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Polynomial inverse images and polynomial inequalities

by

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Dedicated to Ronald A. DeVore on his 60th birthday

1. Introduction

Let $\|\cdot\|_E$ denote the supremum norm on the set E. Two of the most used inequalities for the derivatives of polynomials are the Bernstein inequality

$$|P_n'(x)| \leqslant \frac{n}{\sqrt{1-x^2}} \|P_n\|_{[-1,1]}, \quad x \in [-1,1],$$

and the Markoff inequality

$$\|P_n'\|_{[-1,1]} \leqslant n^2 \|P_n\|_{[-1,1]},$$

valid for polynomials P_n of degree at most n. In this paper we are primarily interested in what form these inequalities take on several intervals. We shall see that the extension to general sets involves the equilibrium measure of these sets. We shall give the precise form of the Bernstein inequality for arbitrary compacts, and an asymptotically best form of the Markoff inequality for sets consisting of finitely many intervals. Actually, in this case we shall prove different Markoff inequalities one-one-associated with each one of the endpoints of the system of intervals.

The proofs will heavily use sets that are obtained as the inverse images of intervals under (special) polynomial mappings. We shall see that the original Bernstein and

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Markoff inequalities instantly transfer to such polynomial inverse images (at least for some special polynomials). From here we shall get the extension to more general sets and more general polynomials by approximation. The approximation is based on the density of polynomial inverse images in the family of compact sets, and we shall also verify this density property.

We shall start with the just mentioned density property in the next section. Then in §3 we shall consider the extension of the Bernstein inequality. Finally, the extension of the Markoff inequality will be done in the last section.

2. Polynomial inverse images of intervals

Let T be a polynomial of degree $N\geqslant 2$ with real and simple zeros $X_1<X_2<\ldots<X_N$. Let $Y_1<Y_2<\ldots<Y_{N-1}$ be the zeros of T', and assume that $|T(Y_j)|\geqslant 1$, $j=1,\ldots,N-1$ (note that $T(Y_j)$ are the local extrema of T). Then it is elementary (see [6, Lemma 1]) that there exists a unique sequence of closed intervals $E_1,\ldots,E_n$ such that for all $1\leqslant i\leqslant N$ we have $T(E_i)=[-1,1]$, $X_i\in E_i$ and for $1\leqslant i\leqslant N-1$ the set $E_i\cap E_{i+1}$ contains at most one point. We call any such polynomial *admissible*, and we are interested in the inverse image $T^{-1}([-1,1])=\bigcup_{i=1}^N E_i$. We denote by T_i^{-1} that branch of T^{-1} that maps $[-1,1]$ into E_i, and if ν is a measure on $[-1,1]$, we set

$$T^{-1}(\nu)(A):=\nu(T(A)) \quad \text{for } A\subset E_i,\ i=1,\ldots,N.$$

Polynomial inverse images of intervals, i.e. sets of the form $T^{-1}([-1,1])$ with admissible T, have many interesting properties. They are the sets that support weights for which the recurrence coefficients of the associated orthogonal polynomials are periodic, or they are the sets $\Sigma=\bigcup_{i=1}^l [a_i,b_i]$ for which the Pell equation

$$P^2(z)-Q(z)S^2(z)=1 \quad \text{with } Q(x)=\prod_{i=1}^{l}(x-a_i)(x-b_i),$$

that goes back to N. H. Abel, has polynomial solutions P and Q. They are also connected with continued fractions and Toda lattices. For all these and many more interesting results connected with polynomial inverse images see the papers [8]–[11], [13] by F. Peherstorfer and the references there (see also [14]). However, the question if these sets are dense among all sets consisting of finitely many intervals has been open. In this section we prove this density, and in the subsequent sections we shall apply this result to polynomial inequalities on several intervals.

