotes a set of cardinality one. It only remains to check that the canonical map $Z^\sharp(L)(S) \rightarrow \prod_{s\in S} Z^\sharp(L)(s)$ is a homotopy equivalence (Segal's condition).

Let $\mathcal{I}'(L,S)$ be the full subcategory of $\mathcal{I}(L,S)$ consisting of all objects

$$S \xleftarrow{f} L' \xrightarrow{g} L$$

where g restricted to $f^{-1}(s)$ is a homotopy equivalence (see Definition 9.2) for every $s\in S$. Note that $\mathcal{I}(L,S)$ and consequently $\mathcal{I}'(L,S)$ are equivalent to posets, i.e., between any two objects there is at most one morphism. We now find it convenient to denote objects by single letters $c, d, \ldots$ and morphisms by a symbol $\leqslant$, as in $c \leqslant d$. It is sufficient to observe the following. For every finite subposet $\mathcal{K}$ of $\mathcal{I}(L,S)$ there exist a functor $\varphi\colon \mathcal{K} \rightarrow \mathcal{I}(L,S)$ and an object d in $\mathcal{I}'(L,S)$ with the following properties: $\varphi(c) \leqslant c$ for all $c\in\mathcal{K}$ and $\varphi(c) \leqslant d$. □

10. Converse Riemann–Roch for topological manifolds

Index-theoretic view of topological h-cobordisms

Overview. Using relative excisive characteristics we construct a map from $\mathcal{H}_\$(L)$ to $A^\%(L)$ which fits into a homotopy fiber sequence $\mathcal{H}_\$(L) \rightarrow A^\%(L) \rightarrow A(L)$. Of course we already have such a homotopy fiber sequence from §9; but here we want to rebuild it with index-theoretic methods.

We begin with the construction of a map $\varkappa\colon \mathcal{P}_\$(L) \rightarrow A^\%(L)$ which refines the map $\varkappa_h\colon \mathcal{P}_\$(L) \rightarrow A(L)$ of §9. The idea is not difficult. For a retractive manifold N over L, we form $(N,L)^!$, the pushout of $N \hookleftarrow L \hookrightarrow N$. The underlying set is the union of $\{1\}\times N$ and $\{-1\}\times(N\setminus L)$. The projection to N and the section $x \mapsto (1,x)$ from N make $(N,L)^!$ into a retractive space over N and an object of

$$\mathcal{R}^{fd}(N) \cap \mathcal{V}(N) \subset \mathcal{R}^{ld}(\mathbb{J}N),$$

see (8.6). Hence $(N,L)^!$ determines a point in $F'(N) = A^\%(N)$; see Lemma 8.7. We denote this by $\chi(N,L)$. Its image in $A^\%(L)$ under the map $r_*\colon A^\%(N) \rightarrow A^\%(L)$ induced by the retraction $r\colon N \rightarrow L$ is by definition $\varkappa(N)$. This procedure will also give us, for a retractive manifold N over $L\times I^s$, an element in $A^\%(L\times I^s)$; we push it forward to $A^\%(L)$ using the projection-induced map and call the result $\varkappa(N) \in A^\%(L)$.

In trying to promote the assignments $N \mapsto \varkappa(N)$ to a map defined on $\mathcal{P}_\$(L)$, we encounter the usual difficulties. Namely, it is not easy to evaluate excisive characteristics on families (bundles) with nondiscrete structure group. It will be solved in the usual manner, compare Theorem 2.5, Corollary 2.7. Thus the key statement is that a certain map

$$\operatorname*{holim}_{(s,N)} A^{\%}(N) \to \operatorname*{holim}_{(s,N,\theta)} A^{\%}(N^\theta) \tag{10.1}$$

analogous to (2.6) is a homotopy equivalence. The statement involves two indexing categories which we now define:

(1) The first has objects (s,N) where N is a simplex in $\mathfrak{P}(L\times I^s)$; see Definition 9.1. A morphism from (s,N) to (t,N') is a morphism from N to $N'\times I^{s-t}$ in $\operatorname{simp}(\mathfrak{P}(L\times I^s))$.

(2) The second has objects (s,N,θ) with s and N as before; the symbol θ is used, as in Theorem 2.5, for an equivalence relation on N, with quotient space N^θ, such that the projections $N\to\Delta^k$ and $N\to N^\theta$ define a homeomorphism $N\to\Delta^k\times N^\theta$. We also require that the fibers of the projection $N\to(L\times I^s)\times I$ over points $((x,z),t)$ with sufficiently small $t\in I$ are equivalence classes of θ. A morphism from (s,N,θ) to (t,N',θ') is a morphism $(s,N)\to(t,N')$ as defined above, respecting the equivalence relations.

The rules $(s,N)\mapsto A^{\%}(N)$ and $(s,N,\theta)\mapsto A^{\%}(N^\theta)$ are functors on these categories. The projections $N\to N^\theta$ induce the map (10.1). It is indeed a homotopy equivalence (compare Corollary 2.7).

Definition 10.2. Let C be the (contractible) homotopy fiber of (10.1) over the point determined by the relative characteristics $\chi(N^\theta, L\times I^s)$ for all (s,N,θ). Every choice of point in C determines a map $\varkappa\colon \mathcal{P}(L)\to A^{\%}(L)$, the image of that point under the composition

$$C \xrightarrow{\text{proj}} \operatorname*{holim}_{(s,N)} A^{\%}(N) \xrightarrow{\text{proj}} \operatorname{map}(\mathcal{P}_\$(L), A^{\%}(L)).$$

The second arrow here uses the projection-induced maps $A^{\%}(N)\to A^{\%}(L)$ and the fact that the classifying space of the category with objects (s,N) contains $\mathcal{P}_\$(L)$. Compare Definition 9.1.

PROPOSITION 10.3. *The map $\varkappa$ in Definition 10.2 respects the addition laws.*

A more detailed formulation and an extremely technical proof will be given in the next subsection.

PROPOSITION 10.4. *The following commutative square resulting from Definition*

10.2 *is homotopy cartesian:*

$$\begin{tikzcd} \mathcal{H}_\$(L) \arrow[r, "\varkappa"] \arrow[d] & A^{\%}(L) \arrow[d, "\alpha"] \\ \mathrm{cone}(\mathcal{H}_\$(L)) \arrow[r] & A(L). \end{tikzcd}$$

Explanation and proof. The upper horizontal arrow is $\varkappa$ of (10.2), restricted to $\mathcal{H}_\$(L)$. The composition $\alpha\varkappa$ from $\mathcal{H}_\$(L)$ to $A(L)$ is canonically homotopic to $\varkappa_h$ (verification left to the reader), hence canonically nullhomotopic on $\mathcal{H}_\$(L)$. This gives the rest of the square. To show that the square is homotopy cartesian, we compare with

$$\begin{tikzcd} \mathcal{H}_\$(L) \arrow[r, "\subset"] \arrow[d] & \mathcal{P}^j_\$(L) \arrow[d, "\alpha\varkappa"] \\ \mathrm{cone}(\mathcal{H}_\$(L)) \arrow[r] & A(L) \end{tikzcd} \tag{10.5}$$

for $j \gg 0$. This is $(j-\varepsilon)$-cartesian after group completion because, up to some homotopies, it is identical with the square in the Fibration Theorem 9.5. Square (10.5) comes with a natural transformation to the square in Proposition 10.4 given by $\varkappa$ on the upper right-hand term and by identity maps on the other terms. The Fibration Theorem 9.5 implies that the map between upper right-hand terms is highly connected after group completion when L is a disk; the Excision Theorem 9.6 then implies that it is always highly connected. More precisely, the connectivity goes to ∞ as j goes to ∞. Therefore the square in Proposition 10.4 is homotopy cartesian. □

Naturality, homotopy ends and addition laws

Here we are concerned with the proof of Proposition 10.3. We know already from Lemma 9.3 that it is enough to promote $\varkappa$ in Definition 10.2 to a natural transformation $\mathcal{P}_\$(L;\partial_0 L)\to A^{\%}(L)$ of functors on $\mathcal{W}$. That is not a trivial matter, however. We need a very homotopy theoretic approach to natural transformations.

