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On Giambelli's theorem on complete correlations

by

D. LAKSOV, A. LASCOUX and A. THORUP

Royal Institute of Technology Stockholm, Sweden

Université Paris VII, Paris, France

University of Copenhagen Copenhagen, Denmark

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1. Introduction

The point of departure for the following work was an attempt to prove a formula of Giambelli which gives explicit expressions for a large family of characteristic classes of complete correlations. This formula of Giambelli together with some related formulas of Schubert stand out in the rich flora of numbers obtained from enumerative geometric problems; they constitute a generel species, solving large classes of enumerative problems, and therefore are of particular interest.

We shall present below a proof of Giambelli's formula for complete correlations and also a similar formula for complete quadrics which we shall make more precise later in this introduction. A special case of Giambelli's formula for complete correlations is a beautiful formula of Schubert for the powers of the first characteristic class (see [S2]). It is interesting to note that in an earlier paper [S1], Schubert expressed the powers of the first characteristic class of complete quadrics in terms of a numeric function ψ_A for which he only had a recursive definition; comparing with the explicit formula obtained for correlations indicated that there should be an explicit formula for ψ_A: "Während aber die Ergebnisse der früheren Untersuchung *noch nicht studierte* aus Binomialcoefficienten zusammengesetzte Ausdrücke sind, so sind die Ergebnisse der neuen Untersuchung elegant gestaltete Determinanten der Binomialcoefficienten." We obtain the explicit formula requested by Schubert as a special case of our results.

The formulas of Giambelli and Schubert, although interesting in themselves, constitute only a minor part of this work. As often happens in enumerative geometry, the verification of numbers, which may be of little interest in themselves, leads to problems of a more general and fundamental character and inspires work in other branches of mathematics. The formula of Giambelli is certainly no exception to this rule.

First of all the rather vague geometric arguments supporting the proofs of Giambelli and Schubert have lead to a sequence of works ([DC-P], [L], [T-K], [U], [V1], [V2], . . .) on the parameter spaces of complete correlations, collineations, quadrics and more general complete objects. We shall not be concerned with this work here, but shall refer to the relevant parts of [T-K] and [L] when needed.

Secondly, a deeper analysis of the formulas reveals that they are intimately related to explicit expressions for the Segre classes of tensor and symmetric products of locally free sheaves in terms of Schur functions. A major part of our work concerns such formulas for the Chern and Segre classes for the tensor product of two locally free sheaves and the second symmetric and exterior powers of a locally free sheaf. Some of the formulas we give can be found scattered in the literature with a variety of proofs (see especially [Lx]). We have not been able to find the explicit formulas for the Segre classes of the second symmetric and exterior powers elsewhere. We shall give a unified proof of all of the formulas.

Thirdly, in the study of characteristic classes recursive formulas for the numeric functions appearing in the expressions of such classes appear naturally. Schubert observed several such formulas. We prove all of them. However, we go much further and prove recurrence relations for quite general alternating functions of power series. This approach to the recurrence relations is perhaps one of the most intriguing parts of our work. In addition to the formulas of Schubert we also obtain some expression in terms of Pfaffians that were given by Pragacz [P]. Although the recursive formulas are often convenient for computational purposes they are not always adequate. As a curiosity, the formula $\psi_{0,1,\dots,r-1}=1$ for all r, which is an immediate consequence of our explicit formula (see the definition of ψ_A below), does not seem to be easily obtainable by recursive means. Schubert wrote: "Der Verfasser hat sich *vergeblich* bemüht, dieses interessante Resultat auch rein arithmetisch, allein aus der Definition von ψ, zu beweisen."

A fourth topic that is indicated by the work of Giambelli and Schubert is that geometry plays a very little role in their deduction of the formulas. Most of the arguments are of a purely algebraic and combinatorial nature. We have therefore systematically stressed the formal sides of the arguments. We have separated out the combinatorial and algebraic arguments in a long appendix and kept the geometric arguments to the article proper.

The contents of the article, section by section, is as follows:

Sections 2, 3 and 4 contain the material that we need from other fields.

In Section 2 we recall the language and results needed from general intersection theory. The references are to [K-T] and [F]. In this section we also interpret the formal results from the Appendix in terms of the Segre classes of locally free sheaves.

The relevant material from the theory of Schubert subvarieties of Grassmannians is collected in Section 3. The main references here are [F] and [K-L].

In Section 4 we recall the material we need about the geometry of complete bilinear forms and complete quadrics. This material is taken from [T-K] and [L].

In Section 5 we prove Giambelli's formula for bilinear forms and in Section 6 we prove the analogous formula for complete quadrics. We shall next give a geometrical interpretation of the latter formula. A similar, but slightly more complicated, interpretation of Giambelli's formula can be given. We leave this matter to the reader.

Assume that we are given a projective space $\mathbf{P}$ and an integer r. The space B of complete quadrics of rank r in $\mathbf{P}$ represents sequences $Q_1 \subset Q_2 \subset \ldots \subset Q_t$ of quadrics in $\mathbf{P}$ (for variable t) such that the linear span E_j of Q_j is the vertex of Q_{j+1} for $j=1,2,\ldots,t-1$ and such that Q_1 is non-singular and the linear span E_t of Q_t has dimension $r-1$. For each integer $i=1,2,\ldots,r$ we denote by μ_i the class in the intersection ring of B representing the locus of complete quadrics $Q_1 \subset Q_2 \subset \ldots \subset Q_t$ such that Q_t is tangent to a given plane in $\mathbf{P}$ of codimension i (when $i=r$ we interpret this as twice the condition that the span of the quadric Q_t meets a given plane of codimension r).

Let $A=(a_1,a_2,\ldots,a_r)$ be a strictly increasing sequence of non-negative integers. Given a flag $L_1 \subset L_2 \subset \ldots \subset L_r$ in $\mathbf{P}$, where the dimension of L_i is equal to a_i for all i, we denote by $[\Omega]$ the class, in the intersection ring of the Grassmannian $T=G_r(\mathbf{P})$ of $(r-1)$-dimensional linear subspaces in $\mathbf{P}$, which represents the locus of spaces L such that $\dim(L \cap L_i) \geqslant i-1$ for $i=1,2,\ldots,r$. There is a canonical map from B to T and we denote further by ω_A the preimage of $[\Omega]$ in the intersection ring of B. With this notation the class

$$\mu_1^{m_1}\mu_2^{m_2}\ldots\mu_r^{m_r} \cap \omega_A$$

in the intersection ring of B is represented by the locus of complete quadrics $Q_1 \subset Q_2 \subset \ldots \subset Q_t$ such that Q_t is tangent to m_i fixed planes of codimension i in general position in $\mathbf{P}$ and such that $\dim(E_i \cap L_i) \geq i-1$ for each member L_i of a fixed flag $L_1 \subset L_2 \subset \ldots \subset L_r$ in general position and with $\dim L_i = a_i$ for $i=1,2,\ldots,r$.

The class

$$\mu_1^{m_1}\mu_2^{m_2}\ldots\mu_r^{m_r} \cap \omega_A$$

represents a locus of dimension $\Sigma_{i=1}^r a_i + r - 1 - \Sigma_{i=1}^r m_i$ and is denoted by

$$(a_0, a_1, \ldots, a_p)\mu_1^{m_1}\mu_2^{m_2}\ldots\mu_r^{m_r}$$

by Schubert. We thus have $r=p+1$ and index the a_i's one higher than he does.

When $\Sigma_{i=1}^r a_i + r - 1 = \Sigma_{i=1}^r m_i$ the class above has dimension zero and its integral represents a number. To state the formula for this number we must define some combinatorial numbers. Let $\varphi^p(k,i)$ be the function defined by

$$\varphi^p(k,i) := \begin{cases} \binom{k}{0} + p\binom{k}{1} + \ldots + p^i\binom{k}{i} & \text{if } i \geq 0 \\ 0 \quad \text{if} \quad i < 0. \end{cases}$$

Moreover, let E^A be the $r \times \infty$ matrix with entries $\binom{a_i}{j}$ for $i=1,\ldots,r$ and $j=0,1,2,\ldots$ and denote by ψ_A the sum of all r by r minors of E^A.

The formula states that if p is a number such that $0 \leq p < r$ and such that the following inequalities are satisfied:

$$\sum_{i=1}^q m_i > \sum_{i=1}^q a_{r-i+1} + q - 1 \quad \text{for} \quad q = 1, \ldots, p-1,$$

then we have that

$$\int \mu_1^{m_1}\ldots\mu_p^{m_p}\mu_{p+1}^{m_{p+1}} \cap \omega_A$$
$$= 1^{m_1}2^{m_2}\ldots p^{m_p}\left((p+1)^{m_{p+1}}\psi_A - \sum_K \varphi^p(m_{p+1}, m_{p+1} - \|K\| - (r-p))\,\varepsilon_K\,\psi_K\,\psi_{\bar{K}}\right),$$

where the sum is over all $K=(k_1,\ldots,k_{r-p})$ that are subsequences of A with $r-p$ elements. Here $\|K\| = \Sigma_{i=1}^{r-p} k_i$, the sequence $\bar{K}$ is the complementary subsequence of K in A and ε_K is the sign of the permutation $(K, \bar{K})$ of A.

For $p=0,1$ and 2 Schubert gives (in [S1]) an explicit formula for $\int \mu_1^{m_1}\ldots\mu_{p+1}^{m_{p+1}} \cap \omega_A$ without any restrictions on the exponents $m_1, \ldots, m_{p+1}$. For $p=0$ and 1, there are no inequalities to be satisfied in our result and our formula coincides with that of Schubert.

In the case $p=2$, however, there is an inequality to be satisfied by the exponents which restricts the validity of our formula and, in fact, Schubert's formula contains an extra term.

Some relevant historical notes on the material mentioned above can be found in [K], [K-T], [L1] and [L2].

2. Notation and conventions

Setup (2.1). We fix a ground scheme S. All schemes will be of finite type over S. By convention, a bundle on a scheme X will be a locally free $\mathcal{O}_X$-module of finite type. We shall throughout assume that S admits an intersection theory in the sense of [F] or [K-T]. (For such a theory to exist it suffices that S is an algebraic scheme over a field.) Then, for all schemes X considered in the following (space of complete forms, degeneration loci, flags, Schubert schemes, ...), there will exist a graded group of (cycle) classes $A(X)$, covariant with respect to proper maps (proper push-forward) and contravariant with respect to flat maps of pure dimension (flat pull-back). As in [F], flat maps are assumed to be of pure dimension. Moreover, for all maps $f: X \to Y$ there will exist a graded group of (bivariant) classes $A^*(f)$ or $A^*(X/Y)$. A (bivariant) class α in $A^i(X/Y)$ is a family $\alpha=(\alpha_W)$, indexed by schemes W/Y, of graded homomorphisms of degree $-i$:

$$\alpha_W: A(W) \to A(X \times_Y W), \quad \text{denoted} \quad z \mapsto \alpha \cap z,$$

commuting with proper push-forward, with flat pull-backs, and with refined Gysin maps of regular embeddings, see [F] or [K-T].

The bivariant classes form a bivariant theory in the sense of Fulton-MacPherson: Let $\alpha=(\alpha_W)$ in $A^*(X/Y)$ be a (bivariant) class. The *product* of α with a (bivariant) class $\beta=(\beta_U)$ in $A^*(Y/Z)$ is the (bivariant) class $\alpha\beta := (\alpha_{U \times_Z Y}\beta_U)$ in $A^*(X/Z)$ defined by composition of families. The proper *push-forward* of α by a proper Y-map $f: X \to V$ is the (bivariant) class $f_*(\alpha) := (f_{W*}\alpha_W)$ in $A^*(V/Y)$. The *pull-back* or *base-change* of α along a map $g: Z \to Y$ is the (bivariant) class $\alpha|Z = g^*(\alpha)$ in $A^*(X \times_Y Z/Z)$ obtained by restricting the family (α_W) to schemes W/Z. These three basic operations satisfy the 7 bivariant axioms listed in [F].

Examples of bivariant classes are: The class defined by flat pull-back of (cycle) classes along a flat map of pure dimension, the refined Gysin class of a regular embedding and Chern and Segre classes of bundles (see below).

Let α and β be (bivariant) classes in $A^*(X/Y)$ and $A^*(Z/Y)$ respectively; denote by g the structure map of Z/Y. Then the *cross product* $\alpha \times \beta$ is the (bivariant) class $g^*(\alpha)\beta$

in $A^*(X\times_Y Z/Z)$. The classes α and β are said to *commute* if $\alpha\times\beta=\beta\times\alpha$. An *orientation* of a map $f: X\to Y$ is a class θ belonging to the subsystem of $A^*(X/Y)$ generated by flat pull-back classes and refined Gysin classes and such that

$$\theta\cap[Y]=[X].$$

Certain maps have natural orientations: Flat maps are oriented by the class of the flat pull-back. Regular embeddings are oriented by the Gysin class and, more generally, maps $f: X\to Y$ that are regular in the sense of [F] have a natural orientation class $[f]$ in $A^*(X/Y)$. Note that the notion of orientation is stable under composition of classes, but not in general under pull-back.

For a proper map $f: X\to Y$ we shall use the following notation:

(1) If z is a (cycle) class in $A(X)$, then we denote by $\int_{X/Y} z$ the image of z in $A(Y)$ under proper push-forward, i.e.,

$$\int_{X/Y} z:=f_*(z)\in A(Y).$$

(2) If θ is a (bivariant) class in $A^*(X/Y)$, then we denote by $\int_{X/Y}\theta$ the (bivariant) image of θ in $A^*(Y/Y)$ under proper push-forward, i.e.,

$$\int_{X/Y}\theta:=f_*(\theta)\in A^*(Y/Y).$$

(3) If f is oriented by a class $[f]$ and α is a (bivariant) class in $A^*(X/X)$, then we denote by $\int_{X/Y}\alpha$ the image of $\alpha[f]$ in $A^*(Y/Y)$ under proper push-forward, i.e.,

$$\int_{X/Y}\alpha:=f_*(\alpha[f])=\int_{X/Y}\alpha[f]\in A(Y/Y).$$

Note that if a flat and proper map $f: X\to Y$ is oriented by its natural orientation class, then we have that

$$\left(\int_{X/Y}\alpha\right)\cap z=f_*(\alpha\cap f^*z).$$

The three notions of integrating classes are connected by the following commutative diagram

$$\begin{tikzcd}
A^*(X/X) \arrow[r, "\cdot[f]"] \arrow[d, "\int_{X/Y}"'] & A^*(X/Y) \arrow[r, "\cap[Y]"] \arrow[d, "\int_{X/Y}"'] & A(X) \arrow[d, "\int_{X/Y}"] \\
A^*(Y/Y) \arrow[r, equal] & A^*(Y/Y) \arrow[r, "\cap[Y]"] & A(Y)
\end{tikzcd}$$

If X/Y is smooth, then the upper left horizontal map, $\alpha \mapsto \alpha[f]$, is an isomorphism by Poincaré duality, see [K-T] or [F]. If Y is *orienting* in the sense of [K-T] (say Y is the spectrum of a field or the spectrum of a regular ring of dimension 1 or smooth over an orienting scheme), then the two horizontal maps to the right, $\beta \mapsto \beta \cap [Y]$, are isomorphisms (see [K-T]). Note that if Y is orienting, then $A^0(Y/Y)=A_{\dim Y}(Y)=\mathbf{Z}$, thus identifying the (bivariant) identity class 1 in $A^0(Y/Y)$ with the (cycle) class $[Y]$ in $A_{\dim Y}(Y)$.

If Y is the spectrum of a field and α is a class in $A^*(X/X)$, then we may interpret $\int_{X/Y}\alpha$ as an integer, since the graded group $A^*(Y/Y)$ has $\mathbf{Z}$ as its only non-zero component (in degree 0). More generally, if Y is orienting and α is a class in $A^*(X/X)$ of degree equal to the relative dimension of X/Y, then we may interpret $\int_{X/Y}\alpha$ as the unique integer n satisfying the equation

$$\int_{X/Y} \alpha = n1 \in A^0(Y/Y).$$

PROJECTION FORMULAS (2.2). *Given proper maps $f: X \to Y$ and $g: Y \to S$. Then:*

(1) *If $z \in A(X)$ is a (cycle) class and $\beta \in A^*(Y/Y)$ is a (bivariant) class, then we have that*

$$\int_{X/S} z = \int_{Y/S}\int_{X/Y} z \quad \textit{and} \quad \int_{X/S} f^*(\beta) \cap z = \int_{Y/S}\left(\beta \int_{X/Y} z\right).$$

(2) *If $\theta' \in A^*(X/Y)$ and $\theta'' \in A^*(Y/S)$ are (bivariant) classes, then we have that*

$$\int_{X/S} \theta'\theta'' = \int_{Y/S}\left(\left(\int_{X/Y} \theta'\right)\theta''\right).$$

(3) *Assume that f and g are oriented. If $\alpha \in A^*(X/Y)$ and $\beta \in A^*(Y/S)$ are (bivariant) classes, then we have that*

$$\int_{X/S} \alpha = \int_{Y/S}\int_{X/Y} \alpha \quad \textit{and} \quad \int_{X/S} f^*(\beta)\,\alpha = \int_{Y/S}\left(\beta \int_{X/Y} \alpha\right).$$

Proof. The formulas are easily verified using the bivariant axioms in Chapter 17.2 of [F].

KÜNNETH FORMULAS (2.3). *Given a cartesian diagram of proper maps*

$$\begin{array}{ccc} X & \xrightarrow{p_1} & X_1 \\ {\scriptstyle p_2}\downarrow & & \downarrow{\scriptstyle f_1} \\ X_2 & \xrightarrow[f_2]{} & Y \end{array}$$

Then:

(1) *Assume that Y is orienting and that at least one of the maps f_i is flat. If $\alpha_i \in A^*(X_i/Y)$ is a (bivariant) class of degree equal to the relative dimension of X_i/Y for $i=1,2$, then we have that*

$$\int_{X/Y} p_1^*(\alpha_1)p_2^*(\alpha_2) = \int_{X_1/Y} \alpha_1 \int_{X_2/Y} \alpha_2.$$

(2) *If $\theta_i \in A^*(X_i/Y)$ for $i=1,2$ are commuting (bivariant) classes, then we have that*

$$\int_{X/Y} \theta_1 \times \theta_2 = \int_{X_1/Y} \theta_1 \int_{X_2/Y} \theta_2.$$

(3) *Assume that X_i/Y is oriented by an orientation $[f_i]$ for $i=1,2$ and at least one of the maps $f_i: X_i \to Y$ is flat. Then the product class $[f_1]\times[f_2]$ is an orientation of X/Y. Moreover, if α_i is a (bivariant) class in $A^*(X/X)$ for $i=1,2$ then we have that*

$$\int_{X/Y} p_1^*(\alpha_1)p_2^*(\alpha_2) = \int_{X_1/Y} \alpha_1 \int_{X_2/Y} \alpha_2.$$

Proof. The formula (2) is a consequence of the bivariant axioms in Chapter 17.2 of [F]. The formulas (1) and (3) are consequences of (2).

Definition (2.4). Let $\mathscr{E}$ be a bundle on S. We define the *total Segre class* $s(\mathscr{E}) = s_0(\mathscr{E})+s_1(\mathscr{E})+\ldots$ of $\mathscr{E}$ by

$$s_i(\mathscr{E}) := \int_{\mathbf{P}(\mathscr{E})/S} c_1(\mathcal{O}(1))^{i+\operatorname{rk}\mathscr{E}-1}. \tag{2.4.1}$$

The *total Chern class* $c(\mathscr{E})$ of $\mathscr{E}$ is defined by the equation

$$s(\mathscr{E})\,c(\mathscr{E}^*)=1.$$

Note that Segre classes defined in [F] differ from ours. The ith Segre class in [F] is equal to $(-1)^i s_i(\mathscr{E})$ in our notation. Our definition of Chern classes is, however, in accordance with [F]. In particular, if $u\colon \mathcal{O}_S\to\mathscr{L}$ is a regular section of a line bundle $\mathscr{L}$ then the first Chern class $c_1(\mathscr{L})$ is determined by

$$c_1(\mathscr{L})\cap[S]=[Z(u)].$$

Here $Z(u)$ is the scheme of zeroes of u.

For a line bundle $\mathscr{L}$, we note that the definitions give

$$s(\mathscr{L})=1+l+l^2+\dots \quad\text{and}\quad c(\mathscr{L})=1+l,$$

where l is the first Chern class of $\mathscr{L}$.