THEOREM 2.1. *Given a system* $\Sigma=\{[a_i,b_i]\}_{i=1}^l$ *of disjoint closed intervals and an* $\varepsilon>0$, *there is another system* $\Sigma'=\{[a_i',b_i']\}_{i=1}^l$ *such that* $\bigcup_{i=1}^l[a_i',b_i']=T^{-1}([-1,1])$ *for some admissible polynomial* T, *and for each* $1\leqslant i\leqslant l$ *we have*

$$|a_i-a_i'|\leqslant\varepsilon,\quad |b_i-b_i'|\leqslant\varepsilon.$$

The theorem immediately implies its strengthened form when we also prescribe if a given a_i' (or b_i') is smaller or bigger than a_i (or b_i). In particular, it is possible to require e.g. that $\Sigma\subset\Sigma'$. The proof also shows that in the theorem we can select $a_i'=a_i$ for all i, and even $b_l'=b_l$. Alternatively we can fix any other b_j.

N. I. Akhiezer [1] described polynomial inverse images of $[-1,1]$ consisting of two intervals via elliptic functions. From here the validity of Theorem 2.1 follows when $l=2$. However, if we use the characterization of polynomial inverse images given in Lemma 2.2, then one can see that the two-interval case ($l=2$ in Theorem 2.1) can be obtained by simply changing continuously one endpoint of one of the intervals. When $l>2$ the situation is more complex.

After having learned of Theorem 2.1, F. Peherstorfer [12] has also given a proof using a completely different approach.

We shall need the following known characterization of polynomial inverse images of intervals (see [2] and also [10]). Since the terminology is somewhat different from those in the papers [2] or [10], for completeness we present a short proof.

LEMMA 2.2. *Let* $\Sigma=\bigcup_{i=1}^l[a_i,b_i]$ *be the disjoint union of* l *intervals. Then* $\Sigma=T^{-1}([-1,1])$ *for some admissible polynomial* T *if and only if each of the numbers* $\mu_\Sigma([a_i,b_i])$ *is rational, where* μ_Σ *denotes the equilibrium measure of the set* Σ.

For the concept of the equilibrium measure and of the logarithmic capacity of a compact set see any text on logarithmic potentials, e.g. [20], [15] or [16]; but actually we shall only use the defining properties (2.1)–(2.2) below.

Proof. If $\Sigma=T^{-1}([-1,1])$ and N is the degree of T, then (see [6, Theorem 11], [15]) $\mu_\Sigma=T^{-1}(\mu_{[-1,1]})/N$. Therefore if $[a_i,b_i]$ consists of l_i subintervals $T_j^{-1}([-1,1])$, then $\mu_\Sigma([a_i,b_i])=l_i/N$, and this proves the necessity of the condition.

Suppose now that each $\mu_\Sigma([a_i,b_i])$ is rational, say $\mu_\Sigma([a_i,b_i])=l_i/N$ for some positive integers N and l_i, $i=1,...,N$. Consider the function

$$H(z)=\exp\left(N\int\log(z-t)\,d\mu_\Sigma(t)-N\log\operatorname{cap}(\Sigma)\right)$$

on the Riemann sphere $\overline{\mathbf{C}}$ cut along Σ, where $\operatorname{cap}(\Sigma)$ denotes the logarithmic capacity of Σ.

We need the following properties of equilibrium measures (see e.g. [15], [20, Theorem III.12] or [16, Theorem I.1.3 and Corollary II.3.4]):

$$\int \log|z-t|\,d\mu_\Sigma(t) = \log \operatorname{cap}(\Sigma), \quad z\in\Sigma, \tag{2.1}$$

$$\int \log|z-t|\,d\mu_\Sigma(t) > \log \operatorname{cap}(\Sigma), \quad z\notin\Sigma, \tag{2.2}$$

and the left-hand side is a continuous function on the whole plane.

