K-theory for certain group C^*-algebras

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Introduction

The use of K-theoretic techniques in C^*-algebras has led to the solution of several outstanding problems, among them the conjecture of R. V. Kadison that $C_r^*(F_n)$, the reduced C^*-algebra of the free group on n generators ($n \geq 2$), has no nontrivial projections. This was resolved affirmatively by Pimsner and Voiculescu [6] as a corollary to a remarkable theorem which describes the K-groups for any reduced crossed product of a C^*-algebra by an action of a free group.

This paper originated in the author's attempt to understand the work of Pimsner and Voiculescu, and in particular to see whether their methods could be used to give a simpler proof that $K_0(C_r^*(F_n))=\mathbf{Z}$. By slightly adapting their approach, we are able to give a description of $K_*(C_r^*(\Gamma))$ for any group Γ which is a free product of countable amenable groups (Corollary 5.5 below). When specialized to the case $\Gamma=F_n$, our results naturally agree with those of Pimsner and Voiculescu. Our proof of their result is not actually much simpler than theirs, given the technical simplifications that accrue from not considering crossed products, but we feel that the structure of the proof becomes clearer when displayed in a more general context.

The K-theory of the full C^*-algebras of some free product groups has been investigated by Cuntz [2] and Rosenberg [7]. Comparison of their results with ours shows that $K_*(C_r^*(\Gamma))$ is the same as $K_*(C^*(\Gamma))$ in all known cases.

A vital element in the work of Pimsner and Voiculescu is the construction of an extension, which they call the Toeplitz extension, of $C_r^*(F_n)$ by the algebra K of compact operators. The Toeplitz extension is intimately tied to the group of integers, and in order to be able to deal with free products of groups other than $\mathbf{Z}$ we have replaced it by another extension of $C_r^*(\Gamma)$ by K which can be constructed for any free product group Γ and which turns out to be somewhat easier to handle than the Toeplitz extension. We describe this extension and some of its properties in section three.

In section two we investigate what seems to us the crucial property of the integers which allows Pimsner and Voiculescu's methods to work. We show that this property holds in any countable amenable group and also in some nonamenable groups.

Another important ingredient of Pimsner and Voiculescu's work is a useful method for constructing homomorphisms between the K-groups of C^*-algebras. We call this construction the "difference map" and give a systematic account of it in section one. We assume a general familiarity with K-theory for C^*-algebras as presented for example in [8] and [9].

For any group G we write $\{\delta(g): g \in G\}$ for the canonical orthonormal basis of $l^2(G)$, so that $\delta(g)$ takes the value 1 at g and 0 elsewhere. We write λ for the left regular representation of G, so that $\lambda(g)\delta(h)=\delta(gh)$. The *reduced C^*-algebra of G, $C_r^*(G)$*, is the C^*-algebra generated by $\{\lambda(g): g \in G\}$. If ψ is a representation of G which is quasi-equivalent to λ then ψ extends to a representation of $C_r^*(G)$. We shall denote this representation also by ψ, so that if $x=\lambda(g)$ then $\psi(x)=\psi(g)$.

If G, G' are groups then we write $\{\delta(g,g'): g \in G,\ g' \in G'\}$ for the canonical orthonormal basis of $l^2(G\times G')$. We habitually identify this space with $l^2(G)\otimes l^2(G')$ so that, for example, if $x \in B(l^2(G))$ then we write $x\otimes 1$ for the operator on $l^2(G\times G')$ which behaves like x on the first coordinate and leaves the second coordinate fixed. When we refer to tensor products of C^*-algebras we always mean the spatial, or minimal, tensor product ([10]).

1. The "difference" map

If A is a C^*-algebra then we denote by $A^\dagger$ the algebra obtained by adjoining an identity 1 to A, unless A is already unital in which case $A^\dagger=A$. Replacing A by $K\otimes A$, we may assume that $K_1(A)$ consists of equivalence classes $[u]_1$ of elements of the unitary group $A_U^\dagger$ of $A^\dagger$. If $\alpha: A\to B$ is a homomorphism between C^*-algebras then α can be extended to a homomorphism, still denoted by α, from $A^\dagger$ to $B^\dagger$. Let $p=\alpha(1)$, so that p is a projection in $B^\dagger$, and write $p^\perp$ for $1-p$. (It is important that we should not assume $\alpha(1)=1$.) The induced map $\alpha_1: K_1(A)\to K_1(B)$ is given by $\alpha_1[u]_1=[\alpha(u)+p^\perp]_1$. The induced map $\alpha_0: K_0(A)\to K_0(B)$ is obtained by taking suspensions. We write α_* for the homomorphism $\alpha_0\oplus\alpha_1$ of graded groups $K_*(A)\to K_*(B)$.

Suppose that α, β are homomorphisms and that J is an ideal (always closed and two-sided) in B such that $\alpha(x)-\beta(x)\in J$ for all x in A. We say that α and β *agree* mod J. Let $p=\alpha(1)$, $q=\beta(1)$. Then $p-q\in J$ and so $p^\perp-q^\perp\in J$. For u in $A_U^\dagger$,

$$(\alpha(u)+p^\perp)(\beta(u^{-1})+q^\perp)=(\alpha(u)-\beta(u))\beta(u^{-1})+(p^\perp-q^\perp)(\beta(u^{-1})+q^\perp)-\alpha(u)(p^\perp-q^\perp)+1\in J^\dagger.$$

Since $(\alpha(u)+p^{\perp})(\beta(u^{-1})+q^{\perp})$ is clearly unitary, we can define a map $(\alpha/\beta)_1$ from $K_1(A)$ to $K_1(J)$ by

$$(\alpha/\beta)_1[u]_1 = [(\alpha(u)+p^{\perp})(\beta(u^{-1})+q^{\perp})]_1.$$

LEMMA 1.1. *The map $(\alpha/\beta)_1$ is a homomorphism from $K_1(A)$ to $K_1(J)$.*

Proof. It is clear that the map $(\alpha/\beta)_1$ is well-defined. Let $w \in J^{\dagger}_U$, $v \in B^{\dagger}_U$. In the algebra $M_2(B^{\dagger})$ of 2×2 matrices over $B^{\dagger}$ we have

$$\begin{pmatrix} v & 0 \\ 0 & v^{-1} \end{pmatrix}\begin{pmatrix} w & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} v^{-1} & 0 \\ 0 & v \end{pmatrix} = \begin{pmatrix} vwv^{-1} & 0 \\ 0 & 1 \end{pmatrix}.$$

There is a homotopy (V_t) in $M_2(B^{\dagger})_U$ from the identity to

$$\begin{pmatrix} v & 0 \\ 0 & v^{-1} \end{pmatrix},$$

so

$$V_t\begin{pmatrix} w & 0 \\ 0 & 1 \end{pmatrix}V_t^{-1}$$

is a homotopy in $M_2(J^{\dagger}_U)$ from

$$\begin{pmatrix} w & 0 \\ 0 & 1 \end{pmatrix} \text{ to } \begin{pmatrix} vwv^{-1} & 0 \\ 0 & 1 \end{pmatrix}.$$

Hence $[w]_1=[vwv^{-1}]_1$ in $K_1(J)$. It follows that

$$[(\alpha(x)+p^{\perp})(\beta(x^{-1})+q^{\perp})]_1 = [(\beta(x^{-1})+q^{\perp})(\alpha(x)+p^{\perp})]_1$$

in $K_1(J)$, for any x in $A^{\dagger}_U$.