It is well known that the definition of Chern and Segre classes extends to all of the Grothendieck group $K(S)$ of bundles on S, and it is a consequence of the splitting principle (see e.g. [F]), that for a bundle $\mathscr{E}$ we may (formally) write

$$c(\mathscr{E})=\prod_{a\in A}(1+a) \quad\text{and}\quad s(\mathscr{E})=\prod_{a\in A}\frac{1}{1-a},$$

where A is a finite family with $\operatorname{rk}\mathscr{E}$ elements.

Let $\mathscr{F}$ be another bundle on S and write

$$s(\mathscr{F})=\prod_{b\in B}\frac{1}{1-b},$$

where B is a family with $\operatorname{rk}\mathscr{F}$ elements. We then have that (see [F])

$$c(\mathscr{E}^*)=\prod_{a\in A}(1-a),\quad s(\mathscr{E}^*)=\prod_{a\in A}\frac{1}{1+a}\quad\text{and}$$

$$c(\mathscr{E}\otimes\mathscr{F})=\prod_{a\in A,\,b\in B}(1+(a+b)),\quad s(\mathscr{E}\otimes\mathscr{F})=\prod_{a\in A,\,b\in B}\frac{1}{1-(a+b)}.$$

Also if $\Lambda^2\mathscr{E}$ and $\mathcal{S}ym^2\mathscr{E}$ are, respectively, the exterior and symmetric square of $\mathscr{E}$ and $A=\{a_1,\dots,a_r\}$, then we have that

$$c(\Lambda^2\mathscr{E}) = \prod_{1\le i<j\le r}(1+a_i+a_j), \quad s(\Lambda^2\mathscr{E}) = \prod_{1\le i<j\le r}\frac{1}{1-(a_i+a_j)} \quad \text{and}$$

$$c(S_{ym}{}^2\mathscr{E}) = \prod_{1\le i\le j\le r}(1+a_i+a_j), \quad s(S_{ym}{}^2\mathscr{E}) = \prod_{1\le i\le j\le r}\frac{1}{1-(a_i+a_j)}.$$

Notation (2.5). In the sequel we shall consider matrices M (possibly with a countable infinity of rows and columns) with coefficients in a ring. It will often be convenient to number rows and columns $0, 1, \dots$, i.e., starting with 0. If I, resp. J, is a sequence of row indices, resp. column indices, we shall denote by M^I, resp. M_J, the matrix whose row indices, resp. column indices, are those of the sequence I, resp. J. If $I=(i_1, \dots, i_r)$ is a finite sequence of non-negative integers, then we denote by $\|I\|$ the *total degree*, that is

$$\|I\| := i_1 + \dots + i_r.$$

A finite set I of integers will always be identified with the strictly increasing sequence whose entries are the elements of I in their natural order. Given a subset I of the integers $0, 1, \dots, n-1$. We denote by $\bar{I}$ the complement of I inside $\{0, 1, \dots, n-1\}$. Moreover, we denote by I^* the image of I under the involution $i \mapsto i^* = n-1-i$ and we let $I' := \bar{I}^*$. We shall write (r) for the set $\{0, 1, \dots, r-1\}$. Thus $M^{(r)}$ denotes the matrix consisting of the first r rows of M, and we have that $\|(r)\| = \binom{r}{2}$ and $(r)' = (n-r)$.

Recall the following about Laplace expansion of an $n\times n$ matrix M: Let r be an integer such that $0\le r\le n$ and let I and J be subsets with r elements of row and column indices respectively. The *cofactor* or *algebraic complement* in the determinant of M to the r by r minor $\det M^I_J$ is the complementary minor multiplied with the signs of the permutations determined by the two subsets I and J with their natural order. That is, it is equal to

$$\operatorname{sign}(I, \bar{I}) \operatorname{sign}(J, \bar{J}) \det M^{\bar{I}}_{\bar{J}}$$

where $\bar{I}$, resp. $\bar{J}$, is the ordered complement of I, resp. J, inside the set of row indices, resp. column indices. Let $\Lambda^r M$ and $\mathrm{V}^r M$ denote the $\binom{n}{r}\times\binom{n}{r}$ matrices indexed by ordered subsets I, J as above and whose IJth entries are, respectively, the minor $\det M^I_J$ and its algebraic complement. Then *Laplace expansion* of the determinant of M along r rows can be expressed by the following equation:

$$(\Lambda^r M)(\mathrm{V}^r M)^{\mathrm{tr}} = (\det M)\,1.$$

When $\det M=1$ it follows that we have the equation $\wedge^r M^{-1}=(\vee^r M)^{\mathrm{tr}}$. Applying the latter equation to M^{-1}, we get the equation

$$\det M^I_J = \operatorname{sign}(I,\tilde I)\operatorname{sign}(J,\tilde J)\det(M^{-1})^{\tilde J}_{\tilde I}. \tag{2.5.1}$$

Definition (2.6). Let $\mathscr{E}$ be a bundle. We denote by $S(\mathscr{E})$ the $\infty\times\infty$ matrix whose ijth entry is the Segre class $s_{j-i}=s_{j-i}(\mathscr{E})$, i.e.,

$$S(\mathscr{E})=\begin{pmatrix} s_0 & s_1 & s_2 & \cdots \\ 0 & s_0 & s_1 & \cdots \\ 0 & 0 & s_0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Let $J=(j_1,\dots,j_r)$ be a sequence of non-negative integers. Then the $r\times r$ matrix $S^{(r)}_J(\mathscr{E})$ is the matrix obtained from $S(\mathscr{E})$ by selecting the first r rows and the r columns corresponding to the indices in J. The determinant of the matrix $S^{(r)}_J(\mathscr{E})$ is called the Jth *Segre class* of the bundle $\mathscr{E}$ and is denoted $s_J(\mathscr{E})$, that is,

$$s_J(\mathscr{E}):=\begin{vmatrix} s_{j_1} & s_{j_2} & \cdots & s_{j_r} \\ s_{j_1-1} & s_{j_2-1} & \cdots & s_{j_r-1} \\ \vdots & \vdots & & \vdots \\ s_{j_1-(r-1)} & s_{j_2-(r-1)} & \cdots & s_{j_r-(r-1)} \end{vmatrix}.$$

The Jth *Chern class* $c_J(\mathscr{E})$ is defined similarly using the Chern classes $c_i=c_i(\mathscr{E})$ for $i=0,1,\dots$. The Segre class $s_J(\mathscr{E})$ and the Chern class $c_J(\mathscr{E})$ are (bivariant) classes of degree $\|J\|-\binom{r}{2}$.

From the Appendix (A.4.3) we get the Complementarity Formula for Chern and Segre classes

$$c_J(\mathscr{E})=s_{J'}(\mathscr{E}). \tag{2.6.1}$$

Notation (2.7). Let t be an integer. We denote by $D(t)$ and E the $\infty\times\infty$ matrices whose ijth entries are, respectively, the binomial coefficients $\binom{i+j+t}{i}$ and $\binom{i}{j}$ for $i,j=0,1,2,\dots$, i.e.,

$$D(t)=\begin{pmatrix} \binom{t}{0} & \binom{1+t}{0} & \binom{2+t}{0} & \cdots \\ \binom{1+t}{1} & \binom{2+t}{1} & \binom{3+t}{1} & \cdots \\ \binom{2+t}{2} & \binom{3+t}{2} & \binom{4+t}{2} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \quad\text{and}\quad E=\begin{pmatrix} 1 & 0 & 0 & 0 & \cdots \\ 1 & 1 & 0 & 0 & \cdots \\ 1 & 2 & 1 & 0 & \cdots \\ 1 & 3 & 3 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Let $I=(i_1, \dots, i_r)$ and $J=(j_1, \dots, j_r)$ be sequences of non-negative integers. We denote by $d_{I,J}(t)$ the determinant of the $r\times r$ matrix $D^I_J(t)$ and we let $d_{I,J} := d_{I,J}(0)$. Assume that the sequences I and J are strictly increasing and identify them with sets of non-negative integers. Let K and L, respectively, be subsets of I and J with the same number of elements. The algebraic complement in the determinant $\det D^I_J(0)$ to the r by r minor $\det D^K_L(0)$ will be denoted $d^{K,L}_{I,J}$, that is

$$d^{K,L}_{I,J} := \operatorname{sign}(K, I\setminus K)\operatorname{sign}(L, J\setminus L)\det D^{I\setminus K}_{J\setminus L}(0).$$

Moreover we denote by ψ_I the sum of all r by r minors of the $r\times\infty$ matrix E^I, i.e.,

$$\psi_I := \sum_K \det E^I_K,$$

where the sum is over all strictly increasing sequences K of r non-negative integers. Finally we denote by α_I the following signed sum

$$\alpha_I := (-1)^{\binom{r}{2}} \sum_K (-1)^{\|K\|} \det E^I_K,$$

where the sum is over all strictly increasing sequences K of r non-negative integers.

PROPOSITION (2.8). *Let $\mathscr{E}$ and $\mathscr{F}$ be bundles of rank r and t respectively. Then the following formulas hold:*

$$s_I(\mathscr{E}^*) = (-1)^{\binom{r}{2}-\|I\|} s_I(\mathscr{E}) \tag{2.8.1}$$

$$s(\mathscr{E}\otimes\mathscr{F}) = \sum_{I,J} s_I(\mathscr{E})\, d_{I,J}(t-r)\, s_J(\mathscr{F}) \tag{2.8.2}$$

$$c(\mathscr{E}\otimes\mathscr{F}) = (-1)^{\binom{r}{2}} \sum_{I,J} (-1)^{\|I\|} s_I(\mathscr{E})\, d_{I,J}(t-r)\, s_{J'}(\mathscr{F}) \tag{2.8.3}$$

$$c(\mathcal{S}ym^2\mathscr{E}) = (-1)^{\binom{r}{2}} 2^{-r(r-1)} \sum_I (-2)^{\|I\|} d_{I,(\mathrm{ev})}(-2r)\, s_I(\mathscr{E}) \tag{2.8.4}$$

$$c(\Lambda^2\mathscr{E}) = (-1)^{\binom{r}{2}} 2^{-r(r-1)} \sum_I (-2)^{\|I\|} d_{I,(\mathrm{ev})}(1-2r)\, s_I(\mathscr{E}) \tag{2.8.5}$$

$$s(\mathcal{S}ym^2\mathscr{E}) = \sum_I \psi_I\, s_I(\mathscr{E}) \tag{2.8.6}$$

$$s(\Lambda^2\mathscr{E}) = \sum_I \alpha_I\, s_I(\mathscr{E}). \tag{2.8.7}$$

The sums are over strictly increasing sequences $I=(i_1, \ldots, i_r)$ and $J=(j_1, \ldots, j_r)$ of non-negative integers and (ev) *denotes the sequence consisting of the r even integers* $0, 2, \ldots, 2r-2$.

Proof. The formulas follow from the formulas of Appendix (A.13) using the splitting principle of (2.4).

Remark (2.9). The functions ψ_I and α_I are studied i more detail in Appendix (A.15) and (A.16).

3. Schubert Calculus

GYSIN FORMULA (3.1). *Let $\mathscr{E}$ be a bundle on S. Denote by $G:=\mathrm{Grass}^q(\mathscr{E})$ the Grassmannian of rank q quotients of $\mathscr{E}$ and let*

$$0 \to \mathscr{K} \to \mathscr{E}_G \to \mathscr{Q} \to 0 \tag{3.1.1}$$

be the tautological sequence on G. Moreover, set $k:=\mathrm{rk}\,\mathscr{E}-q=\mathrm{rk}\,\mathscr{K}$ and let $I=(i_1, \ldots, i_q)$ and $J=(j_1, \ldots, j_k)$ be sequences of non-negative integers. Then we have that

$$\int_{G/S} s_J(\mathscr{K})\, s_I(\mathscr{Q}) = s_{JI}(\mathscr{E}),$$

where JI is the concatenated sequence $(j_1, \ldots, j_k, i_1, \ldots, i_q)$.

Proof. We shall use induction on q. If $q=1$, then $G=\mathbf{P}(\mathscr{E})$ and the tautological sequence is

$$0 \to \mathscr{K} \to \mathscr{E}_{\mathbf{P}(\mathscr{E})} \to \mathscr{L} \to 0, \tag{3.1.2}$$

where $\mathscr{L}=\mathscr{O}_{\mathbf{P}(\mathscr{E})}(1)$ is the tautological line bundle on $\mathbf{P}(\mathscr{E})$. If l is the first Chern class of $\mathscr{L}$, then $s_j(\mathscr{L})=l^j$ and $s(\mathscr{E}_{\mathbf{P}(\mathscr{E})})=s(\mathscr{K})\,s(\mathscr{L})$.

To prove the assertion for $q=1$ we shall use the formula

$$s_{j_1, \ldots, j_k}(\mathscr{K})\, l^i = \det \begin{pmatrix} s_{j_1} & s_{j_2} & \cdots & s_{j_k} & l^{i+k} \\ s_{j_1-1} & s_{j_2-1} & \cdots & s_{j_k-1} & l^{i+k-1} \\ \vdots & \vdots & & \vdots & \vdots \\ s_{j_1-k} & s_{j_2-k} & \cdots & s_{j_k-k} & l^i \end{pmatrix}. \tag{3.1.3}$$

The matrix here is a $(k+1)\times(k+1)$ matrix and the Segre classes occurring in it are the Segre classes of $\mathscr{E}_{\mathbf{P}(\mathscr{E})}$.

To prove the formula, change the matrix by subtracting from each row the following row multiplied by l, beginning with the top row, using the relation

$$s_j(\mathscr{H})=s_j-s_{j-1}\,l.$$

The assertion in the case $q=1$ follows immediately from (3.1.3), the Projection Formula (2.2)(3), and Definition (2.4.1).

Assume that the assertion holds for $q-1$. We consider $G=\mathrm{Grass}^q(\mathscr{E})$ with its tautological sequence (3.1.1) and $\mathbf{P}(\mathscr{E})$ with its tautological sequence (3.1.2). Let X be the incidence correspondence

$$X:=\mathbf{P}(\mathscr{Q})=\mathrm{Grass}^{q-1}(\mathscr{H})$$

corresponding to the commutative diagram

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \mathscr{K}_X \arrow[r] \arrow[d, equal] & \mathscr{H}_X \arrow[r] \arrow[d] & \mathscr{R} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathscr{K}_X \arrow[r] & \mathscr{E}_X \arrow[r] \arrow[d] & \mathscr{Q}_X \arrow[r] \arrow[d] & 0 \\
 & & \mathscr{L}_X \arrow[r, equal] \arrow[d] & \mathscr{L}_X \arrow[d] & \\
 & & 0 & 0 &
\end{tikzcd}$$

where the bundle $\mathscr{R}$ on X is the unique bundle defined by the diagram. It is the tautological rank $q-1$ quotient of $\mathscr{H}_X$ on X over G or, equivalently, the tautological corank 1 subbundle of $\mathscr{Q}_X$ on X over $\mathbf{P}(\mathscr{E})$.

The proof is now an elementary calculation. From the case $q=1$ and the projection formula we obtain the equalities

$$\begin{aligned}\int_{G/S} s_J(\mathscr{K})\,s_I(\mathscr{Q}) &= \int_{G/S} s_J(\mathscr{K}) \int_{X/G} s_{i_1,\dots,i_{q-1}}(\mathscr{R})\,s_{i_q}(\mathscr{L}_X)\\ &= \int_{X/S} s_J(\mathscr{K}_X)\,s_{i_1,\dots,i_{q-1}}(\mathscr{R})\,s_{i_q}(\mathscr{L}_X).\end{aligned}$$

The last integral can also be computed via the scheme $\mathbf{P}(\mathscr{E})$. By the projection formula it is equal to

$$\int_{\mathbf{P}(\mathscr{E})/S}\left(\int_{X/\mathbf{P}(\mathscr{E})} s_J(\mathscr{K}_X)\, s_{i_1,\dots,i_{q-1}}(\mathscr{R})\right) s_{i_q}(\mathscr{L})$$

and by the induction hypothesis the latter expression is equal to the integral

$$\int_{\mathbf{P}(\mathscr{E})/S} s_{J,i_1,\dots,i_{q-1}}(\mathscr{K})\, s_{i_q}(\mathscr{L}).$$

Again, by the case $q=1$, this integral is equal to

$$s_{J,i_1,\dots,i_{q-1},i_q}(\mathscr{E}).$$

Hence the assertion of the Proposition follows.

The following result was given in [Lx2].

COROLLARY (3.2). *Under the conditions of* (3.1) *assume that the bundle* $\mathscr{E}$ *is free. Moreover, let* $A=(a_1,\dots,a_q)$ *be a strictly increasing sequence of non-negative integers such that* $a_q<\operatorname{rk}\mathscr{E}$ *and let* J *be the strictly increasing sequence obtained from the set* $J:=\{0,\dots,\operatorname{rk}\mathscr{E}-1\}\setminus\{a_1,\dots,a_q\}$. *Then for every increasing sequence* I *of non-negative integers with* q *elements we have that*

$$\int_{G/S} s_I(\mathscr{Q})\, s_J(\mathscr{K}^*) = \begin{cases} 1 & \text{if } I=A \\ 0 & \text{otherwise} \end{cases}. \tag{3.2.1}$$

Proof. Set $k:=\operatorname{rk}\mathscr{E}-q$ and let J, ordered increasingly, be $J=(j_1,\dots,j_k)$. By the Gysin formula (3.1) we have that

$$\begin{aligned}\int_{G/S} s_I(\mathscr{Q})\, s_J(\mathscr{K}^*) &= (-1)^{j_1+j_2-1+\dots+j_k-(k-1)}\int_G s_I(\mathscr{Q})\, s_J(\mathscr{K}) \\ &= (-1)^{j_1+j_2-1+\dots+j_k-(k-1)} s_{JI}(\mathscr{E}).\end{aligned}$$

The concatenated sequence JI has $\operatorname{rk}\mathscr{E}$ elements and $\mathscr{E}$ is free so that $s(\mathscr{E})=1$. Hence

$$s_{JI}(\mathscr{E}) = \begin{cases} \operatorname{sign}(JI) & \text{if } JI \text{ is a permutation of } (0,\dots,\operatorname{rk}\mathscr{E}-1) \\ 0 & \text{otherwise.} \end{cases}$$

Clearly the first condition is satisfied if and only if $I=A$. In that case the sequence JA may be ordered increasingly using

$$j_k-(k-1)+\ldots+j_2-1+j_1$$

transpositions. Hence the assertion of the Corollary follows.

Definition (3.3). Fix a bundle $\mathcal{U}$ on S and an integer $r>0$. Let $G:=\mathrm{Grass}^r(\mathcal{U})$ be the Grassmannian of rank r quotients of $\mathcal{U}$ and let

$$0\to\mathcal{K}\to\mathcal{U}_G\to\mathcal{E}\to 0$$

be the tautological sequence. Moreover, let $\{\mathcal{U}_i\}$ be a strictly increasing flag

$$\mathcal{U}_1\subset\mathcal{U}_2\subset\ldots\subset\mathcal{U}_r\subset\mathcal{U}$$

of subbundles of $\mathcal{U}$. The *Schubert scheme* $\Omega=\Omega(\{\mathcal{U}_i\},\mathcal{U})$ is the subscheme of G representing rank r quotients of $\mathcal{U}$ satisfying, for $i=1,\ldots,r$, the following *rank condition:*

$$\text{the composite map } \mathcal{U}_{i,G}\to\mathcal{U}_G\to\mathcal{E} \text{ has rank strictly less than } i. \tag{3.3.1}$$

Let $A=(a_1,\ldots,a_r)$ and $B=(b_1,\ldots,b_r)$ be the strictly increasing sequences of integers defined by

$$b_i:=\mathrm{rk}\,\mathcal{U}_i \quad\text{and}\quad a_i:=\mathrm{rk}\,\mathcal{U}/\mathcal{U}_{r-i+1}-1 \quad\text{for}\quad i=1,\ldots,r.$$

It is well known, see e.g. [F] or [K-L], that the Schubert scheme Ω is equidimensional over the base and that its relative dimension, resp. codimension in G, is

$$\sum_{i=1}^{r}a_i-(i-1)=\sum_{i=1}^{r}a_i-\binom{r}{2},\quad\text{resp.}\quad\sum_{i=1}^{r}b_i-(i-1)=\sum_{i=1}^{r}b_i-\binom{r}{2}. \tag{3.3.2}$$

PROPOSITION (3.4) (Giambelli). *Keep the above notation. Assume that the bundles $\mathcal{U}_i$ are free. Then the following equations hold in $A(G)$:*

$$[\Omega]=s_B(\mathcal{E})\cap[G]=s_J(\mathcal{K}^*-\mathcal{U}^*)\cap[G], \tag{3.4.1}$$

where J is the strictly increasing sequence of non-negative integers obtained from the set $\{0,\ldots,\mathrm{rk}\,\mathcal{U}-1\}\setminus\{a_1,\ldots,a_r\}$.