Using these and the form of $\mu_\Sigma([a_i,b_i])$ we can infer that $H(z)$ is a single-valued analytic function on $\mathbf{C}\setminus\Sigma$ with modulus 1 on the cut Σ. Thus, it easily follows that

$$G(z) = \frac{1}{2}\left(H(z)+\frac{1}{H(z)}\right)$$

is real-valued on both sides of the cut, and $G(\bar z)=\overline{G(z)}$ is satisfied, hence the reflection principle shows that this function can be continued analytically through each (a_i,b_i). Furthermore, H is bounded away from zero and infinity on compact subsets of the complex plane, so G has a removable singularity at every a_i and b_i. Finally, $H(z)$ has a pole of order N at infinity, therefore the same is true of $G(z)$. In summary, $G(z)$ is an entire function with a pole of order N at infinity, hence $G(z)$ is a polynomial of degree N. Clearly, $G(z)$ is real if $z\in\mathbf{R}$, and since we have $|H(x)|=1$ on Σ, and $|H(x)|>1$ and $H(x)$ is real for all other $x\in\mathbf{R}$, it follows that $-1\leqslant G(x)\leqslant 1$ for $x\in\Sigma$, and $|G(x)|>1$ for $x\in\mathbf{R}\setminus\Sigma$. Thus, $\Sigma=G^{-1}([-1,1])$, and all we have to show is that G is an admissible polynomial. From the construction it is clear that for $x\in\Sigma$

$$G(x) = \cos\left(\operatorname{Arg} N\int \log(x-t)\,d\mu_\Sigma(t)\right) = \cos\left(N\pi\int_x^\infty d\mu_\Sigma(t)\right),$$

from which it is clear that G has N zeros in Σ, and from the same formula the admissibility of G also easily follows. □

Next we discuss the properties of the mapping

$$\{a_1,b_1,\dots,a_l,b_l\}\to\{\mu_\Sigma([a_1,b_1]),\dots,\mu_\Sigma([a_l,b_l])\},$$

where $\Sigma=\bigcup_1^l[a_i,b_i]$, and this is a disjoint union. This is a mapping of a subset of $\mathbf{R}^{2l}$ to $\mathbf{R}^l$, so it is singular. It is singular even if we fix, say, the left endpoints (to obtain a mapping from $\mathbf{R}^l$ to $\mathbf{R}^l$), namely the image set is on a hyperplane, for the sum of the coordinates in the image set is 1. Now we show that if we also fix b_l, thereby obtaining a mapping from $\mathbf{R}^{l-1}$ into the hyperplane mentioned before, then this mapping is nonsingular.

Thus, let

$$\Sigma(x_1,\dots,x_{l-1})=[a_1,b_1+x_1]\cup[a_2,b_2+x_2]\cup\dots\cup[a_{l-1},b_{l-1}+x_{l-1}]\cup[a_l,b_l],$$

and we consider this set for $(x_1,\dots,x_{l-1})$ lying in a neighborhood U of the origin in $\mathbf{R}^{l-1}$ which is so small that in that neighborhood we have $a_j<b_j+x_j<a_{j+1}$ for all $1\leqslant j<l$. Let

$$M(x_1,\dots,x_{l-1})=(\mu_{\Sigma(x_1,\dots,x_{l-1})}([a_1,b_1+x_1]),\dots,\mu_{\Sigma(x_1,\dots,x_{l-1})}([a_{l-1},b_{l-1}+x_{l-1}])). \quad (2.3)$$

Then $M\colon U\to\mathbf{R}^{l-1}$, and we are going to show that M is a nonsingular C^∞-mapping, hence in particular it is an open mapping. Thus, we can find arbitrarily close to the origin points $x_1,\dots,x_{l-1}$ so that all the numbers

$$\mu_{\Sigma(x_1,\dots,x_{l-1})}([a_j,b_j+x_j]),\quad 1\leqslant j<l,$$

are rational. Then, however,

$$\mu_{\Sigma(x_1,\dots,x_{l-1})}([a_l,b_l])$$

is also rational, for it complements the sum of the preceding numbers to 1. These facts together with Lemma 2.2 prove Theorem 2.1.