Let $\mathcal{C}$ be a small category and let J be a functor from $\mathcal{C}^{\mathrm{op}}\times\mathcal{C}$ to spaces. It is customary to define the *end* or *diagonal limit* of J as the subspace of

$$\prod_{c \text{ in } \mathcal{C}} J(c,c)$$

consisting of all points (x_c) which satisfy $f_*(x_c)=f^*(x_d)\in J(c,d)$ for every morphism $f\colon c\to d$ in $\mathcal{C}$. See [Mac]. For a standard example which is quite relevant here: suppose that U and V are functors from $\mathcal{C}$ to spaces. Let $J(c,d)=\operatorname{map}(U(c),V(d))$. Then the end of J can be identified with the space of natural transformations from U to V.

We adopt a slightly different point of view where $\mathcal{C}^{\mathrm{op}}\times\mathcal{C}$ gets replaced by another category $\mathcal{C}'$. The objects of $\mathcal{C}'$ are the morphisms of $\mathcal{C}$; a morphism from $f_1\colon c_1\to d_1$ to $f_2\colon c_2\to d_2$ is a commutative diagram in $\mathcal{C}$ of the form

$$\begin{tikzcd} c_1 \arrow[r, "f_1"] & d_1 \arrow[d] \\ c_2 \arrow[u] \arrow[r, "f_2"] & d_2. \end{tikzcd}$$

There is a forgetful functor $\mathcal{C}'\to\mathcal{C}^{\mathrm{op}}\times\mathcal{C}$ which takes an object $f\colon c\to d$ in $\mathcal{C}'$ (alias morphism in $\mathcal{C}$) to the pair (c,d). Our point of view is that what we need to make an end is (only) a functor J from $\mathcal{C}'$ to spaces. The end of J is then defined as the subspace of

$$\prod_{c\text{ in }\mathcal{C}} J(\mathrm{id}_c)$$

consisting of all points (x_c) which satisfy $f_*(x_c)=f^*(x_d)\in J(f)$ for every morphism $f\colon c\to d$ in $\mathcal{C}$, alias object f of $\mathcal{C}'$. Here $f_*\colon J(\mathrm{id}_c)\to J(f)$ and $f^*\colon J(\mathrm{id}_d)\to J(f)$ are the maps which J associates to the following morphisms in $\mathcal{C}'$, respectively:

$$\begin{tikzcd} c \arrow[r, "\mathrm{id}"] & c \arrow[d, "f"] \\ c \arrow[u, "\mathrm{id}"] \arrow[r, "f"] & d, \end{tikzcd} \qquad \begin{tikzcd} d \arrow[r, "\mathrm{id}"] & d \arrow[d, "\mathrm{id}"] \\ c \arrow[u, "f"] \arrow[r, "f"] & d. \end{tikzcd}$$

The end of a functor from $\mathcal{C}^{\mathrm{op}}\times\mathcal{C}$ can be (re)defined by composing with the forgetful functor $\mathcal{C}'\to\mathcal{C}^{\mathrm{op}}\times\mathcal{C}$.

In general, the end of a functor J from $\mathcal{C}'$ or $\mathcal{C}^{\mathrm{op}}\times\mathcal{C}$ to spaces has a fairly unpredictable homotopy type. One is led to look for something like a *homotopy end* of J, with a more predictable homotopy type.

Definition 10.6. The *homotopy end* of a functor J from $\mathcal{C}'$ to spaces, in symbols $\operatorname{hoend} J$, is the corealization (Tot) of the cosimplicial space

$$k\mapsto \prod_{c_0\to c_1\to\ldots\to c_k} J(c_0\to c_k).$$

Here the product is indexed by the strings $c_0\to c_1\to\ldots\to c_k$ of k composable morphisms in $\mathcal{C}$. The factor corresponding to such a string is J evaluated on the composite morphism $c_0\to c_k$. For $k=0$ the strings reduce to single objects in $\mathcal{C}$; the factor corresponding to such an object c is $J(\mathrm{id}_c)$. The simplicial operators are obvious. *Remark*: It can be shown [GoKW] that $\operatorname{hoend} J$ is homeomorphic to $\operatorname{holim} J$.

Example. Suppose again that U and V are functors from $\mathcal{C}$ to spaces. Define J from $\mathcal{C}'$ to spaces by $J(f\colon c\to d):=\mathrm{map}(U(c),V(d))$. One finds easily that hoend J can be identified with the space of natural transformations from U^{fat} to V, where U^{fat} is defined by

$$U^{\mathrm{fat}}(d)=\operatorname*{hocolim}_{c\to d} U(c).$$

Here the homotopy colimit is taken over the category of $\mathcal{C}$-objects over d. Note that there is a natural projection $U^{\mathrm{fat}}(d)\to U(d)$ which is a homotopy equivalence; the inclusion of $U(d)$ in $U^{\mathrm{fat}}(d)$ is a nonnatural homotopy inverse.

The application to Proposition 10.3 is as follows, in outline. We take $\mathcal{C}=\mathcal{W}$, hence $\mathcal{C}'=\mathcal{W}'$. We will construct two functors J_1 and J_2 from $\mathcal{W}'$ to spaces such that for example

$$J_1(\mathrm{id}_L)=\operatorname*{holim}_{(s,N)} A^{\%}(N),$$
$$J_2(\mathrm{id}_L)=\operatorname*{holim}_{(s,N,\theta)} A^{\%}(N^{\theta})$$

for any $L=(L;\partial_0L,\partial_1L)$ in $\mathcal{W}$ with empty ∂_0L; the variables (s,N) and (s,N,θ) have the same meaning as in (10.1). The map (10.1) generalizes to a natural transformation $J_1\to J_2$ which, evaluated at any object of $\mathcal{W}'$, is a homotopy equivalence. Hence we obtain an induced homotopy equivalence

$$\mathrm{hoend}\,J_1\to\mathrm{hoend}\,J_2.$$

We also need the functor J_3 from $\mathcal{W}'$ to spaces whose value on a morphism

$$f\colon(L';\partial_0L',\partial_1L')\to(L;\partial_0L,\partial_1L)$$

is the space of maps $\mathcal{P}_{\$}(L';\partial_0L')\to A^{\%}(L)$. Now the diagram in Definition 10.2 has a refinement or generalization of the following sort:

$$C_u\xrightarrow{\mathrm{proj}}\mathrm{hoend}\,J_1\to\mathrm{hoend}\,J_3.$$

Here C_u is the (contractible) homotopy fiber of the above homotopy equivalence hoend $J_1\to$hoend J_2 over a certain point in hoend J_2 determined, much as in Definition 10.2, by certain relative characteristics. The map hoend $J_1\to$hoend J_3 is induced by a natural transformation $J_1\to J_3$ which refines/generalizes the second arrow in the diagram in Definition 10.2. In conclusion, any choice of point in C_u determines a point in hoend J_3; by the example right after Definition 10.6, this is all we want. □

Some details. Let f be a morphism in $\mathcal{W}$ from $(L';\partial_0 L',\partial_1 L')$ to $(L;\partial_0 L,\partial_1 L)$; so f is a codimension-zero embedding $L'\to L\times I^k$ subject to a condition (see Definition 9.2). We define

$$J_1(f):=\operatorname*{holim}_{(s,N)} A^{\%}(f_*N_\sim),$$
$$J_2(f):=\operatorname*{holim}_{(s,N,\theta)} A^{\%}(f_*N^\theta_\sim),$$

where s stands for a nonnegative integer and where N is a j-simplex (for some j) in $\mathfrak{P}(L'\times I^s;\partial_0 L'\times I^s)$. By $N_\sim$ we mean the retractive space over $L'\times I^s$ associated with N according to Definition 9.2: the pushout of

$$N\hookleftarrow \Delta^j\times((L'\times I^s\times 0)\cup(\partial_0 L'\times I^s\times I))\xrightarrow{\text{proj}} L'\times I^s.$$

By $f_*N_\sim$ is meant the retractive space over $L\times I^{k+s}$ obtained from $N_\sim$ by pushforward along the embedding $f\times\mathrm{id}$ from $L'\times I^s$ to $L\times I^{k+s}$. It is regarded here as a space in its own right. In the definition of $J_2(f)$, we also have an equivalence relation θ on N, giving a product decomposition $N\cong\Delta^j\times N^\theta$, prescribed near $\Delta^j\times((L'\times I^s\times 0)\cup(\partial_0 L'\times I^s\times I))$. By $N^\theta_\sim$ is meant the pushout of

$$N^\theta\hookleftarrow (L'\times I^s\times 0)\cup(\partial_0 L'\times I^s\times I)\xrightarrow{\text{proj}} L'\times I^s.$$

This contains a copy of $L'\times I^s$, but does not have a preferred retraction to $L'\times I^s$. By $f_*N^\theta_\sim$ is meant the pushforward of $N^\theta_\sim$ along the embedding $f\times\mathrm{id}$ from $L'\times I^s$ to $L\times I^{k+s}$. This contains a copy of $L\times I^{k+s}$, but does not have a preferred retraction to $L\times I^{k+s}$.