Assume in addition that $\mathcal{U}$ is free. Then, for every sequence $I=(i_1,\ldots,i_r)$ with $0\leq i_1<\ldots<i_r$, the following equation holds in $A(S)$:

$$\int_G s_I(\mathcal{E})\cap[\Omega]=\begin{cases}[S] & \text{if } I=A\\ 0 & \text{otherwise.}\end{cases} \tag{3.4.2}$$

Proof. The first equation in (3.4.1) is proved in [K–L] or [F]. (Note that their setup is the dual. Their formula expresses the class $[\Omega]$ in terms of the total Chern classes $c(\mathcal{K}^*-(\mathcal{U}/\mathcal{U}_i)^*_G)=c(\mathcal{U}^*_{i,G}-\mathcal{E}^*)$. However, $\mathcal{U}_i$ is free, so $c(\mathcal{U}^*_i-\mathcal{E}^*)=s(\mathcal{E})$.) From the Complementarity Formula (2.6.1) we have that $s_B(\mathcal{E})=c_{B'}(\mathcal{E})$. However, $J=\tilde{A}=\tilde{B}^*=B'$ and we have the relation $c(\mathcal{E})=s(-\mathcal{E}^*)=s(\mathcal{K}^*-\mathcal{U}^*)$. Hence we have that $s_B(\mathcal{E})=s_J(\mathcal{K}^*-\mathcal{U}^*)$ and we have proved the first part of the Proposition.

The equation (3.4.2) follows immediately from (3.4.1) using the Corollary to the Gysin Formula (3.2).

Notation (3.5). As in [T–K] we shall use the following compact notation for pairs: If $\mathbf{U}=(\mathcal{U}',\mathcal{U}'')$ is a pair of sheaves, then we denote by $\Lambda^r\mathbf{U}$ the pair $(\Lambda^r\mathcal{U}',\Lambda^r\mathcal{U}'')$ and by $\mathbf{U}^{\otimes}$ the sheaf $\mathcal{U}'\otimes\mathcal{U}''$. If $\mathbf{I}=(I',I'')$ is a pair of sequences $I'=(i'_1,..,i'_r)$ and $I''=(i''_1,\dots,i''_r)$ of non-negative integers, we let

$$\|\mathbf{I}\|:=i'_1+\dots+i'_r+i''_1+\dots+i''_r.$$

Let $\mathbf{U}=(\mathcal{U}',\mathcal{U}'')$ be a pair of bundles. The Segre class $s_{\mathbf{I}}(\mathbf{U})$ is defined as the product of the Segre classes of the two coordinates, i.e.,

$$s_{\mathbf{I}}(\mathbf{U}):=s_{I'}(\mathcal{U}')\,s_{I''}(\mathcal{U}'').$$

We denote by $d_{\mathbf{I}}$ the determinant $d_{I',I''}=\det D^{I'}_{I''}(0)$ introduced in (2.7) (see also (A.7)). If the two bundles $\mathcal{U}'$ and $\mathcal{U}''$ of $\mathbf{U}$ have the same rank, then the formula (2.8.2) for the Segre class of a tensor product with this notation takes the form

$$s(\mathbf{U}^{\otimes})=\sum_{\mathbf{I}} d_{\mathbf{I}}\,s_{\mathbf{I}}(\mathbf{U}),\tag{3.5.1}$$

where the sum is over all pairs of increasing sequences of non-negative integers that each has a number of elements equal to the common rank of the bundles in $\mathbf{U}$. Note that the degree of the Segre class $s_{\mathbf{I}}(\mathbf{U})$ in the above notation is $\|\mathbf{I}\|-r(r-1)$.

The *Grassmann scheme* $G=\operatorname{Grass}^r(\mathbf{U})$ is the scheme representing pairs of rank r quotients of $\mathbf{U}$. That is, G is equal to the product $\operatorname{Grass}^r(\mathcal{U}')\times\operatorname{Grass}^r(\mathcal{U}'')$ of the two Grassmannians formed from the components of $\mathbf{U}$, and the tautological sequence of pairs

$$0\to\mathbf{K}\to\mathbf{U}_G\to\mathbf{E}\to 0$$

consists of the pull-backs along the projections of the tautological sequences on the two factors.

Let $\{\mathbf{U}_i\}=\{(\mathcal{U}_i', \mathcal{U}_i'')\}$ be a flag of pairs of subbundles of $\mathbf{U}$ as in (3.3), that is

$$\mathcal{U}_1' \subset \mathcal{U}_2' \subset \ldots \subset \mathcal{U}_r' \subset \mathcal{U}' \quad \text{and} \quad \mathcal{U}_1'' \subset \mathcal{U}_2'' \subset \ldots \subset \mathcal{U}_r'' \subset \mathcal{U}''.$$

The corresponding *Schubert scheme* $\Omega=\Omega(\{\mathbf{U}_i\},\mathbf{U})$ is the subscheme of $\operatorname{Grass}^r(\mathbf{U})$ representing pairs of rank r quotients of $\mathbf{U}$ satisfying the rank condition (3.3.1) in each coordinate for $i=1,\ldots,r$. That is, Ω is equal to the product $\Omega(\{\mathcal{U}_i'\},\mathcal{U}')\times\Omega(\{\mathcal{U}_i''\},\mathcal{U}'')$ of the Schubert schemes formed from each of the two coordinates. From the flags $\{\mathcal{U}_i'\}$ and $\{\mathcal{U}_i''\}$ we obtain pairs of sequences $\mathbf{A}=(A',A'')$ and $\mathbf{B}=(B',B'')$ of non-negative integers using the definition in (3.4) in each coordinate. By (3.3.2) the relative dimension of the Schubert scheme $\Omega=\Omega(\{\mathbf{U}_i\},\mathbf{U})$ is equal to

$$\dim_S \Omega = \sum_{i=1}^{r} (a_i'+a_i'')-r(r-1). \tag{3.5.2}$$

Note that with the above notation this dimension is equal to $\|\mathbf{A}\|-r(r-1)$.

PROPOSITION (3.7). *Keep the notation of* (3.5), *and assume that the pairs* $\{\mathbf{U}_i\}$ *and* $\mathbf{U}$ *are pairs of free bundles. Let* $G=\operatorname{Grass}^r(\mathbf{U})$ *be the Grassmannian and*

$$0\to\mathbf{K}\to\mathbf{U}_G\to\mathbf{E}\to 0$$

the tautological sequence on G. If $\mathbf{I}$ *is a pair of increasing sequences with r elements, then*

$$\int_{G/S} s_{\mathbf{I}}(\mathbf{E})\cap[\Omega]=\begin{cases}[S] & \text{if } \mathbf{I}=\mathbf{A},\\ 0 & \text{otherwise.}\end{cases}$$

Proof. Let $\mathbf{J}=(J',J'')$ be the pair of increasing sequences obtained from the pair $\mathbf{A}=(A',A'')$ using the definition in Proposition (3.4) in each coordinate. If T is an S-scheme, then clearly $\Omega(\{\mathcal{U}_i'\},\mathcal{U}')\times_S T=\Omega(\{\mathcal{U}_{i,T}'\},\mathcal{U}_T')$. Therefore, by (3.4.1), we have that

$$[\Omega(\{\mathcal{U}_i'\},\mathcal{U}')\times_S T]=s_{J'}(\mathcal{K}'^*\otimes_S \mathcal{O}_T)\cap[\operatorname{Grass}^r(\mathcal{U}_T')]. \tag{3.6.1}$$

Set $\Omega'':=\Omega(\{\mathcal{U}_i''\},\mathcal{U}'')$. Substituting $T:=\Omega''$ in the above equation, we obtain that

$$[\Omega]=s_{J'}(\mathcal{K}'^*\otimes_S \mathcal{O}_{\Omega''})\cap[\operatorname{Grass}^r(\mathcal{U}_{\Omega''}')]. \tag{3.6.2}$$

Set $G':=\operatorname{Grass}^r(\mathcal{U}')$. Then $[\operatorname{Grass}^r(\mathcal{U}_{\Omega''}')]=[G'\times_S\Omega(\{\mathcal{U}_i''\},\mathcal{U}'')]$ and applying (3.6.1) to the latter expression we obtain that

$$[\mathrm{Grass}^r(\mathcal{U}'_{\Omega''})]=s_{r}(\mathcal{O}_{G'}\otimes_S\mathcal{K}''^*)\cap[G'\times_S\mathrm{Grass}^r(\mathcal{U}'')]=s_{r}(\mathcal{O}_{G'}\otimes_S\mathcal{K}''^*)\cap[G]. \quad (3.6.3)$$

From (3.6.2) and (3.6.3) it follows that $[\Omega]=s_{\mathbf{J}}(\mathbf{K}^*)\cap[G]$ and hence we have that

$$\int_{G/S}s_{\mathbf{I}}(\mathbf{E})\cap[\Omega]=\int_{G/S}s_{\mathbf{I}}(\mathbf{E})\,s_{\mathbf{J}}(\mathbf{K}^*)\cap[G]=\left(\int_{G/S}s_{\mathbf{I}}(\mathbf{E})\,s_{\mathbf{J}}(\mathbf{K}^*)\right)\cap[S]. \quad (3.6.4)$$

By the Künneth formula (2.3) (3), the integral $\int_{G/S}s_{\mathbf{I}}(\mathbf{E})\,s_{\mathbf{J}}(\mathbf{K}^*)$ is equal to a product of two integrals, one from each factor of G. Evaluating each of these integrals using the Corollary to the Gysin Formula (3.2), the assertion follows from (3.6.4).

4. Intersection on the space of complete forms

Setup (4.1). We shall in this section recall some notation and definitions from the works [T-K] of Thorup-Kleiman and [L] of Laksov.

Fix a pair $\mathbf{E}$ of bundles such that the minimum of the ranks of the two components is $r\geq 1$. Let B be the space of complete forms on $\mathbf{E}$. In the notation of [T–K], the space B is the scheme $B_r(\mathbf{E})$ of projectively r-complete forms on $\mathbf{E}$. When $\mathbf{E}$ has the form $\mathbf{E}=(\mathscr{E},\mathscr{F}^*)$, the space B is the space of complete collineations between $\mathscr{E}$ and $\mathscr{F}$ denoted by $CL(\mathscr{E},\mathscr{F})$ in [L].

By [T-K] or [L], the formation of $B_r(\mathbf{E})$ is functorial in the sense that for every scheme T/S we have that $B_r(\mathbf{E}_T)=B_r(\mathbf{E})\times_S T$.

Moreover, the scheme B is smooth over S and of relative dimension $\mathrm{rk}\,\mathbf{E}^{\otimes}-1$. In fact it can be covered by open subsets which are isomorphic to affine spaces of this dimension over open subsets of S (see [T-K] or [L]). On the space B there is a canonical surjective form $w\colon\mathbf{E}_B^{\otimes}\to\mathcal{O}_B(1)$. This form defines a map $B\to\mathbf{P}(\mathbf{E}^{\otimes})$. The latter map can be described as a sequence of monoidal transformations with centers on regular subschemes, see [T-K] or [L]. The canonical form $w\colon\mathbf{E}_B^{\otimes}\to\mathcal{O}_B(1)$ is *r-divisorial* in the sense of [T-K]; that is, its exterior powers $\wedge^i w\colon(\wedge^i\mathbf{E}_B)^{\otimes}\to\mathcal{O}_B(i)$ have line bundles $\mathcal{M}_i$ as images for $i=1,\dots,r$. The surjections from $(\wedge^i\mathbf{E}_B)^{\otimes}$ to $\mathcal{M}_i$ are called *modified exterior powers* in [T-K]. In [L] the corresponding quotients are denoted by $\wedge^i\mathscr{E}_B\otimes\wedge^i\mathscr{F}^*_B\to\mathscr{L}(i)$ and are called *characteristic maps*. The collection of these maps defines a closed embedding

$$B\hookrightarrow\mathbf{P}(\mathbf{E}^{\otimes})\times\mathbf{P}((\wedge^2\mathbf{E})^{\otimes})\times\ldots\times\mathbf{P}((\wedge^r\mathbf{E})^{\otimes}).$$

(see [T-K] or [L]).

For convenience, set $\mathcal{M}_0:=\mathcal{O}_B$. The linebundles $\mathscr{L}_i:=\mathcal{M}_{i+1}\otimes\mathcal{M}_i^{-1}$ fit into a chain of injective maps:

$$\mathscr{L}_1 \hookleftarrow \mathscr{L}_2 \hookleftarrow \dots \hookleftarrow \mathscr{L}_r. \tag{4.1.1}$$

For $i=1,\dots,r$ the scheme of zeros of the ith map $\mathscr{L}_{i+1} \to \mathscr{L}_i$ is a Cartier divisor denoted by V_i in [T-K] and by $CL(r-1, r-1-i)$ in [L]. The corresponding invertible ideal is

$$\mathscr{I}_i := \mathscr{L}_{i+1} \otimes \mathscr{L}_i^{-1} = \mathscr{M}_{i+1} \otimes \mathscr{M}_i^{\otimes -2} \otimes \mathscr{M}_{i-1} \tag{4.1.2}$$

and the associated linebundle is $\mathscr{O}(V_i) = \mathscr{I}_i^{-1}$.

Definition (4.2). The Chern classes

$$\mu_i := c_1(\mathscr{M}_i) \quad \text{for} \quad i=1,\dots,r \quad \text{and}$$

$$\delta_i := c_1(\mathscr{O}(V_i)) = -c_1(\mathscr{I}_i) \quad \text{for} \quad i=1,\dots,r-1,$$

are called the *characteristic classes*, respectively *degeneration classes*, of $\mathbf{E}$. For convenience we define μ_0 to be equal to $c_1(\mathscr{M}_0) = c_1(\mathscr{O}) = 0$.

THEOREM (4.3). *Let* $\mathbf{E}$ *be a pair of bundles on* S *such that the minimum of the ranks of the two components is* $r \geq 1$. *Let* $B = B_r(\mathbf{E})$ *be the space of complete forms on* $\mathbf{E}$ *and* $f: B \to S$ *the structure map. Then:*

(1) *The orientation class* $[f] \in A^*(B/S)$ *is determined by*

$$[f][Z] = [B_r(\mathbf{E}_Z)] \quad \textit{for all schemes } Z/S.$$

(2) *Each class* $\alpha \in A^1(B)$ *can modulo elements in* $f^*A^1(S)$ *be written as*

$$\alpha = \sum_{i=1}^{r} \beta_i \mu_i \quad \textit{with} \quad \beta_i \in A^0(S).$$

If the two components of $\mathbf{E}$ *have different rank, then this expression is unique. If the two components have the same rank we may take* $\beta_r = 0$ *and then the expression is unique*

(3) *If the two components,* $\mathscr{E}'$ *and* $\mathscr{E}''$, *of* $\mathbf{E}$ *have the same rank, then*

$$\mu_r = c_1(\mathscr{E}') + c_1(\mathscr{E}'').$$

(4) *The following equivalent set of equations hold in* $A^1(B)$:

$$\delta_i = -\mu_{i+1} + 2\mu_i - \mu_{i-1} \quad \text{for} \quad i=1,\dots,r-1 \tag{4.3.1}$$

$$\mu_{i+1} = -\delta_i - 2\delta_{i-1} - \dots - i\delta_1 + (i+1)\mu_1 \quad \text{for} \quad i=1,\dots,r-1. \tag{4.3.2}$$

Proof. (1) Since the structure map f is smooth, its orientation class is given by the flat pull-back, i.e., by $[f][Z]=[B_r(\mathbf{E})\times_S Z]$. The equation follows because formation of $B_r(\mathbf{E})$ commutes with base change, see (4.1).

(2) Since the scheme B can be obtained from $\mathbf{P}(\mathbf{E}^{\otimes})$ as a sequence of monoidal transformations with centers on regular subschemes, the assertion is a consequence of standard results on the behavior of the Picard group under blow ups, see e.g. [F], Proposition 6.7 (e), p. 115 or Example 17.5.1 (c), p. 333.

We remark that in [T-K] and [L] there is given a very precise description of the schemes V_i, resp. $CL(r-1,r-1+i)$, mentioned above and of their intersections. From this description one may obtain considerable additional information about the Picard group. The statement of (2) is however sufficient for our purpose.

(3) Under the given assumption, we have that $\mathcal{M}_r=(\Lambda^r\mathbf{E}_B)^{\otimes}=\Lambda^r\mathcal{E}'_B\otimes\Lambda^r\mathcal{E}''_B$. The assertion follows because the first Chern class of a bundle is equal to the first Chern class of its determinant.

(4) The equivalence of the two sets of equations is easily checked. The first set of equations follow from the definition of δ_i and the equations (4.1.2).

PROPOSITION (4.4). *Keep the notation of* (4.1). *Assume that the two components of* $\mathbf{E}$ *have the same rank* $r\geqslant 1$. *Then, in the notation of* (3.5), *the following holds:*

(1) *For all non-negative integers* m *we have that*

$$\int_{B/S}\mu_1^m=\sum_{\mathbf{I}} d_{\mathbf{I}}\, s_{\mathbf{I}}(\mathbf{E}),$$

where the sum is over all $\mathbf{I}$ *that are pairs of increasing sequences of non-negative integers with* r *elements such that* $\|\mathbf{I}\|=m-r+1$.

(2) *Let* p *be an integer such that* $0<p<r$ *and let* $\bar{\mu}_1$ *be the first Chern class* $\bar{\mu}_1=c_1(\mathcal{L}_{p+1})=c_1(\mathcal{M}_{p+1}\otimes\mathcal{M}_p^{-1})$ *(see* (4.1)). *For all non-negative integers* m, k *we have that*

$$\int_{B/S}\mu_1^m\bar{\mu}_1^k\delta_p=\sum_{\mathbf{H},\mathbf{K}} d_{\mathbf{K}}\, d_{\mathbf{H}}\, s_{\mathbf{KH}}(\mathbf{E}),$$

where the sum is over all $\mathbf{K}$, *resp.* $\mathbf{H}$, *that are pairs of strictly increasing sequences of non-negative integers with* $r-p$, *resp.* p, *elements such that* $\|\mathbf{K}\|=k-(r-p)+1$, *resp.* $\|\mathbf{H}\|=m-p+1$.

(3) *For all sequences* $m_1,\dots,m_r$ *of non-negative integers there exists an algorithm for determining the integer coefficients* $i(m_1,\dots,m_r,\mathbf{I})$ *in the sum*

$$\int_{B/S} \mu_1^{m_1} \ldots \mu_r^{m_r} = \sum_{\mathbf{I}} i(m_1, \ldots, m_r, \mathbf{I})\, s_{\mathbf{I}}(\mathbf{E}),$$

where $\mathbf{I}$ *is a pair of increasing sequences of non-negative integers each with* r *elements.*

(4) *Let* $c_{n_1,\ldots,n_r}^{m_1,\ldots,m_r}$ *for sequences* $(m_1, \ldots, m_r)$ *and* $(n_1, \ldots, n_r)$ *such that* $m_1+\ldots+m_r = n_1+\ldots+n_r$ *be a system of integers satisfying the equation*

$$(\mu_{r-1}-\mu_r)^{m_1} \ldots (\mu_1-\mu_r)^{m_{r-1}}(-\mu_r)^{m_r} = \sum_{n_1,\ldots,n_r} c_{n_1,\ldots,n_r}^{m_1,\ldots,m_r} \mu_1^{n_1} \ldots \mu_{r-1}^{n_{r-1}} \mu_r^{n_r}.$$

Then the intersection coefficients $i(m_1, \ldots, m_r, \mathbf{I})$ *of* (3) *satisfy the following equation:*

$$i(m_1, \ldots, m_r, \mathbf{I}) = (-1)^{\|\mathbf{I}\|} \sum_{n_1,\ldots,n_r} c_{n_1,\ldots,n_r}^{m_1,\ldots,m_r} i(n_1, \ldots, n_r, \mathbf{I}).$$

Proof. (1) As mentioned above the structure map $B \to S$ can be factored via a map $B \to \mathbf{P}(\mathbf{E}^{\otimes})$ such that μ_1 is the pullback to B of the Chern class $c_1(\mathcal{O}(1))$ and the map $B \to \mathbf{P}(\mathbf{E}^{\otimes})$ is a composite of blow-ups along regularly embedded subschemes. Hence, by Proposition 17.5 (a), p. 332 in [F], we have that $\int_{B/\mathbf{P}(\mathbf{E}^{\otimes})} 1 = 1$. By (2.2)(3), we therefore have that

$$\int_{B/S} \mu_1^m = \int_{\mathbf{P}(\mathbf{E}^{\otimes})/S} c_1(\mathcal{O}(1))^m.$$

Moreover, by the definition of Segre classes (2.4.1) we have that the right hand side of the latter equation is equal to the class

$$s_{m-\operatorname{rk}\mathbf{E}^{\otimes}+1}(\mathbf{E}^{\otimes})$$

and by the product formula (3.5.1) this class is equal to the class

$$\sum_{\mathbf{I}} d_{\mathbf{I}}\, s_{\mathbf{I}}(\mathbf{E}).$$

Here the sum is over all $\mathbf{I}$ such that

$$\|\mathbf{I}\| - r(r-1) = m - \operatorname{rk}\mathbf{E}^{\otimes} + 1.$$

Since $\operatorname{rk}\mathbf{E}^{\otimes} = r^2$, the latter equation gives the asserted condition on $\mathbf{I}$ and we have proved assertion (1).