Thus for x, y in $A^{\dagger}_U$ we have

$$\begin{aligned}(\alpha/\beta)_1[xy]_1 &= [(\alpha(xy)+p^{\perp})(\beta(y^{-1}x^{-1})+q^{\perp})]_1 \\ &= [(\alpha(x)+p^{\perp})(\alpha(y)+p^{\perp})(\beta(y^{-1})+q^{\perp})(\beta(x^{-1})+q^{\perp})]_1 \\ &= [(\beta(x^{-1})+q^{\perp})(\alpha(x)+p^{\perp})(\alpha(y)+p^{\perp})(\beta(y^{-1})+q^{\perp})]_1 \\ &= (\alpha/\beta)_1[x]_1(\alpha/\beta)_1[y]_1.\end{aligned}$$

Thus $(\alpha/\beta)_1$ is a homomorphism.

The difference map $(\alpha/\beta)_0$ from $K_0(A)$ to $K_0(J)$ is defined by taking suspensions. We write $(\alpha/\beta)_*$ for the map

$$(\alpha/\beta)_0 \oplus (\alpha/\beta)_1 : K_*(A) \to K_*(J).$$

Difference maps were introduced by Pimsner and Voiculescu in [6] for the following reason. Given a homomorphism $\alpha: A \to B$ of C^*-algebras, it can sometimes happen that the induced map $\alpha_*: K_*(A) \to K_*(B)$ is an isomorphism even though α itself is not. In such cases, the inverse map α_*^{-1} does not necessarily lift to the algebras. In other words, there need not exist any homomorphism $\beta: B \to A$ such that $\beta_* = \alpha_*^{-1}$. What Pimsner and Voiculescu discovered was that in such circumstances it is sometimes possible to embed A (or rather the stably isomorphic algebra $K \otimes A$) as an ideal in a C^*-algebra C in such a way that there are homomorphisms β, $\gamma: B \to C$ which agree mod $K \otimes A$, with $(\beta/\gamma)_* = \alpha_*^{-1}$. Thus although α_*^{-1} does not lift in the usual way, it does lift as the difference between two homomorphisms.

The construction of the difference map seems to be special to C^*-algebraic K-theory. There does not appear to be any direct way of constructing $(\alpha/\beta)_0$ without using suspensions and thereby invoking Bott periodicity, so it is hard to see how one could define difference maps in the setting of algebraic K-theory.

The following five lemmas give some of the elementary properties of difference maps. In each case $\alpha, \beta: A \to B$ are homomorphisms which agree mod J. The proofs are trivial and are omitted.

LEMMA 1.2. $(\alpha/\beta)_* = -(\beta/\alpha)_*$.

LEMMA 1.3. *If $\gamma: C \to A$ is a homomorphism of C^*-algebras then*

$$(\alpha\gamma/\beta\gamma)_* = (\alpha/\beta)_* \gamma_*.$$

LEMMA 1.4. *Suppose $\delta: J \to D$ is a homomorphism of C^*-algebras. Suppose also that F is a C^*-algebra containing D as an ideal and that $\bar{\delta}: \beta \to F$ is a homomorphism which extends δ. Then*

$$(\bar{\delta}\alpha/\bar{\delta}\beta)_* = \delta_*(\alpha/\beta)_*.$$

For the next lemma, suppose that $\gamma: A \to B$ is a homomorphism such that $\alpha(1)\gamma(1) = 0$. Then we can define a homomorphism $\alpha \oplus \gamma: A \to B$ by

$$(\alpha \oplus \gamma)(x) = \alpha(x) + \gamma(x) \quad (x \in A).$$

LEMMA 1.5. *Suppose that $\gamma: A \to B$ is a homomorphism with $\alpha(1)\gamma(1)=\beta(1)\gamma(1)=0$. Then*

$$(\alpha\oplus\gamma/\beta\oplus\gamma)_* = (\alpha/\beta)_*.$$

LEMMA 1.6. *Suppose that α, β, $\gamma: A \to B$ are homomorphisms which all agree* mod J. *Then*

$$(\alpha/\gamma)_* = (\alpha/\beta)_* + (\beta/\gamma)_*.$$

Let α, $\beta: A \to B$ be homomorphisms which agree mod J. We say that α and β are *J-homotopic* if there is a path $\{\gamma_t: 0 \leq t \leq 1\}$ of homomorphisms from A to B such that

(i) (γ_t) is continuous (in the sense that $t \mapsto \gamma_t(x)$ is continuous, for each x in A),
(ii) $\gamma_0=\alpha$, $\gamma_1=\beta$,
(iii) each γ_t agrees with α (or β) mod J.

The proof of the next lemma comes in Lemmas 2.1 and 2.2 of [6].

LEMMA 1.7. *Let J be an ideal in B, and suppose α, $\beta: A \to B$ and $\gamma: A \to J$ are homomorphisms with $\beta(1)\gamma(1)=0$ such that $\beta\oplus\gamma$ and α are J-homotopic. Then $\gamma_*=(\alpha/\beta)_*$.*

Proof ([6]). Suppose $(\sigma_t: 0 \leq t \leq 1)$ is a homotopy of homomorphisms from A to B which all agree mod J, with $\sigma_0=\beta\oplus\gamma$, $\sigma_1=\alpha$. Let $p_t=\sigma_t(1)$, $q=\beta(1)$. For u in $A_U^\dagger$ let

$$w_t = (\sigma_t(u)+p_t^\perp)(\beta(u^{-1})+q^\perp).$$

Then (w_t) is a continuous path of unitaries in $J^\dagger$ with

$$[w_1]_1 = (\alpha/\beta)_1 [u]_1$$

and

$$\begin{aligned} w_0 &= (\beta(u)+\gamma(u)+p_0^\perp)(\beta(u^{-1})+q^\perp) \\ &= \gamma(u)+q+p_0^\perp \end{aligned}$$

so that $[w_0]_1=\gamma_1[u]_1$. Thus $(\alpha/\beta)_1=\gamma_1$. It follows that $(\alpha/\beta)_0=\gamma_0$ by suspension.

2. Groups with property Λ

Throughout this section, λ_0 will denote the left regular representation of a group G. We say that a representation λ_1 of G on $l^2(G)$ has a fixed point if, for some unit vector ξ in $l^2(G)$, $\lambda_1(g)\xi=\xi$ for all g in G. Let K be the ideal of compact operators in $B(l^2(G))$.

Definition. The group G has property Λ if λ_0 (considered as a representation of the full C^*-algebra $C^*(G)$) is K-homotopic to a representation λ_1 which has a fixed point.

THEOREM 2.1. *Any countable amenable group has property* Λ.

Proof. To say that λ_1 has a fixed point is the same as to say that λ_1 contains the trivial representation τ of G as a subrepresentation. If G is finite then λ_0 contains τ so the theorem clearly holds.

Suppose then that G is infinite and let $\{g_j : j\geq 0\}$ be an enumeration of the elements of G. Observe first that if A, B are finite subsets of G then we can find a right translate of B which is disjoint from A (choose $x\notin B^{-1}A$: then $A\cap Bx$ is empty). Using Følner's condition ([4], p. 64), we can find finite nonempty subsets $K_0, K_1, \ldots$ of G with the property that

$$|K_n|^{-1}|g_jK_n\cap K_n|>1-2^{1-2n} \quad (j\leq n).$$

By the above observation we may assume that the sets K_n are all disjoint. Define unit vectors $\xi_0, \xi_1, \ldots$ in $l^2(G)$ by

$$\xi_n=|K_n|^{-1/2}\sum_{g\in K_n}\delta(g).$$

Then $\{\xi_n\}$ is an orthonormal set, spanning a subspace M of $l^2(G)$, and

$$\|\lambda_0(g_j)\xi_n-\xi_n\|<2^{-n} \quad (j\leq n).$$

For $t\geq 0$, let n be the integer part of t and write $\theta=(\pi/2)(t-n)$. Define an isometry v_t on $l^2(G)$ as follows:

$$v_t(\xi_i)=\xi_i \quad (i<n),$$

$$v_t(\xi_n)=\cos\theta\cdot\xi_n+\sin\theta\cdot\xi_{n+1},$$

$$v_t(\xi_i)=\xi_{i+1} \quad (i>n),$$

$$v_t(\zeta)=\zeta \quad (\zeta\in M^{\perp}).$$

Let $p_t = 1 - v_t v_t^*$, so that p_t is the projection onto the one-dimensional subspace spanned by $\sin\theta \cdot \xi_n - \cos\theta \cdot \xi_{n+1}$.

