A COMPLEX TENSOR CALCULUS FOR KÄHLER MANIFOLDS.

By

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Introduction.

In this paper we develop a complex tensor calculus for Kähler manifolds and apply it to obtain results concerning analytic p-vectors on such manifolds. The Stokes and Brouwer operators d and δ are real operators in the sense that they send real p-vectors into real ones. We define complex analogues of these operators in terms of which the classical Laplace-Beltrami operator Δ for p-forms is split into a complex Laplace-Beltrami operator $\Box$ and its conjugate $\bar{\Box}$. In the case of scalars on Kähler manifolds and in the case of p-vectors of arbitrary degree p in Euclidean space we have $\Box = \bar{\Box} = \frac{1}{2}\Delta$.[1]

In Section 4 the complex operators are defined for currents of degree p (in the sense of de Rham). In Section 6 a method is given in terms of the complex operators for finding the complex-analytic projection of an arbitrary norm-finite p-vector, and in Section 7 this method is extended to currents. In Sections 8 and 9 the real operator Δ is investigated on submanifolds of the given manifold. Here the Kähler property of the metric is not used; therefore the results of Sections 8 and 9 are valid for Riemannian manifolds. In particular, it is established that every finite submanifold possesses a singular kernel $g_p(x, y)$ satisfying $\Delta_x g_p(x, y) = -\beta_p(y, x)$ where β_p is the reproducing kernel for harmonic p-forms (the existence of $g_p(x, y)$ on compact manifolds has been proved by de Rham).

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[1] (Added in proof.) The complex operators as defined below were introduced by the authors in a report having the same title as this paper [Technical Report No. 17, Stanford University, California (May 21, 1951)]. The same operators, in a different notation, were introduced independently by Hodge [Proc. Cambridge Phil. Soc., 47 (July 1951)] who proved the equality of the operators $\Box$ and $\bar{\Box}$ in all cases. Since the present paper was submitted for publication before the appearance of Hodge's paper, we have not been able to use this identity to simplify some of the later portions of this paper. However, we remark that the identity $\Box = \bar{\Box}$ follows readily from formula (2.26) below and from Ricci's identity.

In Section 10 we return to the complex operators in order to establish certain relations between harmonic forms and complex-analytic ones. In Section 11 we derive properties for the complex Laplace-Beltrami operator $\square$ analogous to those derived in the earlier sections for the real operator Δ. In Section 12 we solve a generalized version of the complex boundary value problem treated in [8] for Euclidean space and we show that this problem has a unique solution for every finite submanifold of a compact Kähler manifold.

In Section 13 we establish the existence of Kodaira's fundamental singularity for real harmonic fields on an arbitrary Riemannian manifold of class C^∞ by a method which is different from that of Kodaira and which seems to us to lie closer to the classical method used on Riemann surfaces. In Section 14 we construct the corresponding singularity for complex harmonic fields, but here we have to assume that the manifold is a subdomain of a compact Kähler manifold. Finally, in Section 15 we make a remark concerning the Cousin problem for complex harmonic fields on a compact Kähler manifold.

1. Complex manifolds.

For the sake of completeness, we bring together in this section various known properties of complex manifolds.

A complex manifold M^k of complex dimension k is a Hausdorff space to each point p of which there is associated a neighborhood $N(p)$ which is mapped topologically onto a subdomain of the Euclidean space of the complex variables $z^1, \ldots, z^k$. If $q \in N(p)$, the coordinates of q will be denoted by $z^i(q)$, $i = 1, 2, \ldots, k$. Wherever two neighborhoods intersect, the coordinates are connected by a pseudo-conformal mapping.

Following [5] we introduce a conjugate manifold $\bar{M}^k$ which is a homeomorphic image of M^k in which the point p of M^k corresponds to the point $\bar{p}$ of $\bar{M}^k$ and the neighborhood $N(p)$ to $\bar{N}(\bar{p})$. Let Latin indices run from 1 to $2k$, and let

$$\bar{\imath} = 1 + k \pmod{2k}. \tag{1.1}$$

If $\bar{q} \in \bar{N}(\bar{p})$, we define

$$z^{\bar{\imath}}(\bar{q}) = (z^i(\bar{q}))^-, \tag{1.2}$$

where $(z)^-$ denotes the complex conjugate of the quantity z. By means of (1.2) the neighborhood $\bar{N}(\bar{p})$ is mapped onto a domain in the space of the variables $z^{\bar{\imath}} = \bar{z}^i$ $(i = 1, 2, \ldots, k)$.

Now consider the product manifold $M^k \times \bar{M}^k$ whose points are the ordered pairs $(p, \bar{q})$, and let

$$z^i(p, \bar{q}) = \begin{cases} z^i(p), & i = 1, 2, \ldots, k \\ z^{\bar{i}}(\bar{q}) = (z^{\bar{i}}(\bar{q}))^-, & i = k+1, \ldots, 2k. \end{cases} \tag{1.3}$$

Then

$$z^i(p, \bar{q}) = (z^{\bar{i}}(q, \bar{p}))^-, \quad i = 1, 2, \ldots, 2k. \tag{1.4}$$

The product manifold $M^k \times \bar{M}^k$ is covered by the coordinates $z^i(p, \bar{q})$ $i = 1, 2, \ldots, 2k$. Introduce coordinates $x^i(p, \bar{q})$ by the formulas

$$z^i = \frac{1+\sqrt{-1}}{2} x^i + \frac{1-\sqrt{-1}}{2} x^{\bar{i}}, \quad x^i = \frac{1-\sqrt{-1}}{2} z^i + \frac{1+\sqrt{-1}}{2} z^{\bar{i}}, \tag{1.5}$$

$$i = 1, 2, \ldots, 2k.$$

Then

$$x^i(p, \bar{q}) = (x^i(q, \bar{p}))^-, \quad i = 1, 2, \ldots, 2k. \tag{1.6}$$

On the diagonal manifold D^k of $M^k \times \bar{M}^k$ where $p = q$, we have

$$z^i = z^i(p, \bar{p}) = (z^{\bar{i}})^-, \quad x^i = x^i(p, \bar{p}) = (x^i)^-. \tag{1.7}$$

Thus D^k is covered either by the self-conjugate coordinates z^i, $z^{\bar{i}} = \bar{z}^i$, $i = 1, 2, \ldots, 2k$, or by the real coordinates x^i.

We shall be concerned mainly with the diagonal space D^k. A tensor on D^k whose components are real when they are expressed in the real coordinates x^i will be called a real tensor. A real tensor T when expressed in self-conjugate coordinates z^i satisfies

$$T_{i \ldots j}{}^{l \ldots m} = (T_{\bar{i} \ldots \bar{j}}{}^{\bar{l} \ldots \bar{m}})^-. \tag{1.8}$$

Let unbarred Greek indices run from 1 to k, and write

$$\bar{\alpha} = \alpha + k, \quad \bar{\bar{\alpha}} = \alpha. \tag{1.8'}$$

Then (1.8) can also be written

$$T_{\alpha\bar{\beta} \ldots \gamma}{}^{\bar{\mu}\nu \ldots \lambda} = (T_{\bar{\alpha}\beta \ldots \bar{\gamma}}{}^{\mu\bar{\nu} \ldots \bar{\lambda}})^-. \tag{1.9}$$

The tensors properly associated with the original manifold M^k are the complex analytic ones whose indices range over values from 1 to k (Section 6).

On D^k there is a "quadrantal versor" which is a real tensor $h_i{}^j$ satisfying

$$h_i{}^j h_j{}^l = \begin{cases} -1, & i = l \\ 0, & i \neq l. \end{cases} \tag{1.10}$$

In self-conjugate coordinates z^i this tensor has the components

$$h_i{}^j(z) = \begin{cases} \sqrt{-1}, & 1 \le i = j \le k \\ -\sqrt{-1}, & k+1 \le i = j \le 2k \\ 0, & i \ne j \end{cases} \tag{1.11}$$

or, in the real coordinates x^i,

$$h_i{}^j(x) = \begin{cases} 1, & i = \bar{j},\ i \le i \le k \\ -1, & i = \bar{j},\ k+1 \le i \le 2k \\ 0, & i \ne \bar{j}. \end{cases} \tag{1.11'}$$

The values (1.11), (1.11′) are pseudo-conformal invariants.

Given a vector φ_i, let

$$(I\varphi)_i = \varphi_i \tag{1.12}$$

be the identity transformation, and let

$$(h\varphi)_i = h_i{}^j \varphi_j \tag{1.13}$$

be rotation through a "quadrant". Given real numbers a and b, the operation $aI + bh$ applied to vectors corresponds to complex multiplication in which the reality of the vector is preserved. We have

$$(aI + bh)(cI + dh) = (ac - bd)I + (ad + bc)h. \tag{1.14}$$

In other words, the field obtained from the real vectors by adjoining the operator h is isomorphic to the complex number field.

Now suppose that D^k carries a Kähler metric g_{ij}. A Kähler metric is a Riemannian metric which satisfies the following two conditions:

a) $g_{ij} = g_{pq} h_i{}^p h_j{}^p$,

b) $D_p(h_i{}^j \varphi_j) = h_i{}^j D_p \varphi_j$.

Here

$$D_p \varphi_j = \frac{\partial \varphi_j}{\partial z^p} - \begin{Bmatrix} q \\ j\ p \end{Bmatrix} \varphi_q \tag{1.15}$$

denotes covariant differentiation, $\begin{Bmatrix} q \\ j\ p \end{Bmatrix}$ being the coefficients of affine connection. Condition a) states that the vectors φ_i and $(h\varphi)_i$ have the same length, while b) states that the operators h and D commute: $Dh = hD$.

Let

$$h_{ij} = g_{jl}\, h_i{}^l. \tag{1.16}$$

Multiplying both sides of a) by $h_r{}^j$ and summing on j from 1 to $2k$, we obtain

$$h_{ri} = g_{ij}\, h_r{}^j = g_{pq}\, h_i{}^p\, h_j{}^q\, h_r{}^j = -g_{pr}\, h_i{}^p = -h_{ir}. \tag{1.17}$$

Thus h_{ij} is skew-symmetric, and hence by (1.16)

$$h_{ij}\, h_p{}^i\, h_q{}^j = g_{jl}\, h_i{}^l\, h_p{}^i\, h_q{}^j = -g_{jp}\, h_q{}^j = -h_{qp} = h_{pq};$$

that is,

$$h_{pq} = h_{ij}\, h_p{}^i\, h_q{}^j. \tag{1.18}$$

In terms of self-conjugate coordinates z^i, the formula (1.18) shows by (1.11) that any non-zero component of h_{pq} is necessarily of the form $h_{\alpha\bar\beta}$ or $h_{\bar\alpha\beta}$. In other words, $h_{pq}=0$ unless p and q are indices of opposite parity with respect to conjugation.

Condition b) gives

$$h_p{}^q \begin{Bmatrix} p \\ i\ \ j \end{Bmatrix} = h_i{}^p \begin{Bmatrix} q \\ p\ \ j \end{Bmatrix}. \tag{1.19}$$

Taking $q=\alpha$, $i=\bar\beta$ and using self-conjugate coordinates, we obtain

$$\sqrt{-1} \begin{Bmatrix} \alpha \\ \bar\beta\ \ j \end{Bmatrix} = -\sqrt{-1} \begin{Bmatrix} \alpha \\ \bar\beta\ \ j \end{Bmatrix},\ j = 1, 2, \ldots, 2k.$$

Hence

$$\begin{Bmatrix} \alpha \\ \bar\beta\ \ j \end{Bmatrix} = \left(\begin{Bmatrix} \bar\alpha \\ \beta\ \ \bar j \end{Bmatrix} \right)^{-} = 0,\ j = 1, 2, \ldots, 2k, \tag{1.19'}$$

and therefore the only non-zero components of the coefficients of affine connection are those with all three indices of the same parity. Since

$$\begin{Bmatrix} p \\ i\ \ j \end{Bmatrix} = \tfrac{1}{2} g^{pq} \left[\frac{\partial g_{iq}}{\partial z^j} + \frac{\partial g_{jq}}{\partial z^i} - \frac{\partial g_{ij}}{\partial z^q} \right],$$

we conclude that

$$\frac{\partial g_{\alpha\bar\beta}}{\partial z^\gamma} = \frac{\partial g_{\gamma\bar\beta}}{\partial z^\alpha} \quad \text{or} \quad \frac{\partial h_{\alpha\bar\beta}}{\partial z^\gamma} = \frac{\partial h_{\gamma\bar\beta}}{\partial z^\alpha}. \tag{1.20}$$

A 1-form φ on D^k is a differential form of the first degree

$$\varphi = \varphi_i\, d z^i,$$

where φ_i are the components of a covariant vector, the summation convention being used. A p-form, or exterior differential form of degree p, $p>1$, is a sum of exterior products of 1-forms. Exterior multiplication, represented by the symbol $\wedge$, is associative, distributive, and satisfies (see [11]).

$$dz^i \wedge dz^j = -dz^j \wedge dz^i, \quad dz^i \wedge dz^i = 0$$
$$a \wedge dz^i = \quad dz^i \wedge a = a\, dz^i$$
$$dz^i \wedge a\, dz^j = a\, dz^i \wedge dz^j,$$

where a denotes a scalar. A p-form φ may be written in the form

$$\varphi = \varphi_{(i_1 \ldots i_p)}\, dz^{i_1} \wedge dz^{i_2} \wedge \cdots \wedge dz^{i_p} \tag{1.21}$$
$$= \sum_{i_1 < \ldots < i_p} \varphi_{i_1 \ldots i_p}\, dz^{i_1} \wedge dz^{i_2} \wedge \cdots \wedge dz^{i_p},$$

where $\varphi_{i_1 \ldots i_p}$ is a skew-symmetric covariant tensor of rank p or p-vector in the language of E. Cartan, and where the parenthesis indicates that the indices are ordered according to magnitude.

Let

$$\Gamma_{i_1 \ldots i_p, j_1 \ldots j_p} = \begin{vmatrix} g_{i_1 j_1} & \cdots & g_{i_p j_1} \\ g_{i_1 j_p} & \cdots & g_{i_p j_p} \end{vmatrix}. \tag{1.22}$$

Then

$$\Gamma_{i_1 \ldots i_p}{}^{j_1 \ldots j_p} = \begin{vmatrix} g_{i_1}{}^{j_1} & \cdots & g_{i_p}{}^{j_1} \\ g_{i_1}{}^{j_p} & \cdots & g_{i_p}{}^{j_p} \end{vmatrix} \tag{1.23}$$

is just the Kronecker symbol which is usually denoted by $\delta^{j_1 \ldots j_p}_{i_1 \ldots i_p}$. We depart from the conventional notation in this instance for reasons of notational symmetry.

The differential $d\varphi$ of a p-form is the $(p+1)$-form

$$d\varphi = (d\varphi)_{(i_1 \ldots i_{p+1})}\, dz^{i_1} \wedge \cdots \wedge dz^{i_{p+1}}, \tag{1.24}$$

where

$$(d\varphi)_{i_1 \ldots i_{p+1}} = \Gamma_{i_1 \ldots i_{p+1}}{}^{j(j_1 \ldots j_p)}\, D_j \varphi\, (j_1 \ldots j\, p) \tag{1.25}$$
$$= \frac{1}{p!} \Gamma_{i_1 \ldots i_{p+1}}{}^{j j_1 \ldots j_p}\, D_j \varphi\, (j_1 \ldots j\, p).$$

Here

$$D_j \varphi_{j_1 \ldots j_p} = \frac{\partial \varphi_{j_1 \ldots j_p}}{\partial z_j} = \sum_{\mu=1}^{p} \begin{Bmatrix} i \\ j\ j_\mu \end{Bmatrix} \varphi_{j_1 \ldots j_{\mu-1} i j_{\mu+1} \ldots j_p},$$

and we observe that

$$\Gamma_{i_1 \ldots i_{p+1}}{}^{j j_1 \ldots j_p} \begin{Bmatrix} q \\ j\ j_\mu \end{Bmatrix} = 0,$$

since $\left\{ {q \atop j\ j_\mu} \right\} = \left\{ {q \atop j_\mu\ j} \right\}$. Hence in (1.28) we may replace covariant differentiation D_j by ordinary differentiation $\partial/\partial z_j$. We have

$$d^2 \varphi = d(d\varphi) = 0. \tag{1.26}$$

A form φ satisfying $d\varphi = 0$ is said to be closed, and a form $\varphi = d\psi$ is said to be exact. Formula (1.26) therefore states that an exact form is closed.

Let

$$e_{i_1 \dots i_{2k}} = \Gamma_{i_1 \dots i_{2k}}{}^{12 \dots 2k} \sqrt{\Gamma_{12 \dots 2k,\, 12 \dots 2k}}, \tag{1.27}$$

and after de Rham [11] set

$$* \varphi = (* \varphi)_{j_1 \dots j_{2k-p}} dz^{j_1} {}_\wedge \cdots {}_\wedge dz^{j_{2k-p}}, \tag{1.28}$$

where

$$(* \varphi)_{j_1 \dots j_{2k-p}} = e_{(i_1 \dots i_p) j_1 \dots j_{2k-p}} \varphi^{(i_1 \dots i_p)}. \tag{1.29}$$

We verify that

$$* * \varphi = (-1)^p \varphi, \tag{1.30}$$

and for the scalar 1

$$* 1 = e_{12 \dots 2k} dz^1 {}_\wedge \cdots {}_\wedge dz^{2k}. \tag{1.31}$$

Thus $*1$ is just the volume element.

The co-differential $\delta\varphi$ of a p-form φ is

$$\delta \varphi = (\delta \varphi)_{(i_1 \dots i_{p-1})} dz^{i_1} {}_\wedge \cdots {}_\wedge dz^{i_{p-1}}, \tag{1.32}$$

where

$$(\delta \varphi)_{i_1 \dots i_{p-1}} = -(* d * \varphi)_{i_1 \dots i_{p-1}} = -g^{ij} D_j \varphi_{i i_1 \dots i_{p-1}}. \tag{1.33}$$

In contrast with the differential $d\varphi$, the co-differential involves the metric structure of the manifold in an essential way. We have

$$\delta^2 \varphi = \delta(\delta \varphi) = 0. \tag{1.34}$$

A form φ satisfying $\delta\varphi = 0$ is called co-closed; a form $\varphi = \delta\psi$ is said to be co-exact.