(2) Let V_p be the scheme of zeros of the map $\mathscr{L}_{p+1} \to \mathscr{L}_p$ of (4.1). Then, since $\delta_p = [V_p]$ by definition, it follows from the second formula of (2.2)(3) that

$$\int_{B/S} \mu_1^m \bar{\mu}_1^k \delta_p = \int_{V_p/S} \mu_1^m \bar{\mu}_1^k. \tag{4.4.1}$$

Let $G := \mathrm{Grass}^p(\mathbf{E})$ be the Grassmannian of pairs of rank p quotients of $\mathbf{E}$ and

$$0 \to \mathbf{S} \to \mathbf{E}_G \to \mathbf{R} \to 0$$

the tautological sequence. Then by [T-K] or [L] there exists a cartesian diagram

$$\begin{array}{ccc} V_p & \longrightarrow & B_p(\mathbf{R}) \\ \downarrow & & \downarrow \\ B_{r-p}(\mathbf{S}) & \longrightarrow & G \end{array} \tag{4.4.2}$$

such that the classes μ_1 and $\bar{\mu}_1$ are the pullbacks to V_p of the corresponding classes on $B_p(\mathbf{R})$ and $B_{r-p}(\mathbf{S})$ respectively. Therefore, by the Künneth formula (2.3)(3), the integral $\int_{V_p/G} \mu_1^m \bar{\mu}_1^k$ is equal to a product of two integrals coming from the factors $B_{r-p}(\mathbf{S})$ and $B_p(\mathbf{R})$ respectively. These two integrals may be evaluated using assertion (1) on the schemes $B_{r-p}(\mathbf{S})/G$ and $B_p(\mathbf{R})/G$ respectively. Hence we obtain that

$$\int_{V_p/G} \mu_1^m \bar{\mu}_1^k = \sum_{\mathbf{K},\mathbf{H}} d_{\mathbf{H}} d_{\mathbf{K}} s_{\mathbf{K}}(\mathbf{S}) s_{\mathbf{H}}(\mathbf{R}), \tag{4.4.3}$$

where the sum is over all $\mathbf{K}, \mathbf{H}$ as indicated in statement (2) of the Proposition.

The scheme G is a product of two Grassmannians. Therefore, by the Künneth formula (2.3)(3), the integral $\int_{G/S} s_{\mathbf{K}}(\mathbf{S}) s_{\mathbf{H}}(\mathbf{R})$ is equal to a product of two integrals, one from each factor of G. Each of these integrals may be evaluated using the Gysin formula (3.1). Hence we obtain that

$$\int_{G/S} s_{\mathbf{K}}(\mathbf{S}) s_{\mathbf{H}}(\mathbf{R}) = s_{\mathbf{KH}}(\mathbf{E}). \tag{4.4.4}$$

Combining (4.4.1), (4.4.3) and (4.4.4), the second assertion of the Proposition follows.

(3) We order the sequences $(m_1, \ldots, m_r) \in \mathbf{Z}_+^r$ in the following way: First we order them after the total degree $m_1 + \ldots + m_r$ and secondly we order sequences of the same total degree lexicographically. The proof of assertion (3) proceeds by descending induction on this ordering.

Let $p<r$ be the largest integer such that $m_{p+1}>0$. If $p=0$, we even have the explicit expression of assertion (1). Hence we may assume that $p>0$. From (4.3.1) we get

$$\mu_1^{m_1}\ldots\mu_r^{m_r}=\mu_1^{m_1}\ldots\mu_p^{m_p}\mu_{p+1}^{m_{p+1}-1}(2\mu_p-\mu_{p-1}-\delta_p).$$

We integrate both sides and get three terms on the right hand side. The first two of these can be determined algorithmically by the induction hypothesis. To determine the third term

$$\int_{B/S}\mu_1^{m_1}\ldots\mu_p^{m_p}\mu_{p+1}^{m_{p+1}-1}\delta_p,$$

we use the expression $\mu_{p+1}=\bar{\mu}_1+\mu_p$ resulting from the definition in assertion (2). Inserting this expression in the integral above, we get a sum of terms of the form

$$\int_{B/S}\mu_1^{m_1}\ldots\mu_{p-1}^{m_{p-1}}\mu_p^n\bar{\mu}_1^k\delta_p.$$

Arguing as in part (2) above, this integral is equal to

$$\int_{G/S}\int_{V_p/G}\mu_1^{m_1}\ldots\mu_{p-1}^{m_{p-1}}\mu_p^n\bar{\mu}_1^k.$$

The inner integral may be evaluated using the argument in (2) and the induction hypothesis on the scheme $B_p(\mathbf{R})$ and the sequence of integers $(m_1,\ldots,m_{p-1},n)$ (note that the total degree of this sequence is at least 1 less than the total degree of $(m_1,\ldots,m_{p-1},m_p)$). We obtain an equation similar to (4.4.3), but with algorithmically determined integer coefficients in the expression on the right hand side. The latter expression may now be integrated from G to S using the equation (4.4.4). Thus the third assertion of the Proposition follows.

(4) Denote by B^* the space of complete forms on the pair of dual bundles $\mathbf{E}^*$ and by $\check{\mu}_i$ for $i=1,\ldots,r$ the corresponding characteristic classes. Then, by the duality results in Section 4 of [T-K], there is a canonical isomorphism of schemes $B=B^*$ and we have that $\check{\mu}_i=\mu_{r-i}-\mu_r$ for $i=1,\ldots,r$. The assertion follows easily, noting that $s_{\mathbf{I}}(\mathbf{E}^*)=(-1)^{\|\mathbf{I}\|}s_{\mathbf{I}}(\mathbf{E})$ by (2.8.1). Thus the Proposition is proved.

5. Giambelli's formula

Setup (5.1). Assume that ground scheme S is the spectrum of a field and fix a pair $\mathbf{U}$ of vectorspaces of the same dimension over the field. Let $r\geq 1$ be an integer and $\{\mathbf{U}_i\}$ a strictly increasing flag of pairs of subspaces:

$$\mathbf{U}_1 \subset \mathbf{U}_2 \subset \ldots \subset \mathbf{U}_r \subset \mathbf{U}$$

and let $\mathbf{A}=(A',A'')$ be the pair of increasing sequences

$$A' = (a'_1, \ldots, a'_r) \quad \text{and} \quad A'' = (a''_1, \ldots, a''_r)$$

of non-negative integers defined from this flag in (3.5).

Let $T:=\mathrm{Grass}^r(\mathbf{U})$ be the Grassmannian of pairs of rank r quotients of $\mathbf{U}$ and

$$\mathbf{U}_T \twoheadrightarrow \mathbf{E}$$

the tautological quotient. Thus $\mathbf{E}$ is a pair of bundles of rank r on T. Corresponding to the flag $\{\mathbf{U}_i\}$ we introduced in (3.5) a Schubert scheme $\Omega=\Omega(\{\mathbf{U}_i\},\mathbf{U})$ which is a subscheme of T.

Let $B:=B_r(\mathbf{E})$ be the space of r-complete forms on $\mathbf{E}$ and $f\colon B\to T$ the structure map. From the functoriality of the space of r-complete forms it follows that the scheme $f^{-1}(\Omega)$ is equal to $B_r(\mathbf{E}_\Omega)$. We have that

$$N := \dim f^{-1}(\Omega) = \dim B_r(\mathbf{E}_\Omega) = \sum_{i=1}^{r} (a'_i+a''_i)+r-1. \tag{5.1.1}$$

Indeed, by (3.5.2) we get

$$\dim B_r(\mathbf{E}_\Omega) = \dim_\Omega B_r(\mathbf{E}_\Omega)+\dim \Omega = (r^2-1)+\sum_{i=1}^{r} (a'_i+a''_i)-r(r-1)$$

which is the desired equation.

We denote by $[\Omega]$ the class of Ω in $A(T)$. The class $[\Omega]$ depends on the sequences $\mathbf{A}=(A',A'')$ only. Moreover, it follows from (3.5.2) that the class of $f^{-1}(\Omega)$ in $A(B)$ is equal to the inverse image $[f]([\Omega])$ of $[\Omega]$ under the orientation class. We denote this class by $\omega_{\mathbf{A}}$, that is

$$\omega_{\mathbf{A}} := [f^{-1}(\Omega)] = [f]([\Omega]).$$

PROPOSITION (5.2). *Let $N:=\Sigma_{i=1}^r(a'_i+a''_i)+r-1$ be the dimension of $f^{-1}(\Omega)$. Then:*
(1) (*Schubert*)

$$\int_B \mu_1^N \cap \omega_{\mathbf{A}} = d_{\mathbf{A}}.$$

(2) *Let p and k be integers such that $0<p<r$ and $0\leq k<N$. Set*

$$\bar{\mu}_1 := c_1(\mathscr{L}_{p+1}) = c_1(\mathscr{M}_{p+1}\otimes\mathscr{M}_p^{-1}) = \mu_{p+1}-\mu_p.$$

Then we have that

$$\int_B \mu_1^{N-k-1}\bar{\mu}_1^k\delta_p \cap \omega_{\mathbf{A}} = \sum_{\mathbf{K}} d_{\mathbf{K}}\, d_{\mathbf{A}}^{\mathbf{K}},$$

where the sum is over all $\mathbf{K}$ that are pairs of subsequences of $\mathbf{A}$ with $r-p$ elements each and such that $\|\mathbf{K}\|=k-(r-p)+1$. Here $d_{\mathbf{A}}^{\mathbf{K}}$ is the algebraic complement in $d_{\mathbf{A}}$ to the minor $d_{\mathbf{K}}$ as defined in (2.7).

Proof. (1) It follows from the projection formula and from (4.4)(1) that

$$\begin{aligned}\int_B \mu_1^N \cap \omega_{\mathbf{A}} &= \int_T \left(\int_{B/T} \mu_1^N\right) \cap \omega_{\mathbf{A}}\\ &= \int_T \sum_{\mathbf{I}} d_{\mathbf{I}}\, s_{\mathbf{I}}(\mathbf{E}) \cap \omega_{\mathbf{A}},\end{aligned}$$

where the sum is over all $\mathbf{I}$ that are pairs of increasing sequences of non-negative integers with $\|\mathbf{I}\|=N-r+1$. By (3.6), the only non vanishing term in the last sum is equal to $d_{\mathbf{A}}$. This proves the first assertion of the Proposition.

(2) It follows from the projection formula and from (4.4)(2) that

$$\begin{aligned}\int_B \mu_1^{N-k-1}\bar{\mu}_1^k\delta_p \cap \omega_{\mathbf{A}} &= \int_T \left(\int_{B/T} \mu_1^{N-k-1}\bar{\mu}_1^k\delta_p\right) \cap \omega_{\mathbf{A}}\\ &= \int_T \sum_{\mathbf{K},\mathbf{H}} d_{\mathbf{K}}\, d_{\mathbf{H}}\, s_{\mathbf{KH}}(\mathbf{E}) \cap \omega_{\mathbf{A}},\end{aligned}$$

where the sum is over all $\mathbf{K}$, resp. $\mathbf{H}$, that are pairs of increasing sequences of non-negative integers with $r-p$, resp. p, elements such that $\|\mathbf{K}\|=k-(r-p)+1$, resp. $\|\mathbf{H}\|=N-k-p$. By (3.6), in the last sum the term corresponding to $\mathbf{K},\mathbf{H}$ vanishes except when the concatenated pair $\mathbf{KH}$ ordered increasingly is equal to $\mathbf{A}$. The exception occurs precisely when $\mathbf{K}$ is a pair of subsequences of $\mathbf{A}$ and $\mathbf{H}$ is its complement. In this case we have that

$$s_{\mathbf{KH}}(\mathbf{E}) = \operatorname{sign}(\mathbf{KH})\, s_{\mathbf{A}}(\mathbf{E}) \quad\text{and}\quad d_{\mathbf{H}} = \operatorname{sign}(\mathbf{KH})\, d_{\mathbf{A}}^{\mathbf{K}},$$

and, by (3.6), the corresponding term in the sum is therefore equal to $d_{\mathbf{K}}\, d_{\mathbf{A}}^{\mathbf{K}}$. Therefore the second assertion of the Proposition holds.

LEMMA (5.3). *Let $m_1, \dots, m_r$ be non-negative integers and consider for $1 \leq q \leq r$ the inequality*

$$\sum_{i=1}^{q} m_i > \sum_{i=1}^{q} (a'_{r-i+1} + a''_{r-i+1}) + q - 1. \tag{5.3.q}$$

Let α be a class in $A^*(B)$. Then:

(1) *If (5.3.q) holds for some $q<r$, then*

$$\int_B \mu_1^{m_1} \mu_2^{m_2} \dots \mu_q^{m_q} \alpha \delta_q \cap \omega_{\mathbf{A}} = 0.$$

(2) *If $p \leq r$ and (5.3.q) holds for all $q<p$, then*

$$\int_B \mu_1^{m_1} \mu_2^{m_2} \dots \mu_p^{m_p} \alpha \cap \omega_{\mathbf{A}} = 1^{m_1} 2^{m_2} \dots p^{m_p} \int_B \mu_1^{m_1+m_2+\dots+m_p} \alpha \cap \omega_{\mathbf{A}}.$$

Proof. (1) Using the notation of (4.1) and (4.2) we have that

$$\delta_q \cap \omega_{\mathbf{A}} = \delta_q \cap [B_r(\mathbf{E}_\Omega)] = [V_q(\mathbf{E}_\Omega)].$$

It follows from the projection formula (2.2) (1) that

$$\int_B \mu_1^{m_1} \mu_2^{m_2} \dots \mu_q^{m_q} \alpha \delta_q \cap \omega_{\mathbf{A}} = \int_{V_q(\mathbf{E}_\Omega)} \mu_1^{m_1} \mu_2^{m_2} \dots \mu_q^{m_q} \cap z, \tag{5.3.1}$$

where z is the class $\alpha \cap [V_q(\mathbf{E}_\Omega)]$ in $A(V_q(\mathbf{E}_\Omega))$. In the notation of (4.4) with $p:=q$ and $S:=\Omega$, the cartesian diagram (4.4.2) becomes:

$$\begin{tikzcd} V_q(\mathbf{E}_\Omega) \arrow[r] \arrow[d] & B_q(\mathbf{R}) \arrow[d] \\ B_{r-q}(\mathbf{S}) \arrow[r] & \mathrm{Grass}^q(\mathbf{E}_\Omega) \end{tikzcd}$$

Here $\mathbf{R}$ and $\mathbf{S}$ are the tautological pairs of rank q quotients and rank $r-q$ subbundles respectively on $G:=\mathrm{Grass}^q(\mathbf{E}_\Omega)$ and the restricted classes $\mu_1, \dots, \mu_q$ are the pull-backs of the corresponding classes on $B_q(\mathbf{R})$.

Let $\mathbf{U}_G \twoheadrightarrow \mathbf{E}_G$ be the surjection identifying $\mathbf{E}_G$ with a pair of quotients of $\mathbf{U}_G$. Then the composite maps of pairs

$$\mathbf{U}_{i,G} \to \mathbf{U}_G \to \mathbf{E}_G$$

satisfy, in each coordinate, the rank condition (3.3.1) for $i=1,\dots,r$, because G is an Ω-scheme. It follows that the composite map $\mathbf{U}_G \twoheadrightarrow \mathbf{E}_G \twoheadrightarrow \mathbf{R}$ defines a pair of rank q quotients of $\mathbf{U}_G$ and that the composite maps of pairs

$$\mathbf{U}_{i,G} \to \mathbf{U}_G \to \mathbf{E}_G \to \mathbf{R}$$

satisfy, in each coordinate, the rank condition (3.3.1) for $i=1,\dots,q$. Therefore, if we let $\bar{\Omega}:=\Omega(\{\mathbf{U}_i\}_{i=1}^q, \mathbf{U})$ denote the Schubert scheme corresponding to the truncated flag of pairs $\{\mathbf{U}_i\}_{i=1}^q$ in $\mathbf{U}$ and by $\bar{\mathbf{E}}_{\bar{\Omega}}$ the tautological pair of rank q quotients of $\mathbf{U}_{\bar{\Omega}}$, then there exists a natural map from G to $\bar{\Omega}$ such that $\bar{\mathbf{E}}_{\bar{\Omega}}$ pulls back to $\mathbf{R}$. Hence, by the functoriality of the space of complete forms we have a cartesian diagram

$$\begin{array}{ccc} B_q(\mathbf{R}) & \longrightarrow & B_q(\bar{\mathbf{E}}_{\bar{\Omega}}) \\ \downarrow & & \downarrow \\ \mathrm{Grass}^q(\mathbf{E}_\Omega) & \longrightarrow & \bar{\Omega} \end{array}$$

and the classes $\mu_1,\dots,\mu_q$ on $B_q(\mathbf{R})$ are the pull-backs of the corresponding classes on $B_q(\bar{\mathbf{E}}_{\bar{\Omega}})$. In particular the composite of the two upper horizontal maps in the two preceeding diagrams gives a map $V_q(\mathbf{E}_\Omega) \to B_q(\bar{\mathbf{E}}_{\bar{\Omega}})$ such that the classes $\mu_1,\dots,\mu_q$ on $V_q(\mathbf{E}_\Omega)$ are the pull-backs of the corresponding classes on $B_q(\bar{\mathbf{E}}_{\bar{\Omega}})$. Assertion (1) of the Proposition now follows from equation (5.3.1) and the projection formula, since the assumed inequality (5.3.q) asserts precisely that the degree of $\mu_1^{m_1}\dots\mu_q^{m_q}$ is bigger than the dimension, $\bar{N}:=\Sigma_{i=1}^q(a'_{r-i+1}+a''_{r-i+1})+q-1$, of $B_q(\bar{\mathbf{E}}_{\bar{\Omega}})$ (see (5.1.1)).

(2) We shall proceed by induction on m_p. The assertion is trivial if $p=1$, so we may assume that $p>1$. We may clearly also assume that $m_p>0$. By (4.3.2) we have that

$$\mu_1^{m_1}\dots\mu_p^{m_p} = \mu_1^{m_1}\dots\mu_{p-1}^{m_{p-1}}\mu_p^{m_p-1}(p\mu_1 - \sum_{q<p}(p-q)\delta_q).$$

We multiply by $\alpha \cap \omega_{\mathbf{A}}$ and integrate. It follows from assertion (1) that the resulting sum on the right hand side is equal to its first term

$$p\int_B \mu_1^{m_1+1}\mu_2^{m_2}\dots\mu_{p-1}^{m_{p-1}}\mu_p^{m_p-1}\,\alpha \cap \omega_{\mathbf{A}}.$$

By the induction hypothesis we have that the latter term is equal to

$$1^{m_1}2^{m_2}\dots(p-1)^{m_{p-1}}p^{m_p-1}p\int_B \mu_1^{m_1+m_2+\dots+m_p}\alpha\cap\omega_{\mathbf{A}}$$

and we have proved assertion (2).

GIAMBELLI'S THEOREM (5.4). *We keep the notation of* (5.1). *In particular, we let* $\mathbf{U}$ *be a pair of vectorspaces of the same dimension and* $\{\mathbf{U}_i\}_{i=1}^r$ *a strictly increasing flag of pairs of subspaces of* $\mathbf{U}$. *We denote by* $T := \mathrm{Grass}^r(\mathbf{U})$ *the Grassmannian of pairs of rank r quotients of* $\mathbf{U}$ *and by* $\mathbf{E}$ *the tautological pair of rank r quotients of* $\mathbf{U}_T$. *Moreover, we denote by* $B=B_r(\mathbf{E})$ *the space of r-complete forms on* $\mathbf{E}$ *with the characteristic classes* $\mu_1,\dots,\mu_r$ *in* $A^*(B)$ *and by* $f\colon B\to T$ *the structure map. Furthermore, let* $\mathbf{A}$ *be the pair of strictly increasing sequences* $(a_1',\dots,a_r')$ *and* $(a_1'',\dots,a_r'')$ *defined by*

$$a_i' := \mathrm{rk}\, U'/U'_{r-i+1}-1 \quad\text{and}\quad a_i'' := \mathrm{rk}\, U''/U''_{r-i+1}-1$$

for $i=1,\dots,r$. *Finally, we denote by* $\Omega := \Omega(\{\mathbf{U}_i\},\mathbf{U})$ *the Schubert subscheme of* T *defined by the flag* $\{\mathbf{U}_i\}_{i=1}^r$ *and by* $\omega_{\mathbf{A}} := [f^{-1}\Omega]$ *the class of* $f^{-1}\Omega$ *in* $A(B)$.