Suppose $k \geq \max\{n+2, j\}$. Then

$$\|(\lambda_0(g_j) - v_t \lambda_0(g_j) v_t^*)\xi_k\| = \|\lambda_0(g_j)\xi_k - \xi_k - v_t(\lambda_0(g_j)\xi_{k-1} - \xi_{k-1})\|$$
$$< 2^{-k+2}.$$

This shows that the restriction of $\lambda_0(g_j) - v_t \lambda_0(g_j) v_t^*$ to M is Hilbert–Schmidt and hence compact. Let p_M denote the projection of $l^2(G)$ onto M and write $p_M^\perp$ for $1 - p_M$. We have shown that $(\lambda_0(g_j) - v_t \lambda_0(g_j) v_t^*) p_M$ is compact. Replacing g_j by its inverse and taking adjoints, we see that $p_M(\lambda_0(g_j) - v_t \lambda_0(g_j) v_t^*)$ is compact. Since v_t is the identity on $M^\perp$,

$$p_M^\perp(\lambda_0(g_j) - v_t \lambda_0(g_j) v_t^*) p_M^\perp = 0,$$

and it follows that $\lambda_0(g_j) - v_t \lambda_0(g_j) v_t^*$ is compact, for all j and all t.

Fix j, and let $g_i = g_j^{-1}$. We suppose that t is large, so that $n \geq \max\{i, j\}$, and we wish to estimate the norm of the compact operator

$$k_t = \lambda_0(g_j) - p_t - v_t \lambda_0(g_j) v_t^*.$$

We begin by calculating the effect of k_t on a set of basis vectors for the subspace M_n spanned by $\{\xi_i : i \geq n\}$. The computation in the previous paragraph shows that

$$\|k_t(\cos\theta \cdot \xi_n + \sin\theta \cdot \xi_{n+1})\| = \|(\lambda_0(g_j) v_t \xi_n - v_t \xi_n) + v_t(\lambda_0(g_j)\xi_n - \xi_n)\|$$
$$< 2^{-n}(\cos\theta + \sin\theta + 1),$$

$$\|k_t(\sin\theta \cdot \xi_n - \cos\theta \cdot \xi_{n+1})\| = \|(\lambda_0(g_j) - 1)(\sin\theta \cdot \xi_n - \cos\theta \cdot \xi_{n+1})\|$$
$$< 2^{-n}(\sin\theta + \cos\theta),$$

$$\|k_t \xi_j\| < 2^{-j} \quad (j \geq n+2).$$

Thus the Hilbert–Schmidt norm, and hence the operator norm, of k_t restricted to M_n is at most $c2^{-n}$, for some constant c. Since v_t is the identity on $M_n^\perp$, the argument in the previous paragraph shows that $\|k_t\| < c' 2^{-n}$ for some constant c', and so $k_t \to 0$ as $t \to \infty$.

Define a path $\{u_t : 0 \leq t < 1\}$ of unitary operators from $l^2(G) \oplus \mathbf{C}$ onto $l^2(G)$ as follows. Let $s = \tan(\pi t/2)$, let $n = [s]$, let $\theta = (\pi/2)(s - n)$ and let

$$u_t(\xi, \alpha) = v_s(\xi) + \alpha(\sin\theta \cdot \xi_n - \cos\theta \cdot \xi_{n+1}).$$

Define a path of representations μ_t of G on $l^2(G)$ for $0 \leq t < 1$ by

$$\mu_t(g) = u_t(\lambda_0 \oplus \tau)(g)\, u_t^*.$$

It is easy to check that $\lambda_0(g_j) - \mu_t(g_j) = k_s$, which is compact and tends to zero in norm as $t \to 1$. If we let $\mu_1 = \lambda_0$ then (μ_t) is a K-homotopy joining μ_0, which evidently contains τ as a subrepresentation, to λ_0.

The proof of the above theorem is modelled on Arveson's approach to Voiculescu's Weyl–von Neumann theorem ([1], Theorem 4; [11]). Voiculescu defines two representations λ, μ of a C^*-algebra A on a Hilbert space H to be approximately equivalent if there is a sequence of unitary operators u_n on H such that λ and $u_n\mu(\cdot)u_n^*$ agree mod K for all n and $u_n\mu(x)\, u_n^* \to \lambda(x)$ as $n \to \infty$ for all x in A. It is evident that the concepts of K-homotopy and approximate equivalence are quite similar. It is not hard to show that a countable group G is amenable if and only if its left regular representation is approximately equivalent to a representation with a fixed point. In fact, if G is amenable then an argument like that used to prove Theorem 2.1, but simpler, shows that λ_0 is approximtely equivalent to a representation of the form $\mu \oplus \tau$. Conversely, if $\mu \oplus \tau$ is approximately equivalent to λ_0 then it is easy to see that the state associated with τ is a weak*-limit of vector states associated with λ_0, so that τ is weakly contained in λ_0 and therefore G is amenable ([4]; Proposition 18.3.6 of [3]). It will follow from Proposition 2.2, however, that property Λ does not imply amenability. The reason is that no unitary equivalences are assumed among the representations in a K-homotopy.

For a specific group, it is often possible to exhibit a much simpler K-homotopy between λ_0 and a representation with a fixed point than that provided by Theorem 2.1. For $G = \mathbf{Z}$, Pimsner and Voiculescu construct a K-homotopy as follows. Given a 2×2 unitary matrix u, we can associate with u a unitary operator $\hat{u}$ on $l^2(\mathbf{Z})$ by making u act on the two-dimensional subspace generated by $\delta(0)$ and $\delta(1)$ and leaving the orthogonal complement fixed. The left regular representation of $\mathbf{Z}$ is generated by $\lambda_0(1)$, which is just the bilateral shift on $l^2(\mathbf{Z})$. Let (u_t) be a continuous path of 2×2 matrices joining the identity to

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

for example we could take

$$u_t = \begin{pmatrix} c_t & s_t \\ s_t & c_t \end{pmatrix},$$

where $c_t=(1+e^{\pi ti})/2$, $s_t=(1-e^{\pi ti})/2$, and define $\lambda_t(1)=\hat{u}_t\lambda_0(1)$. This specifies a K-homotopy λ_t of representations of $\mathbf{Z}$ joining λ_0 to a representation which fixes $\delta(0)$.

PROPOSITION 2.2. *The free group F_n has property Λ.*

Proof. Denote by $\alpha_1, \ldots, \alpha_n$ the generators of F_n and for $1 \leq j \leq n$ let G_j be the subgroup generated by α_j. Let λ_0 be the left regular representation of F_n and define unitary operators $\lambda_t(\alpha_j)$ $(0 \leq t \leq 1)$ as follows. On the subspace $l^2(G_j) \cong l^2(\mathbf{Z})$, $\lambda_t(\alpha_j)$ is a copy of the operator $\lambda_t(1)$ constructed above and on the orthogonal subspace $l^2(F_n \setminus G_j)$ the operator $\lambda_t(\alpha_j)$ is equal to $\lambda_0(\alpha_j)$. Clearly $\lambda_t(\alpha_j)$ is unitary. Thus λ_t extends in a unique way to a representation of F_n, which gives a K-homotopy from λ_0 to a representation which fixes $\delta(e)$.

The above argument shows in fact that the class of groups with property Λ is closed under the formation of free products. However, there are groups which fail to have property Λ, as the next result shows. For the definition of property T, see [5].