Let

$$\omega = h_{(ij)} dz^i {}_\wedge dz^j. \tag{1.35}$$

The second condition (1.20) expresses that ω is closed:

$$(1.36) \qquad d\omega = 0.$$

The condition (1.19), on the other hand, asserts that $D_l h_{ij} = 0$ and hence

$$(1.36') \qquad \delta\omega = 0.$$

Thus the form ω is both closed and co-closed.

The classical Laplace-Beltrami operator for p-forms is

$$(1.37) \qquad \Delta = \Delta' + \Delta'',$$

where

$$(1.38) \qquad \Delta' = \delta d, \quad \Delta'' = d\delta, \quad \Delta'\Delta'' = \Delta''\Delta' = 0.$$

A p-form φ satisfying $\Delta\varphi = 0$ is said to be harmonic, and one satisfying $d\varphi = \delta\varphi = 0$ is said to be a harmonic field. From (1.36) and (1.36′) we see that the 2-form ω is a harmonic field.

We recall that the Riemann curvature tensor

$$(1.39) \qquad R^m{}_{ijl} = \frac{\partial}{\partial z^j}\begin{Bmatrix} m \\ i \;\; l \end{Bmatrix} - \frac{\partial}{\partial z^l}\begin{Bmatrix} m \\ i \;\; j \end{Bmatrix} + \begin{Bmatrix} p \\ i \;\; l \end{Bmatrix}\begin{Bmatrix} m \\ p \;\; j \end{Bmatrix} - \begin{Bmatrix} p \\ i \;\; j \end{Bmatrix}\begin{Bmatrix} m \\ p \;\; l \end{Bmatrix}$$

has the symmetries

$$(1.40) \qquad \begin{cases} R_{hijl} = -R_{ihjl} = -R_{hilj} \\ R_{hijl} = R_{jlhi}. \end{cases}$$

It also satisfies the Bianchi identity

$$(1.41) \qquad R_{hijl} + R_{hjli} + R_{hlij} = 0.$$

The non-commutativity of covariant differentiation is expressed by the Ricci identity

$$(1.42) \qquad (D_i D_j - D_j D_i)\,\varphi_{i_1 \ldots i_p} = \sum_{\mu=1}^{p} \varphi_{i_1 \ldots i_{\mu-1} h i_{\mu+1} \ldots i_p} R^h{}_{i_\mu ij}.$$

In terms of geodesic coordinates y^i,

$$(1.39') \qquad R^m{}_{ijl} = \frac{\partial}{\partial y^j}\begin{Bmatrix} m \\ i \;\; l \end{Bmatrix} - \frac{\partial}{\partial y^l}\begin{Bmatrix} m \\ i \;\; j \end{Bmatrix}.$$

If the metric is Kähler, then by (1.19)

$$(1.43) \qquad h_m{}^n R^m{}_{ijl} = h_i{}^m R^n{}_{mjl}.$$

Thus, in self-conjugate coordinates $R^m{}_{ijl}$ is zero unless m and i have the same parity. In other words, $R_{hijl} = 0$ unless h, i are of different parity and also j, l. From (1.41) if follows that

$$(1.44) \qquad R_{\alpha\bar\beta\gamma\bar\delta} = R_{\alpha\bar\delta\gamma\bar\beta} = R_{\gamma\bar\delta\alpha\bar\beta}.$$

In other words, indices of the same parity commute. Finally, any non-zero component of the Ricci tensor

$$(1.45) \qquad R_{ij} = R^l{}_{ijl} = R_{ji}$$

has indices of opposite parity.

2. Complex tensors.

The tensors and operators considered in Section 1 are all real; in other words, the operators send a real tensor into a real tensor. Now we introduce complex tensors and operators.

As in [5] let

$$(2.1) \qquad \prod_{1,0}{}_i{}^j = \tfrac{1}{2}(g_i{}^j - \sqrt{-1}\, h_i{}^j).$$

The conjugate tensor is

$$(2.2) \qquad \prod_{0,1}{}_i{}^j = \overline{\prod_{1,0}{}_i{}^j} = \tfrac{1}{2}(g_i{}^j + \sqrt{-1}\, h_i{}^j).$$

conjugates always being defined in terms of a real coordinate system. Let $\varrho + \sigma = p$, $\varrho \geq 0$, $\sigma \geq 0$, and set (compare [5])

$$(2.3) \qquad \prod_{\varrho,\sigma}{}_{i_1 \ldots i_p}{}^{j_1 \ldots j_p} = \Gamma_{i_1 \ldots i_p}{}^{m_1 \ldots m_\varrho n_1 \ldots n_\sigma} \prod_{1,0}{}_{m_1}{}^{r_1} \cdots$$
$$\cdot \prod_{1,0}{}_{m_\varrho}{}^{r_\varrho} \prod_{0,1}{}_{n_1}{}^{s_1} \cdots \prod_{0,1}{}_{n_\sigma}{}^{s_\sigma} \Gamma_{(r_1 \ldots r_\varrho)(s_1 \ldots s_\sigma)}{}^{j_1 \ldots j_p}.$$

In self-conjugate coordinates

$$(2.4) \qquad \prod_{1,0}{}_i{}^j = \begin{cases} 1, & 1 \leq i = j \leq k \\ 0, & \text{otherwise}. \end{cases}$$

Therefore, any non-zero component of the tensor

$$(\prod_{\varrho,\sigma} \varphi)_{i_1 \ldots i_p} = \prod_{\varrho,\sigma}{}_{i_1 \ldots i_p}{}^{(j_1 \ldots j_p)} \varphi_{(j_1 \ldots j_p)}$$

has precisely ϱ indices between 1 and k and σ indices between $k+1$ and $2k$. In other words,

$$(2.5) \qquad \prod_{\varrho,\sigma} \varphi = \varphi_{(\alpha_1 \ldots \alpha_\varrho)(\bar\beta_1 \ldots \bar\beta_\sigma)}\, dz^{\alpha_1} \wedge \cdots \wedge dz^{\alpha_\varrho}\, dz^{\bar\beta_1} \wedge \cdots \wedge dz^{\bar\beta_\sigma}.$$

If $\varrho + \sigma = p > 2k$ or if either $\varrho < 0$ or $\sigma < 0$, we define $\prod_{\varrho,\sigma}$ to be zero. We plainly have

(2.6) $$\sum_{\varrho+\sigma=p} \prod_{\varrho,\sigma} = \Gamma$$

and

(2.7) $$\prod_{\varrho,\sigma} \prod_{\varrho',\sigma'} = \begin{cases} \prod_{\varrho,\sigma}, & \varrho=\varrho',\ \sigma=\sigma' \\ 0, & \text{otherwise.} \end{cases}$$

Thus (2.6) is an orthogonal decomposition of the identity operator Γ. Since

(2.8) $$h_i{}^j = g^{jp}\, h_{ip} = -g^{jp}\, h_{pi} = -h^j{}_i$$

we have

(2.9) $$\prod_{\varrho,\sigma} {}_{i_1 \ldots i_p, j_1 \ldots j_p} = \left(\prod_{\varrho,\sigma} {}_{j_1 \ldots j_p, i_1 \ldots i_p}\right)^{-} = \prod_{\sigma,\varrho} {}_{j_1 \ldots j_p, i_1 \ldots i_p}.$$

We define next a complex covariant differentiator, namely

(2.10) $$\mathcal{D}_i = \prod_{1,0} {}_i{}^j D_j.$$

The corresponding contravariant differentiator is

(2.10′) $$\mathcal{D}^i = g^{il}\, \mathcal{D}_l = \prod_{0,1} {}_j{}^i D^j = \overline{\prod_{1,0}} {}_j{}^i D^j.$$

The conjugate operators are

(2.11) $$\bar{\mathcal{D}}_i = \prod_{0,1} {}_i{}^j D_j,$$

(2.11′) $$\bar{\mathcal{D}}^i = \prod_{1,0} {}_j{}^i D^j.$$

In the complex tensor calculus which we propose to use the Hermitian operator $\prod$ replaces the symmetric identity operator Γ, and the complex differentiator $\mathcal{D}$ replaces D.

Formulas (1.25) and (1.33) may be written

(2.12) $$\begin{aligned} (d\varphi)_{i_1 \ldots i_{p+1}} &= \Gamma_{i_1 \ldots i_{p+1}}{}^{j(j_1 \ldots j_p)} D_j\, \varphi_{(j_1 \ldots j_p)} \\ &= \Gamma_{i_1 \ldots i_{p+1}, j(j_1 \ldots j_p)} D^j \varphi^{(j_1 \ldots j_p)}, \\ (\delta\varphi)_{i_1 \ldots i_{p-1}} &= -\Gamma_{i i_1 \ldots i_{p-1}}{}^{(j_1 \ldots j_p)} D^i \varphi_{(j_1 \ldots j_p)} \\ &= -\Gamma_{i i_1 \ldots i_{p-1}, (j_1 \ldots j_p)} D^i \varphi^{(j_1 \ldots j_p)}. \end{aligned}$$

The complex analogues of these operators are

(2.13) $$\begin{aligned} (\partial\varphi)_{i_1 \ldots i_{p+1}} &= \prod_{\varrho+1,\sigma} d \prod_{\varrho,\sigma} = \prod_{\varrho+1,\sigma} {}_{i_1 \ldots i_{p+1}}{}^{j(j_1 \ldots j_p)} \mathcal{D}_j\, \varphi_{(j_1 \ldots j_p)} \\ &= \prod_{\varrho+1,1} {}_{i_1 \ldots i_{p+1}, j(j_1 \ldots j_p)} \mathcal{D}^j \varphi^{(j_1 \ldots j_p)}, \end{aligned}$$

$$(2.13')\qquad (\mathfrak{d}\,\varphi)_{i_1\ldots i_{p-1}} = \prod_{\varrho,\sigma-1}\delta\prod_{\varrho,\sigma} = -\prod_{\varrho,\sigma}{}_{i i_1\ldots i_{p-1}}{}^{(j_1\ldots j_p)}\mathcal{D}^i\varphi_{(j_1\ldots j_p)}$$
$$= -\prod_{\varrho,\sigma}{}_{i i_1\ldots i_{p-1},(j_1\ldots j_p)}\mathcal{D}^i\varphi^{(j_1\ldots j_p)}.$$

The conjugate operators have the forms

$$(2.14)\qquad (\bar{\partial}\,\varphi)_{i_1\ldots i_{p+1}} = \prod_{\varrho,\sigma+1}d\prod_{\varrho,\sigma} = \prod_{\varrho,\sigma+1}{}_{i_1\ldots i_{p+1},j(j_1\ldots j_p)}\bar{\mathcal{D}}^j\varphi^{(j_1\ldots j_p)},$$

$$(2.14')\qquad (\bar{\mathfrak{d}}\,\varphi)_{i_1\ldots i_{p-1}} = \prod_{\varrho-1,\sigma}\delta\prod_{\varrho,\sigma} = -\prod_{\varrho,\sigma}{}_{i i_1\ldots i_{p-1},j(j_1\ldots j_p)}\bar{\mathcal{D}}^i\varphi^{(j_1\ldots j_p)}.$$

The following identities are readily verified:

$$(2.15)\qquad *\prod_{\varrho,\sigma} = \prod_{k-\sigma,k-\varrho}*,$$

$$(2.16)\qquad *\,\partial = (-1)^{p+1}\mathfrak{d}\,*,\quad *\,\mathfrak{d} = (-1)^p\partial\,*,$$

$$(2.17)\qquad \partial^2 = 0,\ \mathfrak{d}^2 = 0,$$

$$(2.18)\qquad \begin{aligned}\partial\bar{\partial}+\bar{\partial}\partial &= 0\\ \mathfrak{d}\bar{\mathfrak{d}}+\bar{\mathfrak{d}}\mathfrak{d} &= 0.\end{aligned}$$

A calculation gives

$$(2.19)\qquad (\Delta\,\varphi)_{i_1\ldots i_p} = -\Gamma_{i_1\ldots i_p}{}^{(j_1\ldots j_p)}D^iD_i\varphi_{(j_1\ldots j_p)} +$$
$$+\sum_{\mu=1}^{p}R_{i_\mu}{}^{h}\,\Gamma_{i_1\ldots i_{\mu-1}h i_{\mu+1}\ldots i_p}{}^{(j_1\ldots j_p)}\varphi_{(j_1\ldots j_p)} +$$
$$+\sum_{\mu,\nu=1}^{p}R_{i_\mu i_\nu}{}^{hl}\,\Gamma_{i_1\ldots i_{\mu-1}h i_{\mu+1}\ldots i_{\nu-1}l i_{\nu+1}\ldots i_p}{}^{(j_1\ldots j_p)}\varphi_{(j_1\ldots j_p)}.$$

In view of the properties of the curvature tensor for a Kähler metric, we have

$$(2.20)\qquad \Delta\prod_{\varrho,\sigma} = \prod_{\varrho,\sigma}\Delta.$$

Now we introduce a complex Laplace-Beltrami operator

$$(2.21)\qquad \Box = \Box' + \Box'',$$

where

$$(2.22)\qquad \Box' = \bar{\mathfrak{d}}\,\partial,\ \Box'' = \partial\,\bar{\mathfrak{d}},\ \Box'\,\Box'' = \Box''\,\Box' = 0.$$

Then

$$(2.23)\qquad \Delta\prod_{\varrho,\sigma} = \Box + \bar{\Box}.$$

where

$$(2.24)\qquad \bar{\Box} = \mathfrak{d}\,\bar{\partial} + \bar{\partial}\,\mathfrak{d}.$$

The following identities are readily seen to be valid:

$$\left\{\begin{array}{l} * \square = \bar{\square} *, \quad \square = (-1)^p * \bar{\square} *, \quad \square' * = * \bar{\square}'' \\ \partial \square = \square \partial, \quad \bar{\mathfrak{d}} \square = \square \bar{\mathfrak{d}}. \end{array}\right. \tag{2.25}$$

We note the identity

$$\prod_{\varrho,\sigma} \square = \square \prod_{\varrho,\sigma},$$

which, in contrast with (2.20), is trivial. A calculation gives

$$(\square \varphi)_{i_1 \ldots i_p} = -\prod_{\varrho,\sigma}{}_{i_1 \ldots i_p}{}^{(j_1 \ldots j_p)} \bar{\mathcal{D}}^l \mathcal{D}_l \varphi_{(j_1 \ldots j_p)} + \tag{2.26}$$

$$+ \sum_{\mu=1}^{p} \prod_{1,0}{}_{i_\mu}{}^l R_l{}^h \prod_{\varrho,\sigma}{}_{i_1 \ldots i_{\mu-1} h i_{\mu+1} \ldots i_p}{}^{(j \ldots j_p)} \varphi_{(j_1 \ldots j_p)} +$$

$$+ \tfrac{1}{2} \cdot \sum_{\mu,\nu=1}^{p} \prod_{1,0}{}_{i_\mu}{}^l \prod_{1,0}{}_q{}^h R_{l i_\nu}{}^{q t} \cdot$$

$$\cdot \prod_{\varrho,\sigma}{}_{i_1 \ldots i_{\mu-1} h i_{\mu+1} \ldots i_{\nu-1} t i_{\nu+1} \ldots i_p}{}^{(j_1 \ldots j_p)} \varphi_{(j_1 \ldots j_p)}.$$

We define the scalar product of two p-forms φ and ψ over a subdomain B of D^k to be

$$(\varphi, \psi) = \int_B \varphi_\wedge * \bar{\psi}. \tag{2.27}$$

By (2.15)

$$\prod_{\varrho,\sigma} \varphi_\wedge * \bar{\psi} = \varphi_\wedge \prod_{k-\varrho,k-\sigma} * \bar{\psi} = \varphi_\wedge * \prod_{\sigma,\varrho} \bar{\psi} = \varphi_\wedge \left(\prod_{\varrho,\sigma} \psi\right)^-.$$

Thus $\prod_{\varrho,\sigma}$ is self-adjoint; that is,

$$\left(\prod_{\varrho,\sigma} \varphi, \psi\right) = \left(\varphi, \prod_{\varrho,\sigma} \psi\right). \tag{2.28}$$

If $p \geq 2$ define

$$(\Lambda \varphi)_{i_1 \ldots i_{p-2}} = -h^{(ij)} \varphi_{(ij) i_1 \ldots i_{p-2}}, \tag{2.29}$$

while if $p = 0$ or 1 set $\Lambda \varphi = 0$. Then

$$\begin{aligned} ((\Lambda \partial - \partial \Lambda) \varphi)_{i_2 \ldots i_p} &= -h_l{}^{i_1} \mathcal{D}^l \varphi_{i_1 \ldots i_p} \\ &= \sqrt{-1}\, \mathcal{D}^{i_1} \varphi_{i_1 \ldots i_p} \\ &= -\sqrt{-1}\, (\mathfrak{d} \varphi)_{i_2 \ldots i_p}. \end{aligned}$$

That is,

$$\Lambda \partial - \partial \Lambda = -\sqrt{-1}\, \mathfrak{d}. \tag{2.30}$$

This is the complex analogue of the well-known formula

$$\Lambda d - d \Lambda = h^{-1} \, \delta h, \tag{2.30'}$$

where, in this formula, h is the operator defined by

$$h_{i_1 \dots i_p . j_1 \dots j_p} = \begin{vmatrix} h_{i_1 j_1} & \dots & h_{i_p j_1} \\ & \dots & \\ h_{i_1 j_p} & \dots & h_{i_p j_p} \end{vmatrix}. \tag{2.31}$$