The pairs (s,N) and the triples (s,N,θ) form categories, as in Definition 10.2 and preceding definitions. The relative characteristics $\chi(f_*N^\theta_\sim, L\times I^{k+s})$ are sufficiently compatible to give us a distinguished point y_f in $J_2(f)$, for each f. Furthermore, the points y_f for the various f in $\mathcal{W}'$ are sufficiently compatible to give us a distinguished point y in hoend J_2. We leave the verifications to the reader, with regrets and apologies. □

Trimmings

Let X be an n-manifold without boundary, possibly noncompact. A *trimming* of X is a subset $N\subset X$ which is a compact topological n-manifold in its own right, with the property that $N\to X$ is a homotopy equivalence. More generally, a compact subset $N\subset\Delta^j\times X$ is a *family of trimmings of X parametrized by* Δ^j if the projection $N\to\Delta^j$ is a bundle of compact n-manifolds and the inclusion $N\to\Delta^j\times X$ is a homotopy equivalence. Such families are the j-simplices of a simplicial set $\mathfrak{T}(X)$. Let $\mathcal{T}(X)=|\mathrm{simp}(\mathfrak{T}(X))|$.

There is an obvious stabilization map $\mathcal{T}(X)\to\mathcal{T}(X\times\mathbb{R})$; here we are using the inclusion $I\to\mathbb{R}$. We let

$$\mathcal{T}_{\$}(X):=\operatorname*{hocolim}_{r}\mathcal{T}(X\times\mathbb{R}^r).$$

Let $N\subset X$ be any trimming.

There is an important inclusion map $\mathcal{H}(N)\hookrightarrow\mathcal{T}(X\times\mathbb{R})$ because, with our conventions, any h-bordism over N is a subset of $N\times I\subset X\times\mathbb{R}$. This stabilizes to an inclusion $\mathcal{H}_{\$}(N)\hookrightarrow\mathcal{T}_{\$}(X\times\mathbb{R})\hookrightarrow\mathcal{T}_{\$}(X)$.

PROPOSITION 10.7. *The inclusion $\mathcal{H}_{\$}(N)\to\mathcal{T}_{\$}(X)$ is a homotopy equivalence, for any trimming N of X.*

Proof. Let $\mathcal{T}'(X\times\mathbb{R}^s)\subset\mathcal{T}(X\times\mathbb{R}^s)$ be the subspace determined by the trimmings L of $X\times\mathbb{R}^s$ which are contained in $N\times I^s$ and for which the inclusion $\partial L\to L$ is 1-connected. Note that $\mathcal{T}'(X\times\mathbb{R}^s)$ contains $\mathcal{H}(N\times I^{s-1})$ for $s>0$. Topological immersion theory [Ga] and general position arguments show that the inclusion

$$\operatorname*{hocolim}_{s\geqslant 1}\mathcal{T}'(X\times\mathbb{R}^s)\to\operatorname*{hocolim}_{s\geqslant 0}\mathcal{T}(X\times\mathbb{R}^s)=\mathcal{T}_{\$}(X)$$

is a homotopy equivalence. By inspection, the stabilization map from $\mathcal{T}'(X\times\mathbb{R}^{s-1})$ to $\mathcal{T}'(X\times\mathbb{R}^s)$ factors through $\mathcal{H}(N\times I^{s-1})$, up to homotopy. Therefore the following inclusion is also a homotopy equivalence:

$$\operatorname*{hocolim}_{s\geqslant 1}\mathcal{H}(N\times I^{s-1})\to\operatorname*{hocolim}_{s\geqslant 1}\mathcal{T}'(X\times\mathbb{R}^s).\qquad\square$$

Trimmings and their characteristics

Suppose that X is homotopy finitely dominated (and a manifold without boundary, as in the previous subsection). We want to construct a map from $\mathcal{T}(X)$ to the homotopy fiber of the assembly $\alpha\colon A^{\%}(X)\to A(X)$ over the point $\chi_h(X)$. (Here $\chi_h(X)$ is the 'homotopy invariant' characteristic of X, represented by the retractive space $X^!:=\mathbb{S}^0\times X$.) More precisely, we will make a commutative square

$$\begin{array}{ccc}\mathcal{T}_{\$}(X) & \xrightarrow{\chi} & A^{\%}(X)\\ \big\downarrow{\scriptstyle\subset} & & \big\downarrow{\scriptstyle\alpha}\\ \operatorname{cone}(\mathcal{T}_{\$}(X)) & \longrightarrow & A(X)\end{array}\tag{10.8}$$

using appropriate models of $A^{\%}(X)$ and $A(X)$ which we are about to describe. Namely, we interpret $A^{\%}(X)$ as $F'(X)$, with F' as in Observation 8.8. Strictly speaking $F'(X)$

is not clearly defined in or around Observation 8.8 since X might not be compact; we decree however that $F'(X) := \operatorname{colim}_{M\subset X} F'(M)$ where M runs through the codimension-zero compact submanifolds of M.

The construction of (10.8) is similar to that of the square in Proposition 10.4. Moreover it is essentially a special case of §2.8, so we will be very brief. We begin by writing down a homotopy equivalence

$$\operatorname*{holim}_{(s,N)} A^{\%}(N) \to \operatorname*{holim}_{(s,N,\eta)} A^{\%}(N^{\theta}) \tag{10.9}$$

where $s \geqslant 0$ and $N \subset \Delta^k \times X \times \mathbb{R}^s$ is a k-parameter family of trimmings of $X \times \mathbb{R}^s$ for some $k \geqslant 0$ and θ is an equivalence relation on N giving a product structure $N \cong \Delta^k \times N^{\theta}$. The excisive characteristics $\chi(N^{\theta})$, etc., determine a point in the codomain of (10.9); let C be the homotopy fiber of (10.9) over that point. Now any choice of point $* \in C$ determines a map $\mathcal{T}_{\$}(X) \to A^{\%}(X)$. This is the image of $*$ under the composition

$$C \xrightarrow{\text{proj}} \operatorname*{holim}_{(s,N)} A^{\%}(N) \to \operatorname*{holim}_{(s,N)} A^{\%}(X) \xrightarrow{\text{proj}} \operatorname{map}(\mathcal{T}_{\$}(N), A^{\%}(X)).$$

The composition of $\mathcal{T}_{\$}(X) \to A^{\%}(X)$ with assembly is canonically homotopic to the constant map with value $\chi_h(X)$. This gives the rest of (10.8).

THEOREM 10.10. *Square* (10.8) *is homotopy cartesian.*

Proof. First, assume that $\mathcal{T}_{\$}(X)$ is empty. Then we have to show that the homotopy fiber of the assembly map $A^{\%}(X) \to A(X)$ over the point $\chi_h(X)$ is also empty; equivalently, we have to show that the Wall finiteness obstruction of X, the class of $\chi_h(X)$ in the cokernel of the assembly $\pi_0 A^{\%}(X) \to \pi_0 A(X)$, is nonzero. Suppose, if possible, that the Wall finiteness obstruction of X is zero. Then there exists a homotopy equivalence $g: K \to M$ where K is a finite simplicial complex. We may assume that K is a compact manifold also. Replacing X by $X \times \mathbb{R}^s$ with $s \gg 0$ if necessary and deforming g if necessary, we may also assume that g is a locally flat embedding with a normal disk bundle $N(K) \subset X$. Now $N(K)$ is a trimming of X, contradicting the assumption.