Let p be an integer such that $0\leq p<r$ *and* $m_1,\dots,m_{p+1}$ *a sequence of non-negative integers such that*

$$\sum_{i=1}^{p+1} m_i=\sum_{i=1}^{r}(a_i'+a_i'')+r-1=\dim f^{-1}(\Omega)$$

and satisfying the following inequalities:

$$\sum_{i=1}^{q} m_i>\sum_{i=1}^{q}(a'_{r-i+1}+a''_{r-i+1})+q-1 \quad\text{for}\quad q=1,\dots,p-1.$$

Moreover, let $\varphi^p(k,i)$ *be the function defined by*

$$\varphi^p(k,i)=\begin{cases}\binom{k}{0}+p\binom{k}{1}+\dots+p^i\binom{k}{i} & \text{if } i\geq 0\\ 0 \quad \text{if } i<0.\end{cases}$$

Then we have the following formula:

$$\int \mu_1^{m_1}\dots\mu_p^{m_p}\mu_{p+1}^{m_{p+1}}\cap\omega_{\mathbf{A}}$$
$$=1^{m_1}2^{m_2}\dots p^{m_p}\left((p+1)^{m_{p+1}}d_{\mathbf{A}}-\sum_{\mathbf{K}}\varphi^p(m_{p+1},m_{p+1}-\|\mathbf{K}\|-(r-p))\,d_{\mathbf{K}}\,d_{\mathbf{A}}^{\mathbf{K}}\right),$$

where the sum is over all $\mathbf{K}$ *that are pairs of subsequences* $(k'_1, \ldots, k'_{r-p})$ *and* $(k''_1, \ldots, k''_{r-p})$ *of* $\mathbf{A}$. *Here* $\|\mathbf{K}\| := \Sigma_{i=1}^{r-p}(k'_i + k''_i)$ *and* $d_{\mathbf{A}}^{\mathbf{K}}$ *is the algebraic complement in* $d_{\mathbf{A}}$ *to the minor* $d_{\mathbf{K}}$ *as defined in* (2.7).

Remark (5.5). By definition of the function $\varphi^p(k, i)$ the summation in Giambelli's formula may be restricted to pairs of subsequences $\mathbf{K}$ each with $r-p$ elements such that $\|\mathbf{K}\| \leq m_{p+1} - (r-p)$. When $p=0$, the only possible subsequence is $\mathbf{K}=\mathbf{A}$. The sequence $\mathbf{K}=\mathbf{A}$ does not satisfy the latter inequality, because we have $\|\mathbf{A}\| = \Sigma_{i=1}^{p+1} m_i - (r-1) > m_{p+1} - (r-p)$ when $p=0$. Hence Giambelli's formula in the case $p=0$ reduces to Schubert's formula (5.2) (1).

Our presentation of the formula is very close to Giambelli's presentation in [G]. To translate his notation to ours one should replace his numbers d, n, s and r by $r-1$, $\mathrm{rk}\,\mathrm{U}-1, p$ and q respectively. Moreover, his characteristic classes μ_{d-i} for $i=1, \ldots, d$ are our characteristic classes μ_i for $i=1, \ldots, r-1$, his subsequences are defined by subsets of the indices of $\mathbf{A}$ and the A_d and B_d in his terminology is $d_{\mathbf{K}}$ and $d_{\mathbf{A}}^{\mathbf{K}}$ in our. Note finally that our characteristic class μ_r does not occur in his result. The class μ_r is exceptional in the sense that it comes from T and occurs in the formula of (5.4) only in the limiting case $p=r-1$, a case not covered by the original formula in [G].

Proof of (5.4): By the above Remark (5.5) we may assume that $p>0$. Let $N := \Sigma_{i=1}^{r}(a'_i + a''_i) + r - 1 = \Sigma_{i=1}^{p+1} m_i$. By (5.3)(2) we have that

$$\begin{aligned} \int_B \mu_1^{m_1} \ldots \mu_p^{m_p} \mu_{p+1}^{m_{p+1}} \cap \omega_{\mathbf{A}} &= 1^{m_1} 2^{m_2} \ldots p^{m_p} \int_B \mu_1^{m_1+m_2+\ldots+m_p} \mu_{p+1}^{m_{p+1}} \cap \omega_{\mathbf{A}} \\ &= 1^{m_1} 2^{m_2} \ldots p^{m_p} P(m_{p+1}) \end{aligned} \tag{5.4.1}$$

where the function P is defined by:

$$P(k) := \int_B \mu_1^{N-k} \mu_{p+1}^{k} \cap \omega_{\mathbf{A}} \quad \text{for} \quad 0 \leq k \leq N.$$

We shall consider the function $P(k)$ for $0 \leq k \leq m_{p+1}$. By (5.2)(1) we have that

$$P(0) = d_{\mathbf{A}}. \tag{5.4.2}$$

Assume next that $k>0$. From the equations (4.3.1) and (5.3)(2) we get

$$P(k)=\int_B \mu_1^{N-k}\mu_{p+1}^{k-1}(2\mu_p-\mu_{p-1}-\delta_p)\cap\omega_{\mathbf{A}}$$
$$=\int_B \mu_1^{N-k}\mu_{p+1}^{k-1}(2p\mu_1-(p-1)\mu_1-\delta_p)\cap\omega_{\mathbf{A}}.$$

Hence we obtain a recurrence formula

$$P(k)=(p+1)P(k-1)-\int_B \mu_1^{N-k}\mu_{p+1}^{k-1}\delta_p\cap\omega_{\mathbf{A}}.$$

Set $\bar{\mu}_1:=\mu_{p+1}-\mu_p$ as in (5.2) (2). Then we have that

$$\int_B \mu_1^{N-k}\mu_{p+1}^{k-1}\delta_q\cap\omega_{\mathbf{A}}=\sum_{i=0}^{k-1}\binom{k-1}{i}\int_B \mu_1^{N-k}\mu_p^i\bar{\mu}_1^{k-i-1}\delta_p\cap\omega_{\mathbf{A}}.$$

It follows from (5.3)(2) that the right hand side of the last equation is equal to

$$\sum_{i=0}^{k-1}\binom{k-1}{i}p^i\int_B \mu_1^{N-k+i}\bar{\mu}_1^{k-i-1}\delta_p\cap\omega_{\mathbf{A}}.$$

By (5.2)(2), the latter sum is equal to the sum

$$\sum_{i=0}^{k-1}\binom{k-1}{i}p^i\sum_{\|\mathbf{K}\|=k-(r-p)-i}d_{\mathbf{K}}d_{\mathbf{A}}^{\mathbf{K}},$$

where the sum is over $\mathbf{K}$ that are pairs of subsequences of $\mathbf{A}$ each with $r-p$ elements. Hence

$$P(k)-(p+1)P(k-1)=-\sum_{i=0}^{k-1}\binom{k-1}{i}p^i\sum_{\|\mathbf{K}\|=k-(r-p)-i}d_{\mathbf{K}}d_{\mathbf{A}}^{\mathbf{K}}. \tag{5.4.3}$$

The function φ^p of (5.4) satisfies the recurrence relation

$$\varphi^p(k,i)-(p+1)\varphi^p(k-1,i-1)=p^i\binom{k-1}{i}\quad\text{for}\quad k>i\geqslant 0\quad\text{and}\quad \varphi^p(k,-1)=0,$$

as is seen from the identity $\binom{k}{j}=\binom{k-1}{j}+\binom{k-1}{j-1}$. Consequently, if we introduce the function

$$Q(k):=-\sum_{i=0}^{k-1}\varphi^p(k,i)\sum_{\|\mathbf{K}\|=k-(r-p)-i}d_{\mathbf{K}}d_{\mathbf{A}}^{\mathbf{K}}, \tag{5.4.4}$$

then it follows from (5.4.3) and a short computation that we have

$$P(k)-(p+1)P(k-1)=Q(k)-(p+1)Q(k-1).$$

Therefore we get that $P(k)-(p+1)^kP(0)=Q(k)-(p+1)^kQ(0)=Q(k)$. From (5.4.2) and (5.4.4) we get that

$$\begin{aligned} P(k) &= (p+1)^kP(0)+Q(k) \\ &= (p+1)^k d_{\mathbf{A}} - \sum_{i=0}^{k-1} \varphi^p(k,i) \sum_{\|\mathbf{K}\|=k-(r-p)-i} d_{\mathbf{K}}\, d_{\mathbf{A}}^{\mathbf{K}}, \end{aligned}$$

Interchanging the order of summation in the last sum we obtain that

$$P(k) = (p+1)^k d_{\mathbf{A}} - \sum_{\mathbf{K}} \varphi^p(k, k-\|\mathbf{K}\|-(r-p))\, d_{\mathbf{K}}\, d_{\mathbf{A}}^{\mathbf{K}}, \tag{5.4.5}$$

where the sum is over all $\mathbf{K}$ that are pairs of subsequences of $\mathbf{A}$ with $r-p$ elements each.

Equation (5.4.5) was derived under the assumption that $k>0$. However, equation (5.4.5) is also true if $k=0$. Indeed, in this case all the terms in the sum on the right hand side vanish by the definition of φ^p and the assertion follows from (5.4.2). Therefore, Giambelli's formula follows from (5.4.1) and (5.4.5).

6. The formula for quadrics

Setup (6.1). Assume that the ground scheme S is the spectrum of a field of characteristic different from 2 and fix a finite dimensional vectorspace $\mathcal{U}$ over the field. Let $r\geq 1$ be an integer and $\{\mathcal{U}_i\}$ a strictly increasing flag of subspaces:

$$\mathcal{U}_1 \subset \mathcal{U}_2 \subset \ldots \subset \mathcal{U}_r \subset \mathcal{U}$$

and let A be the strictly increasing sequence $A=(a_1, \ldots, a_r)$ of integers defined by $a_i := \dim \mathcal{U}/\mathcal{U}_{r-i+1} - 1$ for $i=1, \ldots, r$ as in (3.3).

Let $T := \operatorname{Grass}^r(\mathcal{U})$ be the Grassmannian of rank r quotients of $\mathcal{U}$, and

$$\mathcal{U}_T \twoheadrightarrow \mathcal{E}$$

the tautological quotient. Thus $\mathcal{E}$ is a bundle of rank r on T. Corresponding to the flag $\{\mathcal{U}_i\}$ we introduced in (3.3) the Schubert scheme $\Omega=\Omega(\{\mathcal{U}_i\}, \mathcal{U})$ which is a subscheme of T.

Let B be the space of complete symmetric forms on $\mathcal{E}$. In the notation of [T-K], the space B is the scheme $B := B_r^{\mathrm{sym}}(\mathcal{E}, \mathcal{E})$ of (projectively) r-complete symmetric forms on the pair $(\mathcal{E}, \mathcal{E})$. The theory of complete symmetric forms, completely parallel to the theory described in Section 4, may be found in [T-K], and of its notations and results we shall freely use the parallel symmetric versions of (4.1), (4.2) and (4.3). In particular, the structure map $f: B \to T$ factors through the characteristic map $B \to \mathbf{P}(\mathcal{S}ym^2 \mathscr{E})$ and the latter map can be described as a sequence of monoidal transformations with centers on regular subschemes. The relative dimension of B/T is therefore equal to $\binom{r+1}{2}-1$.

The scheme $B_\Omega := f^{-1}(\Omega)$ is the space of complete symmetric forms on the bundle $\mathscr{E}_\Omega$. We have that

$$N := f^{-1}(\Omega) = \dim B_\Omega = \sum_{i=1}^{r} a_i + r - 1. \tag{6.1.1}$$

Indeed, by (3.3.2) we get

$$\dim B_\Omega = \dim_\Omega B_\Omega + \dim \Omega = \binom{r+1}{2} - 1 + \sum_{i=1}^{r} a_i - \binom{r}{2},$$

which is the desired equation.

We denote by $[\Omega]$ the class of Ω in $A(T)$. The class $[\Omega]$ depends on the sequence A only. Moreover, it follows from (3.5.2) that the class of $f^{-1}(\Omega)$ in $A(B)$ is equal to the inverse image $[f]([\Omega])$ of $[\Omega]$ under the orientation class. We denote this class by ω_A, that is

$$\omega_A := [f^{-1}(\Omega)] = [f]([\Omega]).$$

PROPOSITION (6.2). *Let $N := \Sigma_{i=1}^{r} a_i + r - 1$ be the dimension of $f^{-1}(\Omega)$. Then:*
(1)

$$\int_B \mu_1^N \cap \omega_A = \psi_A.$$

(2) *Let p and k be integers such that $0<p<r$ and $0 \le k < N$. Set $\bar\mu_1 := c_1(\mathscr{L}_{p+1}) = c_1(\mathscr{M}_{p+1} \otimes \mathscr{M}_p^{-1}) = \mu_{p+1} - \mu_p$. Then we have that*

$$\int_B \mu_1^{N-k-1} \bar\mu_1^k \delta_p \cap \omega_A = \sum_K \operatorname{sign}(K, \check K)\, \psi_K \psi_{\check K},$$

where the sum is over all K that are subsequences of A with $r-p$ elements and such that $\|\mathbf{K}\| = k-(r-p)+1$. Here $\check K$ is the complementary subsequence of A and the sign is the sign of the permutation $(K, \check K)$ of A.

The integers ψ_K are defined in (2.7) (*see also Appendix* (A.15)).

Remark. As mentioned in the introduction, formula (1) was obtained by Schubert, but he only had the recursive formulas (A.15.3), (A.15.6) and (A.15.7) for ψ_A, while we have defined ψ_A by the formula (A.15.2) which is the explicit expression requested by Schubert.

Proof. (1) The proof is similar to the proof of (5.2)(1), using the parallel symmetric version of (4.4)(1).

(2) The proof is similar to the proof of (5.2)(2) using the parallel symmetric version of (4.4)(2).

LEMMA (6.3). *In the setup of* (6.1), *the statements of Lemma* (5.3) *are valid if the inequality* (5.3.q) *is replaced by the following:*

$$\sum_{i=1}^{q} m_i > \sum_{i=1}^{q} a_{r-i+1} + q - 1.$$

Proof. The proof is similar to the proof of (5.3), using the parallel symmetric version of the cartesian diagram of (4.4.2).

(6.4). We are now ready to give the formula for complete quadrics that is similar to Giambelli's formula for complete bilinear forms.

FORMULA FOR COMPLETE QUADRICS. *We keep the notation of* (6.1). *In particular, we let* $\mathcal{U}$ *be a vectorspace and* $\{\mathcal{U}_i\}_{i=1}^{r}$ *a strictly increasing flag of subspaces of* U. *We denote by* $T := \mathrm{Grass}^r(\mathcal{U})$ *the Grassmannian of rank* r *quotients of* $\mathcal{U}$ *and by* $\mathcal{E}$ *the tautological rank* r *quotient of* $\mathcal{U}_T$. *Moreover, we denote by* $B = B_r^{\mathrm{sym}}(\mathcal{E}, \mathcal{E})$ *the space of complete symmetric forms on* $\mathcal{E}$ *with the characteristic classes* $\mu_1, \ldots, \mu_r$ *in* $A^*(B)$ *and by* $f: B \to T$ *the structure map. Furthermore, let* A *be the strictly increasing sequence* $(a_1, \ldots, a_r)$ *defined by*

$$a_i := \mathrm{rk}\, \mathcal{U}/\mathcal{U}_{r-i+1} - 1 \quad \textit{for} \quad i = 1, \ldots, r.$$

Finally, we denote by $\Omega := \Omega(\{\mathcal{U}_i\}, \mathcal{U})$ *the Schubert subscheme of* T *defined by the flag* $\{\mathcal{U}_i\}_{i=1}^{r}$ *and by* $\omega_A := [f^{-1}\Omega]$ *the class of* $f^{-1}\Omega$ *in* $A(B)$.

Let p *be an integer such that* $0 \leqslant p < r$ *and* $m_1, \ldots, m_{p+1}$ *a sequence of non-negative integers such that*

$$\sum_{i=1}^{p+1} m_i = \sum_{i=1}^{r} a_i + r - 1 = \dim f^{-1}\Omega$$

and satisfying the following inequalities:

$$\sum_{i=1}^{q} m_i > \sum_{i=1}^{q} a_{r-i+1} + q - 1 \quad \text{for} \quad q = 1, \ldots, p-1.$$

Moreover, let $\varphi^p(k,i)$ be the function defined by

$$\varphi^p(k,i) = \begin{cases} \binom{k}{0} + p\binom{k}{1} + \ldots + p^i\binom{k}{i} & \text{if } i \geqslant 0 \\ 0 \quad \text{if } i < 0. \end{cases}$$

Then we have the following formula:

$$\int \mu_1^{m_1} \ldots \mu_p^{m_p} \mu_{p+1}^{m_{p+1}} \cap \omega_A$$
$$= 1^{m_1} 2^{m_2} \ldots p^{m_p} \left((p+1)^{m_{p+1}} \psi_A - \sum_K \varphi^p(m_{p+1}, m_{p+1} - \|K\| - (r-p))\, \varepsilon_K \psi_K \psi_{\bar{K}} \right),$$

where the sum is over all $K = (k_1, \ldots, k_{r-p})$ that are subsequences of A with $r-p$ elements. Here $\|K\| = \Sigma_{i=1}^{r-p} k_i$, the sequence $\bar{K}$ is the complementary subsequence of K in A, and ε_K is the sign of the permutation $(K, \bar{K})$ of A. The integers ψ_K are defined in (2.7) *(see also Appendix* (A.15)*).*

Remark (6.5). By definition of the function $\varphi^p(k,i)$, the summation in Giambelli's formula may be restricted to subsequences K with $r-p$ elements such that $\|K\| \leqslant m_{p+1} - (r-p)$. When $p=0$, the only possible subsequence is $K=A$. The sequence $K=A$ does not satisfy the latter inequality, because we have

$$\|A\| = \Sigma_{i=1}^{p+1} m_i - (r-1) > m_{p+1} - (r-p) \quad \text{when} \quad p=0.$$

Hence the formula in the case $p=0$ reduces to Schubert's formula (6.2)(1).

Our presentation of the formula is very close to the presentation of Giambelli's Formula for complete correlations in (5.4). It should be emphasized, however, that the formula above was never stated, nor indicated, by Giambelli. As the statement (and its proof, see below) is so parallel to the statement of Giambelli, an explanation may simply lie in the fact that an explicit formula for the function ψ_A was not known.

Proof of (6.4). The proof is entirely identical to the proof of (5.4), using the symmetric versions of (4.3.1), and (6.2) and (6.3), instead of (5.2) and (5.3).

Remark (6.6). The parameter space B and the intersection numbers $\int \mu_1^{m_1} \ldots \mu_r^{m_r} \cap \omega_A$

have other interpretations than that given in the Introduction. As for the latter, there are similar interpretations for complete bilinear forms.

An algebraic interpretation is as follows: The parameter space B is a compactification of the set of symmetric forms (up to scalar multiplication) of rank r on $\mathcal{U}$. The null space of such a form is a (vector) subspace $\mathcal{V}$ of corank r in $\mathcal{U}$ and the Schubert condition corresponding to the flag $\{\mathcal{U}_i\}$ in (6.1) is translated to the following: the quotient $(\mathcal{U}_i+\mathcal{V})/\mathcal{V}$ has rank less than i for $i=1,\ldots,r$. The condition μ_i is represented as follows: Fix a rank i (vector) subspace $\mathcal{W}$ of $\mathcal{U}$. Then the composite map $\mathcal{S}ym^2\Lambda^i\mathcal{W}_B\to\mathcal{S}ym^2\Lambda^i\mathcal{U}_B\to\mathcal{S}ym^2\Lambda^i\mathcal{E}_B\to\mathcal{M}_i$ defines a section of $\mathcal{M}_i$, and its (scheme of) zeroes correspond to the symmetric forms on $\mathcal{U}$ whose restriction to $\mathcal{W}$ has rank less than i, i.e. whose restriction to $\mathcal{W}$ is singular. (When $i=r$ this has to be interpreted as twice the condition that the null space meets $\mathcal{W}$ non-trivially.)