3. Green's formulas.

If C^p is a p-chain on D^k with real coefficients and if φ is a $(p-1)$-form, we have the well-known Stokes' formula

$$\int_{C^p} d\varphi = \int_{bC^p} \varphi, \tag{3.1}$$

where bC^p denotes the boundary of C^p. In particular, if we take $p = 2k$ and $C^p = C^{2k} = B$, where B is a subdomain of D^k, then for a p-form φ and a $(p+1)$-form ψ we have

$$\int_B d(\varphi_\wedge * \bar{\psi}) = \int_B (d\varphi_\wedge * \bar{\psi} + (-1)^p \varphi_\wedge d * \bar{\psi}) = \int_B (d\varphi_\wedge * \bar{\psi} - \varphi_\wedge * \delta \bar{\psi}) = \int_{bB} \varphi_\wedge * \bar{\psi}.$$

Thus

$$(d\varphi, \psi) - (\varphi, \delta\psi) = \int_{bB} \varphi_\wedge * \bar{\psi}. \tag{3.2}$$

By specializing φ and ψ we derive at once from (3.2) the following well-known "real" Green's formulas:

$$\left\{\begin{aligned}
&(d\varphi, d\psi) - (\varphi, \Delta'\psi) = \int_{bB} \varphi_\wedge * d\bar{\psi} \\
&(\Delta''\varphi, \psi) - (\delta\varphi, \delta\psi) = \int_{bB} \delta\varphi_\wedge * \bar{\psi} \\
&(\Delta'\varphi, \psi) - (\varphi, \Delta'\psi) = \int_{bB} (\varphi_\wedge * d\bar{\psi} - \bar{\psi}_\wedge * d\varphi) \\
&(\Delta''\varphi, \psi) - (\varphi, \Delta''\psi) = \int_{bB} (\delta\varphi_\wedge * \bar{\psi} - \delta\bar{\psi}_\wedge * \varphi) \\
&(\Delta\varphi, \psi) - (\varphi, \Delta\psi) = \int_{bB} (\varphi_\wedge * d\bar{\psi} - \bar{\psi}_\wedge * d\varphi + \delta\varphi_\wedge * \bar{\psi} - \delta\bar{\psi}_\wedge * \varphi).
\end{aligned}\right. \tag{3.3}$$

Taking $\varphi = \prod_{\varrho, \sigma} \varphi$, $\psi = \prod_{\varrho+1, \sigma} \psi$ in (3.2), we obtain its complex analogue:

$$(\partial\varphi, \psi) - (\varphi, \bar{\mathfrak{d}}\psi) = \int_{bB} \varphi_\wedge * \bar{\psi}. \tag{3.4}$$

From (3.4) we derive immediately the "complex" Green's formulas:

$$
(3.5) \qquad \begin{cases} (\partial\varphi, \partial\psi) - (\varphi, \Box'\psi) = \int\limits_{bB} \varphi_\wedge * (\partial\psi)^- \\ (\Box''\varphi, \psi) - (\bar{\mathfrak{d}}\varphi, \bar{\mathfrak{d}}\psi) = \int\limits_{bB} \bar{\mathfrak{d}}\varphi_\wedge * \bar{\psi} \\ (\Box'\varphi, \psi) - (\varphi, \Box'\psi) = \int\limits_{bB} (\varphi_\wedge * (\partial\psi)^- - \bar{\psi}_\wedge * \partial\varphi) \\ (\Box''\varphi, \psi) - (\varphi, \Box''\psi) = \int\limits_{bB} (\bar{\mathfrak{d}}\varphi_\wedge * \bar{\psi} - (\bar{\mathfrak{d}}\psi)^-_\wedge * \varphi) \\ (\Box\varphi, \psi) - (\varphi, \Box\psi) = \int\limits_{bB} (\varphi_\wedge * (\partial\psi)^- - \bar{\psi}_\wedge * \partial\varphi + \bar{\mathfrak{d}}\psi_\wedge * \bar{\psi} - (\bar{\mathfrak{d}}\psi)^-_\wedge * \varphi). \end{cases}
$$

In (3.4) and (3.5) the forms φ and ψ are assumed to satisfy $\varphi = \prod_{\varrho,\sigma} \varphi$, $\psi = \prod_{\varrho,\sigma} \psi$.

In applying the above Green's identities we suppose that B is a compact subdomain of D^k satisfying the following condition. At each boundary point p of B there is a full neighborhood $N(p)$ of p in D^k and real coordinates $u^1, \ldots, u^{2k}$ which are functions of the x^i (defined by (1.5)) of class C^∞. We suppose that the u^{2k}-curve is orthogonal to the u^i-curves, $1 \le i \le 2k-1$. The intersection $N(p) \cap B$ is mapped topologically onto a hemisphere

$$\sum_{i=1}^{2k} (u^i)^2 < \delta^2, \ u^{2k} > 0,$$

the base $u^{2k}=0$ of the hemisphere corresponding to the boundary of B. Thus the u^i, $1 \le i \le 2k-1$, constitute a set of local parameters for the boundary. The coordinates $u^1, u^2, \ldots, u^{2k}$ will be called boundary coordinates.

Let φ be a p-form expressed in terms of boundary coordinates u^i:

$$(3.6) \qquad \varphi = \varphi_{(i_1 \ldots i_p)} du^{i_1}{}_\wedge \cdots {}_\wedge du^{i_p} = \sum_{i_1 < \cdots < i_p} \varphi_{i_1 \ldots i_p} du^{i_1}{}_\wedge \cdots {}_\wedge du^{i_p}.$$

We define

$$(3.7) \qquad \begin{cases} t\varphi = \sum\limits_{i_1 < \cdots < i_p < 2k} \varphi_{i_1 \ldots i_p} du^{i_1}{}_\wedge \cdots {}_\wedge du^{i_p} \\ n\varphi = \varphi - t\varphi. \end{cases}$$

Then

$$(3.8) \qquad *t = n*, \ t* = *n.$$

In fact, since the u^{2k}-curves are orthogonal to the boundary of B, we have on bB

$$(3.9) \qquad g_{i,2k} = g^{i,2k} = 0, \ i = 1, 2, \ldots, 2k-1,$$

and (3.8) follows from (1.29). Moreover,

$$h_{2k}{}^{2k} = h_{2k}{}^{2k}(u) = g^{2k,1} h_{2k,1} = 0. \tag{3.10}$$

Therefore, if φ is a 1-form,

$$nh\,\varphi = h_{2k}{}^{i}\,\varphi_i = nht\,\varphi. \tag{3.11}$$

At a point p of D^k let $\xi^1, \ldots, \xi^{2k-1}$, r be geodesic coordinates, where the ξ^i are direction parameters of the geodesics issuing from p and r is the geodesic distance from p. Let B be the geodesic sphere $r < r_0$. Then $u^i = \xi^i$, $i = 1, 2, \ldots, 2k-1$, and $u^{2k} = r_0 - r$ are boundary coordinates for B. In these coordinates we plainly have

$$g_{2k,2k} = 1, \; g^{2k,2k} = 1. \tag{3.12}$$

4. Currents.

The carrier of a p-form φ is the set of points where φ is different from zero. If φ vanishes identically outside some compact subdomain of the manifold, φ is said to have a compact carrier.

G. de Rham [11] has introduced the concept of current in order to bring differential forms and topological chains under a single theory. A current $T[\varphi]$ is a linear functional over the space of p-forms φ with compact carriers and of class C^∞; that is

$$T[a_1\varphi_1 + a_2\varphi_2] = a_1 T[\varphi_1] + a_2 T[\varphi_2]$$

for any constants a_1, a_2. Moreover, T is assumed to be continuous in the following sense: Let φ_μ be a sequence of forms of the linear space whose carriers are contained in a compact set lying in the interior of a domain of a self-conjugate coordinate system $z^1, \ldots, z^{2k}$ and suppose that the partial derivatives of the coefficients of the φ_μ with respect to the z^i tend uniformly to zero as μ approaches infinity; then $T[\varphi_\mu]$ tends to zero. T is said to be of dimension p and degree $2k-p$.

Let α be a form of degree $2k-p$ whose coefficients are locally integrable. Then

$$\alpha[\varphi] = \int_{D^k} \alpha_\wedge \bar{\varphi} \tag{4.1}$$

defines a current, and this current is said to be equal to the form α.

A p-chain C^p defines a current of degree $2k-p$ and dimension p:

$$C^p[\varphi] = \int_{C^p} \bar{\varphi}. \tag{4.2}$$

This current is said to equal the chain C^p.

If $v_{i_1 \dots i_p}$ is a p-vector at a point, then

$$v[* \varphi] = v_{(i_1 \dots i_p)} \overline{\varphi}^{(i_1 \dots i_p)} = v^{(i_1 \dots i_p)} \overline{\varphi}_{(i_1 \dots i_p)} \tag{4.3}$$

defines a currrent.

The currents (4.1), (4.2) and (4.3) were introduced by de Rham on real Riemannian manifolds. For completeness, we summarize briefly the various relevant definitions relating to real currents; these definitions are essentially those given by de Rham [11]. Then we indicate briefly the corresponding definitions for complex currents.

A current is zero in an open set if it vanishes for each form φ with compact carrier contained in the open set. The carrier of a current T is the complement of the largest open set in which $T = 0$. For example, the carrier of the current (4.3) is a single point.

Let T be a current of degree p, φ a q-form, and set

$$T_\wedge \varphi [\psi] = T[\varphi_\wedge \psi], \; \varphi_\wedge T = (-1)^{pq} T_\wedge \varphi. \tag{4.4}$$

Further let

$$\int_{D^k} T_\wedge \overline{\varphi} = T[\varphi]. \tag{4.5}$$

If T is of degree p, we define

$$* T[\varphi] = (-1)^p T[* \varphi] = (-1)^p (T, \varphi) = (-1)^p ((\varphi, T))^-, \tag{4.6}$$

$$d T[\varphi] = (-1)^{p+1} T[d \varphi], \tag{4.7}$$

$$\delta T[\varphi] = (-1)^p T[\delta \varphi], \; \delta T = - * d * T. \tag{4.7'}$$

Then

$$d^2 T = \delta^2 T = 0 \tag{4.8}$$

and

$$(* T, * \varphi) = (T, \varphi), \; * * T = (-1)^p T, \; (T, d \varphi) = (\delta T, \varphi), \; (d T, \varphi) = (T, \delta \varphi). \tag{4.9}$$

A current is said to be harmonic if $\Delta T = 0$, $\Delta = d \delta + \delta d$.

In particular, if T is a chain C^p, φ a $(p-1)$-form of the space, we have

$$d C^p [\varphi] = (-1)^{p+1} C^p [d \varphi] = (-1)^{p+1} \int_{C^p} d \overline{\varphi} = (-1)^{p+1} \int_{b C^p} \overline{\varphi} = (-1)^{p+1} b C^p [\varphi].$$

That is,

$$dC^p = (-1)^{p+1}\, b\, C^p. \tag{4.10}$$

In an analogous fashion we introduce complex currents. Let T be of degree p, $\varrho + \sigma = p$, and define

$$\prod_{\varrho,\sigma} T[\varphi] = T[\prod_{k-\sigma,\, k-\varrho} \varphi]. \tag{4.11}$$

By (2.15) and (4.6):

$$(\prod_{\varrho,\sigma} T, \varphi) = (T, \prod_{\varrho,\sigma} \varphi). \tag{4.12}$$

Further, in complete analogy with (4.7) and (4.7′) we set

$$\partial T[\varphi] = (-1)^{p+1} T[\bar\partial \varphi],\ \mathfrak{b} T[\varphi] = (-1)^p T[\bar{\mathfrak{b}} \varphi],\ \mathfrak{b} T = -\ast \partial \ast T. \tag{4.13}$$

We have

$$(\bar{\mathfrak{b}} T, \varphi) = (T, \partial \varphi),\ (\partial T, \varphi) = (T, \bar{\mathfrak{b}} \varphi). \tag{4.14}$$

Finally let

$$\Lambda T[\varphi] = T[\Lambda \varphi], \tag{4.15}$$

where Λ is the operator (2.29).

5. The parametrix.

Let $x = (x^1, \ldots, x^{2k})$ denote the point with real coordinates x^i, and denote the geodesic distance from x to y by $r(x, y)$. To each compact subdomain K of D^k there corresponds a positive number η_0 such that if $r(x, y) < \eta_0$, $y \in K$, there is a unique geodesic from x to y whose length is $r(x, y)$.

Let $\varrho = \varrho(\sigma)$ be a function C^∞ of the real variable σ which satisfies $0 \le \varrho \le 1$, $\varrho = 1$ if $\sigma < \frac{1}{2}$ and $\varrho = 0$ if $\sigma > 1$. Moreover, let

$$a = -\tfrac{1}{2} r^2(x, y) \tag{5.1}$$

and define

$$a_{ij} = \frac{\partial^2 a}{\partial x^i\, \partial y^j}, \tag{5.2}$$

$$a_{i_1 \ldots i_p,\, j_1 \ldots j_p} = \begin{vmatrix} a_{i_1 j_1} \cdots a_{i_1 j_p} \\ a_{i_p j_1} \cdots a_{i_p j_p} \end{vmatrix}. \tag{5.3}$$

We denote the volume of the unit $(2k-1)$-sphere by s_k and we set

$$\omega(x, y) = \omega_p(x, y) = \tag{5.4}$$

$$= \frac{\varrho(r/\eta_0)}{2(k-1)s_k r^{2(k-1)}} a_{(i_1 \ldots i_p),(j_1 \ldots j_p)} dx^{i_1} \wedge \cdots \wedge dx^{i_p} \cdot dy^{j_1} \wedge \cdots \wedge dy^{j_p}$$

if $k > 1$. If $k = 1$ we replace $r^{-2(k-1)}/2(k-1)$ by $-\log r$. The expression (5.4) is called a "parametrix", and it is clearly symmetric:

$$\omega(x, y) = \omega(y, x). \tag{5.5}$$

Now let B be a finite submanifold of the type described at the end of Section 3. If D^k is compact, then $B = B^k$ may coincide with D^k; otherwise B is a proper submanifold with boundary. In the following all scalar products involve integration only over B.

After Bidal-de Rham [3] we write

$$\Omega \varphi = (\varphi(y), \ \omega(x, y)), \tag{5.6}$$

$$q(x, y) = -\Delta_x \omega(x, y). \tag{5.7}$$

Ω is self-adjoint:

$$(\Omega \varphi, \psi) = (\varphi, \Omega \psi). \tag{5.8}$$

Moreover, for x near y the form $q(x, y) = O(r^{2(1-k)})$ (see [3]). Let

$$Q \varphi = (\varphi(y), q(x, y)), \ Q' \varphi = (\varphi(y), q(y, x)). \tag{5.9}$$

Then if $\varphi \in C^1$ in the closure of B we have

$$\Omega \Delta \varphi = \{\varphi\} - Q' \varphi + \int_{bB} \{\varphi_\wedge * d\omega - \omega_\wedge * d\varphi + \delta\varphi_\wedge * \omega - \delta\omega_\wedge * \varphi\}, \tag{5.10}$$

where $\{\varphi\}$ is equal to φ in B and equal to zero outside B. Further

$$\Delta \Omega \varphi = \{\varphi\} - Q \varphi. \tag{5.11}$$

For a proof of these formulas see [3], where, however, the boundary integral in (5.10) is absent because B is assumed to be compact.

For completeness, we sketch here the proof given in [3] that a fundamental singularity exists if B is "small enough" such that

$$|Q \varphi| \leq k \max |\varphi|, \tag{5.12}$$

where $0 < k < 1$. In fact, consider the equation

$$\Delta \mu = \beta. \tag{5.13}$$

Taking $\mu = \Omega\,\xi$, we have by (5.11)

$$\xi - Q\,\xi = \beta. \tag{5.14}$$

Under the assumption (5.12) this integral equation has a solution

$$\xi = \beta + P\,\beta. \tag{5.15}$$

where P is an integral operator with kernel

$$p\,(x, y) = q\,(x, y) + Q\,q\,(x, y) + \cdots + Q^k\,q\,(x, y) + \cdots. \tag{5.16}$$

Then

$$\mu = \Omega\,\xi = (\Omega + \Omega\,P)\,\beta \tag{5.17}$$

is a solution of (5.13), and the kernel $\gamma\,(x, y)$ of the operator $\Gamma = \Omega + \Omega\,P$ has essentially the same singularity as $\omega\,(x, y)$. We therefore have

$$\Delta_x\,(\beta\,(y),\, \gamma\,(x, y)) = \beta\,(x) \tag{5.18}$$

or, if $\Gamma\,(x)$ is the current which is equal to $\gamma\,(x, y)$ in B,

$$\Delta\,\Gamma\,\beta = \beta. \tag{5.19}$$

It follows readily that $\Delta_x\,\gamma\,(x, y) = 0$ for $x \neq y$. Let u be the current defined by

$$u^{i_1 \cdots i_p} = d\,y^{i_1} \ldots d\,y^{i_p} \tag{5.20}$$

at the point y of the manifold. Then $(\varphi, u) = \varphi$ and (5.19) may be written

$$\Delta\,\Gamma = u. \tag{5.19'}$$

If there is a current Γ in B such that (5.19′) is satisfied, we say that B possesses a fundamental singularity. The above method shows that a fundamental singularity exists over every finite submanifold for which (5.12) holds.

The existence of a local fundamental singularity implies the following theorem of de Rham [11]:

Theorem 5.1. *If $\Delta\,T$ is equal to a form C^∞ in a domain B, T is also equal to a form C^∞ in B. In particular, a harmonic current is equal to a harmonic form.*

6. Analytic and harmonic p-forms.

We consider pure p-forms φ which satisfy

$$\varphi = \prod_{p,0} \varphi, \quad 0 \leq p \leq k. \tag{6.1}$$

For such forms we have the formula

$$\bar{D}_i\, \varphi_{i_1 \ldots i_p} = \prod_{0,1}{}_i{}^j \frac{\partial \varphi_{i_1 \ldots i_p}}{\partial z^j}. \tag{6.2}$$

Thus the covariant derivative of a pure p-vector in a direction $z^{\bar{a}}$ coincides with its ordinary derivative, that is

$$D_{\bar{a}}\, \varphi_{a_1 \ldots a_p} = \frac{\partial \varphi_{a_1 \ldots a_p}}{\partial \bar{z}^a}. \tag{6.2'}$$

It follows from (2.14) that a pure p-form is complex analytic if

$$\bar{\partial}\, \varphi = 0. \tag{6.3}$$

If $p = k$, the condition (6.3) is equivalent to

$$\bar{\mathfrak{d}}\, \varphi = 0. \tag{6.3'}$$

If $0 \leq p < k$, the condition (6.3′) is necessary but not sufficient for complex analyticity. If $k+1 \leq p \leq 2k$, then φ may be called "complex analytic" if (6.3′) is satisfied. It follows that if $\varphi = \prod_{p,0} \varphi$ is analytic, then so is $*\,\varphi$ and conversely.