We assume from now on that $\mathcal{T}_{\$}(X)$ is not empty. Then we can easily reduce to the case where $\mathcal{T}(X)$ is nonempty; so choose a vertex L in $\mathcal{T}(X)$. Now we have a diagram

$$\begin{tikzcd}
\mathcal{H}_{\$}(L) \arrow[r, "\varkappa"] \arrow[d, "\simeq"] & A^{\%}(L) \arrow[d, "+\chi(L)"] \\
\mathcal{T}_{\$}(X) \arrow[r, "\chi"] \arrow[d] & A^{\%}(X) \arrow[d] \\
\operatorname{cone}(\mathcal{T}_{\$}(X)) \arrow[r] & A(X)
\end{tikzcd} \tag{10.11}$$

where the lower square is (10.8) and where the upper one uses the inclusion map of Proposition 10.7. The arrow labelled $+\chi(L)$ is given by addition of the constant $\chi(L)$ using the addition law in $A^{\%}(L)$, followed by the inclusion-induced map from $A^{\%}(L)$ to $A^{\%}(X)$. By an inspection which we leave to the reader, the upper square in (10.11) is commutative up to a canonical homotopy (and the vertical arrows in it are homotopy equivalences). Now the outer diagram in (10.11), obtained by deleting the middle horizontal arrow, is homotopy cartesian because it is identical with the one in Proposition 10.4, up to some evident homotopy equivalences. □

Fiberwise trimmings

Overview. We generalize our results about $\mathcal{T}_\$(X)$ to bundles of manifolds without boundary. Let $p\colon E \to B$ be a fiber bundle where the fibers are n-manifolds without boundary, for some n, which are finitely dominated. We will define $\mathcal{T}(p)$, the space of trimmings of p, and a stable version $\mathcal{T}_\$(p)$. Our goal is to obtain an index-theoretic description of the homotopy type of $\mathcal{T}_\$(p)$ when B is also finitely dominated. In that case $\mathcal{T}_\$(p)$ is homotopy equivalent to

$$\Gamma^{\sim}(\mathcal{T}_{\$B}(E) \to B),$$

the space of sections of the fibration associated with the quasifibration $\mathcal{T}_{\$B}(E) \to B$ obtained essentially by applying $\mathcal{T}_\$$ fiberwise (to the fibers of p). We shall show that there is a homotopy cartesian square of spaces over B,

$$\begin{array}{ccc} \mathcal{T}_{\$B}(E) & \xrightarrow{\ \chi\ } & A^{\%}_B(E) \\ \downarrow & & \downarrow{\scriptstyle \alpha} \\ \mathrm{cone}_B(\mathcal{T}_{\$B}(E)) & \longrightarrow & A_B(E). \end{array} \tag{10.12}$$

The lower left-hand term is the mapping cylinder of the projection from $\mathcal{T}_{\$B}(E)$ to B. The map from it to $A_B(E)$ which appears in the square extends the fiberwise (homotopy invariant) characteristic $\chi_h(p)\colon B \to A_B(E)$. (Strictly speaking we will not work with B but with $|\mathrm{simp}(\mathfrak{B})|$, assuming that B is the geometric realization of a simplicial set $\mathfrak{B}$. Compare §1.6.)

Definition 10.13. A trimming of $p\colon E \to B$ is a subbundle p_t of p with the property that the inclusion $p_t^{-1}(b) \to p^{-1}(b)$ is a trimming of $p^{-1}(b)$ for each $b \in B$. A *family of trimmings of p parametrized by Δ^j* is a trimming of

$$\mathrm{id} \times p\colon \Delta^j \times E \to \Delta^j \times B.$$

Such families are the j-simplices of a simplicial set $\mathfrak{T}(p)$. Let $\mathcal{T}(p)$ be the geometric realization of $\mathrm{simp}(\mathfrak{T}(p))$. Let $\sigma^i p$ be the composition of p with the projection $E\times\mathbb{R}^i\to E$ and let

$$\mathcal{T}_\$(p) := \operatorname*{hocolim}_i \mathcal{T}(\sigma^i p).$$

Suppose now that B is the geometric realization of a simplicial set $\mathfrak{B}$. For x in $\mathrm{simp}(\mathfrak{B})$ let $p_x\colon E_x\to\Delta^{|x|}$ be the pullback of p under the characteristic map $\Delta^{|x|}\to B$ of x. Restriction of trimmings gives canonical maps

$$\mathcal{T}(p)\to\operatorname*{holim}_x \mathcal{T}(p_x),$$
$$\mathcal{T}_\$(p)\to\operatorname*{holim}_x \mathcal{T}_\$(p_x).$$

Induction over skeletons shows that the first one is always a homotopy equivalence and the second one is a homotopy equivalence if B is finitely dominated. On the other hand, we know (compare Remark 1.5) that $\mathrm{holim}_x\,\mathcal{T}_\$(p_x)$ is homotopy equivalent to $\Gamma^{\sim}$ of the projection

$$\operatorname*{hocolim}_x \mathcal{T}_\$(p_x)\to|\mathrm{simp}(\mathfrak{B})|.$$

We shall produce homotopy cartesian squares, one for each x in $\mathrm{simp}(\mathfrak{B})$, naturally in x:

$$\begin{array}{ccc}
\mathcal{T}_\$(p_x) & \xrightarrow{\chi} & A^{\%}(E_x) \\
\big\downarrow{\scriptstyle\subset} & & \big\downarrow{\scriptstyle\alpha} \\
\mathrm{cone}(\mathcal{T}(p_x)) & \longrightarrow & A(E_x).
\end{array} \tag{10.14}$$

Then, by taking homotopy colimits over all x in $\mathrm{simp}(\mathfrak{B})$, we will have the precise version of (10.12):

$$\begin{array}{ccc}
\mathrm{hocolim}_x\,\mathcal{T}_\$(p_x) & \xrightarrow{\chi} & \mathrm{hocolim}_x\,A^{\%}(E_x) \\
\big\downarrow{\scriptstyle\subset} & & \big\downarrow{\scriptstyle\alpha} \\
\mathrm{hocolim}_x\,\mathrm{cone}(\mathcal{T}(p_x)) & \longrightarrow & \mathrm{hocolim}_x\,A(E_x).
\end{array} \tag{10.15}$$

For the construction of (10.14), simultaneously for all x, we make a contractible choice, namely, a choice of point in a certain homotopy fiber C of a certain map

$$\operatorname*{holim}_x \operatorname*{holim}_{(s,N)} A^{\%}(N)\to\operatorname*{holim}_x \operatorname*{holim}_{(s,N,\theta)} A^{\%}(N^\theta) \tag{10.16}$$

where $s\geqslant 0$ and $N\subset\Delta^k\times E_x\times\mathbb{R}^s$ is the total space of a trimming of the bundle

$$\Delta^k\times E_x\times\mathbb{R}^s\to\Delta^k\times\Delta^{|x|}$$

and θ is an equivalence relation on N, with quotient space N^θ, giving a homeomorphism $N \cong N^\theta \times \Delta^k \times \Delta^{|x|}$. The map (10.16) is induced by the projections $N \to N^\theta$, and C is its homotopy fiber over the point determined by the (excisive) characteristic χ. Once $* \in C$ has been selected, we obtain the upper horizontal arrow in (10.14) as the image of $*$ under the composition

$$C \xrightarrow{\text{proj}} \operatorname*{holim}_{(s,N)} A^{\%}(N) \hookrightarrow \operatorname*{holim}_{(s,N)} A^{\%}(E_x) \xrightarrow{\text{proj}} \operatorname{map}(\mathcal{T}_{\$}(p_x), A^{\%}(E_x))$$

where $s \geqslant 0$ and N denotes simplices in $\mathcal{T}(\sigma^s p_x)$.