We shall actually show that the Jacobian of $M(x_1,\dots,x_{l-1})$ is diagonally dominant—this is enough to conclude the nonsingularity of M. It is known (see e.g. [17, Lemma 4.4.1], cf. also [21]) that μ_Σ is of the form

$$\frac{d\mu_\Sigma(t)}{dt}=\frac{\prod_{j=1}^{l-1}|t-\tau_j|}{\pi\prod_{j=1}^{l}|(t-a_j)(t-b_j)|^{1/2}} \quad (2.4)$$

for $t\in\Sigma$, where the numbers τ_j lie in the intervals (b_j,a_{j+1}). We shall need more precise information on where these numbers lie, therefore we derive again this formula in a way that also supplies this additional information. Actually, we need the same form and information on the balayage measure of a Dirac mass δ_a, $a\in\mathbf{R}\setminus\Sigma$, onto Σ. This measure is defined as the unique measure ν on Σ that has total mass 1 and for which

$$\int\log|x-t|\,d\nu(t)=\log|x-a|+\text{const},\quad x\in\Sigma, \quad (2.5)$$

(for the existence and properties see e.g. [16, §II.4] or [7]). Since the balayage of δ_∞ onto Σ is μ_Σ, and since a fractional linear transformation preserves the balayage measure (see below), these two questions are basically the same. We set $\overline{\mathbf{R}}=\mathbf{R}\cup\{\infty\}$, and identify $-\infty$ with ∞.

LEMMA 2.3. *Let* $\Sigma=\bigcup_{i=1}^{l}[a_i,b_i]$, $(b_l,a_{l+1})=(b_l,\infty]\cup(-\infty,a_1)$, *and for* $a\in\overline{\mathbf{R}}\setminus\Sigma$ *let* $i(a)$ *be that index* $1\leqslant i\leqslant l$ *for which* $a\in(b_i,a_{i+1})$. *The density of the balayage measure* $\hat{\delta}_a$ *of the Dirac delta mass* δ_a *onto* Σ *is given by*

$$\frac{1}{\pi}\frac{\prod_1^l|(a-a_j)(a-b_j)|^{1/2}}{\prod_1^l|(t-a_j)(t-b_j)|^{1/2}}\frac{|P_{l-1}(t)|}{|P_{l-1}(a)|}\frac{1}{|t-a|},\quad t\in\Sigma, \tag{2.6}$$

where the polynomial

$$P_{l-1}(t)=\prod_{\substack{1\leqslant i\leqslant l\\ i\neq i(a)}}(t-\tau_i)$$

satisfies for all $1\leqslant i\leqslant l$, $i\neq i(a)$ *the condition*

$$\int_{b_i}^{a_{i+1}}\frac{\prod_1^l|(a-a_j)(a-b_j)|^{1/2}}{\prod_1^l|(t-a_j)(t-b_j)|^{1/2}}\frac{P_{l-1}(t)}{P_{l-1}(a)}\frac{1}{|t-a|}\,dt=0. \tag{2.7}$$

This system of equations uniquely determines each τ_i, $i\neq i(a)$, *and we have* $\tau_i\in(b_i,a_{i+1})$.

In particular, for $a\to\infty$ *we get that the equilibrium measure* μ_Σ *is of the form* (2.4) *with* $\tau_1,\dots,\tau_{l-1}$ *satisfying for all* $1\leqslant i\leqslant l-1$

$$\int_{b_i}^{a_{i+1}}\frac{\prod_1^{l-1}(t-\tau_j)}{\prod_1^l|(t-a_j)(t-b_j)|^{1/2}}\,dt=0. \tag{2.8}$$

Note that the first numerator and second denominator in (2.7) are constant, so they could be omitted from the formulae. However, it may well happen that τ_l equals ∞, in which case the factor $(t-\tau_l)/(a-\tau_l)$ should be omitted from all formulae. This difficulty can be overcome by applying a fractional linear transformation as will be done in the proof. The same remark applies if we want to speak of the balayage of δ_∞, in which case all the terms that contain a should be dropped.