To prove (6.3′) we have only to observe that

$$\bar{D}_i\, \varphi_{i_1 \ldots i_p} = \prod_{0,1}{}_i{}^j \frac{\partial \varphi_{i_1 \ldots i_p}}{\partial z^j} - \sum_{\mu=1}^{p} \prod_{0,1}{}_i{}^j \left\{ {q \atop i_\mu\ j} \right\} \varphi_{i_1 \ldots i_{\mu-1} q i_{\mu+1} \ldots i_p}$$

where

$$\prod_{0,1}{}_i{}^j \left\{ {q \atop i_\mu\ j} \right\} = 0$$

by (1.19).

If φ is a pure p-form, then by (2.13′)

$$\mathfrak{d}\, \varphi = 0, \tag{6.4}$$

since $\prod_{p,-1} = 0$. Thus by (2.22) and (2.23)

$$\Delta\, \varphi = (\Box + \bar{\Box})\, \varphi = (\bar{\mathfrak{d}}\, \partial + \partial\, \bar{\mathfrak{d}} + \mathfrak{d}\, \bar{\partial})\, \varphi. \tag{6.5}$$

If φ is analytic, so is $\partial\, \varphi$. Hence by (6.3) and (6.3′) we see that $\Delta\, \varphi = 0$. Thus analytic forms are harmonic with respect to an arbitrary Kähler metric.

Conversely, any pure harmonic p-form on a compact manifold is necessarily complex analytic. For

$$0 = (\Delta\, \varphi, \varphi) = (\delta\, d\, \varphi, \varphi) + (d\, \delta\, \varphi, \varphi) = (d\, \varphi, d\, \varphi) + (\delta\, \varphi, \delta\, \varphi),$$

and therefore φ is both closed and co-closed. But a pure closed form is obviously complex analytic. If the p-form φ is harmonic, then by (2.20) so is its pure component. Therefore the pure component of any harmonic p-form on a compact manifold is complex analytic.

If φ is a scalar, we have as a consequence of (6.2)

$$\bar{\mathcal{D}}_j \mathcal{D}_i \varphi = \mathcal{D}_i \bar{\mathcal{D}}_j \varphi. \tag{6.6}$$

It follows that

$$\square \varphi = \square' \varphi = \bar{\mathfrak{d}} \partial \varphi = \delta \prod_{1,0} d\varphi = -\bar{\mathcal{D}}^i \mathcal{D}_i \varphi = -g^{ij} \bar{\mathcal{D}}_j \mathcal{D}_i \varphi = -\tfrac{1}{2} g^{ij} \frac{\partial^2 \varphi}{\partial z^i \partial \bar{z}^j} = \tfrac{1}{2} \Delta \varphi;$$

that is

$$\square \varphi = \bar{\square} \varphi = \tfrac{1}{2} \Delta \varphi. \tag{6.7}$$

Now let P be the space of forms φ satisfying (6.1) which have finite norms over D^k:

$$N(\varphi) = (\varphi, \varphi) < \infty. \tag{6.8}$$

By the Riesz-Fischer theorem the space P is complete.

Let Ψ be the space of p-forms ψ,

$$\psi = \mathfrak{d} \chi, \tag{6.9}$$

where χ has a compact carrier and

$$\chi = \prod_{p,1} \chi, \quad \chi \in C^\infty. \tag{6.10}$$

We denote the closure of the space Ψ (in the sense of the scalar product) by Q. Thus Q is the closure of the space of complex co-differentials $\mathfrak{d}\chi$, $\chi \in C^\infty$, where χ has a compact carrier. By (2.13′) we obviously have

$$\psi = \prod_{p,0} \psi. \tag{6.11}$$

Let A be the subspace of P composed of pure complex analytic p-forms φ:

$$\varphi = \prod_{p,0} \varphi, \quad \bar{\partial} \varphi = 0. \tag{6.12}$$

We have the decomposition formula

$$P = Q + A, \tag{6.13}$$

where the spaces Q and A are clearly orthogonal. We omit a proof of this formula.

Non-trivial complex analytic p-forms φ with finite norms over D^k will exist if and only if Q is a proper subspace of P. Since there are manifolds which fail to have analytic p-forms for certain values of p, $0 \leq p \leq k$, the question whether Q coincides with P or not cannot be decided by general arguments which do not take into account the particular structure of the manifold under consideration.

Finally let L be the space of all forms which have finite norms over D^k, and let M be the closure of the space of p-forms ψ,

$$\psi = \Delta \chi, \tag{6.14}$$

where χ is of class C^∞ and has a compact carrier. Then

$$L = M + H, \tag{6.15}$$

where H is the space of harmonic forms α satisfying

$$\Delta \alpha = 0. \tag{6.16}$$

For let $\varphi \in L$, and let ψ be the element of M which minimizes $N(\varphi - \psi)$. Writing $\beta = \varphi - \psi$, we have

$$(\beta, \psi) = 0, \ \psi \in M. \tag{6.17}$$

In particular, choosing $\psi = \Delta \chi$, χ of class C^∞ with compact carrier, we obtain

$$(\beta, \Delta \chi) = 0. \tag{6.17'}$$

From Theorem 5.1 it follows that β is harmonic.

7. Decomposition of currents.

We define an analytic current T to be a current satisfying

$$T = \prod_{p,0} T, \ \bar{\partial} T = 0, \ T \text{ of degree } p, \ 0 \leq p \leq k, \tag{7.1}$$

$$T = \prod_{k,k-p} T, \ \bar{\mathfrak{b}} T = 0, \ T \text{ of degree } 2k - p, \ 0 \leq p \leq k. \tag{7.1'}$$

Since

$$\Lambda \prod_{p,0} = 0, \tag{7.2}$$

we see from (2.30) that (7.1) implies $\bar{\mathfrak{b}} T = 0$. As in Section 6 we therefore conclude that an analytic current T is harmonic. By Theorem 5.1 a harmonic current is a harmonic form and therefore an analytic current is an analytic form.

Given an open covering $\{U_i\}$ of D^k, there is a set of functions φ_j such that

$$1 = \sum_j \varphi_j, \tag{7.3}$$

where (*i*) $\varphi_j \in C^\infty$, $0 \le \varphi_j \le 1$, the carrier of φ_j is compact and contained in one of the open sets U_i; (*ii*) every point of D^k has a neighborhood which is met by only a finite number of the carriers of the φ_j. The formula (5.18) gives a "partition of unity".

If $\varphi \in C^\infty$ and if φ has a non-compact carrier, we say that $T[\varphi]$ is convergent and that

$$T[\varphi] = \sum_i T[\varphi_i \varphi]$$

if the series on the right is convergent for each partition (7.3). Then the sum is absolutely convergent and its value is independent of the partition used to define it.

Let us consider the class C of currents $T = \prod_{p,0} T$ such that $T[\varphi]$ is convergent for every form φ which is of class C^∞ and has a finite norm. By (6.13) we know that any φ of the space P has the decomposition

$$\varphi = \psi + \alpha \tag{7.4}$$

where $\psi \in Q$ and α is analytic, $\alpha \in A$. If $\varphi \in C^\infty$, then since α is analytic it follows that $\psi \in C^\infty$. Now define currents E, F by the formulas

$$(E, \varphi) = (T, \psi), \quad (F, \varphi) = (T, \alpha). \tag{7.5}$$

Clearly

$$T = E + F, \tag{7.6}$$

where

$$(F, \mathfrak{d}\chi) = (\bar{\partial} F, \chi) = 0$$

for any form $\chi = \prod_{p,1} \chi$ of class C^1 with compact carrier. Hence F is complex analytic, and formula (7.6) gives the orthogonal decomposition of the class C into the space of analytic forms and an orthogonal space. We call F the analytic projection AT of T.

In particular, the current

$$v^{i_1 \cdots i_p} = d\bar{\zeta}^{i_1} \wedge \cdots \wedge d\bar{\zeta}^{i_p} \tag{7.7}$$

is a current of class C whose carrier is the point ζ of the manifold, and it satisfies

$$(\varphi, v) = \varphi. \tag{7.8}$$

By the orthogonal decomposition we have

$$v = \psi + \alpha, \tag{7.9}$$

where $\alpha = \alpha_p(z, \bar{\zeta})$ is the reproducing kernel for complex analytic p-forms:

$$(\varphi(z), \alpha_p(z, \bar{\zeta})) = \varphi(\zeta), \ \varphi \text{ complex analytic.} \tag{7.10}$$

More generally, given any current of class C, its projection on the space of analytic p-forms is given by

$$A\,T = (T(z), \ \alpha_p(z, \bar{\zeta})). \tag{7.11}$$

In particular,

$$\alpha_p(z, \bar{\zeta}) = A\,v. \tag{7.12}$$

We observe that

$$(\alpha_p(z, \bar{\zeta}), \ \alpha_p(z, \bar{w})) = \alpha_p(w, \bar{\zeta}) = (\alpha_p(\zeta, \bar{w}))^{-}. \tag{7.13}$$

Finally, let B be the class of real currents T such that $T[\varphi]$ is convergent for every form φ which is of class C^∞ and of finite form. Given $\varphi \in L$, we have by Section 6

$$\varphi = \psi + \beta, \tag{7.14}$$

where $\psi \in M$ and β is harmonic. As above, define currents X, Y by the formulas

$$(X, \varphi) = (T, \psi), \ \ (Y, \varphi) = (T, \beta). \tag{7.15}$$

Then

$$T = X + Y, \tag{7.16}$$

where

$$(Y, \Delta\,\chi) = (\Delta\,Y, \chi) = 0$$

for every form χ of class C^∞ with compact carrier. Hence Y is harmonic, and therefore Y is a harmonic form. We call Y the harmonic projection $H\,T$ of T. If u is the current defined by (5.20) we have by (7.16)

$$u = \psi + \beta_p, \tag{7.17}$$

where $\beta_p(x, y)$ is the reproducing kernel for harmonic p-forms,

$$(\beta(x), \ \beta_p(x, y)) = \beta(y), \ \beta \text{ harmonic.} \tag{7.18}$$

Since u is a real tensor, we have in place of (7.13) the symmetry

$$\beta_p(x, y) = \beta_p(y, x). \tag{7.19}$$

By the Schwarz inequality applied to (7.18):

$$|\beta(y)| \leq K(y)\sqrt{N(\beta)}, \quad |\beta(y)| = \sqrt{\beta_{(i_1 \dots i_p)}\bar{\beta}^{(i_1 \dots i_p)}}, \tag{7.20}$$

where $K(y)$ is a positive number depending on y but not on the particular harmonic form β. Indeed, the condition (7.20) is necessary and sufficient for the existence of a reproducing kernel in a Hilbert space. It follows in particular from (7.20) that convergence in norm implies point-wise convergence. The inequality (7.20) is established in a different way in [10].

8. Finite submanifolds which possess a fundamental singularity.

Let B be a finite submanifold with boundary, and suppose that there is a fundamental singularity $\gamma(x, y) = \gamma_p(x, y)$ which is defined for x and y in some domain containing B in its interior. If B is sufficiently small, such a $\gamma(x, y)$ will exist.

Let

$$\mu(x, y) = \gamma(x, y) - (\beta(z, y), \gamma(x, z)), \tag{8.1}$$

where the integration in the scalar product is extended over B and where $\beta = \beta_p$ is the harmonic reproducing kernel for B. By (5.18)

$$\Delta_x \mu(x, y) = -\beta(x, y), \; x \neq y, \tag{8.2}$$

and

$$\Delta_x(\varphi(y), \mu(x, y)) = \varphi(x) - H\varphi(x). \tag{8.3}$$

We set

$$g_p(x, y) = \mu(x, y) - (\mu(t, y), \beta(t, x)), \tag{8.4}$$

and then

$$\Delta_x g_p(x, y) = -\beta_p(x, y), \; (H\varphi(x), g_p(x, y)) = 0. \tag{8.5}$$

In particular, the relations (8.5) define $g_p(x, y)$ uniquely.

We write

$$G\varphi = (\varphi(y), g_p(x, y)); \tag{8.6}$$

then

$$\Delta G\varphi = \varphi - H\varphi, \; HG\varphi = 0. \tag{8.7}$$

We note the trivial relations

$$(H\varphi, \psi) = (\varphi, H\psi) \tag{8.8}$$

and

$$GH = HG = 0. \tag{8.9}$$

Let us assume that the form φ, in addition to being of class C^∞, also has a compact carrier with respect to B; that is, φ vanishes outside a compact set lying in the interior of B. Then

$$\Delta H\varphi = H\Delta\varphi = 0, \tag{8.10}$$

so

$$\Delta\Delta G\varphi = \Delta\varphi - \Delta H\varphi = \Delta\varphi,\ \ \Delta G\Delta\varphi = \Delta\varphi - H\Delta\varphi = \Delta\varphi,\ \ \Delta(\Delta G - G\Delta)\varphi = 0.$$

Therefore $(\Delta G - G\Delta)\varphi$ is harmonic. But $H\Delta G\varphi = H\varphi - H^2\varphi = 0$ and $HG\Delta\varphi = 0$, so $(\Delta G - G\Delta)\varphi = 0$. Thus

$$\Delta G\varphi = G\Delta\varphi. \tag{8.11}$$

At the end of Section 6 we observed that a form φ with finite norm over B can be decomposed into a harmonic component $H\varphi$ and a component which is the limit in the sense of the norm of elements $\Delta\chi$, χ of class C^∞ with compact carrier. The latter component is here to be identified with $\Delta G\varphi$. Let χ_μ be a sequence of forms with compact carriers such that

$$N(\Delta G\varphi - \Delta\chi_\mu)$$

tends to zero. We have

$$\chi_\mu - \chi_\nu = (\Delta_x(\chi_\mu - \chi_\nu),\ \gamma(x, y))$$

and therefore $N(\chi_\mu - \chi_\nu)$ converges to zero as μ, ν tend to infinity. Hence there is a form χ, $N(\chi) < \infty$, such that $N(\chi - \chi_\mu)$ tends to zero. On the other hand, if ψ has a compact carrier, then

$$\begin{aligned}(G\varphi - \chi, \Delta\psi) &= \lim\,(G\varphi - \chi_\mu, \Delta\psi) \\ &= \lim\,(\Delta G\varphi - \Delta\chi_\mu, \psi) = 0.\end{aligned}$$

Thus $G\varphi - \chi$ is a harmonic current, whence by Theorem 4.1 it is equal to a harmonic form. Therefore $G\varphi = \chi - H\chi$.

Now let φ and ψ be any two forms C^∞ with compact carriers; we have

$$\begin{aligned}(G\psi, \varphi) &= (G\psi, \Delta G\varphi) = \lim\,(G\psi, \Delta\chi_\mu) = \lim\,(\Delta G\psi, \chi_\mu) \\ &= (\Delta G\psi, G\varphi) = (\psi, G\varphi).\end{aligned}$$

That is

$$(8.12) \qquad (G\varphi, \psi) = (\varphi, G\psi),$$

or, in other words,

$$((\varphi(y), g_p(x, y)), \psi(x)) = (\varphi(y), (\psi(x), g_p(x, y)))$$
$$= (\varphi(y), (\psi(x), g_p(y, x))).$$

Since φ is an arbitrary form with compact carrier, we conclude that

$$(\psi(x), g_p(x, y)) = (\psi(x), g_p(y, x)),$$

and then, since ψ is arbitrary,

$$(8.13) \qquad g_p(x, y) = g_p(y, x).$$

Let

$$\eta(x, y) = g_p(x, y) + (\beta(t, y), \gamma(x, t)) + (\beta(t, x), \gamma(y, t)).$$

Then $\eta(x, y) = \eta(y, x)$ and

$$\Delta_x \eta(x, y) = \Delta_y \eta(x, y) = 0.$$

We have therefore defined a symmetric fundamental singularity $\eta(x, y)$ in B. We may therefore always suppose that the fundamental singularity is symmetric.

Let φ be a p-form which is of class C^∞ in the closure of B and with $N(\Delta\varphi) < \infty$. We seek a harmonic p-form β such that

$$(8.14) \qquad t(\beta - \varphi) = n(\beta - \varphi) = 0$$

on the boundary of B. If a form β satisfying (7.14) exists, it is unique. For the difference λ of two such forms satisfies

$$t\lambda = n\lambda = 0$$

on the boundary. Hence by Green's formula

$$(8.15) \qquad N(d\lambda) + \mathrm{N}(\delta\lambda) = 0.$$

That is $d\lambda = \delta\lambda = 0$ and λ is closed and co-closed. Therefore

$$(8.16) \qquad 0 = (d_x\lambda, d_x\gamma) + (\delta_x\lambda, \delta_x\gamma)$$
$$= \lambda + \int_{bB} (\lambda_\wedge * d_x\gamma - \delta_x\lambda_\wedge * \lambda) = \lambda.$$

The boundary value problem (8.14) can be established on a rigorous basis by minimizing the expression

$$N(d_x(\varphi - \alpha)) + N(\delta_x(\varphi - \alpha))$$

with respect to all harmonic p-forms α. The proof, which does not differ essentially from that used in the case $k=1$ (see [12]) will be omitted.