OBSERVATION 10.17. *Square* (10.15) *alias* (10.12) *is homotopy cartesian.*

Proof. It is enough to show that (10.14) is homotopy cartesian for each x. Naturality in x reduces this to the claim that (10.14) is homotopy cartesian for each x with $|x|=0$. This is identical with Theorem 10.10. □

COROLLARY 10.18 (Riemann–Roch with converse for topological manifolds). *Suppose that $p: E \to B$ is a fibration with finitely dominated fibers and base. The following are equivalent:*

(1) *p is fiber homotopy equivalent to a bundle of compact topological n-manifolds on B, for some n;*

(2) *the component of the fiberwise characteristic $\chi_h(p)$ is in the image of the map induced by fiberwise assembly,*

$$\begin{array}{c} \pi_0 \Gamma^{\sim}(A^{\%}_B(E) \to B) \\ \downarrow \\ \pi_0 \Gamma^{\sim}(A_B(E) \to B). \end{array}$$

Proof. $\Rightarrow$ is clear from Observation 10.17 and was also established in Parts I and II. $\Leftarrow$ is also clear from Observation 10.17 if p is fiber homotopy equivalent to a bundle of open manifolds. It turns out that p is always fiber homotopy equivalent to a bundle of open manifolds. Namely, Casson and Gottlieb [CaG] have shown that p is fiber homotopy equivalent to a bundle of open manifolds if the fibers of p are homotopy equivalent to compact CW-spaces. If the fibers of p do not satisfy this condition, we can still apply the Casson–Gottlieb theorem to $pq: \mathbb{S}^1 \times E \to B$, where $q: \mathbb{S}^1 \times E \to E$ is the projection. Then the Wall finiteness obstructions of the fibers of pq vanish. Hence pq is fiber homotopy equivalent to a bundle of open manifolds. By taking suitable infinite cyclic covers we deduce that p is fiber homotopy equivalent to a bundle of open manifolds. □

11. Stability matters

Stability of the index theorem

We return to the abstract setting of §3 in order to make precise how the Index Theorem 3.11 and Remark 3.14 for a compact manifold M^n with boundary is related to the same index theorem for $M\times I$. We still think of the index theorem for M as a point $(\bar{e}_n, \omega(M), \chi(M))$ in the homotopy pullback of diagram (3.16). Consequently the relationship that we are after should involve a suitable commutative diagram whose upper row is (3.16) for M and whose lower row is (3.16) for $M\times I$. Such a diagram is easy to set up: it is

$$\begin{tikzcd}
\operatorname{holim}_{\mathcal{GB}(n)} F \arrow[r] & \operatorname{holim}_{\mathcal{O}(M)} F & F(M) \arrow[l, "\bar{\wp}"'] \\
\operatorname{holim}_{\mathcal{GB}(n)} F(-\times I) \arrow[r] \arrow[u] \arrow[d] & \operatorname{holim}_{\mathcal{O}(M)} F(-\times I) \arrow[u] \arrow[d] & F(M\times I) \arrow[l, "\bar{\wp}"'] \arrow[u] \arrow[d, "="] \\
\operatorname{holim}_{\mathcal{GB}(n)\times\mathcal{O}(I)} F \arrow[r] & \operatorname{holim}_{\mathcal{O}(M)\times\mathcal{O}(I)} F & F(M\times I) \arrow[l] \\
\operatorname{holim}_{\mathcal{GB}(n+1)} F \arrow[r] \arrow[u, "\beta"'] & \operatorname{holim}_{\mathcal{O}(M\times I)} F \arrow[u] & F(M\times I). \arrow[l, "\bar{\wp}"'] \arrow[u, "="']
\end{tikzcd} \tag{11.1}$$

(For typographical reasons the indexing categories appear with the holim sign, not to the right of it in the name of the functor.) All arrows in the diagram are obvious. All vertical arrows except possibly the one labelled β are homotopy equivalences. In particular, the first three rows of the diagram are essentially identical.

THEOREM 11.2 (stability). *The characteristic χ determines a point in the homotopy limit of* (11.1) *which projects to $(\bar{e}_n, \omega(M), \chi(M))$ in the homotopy limit of the upper row and to $(\bar{e}_{n+1}, \omega(M\times I), \chi(M\times I))$ in the homotopy limit of the lower row.*

The proof of Theorem 11.2 is a formality. It resembles the proof of Theorem 3.11 and can be seen as an application of Remark 4.15. We leave it to the reader. We also leave to the reader to prove a family version of Theorem 11.2.

Stability of the vanishing theorem

In Theorem 4.10, we saw that the characteristic classes $[\bar{e}_n]$ and $[\bar{b}_n]$ agree on n-disk bundles. A sharper version is Theorem 4.13. The 'vanishing theorem' terminology comes from around (0.5). We will now establish a relationship between the dimension n case and the dimension $n+1$ case of Theorem 4.13. To describe the relationship, we need the commutative diagram

$$\begin{tikzcd}
\operatorname{holim}_{(D,H)} Q^\bullet(H) \arrow[r, "\eta"] & \operatorname{holim}_{(D,H)} F(H) \\
\operatorname{holim}_{(D,H)} Q^\bullet(H\times I) \arrow[r, "\eta"] \arrow[u, "\simeq"] \arrow[d, "\simeq"] & \operatorname{holim}_{(D,H)} F(H\times I) \arrow[u, "\simeq"] \arrow[d, "\simeq"] \\
\operatorname{holim}_{(D,H),V} Q^\bullet(H\times V) \arrow[r, "\eta"] & \operatorname{holim}_{(D,H),V} F(H\times V) \\
\operatorname{holim}_{(D',H')} Q^\bullet(H') \arrow[r, "\eta"] \arrow[u] & \operatorname{holim}_{(D',H')} F(H') \arrow[u]
\end{tikzcd} \tag{11.3}$$

where (D,H) runs through $\mathcal{OD}(n)$, (D',H') runs through $\mathcal{OD}(n+1)$, and V runs through $\mathcal{O}(I)$. The vertical arrows from the last row to the next one above are induced by the functor $(D,H),V\mapsto(D\times I,H\times V)$ from $\mathcal{OD}(n)\times\mathcal{O}(I)$ to $\mathcal{OD}(n+1)$.

THEOREM 11.4 (construction). *We construct a point in the homotopy inverse limit of* (11.3) *projecting to*

(1) $(\iota^*\bar{b}_n, v_n, \iota^*\bar{e}_n)$ *in the* holim *of the upper row* (*see Theorem* 4.13);

(2) $(\iota^*\bar{b}_{n+1}, v_{n+1}, \iota^*\bar{e}_{n+1})$ *in the* holim *of the lower row*;

(3) *the point determined by* χ *in the* holim *of the right-hand column* (*see Remark* 4.15).

Details of the construction. The natural transformation η from $Q^\bullet$ to F gives us a map between the following (3×4)-diagrams:

$$\begin{tikzcd}
\operatorname{holim}_{(D,H)} Q^\bullet(H) & \operatorname{holim}_D Q^\bullet(D) \arrow[l, "\simeq"'] \arrow[r, "\simeq"] & \operatorname{holim}_D Q^\bullet(*) \\
\operatorname{holim}_{(D,H)} Q^\bullet(H\times I) \arrow[u, "\simeq"] \arrow[d, "\simeq"] & \operatorname{holim}_D Q^\bullet(D\times I) \arrow[l, "\simeq"'] \arrow[r, "\simeq"] \arrow[u, "\simeq"] \arrow[d, "="] & \operatorname{holim}_D Q^\bullet(*) \arrow[u, "="] \arrow[d, "="] \\
\operatorname{holim}_{(D,H),V} Q^\bullet(H\times V) & \operatorname{holim}_D Q^\bullet(D\times I) \arrow[l, "\simeq"'] \arrow[r, "\simeq"] & \operatorname{holim}_D Q^\bullet(*) \\
\operatorname{holim}_{(D',H')} Q^\bullet(H') \arrow[u] & \operatorname{holim}_{D'} Q^\bullet(D') \arrow[l, "\simeq"'] \arrow[r, "\simeq"] \arrow[u] & \operatorname{holim}_{D'} Q^\bullet(*), \arrow[u]
\end{tikzcd}$$

$$\begin{tikzcd}
\operatorname{holim}_{(D,H)} F(H) & \operatorname{holim}_D F(D) \arrow[l, "\simeq"'] \arrow[r, "\simeq"] & \operatorname{holim}_D F(*) \\
\operatorname{holim}_{(D,H)} F(H\times I) \arrow[u, "\simeq"] \arrow[d, "\simeq"] & \operatorname{holim}_D F(D\times I) \arrow[l, "\simeq"'] \arrow[r, "\simeq"] \arrow[u, "\simeq"] \arrow[d, "="] & \operatorname{holim}_D F(*) \arrow[u, "="] \arrow[d, "="] \\
\operatorname{holim}_{(D,H),V} F(H\times V) & \operatorname{holim}_D F(D\times I) \arrow[l, "\simeq"'] \arrow[r, "\simeq"] & \operatorname{holim}_D F(*) \\
\operatorname{holim}_{(D',H')} F(H') \arrow[u] & \operatorname{holim}_{D'} F(D') \arrow[l, "\simeq"'] \arrow[r, "\simeq"] \arrow[u] & \operatorname{holim}_{D'} F(*). \arrow[u]
\end{tikzcd}$$