PROPOSITION 2.3. *A nonamenable group which has property T cannot have property Λ.*

Proof. Suppose G has property T. There exist a finite subset F of G and $\varepsilon > 0$ with the following property. If ψ is a representation of G on a Hilbert space H and ξ is a unit vector in H such that

$$\|\psi(g)\xi - \xi\| < \varepsilon \quad (g \in F)$$

then ψ contains τ. It follows from [3], Proposition 3.4.2 (ii), that if ψ contains τ weakly then ψ contains τ.

Suppose that G also has property Λ, and let (λ_t) be a K-homotopy joining the left regular representation of G to a representation which contains τ. Let $T=\{t: \lambda_t$ contains $\tau\}$. Since G has property T, the set T is open. On the other hand, it is easily seen that the set of all t for which λ_t weakly contains τ is closed, so that T is closed. Since T contains 1, it also contains 0, and so G is amenable.

We conclude this section with a simple result which makes it easier to handle representations with a fixed point.

LEMMA 2.3. *A representation μ of G on $l^2(G)$ which has a fixed point ξ is K-homotopic to a representation which has $\delta(e)$ as a fixed point.*

Proof. Let M be the two-dimensional subspace of $l^2(G)$ generated by $\delta(e)$ and ξ, and let (u_t) be a continuous path of unitary operators on $l^2(G)$ such that u_0 is the identity, u_t is the identity on $M^\perp$ and $u_1\delta(e)=\xi$. Then define $\mu_t(g)=u_t\mu(g)u_t^*(g\in G)$. It is clear that (μ_t) gives the required K-homotopy.

In future, given a group G with property Λ, we shall always assume that (λ_t) is a K-homotopy joining λ_0 to a representation λ_1 which fixes $\delta(e)$.

3. The extension algebra

Let G, S be groups. We use e to denote the identity element of any group and we write $G^*=G\setminus\{e\}$, $S^*=S\setminus\{e\}$. We require that all groups considered should be nontrivial, so G^* and S^* are nonempty.

Let $\Gamma=G*S$ be the free product of G and S. In the usual way, we express each element of Γ as a reduced word in G^* and S^* (with e corresponding to the empty word). We say that a word w in Γ *ends in* G if $w=\dots g_2s_2g_1$ (with $g_1\in G^*$). Let Γ_1^* be the set of all nonempty words in Γ which end in G and let $\Gamma_1=\Gamma_1^*\cup\{e\}$. Similarly, let Γ_2^* be the set of all nonempty words ending in S, Γ^{1*} the set of all nonempty words beginning with G, Γ^{2*} the set of all nonempty words beginning with S,

$$\Gamma^2=\Gamma^{2*}\cup\{e\},\quad \Gamma_1^2=\Gamma^2\cap\Gamma_1,\quad \Gamma_1^{2*}=\Gamma^{2*}\cap\Gamma_1^*$$

and so on.

For $M\subseteq\Gamma$ let $q(M)$ be the projection from $l^2(\Gamma)$ onto $l^2(M)$. Most of what follows will be concerned with the space $l^2(\Gamma_1)$, and if $M\subseteq\Gamma_1$ then we shall also use $q(M)$ to denote the projection from $l^2(\Gamma_1)$ onto $l^2(M)$ where the context makes it clear what is happening. We write q_w for $q(\{w\})$ $(w\in\Gamma)$.

Notice that $\lambda(g)$ leaves $l^2(\Gamma_1)$ invariant for g in G (where λ is the left regular representation of Γ) and $\lambda(s)$ leaves $l^2(\Gamma_1^*)$ invariant for s in S. For g in G let $\mu(g)$ be the restriction of $\lambda(g)$ to $l^2(\Gamma_1)$, and for s in S let $\nu(s)$ be the restriction of $\lambda(s)q(\Gamma_1^*)$ to $l^2(\Gamma_1)$. Then μ is a representation of G on $l^2(\Gamma_1)$ and ν is a nonunital representation of S on $l^2(\Gamma_1)$.

Write $A=C_r^*(G)$, $B=C_r^*(S)$. Then μ, ν extend to representations (which we still denote by μ, ν) of A, B respectively on $l^2(\Gamma_1)$. Let E be the C^*-algebra generated by $\mu(A)$ and $\nu(B)$. Notice that $q_e=\mu(1)-\nu(1)\in E$. For any word $w=\dots s_{-1}g_0s_0g_1\dots$ in Γ let

$$\sigma(w)=\dots\nu(s_{-1})\mu(g_0)\nu(s_0)\mu(g_1)\dots$$

be the corresponding element of E (with $\sigma(e)=1$). If $w\in\Gamma_1$ then $\sigma(w)q_e$ is the rank one operator $\xi \mapsto \langle \xi, \delta(e)\rangle \delta(w)$ in $B(l^2(\Gamma_1))$. Let K be the ideal in E generated by q_e. Then it is clear from the above that K is just the compact operators on $l^2(\Gamma_1)$. The following result is Lemma 1.1 of [6].

LEMMA 3.1. *There is a homomorphism π from E onto $C_r^*(\Gamma)$, with kernel K, such that $\pi(\mu(g))=\lambda(g)$ $(g \in G)$ and $\pi(\nu(s))=\lambda(s)$ $(s \in S)$.*

Proof ([6]). Fix h in G^*, t in S^*. For $n \geq 1$ let $M_n=\{w(ht)^n : w \in \Gamma_1\}$. Then $M_n \uparrow \Gamma$ as $n \to \infty$. Define $v_n : l^2(\Gamma_1) \to l^2(\Gamma)$ by

$$v_n \delta(w) = \delta(w(ht)^n) \quad (w \in \Gamma_1).$$

Then v_n maps $l^2(\Gamma_1)$ isometrically onto the range of $q(M_n)$, so $v_n v_n^* \to 1$ strongly as $n \to \infty$. Since right multiplication commutes with left multiplication it is easily verified that

$$v_n \mu(g) v_n^* \to \lambda(g) \text{ strongly} \quad (g \in G),$$

$$v_n \nu(s) v_n^* \to \lambda(s) \text{ strongly} \quad (s \in S).$$

Hence the strong limit $\pi(x)=\lim_{n\to\infty} v_n x v_n^*$ exists for each x in E. Obviously π maps E homomorphically onto $C_r^*(\Gamma)$ and since $\pi(q_e)=0$ it is clear that K is contained in the kernel of π.

To complete the proof it only remains to show that $\ker \pi \subseteq K$. For x in $C_r^*(\Gamma)$ let $p(x)$ be the restriction of $q(\Gamma_1)x$ to $l^2(\Gamma_1)$. Then the linear mapping p takes $\lambda(g)$ to $\mu(g)$ $(g \in G)$ and $\lambda(s)$ to $\nu(s)$ $(s \in S)$. It is easy to check by induction on the length of w that

$$p\pi\sigma(w)-\sigma(w)=p\lambda(w)-\sigma(w) \in K \quad (w \in \Gamma).$$

It follows that $p\pi(y)-y \in K$ $(y \in E)$ so that if $\pi(y)=0$ then $y \in K$ as required.

Thus E is an extension of $C_r^*(\Gamma)$ by the compact operators. In the following sections we shall see that if G has property Λ then one can compute the K-groups of E and thereby those of $C_r^*(\Gamma)$.

4. Construction of various homomorphisms

We suppose throughout this section that G has property Λ. Any element of $\Gamma=G*S$ can be uniquely written in the form wg, with w in Γ_2 and g in G, and any element of Γ_1^* can be uniquely written in the form wh, with w in Γ_2 and h in G^*. Define $u: l^2(\Gamma \times G^*) \to l^2(\Gamma_1^* \times G)$ by

$$u\delta(wg,h)=\sum_{k\in G^*}\langle \lambda_1(gh)\,\delta(h^{-1}),\delta(k)\rangle\,\delta(wk,k^{-1}gh),$$

where λ_1 is a representation of G on $l^2(G)$, K-homotopic to the left regular representation λ_0, which leaves $\delta(e)$ fixed.