By subtracting a suitable harmonic p-form from the fundamental singularity $\gamma(x, y)$, we obtain a Green's form $G_p(x, y)$ which satisfies $\Delta_x G_p(x, y)=0$, $x \neq y$, in B and on the boundary

$$t(x) G_p(x, y) = n(x) G_p(x, y) = 0. \tag{8.17}$$

We obtain from Green's formula in the usual way the symmetry relation

$$G_p(x, y) = G_p(y, x). \tag{8.18}$$

In terms of Green's form the solution of the boundary value problem is given by

$$\beta(y) = -\int_{bB} \{\varphi_\wedge * d_x G_p(x, y) - \delta_x G_p(x, y)_\wedge * \varphi\}. \tag{8.19}$$

9. The existence of a fundamental singularity on an arbitrary submanifold with boundary.

Let B be a finite submanifold with boundary imbedded in a larger Riemannian manifold. Then it is not difficult to show that B can be imbedded isometrically in a compact Riemannian manifold M of class C^∞.

Let $\mathfrak{E}$ be the space of harmonic p-forms on M which vanish identically outside B, and let $\mathfrak{E}_1$ be the orthogonal complement of $\mathfrak{E}$ in the space of all harmonic p-forms on M. Any linear combination of forms in $\mathfrak{E}_1$ which vanishes identically outside B must necessarily vanish throughout B, and we can therefore find a basis $\{\varphi_i\}$ for $\mathfrak{E}_1$ which is orthonormal over $M-B$. Let y be a fixed point of B, set

$$e_1(x, y) = \begin{cases} \Sigma\, \varphi_i(x)\, \varphi_i(y), & x \in M-B \\ 0, & x \in B, \end{cases}$$

and write

$$E_1 \varphi = (\varphi(x),\ e_1(y, x)).$$

We denote orthogonal projection onto $\mathfrak{E}$ by E, and we define $Q = I - E - E_1$ where I is the identity operator at the point y, $I\varphi = \varphi(y)$. If H is the projection operator into the space of harmonic p-forms in M, we clearly have $HQ=0$. Now let G be de Rham's Green's operator for the compact manifold M (see [11]), and define $G_1 = GQ$. Then

$$\Delta G_1 = I - E - E_1.$$

Since the carrier of E_1 is contained in $M-B$, we see that $\Delta G_1 = I - E$ in B and we have thus proved the following result:

Theorem 9.1. *A finite manifold with boundary possesses a fundamental singularity if and only if* $E=0$.

In any case, we always have a Green's operator G on B which satisfies $\Delta G\varphi = \varphi - H\varphi$ where H now denotes projection onto the space of harmonic p-forms in B.

10. Relations between harmonic and analytic p-vectors on a finite manifold.

On a compact (closed) Kähler manifold a pure harmonic p-form φ, $\varphi = \prod_{\varrho,\sigma}\varphi$, is complex-analytic. Since the operators $\prod_{\varrho,\sigma}$ and Δ commute, the pure component $\prod_{\varrho,\sigma}\varphi$ of a harmonic p-form φ is harmonic and therefore analytic. Let $\alpha_p(z,\bar{\zeta})$ and $\beta_p(z,\zeta)$ be the complex-analytic and harmonic reproducing kernels, respectively, both written in self-conjugate coordinates. Then

$$\alpha_p(z,\bar{\zeta}) = \prod_{p,0}(z)\prod_{0,p}(\zeta)\,\beta_p(z,\zeta)\,. \tag{10.1}$$

Here $0\le p\le k$, but if $p=0$ the relation (9.1) is trivial since we then have

$$\alpha_p(z,\bar{\zeta}) = \beta_p(z,\zeta) = \frac{1}{V}\,, \tag{10.2}$$

where V denotes the volume of the manifold. Since

$$\alpha_{2k-p}(z,\bar{\zeta}) = *_z *_\zeta \alpha_p(z,\bar{\zeta}),\quad \beta_{2k-p}(z,\zeta) = *_z *_\zeta \beta_p(z,\zeta), \tag{10.3}$$

the relation (10.1) for $k+1\le p\le 2k$ becomes

$$\alpha_{2k-p}(z,\bar{\zeta}) \prod_{k,k-p}(z)\prod_{k-p,p}(\zeta)\,\beta_{2k-p}(z,\zeta)\,. \tag{10.1'}$$

To prove (10.1) we observe that $\alpha_p(z,\bar{\zeta})$ is analytic in z and in $\bar{\zeta}$. If φ is complex analytic, we have

$$\begin{aligned}\prod_{p,0}(\zeta)\,\varphi(\zeta) &= (\varphi(z),\ \beta_p(z,\zeta))\\ &= (\prod_{p,0}(z)\,\varphi(z),\ \beta_p(z,\zeta))\\ &= (\varphi(z),\ \prod_{p,0}(z)\,\beta_p(z,\zeta))\\ &= (\varphi(z),\ \prod_{p,0}(z)\prod_{0,p}(\zeta)\,\beta_p(z,\zeta))\,.\end{aligned}$$

Since the kernel $g_p(z,\zeta)$ exists on a compact manifold and satisfies

$$\beta_p(z,\zeta) = -\Delta_z\, g_p(z,\zeta)\,,$$

we have

$$\alpha_p(z, \bar{\zeta}) = -\Delta_z f_p(z, \bar{\zeta}), \tag{10.4}$$

where

$$f_p(z, \bar{\zeta}) = \prod_{p,0}(z) \prod_{0,p}(\zeta)\, g_p(z, \zeta). \tag{10.5}$$

We normalize $f_p(z, \bar{\zeta})$ by the condition

$$(A\varphi(z),\ f_p(z, \bar{\zeta})) = 0, \tag{10.6}$$

where $A\varphi$ denotes the analytic projection of φ. Then $f_p(z, \bar{\zeta})$ is unique. The Hermitian symmetry

$$f_p(z, \bar{\zeta}) = (f_p(\zeta, \bar{z}))^- \tag{10.7}$$

is readily verified by means of Green's formula.

The relation between harmonic and analytic p-forms on a finite submanifold B with boundary is more complicated. By (3.5)

$$(\bar{\partial}\varphi, \bar{\partial}\psi) + (\mathfrak{d}\varphi, \mathfrak{d}\psi) - (\varphi, \bar{\Box}\psi) = \int_{bB} [\varphi_\wedge * (\partial\psi)^- - (\mathfrak{d}\psi)^-{}_\wedge * \varphi]. \tag{10.8}$$

If $\varphi = \prod_{p,0}\varphi$, $\psi = \prod_{p,0}\psi$, then clearly

$$\mathfrak{d}\varphi = \mathfrak{d}\psi = 0 \tag{10.9}$$

and we have simply

$$(\bar{\partial}\varphi, \partial\psi) - (\varphi, \bar{\Box}\psi) = \int_{bB} \varphi_\wedge * (\partial\psi)^-. \tag{10.8'}$$

In the case of scalars ($p = 0$) we have by (6.7)

$$\Box\varphi = \bar{\Box}\varphi = \tfrac{1}{2}\Delta\varphi. \tag{10.10}$$

If the space is flat, then by (2.26) for any p, $0 \le p \le k$,

$$\begin{aligned}(\Box\varphi)_{i_1 \ldots i_p} &= -\prod_{\varrho,\sigma}{}_{i_1 \ldots i_p}{}^{(j_1 \ldots j_p)} \bar{\mathcal{D}}^i \mathcal{D}_i \varphi_{(j_1 \ldots j_p)} \\ &= -\prod_{\varrho,\sigma}{}_{i_1 \ldots i_p}{}^{(j_1 \ldots j_p)} \mathcal{D}^i \bar{\mathcal{D}}_i \varphi_{(j_1 \ldots j_p)} \\ &= (\bar{\Box}\varphi)_{i_1 \ldots i_p}.\end{aligned}$$

Hence by (2.23)

$$\Box\varphi = \bar{\Box}\varphi = \tfrac{1}{2}\prod_{\varrho,\sigma}\Delta\varphi. \tag{10.10'}$$

Thus for scalars on an arbitrary Kähler submanifold or for pure p-vectors on a Euclidean manifold we have the Green's formula

$$(\bar{\partial}\varphi, \bar{\partial}\psi) - \tfrac{1}{2}(\varphi, \Delta\psi) = \int_{bB} \varphi_\wedge * (\bar{\partial}\psi)^-. \tag{10.11}$$

Hence harmonic scalars on a Kähler space or pure harmonic p-vectors on Euclidean space satisfy the simple equation

$$(\bar{\partial}\varphi, \bar{\partial}\psi) = \int_{bB} \varphi_\wedge * (\bar{\partial}\psi)^-. \tag{10.12}$$

Therefore a harmonic scalar φ is complex analytic if

$$n\bar{\partial}\varphi = 0 \tag{10.13}$$

on the boundary. This boundary condition in the case of Euclidean space was investigated in [7 b]. If $0 < p \leq k$, pure harmonic p-vectors φ in Euclidean space are complex analytic if (9.13) is satisfied or if

$$t\varphi = 0 \tag{10.14}$$

on the boundary. But (10.14) cannot always be realized unless φ is zero.

We obtain a finite Kähler submanifold B by removing a small cell from a compact Kähler manifold, and any complex analytic function on B is then necessarily continuable as a complex analytic function throughout the cell and is therefore equal to a constant. Hence the only complex analytic scalar satisfying the boundary condition (10.13) on such a B is a constant.

In the case of Euclidean manifolds a method based on (10.11) for determining scalars φ with $n\bar{\partial}\varphi$ prescribed on the boundary has been given in [7], and this method carries over to finite Kähler submanifolds. It involves a modification of the projection procedure used at the end of Section 6 for determining the harmonic projection of a given φ. Let φ be a given scalar of class C^∞ in the closure of B, and let M_0 be the closure of the space of scalars $\Delta\chi$ where χ is of class C^∞ in the closure of B and

$$n\bar{\partial}\chi = 0 \tag{10.15}$$

on the boundary. Let ψ_0 minimize $N(\varphi - \psi)$ among all elements ψ of M_0. Since $M \subseteq M_0$, where M is the closure of the space of scalars $\Delta\chi$, χ of class C^∞ with compact carrier, we see that

$$\alpha = \varphi - \psi_0 \tag{10.16}$$

is harmonic.

Let local boundary coordinates $u^1, \ldots, u^{2k}$ be introduced. We have, for a scalar λ,

$$n\bar{\partial}\lambda = \tfrac{1}{2}\left(\frac{\partial\lambda}{\partial u^{2k}} + \sqrt{-1}\, h_{2k}{}^{i}\frac{\partial\lambda}{\partial u^{i}}\right), \tag{10.17}$$

where $h_{2k}{}^{2k} = 0$ by (3.10). Hence, given $t\lambda$, $n\bar{\partial}\lambda$ can be prescribed arbitrarily. Using (10.17) and the formula (10.11), it can be established formally that

$$\bar{\partial}\alpha = 0. \tag{10.18}$$

In other words, α as defined by (10.16) is complex analytic.

Taking, in particular, $\varphi = \Delta\gamma$, where γ is of class C^∞ in the closure of B, the above problem reduces to that of minimizing $N(\Delta\sigma)$ with

$$n\bar{\partial}\sigma = n\bar{\partial}\gamma \tag{10.19}$$

on the boundary. We are thus led to conclude that the solution σ of the minimum problem satisfies (10.19) on the boundary and $\bar{\partial}\Delta\sigma = 0$ in B – that is to say, $\Delta\sigma$ is analytic in B. The scalar σ is made unique by the requirement that it is orthogonal to all analytic functions in B. We have, of course, no assurance that $\Delta\sigma$ is not equal to a constant throughout B.

Now let γ_0 be a fundamental singularity defined in some domain containing B in its interior. We suppose that $\gamma = \gamma(z, \bar{\zeta})$ is expressed in terms of self-conjugate coordinates and, in analogy with (8.1), we set

$$\mu(z, \bar{\zeta}) = \gamma(z, \bar{\zeta}) - (\alpha(w, \bar{\zeta}), \gamma(w, \bar{z})), \tag{10.20}$$

where $\alpha(w, \bar{\zeta})$ is the reproducing kernel for complex analytic functions in B. Then

$$\Delta_z\mu(z, \bar{\zeta}) = -\alpha(z, \bar{\zeta}), \quad z \neq \zeta, \tag{10.21}$$

and

$$\Delta_z(\varphi(\zeta), \mu(\zeta, \bar{z})) = \varphi(z) - A\varphi(z). \tag{10.22}$$

We define

$$h(z, \bar{\zeta}) = \mu(z, \bar{\zeta}) - (\mu(w, \bar{\zeta}), \alpha(w, \bar{z})), \tag{10.23}$$

and then

$$\Delta_z h(z, \bar{\zeta}) = -\alpha(z, \bar{\zeta}), \quad (A\varphi(z), h(z, \bar{\zeta})) = 0. \tag{10.24}$$

The properties (10.24) uniquely determine $h(z, \bar{\zeta})$ up to a harmonic function with vanishing analytic projection. If we suppose that the boundary-value problem (10.19) is solvable, then there is a scalar $\sigma(z, \bar{\zeta})$ which is harmonic with respect to z and which satisfies

$$n\,\bar{\partial}_z\,\sigma(z,\bar{\zeta}) = n\,\bar{\partial}_z\,h(z,\bar{\zeta}) \tag{10.25}$$

on the boundary.

In fact, let $\sigma(z,\bar{\zeta})$ be a scalar such that

$$n\,(\bar{\partial}_z\,\sigma(z,\bar{\zeta})) = n\,(\bar{\partial}_z\,h(z,\bar{\zeta})) \tag{10.26}$$

on the boundary and with $\bar{\partial}_z\,\Delta_z\,\sigma(z,\bar{\zeta}) = 0$, $A_z\,\sigma(z,\bar{\zeta}) = 0$ in B. Let

$$f(z,\bar{\zeta}) = h(z,\bar{\zeta}) - \sigma(z,\bar{\zeta}). \tag{10.27}$$

Then $f(z,\bar{\zeta})$ is uniquely determined, and it satisfies

$$n\,\bar{\partial}_z\,f(z,\bar{\zeta}) = 0 \tag{10.28}$$

for z on the boundary of B;

$$\bar{\partial}_z\,\Delta_z\,f(z,\bar{\zeta}) = 0 \tag{10.29}$$

in B;

$$(A\,\varphi(z), f(z,\bar{\zeta})) = 0. \tag{10.30}$$

It is shown in [7 b] that an $f(z,\bar{\zeta})$ satisfying (10.28)—(10.30) necessarily satisfies the two further conditions

$$\Delta_z\,f(z,\bar{\zeta}) = -\,\alpha(z,\bar{\zeta}), \tag{10.31}$$

$$f(z,\bar{\zeta}) = (f(\zeta, z))^-. \tag{10.32}$$

The proof turns on the Green's formula

$$(\Delta\,\varphi, \psi) - (\varphi, \Delta\,\psi) = 2\int_{bB} [\varphi_\wedge * (\partial\,\psi)^- - \bar{\psi}_\wedge * (\partial\,\varphi)], \tag{10.33}$$

which is obtained from (10.11) by interchanging φ and ψ, taking the conjugate of the resulting equation and subtracting it from the original expression (10.11). Hence the proof given in [7 b] applies here, and we conclude that $f(z,\bar{\zeta})$ satisfies (10.31), (10.32) on the Kähler submanifold. Comparing with (10.24) we see that $\sigma(z,\bar{\zeta})$ must be harmonic. As in [7 b] we may also show that the solution of the boundary value problem (10.19) is given by

$$\sigma(z) = \int_{bB} f(z,\bar{\zeta})_\wedge * \bar{\partial}\,\gamma(\zeta). \tag{10.34}$$

11. The complex Laplace-Beltrami operator on a curved space.

We now investigate the properties of the complex Laplace-Beltrami operator $\square$ defined by (2.21) and (2.22), and we show that the results obtained in the preceding sections for the real operator Δ are also valid for the complex operator. If the space is flat, then by (10.8′) the operator $\square$ coincides essentially with Δ.