A translation to geometry is as follows: A symmetric form of rank r on $\mathcal{U}$ corresponds to an element of $\mathcal{S}ym^2\mathcal{U}^*$, i.e., a global section of $\mathcal{O}_{\mathbf{P}}(2)$, where $\mathbf{P}:=\mathbf{P}(\mathcal{U}^*)$. The scheme of zeroes of the latter section is a quadratic hypersurface in $\mathbf{P}$. The null space $\mathcal{V}$ of the form corresponds to the vertex V of codimension r in $\mathbf{P}$ of the quadratic hypersurface. The parameter space B is therefore the compactification of the set of rank r quadratic hypersurfaces $\mathbf{P}$. The points added on the boundary correspond to finite sequences consisting of a quadratic hypersurface in $\mathbf{P}$ with vertex V_1, a quadratic hypersurface in V_1 with vertex $V_2,\ldots,$ a quadratic hypersurface in V_{t-1} vertex V_t, such that the last vertex V_t has codimension r in $\mathbf{P}$.

The dual of the flag $\{\mathcal{U}_i\}$ corresponds to a strictly decreasing flag $\mathbf{P}\supset L_1\supset\ldots\supset L_r$ of linear subspaces (and $a_i=\operatorname{codim} L_i-1$), and the Schubert condition on the vertex V is that the span L_i+V has codimension at least i in $\mathbf{P}$ for $i=1,\ldots,r$.

The class μ_i may be represented by the condition for quadratic hypersurfaces to be tangent to a given $(i-1)$-plane in $\mathbf{P}$ (corresponding to the dual vectorspace of $\mathcal{W}\subseteq\mathcal{U}$). (When $i=r$ this has to be interpreted as twice the condition that the vertex meets a given $(r-1)$-plane.)

Note finally that the interpretation given in the introduction is the dual of the geometrical interpretation above.

Appendix: Schur functions

Setup (A.1). Consider a graded ring $R=R_0\oplus R_1\oplus R_2\oplus\ldots$ and associate with it its completion, $\hat{R}$, with respect to the filtration determined by the grading, i.e., $\hat{R}:=R_0\times R_1\times R_2\times\ldots$. Elements in the ideal $R_k\times R_{k+1}\times\ldots$ of $\hat{R}$ will be said to have *order* at least k. We shall consider power series in $\hat{R}[[T]]$, and for p in $\hat{R}[[T]]$ we shall write p_i for the ith coefficient and indicate this by writing $p=p(T)=p_0+p_1T+p_2T^2+\ldots$. The power series $p(T)$ is *convergent* if the order of p_i goes to infinity with i. Note that we may substitute an element $z \in \hat{R}$ in the power series $p(T)$ if the power series $p(T)$ is convergent or if the element z has positive order. In any case substitution yields the element $p(z)=\Sigma_{i=0}^{\infty} p_i z^i$ in $\hat{R}$.

Notation (A.2). If $p=p_0+p_1T+p_2T^2+\ldots$ is a power series in $\hat{R}[[T]]$, then we denote by $\langle p \rangle$ the infinite row of its coefficients, i.e.,

$$\langle p \rangle := (p_0, p_1, p_2, \ldots).$$

More generally, if $(p^1, p^2, \ldots)$ is a finite or infinite sequence of r power series in $\hat{R}[[T]]$ and $1 \leq r \leq \infty$, then we denote by $\langle p^1, p^2, \ldots \rangle$ the $r \times \infty$ matrix whose ith row is $\langle p^i \rangle$. If r is finite and the power series $p^1, \ldots, p^r$ are convergent, then we denote by $[p^1, \ldots, p^r]$ the sum of all r by r minors of the matrix $\langle p^1, \ldots, p^r \rangle$. That is,

$$[p^1, \ldots, p^r] := \sum_J \det\langle p^1, \ldots, p^r \rangle_J,$$

where the sum is over all strictly increasing sequences $J=(j_1, \ldots, j_r)$ of non-negative integers. The notation for submatrices here and in the following is that of (2.5).

If p is a power series, then we denote by $M(p)$ the $\infty \times \infty$ matrix whose ith row consists of the coefficients of the power series $T^i p(T)$, i.e.,

$$M(p) := \langle p, Tp, T^2p, \ldots \rangle = \begin{pmatrix} p_0 & p_1 & p_2 & \cdots \\ 0 & p_0 & p_1 & \cdots \\ 0 & 0 & p_0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

For every sequence $J=(j_1, \ldots, j_r)$ of r non-negative integers the $r \times r$ matrix $M_J^{(r)}(p)$ is the matrix obtained from $M(p)$ by selecting the first r rows and the r columns corresponding to the indices in J, that is,

$$M_J^{(r)}(p) = \begin{pmatrix} p_{j_1} & p_{j_2} & \cdots & p_{j_r} \\ p_{j_1-1} & p_{j_2-1} & \cdots & p_{j_r-1} \\ \vdots & \vdots & & \vdots \\ p_{j_1-(r-1)} & p_{j_2-(r-1)} & \cdots & p_{j_r-(r-1)} \end{pmatrix}.$$

The determinant of the latter matrix will be denoted $p_J := \det M_J^{(r)}(p)$.

Moreover, given an ordered set $(z_1, \ldots, z_r)$ of elements in $\hat{R}$, we let $V(z_1, \ldots, z_r)$ denote the $r \times \infty$ matrix whose rows are the coefficients of the power series $(1-z_i T)^{-1}$ for $i=1, \ldots, r$, that is

$$V(z_1, \ldots, z_r) := \left\langle \frac{1}{1-z_1 T}, \ldots, \frac{1}{1-z_r T} \right\rangle = \begin{pmatrix} 1 & z_1 & z_1^2 & \cdots \\ 1 & z_2 & z_2^2 & \cdots \\ \vdots & \vdots & \vdots & \\ 1 & z_r & z_r^2 & \cdots \end{pmatrix}.$$

If J is a sequence of r non-negative integers, then we shall denote by $\Delta_J(z_1, \ldots, z_r)$ the determinant $\Delta_J(z_1, \ldots, z_r) := \det V_J(z_1, \ldots, z_r)$. In particular, for $J=(r)$, we have that

$$\Delta(z_1, \ldots, z_r) := \Delta_{(r)}(z_1, \ldots, z_r) = \begin{vmatrix} 1 & z_1 & \cdots & z_1^{r-1} \\ \vdots & \vdots & & \vdots \\ 1 & z_r & \cdots & z_r^{r-1} \end{vmatrix}$$

is the usual Vandermonde determinant.

LEMMA (A.3). *Let p and q be power series in $\hat{R}[[T]]$ and z an element of $\hat{R}$. Then:*
(1) *For every integer $n \geq 1$ the following matrix equations hold:*

$$\langle p \rangle M(q) = \langle pq \rangle, \quad M(p)M(q) = M(pq) \quad \text{and} \quad M_{(n)}^{(n)}(p) M_{(n)}^{(n)}(q) = M_{(n)}^{(n)}(pq).$$

(2) *If the power series p is convergent or the element z has positive order, then we have that*

$$V(z)\langle p \rangle^{\mathrm{tr}} = p(z) \quad \text{and} \quad V(z) M(p)^{\mathrm{tr}} = p(z) V(z).$$

Proof. (1) The first equation is trivially verified. The second follows by applying the first to the power series $T^i p(T)$ and $q(T)$ for $i=0, 1, \ldots$. The third follows from the second since the matrices in question are upper triangular matrices.

(2) The first equation is just the definition $\Sigma_{j=0}^{\infty} z^j p_j = p(z)$ of $p(z)$. The second follows by applying the first to the power series $T^i p(T)$ for $i=0, 1 \ldots$.

LEMMA (A.4). *Let $p(T) = p_0 + p_1 T + p_2 T^2 + \ldots$ be a power series with constant term $p_0 = 1$ and let I and J be subsets of $\{0, 1, \ldots, n-1\}$ with r elements each. Moreover, let z be an element of $\hat{R}$ and let c be the power series defined by the equation*

$$p(T)\, c(-T) = 1.$$

Then the following equations hold:

(1) (Symmetry)

$$\det M_{I^*}^{J^*}(p(T)) = \det M_J^I(p(T)).$$

(2) (Homogeneity)

$$\det M_J^I(p(zT)) = z^{\|J\|-\|I\|} \det M_J^I(p(T)).$$

(*Where both sides are equal to* 0 *if* $\|J\|<\|I\|$.)

(3) (Complementarity)

$$\det M_{J'}^{I'}(c(T)) = \det M_J^I(p(T)).$$

Proof. Let $(i_1, \dots, i_r)$ and $(j_1, \dots, j_r)$ be the increasing sequences determined by the subsets I and J respectively.

(1) The matrix $M_J^I(p)$ is equal to the transpose of the matrix $M_{i_1^*,\dots,i_r^*}^{j_1^*,\dots,j_r^*}(p)$, because $i-j=j^*-i^*$. The latter matrix can be transformed into $M_{I^*}^{J^*}(p)$ by reversing the order first of its rows and next of its columns. Therefore the asserted equality of determinants holds.

(2) The determinant $\det M_J^I(p(zT))$ is a sum of products of the form

$$\pm z^{k_1-i_1+\dots+k_r-i_r} p_{k_1-i_1} \cdots p_{k_r-i_r},$$

where $(k_1, \dots, k_r)$ is a permutation of $(j_1, \dots, j_r)$. The latter product is only non-zero if all the inequalities $k_1 \geq i_1, \dots, k_r \geq i_r$ are satisfied. As $\|K\|=\|J\|$ it follows that the power of z is equal to $z^{\|J\|-\|I\|}$ (and that the determinant is equal to 0 if $\|J\|<\|I\|$).

(3) Let $u=c(-T)$ in $\hat{R}[[T]]$ be the inverse of p and let $P:=M_{(n)}^{(n)}(p)$ and $U:=M_{(n)}^{(n)}(u)$. From the third equation of (A.3)(1) it follows that $PU=1$. By the Laplace expansion (2.5.1), we have that $\det P_J^I = \operatorname{sign}(I, \tilde{I}) \operatorname{sign}(J, \tilde{J}) \det U_{\tilde{I}}^{\tilde{J}}$, that is,

$$\det M_J^I(p) = \operatorname{sign}(I, \tilde{I}) \operatorname{sign}(J, \tilde{J}) \det M_{\tilde{I}}^{\tilde{J}}(u). \tag{A.4.4}$$

A permutation $(I, \tilde{I})$ may clearly be put in increasing order by $i_r-(r-1)+\dots+i_2-1+i_1$ transpositions. It follows that the product of the two signs on the right hand side of equation (A.4.4) is equal to $(-1)^{\|J\|-\|I\|}=(-1)^{\|\tilde{I}\|-\|\tilde{J}\|}$. By assertion (2) with $z:=-1$ we therefore obtain that $\det M_J^I(p)=\det M_{\tilde{I}}^{\tilde{J}}(c)$. Finally it follows from assertion (1) that $\det M_{\tilde{I}}^{\tilde{J}}(c)=\det M_{J'}^{I'}(c)$ and we have proved assertion (3).

Definition (A.5). For a finite family A of elements in $\hat{R}$, we denote by $c(AT)$ and $s(AT)$ the power series defined by

$$c(AT):=\prod_{a\in A}(1+aT) \quad \text{and} \quad s(AT):=\prod_{a\in A}\frac{1}{1-aT}.$$

The coefficients of $c(AT)=c_0+c_1T+\ldots$ and $s(AT)=s_0+s_1T+\ldots$ are the *elementary symmetric functions* and *complete symmetric functions* respectively of the finite family A. The matrices $M(c(AT))$ and $M(s(AT))$ will be denoted by $C(A)$ and $S(A)$ respectively. For a finite sequence $J=(j_1,\ldots,j_r)$ of non-negative integers, the determinant $s_J(A):=\det S_J^{(r)}(A)$ is called the Jth *Schur function* of the family A. If the elements of A have positive order, then the power series $c(AT)$ and $s(AT)$ are convergent. Substitution of $T=1$ yields in this case the values

$$c(A):=\prod_{a\in A}(1+a) \quad \text{and} \quad s(A):=\prod_{a\in A}\frac{1}{1-a}.$$

If the elements of A are homogeneous of degree 1, i.e., if they are elements of R_1, then clearly $s_J(A)$ is homogeneous of degree $j_1+(j_2-1)+\ldots+(j_r-(r-1))$ in R.

A detailed treatment of Schur functions (with a change in indexation) is given in [M].

LEMMA (A.6) (Jacobi-Trudi). *Let $(a_1,\ldots,a_r)$ be an ordering of a finite family A of elements of positive order in $\hat{R}$ and for $k=1,\ldots,r$ denote by c^k the polynomial $c^k(T):=\Pi_{i\neq k}(1-a_iT)$. Moreover, denote by $W(a_1,\ldots,a_r)$ the $r\times r$ matrix $\langle c^1,\ldots,c^r\rangle_{(r)}$. Then the following formulas hold:*

$$W(a_1,\ldots,a_r)\,S^{(r)}(A)=V(a_1,\ldots,a_r), \tag{A.6.1}$$

$$W(a_1,\ldots,a_r)=V_{(r)}(a_1,\ldots,a_r)\,S_{(r)}^{(r)}(A)^{-1}, \tag{A.6.2}$$

$$\det W(a_1,\ldots,a_r)=\Delta(a_1,\ldots,a_r). \tag{A.6.3}$$

Moreover, if J is a sequence of r non-negative integers, then we have that

$$\Delta_J(a_1,\ldots,a_r)=\Delta(a_1,\ldots,a_r)\,s_J(A). \tag{A.6.4}$$

Proof. From the definitions we get that

$$c^k(T)\,s(AT)=\frac{1}{1-a_kT} \quad \text{for} \quad k=1,\ldots,r.$$

Therefore, applying the first equation in (A.3)(1) to r rows, we have that

$$\langle c^1(T), \ldots, c^r(T)\rangle\, M(s(AT)) = \left\langle \frac{1}{1-a_1T}, \ldots, \frac{1}{1-a_rT}\right\rangle.$$

The matrix on the right hand side of the above equation is the matrix on the right hand side of (A.6.1). The first factor on the left hand side of the above equation has only its first r columns non-vanishing, because the $c^k(T)$'s are polynomials of degree less than r. Therefore the product on the left hand side is equal to the product of the two matrices obtained from the first r columns of the first factor and the first r rows of the second factor, that is, the left hand side of the above equation is equal to the left hand side of (A.6.1). Thus equation (A.6.1) holds.

The matrix $S^{(r)}_{(r)}(A)$ is an upper triangular matrix with 1 in the diagonal and consequently it has determinant equal to 1. Therefore, equation (A.6.2) follows from equation (A.6.1) by selecting the first r columns in (A.6.1). Equation (A.6.3) follows from taking the determinants of two sides of (A.6.2). Finally, the last formula follows from the first and the third.

Definition (A.7). For an integer t, let $D(t)$ and $E(t)$ be the $\infty\times\infty$ matrices given by

$$D(t) := \left\langle \frac{1}{(1-T)^{t+1}}, \frac{1}{(1-T)^{t+2}}, \frac{1}{(1-T)^{t+3}}, \ldots\right\rangle^{\mathrm{tr}}$$

and

$$E(t) := \left\langle \frac{1}{(1-T)^{t+1}}, \frac{T}{(1-T)^{t+2}}, \frac{T^2}{(1-T)^{t+3}}, \ldots\right\rangle^{\mathrm{tr}}.$$

Thus the ijth entries are, respectively, the binomial coefficients

$$\binom{-t-1-j}{i}(-1)^i = \binom{i+j+t}{i} \quad \text{and} \quad \binom{-t-1-j}{i-j}(-1)^{i-j} = \binom{i+t}{i-j} \quad \text{for} \quad i,j = 0,1,2,\ldots,$$

i.e.,

$$D(t) = \begin{pmatrix} \binom{t}{0} & \binom{1+t}{0} & \binom{2+t}{0} & \cdots \\ \binom{1+t}{1} & \binom{2+t}{1} & \binom{3+t}{1} & \cdots \\ \binom{2+t}{2} & \binom{3+t}{2} & \binom{4+t}{2} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \quad \text{and} \quad E(t) = \begin{pmatrix} \binom{t}{0} & 0 & 0 & \cdots \\ \binom{1+t}{1} & \binom{1+t}{0} & 0 & \cdots \\ \binom{2+t}{2} & \binom{2+t}{1} & \binom{2+t}{0} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

LEMMA (A.8). *Let a be an element of positive order in $\hat{R}$ and set $z := (1-a)^{-1} \in \hat{R}$.*

Then we have that

$$V(a)\,D(t) = z^{t+1}V(z) \quad \textit{and} \quad V(a)\,E(t) = z^{t+1}V(z-1). \tag{A.8.1}$$

Proof. The equations follow from the first equation in (A.3)(2) and the definitions of $D(t)$ and $E(t)$.

Remark (A.9). The ijth entry in $E(0)$ is $\binom{i}{j}$ and therefore we have that

$$E(0) = \langle 1, 1+T, (1+T)^2, \ldots \rangle.$$

Hence, by (A.3)(2) and (A.8), for every a of positive order (and $z := (1-a)^{-1}$) we have that

$$\begin{aligned} V(a)\,E(t)\,E(0)^{\mathrm{tr}} &= z^{t+1}V(z-1)\,E(0)^{\mathrm{tr}} = z^{t+1}(1, z, z^2, \ldots) \\ &= z^{t+1}V(z) = V(a)\,D(t). \end{aligned}$$

It follows easily that $D(t) = E(t)\,E(0)^{\mathrm{tr}}$. Consequently the minors of $D(t)$ can be expressed as sums of minors of $E(t)$ with integer coefficients.

Remark. The matrix equation $D(t) = E(t)\,E(0)^{\mathrm{tr}}$ is a special case of a general equation. Namely, if p, q and u are power series such that u has constant term equal to 0, then for the composite power series $p \circ u$ we have that

$$\langle p \rangle \langle q, uq, u^2 q, \ldots \rangle = \langle (p \circ u)\, q \rangle.$$

In fact, if a is an element of positive order, then

$$\begin{aligned} V(a)\,\langle q, uq, u^2 q, \ldots \rangle^{\mathrm{tr}} \langle p \rangle^{\mathrm{tr}} &= q(a)\,(1, u(a), u(a)^2, \ldots)\,\langle p \rangle^{\mathrm{tr}} \\ &= q(a)\,p(u(a)) = V(a)\,\langle (p \circ u)\, q \rangle^{\mathrm{tr}}, \end{aligned}$$

and this implies the assertion.

LEMMA (A.10). *Let $(a_1, \ldots, a_r)$ be an ordering of a finite family A of elements of positive order in $\hat{R}$, and set $z_i := (1-a_i)^{-1}$ for $i = 1, \ldots, r$. Moreover, let $W(a_1, \ldots, a_r)$ be the $r \times r$ matrix of Lemma* (A.6). *Then we have that*

$$\prod_{i=1}^{r} z_i = s(A) \quad \textit{and} \quad \Delta(z_1, \ldots, z_r) = \Delta(z_1 - 1, \ldots, z_r - 1) = s(A)^{r-1}\Delta(a_1, \ldots, a_r), \tag{A.10.0}$$

$$W(a_1, \ldots, a_r)\, S^{(r)}(A)\, D(t) = \mathrm{diag}(z_1^{t+1}, \ldots, z_r^{t+1})\, V(z_1, \ldots, z_r), \tag{A.10.1}$$

$$W(a_1, \dots, a_r)\, S^{(r)}(A)\, E(t) = \operatorname{diag}(z_1^{t+1}, \dots, z_r^{t+1})\, V(z_1-1, \dots, z_r-1). \qquad \text{(A.10.2)}$$

Moreover, for every power series p in $\hat{R}[[T]]$ we have that

$$W(a_1, \dots, a_r)\, S^{(r)}(A)\, D(t)\, M(p)^{\mathrm{tr}} = \operatorname{diag}(z_1^{t+1} p(z_1), \dots, z_r^{t+1} p(z_r))\, V(z_1, \dots, z_r). \qquad \text{(A.10.3)}$$

Proof. The first equation in (A.10.0) follows from the definition of $s(A)$ in (A.5). Moreover, we have that

$$\begin{aligned} s(A)^{r-1}\Delta(a_1, \dots, a_r) &= s(A)^{r-1}\Delta(a_1-1, \dots, a_r-1) \\ &= s(A)^{r-1}\Delta(-z_1^{-1}, \dots, -z_r^{-1}) \\ &= \left(\prod z_i^{r-1}\right)\Delta(-z_1^{-1}, \dots, -z_r^{-1}), \end{aligned}$$

where the first equation follows from the usual expression for the Vandermonde determinant $\Delta(a_1, \dots, a_r)=\Pi_{i<j}(a_j-a_i)$. Multiplying out the last of the above expressions and switching columns in the resulting matrix, we see that it is equal to $\Delta(z_1, \dots, z_r)$. The remaining equation in (A.10.0) is trivial.