LEMMA 4.1. *The mapping u is isometric from $l^2(\Gamma\times G^*)$ onto $l^2(\Gamma_1^*\times G)$, with inverse given by*

$$u^*\delta(wh,g)=\sum_{k\in G^*}\langle\,\delta(h),\lambda_1(hg)\,\delta(k)\,\rangle\,\delta(whgk,k^{-1})\quad(w\in\Gamma_2,h\in G^*,g\in G).$$

Proof. Initially, u is defined only as a mapping between the prehilbert spaces spanned algebraically by the basis vectors. It is easy to check that the adjoint operator is as given in the statement of the lemma.

Fix l in G. Then $\{\lambda_1(l)\,\delta(p):p\in G\}$ is an orthonormal basis for $l^2(G)$. If h, $k\in G$ then it follows that

$$\sum_{p\in G}\langle\,\delta(h),\lambda_1(l)\,\delta(p)\,\rangle\,\langle\,\lambda_1(l)\,\delta(p),\delta(k)\,\rangle=\langle\,\delta(h),\delta(k)\,\rangle=\begin{cases}1 & \text{if } h=k\\ 0 & \text{if } h\neq k.\end{cases}$$

Suppose now that h, $k\in G^*$. Since $\lambda_1(l)$ fixes $\delta(e)$, we have

$$\langle\,\delta(h),\lambda_1(l)\,\delta(e)\,\rangle=\langle\,\lambda_1(l)\,\delta(e),\delta(k)\,\rangle=0$$

and so the above equation remains true if we sum over p in G^* (rather than over all p in G).

It follows that, for w in Γ_2, h in G^* and g in G,

$$\begin{aligned}uu^*\delta(wh,g)&=\sum_{k,l\in G^*}\langle\,\delta(h),\lambda_1(hg)\,\delta(l)\,\rangle\,\langle\,\lambda_1(hg)\,\delta(l),\delta(k)\,\rangle\,\delta(wk,k^{-1}hg)\\&=\delta(wh,g).\end{aligned}$$

Thus uu^* (and similarly u^*u) is the identity, and so u extends by continuity to a unitary operator from $l^2(\Gamma\times G^*)$ onto $l^2(\Gamma_1^*\times G)$.

We now wish to define a nonunital representation of E on the space $l^2(\Gamma_1\times G)$. To do this, we identify $l^2(\Gamma_1^*\times G)$ with the obvious subspace of $l^2(\Gamma_1\times G)$ and regard u as a map into $l^2(\Gamma_1\times G)$. For x in E let $\psi(x)=u(\pi(x)\otimes 1)\,u^*$, where π is as in the previous section. For l in G, a calculation like that in the proof of Lemma 4.1 shows that

$$\psi(\mu(l))\,\delta(wh,g)=\begin{cases}\delta(lwh,g) & \text{if } w\neq e\\ \sum_{k\in G}\langle\,\lambda_1(l)\,\delta(h),\delta(k)\,\rangle\,\delta(k,k^{-1}lhg) & \text{if } w=e.\end{cases}\tag{1}$$

As before, we write A for the C^*-algebra $C_r^*(G)$ acting on $l^2(G)$. Define $\psi'\colon E\to E\otimes A\subseteq B(l^2(\Gamma_1\times G))$ by $\psi'(x)=x\otimes 1$.

LEMMA 4.2. *For l in G,*

(i) $\psi(\mu(l))\in E\otimes A$,
(ii) $\psi(\mu(l))-\psi'(\mu(l))\in K\otimes A$.

Proof. Since (i) obviously follows from (ii), we prove (ii). Observe first that

$$\psi'(\mu(l))\,\delta(wh,g)=\delta(lwh,g)\quad(wh\in\Gamma_1,\,g\in G).$$

From (1) we see that

$$(q(\Gamma_1\setminus G)\otimes 1)\,(\psi(\mu(l))-\psi'(\mu(l)))=0.$$

Since $q_e\otimes 1\in K\otimes A$, the proof will be complete if we show that

$$(q(G^*)\otimes 1)\,(\psi(\mu(l))-\psi'(\mu(l)))\in K\otimes A.$$

For h in G^* and g in G,

$$\psi'(\mu(l))\,\delta(h,g)=\delta(lh,g)=\sum_{k\in G}\langle\,\lambda_0(l)\,\delta(h),\delta(k)\,\rangle\,\delta(k,k^{-1}lhg).$$

Thus, from (1),

$$(\psi(\mu(l))-\psi'(\mu(l)))\,\delta(h,g)=\sum_{k\in G}\langle\,(\lambda_1(l)-\lambda_0(l))\,\delta(h),\delta(k)\,\rangle\,\delta(k,k^{-1}lhg).$$

Define a unitary operator U on $l^2(G^*\times G)$ by $U\delta(h,g)=\delta(h,hg)$. Clearly U is in the multiplier algebra of $K\otimes A$, and U^* is given by $U^*\delta(h,g)=\delta(h,h^{-1}g)$. Since λ_1, λ_0 are K-homotopic, $(\lambda_1(l)-\lambda_0(l))\otimes\lambda_0(l)$, and hence also $U^*((\lambda_1(l)-\lambda_0(l))\otimes\lambda_0(l))U$, is in $K\otimes A$. But a routine computation shows that the restriction of the latter operator to the range of $q(G^*)\otimes 1$ is equal to

$$(q(G^*)\otimes 1)\,(\psi(\mu(l))-\psi'(\mu(l))),$$

so the lemma is proved.

So far, we have only considered the effect of ψ and ψ' on elements of the form $\mu(l)$. But it is easy to check that

$$\psi(\nu(s)) = \psi'(\nu(s)) = \nu(s) \otimes 1 \quad (s \in S).$$

Since elements of the form $\mu(l)$ or $\nu(s)$ generate the C^*-algebra E, we conclude that $\psi(x)-\psi'(x) \in K \otimes A$ for all x in E. Thus ψ and ψ' are homomorphisms from E to $E \otimes A$ which agree mod $K \otimes A$.

Any element of Γ can be uniquely written in the form ws, with w in Γ_1 and s in S. Define $v: l^2(\Gamma) \to l^2(\Gamma_1 \times S)$ by $v\delta(ws)=\delta(w, s)$ and let $\theta(x)=v\pi(x)v^*$ $(x \in E)$. For t in S,

$$\theta(\nu(t))\,\delta(w, s) = v\delta(tws) = \begin{cases} \delta(tw, s) & \text{if } w \neq e \\ \delta(e, ts) & \text{if } w = e. \end{cases} \tag{2}$$

Thus $\theta(\nu(t))=q(\Gamma_1^\dagger)\,\nu(t) \otimes 1 + q_e \otimes \nu(t) \in E \otimes B$ (with $B=C_r^*(S)$). Similarly $\theta(\mu(l))= \mu(l) \otimes 1 \in E \otimes B$ for l in G. So θ is a homomorphism from E to $E \otimes B$.

Define $\theta': E \to E \otimes B$ by $\theta'(x)=x \otimes 1$. It is clear that θ and θ' agree mod $K \otimes B$.

If we identify A and B with the direct summands of $A \oplus B$ then we may regard any homomorphism into or out of A (or B) as being a homomorphism to or from $A \oplus B$. With this convention, we have

$$\mu_* + \nu_*: K_*(A \oplus B) \to K_*(E),$$

$$(\theta/\theta')_* - (\psi/\psi')_*: K_*(E) \to K_*(A \oplus B).$$

We aim to show that these maps are inverses of each other, so that $K_*(E) \cong K_*(A) \oplus K_*(B)$.