Let $\omega(z, \zeta) = \omega_p(z, \zeta)$ be the parametrix (5.4) expressed in self-conjugate coordinates, and write (compare Section 5)

$$(11.1) \qquad \begin{cases} \Omega\varphi = (\varphi(\zeta), \ \omega(z, \zeta)) \\ q(z, \zeta) = -\square_z \omega(z, \zeta) \\ Q\varphi = (\varphi(\zeta), q(z, \zeta)), \ Q'\varphi = (\varphi(\zeta), q(\zeta, z)), \end{cases}$$

where the integrations are over the finite submanifold B of M^k. If $\varphi = \prod_{\varrho,\sigma} \varphi$, $\varrho + \sigma = p$, we have, in analogy with (5.10),

$$(11.2) \qquad \Omega \square \varphi = \tfrac{1}{2}\{\varphi\} - Q'\varphi + \int_{bB} \{\varphi_\wedge * (\partial\omega)^- - \omega_\wedge * \partial\varphi + \bar{\mathfrak{b}}\varphi_\wedge * \omega - (\bar{\mathfrak{b}}\omega)^-{}_\wedge * \varphi\}.$$

To prove (11.2) we remove a small geodesic sphere S about the point ζ of B and apply the Green's formula

$$(\square\varphi, \psi) - (\varphi, \square\psi) = \int_{bB} \{\varphi_\wedge * (\partial\psi)^- - \bar{\psi}_\wedge * \partial\varphi + \bar{\mathfrak{b}}\varphi_\wedge * \bar{\psi} - (\bar{\mathfrak{b}}\psi)^-{}_\wedge * \varphi$$

with $\psi = \omega$. Formula (11.2) will follow if

$$(11.3) \qquad \int_{bB} \{\varphi_\wedge * (\partial\omega)^- - \omega_\wedge * \partial\varphi + \bar{\mathfrak{b}}\varphi_\wedge * \omega - (\bar{\mathfrak{b}}\omega)^-{}_\wedge * \varphi\} \to -\tfrac{1}{2}\varphi(\zeta)$$

as the radius of S tends to zero. This statement is a purely local one and it is therefore sufficient to prove (11.3) for the osculating Euclidean space, in which case

$$\square = \bar{\square} = \tfrac{1}{2} \prod_{\varrho,\sigma} \Delta.$$

Let $\gamma = \gamma_p(z, \zeta)$ be the Euclidean fundamental singularity for Δ:

$$\gamma_p(z, \zeta) = \frac{1}{2(k-1)s_k r^{2(k-1)}} \Gamma_{(i_1 \ldots i_p),(j_1 \ldots j_p)} dz^{i_1}{}_\wedge \ldots {}_\wedge dz^{i_p} . d\zeta^{j_1}{}_\wedge \ldots {}_\wedge d\zeta^{j_p}.$$

If the radius of S is sufficiently small and if z and ζ are points in a sufficiently small neighborhood N of S, we have for Euclidean space the relation $\omega = \gamma$. Hence

$$(11.4) \qquad \square(\varphi, \omega)_S = \tfrac{1}{2} \prod_{\varrho,\sigma} \Delta(\varphi, \gamma) = \begin{cases} \tfrac{1}{2}\varphi \text{ in } S \\ 0 \ \text{ in } N - S. \end{cases}$$

In Euclidean space we have also

$$(11.5)\qquad \partial_z \gamma_p(z,\zeta) = \mathfrak{d}_\zeta \gamma_{p+1}(z,\zeta),\ \partial_z = \prod_{\varrho+1,\sigma} d \prod_{\varrho,\sigma},\ \mathfrak{d}_\zeta = \prod_{\sigma,\varrho} \delta \prod_{\sigma,\varrho+1}.$$

In fact

$$\prod_{\varrho,\sigma}{}_z \gamma_p(z,\zeta) = \frac{1}{2(k-1)s_k r^{2(k-1)}} \prod_{\varrho,\sigma}{}_{(i_1\ldots i_p),(j_1\ldots j_p)} dz^{i_1}{}_\wedge\ldots{}_\wedge dz^{i_p}\,.\,d\zeta^{j_1}{}_\wedge\ldots{}_\wedge d\zeta^{j_p}$$

$$= (\prod_{\varrho,\sigma}{}_\zeta \gamma_p(z,\zeta))^- = \prod_{\sigma,\varrho}{}_\zeta \gamma_p(z,\zeta).$$

Hence

$$(\partial_z \gamma_p(z,\zeta))_{i_1\ldots i_{p+1},j_1\ldots j_p} = \prod_{\varrho+1,\sigma}{}_{i_1\ldots i_{p+1},jj\ldots j_p} \mathcal{D}_z{}^j \left\{\frac{1}{2(k-1)s_k r^{2(k-1)}}\right\},$$

$$(\mathfrak{d}_\zeta \gamma_{p+1}(z,\zeta))_{i_1\ldots i_{p+1},j_1\ldots j_p} = -\prod_{\varrho+1,\sigma}{}_{i_1\ldots i_{p+1},jj_1\ldots j_p} \mathcal{D}_\zeta{}^j \left\{\frac{1}{2(k-1)s_k r^{2(k-1)}}\right\}.$$

Since

$$\mathcal{D}_z{}^j \left\{\frac{1}{2(k-1)s_k r^{2(k-1)}}\right\} = -\mathcal{D}_\zeta{}^j \left\{\frac{1}{2(k-1)s_k r^{2(k-1)}}\right\},$$

we obtain (11.5). Formula (11.5) may also be written

$$(11.5')\qquad \partial_z \gamma_p(z,\zeta) = (\bar{\mathfrak{d}}_\zeta \gamma_{p+1}(z,\zeta))^-,\ \bar{\mathfrak{d}}_\zeta = \prod_{\varrho,\sigma} \delta \prod_{\varrho+1,\sigma}.$$

Thus

$$\begin{aligned}\Box(\varphi,\omega)_S = \Box(\varphi,\gamma)_S &= (\bar{\mathfrak{d}}\,\partial + \partial\,\bar{\mathfrak{d}})(\varphi,\gamma)_S \\ &= \bar{\mathfrak{d}}(\varphi,\bar{\mathfrak{d}}\gamma_{p+1})_S + \partial(\varphi,\partial\gamma_{p-1})_S \\ &= \partial\int_{bS}\gamma_{p-1}{}_\wedge * \gamma - \bar{\mathfrak{d}}\int_{bS}\varphi_\wedge * \gamma_{p+1} + \partial(\bar{\mathfrak{d}}\varphi,\gamma_{p-1}) \\ &\qquad + \bar{\mathfrak{d}}(\partial\varphi,\gamma_{p+1})_S \\ &= \int_{bS}(\bar{\mathfrak{d}}\gamma_p)^-{}_\wedge * \varphi - \int_{bS}\varphi_\wedge * (\partial\gamma_p)^- + o(1)\end{aligned}$$

as the radius of S tends to zero, and this proves (11.3).

Next, in analogy with (5.11),

$$(11.6)\qquad \Box\,\Omega\,\varphi = \tfrac{1}{2}(\{\varphi\} - Q\varphi).$$

In fact, if φ has a compact carrier with respect to B, then

$$(\Omega\,\Box\,\varphi,\psi) = (\varphi,\Box\,\Omega\,\psi),$$

and therefore

$$(\tfrac{1}{2}\varphi - Q'\varphi,\psi) = (\tfrac{1}{2}\varphi,\psi) - (\tfrac{1}{2}\varphi,Q\psi) = \tfrac{1}{2}(\varphi,\psi - Q\psi) = (\varphi,\Box\,\Omega\,\psi).$$

Since this formula holds for arbitrary φ with compact carrier, we have (11.6).

If B is sufficiently small in the sense that

$$|Q\varphi| \leq k \max |\varphi|, \quad 0 < k < 1, \tag{11.7}$$

we can repeat the reasoning of Section 6 and we see that the operator $\square$ has a local fundamental singularity $\theta(z, \bar{\zeta})$ (with the same asymptotic behavior at ζ as $\gamma(z, \zeta)$). In particular,

$$\square_z(\beta(\zeta), \theta(\zeta, \bar{z})) = \tfrac{1}{2}\beta(z). \tag{11.8}$$

The proof given by de Rham [11] for Theorem 5.1 can now be applied to obtain a similar theorem for $\square$, namely:

Theorem 11.1. *If $\square T$ is equal to a form C^∞ in a domain B, then T is equal to a form C^∞ in B.*

In fact, the proof depends only on the existence of a local fundamental singularity with the proper singularity at $z = \zeta$.

Using Theorem 11.1 we can show by the same reasoning as before that if $\varphi = \prod_{\varrho, \sigma} \varphi$ has a finite norm over B, then

$$\varphi = \psi + \beta, \tag{11.9}$$

where ψ belongs to the closure of the space of elements $\square \chi$, χ of class C^∞ with compact carrier, and where $\square \beta = 0$. We write $\beta = C\varphi$, where C is the projection operator onto the space of complex-harmonic forms β satisfying $\square \beta = 0$. If T is a current, $T = \prod_{\varrho, \sigma} T$, which satisfies the condition that (T, φ) is convergent for every form $\varphi = \prod_{\varrho, \sigma} \varphi$ of class C^∞ with finite norm, we define two new currents X and Y by the formulas

$$(X, \varphi) = (T, \psi), \quad (Y, \varphi) = (T, C\varphi). \tag{11.10}$$

Then, in analogy with (7.16),

$$T = X + Y, \tag{11.11}$$

where Y is a complex-harmonic form, $Y = CT$, and X is orthogonal to all complex-harmonic forms $\varphi = C\varphi$. Now let

$$u^{i_1 \cdots i_p} = \prod_{\varrho, \sigma} (j_1 \ldots j_p)^{i_1 \cdots i_p} \, d\zeta^{j_1} \wedge \ldots \wedge d\zeta^{j_p} \tag{11.12}$$

at the point ζ of the manifold. Then Cu is the kernel of the projection operator C, that is, $Cu = \varkappa_p(z, \bar{\zeta})$, where $\varkappa_p$ is the reproducing kernel for complex-harmonic forms $\varphi = C\varphi$:

(11.13) $$(\varphi(z), \varkappa_p(z, \bar{\zeta})) = \varphi(\zeta), \quad \varkappa_p(z, \bar{\zeta}) = (\varkappa_p(\zeta, \bar{z}))^{-}.$$

Once in possession of the reproducing kernel, it follows that

(11.14) $$|\varphi(\zeta)| \leq K(\zeta)\sqrt{N(\varphi)},$$

where $K(\zeta)$ is a positive quantity depending only on the point ζ.

If the domain B possesses a fundamental singularity θ for the operator $\Box$, we can repeat the reasoning of Section 8 and show that

(11.15) $$\theta(z, \bar{\zeta}) = (\theta(\zeta, \bar{z}))^{-}$$

and that there exists an operator F satisfying

(11.16) $$\Box F\varphi = \tfrac{1}{2}(\varphi - C\varphi), \quad CF\varphi = 0.$$

The kernel $f_p(z, \bar{\zeta})$ of F has the property that

(11.17) $$\Box_z f_p(z, \bar{\zeta}) = -\tfrac{1}{2}\varkappa_p(z, \bar{\zeta}).$$

Moreover,

(11.18) $$f_p(z, \bar{\zeta}) = \prod_{\varrho, \sigma}{}_z \prod_{\sigma, \varrho}{}_\zeta f_p(z, \bar{\zeta});$$

$$f_p(z, \bar{\zeta}) = (f_p(\zeta, \bar{z}))^{-}, \quad *_z * f_p(z, \bar{\zeta}) = (f_{2k-p}(z, \bar{\zeta}))^{-}.$$

Let $\mathfrak{E}(\varrho, \sigma)$ be the space obtained from $\mathfrak{E}$ by applying the operator $\prod_{\varrho, \sigma}$ to each of its elements. The following result is then an immediate consequence of Theorem 9.1:

Theorem 11.2. *A finite Kähler manifold with boundary possesses a fundamental singularity for $\Box$ if and only if $E(\varrho, \sigma) = 0$.*

If $\Box\varphi = 0$, $t\varphi = n\varphi = 0$ on bB, then by (3.5)

$$(\partial\varphi, \partial\varphi) + (\bar{\mathfrak{d}}\varphi, \bar{\mathfrak{d}}\varphi) = 0$$

so $\partial\varphi = \bar{\mathfrak{d}}\varphi = 0$. We remark that, even if B is a cell, we do not know that $\partial\varphi = 0$ in B implies $\varphi = \partial\psi$ (complex analogue of de Rham's theorem). But, what is more significant, if $\Box\varphi = 0$, $t\varphi = n\varphi = 0$ on bB, then by (11.2)

$$\tfrac{1}{2}\{\varphi\} = Q'\varphi$$

in B. It follows that φ is of class C^1 over the whole manifold M, $\varphi = 0$ in $M - B$, $\partial\varphi = \bar{\mathfrak{d}}\varphi = 0$ in M. If $\varrho = k$ (in which case $\partial\varphi = 0$ automatically), the condition $\bar{\mathfrak{d}}\varphi = 0$ implies that φ is complex-analytic and, since φ vanishes identically outside B, we conclude that it vanishes everywhere. Thus if $\varrho = k$, $0 \leq \sigma \leq k$, we always have the desired uniqueness property.

Analogous statements are valid for the conjugate operator $\bar{\Box}$. If $\varrho = p$, $\sigma = 0$ (that is, if the forms are pure), we have formula (10.8′) and hence such forms satisfying

$$\bar{\Box}\,\varphi = 0\,, \quad n\,\bar{\partial}\,\varphi = 0 \text{ on } bB\,, \tag{11.19}$$

are complex-analytic. On a compact manifold, forms $\varphi = \prod_{p,0} \varphi$ satisfying $\bar{\Box}\,\varphi = 0$ must also satisfy $\bar{\partial}\,\varphi = 0$: pure forms φ, $\bar{\Box}\,\varphi = 0$, on a compact space are complex-analytic.

If there are no complex-harmonic forms $\varphi = \prod_{\varrho,\sigma} \varphi$, $\Box\,\varphi = 0$, on a domain B, then by (11.17)

$$\Box_z\, f_p(z, \bar{\zeta}) = 0\,.$$

In this case $f_p(z, \bar{\zeta}) = \theta_p(z, \bar{\zeta})$ is a fundamental singularity for $\Box$. We note, however, that if B is compact, we cannot conclude that $\bar{\mathfrak{d}}\, f_p = 0$. For $(\bar{\mathfrak{d}}\, f_p, \bar{\mathfrak{d}}\, f_p)$ does not exist because of the singularity of f_p.

We remark finally that for complex fields φ satisfying $\bar{\partial}\,\varphi = \mathfrak{d}\,\varphi = 0$ (or $\partial\,\varphi = \bar{\mathfrak{d}}\,\varphi = 0$) we have a Cauchy's formula. A Cauchy's formula expresses the value of the field at an interior point of a domain in terms of an integral of its values on the boundary, and the formula should satisfy the following two requirements: (i) only the values of the field on the boundary are involved and not the values of any derivatives of the field; (ii) the Cauchy kernel is independent of the domain.

If B is an arbitrary subdomain of a submanifold B_1 which possesses a fundamental singularity $\theta(z, \bar{\zeta})$ for the operator $\bar{\Box}$, then we have (using (11.3))

$$\tfrac{1}{2}\{\varphi(\zeta)\} = -\int_{bB} [\varphi_\wedge * (\bar{\partial}\,\theta)^- - (\mathfrak{d}\,\theta)^-{}_\wedge * \varphi - \bar{\theta}_\wedge * \bar{\partial}\,\varphi + \mathfrak{d}\,\varphi_\wedge * \bar{\theta}] \tag{11.20}$$

where $\{(\zeta)\}$ is $\varphi(\zeta)$ if ζ is in B and is equal to zero if ζ is outside B. If $\bar{\partial}\,\zeta = \mathfrak{d}\,\zeta = 0$, we have simply

$$\tfrac{1}{2}\{\varphi(\zeta)\} = -\int_{bB} [\varphi_\wedge * (\bar{\partial}\,\theta)^- - (\mathfrak{d}\,\theta)^-{}_\wedge * \varphi]\,. \tag{11.21}$$

This is Cauchy's formula for complex fields φ, $\bar{\partial}\,\varphi = \mathfrak{d}\,\varphi = 0$, and it clearly satisfies the the conditions (i) and (ii).

In the ordinary case $k = 1$, $p = 0$, of Euclidean space we have

$$\theta = \frac{1}{2\pi} \log \frac{1}{r}\,\cdot$$

We observe that

$$\theta = -\frac{1}{4\pi}\left(\frac{dz}{z-\zeta}\right)^{-}, \ \ast(\bar{\partial}\theta)^{-} = \frac{i}{4\pi}\frac{dz}{z-\zeta};$$

hence (11.21) in this case is the classical formula

$$\{\varphi(\zeta)\} = \frac{1}{2\pi i}\int_{bB} \varphi(z)\frac{dz}{z-\zeta}.$$

12. A complex boundary-value problem for submanifolds of a compact Kähler space.

In [8] essentially the following boundary-value problem for forms $\varphi = \prod_{\varrho, k} \varphi$ on Euclidean submanifolds B was investigated:

$$\begin{cases} \Box' \varphi = \bar{\mathfrak{d}}\,\partial\varphi = 0, \ \varphi = \bar{\mathfrak{d}}\,\psi \text{ in } B \\ t\,\varphi \text{ prescribed on } bB. \end{cases} \tag{12.1}$$

It was there shown that the equations (12.1) have a unique solution φ. We now show that the same result is valid for forms $\varphi = \prod_{\varrho, k} \varphi$, $\varrho + k = p$, on arbitrary finite submanifolds of a given compact Kähler space. The method of proof parallels [8] once a suitable singularity has been defined.

The boundary-value problem dual to (12.1) is

$$\begin{cases} \Box'' \varphi = \partial\,\bar{\mathfrak{d}}\,\varphi = 0, \ \varphi = \partial\psi \text{ in } B, \ \varphi = \prod_{\varrho, 0} \varphi \\ n\,\varphi \text{ prescribed on } bB. \end{cases} \tag{12.2}$$

A similar treatment shows that the equations (12.2) have a unique solution.

We remark that if $\varrho = k$ in (12.1), then φ is identically zero. For $\bar{\mathfrak{d}}\,\psi = \prod_{\varrho, k} \bar{\mathfrak{d}}\,\psi$, where $0 \leq \varrho \leq k-1$, by (2.14′). The vanishing of φ when $\varrho = k$ is obviously necessary; otherwise φ would be complex-analytic and this would lead to contradictions.

Let B be a finite submanifold with boundary of a given compact Kähler manifold $M = M^k$, and let $f_p(z, \bar{\zeta})$ be the kernel for M. Then

$$\partial_z f_p(z, \bar{\zeta}) = \mathfrak{d}_\zeta f_{p+1}(z, \bar{\zeta}); \tag{12.3}$$
$$\partial_z = \prod_{\varrho+1, \sigma} d \prod_{\varrho, \sigma}, \ \mathfrak{d}_\zeta = \prod_{\sigma, \varrho} \delta \prod_{\sigma, \varrho+1}.$$

In other words, the relation (11.5) is valid for f. In fact,

$$F\,\partial = \partial\,F. \tag{12.4}$$

For

$$\partial \square = \partial \bar{\mathfrak{d}} \partial = \square \partial ;$$

hence, applying the operator ∂ to both sides of the equation

$$\square F \varphi = \varphi - C \varphi , \quad C F \varphi = 0 , \tag{12.5}$$

we have, since $\partial C \varphi = 0$,

$$\square \partial F \varphi = \partial \square F \varphi = \partial \varphi .$$

On the other hand,

$$C \partial \varphi = 0 ,$$

since $\partial \varphi$ is orthogonal to complex-harmonic forms. Therefore, replacing φ by $\partial \varphi$ in (12.5), we obtain

$$\square F \partial \varphi = \partial \varphi .$$

Thus

$$\square (\partial F - F \partial) \varphi = 0 .$$

But since $C F = C \partial = 0$, we conclude that $\partial F - F \partial = 0$. Similarly

$$F \bar{\mathfrak{d}} = \bar{\mathfrak{d}} F . \tag{12.4'}$$

If φ is a $(p+1)$-form, we have by (12.4')

$$(\bar{\mathfrak{d}}_z \varphi, f_p (z, \bar{\zeta}))_M = (\varphi, \mathfrak{d}_\zeta f_{p+1} (z, \bar{\zeta}))_M .$$

Since

$$(\bar{\mathfrak{d}}_z \varphi , f_p (z, \bar{\zeta}))_M = (\varphi, \partial_z f_p (z, \bar{\zeta}))_M ,$$

the relation (12.3) follows.