We can view the map as a $(2\times 3\times 4)$-diagram $\mathcal{A}$ with vertices v_{ijk}, where $i\in\{1,2\}$, $j\in\{1,2,3\}$ and $k\in\{1,2,3,4\}$. The hypotheses and contractible choices made so far (in particular, z_3 from the proof of Theorem 4.13) give us a point y in the homotopy limit of the subdiagram $\mathcal{B}$ obtained by deleting the vertices v_{ijk} with $i=1$, $j\in\{1,2\}$ and $k\in\{2,3\}$; in other words, by deleting the subdiagram

$$\begin{tikzcd}
\operatorname{holim}_{(D,H)} Q^{\bullet}(H\times I) \arrow[d, "\simeq"] & \operatorname{holim}_D Q^{\bullet}(D\times I) \arrow[l, "\simeq"'] \arrow[d, "="] \\
\operatorname{holim}_{(D,H),V} Q^{\bullet}(H\times V) & \operatorname{holim}_D Q^{\bullet}(D\times I). \arrow[l, "\simeq"']
\end{tikzcd}$$

The projection from the homotopy limit of $\mathcal{A}$ to the homotopy limit of $\mathcal{B}$ is a fibration and a homotopy equivalence. Its fiber over y is therefore contractible. Choose a point y' in there. Let y'' be the image of y' under projection to the homotopy limit of the subdiagram spanned by the vertices v_{ijk} with $j=1$, which is identical with (11.3). Then y'' has all the required properties. $\square$

12. Converse Riemann–Roch for smooth manifolds

This chapter is somewhat analogous to §10. We will however need to use the index theorem and the vanishing theorem of Part I, which we did not need in §10. They will be used in a stable form; for that we rely on §11. The exposition will be kept informal; all the truly technical points have been taken care of in previous chapters.

Index-theoretic view of regular h-cobordisms

Let L be a compact regular manifold. By analogy with §10, we begin with the construction of a map $\bar{\varkappa}\colon d\mathcal{P}_{\$}(L)\to Q(L_+)$ which fits into a commutative square

$$\begin{array}{ccc} d\mathcal{P}_{\$}(L) & \xrightarrow{\bar{\varkappa}} & Q(L_+) \\ \downarrow{\scriptstyle \text{forget}} & & \downarrow{\scriptstyle \eta} \\ \mathcal{P}_{\$}(L) & \xrightarrow{\varkappa} & A^{\%}(L). \end{array} \tag{12.1}$$

The lower horizontal arrow $\varkappa$ in (12.1) is canonically homotopic to

$$r_*\chi-\chi(L),$$

the map which takes a retractive manifold N over $L\times I^q$ (for some $q\geqslant 0$) to the difference $r_*\chi(N)-\chi(L)$. Here $r\colon N\to L\times I^s$ is the retraction. We think of it as a map from N to L by dropping the second coordinate. The Index Theorem 3.11 and Remark 3.14 now tell us that it is also canonically homotopic to

$$r_*\wp(e_{\mathrm{dim}}(\tau,\tau_\partial))-\chi(L)$$

where (τ,τ_∂) denotes the tangent bundle pair of a (variable) regular retractive manifold over $L\times I^q$ for some q and e_{dim} means $e_{\mathrm{dim}(N)}$. The Vanishing Theorem 4.10 and Theorem 4.13, coupled with the stability result Theorem 11.4, show that the map out of $\mathcal{P}_{\$}(L)$ which we have called $r_*\wp(e_{\mathrm{dim}}(\tau,\tau_\partial))$ is canonically homotopic to $r_*\eta\wp(b_{\mathrm{dim}}(\tau,\tau_\partial))$ on $d\mathcal{P}_{\$}(L)$. Hence we may define $\bar{\varkappa}$ by the formula

$$r_*\wp(b_{\mathrm{dim}}(\tau,\tau_\partial))-\chi(L)\colon d\mathcal{P}_{\$}(L)\to Q(L_+)$$

bearing in mind that $\chi(L)$ also has a canonical lift to $Q(L_+)$ by the same reasoning, or directly by Thereom 5.7.

Remark. This is a definition of $\bar{\varkappa}$ up to contractible choice. It makes (12.1) commutative up to a contractible choice of preferred homotopies. These shortfalls could be fixed in the usual way, by enlarging the spaces involved without changing their homotopy types.

PROPOSITION 12.2. *Up to natural homotopy equivalences, the map $\bar{\varkappa}$ in* (12.1) *extends to a natural transformation between functors on $\mathcal{W}$* (*compare Lemma* 9.3).

Proof in outline. Up to a canonical homotopy, $\bar{\varkappa}$ can be written as a composition

$$\begin{array}{ccc} d\mathcal{P}_{\$}(L) & & Q(L_+) \\ \downarrow & & \uparrow \\ \mathcal{P}_{\$}(L) & \xrightarrow{\varkappa} & A^{\%}(L) \end{array}$$

where the right-hand vertical arrow is a left homotopy inverse to η in (12.1). But each arrow in this composition extends to a natural transformation between functors on $\mathcal{W}$. $\square$

COROLLARY 12.3. *The following commutative square is homotopy cartesian:*

$$\begin{tikzcd} d\mathcal{H}_\$(L) \arrow[r, "\bar{\varkappa}"] \arrow[d, "\text{forget}"] & Q(L_+) \arrow[d, "\eta"] \\ \mathcal{H}_\$(L) \arrow[r, "\varkappa"] & A^{\%}(L). \end{tikzcd}$$

Proof. We can enlarge the square to a commutative diagram

$$\begin{tikzcd} d\mathcal{H}_\$(L) \arrow[r, "\subset"] \arrow[d, "\text{forget}"] & d\mathcal{P}^j_\$(L) \arrow[r, "\bar{\varkappa}"] \arrow[d, "\text{forget}"] & Q(L_+) \arrow[d, "\eta"] \\ \mathcal{H}_\$(L) \arrow[r, "\subset"] & \mathcal{P}^j_\$(L) \arrow[r, "\varkappa"] & A^{\%}(L) \end{tikzcd} \tag{12.4}$$

with $j \gg 0$. Here the left-hand square becomes $(j-\varepsilon)$-cartesian after group completion by smoothing theory (compare §9, proof of the vanishing theorem). Now it is sufficient to show that the horizontal arrows in the right-hand square are $(j-\varepsilon)$-connected. The Excision Theorem 9.6 and its smooth alias regular version (see the proof of the vanishing theorem in §9) reduce this to the case where $L = *$. We have $A^{\%}(*) \simeq A(*)$, so the lower horizontal arrow in the right-hand square is indeed $(j-\varepsilon)$-connected for $L = *$ (after group completion) by the Fibration Theorem 9.5. As regards the upper horizontal arrow in the right-hand square, we can first use the smooth version of Theorem 9.5 to show that upper stabilization

$$\Omega B(d\mathcal{P}^j_\$(*)) \to \Omega B(d\mathcal{P}^{j+1}_\$(I))$$

is $(j-\varepsilon)$-connected; then we have Corollary 9.9 to deduce that $\Omega B(d\mathcal{P}^j_\$(*))$ has the $(j-\varepsilon)$-type of $Q\mathbb{S}^0$ as an infinite loop space. Since

$$\bar{\varkappa}\colon d\mathcal{P}^j_\$(*) \to Q(*_+)$$

becomes a map of infinite loop spaces after group completion, it suffices to show that the induced homomorphism on π_0, which has the form $\mathbb{N} \to \mathbb{Z}$ before group completion, takes 1 to ± 1. But this is a consequence of commutativity of the right-hand square in (12.4). □

Regular trimmings

Let X be a regular n-manifold without boundary, possibly noncompact. Let (q, U, j) be the regular structure on X. A *regular trimming* of X is a trimming $N \subset X$ with

(1) a regular structure (q', U', j') on N;

(2) a one-parameter family, parametrized by $[0, 1]$, of regular structures on $\mathrm{int}(N)$, specializing to the restrictions of (q', U', j') and (q, U, j) for the parameter values 0 and 1, respectively.