The kth row of the left hand side of (A.10.1) is, by Jacobi-Trudi's Lemma (A.6.1), equal to $V(a_k)D(t)$. The latter row is by the first equation in (A.8.1) equal to the kth row of the right hand side of (A.10.1). The proof of (A.10.2) is similar, using the second equation in (A.8.1).

By Lemma (A.3)(2), the equation (A.10.3) results from (A.10.1) upon multiplication by the matrix $M(p)^{\mathrm{tr}}$.

Remark. The equations involving discriminants follow also from (A.8): Apply (A.8) to r elements $(a_1, \dots, a_r)$ with $t:=-r$ and select the first r columns. The first r columns in $D(-r)$, resp. $E(-r)$, are the coefficients in the polynomials $(1-T)^{r-1}, \dots, 1-T, 1$, resp. $(1-T)^{r-1}, \dots, T^{r-2}(1-T), T^{r-1}$. Since these polynomials have degree less than r, only the first r rows in $D(-r)_{(r)}$, resp. $E(-r)_{(r)}$, are non-vanishing. Therefore, with $D:=D(t)_{(r)}^{(r)}$ the following matrix equation results:

$$V_{(r)}(a_1, \dots, a_r)\, D = \operatorname{diag}(z_1^{-r+1}, \dots, z_r^{-r+1})\, V_{(r)}(z_1, \dots, z_r),$$

resp. a similar equation results with D replaced by E. Clearly D has the elements $(-1)^{r-1}, (-1)^{r-2}, \dots, 1$ in the antidiagonal and zeros below, and E has 1's in the diagonal and zeros above. Thus $\det D=\det E=1$ and the required equations follow from taking determinants.

LEMMA (A.11). *The functions* $[p^1,\ldots,p^r]$ *of r convergent power series* $p^1,\ldots,p^r$ *in* $\hat{R}[[T]]$ *defined in* (A.2) *for* $r=1,2,\ldots$ *are* $\hat{R}$*-multilinear and alternating and satisfy the following equations:*

$$[p]=p(1) \quad \textit{and} \quad [1,p]=p(1)-p(0). \tag{A.11.0}$$

$$[Tp^1,\ldots,Tp^r]=[1,Tp^1,\ldots,Tp^r]=[p^1,\ldots,p^r]. \tag{A.11.1}$$

$$[p^1,\ldots,p^r]-[1,p^1,\ldots,p^r]=\sum_{k=1}^{r}(-1)^{k-1}p^k(0)[p^1,\ldots,\widehat{p^k},\ldots,p^r], \tag{A.11.2}$$

Here the $\frown$ *indicates an omitted argument. Moreover these functions are determined by the functions* $[p]=p(1)$ *and* $[p,q]$ *through the following recurrence formula:*

$$[p^1,\ldots,p^r]=\begin{cases}\sum_{k=1}^{r}(-1)^{k-1}[p^k]\,[p^1,\ldots,\widehat{p^k},\ldots,p^r], & \textit{if } r \textit{ is odd.}\\ \sum_{k=2}^{r}(-1)^{k}[p^1,p^k]\,[p^2,\ldots,\widehat{p^k},\ldots,p^r], & \textit{if } r \textit{ is even.}\end{cases} \tag{A.11.3}$$

Finally the following explicit formula holds:

$$[p^1,\ldots,p^r]=\begin{cases}\mathrm{Pf}(([p^i,p^j])_{i,j=1,\ldots,r} & \textit{if } r \textit{ is even,}\\ \mathrm{Pf}(([p^i,p^j])_{i,j=0,\ldots,r} & \textit{if } r \textit{ is odd,}\end{cases} \tag{A.11.4}$$

where Pf *denotes the Pfaffian of the alternating matrix and where, in case of odd r, we interpret* $[p^0,p^k]$ *as* $[p^k]$.

Proof. From the definitions it is obvious that the functions $[p^1,\ldots,p^r]$ are $\hat{R}$-multilinear and alternating and that the formulas of (A.11.0) hold.

The matrix $\langle Tp^1,\ldots,Tp^r\rangle$ has its first column equal to zero and clearly the same nonvanishing minors as $\langle p^1,\ldots,p^r\rangle$. Therefore the equations of (A.11.1) hold.

To prove equations (A.11.2) and (A.11.3) we proceed by induction. We prove first that the two right hand sides are alternating. As $\hat{R}$-linearity is obvious, it suffices to prove that the right hand sides vanish if two consecutive arguments are equal. This is easily verified in all cases except for the right hand side of (A.11.3) when r is even and the two equal arguments are the first two. To treat the exceptional case in $r+2$ variables, we evaluate the right hand side of (A.11.3) on a sequence of power series $(p,p,p^1,\ldots,p^r)$ with r even. The result is the following sum:

$$[p,p]\,[p^1,\ldots,p^r]+\sum_{k=1}(-1)^k[p,p^k]\,[p,p^1,\ldots,\widehat{p^k},\ldots,p^r].$$

The first term in the sum is clearly equal to zero. By induction we may assume that (A.11.3) holds for r variables. Therefore the kth term in the sum is equal to the following expression:

$$\sum_{j<k}(-1)^{k+j-1}[p,p^k]\,[p,p^j]\,[p^1,\ldots,\widehat{p^j},\ldots,\widehat{p^k},\ldots,p^r]$$
$$+\sum_{j>k}(-1)^{k+j}[p,p^k]\,[p,p^j]\,[p^1,\ldots,\widehat{p^k},\ldots,\widehat{p^j},\ldots,p^r].$$

For $k=1,\ldots,r$ we insert the latter expression in the sum. The vanishing follows easily.

Next we note that, since the power series are convergent, for every n there is only a finite number of terms of order less than n in the sum defining $[p^1,\ldots,p^r]$. Thus it suffices to prove the equations for polynomials. Finally, since the expressions are alternating and the polynomials in $\hat{R}[[T]]$ have the sequence $1,T,T^2,\ldots$ as a free $\hat{R}$-basis, it suffices to verify the equations on sets of the form $(T^{j_1},\ldots,T^{j_r})$, where $0\leqslant j_1<\ldots<j_r$. The verification is immediate, noting that $[T^{j_1},\ldots,T^{j_r}]=1$, because the only $r\times r$ submatrix of $\langle T^{j_1},\ldots,T^{j_r}\rangle$ with all rows non-zero is the unit matrix.

For even r, the formula (A.11.3) is the Laplace type expansion of the Pfaffian, and therefore (A.11.4) holds by induction on r. Finally note that (A.11.4) for an odd number r of variables follows from (A.11.4) applied to the $r+1$ variables $1,Tp^1(T),\ldots,Tp^r(T)$ using (A.11.1) (and interpreting $[p^0,p^k]$ as $[p^k]$). Thus Lemma (A.11) is proved.

LEMMA (A.12). *Let $A=(a_1,\ldots,a_r)$ be an ordered family of elements of positive order in $\hat{R}$. Then the following formulas hold:*

$$\sum_J \Delta_J(a_1,\ldots,a_r)=\Delta(a_1,\ldots,a_r)\prod_i\frac{1}{1-a_i}\prod_{i<j}\frac{1}{1-a_ia_j},\tag{A.12.1}$$

$$\sum_J s_J(A)=\prod_i\frac{1}{1-a_i}\prod_{i<j}\frac{1}{1-a_ia_j},\tag{A.12.2}$$

where the sum is over all strictly increasing sequences $J=(j_1,\ldots,j_r)$ of non-negative integers.

Proof. The formulas of the Lemma are of a universal nature. That is, if they hold when the family $A=(a_1,\ldots,a_r)$ is the family of independent variables $(x_1,\ldots,x_r)$ in the ring $R=\mathbf{Z}[x_1,\ldots,x_r]$ of polynomials over $\mathbf{Z}$, then they hold in general. Indeed, the asserted equations can be obtained by substituting $(a_1,\ldots,a_r)$ for variables in polynomial identities. Therefore, in the proof, we may assume that $(a_1,\ldots,a_r)$ is the sequence of independent variables $(x_1, \ldots, x_r)$ in the ring of polynomials $\mathbf{Z}[x_1, \ldots, x_r]$. Thus the ring $\hat{R}$ is the ring $\mathbf{Z}[[x_1, \ldots, x_r]]$ of formal power series.

Denote by $D(x_1, \ldots, x_r)$ the element of $\hat{R}$ obtained by multiplying the left hand side of equation (A.12.1) by $\Pi_i(1-x_i)$. By definition and the multilinearity of (A.11) we have that

$$D(x_1, \ldots, x_r) = \left[\frac{1-x_1}{1-x_1T}, \ldots, \frac{1-x_r}{1-x_rT}\right].$$

Clearly $D(x_1)=1$ and an easy summation gives $D(x_1,x_2)=(x_2-x_1)/(1-x_1x_2)$. Therefore, by (A.11.4), we have that

$$D(x_1, \ldots, x_r) = \mathrm{Pf}(d_{ij}),$$

where the matrix on the right hand side is the even order matrix of the size indicated in (A.11.4) and with coefficients given by

$$d_{ij} = \frac{x_j-x_i}{1-x_ix_j} \quad \text{(and } d_{0j}=1 \text{ if } r \text{ is odd).}$$

It follows from the expression that $D=D(x_1, \ldots, x_r)$ is a rational function of $x_1, \ldots, x_r$ and, more precisely, that the product of D and the polynomial $\Pi_{i<j}(1-x_ix_j)$ is a polynomial. This polynomial is clearly alternating and therefore divisible by the Vandermonde determinant $\Delta(x_1, \ldots, x_r)=\Pi_{i<j}(x_j-x_i)$. Hence we have proved that the equation

$$D(x_1, \ldots, x_r) = C\prod_{i<j}\frac{x_j-x_i}{1-x_ix_j} \tag{A.12.3}$$

holds with a polynomial factor C on the right hand side.

The rational functions $(x_j-x_i)/(1-x_ix_j)$ are clearly invariant under the substitution determined by $x_k \mapsto 1/x_k$ for $k=1, \ldots, r$. Hence it follows from the Pfaffian expression for D and the equation (A.12.3) that the polynomial C is invariant under the latter substitution. Therefore, the polynomial C is an integer constant.

Except for the factor C on the right hand side, the equation (A.12.1) follows from (A.12.3) upon division by $\Pi_i(1-x_i)$. By Jacobi-Trudi's Lemma (A.6.4), a further division by $\Delta(x_1, \ldots, x_r)$ yields equation (A.12.2). Therefore, equations (A.12.1) and (A.12.2) hold except for a universal integer constant C on the right hand side.

To finish the proof we evaluate the two sides of (A.12.2) on the family $A=(0, \ldots, 0)$. Both sides are clearly equal to 1 and therefore $C=1$. Thus the Lemma is proved.

PROPOSITION (A.13). *Let A and B be finite families of elements of positive order in $\hat{R}$, with r and s elements respectively. Moreover, let $(a_1, \ldots, a_r)$ be an ordering of the family A. Then the following formulas hold:*

$$\prod_{a\in A,\, b\in B} \frac{1}{1-(a+b)} = \sum_{I,J} s_I(A) \det D^I_J(s-r)\, s_J(B), \tag{A.13.1}$$

$$\prod_{a\in A,\, b\in B} (1+(a+b)) = (-1)^{\binom{r}{2}} \sum_{I,J} (-1)^{\|I\|} s_I(A) \det D^I_J(-s-r)\, s_{J'}(B), \tag{A.13.2}$$

$$\prod_{i\leqslant j} (1+(a_i+a_j)) = (-1)^{\binom{r}{2}} 2^{-r(r-1)} \sum_I (-2)^{\|I\|} \det D^I_{(\mathrm{ev})}(-2r)\, s_I(A), \tag{A.13.3}$$

$$\prod_{i<j} (1+(a_i+a_j)) = (-1)^{\binom{r}{2}} 2^{-r(r-1)} \sum_I (-2)^{\|I\|} \det D^I_{(\mathrm{ev})}(1-2r)\, s_I(A), \tag{A.13.4}$$

$$\prod_{i\leqslant j} \frac{1}{1-(a_i+a_j)} = \sum_I \left(\sum_J \det E^I_J(0) \right) s_I(A), \tag{A.13.5}$$

$$\prod_{i<j} \frac{1}{1-(a_i+a_j)} = (-1)^{\binom{r}{2}} \sum_I \left(\sum_J (-1)^{\|J\|} \det E^I_J(0) \right) s_I(A). \tag{A.13.6}$$

The sums are over strictly increasing sequences $I=(i_1, \ldots, i_r)$ and $J=(j_1, \ldots, j_r)$ of non-negative integers. The sequence (ev) *is the sequence consisting of the r even integers $0, 2, \ldots, 2r-2$. The notations $\|J\|$ and J' are introduced in* (2.5) *and the matrices $D(t)$ and $E(t)$ are defined in* (A.7).

Proof. The formulas of the Proposition are of a universal nature, cf. the proof of (A.12). We may therefore, in particular, assume that the Vandermonde determinant $\Delta(a_1, \ldots, a_r)$ is a non zero divisor, and we shall freely divide by the latter determinant.

Define the sequence $(z_1, \ldots, z_r)$ by $z_i := (1-a_i)^{-1}$ as in (A.10) and let t be an integer. In the proof we shall use the following three equations

$$\det W(a_1, \ldots, a_r) \det(S^{(r)}(A)\, D_{(\mathrm{ev})}(t)) = \prod z_i^{t+1} \det \Delta_{(\mathrm{ev})}(z_1, \ldots, z_r), \tag{A.13.7}$$

$$\det W(a_1, \ldots, a_r) \det(S^{(r)}(A)\, E_J(t)) = \prod z_i^{t+1} \Delta_J(z_1-1, \ldots, z_r-1), \tag{A.13.8}$$

$$\det W(a_1, \ldots, a_r) \det(S^{(r)}(A)\, D(t)\, M^{(r)}(p)^{\mathrm{tr}}) = \prod z_i^{t+1} \prod p(z_i)\, \Delta(z_1, \ldots, z_r), \tag{A.13.9}$$

where J is a strictly increasing sequence of non-negative integers with r elements and p is a power series in $\hat{R}[[T]]$. The above equations result when the determinant is applied to the matrix equations obtained from equations (A.10.1), (A.10.2) and (A.10.3) by selecting the columns from (ev), the columns from J and the columns from (r) respectively.

Consider first equation (A.13.9). With $p:=s(BT)$ we have that

$$p(z_i)=\prod_b \frac{1}{1-z_i b}=\prod_b \frac{1}{z_i(1-a_i-b)}=z_i^{-s}\prod_b \frac{1}{1-a_i-b}.$$

From (A.10.0) and the above expression for $p(z_i)$ it follows, when $p=s(BT)$, that the right hand side of equation (A.13.9) is equal to the expression

$$\Delta(a_1,\dots,a_r)\,s(A)^{t-s+r}\prod_b\prod_i \frac{1}{1-a_i-b}.$$

By (A.6.3), the left hand side of equation (A.13.9) is equal to the expression

$$\Delta(a_1,\dots,a_r)\det(S^{(r)}(A)\,D(t)\,S^{(r)}(B)^{\mathrm{tr}}).$$

Therefore, using the formula $\det SDM=\Sigma_{I,J}\det S_I \det D_J^I \det M^J$ for the determinant of a product of matrices S, D and M of sizes $r\times\infty$, $\infty\times\infty$ and $\infty\times r$ respectively, the equation (A.13.1) of the Proposition follows when we set $t:=s-r$.

Similarly, when $p=c(BT)$, we have $p(z_i)=\Pi_b(1+z_i b)=z_i^s\Pi_b(1-a_i+b)$ and the right hand side of (A.13.9) is equal to

$$\Delta(a_1,\dots,a_r)\,s(A)^{t+r+s}\prod_i\prod_b(1-a_i+b).$$

By (A.6.3), the left hand side of equation (A.13.9) is equal to the expression

$$\Delta(a_1,\dots,a_r)\det(S^{(r)}(A)\,D(t)\,C^{(r)}(B)^{\mathrm{tr}}).$$

Therefore, applying equation (A.13.9) to a family of the form wA with $w=-1$ and using the homogeneity and complementarity of Lemma (A.4), the equation (A.13.2) of the Proposition follows when we set $t:=-s-r$.

Consider next equation (A.13.7). We have that

$$\Delta_{(\mathrm{ev})}(z_1,\dots,z_r)=\Delta(z_1^2,\dots,z_r^2)=\prod_{i<j}(z_j^2-z_i^2)=\Delta(z_1,\dots,z_r)\prod_{i<j}(z_i+z_j).$$

As $z_i+z_j=z_iz_j(2-a_i-a_j)$ and $\Pi_{i<j} z_i z_j=\Pi_i z_i^{r-1}$, we have that

$$\Delta_{(\mathrm{ev})}(z_1, \dots, z_r) = \Delta(z_1, \dots, z_r) \prod_i z_i^{r-1} \prod_{i<j} (2-a_i-a_j).$$

From (A.10.0) and the above expression for $\Delta_{(\mathrm{ev})}(z_1, \dots, z_r)$ it follows that the right hand side of equation (A.13.7) is equal to the expression

$$\Delta(a_1, \dots, a_r)\, s(A)^{t+2r-1} \prod_{i<j} (2-a_i-a_j). \tag{A.13.10}$$

By (A.6.3), the left hand side of equation (A.13.7) is equal to the expression

$$\Delta(a_1, \dots, a_r) \det(S^{(r)}(A)\, D_{(\mathrm{ev})}(t)).$$

Therefore, applying equation (A.13.7) to a family of the form wA with $w=-2$ and using the homogeneity of Lemma (A.4), the equation (A.13.4) of the Proposition follows when we set $t:=1-2r$.

Since $2-2a_i=2z_i^{-1}$ and $\Pi_i z_i=s(A)$ we may rewrite the expression (A.13.10) for the right hand side of (A.13.7) in the form

$$\Delta(a_1, \dots, a_r)\, s(A)^{t+2r} 2^{-r} \prod_{i\leqslant j} (2-a_i-a_j).$$

Therefore, applying equation (A.13.7) to a family of the form $-2A$ and using the homogeneity of Lemma (A.4), the equation (A.13.3) of the Proposition follows when we set $t:=-2r$.

Consider finally the equation (A.13.8). If we form the sum over all sequences J of the right hand side of equation (A.13.8), then it follows from (A.12) that the sum is equal to the expression

$$\Delta(z_1-1, \dots, z_r-1) \prod_i z_i^{t+1} \prod_i \frac{1}{1-(z_i-1)} \prod_{i<j} \frac{1}{1-(z_i-1)(z_j-1)}.$$

Since $1-(z_i-1)=z_i(1-2a_i)$ and $1-(z_i-1)(z_j-1)=z_iz_j(1-(a_i+a_j))$, the latter expression is equal to the expression

$$\Delta(z_1-1, \dots, z_r-1) \prod_i z_i^{t+1} \prod_i \frac{1}{z_i} \prod_{i<j} \frac{1}{z_i z_j} \prod_{i\leqslant j} \frac{1}{1-(a_i+a_j)}.$$

From (A.10.0) and the above computation it follows that the sum over all J of the right hand side of equation (A.13.8) is equal to the expression

$$\Delta(a_1, \ldots, a_r)\, s(A)^t \prod_{i \leq j} \frac{1}{1-(a_i+a_j)}.$$

By (A.6.3), the sum over all J of the left hand side of equation (A.13.8) is equal to the expression

$$\Delta(a_1, \ldots, a_r) \sum_J \det(S^{(r)}(A)\, E_J(t)).$$

Therefore, evaluating the last determinant and interchanging the order of summation in the resulting sum, the equation (A.13.5) of the Proposition follows when we set $t := 0$.