We observe that

$$\bar{\mathfrak{d}}_\zeta (f_{p+1} (z, \bar{\zeta}))^- = \bar{\mathfrak{d}}_\zeta f_{p+1} (\zeta, \bar{z}) ,$$

$$f_{p+1} (z, \bar{\zeta}) = \prod_{\varrho+1, \sigma}^{z} \ \prod_{\sigma, \varrho+1}^{\zeta} f_{p+1} (z, \bar{\zeta})$$

is a fundamental singularity for the operator $\square' = \bar{\mathfrak{d}} \partial$. In fact,

$$\square'_\zeta \bar{\mathfrak{d}}_\zeta f_{p+1} (\zeta, \bar{z}) = \bar{\mathfrak{d}}_\zeta \square_\zeta f_{p+1} (\zeta, \bar{z}) = 0 .$$

Now let scalar products be extended over B, where B is the given submanifold with boundary. By (11.6) (with ω replaced by f_p)

$$\square (\varphi, f_p (z, \bar{\zeta})) = - \tfrac{1}{2} (\varphi, \varkappa_p (z, \bar{\zeta})) + \tfrac{1}{2} \{\varphi\} , \tag{12.6}$$

where $\varkappa_p (z, \bar{\zeta}) = - \square_z f_p (z, \bar{\zeta})$ is the kernel for M. Since $(\varphi, \varkappa_p (z, \bar{\zeta}))$ is clearly continuous across the boundary of B, we conclude that the left side of (12.6) diminishes by $\frac{1}{2} \varphi$ as the boundary of B is crossed from the interior to the exterior.

On the other hand (compare Section 11),

$$\square\,(\varphi, f_p(z,\bar{\zeta})) = \bar{\mathfrak{d}}\,\partial\,(\varphi, f_p) + \partial\,\bar{\mathfrak{d}}\,(\varphi, f_p) = \bar{\mathfrak{d}}\,(\varphi, \bar{\mathfrak{d}}\, f_{p+1}) + \partial\,(\varphi, \partial\, f_{p-1}) =$$
$$= \partial \int_{bB} (f_{p-1})^{-}{}_{\wedge} * \varphi - \bar{\mathfrak{d}} \int_{bB} \varphi_{\wedge} (* f_{p+1})^{-} + \partial\,(\bar{\mathfrak{d}}\,\varphi, f_{p-1}) + \bar{\mathfrak{d}}\,(\partial\,\varphi, f_{p+1}).$$

Let us now suppose that $\varphi = \prod_{\varrho, k} \varphi$, $f_p(z, \bar{\zeta}) = \prod^{z}_{\varrho, k} \prod^{\zeta}_{k, \varrho} f_p(z, \bar{\zeta})$.

Then $f_{p-1} = \prod^{z}_{\varrho-1, k} \prod^{\zeta}_{k, \varrho-1} f_{p-1}$, and hence $\delta_{\zeta}(f_{p-1})^{-} = d_{\zeta}(f_{p-1})^{-}$. It follows that

$$t\,\partial \int_{bB} (f_{p-1})^{-}{}_{\wedge} * \varphi = t\,d \int_{bB} (f_{p-1})^{-}{}_{\wedge} * \varphi$$

is continuous across bB and therefore

$$t\,\bar{\mathfrak{d}} \int_{bB} \varphi_{\wedge} (* f_{p+1})^{-}$$

must jump by $-\frac{1}{2} t\, \varphi(\zeta)$.

Now let $\tau = \prod_{0, k} \tau$ be a form defined in a complete neighborhood of the boundary of B which is of class C^{∞} and non-vanishing there. A form τ with these properties is readily constructed. On dividing τ by its "length"

$$|\tau| = \sqrt{\bar{\tau}_{1\,2 \ldots k}\, \tau^{1\,2 \ldots k}}$$

we obtain a form τ which satisfies the additional requirement that $|\tau| = 1$ throughout the neighborhood of the boundary. We observe that $* \tau = \prod_{0, k} \tau$.

If $\psi = \prod_{\varrho, 0} \psi$, we note the following two identities:

$$\partial\,(\psi_{\wedge} \tau) = d\,(\psi_{\wedge} \tau) = d\,\psi_{\wedge} \tau + (-1)^{\varrho}\, \psi_{\wedge}\, d\,\tau, \tag{12.7}$$

$$\bar{\tau}_{\wedge} * (\psi_{\wedge} \tau) = * \psi. \tag{12.8}$$

If $\psi = \prod_{0, \varrho} \psi$, the same identities are valid with τ replaced by $\bar{\tau} = \prod_{k, 0} \tau$. Moreover, of $\psi = \prod_{\varrho, k} \psi$, then $* \psi = \prod_{0, k-\varrho} \psi$ and (12.8) gives

$$\tau_{\wedge} * (* \psi_{\wedge} \bar{\tau}) = (-1)^{p}\, \psi, \quad p = k + \varrho. \tag{12.9}$$

We have

$$\bar{\tau}_{\wedge} * \bar{\mathfrak{d}} \int_{bB} \varphi_{\wedge} (* f_{p+1})^{-} = (-1)^{p+1}\, \bar{\tau}_{\wedge}\, \partial * \int_{bB} \varphi_{\wedge} (* f_{p+1})^{-} =$$
$$= (-1)^{p+1}\, \bar{\tau}_{\wedge}\, d * \int_{bB} \varphi_{\wedge} (* f_{p+1})^{-} = (-1)^{\varrho+1} \{ d\,[\bar{\tau}_{\wedge} * \int_{bB} \varphi_{\wedge} (* f_{p+1})^{-}] -$$
$$- d\,\bar{\tau}_{\wedge} * \int_{bB} \varphi_{\wedge} (* f_{p+1})^{-} \}.$$

Thus

$$t\{*(\bar{\mathfrak{d}}\int_{bB}\varphi_{\wedge}(*f_{p+1})^{-})_{\wedge}\bar{\tau}\}$$

is continuous across bB, and it follows that

$$t\{*(\partial\int_{bB}(f_{p-1})^{-}{}_{\wedge}*\varphi)_{\wedge}\bar{\tau}\}$$

must jump by the amount

$$\tfrac{1}{2}t(*\varphi_{\wedge}\bar{\tau}).$$

Writing

$$\lambda_p=-2\bar{\mathfrak{d}}\int_{bB}\varphi_{p\,\wedge}(*f_{p+1})^{-},\ \ \eta_{p+1}=2\partial\int_{bB}(f_p)^{-}{}_{\wedge}*\varphi_{p+1} \tag{12.10}$$

where

$$\varphi_p=\prod_{\varrho,k}\varphi_p\,,\ \ \varphi_{p+1}=\prod_{\varrho+1,k}\varphi_{p+1}\,,$$

we have

Lemma 12.1. *The expressions $t\lambda p$ and $t(*\eta_{p+1\,\wedge}\bar{\tau})$ decrease by amounts $t\varphi_p$ and $t(*\varphi_{p+1\,\wedge}\bar{\tau})$ as the boundary of B is crossed from the interior of B to the exterior.*

On the boundary of B itself we therefore conclude by the usual reasoning that

$$t\lambda_p=-2Pt\bar{\mathfrak{d}}\int_{bB}\varphi_{p\wedge}(*f_{p+1})^{-}+\tfrac{1}{2}t\varphi_p\,, \tag{12.11}$$

$$t(*\eta_{p+1\wedge}\bar{\tau})=2Pt\{(*\partial\int_{bB}(f_p)^{-}{}_{\wedge}*\varphi_{p+1})_{\wedge}\bar{\tau}\}+\tfrac{1}{2}t(*\varphi_{p+1\wedge}\bar{\tau}), \tag{12.12}$$

where the letter P indicates that the principal values of the integrals are understood

We observe that

$$\mathfrak{d}_\zeta(\varphi_{p\,\wedge}*(f_{p+1})^{-})=*\partial_z f_p(\zeta,\bar{z})_{\wedge}\varphi_p\,. \tag{12.13}$$

Consider now the integral equation

$$t\lambda_p=\theta_p\,,\ \ \theta_p=\prod_{\varrho,k}\theta_p\,, \tag{12.14}$$

with singular kernel

$$-2\,t_z\,t_\zeta\,*_z\,\partial_z f_p(\zeta,\bar{z}). \tag{12.15}$$

We have

$$\int_{bB}\theta_{p\,\wedge}*(\varphi_{p+1})^{-}=(-1)^{k\varrho+p+1}P\{\int_{bB}(2*_z\partial_z\int_{bB}f_p(\zeta,\bar{z})_{\wedge}*_\zeta(\varphi_{p+1}(\zeta))^{-}- \\ -\tfrac{1}{2}*_\zeta(\varphi_{p+1}(\zeta))^{-})_{\wedge}\tau_{z\,\wedge}*_z(*\varphi_{p\,\wedge}\bar{\tau})\}. \tag{12.16}$$

Therefore

$$\int_{bB} \theta_p \wedge * (\varphi_{p+1})^- = 0 \tag{12.17}$$

for every solution φ_{p+1} of the equation

$$t(* \eta_{p+1} \wedge \bar{\tau}) = 0 \tag{12.18}$$

in which the boundary is that of the complementary region $\tilde{B} = M - B$. The complement is not necessarily connected, but this does not matter. It may be shown on the basis of the singular Fredholm theory that (12.14) is solvable if and only if the orthogonality condition (12.17) holds for every solution of (12.18).

The remainder of the proof parallels [8]. We observe that f_p is not complex—harmonic but satisfies

$$\Box_z f_p = - \varkappa_p .$$

Thus the kernel $\varkappa_p$ will appear in certain formulas where it does not occur in [8]. However, where it appears, it will be found to cancel.

The same methods apply to the following modified version of the boundary-value problem (12.1): there exists a form $\varphi = \prod_{\varrho, 0} \varphi$ on B satisfying

$$\begin{cases} \Box' \varphi = \bar{\mathfrak{b}}\, \partial \varphi = 0, \quad \varphi = \bar{\mathfrak{b}}\, \psi \text{ in } B \\ \quad t(\varphi_\wedge \tau) \text{ prescribed on } bB. \end{cases} \tag{12.1'}$$

Finally, on the basis of the boundary-value problems (12.1′) and (12.2) we can, as in [8], define Green's and Neumann's forms $G_p(z, \bar{\zeta})$ and $N_p(z, \bar{\zeta})$ in B. The Green's form is characterized by the properties

$$\begin{cases} G_p(z, \bar{\zeta}) = -2\, \bar{\mathfrak{b}}_z \{ f_{p+1}(z, \bar{\zeta}) + \text{regular terms} \}, \\ \Box'_z\, G_p(z, \bar{\zeta}) = 0, \quad z \neq \zeta, \\ t_z(\tau_z \wedge G_p(z, \zeta)) = 0 \quad \text{on} \quad bB, \end{cases} \tag{12.19}$$

while the Neumann's form is characterized by

$$\begin{cases} N_p(z, \bar{\zeta}) = 2\, \partial_z \{ f_{p-1}(z, \bar{\zeta}) + \text{regular terms} \}, \\ \Box''_z\, N_p(z, \bar{\zeta}) = 0, \quad z \neq \zeta, \\ n_z\, N_p(z, \bar{\zeta}) = 0 \quad \text{on} \quad bB. \end{cases} \tag{12.20}$$

Let

$$K_p(z, \bar{\zeta}) = \partial_z\, G_{p-1}(z, \bar{\zeta}) - \bar{\mathfrak{b}}_z\, N_{p+1}(z, \bar{\zeta}). \tag{12.21}$$

Then

$$(12.22) \qquad \partial_z K_p = \bar{\mathfrak{d}}_z K_p = 0\,.$$

If $\alpha_p = \prod_{p,0} \alpha_p$ satisfies $\partial \alpha_p = \bar{\mathfrak{d}} \alpha_p = 0$ in B, we have

$$(\alpha_p\,,\, K_p) = (\alpha_p\,,\, \partial\, G_{p-1}) - (\alpha_p\,,\, \bar{\mathfrak{d}}\, N_{p+1})$$
$$= \alpha_p + \int_{bB} \{ (G_{p-1})^-{}_\wedge * \alpha_p + \alpha_{p\,\wedge} * (N_{p+1})^- \} = \alpha_p\,.$$

Here we use the fact that $(G_{p-1})^-{}_\wedge * \alpha_p = (G_{p-1\,\wedge}\,\tau)^-{}_\wedge * (\alpha_{p\,\wedge}\,\tau)$ has a vanishing tangential component on the boundary of B. Thus

$$(12.23) \qquad (\alpha_p\,,\, K_p) = \alpha_p$$

and K_p is the reproducing kernel for forms $\alpha_p = \prod_{p,0} \alpha_p$ satisfying $\partial \alpha_p = \bar{\mathfrak{d}} \alpha_p = 0$. We observe that

$$(12.24) \qquad K_p\,(z, \bar{\zeta}) = (K_p\,(\zeta, \bar{z}))^-.$$

Finally, by the definition of G_p, there exists a Γ_{p+1} such that

$$(12.25) \qquad \begin{cases} \Gamma_{p+1}\,(z, \bar{\zeta}) = f_{p+1}\,(z, \bar{\zeta}) + \text{regular terms}, \\ G_p\,(z, \bar{\zeta}) = -\,\bar{\mathfrak{d}}_z\,\Gamma_{p+1}\,(z, \bar{\zeta}), \\ (\Gamma_{p+1}\,,\, \chi_{p+1}) = 0 \quad \text{if } \bar{\mathfrak{d}}\,\chi_{p+1} = 0, \end{cases}$$

and similarly, by the definition of N_{p+1}, there is a Θ_p such that

$$(12.26) \qquad \begin{cases} \Theta_p\,(z, \bar{\zeta}) = f_p\,(z, \bar{\zeta}) + \text{regular terms}, \\ N_{p+1}\,(z, \bar{\zeta}) = \partial_z\,\Theta_p\,(z, \bar{\zeta}), \\ (\theta_p\,,\, \chi_p) = 0 \quad \text{if } \partial\,\chi_p = 0. \end{cases}$$

The forms Γ_{p+1} and θ_p are uniquely determined by these conditions. As in [8] we then find that the solutions of the boundary-value problems (12.1′) and (12.2) are given respectively by

$$(12.27) \qquad \tfrac{1}{2}\,\varphi_p = -\,\bar{\mathfrak{d}} \int_{bB} \varphi_{p\,\wedge} * (\Gamma_{p+1})^- = \bar{\mathfrak{d}} \int_{bB} (\varphi_{p\,\wedge}\,\tau)_\wedge * (\Gamma_{p+1\,\wedge}\,\tau)^-$$

$$(12.28) \qquad \tfrac{1}{2}\,\varphi_{p+1} = \partial \int_{bB} (*\,\varphi_{p+1})_\wedge\,(\Theta_p)^-$$

As in [8] we also have the symmetry laws

$$(12.29) \qquad \mathfrak{d}_\zeta\, G_p\,(z, \bar{\zeta}) = \bar{\mathfrak{d}}_z\,(G_p\,(\zeta, \bar{z}))^-,$$

$$(12.30) \qquad \bar{\partial}_\zeta\, N_p\,(z, \bar{\zeta}) = \partial_z\,(N_p\,(\zeta, \bar{z}))^-.$$

We observe that the boundary-value problem conjugate to (12.2) is

$$\begin{cases} \bar{\Box}'' \varphi = \bar{\partial} \mathfrak{d} \varphi = 0, \ \varphi = \bar{\partial} \psi \text{ in } B, \ \varphi = \prod_{0,q} \varphi \\ n \varphi \text{ prescribed on } bB. \end{cases} \tag{12.31}$$

Let $\psi = \prod_{0,0} \psi$, $\varphi = \prod_{0,1} \varphi$. Then

$$\partial \bar{\Box}' \psi = \partial \bar{\Box} \psi = 0,$$

so $\bar{\Box} \psi$ is complex analytic in B and $n \bar{\partial} \psi$ has prescribed values on the boundary. This is the boundary-value problem considered in Section 10. Here again we have no assurance that the form $\bar{\Box} \psi$ is non-trivial.

Finally, we note one or two special cases. Forms

$$\varphi = \varphi_p = \prod_{0,p} \varphi \text{ satisfying } \bar{\partial} \varphi = \mathfrak{d} \varphi = 0$$

have a reproducing kernel K_p which satisfies (compare (12.21))

$$K_p (z, \bar{\zeta}) = \bar{\partial}_z G_{p-1} (z, \bar{\zeta}) = \mathfrak{d}_z N_{p+1} (z, \bar{\zeta}) . \tag{12.18'}$$

In the particular case $p = 0$ this formula becomes

$$K_0 (z, \bar{\zeta}) = -\mathfrak{d}_z N_1 (z, \bar{\zeta}) = -\bar{\Box}_z \Theta_0 (z, \bar{\zeta}) . \tag{12.32}$$

If $p = 0$, $\mathfrak{d} \varphi = 0$ automatically and the condition $\bar{\partial} \varphi = 0$ implies that φ is complex-analytic. Thus K_0 is the reproducing kernel for complex-analytic scalars.

If $p = k$, $\varphi = \prod_{k,0} \varphi$, (12.21) becomes

$$K_k (z, \bar{\zeta}) = \partial_z G_{p-1} (z, \bar{\zeta}) = -\Box_z \Gamma_k (z, \bar{\zeta}) . \tag{12.33}$$

In this case $\partial \varphi = 0$ automatically and the condition $\bar{\mathfrak{d}} \varphi = 0$ implies that φ is complex-analytic. Thus $K_k (z, \bar{\zeta})$ is the kernel for complex-analytic k-forms φ, $\varphi = \prod_{k,0} \varphi$ (complex-analytic densities).