It is straightforward to define families of regular trimmings of X parametrized by Δ^j. Such families are the j-simplices of a simplicial set $d\mathfrak{T}(X)$. Let $d\mathcal{T}(X) = |\mathrm{simp}(d\mathfrak{T}(X))|$. There is an obvious stabilization map $d\mathcal{T}(X) \to d\mathcal{T}(X \times \mathbb{R})$; here we are using the inclusion $I \to \mathbb{R}$. We let

$$d\mathcal{T}_\$(X) := \operatorname*{hocolim}_r d\mathcal{T}(X \times \mathbb{R}^r).$$

Let $N \subset X$ be any trimming. There is an important but obvious inclusion map $d\mathcal{H}(N) \hookrightarrow d\mathcal{T}(X \times \mathbb{R})$. This leads to another inclusion map $d\mathcal{H}_\$(N) \hookrightarrow d\mathcal{T}_\$(X \times \mathbb{R}) \hookrightarrow d\mathcal{T}_\$(X)$.

PROPOSITION 12.5. *The inclusion $d\mathcal{H}_\$(N) \to d\mathcal{T}_\$(X)$ is a homotopy equivalence, for any regular trimming N of X.* □

Regular trimmings and their characteristics

Suppose that X is homotopy finitely dominated (and a regular manifold without boundary, as in the previous subsection). We want to construct a map from $d\mathcal{T}(X)$ to the homotopy fiber of $\alpha\eta: Q(X_+) \to A(X)$ over the point $\chi_h(X)$. More precisely, we will construct a map $\bar{\chi}: d\mathcal{T}_\$(X) \to Q(X_+)$ making the square

$$\begin{array}{ccc} d\mathcal{T}_\$(X) & \xrightarrow{\bar{\chi}} & Q(X_+) \\ \big\downarrow{\scriptstyle\subset} & & \big\downarrow{\scriptstyle\eta} \\ \mathcal{T}_\$(X) & \xrightarrow{\chi} & A^{\%}(X) \end{array} \tag{12.6}$$

commutative up to a preferred homotopy; combining this with (10.8) we get

$$\begin{array}{ccc} d\mathcal{T}_\$(X) & \xrightarrow{\bar{\chi}} & Q(X_+) \\ \big\downarrow{\scriptstyle\subset} & & \big\downarrow{\scriptstyle\alpha\eta} \\ \mathrm{cone}(\mathcal{T}_\$(X)) & \longrightarrow & A(X). \end{array} \tag{12.7}$$

The construction of $\bar{\chi}$ in (12.6) is similar to (but easier than) that of $\bar{\varkappa}$ in (12.1). We note that $\chi: \mathcal{T}_\$(X) \to A^{\%}(X)$ in (12.6) is canonically homotopic to $\iota_*\wp(e_{\dim}(\tau, \partial\tau))$, the map which takes a trimming N of $X \times \mathbb{R}^q$ (for some $q \geqslant 0$) with inclusion $\iota: N \to X \times \mathbb{R}^q$ to

$$\iota_*\wp(e_{\dim}(\tau, \tau_\partial))$$

where (τ, τ_∂) denotes the tangent bundle pair of N. (We have dropped the $\mathbb{R}^q$-coordinate of ι to view ι_* as a map from $A^{\%}(N)$ to $A^{\%}(X)$.) Hence we can define $\bar{\chi}$ in (12.6) by the formula $\iota_*\wp(b_{\dim}(\tau, \tau_\partial))$. □

THEOREM 12.8. *Squares* (12.6) *and* (12.7) *are homotopy cartesian.*

Proof. Square (12.6) is homotopy cartesian by Corollary 12.3 and Proposition 12.5. Square (12.7) is homotopy cartesian because (12.6) and (10.8) are. □

Fiberwise regular trimmings

Overview. We generalize our results about $d\mathcal{T}_{\$}(X)$ to bundles of regular manifolds without boundary. Let $p: E \to B$ be a fiber bundle where the fibers are regular n-manifolds without boundary, for some n, which are finitely dominated. We will define $d\mathcal{T}(p)$, the space of regular trimmings of p, and a stable version $d\mathcal{T}_{\$}(p)$. Our goal is to obtain an index-theoretic description of the homotopy type of $d\mathcal{T}_{\$}(p)$ when B is also finitely dominated. We will not proceed exactly as in §10, subsection on fiberwise trimmings, because we do not have quite so much functoriality. Nevertheless we will produce a homotopy equivalence from $d\mathcal{T}_{\$}(p)$ to the homotopy fiber over $\chi_h(p)$ of

$$\begin{array}{c} \Gamma^{\sim}((Q_+)_B(E_+) \to B) \\ \Big\downarrow{\scriptstyle \alpha\eta} \\ \Gamma^{\sim}(A_B(E) \to B). \end{array}$$

Compare (10.18).

Definition 12.10. A regular trimming of $p: E \to B$ is a trimming p_t of p where each fiber $p_t^{-1}(b)$ is equipped with the structure of a regular trimming of $p^{-1}(b)$, continuously in $b \in B$. A *family of regular trimmings of p parametrized by* Δ^j is a regular trimming of

$$\mathrm{id} \times p: \Delta^j \times E \to \Delta^j \times B.$$

Such families are the j-simplices of a simplicial set which we denote by $d\mathfrak{T}(p)$. Let $d\mathcal{T}(p) := |\mathrm{simp}(d\mathfrak{T}(p))|$. Let $\sigma^i p$ be the composition of p with the projection $E \times \mathbb{R}^i \to E$ and let

$$d\mathcal{T}_{\$}(p) := \operatorname*{hocolim}_i d\mathcal{T}(\sigma^i p).$$

Suppose now that B is the geometric realization of a simplicial set $\mathfrak{B}$. We can make a square, commutative up to preferred homotopy,

$$\begin{array}{ccc} B \times d\mathcal{T}_{\$}(p) & \xrightarrow{\bar{\chi}} & (Q_+)_B(E) \\ \Big\downarrow & & \Big\downarrow{\scriptstyle \alpha\eta} \\ B \times \mathrm{cone}(d\mathcal{T}_{\$}(p)) & \xrightarrow{\chi_h} & A_B(E). \end{array}$$

The upper horizontal arrow is the (refined) Becker–Gottlieb transfer for the tautological bundle of compact regular manifolds on $B \times d\mathcal{T}_\$(p)$. The lower horizontal arrow, restricted to $B \subset B \times \mathrm{cone}(\ldots)$, is the A-theory characteristic for the same bundle. Rewriting this in adjoint form, we obtain

$$\begin{array}{ccc} d\mathcal{T}_\$(p) & \xrightarrow{\tilde{\chi}} & \Gamma^{\sim}((Q_+)_B(E_+) \to B) \\ \big\downarrow & & \big\downarrow{\scriptstyle \alpha\eta} \\ \mathrm{cone}(d\mathcal{T}_\$(p)) & \xrightarrow{\chi_h} & \Gamma^{\sim}(A_B(E) \to B). \end{array} \tag{12.11}$$

THEOREM 12.12. *Square* (12.11) *is homotopy cartesian if B is finitely dominated.*