5. The main theorem

We continue to assume that $\Gamma = G * S$ where G has property Λ, $A=C_r^*(G)$ and $B=C_r^*(S)$. Let (λ_t) be a K-homotopy of representations of G on $l^2(G)$ joining the left regular representation λ_0 to a representation λ_1 which fixes $\delta(e)$.

Recall from the previous section that $\psi: E \to E \otimes A$ is a homomorphism with $\psi(1)=q(\Gamma_1^\dagger) \otimes 1$. Denote by i_1 the map $a \mapsto q_e \otimes a$ from A to $K \otimes A \subseteq E \otimes A$. Then $\psi(1)i_1(1)=0$ so we can form the (unital) homomorphism $\psi\mu \oplus i_1: A \to E \otimes A$.

LEMMA 5.1. *The homomorphisms $\psi\mu \oplus i_1$ and $\psi'\mu$ from A to $E \otimes A$ are $K \otimes A$-homotopic.*

Proof. For $0 \leq t \leq 1$ and l in G define an operator $\varphi_t(l)$ on $l^2(\Gamma_1 \times G)$ by

$$\varphi_t(l)\,\delta(wg,h)=\begin{cases}\delta(lwg,h) & \text{if } w\in\Gamma_2^*, g\in G^*\\ \sum_{k\in G}\langle\lambda_t(l)\,\delta(g),\delta(k)\rangle\,\delta(k,k^{-1}lgh) & \text{if } w=e, g\in G.\end{cases}$$

We wish to show that there is a homomorphism from A to $E\otimes A$ which extends φ_t. To do this, we exhibit a unitary operator u_t on $l^2(\Gamma_1\times G)$ such that

$$\varphi_t(l)=u_t(\mu(l)\otimes 1)\,u_t^*\quad(l\in G).$$

Define u_t on the basis vectors as follows:

$$u_t\delta(g,h)=\sum_{k\in G}\langle\lambda_t(gh)\,\delta(h^{-1}),\delta(k)\rangle\,\delta(k,k^{-1}gh)\quad(g,h\in G),$$

$$u_t\delta(wg,h)=\delta(wg,h)\quad(w\in\Gamma_2^*, g\in G^*, h\in G).$$

It is easily verified that the adjoint operator is the identity on the range of $q(\Gamma_1\diagdown G)\otimes 1$ and is specified on the range of $q(G)\otimes 1$ by

$$u_t^*\delta(g,h)=\sum_{k\in G}\langle\delta(g),\lambda_t(gh)\,\delta(k^{-1})\rangle\,\delta(ghk^{-1},k).$$

As in the proof of Lemma 4.1, one verifies that u_t is unitary and also that $\varphi_t(l)=u_t(\mu(l)\otimes 1)u_t^*$ $(l\in G)$. Thus φ_t extends to a homomorphism from A into $B(l^2(\Gamma_1\times G))$. An argument like that used in the proof of Lemma 4.2 shows that $\varphi_t(x)-\varphi_0(x)\in K\otimes A$ $(x\in A)$. It is clear that $\varphi_0=\psi'\mu$ and $\varphi_1=\psi\mu\oplus i_1$. Thus $\varphi_t(x)\in E\otimes A$ $(0\leq t\leq 1,\ x\in A)$, and (φ_t) is a K-homotopy connecting $\psi\mu\oplus i_1$ to $\psi'\mu$.

The next result which we need is a more complicated version of Lemma 5.1 in which A is replaced by E.

Let j denote the map $x\mapsto q_e\otimes x$ from E to $K\otimes E\subseteq E\otimes E$ and let $\bar\mu=1\otimes\mu\colon E\otimes A\to E\otimes E$. Then $j(1)=q_e\otimes 1$ and $\bar\mu\psi(1)=q(\Gamma_1^*)\otimes 1$, so we may form the (unital) homomorphism $\bar\mu\psi\oplus j\colon E\to E\otimes E$. Also, let k denote the map $x\mapsto x\otimes q_e$ from E to $E\otimes E$ and let $\bar\nu=1\otimes\nu\colon E\otimes B\to E\otimes E$. Then $k(1)=1\otimes q_e$ and $\bar\nu\theta(1)=1\otimes q(\Gamma_1^*)$, so we may form the (unital) homomorphism $\bar\nu\theta\oplus k\colon E\to E\otimes E$.

LEMMA 5.2. *The homomorphisms $\bar\mu\psi\oplus j$ and $\bar\nu\theta\oplus k$ from E to $E\otimes E$ are $K\otimes E$-homotopic.*

Proof. For l in G and $0\leq t\leq 1$, define an operator $\Phi_t(\mu(l))$ on $l^2(\Gamma_1\times\Gamma_1)$ by

$$\Phi_t(\mu(l))\,\delta(w,w') = \begin{cases} \delta(lw,w') & \text{if } w\in\Gamma_1\setminus G \\ \sum_{k\in G}\langle\lambda_t(l)\,\delta(w),\delta(k)\rangle\,\delta(k,k^{-1}lww') & \text{if } w\in G\end{cases}$$

This operator can be described as follows. Any element w' of Γ_1 can be uniquely written as $w'=hw''$ with h in G and w'' in Γ_1^2, and by means of the correspondence $\delta(w,w')\mapsto\delta(w,h,w'')$ we can identify $l^2(\Gamma_1\times\Gamma_1)$ with $\text{card}(\Gamma_1^2)$ copies of $l^2(\Gamma_1\times G)$. Under this identification, $\Phi_t(\mu(l))$ is just the direct sum of $\text{card}(\Gamma_1^2)$ copies of the operator $\varphi_t(l)$ constructed in Lemma 5.1. It follows that $\Phi_t(\mu(l))$ is a unitary element of $E\otimes E$ which depends continuously on t, and also that $\Phi_t\mu$ is a representation of G on $l^2(\Gamma_1\times\Gamma_1)$ which agrees mod $K\otimes E$ with $\Phi_0\mu$.

For s in S and $0\leq t\leq 1$ define $\Phi_t(\nu(s))$ by

$$\Phi_t(\nu(s))\,\delta(w,w') = \begin{cases}\delta(sw,w') & \text{if } w\neq e \\ \delta(e,sw') & \text{if } w=e,\ w'\neq e. \\ 0 & \text{if } w=w'=e\end{cases}$$

Then $\Phi_t(\nu(s))$ (which is obviously independent of t) is a partial isometry in $E\otimes E$ whose initial and final spaces are both equal to $\{\delta(e,e)\}^\perp$, and $\Phi_t\nu$ is a representation of S.

We shall show that for $0\leq t\leq 1$ there is a homomorphism $\Phi_t\colon E\to E\otimes E$ whose values at $\mu(l)$ and $\nu(s)$ are as above. For $t=0$ or $t=1$ there is indeed such a homomorphism, since a routine verification shows that $\Phi_0=\bar{\nu}\theta\oplus k$ and $\Phi_1=\bar{\mu}\psi\oplus j$. If Φ_t exists, it will clearly be unique, will agree mod $K\otimes E$ with Φ_0 and will vary continuously with t. We shall establish the existence of Φ_t, and thereby prove the lemma, by exhibiting a unitary operator m_t on $l^2(\Gamma_1\times\Gamma_1)$ such that $m_t\Phi_0(x)\,m_t^{-1}=\Phi_t(x)$ whenever $x=\mu(l)$ or $x=\nu(s)$.