13. The fundamental singularity for real harmonic fields.

On an arbitrary real orientable Riemannian manifold M, Kodaira [10] has proved that there exists a fundamental singularity $e_{p+1} (x, y)$ for harmonic fields $\varphi = \varphi_{p+1}$, $d \varphi = \delta \varphi = 0$. This singularity satisfies $d_x e_{p+1} = \delta_x e_{p+1} = 0$ except at y where

$$e_{p+1} (x, y) = d_x d_y \gamma_p (x, y) + \text{regular term}, \tag{13.1}$$

γ_p being a local fundamental singularity for Δ valid in a neighborhood N of the point y. Moreover,

$$(e_{p+1}, e_{p+1})_{M-N} < \infty \tag{13.2}$$

for an arbitrary neighborhood N of y and, if η is a $(p+1)$-form of class C^∞, $d\eta=0$, $\eta\equiv 0$ in N, $(\eta, \eta) < \infty$, then

$$(e_{p+1,\eta})_M = 0. \tag{13.3}$$

Kodaira has established the existence of e_{p+1} under the assumption that M is real analytic, but his proof is easily extended to the case where M is C^∞. The conditions (13.1)—(13.3) determine e_{p+1} uniquely.

We now prove the existence of e_{p+1} by a method which is different from that of Kodaira and which seems to us to lie closer to the classical method used for Riemann surfaces.

Every finite submanifold B of the Riemann manifold can be imbedded in a compact (closed) Riemann manifold $F=F(B)$. Hence on any finite submanifold B we have a kernel $g_p(x, y)$ satisfying

$$\Delta_x g_p(x, y) = -\beta_p(x, y) \tag{13.4}$$

where β_p is the reproducing kernel for harmonic p-forms on the closed $F(B)$. The kernel g_p satisfies

$$d_x g_p(x, y) = \delta_y g_{p+1}(x, y) \tag{13.5}$$

and therefore

$$d_x d_y g_p(x, y) = -\delta_x \delta_y g_{p+2}(x, y) - \beta_{p+1}(x, y) \tag{13.6}$$

where β_{p+1} is the kernel for harmonic $(p+1)$-forms on F. In fact,

$$\begin{aligned} 0 = d_x(d_y g_p - \delta_x g_{p+1}) &= d_x d_y g_p - d_x \delta_x g_{p+1} = d_x d_y g_p + \delta_x d_x g_{p+1} + \beta_{p+1} \\ &= d_x d_y g_p + \delta_x \delta_y g_{p+2} + \beta_{p+1}. \end{aligned}$$

Since F is closed, we have $d_x \beta_{p+1} = \delta_x \beta_{p+1} = 0$. Hence for $0 \le p \le n-2$, n the dimension of M, the singularity $-\delta_x \delta_y g_{p+2}$ defines a singular harmonic field satisfying (13.1) and (13.2) on B. If B is closed, this field also satisfies (13.3) and is the desired singularity.

To obtain a singularity satisfying (13.3) when B is not closed, let φ be a $(p+2)$-form satisfying $d\delta\varphi=0$, $\varphi=d\psi$ in B, with

$$n_x \varphi = n_x \delta_y g_{p+2} \tag{13.7}$$

on the boundary of B, and write

$$e_{p+1}(x, y) = -\delta_x (\delta_y g_{p+2}(x, y) - \varphi(x)). \tag{13.8}$$

Then $d_x e_{p+1} = \delta_x e_{p+1} = 0$ and e_{p+1} satisfies (13.1) and (13.2). Let $\eta = \eta_{p+1}$ be of class C^∞ in B, $d\eta = 0$, $N_B(\eta) < \infty$, and suppose that η vanishes identically in a neighborhood N of y. Then

$$(e_{p+1}, \eta)_B = -(\delta_x(\delta_y g_{p+2}(x, y) - \varphi(x)), \eta) = -(\delta_y g_{p+2} - \varphi, d\eta)$$
$$+ \int_{bB - bN} \bar{\eta}_\wedge * (\delta_y g_{p+2} - \varphi) = 0.$$

Thus e_{p+1} satisfies (13.3), and it therefore agrees with the Kodaira singularity for B.

The relation (13.3) is a minimum condition of a type familiar in the theory of Riemann surfaces. In fact, let S be a small geodesic sphere of radius a and center at the point y, and set

$$\sigma_{p+2}(x, y) = \begin{cases} -\delta_y g_{p+2}(x, y) + \varphi(x) \text{ in } S \\ 0 \text{ in } B - S, \end{cases} \tag{13.9}$$

where φ satisfies $d\delta\varphi = 0$, $\varphi = d\psi$, in S with $n_x \varphi = n_x \delta_y g_{p+2}$ on the boundary of S. Then σ_{p+2} is just the generalization to Riemannian manifolds of H. Weyl's familiar local singularity used in establishing Dirichlet's Principle for a Riemann surface. Let V be the closure of the space of $(p+1)$-forms λ satisfying

(i) $\lambda + \delta_x \sigma_{p+2}$ is C^∞ except at y;

(ii) $d\lambda = 0$ in S and also in $B - S$;

(iii) $N_B(\lambda) < \infty$.

We consider the problem of minimizing $N_B(\lambda)$, $\lambda \varepsilon V$. The form

$$\lambda_0 = \begin{cases} e_{p+1} - \delta_x \sigma_{p+2} \text{ in } S \\ e_{p+1} \text{ in } B - S, \end{cases} \tag{13.10}$$

where e_{p+1} is the fundamental singularity for B, clearly satisfies the conditions (i)—(iii). Let η vanish in a neighborhood of y, $\eta \in C^\infty$ and $d\eta = 0$ in B. Since

$$(\delta_x \sigma_{p+2}, \eta) = (\delta_x \sigma_{p+2}, \eta)_S = 0,$$

we see that

$$(\lambda_0, \eta) = (\lambda_0, \eta)_B = 0. \tag{13.11}$$

Since λ_0 is regular at y, (13.11) is valid even if η does not vanish in a neighborhoood of y and it follows that λ_0 solves the minimum problem. For λ_0 differs from an arbitrary form satisfying (i)—(iii) by a form η which is C^∞ and closed in B. Therefore, if $\lambda \varepsilon V$, we have

$$\lambda = \lambda_0 + \eta \tag{13.12}$$

where η belongs to the closure of the space of closed forms of class C^∞. Thus

$$N(\lambda) = N(\lambda_0) + 2(\lambda_0, \eta) + N(\eta) = N(\lambda_0) + N(\eta) > N(\lambda_0)$$

unless η vanishes almost everywhere.

Let

$$d = d(B) = N(\lambda_0). \tag{13.13}$$

We observe that there are forms satisfying (i)—(iii) when B is replaced by the whole manifold M. For in a finite submanifold B containing S in its interior we have the form

$$\lambda = \begin{cases} d_x\, d_y\, g_p(x, y) - \delta_x\, \sigma_{p+2} \text{ in } S \\ d_x\, d_y\, g_p(x, y) \text{ in } B - S \end{cases} \tag{13.14}$$

where g_p is the kernel satisfying (13.6). If φ_0 is a scalar of class C^∞ with compact carrier which is equal to 1 in a neighborhood of S, we replace $d_y\, g_p(x, y)$ in (13.14) by $\varphi_0(x)\, d_y\, g_p(x, y)$ and thus obtain a form λ on M satisfying (i)—(iii). Hence, in particular, the greatest lower bound D of $N(\lambda)$ over M is finite.

Now let $\{B_\mu\}$ be a sequence of finite submanifolds of M, $B_\mu \subset B_{\mu+1}$, where B_μ tends to M as μ becomes infinite, and write

$$d_\mu = d(B_\mu). \tag{13.15}$$

Then clearly

$$d_\mu \leq d_{\mu+1} \leq D \tag{13.16}$$

and hence the d_μ converge to a limit d, $d \leq D < \infty$.

Let norms and scalar products over B_μ be distinguished by a subscript μ, and let λ_μ be the solution of the minimum problem for B_μ, $N_\mu(\lambda_\mu) = d_\mu$. If $\mu < \nu$ we have

$$0 \leq N_\mu(\lambda_\mu - \lambda_\nu) = N_\mu(\lambda_\mu) - 2(\lambda_\mu, \lambda_\nu)_\mu + N_\mu(\lambda_\nu)$$

where

$$(\lambda_\mu, \lambda_\nu)_\mu = (\lambda_\mu, \lambda_\mu + (\lambda_\nu - \lambda_\mu))_\mu = (\lambda_\mu, \lambda_\mu)_\mu = N_\mu(\lambda_\mu)$$

since $(\lambda_\mu, \lambda_\mu - \lambda_\nu)_\mu = 0$. Thus

$$0 \leq N_\mu(\lambda_\mu - \lambda_\nu) = N_\mu(\lambda_\nu) - N_\mu(\lambda_\mu) \leq N_\nu(\lambda_\nu) - N_\mu(\lambda_\mu) = d_\nu - d_\mu,$$

and it follows that

$$N_\mu(\lambda_\mu - \lambda_\nu) \to 0 \tag{13.17}$$

as μ, ν approach infinity, $\mu < \nu$. In particular, if B is any fixed finite submanifold of M, we have

$$\lim_{\mu,\nu\to\infty} N_B(\lambda_\mu - \lambda_\nu) = 0. \tag{13.18}$$

Since $\lambda_\mu - \lambda_\nu$ is harmonic, $\Delta(\lambda_\mu - \lambda_\nu) = 0$, we can apply inequality (7.20) and we see that λ_μ converges uniformly over any compact subdomain B of M to a limit λ satisfying conditions (i) and (ii) for M. But

$$N_B(\lambda) = \lim N_B(\lambda_\mu) \leq \lim N_\mu(\lambda_\mu) = \lim d_\mu = d \leq D.$$

Letting B tend to M, we conclude that $N(\lambda) \leq D < \infty$ and therefore $N(\lambda) = D$. Hence λ solves the minimum problem for M, and

$$e_{p+1} = \lambda + \delta_x \sigma_{p+2}$$

is the desired fundamental singularity for harmonic fields on M. The singularity e_{p+1} is symmetric: $e_{p+1}(x, y) = e_{p+1}(y, x)$.

Kodaira [10] shows that, if there are forms σ, τ such that $\varphi = d\sigma = \delta\tau$, then in terms of e_p the field φ has the representation

$$\varphi(y) = -\int_{bB} \{\sigma_\wedge * e_p + e_{p\,\wedge} * \tau\}. \tag{13.19}$$

Here y is an interior point of the finite submanifold B.

The singularity e_p is used by Kodaira to establish the existence of harmonic fields of the second and third kinds on the Riemannian manifold.

14. The fundamental singularity for complex harmonic fields.

We now construct the fundamental singularity $c_{p+1}(z, \bar{\zeta})$ for complex harmonic fields $\varphi = \prod_{p,0} \varphi$ on an arbitrary finite submanifold of a compact Kähler manifold, $\partial\varphi = \bar{\mathfrak{d}}\varphi = 0$. This singularity satisfies $\partial_z c_{p+1} = \bar{\mathfrak{d}}_z c_{p+1} = 0$ except at ζ where

$$c_{p+1}(z, \bar{\zeta}) = \partial_z \partial_\zeta \theta_p(z, \bar{\zeta}) + \text{regular terms}, \tag{14.1}$$

$$\theta_p = \prod_{p,0}{}_z \prod_{0,p}{}_\zeta \theta_p,$$

θ_p being a local fundamental singularity for the operator $\Box$ which is valid in a neighborhood N of the point ζ. For an arbitrary neighborhood N of ζ we have

$$(c_{p+1},\ c_{p+1})_{M-N} < \infty \tag{14.2}$$

and, if η is a $(p+1)$-form of class C^∞, $\eta = \prod_{\varrho+1,\sigma} \eta$, $\partial\eta = 0$, $\eta \equiv 0$ in a neighborhood of ζ, $(\eta, \eta) < \infty$, then

$$(c_{p+1},\ \eta)_M = 0. \tag{14.3}$$

Since it is not always true that a submanifold B of an arbitrary Kähler manifold can be imbedded in a compact Kähler manifold, we are forced to assume that B is a subdomain of a compact manifold $M = M^k$.

Let $f_p(z, \bar{\zeta})$ be the kernel for the compact Kähler manifold M satisfying

$$\Box_z f_p(z, \bar{\zeta}) = -\tfrac{1}{2}\varkappa_p(z, \bar{\zeta}) \tag{14.4}$$

where $\varkappa_p$ is the reproducing kernel for complex-harmonic forms on M satisfying $\Box\varphi = 0$. Applying (12.3) we have

$$0 = \partial_z(\bar{\partial}_\zeta f_p - \bar{\mathfrak{d}}_z f_{p+1}) = \partial_z\bar{\partial}_\zeta f_p - \partial_z\bar{\mathfrak{d}}_z f_{p+1} = \partial_z\bar{\partial}_\zeta f_p + \bar{\mathfrak{d}}_z\partial_z f_{p+1} + \tfrac{1}{2}\varkappa_{p+1} =$$
$$= \partial_z\bar{\partial}_\zeta f_p + \bar{\mathfrak{d}}_z\mathfrak{d}_\zeta f_{p+2} + \tfrac{1}{2}\varkappa_{p+1};$$

that is,

$$\partial_z\bar{\partial}_\zeta f_p = -\bar{\mathfrak{d}}_z\mathfrak{d}_\zeta f_{p+2} - \tfrac{1}{2}\varkappa_{p+1}. \tag{14.5}$$

Since M is closed, $\partial_z\varkappa_{p+1} = \bar{\mathfrak{d}}_z\varkappa_{p+1} = 0$, and the singularity $-\bar{\mathfrak{d}}_z f_{p+2}$ is a complex-harmonic field satisfying (14.1) and (14.2) on B. If B is closed, in which case it coincides with M, the desired singularity is $-\bar{\mathfrak{d}}_z\mathfrak{d}_\zeta f_{p+2}$. If B is not closed, let φ be a $(p+2)$-form satisfying $\partial\bar{\mathfrak{d}}\varphi = 0$, $\varphi = \partial\psi$ in B, with

$$n_z\varphi = n_z\mathfrak{d}_\zeta f_{p+2} \tag{14.6}$$

on bB, and set

$$c_{p+1}(z, \bar{\zeta}) = -\bar{\mathfrak{d}}_z(\mathfrak{d}_\zeta f_{p+2}(z, \bar{\zeta}) - \varphi(z)). \tag{14.7}$$

Then c_{p+1} satisfies (14.1), (14.2), and (14.3).

15. Remark on the Cousin problem for a compact Kähler manifold.

Let the Kähler manifold be compact, and let E be a closed set of points. To each point q of the manifold let there be a neighborhood $N(q)$ and a current T_q in $N(q)$ of degree p such that:

$$
(15.1)\qquad \begin{cases} \prod_{\varrho,\sigma} T_q = T_q; \\ \square\, T_q = 0 \quad \text{in } N(q) - E; \\ \square\, (T_q - T_r) = 0 \quad \text{in } N(q) \cap N(r). \end{cases}
$$

The Cousin problem in this case is to find a current T such that $T - T_q$ is complex-harmonic for every q.

This problem is solved in a manner like that applied in [11] to the real operator Δ. We set $Z = \square\, T_q$; then Z is a well-determined current and the required current T, if it exists, is a solution of the equation

$$
(15.2)\qquad \square\, T = Z.
$$

This equation is solvable if and only if $CZ = 0$, and if $CZ = 0$, the solution is given by $T = 2FZ$.

Next, suppose that instead of (15.1) we have

$$
(15.3)\qquad \begin{cases} \prod_{\varrho,\sigma} T_q = T_q; \\ \partial\, T_q = \bar{\mathfrak{b}}\, T_q = 0 \quad \text{in } N(q) - E; \\ \partial\, (T_q - T_r) = \bar{\mathfrak{b}}\, (T_q - T_r) = 0 \quad \text{in } N(q) \cap N(r). \end{cases}
$$

A current T is to be found such that $\partial (T - T_q) = \bar{\mathfrak{b}} (T - T_q) = 0$ for every q. In a manner analogous to that of [11], we set $Z_1 = \partial\, T_q$, $Z_2 = \bar{\mathfrak{b}}\, T_q$. The required current T, if it exists, is a solution of the equations

$$
\partial\, T = Z_1, \quad \bar{\mathfrak{b}}\, T = Z_2.
$$

The conditions of solvability are that $CZ_1 = CZ_2 = 0$ or, what is the same thing, that $Z_1 = \partial\, U_1$, $Z_2 = \bar{\mathfrak{b}}\, U_2$, where U_1 and U_2 are currents. If these conditions are satisfied, a solution is given by

$$
(15.4)\qquad T = 2\,(F\,\partial\, Z_2 + F\,\bar{\mathfrak{b}}\, Z_1).
$$

Replacing in (15.3) the operators ∂ and $\bar{\mathfrak{b}}$ by their conjugates $\bar{\partial}$ and $\mathfrak{b}$ and taking $\varrho = p$, $\sigma = 0$, $0 \leq p \leq k$, we have

$$
(15.5) \qquad \begin{cases} \prod\limits_{p,0} T_q = T_q\,; \\ \bar{\partial}\, T_q = 0 \ \text{ in } N(q) - E\,; \\ \bar{\partial}\,(T_q - T_r) = 0 \ \text{ in } N(q) \cap N(r)\,. \end{cases}
$$

Here $\mathfrak{d}\, T_q = 0$ automatically; therefore this condition is omitted. The problem is to find a current T such that $T - T_q$ is regular-analytic for every q; this is the Cousin problem for complex-analytic currents. We set

$$
(15.6) \qquad Z = \bar{\partial}\, T_q\,, \ Z = \prod_{p,1} Z\,.
$$

If $\bar{C}\, Z = 0$, a solution is given by

$$
(15.7) \qquad T = 2\, F\, \mathfrak{d}\, Z\,.
$$

The condition $\bar{C}\, Z = 0$ may be written

$$
(15.8) \qquad (Z, \varkappa_{p+1}) = 0
$$

where $\varkappa_{p+1}$ is the reproducing kernel for forms $\varphi = \prod\limits_{p,1} \varphi$ satisfying $\bar{\partial}\, \varphi = \mathfrak{d}\, \varphi = 0$ on the manifold.

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