Proof. Suppose first that $\mathfrak{B}$ is finitely generated. We proceed by induction on the number of nondegenerate simplices of $\mathfrak{B}$. The case where $\mathfrak{B}$ is a point is covered by Theorem 12.8. Otherwise write $B = B(1) \subset B(2)$ where $B(1)$ and $B(2)$ are the geometric realizations of proper simplicial subsets of $\mathfrak{B}$. By induction, Theorem 12.12 holds for the subspaces $B(1)$, $B(2)$, $B(1) \cap B(2)$ and the appropriate restrictions of p to these subspaces of B. By excision, Theorem 12.12 must also hold for B. — Note in particular the excisive behavior of the upper left-hand term in (12.11): there is a homotopy cartesian square

$$\begin{array}{ccc} d\mathcal{T}_\$(p) & \longrightarrow & d\mathcal{T}_\$(p|B(1)) \\ \big\downarrow & & \big\downarrow \\ d\mathcal{T}_\$(p|B(2)) & \longrightarrow & d\mathcal{T}_\$(p|(B(1) \cap B(2))). \end{array}$$

To deduce the general case in which B is finitely dominated from the case where B is a finite CW-space, use the fact that a (homotopy) retract of a homotopy cartesian square is a homotopy cartesian square. $\square$

COROLLARY 12.13 (Riemann–Roch with converse for regular manifolds). *Let $p\colon E \to B$ be a fibration with finitely dominated fibers and base. The following are equivalent*:

(1) *p is fiber homotopy equivalent to a bundle of compact regular n-manifolds on B, for some n*;

(2) *the component of the fiberwise characteristic $\chi_h(p)$ is in the image of*

$$\begin{array}{c} \pi_0 \Gamma^{\sim}((Q_+)_B(E) \to B) \\ \big\downarrow{\scriptstyle \alpha\eta} \\ \pi_0 \Gamma^{\sim}(A_B(E) \to B). \end{array}$$

Proof. As in the proof of Corollary 10.18, it is enough to show that p is fiber homotopy equivalent to a bundle of regular open manifolds. We may assume that p is a bundle of open manifolds. Let τ_v be the vertical tangent bundle of E. Let ν_v be a euclidean bundle on E which is Whitney sum inverse to τ_v. Then the total space of ν_v is also the total space of a bundle on B which is fiber homotopy equivalent to E and whose fibers are manifolds with trivialized tangent bundles. □

Waldhausen's alternative approach

In the introduction we mentioned an alternative proof of the Riemann–Roch theorems with converse, Corollaries 10.18 and 12.13, due to Waldhausen. This uses descriptions given in [W3] of $A^{\%}(L)$ and $Q(L_+)$ for a manifold L which mimic Waldhausen's definition of $A(X)$ for a space X, replacing however retractive spaces over X by retractive (regular) manifolds over L throughout. Following is a sketch; an elaboration would undoubtedly also fill many pages.

Definition of $A(X)$. Referring to [W2] for details, we recall that Waldhausen defines $A(X)$ as $\Omega|w\mathcal{S}_\bullet(\mathcal{R}^{hf}(X))|$ where

(1) $\mathcal{R}^{hf}(X)$ is the category of retractive spaces over X satisfying suitable finite domination conditions;

(2) up to equivalence, $\mathcal{S}_k(\mathcal{R}^{hf}(X))$ is the category of strings

$$Y_1 \to Y_2 \to \ldots \to Y_k$$

of cofibrations in $\mathcal{R}^{hf}(X)$;

(3) up to equivalence, $w\mathcal{S}_k(\mathcal{R}^{hf}(X))$ is the subcategory of the weak equivalences in $\mathcal{S}_k(\mathcal{R}^{hf}(X))$.

It is of course important here that the $\mathcal{S}_k(\ldots)$ for all k are related by certain face and degeneracy functors, making $\mathcal{S}_\bullet(\ldots)$ and $w\mathcal{S}_\bullet(\ldots)$ into simplicial categories.

The $\mathcal{S}_\bullet$-construction with retractive manifolds. Suppose that L is a compact manifold. We fix $k \geqslant 0$ and introduce strings $N_1 \subset N_2 \subset \ldots \subset N_k$ of retractive manifolds over L. Denote the space of all such strings, suitably defined, by $\mathcal{S}_k\mathcal{P}(L)$. This is intended to mimic $|w\mathcal{S}_k(\mathcal{R}^{hf}(X))|$ above, especially for $X=L$, but it has a major shortcoming: not all the face operators

$$\mathcal{S}_k(\mathcal{R}^{hf}(X)) \to \mathcal{S}_{k-1}(\mathcal{R}^{hf}(X))$$

have analogs $\mathcal{S}_k\mathcal{P}(L) \to \mathcal{S}_{k-1}\mathcal{P}(L)$. To illustrate: there is a face operator $\mathcal{S}_k(\mathcal{R}^{hf}(X)) \to \mathcal{S}_{k-1}(\mathcal{R}^{hf}(X))$ which takes a string $Y_1 \to Y_2$ of cofibrations in $\mathcal{R}^{hf}(X)$ to the retractive space Y_2/Y_1 over X, the pushout of $X \leftarrow Y_1 \to Y_2$. Unfortunately, for a string $N_1 \subset N_2$ of retractive manifolds over L, the retractive space quotient N_2/N_1 is typically not a retractive manifold over L. There are however retractive manifold candidates for something like the stabilization $\sigma(N_2/N_1)$ of N_2/N_1. One such is

$$N_2 \times \left[0, \tfrac{1}{3}\right] \cup N_1 \times I \cup (L \times I) \times \left[\tfrac{2}{3}, 1\right],$$

a subset of $L \times I \times I$ and a retractive manifold over $L \times 0 \times I \cong L \times I$. In this way, although we do not have a simplicial space $\mathcal{S}_k\mathcal{P}(L)$, we do get a simplicial space $\mathcal{S}_k\mathcal{P}_\$(L)$. Waldhausen shows essentially that

$$\begin{aligned} &\Omega\,|\mathcal{S}_{\bullet}\mathcal{P}_\$(L)| \simeq A^{\%}(L), \\ &\Omega\,|\mathcal{S}_{\bullet}d\mathcal{P}_\$(L)| \simeq Q(L_+) \quad \text{for regular } L. \end{aligned}$$

He also identifies the forgetful maps

$$\Omega\,|\mathcal{S}_{\bullet}d\mathcal{P}_\$(L)| \to \Omega\,|\mathcal{S}_{\bullet}\mathcal{P}_\$(L)| \to \Omega\,|\mathbb{S}_{\bullet}\mathcal{R}^{hf}(L)|$$

with η and α, respectively. (Actually Waldhausen prefers to use Thomason's modification of the $\mathbb{S}_{\bullet}$-construction in this context.)

Characteristics of manifolds revisited. To concentrate on the most interesting case, suppose that L is compact and regular. The retractive manifold

$$L \times \left(\left[0, \tfrac{1}{3}\right] \cup \left[\tfrac{2}{3}, 1\right]\right) \subset L \times I$$

is a point in

$$d\mathcal{P}_\$(L) \subset \Omega\,|\mathcal{S}_{\bullet}d\mathcal{P}_\$(L)| \simeq Q(L_+).$$

Its image under $\alpha\eta\colon Q(L_+) \to A(L)$ is clearly related to $\chi_h(L)$ by a canonical path. This argument works well with families of compact regular manifolds and so re-proves the most important part of (0.6).

The Riemann-Roch theorem with converse. It is also possible to prove Corollary 12.13 using this approach. The key step is to identify $d\mathcal{T}_\$(X)$ with the homotopy fiber of $\alpha\eta\colon Q(X_+) \to A(X)$ over $\chi_h(X)$ in the case where X is an open manifold and finitely dominated. The appropriate manifold model for $Q(X_+)$ can be described roughly as

$$\Omega\,|\mathcal{S}_{\bullet}d\mathcal{P}_\$(X;\infty)|$$

where $\mathcal{P}(X;\infty)$ is the space of retractive manifolds over X with *compact support.* (A retractive manifold $N \subset X \times I$ is said to have compact support if the closure of $(X \times I) \setminus N$ in $X \times I$ is compact.) Using this, one does obtain a map from $d\mathcal{T}_\$(X)$ to the homotopy fiber of $\alpha\eta\colon Q(X_+) \to A(X)$ over $\chi_h(X)$. To show that it is a homotopy equivalence, one can use the elementary Proposition 12.5 and the nonelementary Corollary 9.9.

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