The right hand side of (A.13.8) multiplied by $(-1)^{\|J\|}$ is clearly equal to the expression

$$\Delta_J(1-z_1, \ldots, 1-z_r) \prod_i z_i^{t+1}.$$

If we form the sum over all sequences J of the last expressions, then it follows from (A.11) that the sum is equal to the expression

$$\Delta(1-z_1, \ldots, 1-z_r) \prod_i z_i^{t+1} \prod_i \frac{1}{1-(1-z_i)} \prod_{i<j} \frac{1}{1-(1-z_i)(1-z_j)}.$$

Since $1-(1-z_i)=z_i$ and $1-(1-z_i)(1-z_j)=z_i z_j(1-((a_i+a_j))$, the latter expression is equal to the expression

$$\Delta(1-z_1, \ldots, 1-z_r) \prod_i z_i^{t+1} \prod_i \frac{1}{z_i} \prod_{i<j} \frac{1}{z_i z_j} \prod_{i<j} \frac{1}{1-(a_i+a_j)}.$$

Moreover,

$$\Delta(1-z_1, \ldots, 1-z_r) = (-1)^{\binom{r}{2}} \Delta(z_1-1, \ldots, z_r-1).$$

From (A.10.0) and the above computation it follows that the sum over all J of the right hand sides of equation (A.13.8) multiplied by $(-1)^{\|J\|}$ is equal to the expression

$$(-1)^{\binom{r}{2}} \Delta(a_1, \ldots, a_r)\, s(A)^t \prod_{i \leq j} \frac{1}{1-(a_i+a_j)}.$$

By (A.6.3), the sum over all J of the left hand sides of equation (A.13.8) multiplied by $(-1)^{\|J\|}$ is equal to the expression

$$\Delta(a_1, \dots, a_r) \sum_J (-1)^{\|J\|} \det(S^{(r)}(A)\, E_J(t)).$$

Therefore, evaluating the last determinant and interchanging the order of summation in the resulting sum, the equation (A.13.6) of the Proposition follows when we set $t := 0$.

LEMMA (A.14). *Let $p^1, \dots, p^r$ be convergent power series in $\hat{R}[[T]]$. Then:*

(1) *If $q^1, \dots, q^r$ are convergent power series in $\hat{R}[[T]]$ satisfying the symmetry conditions*

$$[p^k, q^j] = [p^j, q^k] \quad \textit{for} \quad k,j = 1, \dots, r, \tag{A.14.1}$$

then the following formulas hold: If r is even, then

$$\sum_{k=1}^{r} [p^1, \dots, q^k, \dots, p^r] = 0 \tag{A.14.2}$$

and

$$\sum_{k=1}^{r} [1, p^1, \dots, q^k, \dots, p^r] = \sum_{k=1}^{r} (-1)^{k-1} [1, q^k]\, [p^1, \dots, \widehat{p^k}, \dots, p^r]$$
$$- \sum_{k=1}^{r} (-1)^{k-1} [q^k]\, [1, p^1, \dots, \widehat{p^k}, \dots, p^r]. \tag{A.14.3}$$

If r is odd, then

$$\sum_{k=1}^{r} [p^1, \dots, q^k, \dots, p^r] = \sum_{k=1}^{r} (-1)^{k-1} [q^k]\, [p^1, \dots, \widehat{p^k}, \dots, p^r] \tag{A.14.4}$$

and

$$\sum_{k=1}^{r} [1, p^1, \dots, q^k, \dots, p^r] = \sum_{k=1}^{r} (-1)^{k-1} [1, q^k]\, [p^1, \dots, \widehat{p^k}, \dots, p^r]. \tag{A.14.5}$$

(2) *If the power series $u^1, \dots, u^r$ defined by $u^k := (1+T)^{-1} p^k$ for $k = 1, \dots, r$ are convergent, then for all r the following formulas hold:*

$$r[p^1, \dots, p^r] - 2 \sum_{k=1}^{r} [p^1, \dots, u^k, \dots, p^r] = 0 \tag{A.14.6}$$

and

$$(r+1)[1,p^1,\dots,p^r]-2\sum_{k=1}^{r}[1,p^1,\dots,u^k,\dots,p^r]=[p^1,\dots,p^r]. \tag{A.14.7}$$

(3) *If the power series $v^1,\dots,v^r$ defined by $v^k:=(1-T)^{-1}p^k$ for $k=1,\dots,r$ are convergent, then the following formulas hold:*

If r is even, then

$$r[p^1,\dots,p^r]-2\sum_{k=1}^{r}[p^1,\dots,v^k,\dots,p^r]=0. \tag{A.14.8}$$

If r is odd, then

$$(r+1)[1,p^1,\dots,p^r]-2\sum_{k=1}^{r}[1,p^1,\dots,v^k,\dots,p^r]=-2\sum_{k=1}^{r}(-1)^{k-1}[v^k][p^1,\dots,\widehat{p^k},\dots,p^r]. \tag{A.14.9}$$

Proof. (1) Reordering the arguments, the left hand side of (A.14.5) is equal to the following sum:

$$-\sum_{k=1}^{r}(-1)^{k-1}[q^k,1,p^1,\dots,\widehat{p^k},\dots,p^r].$$

By (A.11.3) for an even number of variables, the kth term in the sum is equal to the following expression:

$$(-1)^k[q^k,1][p^1,\dots,\widehat{p^k},\dots,p^r]-\sum_{j<k}(-1)^{k+j}[q^k,p^j][1,p^1,\dots,\widehat{p^j},\dots,\widehat{p^k},\dots,p^r]$$
$$+\sum_{j>k}(-1)^{k+j-1}[q^k,p^j][1,p^1,\dots,\widehat{p^k},\dots,\widehat{p^j},\dots,p^r].$$

Replacing for $k=1,\dots,r$ the kth term in the former sum by the latter expression and using the symmetry assumption (A.14.1), the sum is easily reduced to the right hand side of (A.14.5). Therefore, (A.14.5) holds.

By (A.11.1), the kth term in the sum on the left hand side of (A.14.3) can be replaced by $[1,T,Tp^1,\dots,Tq^k,\dots,Tp^r]$. Arguing as above, we see that equation (A.14.3) holds.

Finally, by (A.11.1), the equations (A.14.2) and (A.14.4) follow by applying respectively (A.14.3) and (A.14.5) to the sequences $Tp^1,\dots,Tp^r$ and $Tq^1,\dots,Tq^r$.

(2) Let $q^k := p^k - 2u^k = (-1+T)u^k$ for $k=1,\dots,r$. Then, by (A.11), we have that

$$\begin{aligned}[p^k, q^j] &= [(1+T)u^k, (-1+T)u^j] \\ &= -[u^k, u^j] - [Tu^k, u^j] + [u^k, Tu^j] + [Tu^k, Tu^j] \\ &= [u^k, Tu^j] + [u^j, Tu^k].\end{aligned}$$

The last expression is clearly symmetric in k and j. Thus the conditions (A.14.1) are satisfied and consequently, by assertion (1), the equations (A.14.2-5) hold. The left hand side of (A.14.6) is equal to the left hand side of (A.14.2) or (A.14.4). We have clearly $[q^k]=0$. Thus (A.14.6) follows from (A.14.2) or (A.14.4) and, moreover, it follows from (A.14.3) or (A.14.5) that the left hand side of (A.14.7) is equal to the expression

$$[1, p^1, \dots, p^r] + \sum_{k=1}^{r} (-1)^{k-1} [1, q^k]\, [p^1, \dots, \widehat{p^k}, \dots, p^r].$$

We clearly have $[1, q^k] = [q^k] - q^k(0) = p^k(0)$. From this and (A.11.2) it follows that the latter expression is equal to the right hand side of (A.14.7). Thus (A.14.7) holds.

(3) Let $q^k := p^k - 2v^k = (-1-T)v^k$ for $k=1,\dots,r$. As in (2) it follows that the equations (A.14.2-5) hold. Clearly (A.14.8) follows from (A.14.2). To prove (A.14.9), note that $[1, q^k] = q^k(1) - q^k(0) = p^k(0) - 2v^k(1)$. Thus it follows from (A.14.5) that the left hand side of (A.14.9) is equal to the expression

$$[1, p^1, \dots, p^r] + \sum_{k=1}^{r} (-1)^{k-1} p^k(0) [p^1, \dots, \widehat{p^k}, \dots, p^r] - 2 \sum_{k=1}^{r} (-1)^{k-1} v^k(1) [p^1, \dots, \widehat{p^k}, \dots, p^r].$$

The last sum in the above expression is equal to the right hand side of (A.14.9). The sum of the remaining terms is, by (A.11.2), equal to $[p^1, \dots, p^r]$. We clearly have that $[p^k]=0$ and, therefore, it follows from (A.11.3) with odd r that $[p^1, \dots, p^r]=0$. Thus (A.14.9) holds and the Lemma is proved.

PROPOSITION (A.15). *Let $(a_1, \dots, a_r)$ be an ordering of a finite family A of elements of positive order in $\hat{R}$ and denote by $\psi_{i_1,\dots,i_r}$ the integer coefficient to the Schur function $s_{i_1,\dots,i_r}(A)$, in the expansion of $\Pi_{i\leqslant j}(1-(a_i+a_j))^{-1}$. That is,*

$$\prod_{i\leqslant j} \frac{1}{1-(a_i+a_j)} = \sum_I \psi_I s_I(A),$$

where the sum is over all strictly increasing sequences $I=(i_1, \dots, i_r)$ of non-negative *integers. Then:*

(1) *The functions ψ are given by the following explicit formula:*

$$\psi_I = \sum_J \det E(0)^I_J, \tag{A.15.1}$$

where the sum is over all strictly increasing sequences $J=(j_1,\dots,j_r)$ of non-negative integers.

(2) *If p^i denotes the polynomial $(1+T)^i$ for $i=0,1,2\dots$, then we have that*

$$\psi_{i_1,\dots,i_r} = [p^{i_1},\dots,p^{i_r}]. \tag{A.15.2}$$

(3) *The functions ψ for $r=1,2,\dots$ are determined by the functions ψ_i and $\psi_{i,j}$ through the following recurrence formula:*

$$\psi_{i_1,\dots,i_r} = \begin{cases} \sum_{k=1}^{r} (-1)^{k-1}\psi_{i_k}\psi_{i_1,\dots,\widehat{i_k},\dots,i_r}, & \text{if } r \text{ is odd.} \\ \sum_{k=2}^{r} (-1)^{k}\psi_{i_1,i_k}\psi_{i_2,\dots,\widehat{i_k},\dots,i_r}, & \text{if } r \text{ is even.} \end{cases} \tag{A.15.3}$$

(4) *The functions ψ are given by the following explicit formula:*

$$\psi_{i_1,\dots,i_r} = \begin{cases} \mathrm{Pf}(\psi_{i_k,i_l})_{k,l=1,\dots,r} & \text{if } r \text{ is even,} \\ \mathrm{Pf}(\psi_{i_k,i_l})_{k,l=0,\dots,r} & \text{if } r \text{ is odd,} \end{cases} \tag{A.15.4}$$

where Pf *denotes the Pfaffian and where for odd r we interpret ψ_{i_0,i_l} as ψ_{i_l}.*

(5) *The functions ψ in one and two variables are given by*

$$\psi_i = 2^i \quad \text{and} \quad \psi_{i,j} = \varphi(i+j,j)-\varphi(i+j,i), \tag{A.15.5}$$

where $\varphi(k,i) := \Sigma_{l=0}^{i}\binom{k}{l}$. Moreover, the following recurrence formulas hold for all strictly increasing sequences $0\le i_1<\dots<i_r$:

$$r\psi_{i_1,\dots,i_r} - 2\sum_{k=1}^{r}\psi_{i_1,\dots,i_k-1,\dots,i_r} = 0, \quad \text{if} \quad i_1>0. \tag{A.15.6}$$

$$r\psi_{0,i_2,\dots,i_r} - 2\sum_{k=2}^{r}\psi_{0,i_2,\dots,i_k-1,\dots,i_r} = \psi_{i_2,\dots,i_r}. \tag{A.15.7}$$

Proof. Assertion (1) is the content of Formula (A.13.5).

The matrix $E(0)$ is equal to $\langle p^0, p^1, p^2, \ldots\rangle$ (see (A.9)). Therefore assertion (2) follows from (1) and the definitions; (3) and (4) follow from (2) and (A.11).

To prove assertion (5), note that $p^i=(1+T)p^{i-1}$ when $i>0$. Therefore (A.15.6) and (A.15.7) follow from assertion (2) and Lemma (A.14)(2). The formula for ψ_i follows from the definitions: $\psi_i=[p^i]=p^i(1)=(1+1)^i$. From (A.15.6) we see that the function $\psi_{i,j}$ on the left hand side of the second equation in (A.15.5) satisfies the equations:

$$\psi_{i-1,j}+\psi_{i,j-1}=\psi_{i,j} \quad \text{if} \quad 0<i<j.$$

Clearly $\psi_{0,i}=2^i-1$ if $0<i$. The right hand side of the second equation in (A.15.5) is easily seen to satisfy the same equations. Therefore the second equation in (A.15.5) holds, and thus the Proposition is proved.

PROPOSITION (A.16). *Let $(a_1,\ldots,a_r)$ be an ordering of a finite family A of elements of positive order in $\hat{R}$ and denote by $\alpha_{i_1,\ldots,i_r}$ the integer coefficient to the Schur function $s_{i_1,\ldots,i_r}(A)$ in the expansion of $\Pi_{i<j}(1-(a_i+a_j))^{-1}$. That is,*

$$\prod_{i<j}\frac{1}{1-(a_i+a_j)}=\sum_I \alpha_I s_I(A),$$

where the sum is over all strictly increasing sequences $I=(i_1,\ldots,i_r)$ of non-negative integers. Then:

(1) *The functions α are given by the following explicit formula:*

$$\alpha_I=(-1)^{\binom{r}{2}}\sum_J(-1)^{\|J\|}\det E(0)^I_J, \tag{A.16.1}$$

where the sum is over all strictly increasing sequences $J=(j_1,\ldots,j_r)$ of non-negative integers.

(2) *If p^i denotes the polynomial $(1-T)^i$ for $i=0,1,2\ldots$, then we have that*

$$\alpha_{i_1,\ldots,i_r}=(-1)^{\binom{r}{2}}[p^{i_1},\ldots,p^{i_r}]. \tag{A.16.2}$$

(3) *The functions α for $r=1,2,\ldots$ are determined by the value $\alpha_0=1$ and the function $\alpha_{i,j}$ through the following recurrence formula for sequences $0\le i_1<\ldots<i_r$:*

$$\alpha_{i_1,\ldots,i_r}=\begin{cases}0, & \text{if } r \text{ is odd and } 0<i_1.\\ \alpha_{i_2,\ldots,i_r}, & \text{if } r \text{ is odd and } 0=i_1.\\ \displaystyle\sum_{k=2}^{r}(-1)^k\alpha_{i_1,i_k}\alpha_{i_2,\ldots,\widehat{i_k},\ldots,i_r}, & \text{if } r \text{ is even.}\end{cases} \tag{A.16.3}$$

(4) *The functions α for an even number r of arguments are given by the following explicit formula:*

$$\alpha_{i_1,\dots,i_r} = \mathrm{Pf}(\alpha_{i_k,i_l})_{k,l=1,\dots,r}, \tag{A.16.4}$$

where Pf *denotes the Pfaffian.*

(5) *The function α is in two variables given by*

$$\alpha_{i,j} = \binom{i+j-1}{i} - \binom{i+j-1}{j} = \frac{(i+j-1)!}{i!j!}(j-i). \tag{A.16.5}$$

Moreover, for even r the following recurrence formulas hold for all strictly increasing sequences $0 \leq i_1 < \dots < i_r$:

$$r\alpha_{i_1,\dots,i_r} - 2\sum_{k=1}^{r} \alpha_{i_1,\dots,i_k-1,\dots,i_r} = 0, \quad \text{if } i_1 > 0. \tag{A.16.6}$$

$$r\alpha_{0,i_2,\dots,i_r} - 2\sum_{k=2}^{r} \alpha_{0,i_2,\dots,i_k-1,\dots,i_r} = \begin{cases} 2\alpha_{i_3,\dots,i_r}, & \text{if } i_2 = 1. \\ 0, & \text{if } i_2 > 1. \end{cases} \tag{A.16.7}$$

Proof. Assertion (1) is the content of Formula (A.13.6).

If we multiply the jth column in the matrix $E(0)$ for $j=0,1,\dots$ by $(-1)^j$, then the resulting matrix is clearly equal to $\langle p^0, p^1, p^2, \dots \rangle$ (see (A.9)). Therefore assertion (2) follows from the definitions. By (A.11), assertion (3) follows from (2), since $[p^i]=p^i(1)$ is equal to 1 if $i=0$ and equal to 0 if $i>0$ (note that the sign disappears: If r is odd, then $(-1)^{\binom{r}{2}}=(-1)^{\binom{r-1}{2}}$ and if r is even, then the shift in signs passing from r to $r-2$ is $-1=(-1)^{\binom{2}{1}}$). Assertion (4) follows similarly from (2).

To prove assertion (5), note that $p^i=(1-T)p^{i-1}$ when $i>0$. Moreover, with $v^k:=p^{i_k-1}$, we have that $v^k(1)$ is equal to 1 if $i_k=1$ and equal to 0 if $i_k>0$. Therefore (A.16.6) and (A.16.7) follow from Lemma (A.14)(3).

From (A.16.6) we see that the function $\alpha_{i,j}$ on the left hand side of Equation (A.16.5) satisfies the equations:

$$\alpha_{i-1,j} + \alpha_{i,j-1} = \alpha_{i,j} \quad \text{if} \quad 0 < i < j,$$

and clearly $\alpha_{0,i} = -[1, p^i] = 1$ if $0 < i$. The right hand side of Equation (A.16.5) is easily seen to satisfy the same equations. Therefore Equation (A.16.5) holds, and thus the Proposition is proved.

References

[DC-P] DeConcini, C. & Procesi, C. Complete symmetric varieties, in *Invariant Theory, Proceedings, Montecatini,* ed. F. Gheradelli. Springer Lecture Notes, No. 996, 1983, pp. 1–44; II, Preprint, Fall 1983.

[F] Fulton, W. *Intersection Theory*. Ergebnisse der Mathematik und ihrer Grenzgebiete, 3. Folge, Band 2. Springer-Verlag, Berlin, 1984.

[G] Giambelli, G. Z. Il problema della correlazione negli iperspazi. *Mem. Reale Inst. Lombardo,* 19 (1903), 155–194.

[K-L] Kempf, G. & Laksov, D. The determinantal formula of Schubert calculus. *Acta Math.*, 132 (1973), 153–162.

[K] Kleiman, S. Chasles's enumerative theory of conics: A historical introduction, in *Studies in Algebraic Geometry, MAA Studies in Mathematics, vol.* 20, ed. A. Seidenberg, 1980, pp. 117–138.

[K-T] Kleiman, S. & Thorup, A. Intersection theory and enumerative geometry. A decade in review (Section 3), in *Algebraic Geometry: Bowdoin 1985*, ed. S. J. Bloch. Proc. Sympos. Pure Math., vol. 46-2, 1987, pp. 332–338.

[L] Laksov, D. Complete linear maps. *Ark. Mat.*, 26 (1988), 231-263.

[L1] — Notes on the evolution of complete correlations, in *Enumerative and Classical Algebraic Geometry, Proceedings Nice* 1981, ed. Le Barz and Hervier. Birkhäuser, Progress in Math. 24, 1982, pp. 107–132.

[L2] — Complete quadrics and linear maps, in *Algebraic Geometry: Bowdoin 1985*, ed. S. J. Bloch. Proc. Sympos. Pure Math., vol. 46-2, 1987, pp. 371–387.

[Lx] Lascoux, A. Classes de Chern de la variété de drapeaux. *C.R. Acad. Sci. Paris,* 295 (1982), 393–397.

[Lx2] — Fonctions de Schur et grassmanniennes. *C.R. Acad. Sci. Paris,* 281 (1975), 813–815; II, 851–854.

[M] Macdonald, I. G., *Symmetric functions and Hall polynomials*. Oxford Univ. Press, Oxford, 1979.

[P] Pragacz, P. Enumerative geometry of degeneracy loci. *Ann. École Norm. Sup.*, 3 (1988).

[T-K] Thorup, A. & Kleiman, S. Complete bilinear forms, in *Algebraic Geometry: Sundance 1986*, ed. A. Holme and R. Speiser. Lecture Notes in Mathematics, No. 1311, 1988, pp. 253–320.

[S1] Schubert, H. Allgemeine Anzahlfunctionen für Kegelschnitte, Flächen und Räume zweiten Grades in n Dimensionen. *Math. Ann.*, 45 (1894), 153–206.

[S2] — Correlative Verwandtschaft in n Dimensionen. *Jahresber. Deutsch. Math-Verein,* 4 (1894–95), 158–160.

[U] Uzava, T. Equivariant compactifications of algebraic symmetric spaces. Preprint, Fall 1985.

[V1] Vainsencher, I. Schubert calculus for complete quadrics, in *Enumerative and Classical Algebraic Geometry, Proceedings Nice* 1981, ed. Le Barz and Hervier. Birkhäuser, Progress in Math. 24, 1982, pp. 199–235.

[V2] — Complete collineations and blowing up determinantal ideals. *Math. Ann.*, 267 (1984), 417–432.

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