Let $w=\dots s_{-1}g_0s_0g_1\dots\in\Gamma$ and let $\sigma(w)=\dots\nu(s_{-1})\mu(g_0)\,\nu(s_0)\mu(g_1)\dots$ (as in section three). We define $\Phi_t(\sigma(w))$ in $E\otimes E$ to be the corresponding product

$$\dots\Phi_t(\nu(s_{-1}))\,\Phi_t(\mu(g_0))\,\Phi_t(\nu(s_0))\,\Phi_t(\mu(g_1))\dots.$$

Notice that any element of Γ_1^* can be uniquely expressed in the form shw' where $s\in S$, $h\in G^*$, $w'\in\Gamma_1^2$. Define m_t on the basis elements of $l^2(\Gamma_1\times\Gamma_1)$ by the following formulae:

$$m_t\,\delta(e,e)=\delta(e,e),$$

$$m_t\,\delta(e,shw')=\sum_{k\in G^*}\langle\lambda_t(h)\,\delta(h^{-1}),\delta(k)\rangle\,\delta(sk,k^{-1}hw')+\langle\lambda_t(h)\,\delta(h^{-1}),\delta(e)\rangle\,\delta(e,shw')$$

$$(s\in S,h\in G^*,w'\in\Gamma_1^2),$$

$$m_t\,\delta(w,w')=\Phi_t(\sigma(w))\,m_t\,\delta(e,w')\quad(w\in\Gamma_1^*,w'\in\Gamma_1).$$

By linearity, m_t is defined on the prehilbert space H_0 spanned algebraically by the basis vectors. It is easy to verify that

$$m_t\Phi_0(\mu(l))\,\delta(w,w')=\Phi_t(\mu(l))\,m_t\,\delta(w,w'),$$

$$m_t\Phi_0(\nu(s))\,\delta(w,w')=\Phi_t(\nu(s))\,m_t\,\delta(w,w')\quad(l\in G,s\in S,w,w'\in\Gamma_1).$$

Thus $m_t\Phi_0(x)=\Phi_t(x)\,m_t$ on H_0 whenever $x=\mu(l)$ or $x=\nu(s)$. To complete the proof we have to show m_t is unitary on H_0 and so extends to a unitary operator on $l^2(\Gamma_1\times\Gamma_1)$.

Any element of $\Gamma_1\times\Gamma_1$ can be uniquely written in the form (wg,hw') with $g,h\in G$, $w\in\Gamma_2$, $w'\in\Gamma_1^2$ (and in fact $g\in G^*$ unless $w=e$). Define

$$\Delta_0=\{(wg,hw')\in\Gamma_1\times\Gamma_1: g\neq e,\ gh=e\},\ \Delta_1=(\Gamma_1\times\Gamma_1)\setminus\Delta_0.$$

It is easy to check that m_t leaves $\delta(w,w')$ fixed if $(w,w')\in\Delta_0$. We define an equivalence relation on Δ_1 by

$$(w_1,w_1')\sim(w_2,w_2')\quad\text{if and only if}\quad w_1w_1'=w_2w_2'.$$

Each element of Δ_1 is equivalent to exactly one element of the form (e,w), where

$$w=g_1s_1g_2s_2\ldots g_{n-1}s_{n-1}g_n\ (n\geq 1,g_1\in G,g_i\in G^*\text{ for }i>1,s_i\in S^*\text{ for }i\geq 1),$$

and the equivalence class containing (e,w) consists of

$$\{(g_1s_1\ldots s_{j-1}k,k^{-1}g_js_j\ldots g_n): 1\leq j\leq n, k\in G\text{ if }j=1,k\in G^*\quad\text{if }j>1\}.$$

Fix w in Γ_1 as above, and also fix t in $[0,1]$. Write $\varepsilon(j,k)=\delta(g_1s_1\ldots s_{j-1}k,k^{-1}g_js_j\ldots g_n)$ and denote by H_w the closed subspace of $l^2(\Gamma_1\times\Gamma_1)$ spanned by the $\varepsilon(j,k)$. By inspecting the definitions of m_t and Φ_t, one sees that m_t leaves H_w invariant. We complete the proof by showing that m_t maps H_w isometrically onto itself.

With w fixed as above, define unit vectors η_j $(0 \leqslant j \leqslant n)$ in H_w by

$$\eta_0 = \varepsilon(1, e),$$

$$\eta_j = \sum_{k \in G^*} \langle \lambda_t(g_j)\,\delta(e), \delta(k)\rangle\, \varepsilon(j,k) + \langle \lambda_t(g_j)\,\delta(e), \delta(e)\rangle\, \eta_{j-1} \quad (j > 1).$$

By induction on j one sees that

$$m_t\,\varepsilon(j,k) = \sum_{p \in G^*} \langle \lambda_t(g_j)\,\delta(g_j^{-1}k), \delta(p)\rangle\, \varepsilon(j,p) + \langle \lambda_t(g_j)\,\delta(g_j^{-1}k), \delta(e)\rangle\, \eta_j \quad \text{if } k \neq g_j,$$

$$m_t\,\varepsilon(j,g_j) = \sum_{p \in G^*} \langle \lambda_t(g_{j+1})\,\delta(g_{j+1}^{-1}k), \delta(p)\rangle\, \varepsilon(j+1,p) + \langle \lambda_t(g_{j+1})\,\delta(g_{j+1}^{-1}k), \delta(e)\rangle\, \eta_j, \tag{3}$$

except that if $j=n$ and $k=g_n$ then $m_t(n, g_n)$ is given by the first of these formulae rather than the second (so that in fact $m_t(n, g_n)=\eta_n$).

Since $\{\varepsilon(j,k)\}$ is an orthonormal basis for H_w, it follows that H_w is isomorphic to $l^2(G) \oplus H'$, where H' is the direct sum of $n-1$ copies of $l^2(G^*)$. We wish to define n unitary mappings on H_w each of which perturbs a subspace H_j isomorphic to $l^2(G)$ and leaves its orthogonal complement fixed. To do this, we specify (for $1 \leqslant j \leqslant n$) a unitary map v_j from $l^2(G) \oplus H'$ to H_w which takes $l^2(G)$ to H_j in a manner to be described and maps H' in any unitary manner onto $H_j^{\perp}$. We then define a unitary map u_j on $l^2(G) \oplus H'$ which will be specified on $l^2(G)$ and will be the identity on H', and we form the unitary map $v_j u_j v_j^{-1}$ on H_w. It is clear from this that we only need to describe u_j and v_j on the subspace $l^2(G)$.

We define $u_j = \lambda_t(g_j)\,\lambda_0(g_j^{-1})$ on $l^2(G)$ and we define v_j on $l^2(G)$ by induction as follows:

$$v_1\delta(p) = \varepsilon(1,p) \quad (p \in G),$$

and for $1 < j \leqslant n$

$$v_j\delta(p) = \varepsilon(j,p) \quad (p \in G^*),$$

$$v_j\delta(e) = \eta_{j-1}.$$

Notice that η_{j-1} is in the subspace of H_w spanned by $\{\varepsilon(i,k) : i < j\}$ and is therefore orthogonal to $\varepsilon(j,p)\,(p \in G^*)$. It follows that v_j maps $l^2(G)$ isometrically to a subspace H_j of H_w. Hence $v_j u_j v_j^{-1}$ is unitary.

From the equations (3) we see that

$$v_1u_1v_1^{-1}\varepsilon(1,k)=m_t\varepsilon(1,k)\quad(k\in G, k\neq g_1),$$

$$v_1u_1v_1^{-1}\varepsilon(1,g_1)=\eta_1,$$

and $v_1u_1v_1^{-1}$ leaves $\varepsilon(j,k)$ fixed for $j>1$ and $k\in G^*$. Similarly (by induction on j), for $1<j\leq n$

$$v_ju_jv_j^{-1}\varepsilon(j,k)=m_t\varepsilon(j,k)\quad(k\in G^*, k\neq g_j),$$

$$v_ju_jv_j^{-1}\varepsilon(j,g_j)=\eta_j,$$

$$v_ju_jv_j^{-1}\eta_{j-1}=m_t\varepsilon(j-1,g_{j-1}),$$

while $v_ju_jv_j^{-1}$ leaves $\varepsilon(i,k)$ fixed for $i>j$ and also leaves $m_t\varepsilon(i,k)$ fixed for $i<j$ except when $(i,k)=(j-1,g_{j-1})$.

This shows that the restriction of m_t to H_w is equal to

$$v_nu_nv_n^{-1}v_{n-1}u_{n-1}v_{n-1}^{-1}\dots v_1u_1v_1^{-1}$$

and is therefore unitary, as required.

THEOREM 5.3. *The map $\mu_*+\nu_*$ is an isomorphism from $K_*(A\oplus B)$ onto $K_*(E)$.*

Proof. Recall that $\theta'\nu: B\to E\otimes B$ is a homomorphism with $\theta'\nu(1)=q(\Gamma_1^*)\otimes 1$. Denote by i_2 the map $b\mapsto q_e\otimes b$ from B to $K\otimes B\subseteq E\otimes B$. Then we can form the homomorphism $\theta'\nu\oplus i_2$, and it is easy to check that $\theta'\nu\oplus i_2=\theta\nu$. It follows from Lemmas 1.7 and 1.3 that

$$i_{2*}=(\theta\nu/\theta'\nu)_*=(\theta/\theta')_*\nu_*.$$

Similarly, from Lemmas 5.1 and 1.2,

$$i_{1*}=(\psi'\mu/\psi\mu)_*=(\psi'/\psi)_*\mu_*=-(\psi/\psi')_*\mu_*.$$

It is clear from the definitions of ψ, ψ', θ and θ' that $\psi\nu=\psi'\nu$ and $\theta\mu=\theta'\mu$. Hence $(\psi/\psi')_*\nu_*=(\theta/\theta')_*\mu_*=0$, and so

$$((\theta/\theta')_*-(\psi/\psi')_*)(\mu_*+\nu_*)=i_{1*}+i_{2*}.$$

However, $i_1\oplus i_2$ is just the canonical map from $A\oplus B$ into a corner of $K\otimes(A\oplus B)$, so that $i_{1*}+i_{2*}$ is the identity on $K_*(A\oplus B)$.

Observe next that $\bar{\mu}\psi' = \bar{\nu}\theta' \oplus k$, since it is trivial to verify that both sides give the map $x \mapsto x \otimes 1 : E \to E \otimes E$. From Lemma 5.2 and the lemmas in section one we deduce that

$$\begin{aligned} j_* &= ((\bar{\nu}\theta \oplus k)/\mu\psi)_* \\ &= ((\bar{\nu}\theta \oplus k)/(\bar{\nu}\theta' \oplus k))_* + (\bar{\mu}\psi'/\bar{\mu}\psi)_* \\ &= \nu_*(\theta/\theta')_* - \mu_*(\psi/\psi')_*. \end{aligned}$$

Now ψ and ψ' map E into $E \otimes A \subseteq E \otimes (A \oplus B)$, but (by our convention that maps into or out of A or B are to be identified with maps to or from $A \oplus B$) $\bar{\nu} : E \otimes (A \oplus B) \to E \otimes E$ annihilates the first direct summand of $A \oplus B$. Hence $\bar{\nu}\psi = \bar{\nu}\psi' = 0$ and $\nu_*(\psi/\psi')_* = 0$. Similarly $\mu_*(\theta/\theta')_* = 0$. We conclude that

$$j_* = (\mu_* + \nu_*)((\theta/\theta')_* - (\psi/\psi')_*).$$

Finally, j maps E into a corner of $K \otimes E$ so j_* is the identity map. Thus the maps $\mu_* + \nu_*$ and $(\theta/\theta')_* - (\psi/\psi')_*$ are inverses of each other, as required.

THEOREM 5.4. *Let $\Gamma = G * S$, where G has property Λ, and let A, B be the reduced C^*-algebras of G, S. The K-groups of $C_r^*(\Gamma)$ are given by the short exact sequence*

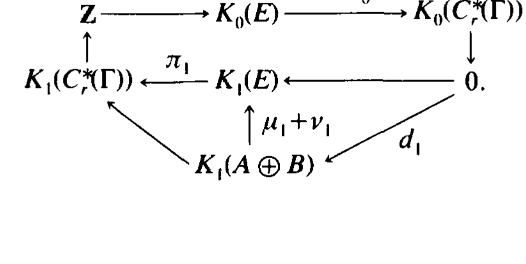
$$0 \to K_*(\mathbf{C}) \xrightarrow{\varkappa_{1*} - \varkappa_{2*}} K_*(A \oplus B) \xrightarrow{\varepsilon_{1*} + \varepsilon_{2*}} K_*(C_r^*(\Gamma)) \to 0,$$

where $\varkappa_1$, $\varkappa_2$ denote the embeddings of $\mathbf{C}$ into the scalar multiples of the identity in A, B, and ε_1, ε_2 are the embeddings of A, B in $C_r^*(\Gamma)$.

Proof. From the short exact sequence

$$0 \to K \to E \to C_r^*(\Gamma) \to 0$$

and the fact that $\mu_* + \nu_*$ is an isomorphism we obtain the following diagram by Bott periodicity.



Since $q_e=\mu(e)-\nu(e)$, it is easily checked that d_0 takes the generator $[q_e]_0$ of $\mathbf{Z}=K_0(K)$ to $[i_1(1)]_0-[i_2(1)]_0$, from which one sees that $d_*=\varkappa_{1*}-\varkappa_{2*}$. Since the algebra A is finite, with a trace given by $x\mapsto\langle x\delta(e),\delta(e)\rangle$, $[i_1(1)]_0$ cannot be the zero element of $K_0(A)$ and therefore d_0 is not the zero mapping. By exactness at $\mathbf{Z}$, it follows that the index map from $K_1(C_r^*(\Gamma))$ to $\mathbf{Z}$ is the zero map. The map from $K_0(C_r^*(\Gamma))$ to 0 is obviously the zero map. So the upper and lower halves of the above diagram can be separated, and the statement of the theorem follows.

COROLLARY 5.5. *If $G_1, G_2, \ldots, G_n$ are nontrivial countable amenable groups and Γ is their free product then*

$$K_0(C_r^*(\Gamma)) = \left\{\left(\bigoplus_{i=1}^{n} K_0(C^*(G_i))\right\}\right/\mathbf{Z}^{n-1},$$

$$K_1(C_r^*(\Gamma)) = \bigoplus_{i=1}^{n} K_1(C^*(G_i)).$$

Proof. This follows from Theorem 5.4 by induction on n: take $G=G_n$, $S=G_1*G_2*\ldots*G_{n-1}$.

Added in proof. Since this work was done there have been several significant developments in this area. J. Cuntz ("K-theoretic amenability for discrete groups", preprint) has used the machinery of Kasparov's KK-theory to give a very neat computation of the K-groups for the reduced C^*-algebras of a class of groups apparently more general than that considered in this paper. In two further preprints ("Generalized homomorphisms between C^*-algebras and KK-theory", "K-theory and C^*-algrebras") he has shown how KK-theory can be developed using what in this paper are called difference maps as the basic elements of the theory. In fact, the difference maps constructed in section four above furnish some instructive examples of elements of certain KK-groups and may be usefully contemplated by anyone wishing to learn KK-theory.

C. Schochet and S. Wassermann (private communications) have pointed out that, contrary to what is implied in section one, the difference map can be defined in a purely algebraic, functorial way. Indeed, the ideas for doing this are essentially present in J. Milnor's book "Introduction to algebraic K-theory" (Princeton, 1971).

Finally, A. Connes ("The Chern character in K-homology", preprint) has given a very short and self-contained proof that the reduced C^*-algebra of the free group on two generators has no nontrivial projections. This proof, although inspired by K-theoretic ideas, uses absolutely none of the heavy machinery of K-theory or KK-theory.

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