(2.7) gives a linear $((l-1)\times(l-1))$-system for the $l-1$ free coefficients of P_{l-1}. This system has a nonsingular matrix (see below), and therefore the solution is unique. It is clear from (2.7) that P_{l-1} must have at least one zero on each (b_i,a_{i+1}), $1\leqslant i\leqslant l$, $i\neq i(a)$, and then it cannot have more than one, so it has exactly one zero in each of these intervals. It also follows that the coefficients of P_{l-1} are C^∞-functions of the endpoints a_j,b_j, which, in view of the fact that the zeros of P_{l-1} are separated, implies that the zeros τ_i are also C^∞-functions of the endpoints a_j,b_j (C^∞ at ∞ should be understood in a proper sense, but we can always speak of normal C^∞ after applying a fractional linear transformation as is done below). Thus, altogether, the density of μ_Σ is a C^∞-function of the endpoints a_j,b_j. By making the substitution $t=a_i+(b_i-a_i)u$ we can see that then each integral

$$\int_{a_i}^{b_i}d\mu_\Sigma(t)=\mu_\Sigma([a_i,b_i])$$

is also a C^∞-function of the variables a_j, b_j, and this verifies that the mapping M from (2.3) is a C^∞-function on U.

Proof of Lemma 2.3. Consider the set Σ^* obtained from Σ via a mapping $x^*=1/(x-\alpha)$, and for a measure ν defined on Σ let ν^* be the image of ν under the same mapping $x^*=1/(x-\alpha)$, i.e. if E^* is the image of a set E, then we set $\nu^*(E^*)=\nu(E)$. Since the balayage measure $\hat{\delta}_a$ of δ_a onto Σ is characterized by the facts (see [16, §II.4 and Theorem II.4.6]) that it is supported on Σ, it has total mass 1 and its logarithmic potential

$$\int \log|z-t|\,\hat{\delta}_a(t)$$

equals a constant plus $\log|z-a|$ on Σ, it is easy to see that ν is the balayage of δ_a onto Σ if and only if ν^* is the balayage of δ_{a^*} onto Σ^*, where $a^*=1/(a-\alpha)$. It is also straightforward to see that this same transformation also preserves the validity of Lemma 2.3, i.e. the lemma is true for Σ and a if and only if it is true for Σ^* and a^*. However, by an appropriate choice of α we can achieve that a^* is bigger than any of the endpoints of Σ^*, hence we may assume from the outset that $a>b_l$. In this case $i(a)=l$.

Consider a polynomial

$$P_{l-1}(x)=x^{l-1}+c_{l-2}x^{l-2}+\ldots+c_0$$

and the system of equations

$$\int_{b_i}^{a_{i+1}} \frac{P_{l-1}(t)}{\prod_1^l |(t-a_j)(t-b_j)|^{1/2}}\,\frac{1}{t-a}\,dt=0, \tag{2.9}$$

$1\leqslant i\leqslant l-1$. Since the leading coefficient of P_{l-1} is fixed to be 1, this is a system of linear equations with matrix

$$\left(\int_{b_i}^{a_{i+1}} \frac{t^{j-1}}{\prod_1^l |(t-a_j)(t-b_j)|^{1/2}}\,\frac{1}{t-a}\,dt\right)_{1\leqslant i,j\leqslant l-1}.$$

If this matrix was singular, then by taking an appropriate linear combination of the rows we would obtain a nonzero polynomial of degree at most $l-2$ that was orthogonal to the denominator in the previous formula on every interval (b_i, a_{i+1}), $1\leqslant i\leqslant l-1$. However, this would mean that this polynomial has at least one zero on each of these intervals, which is not possible. Thus, the above matrix is nonsingular, and the system (2.9) has a unique solution. Clearly, the solution polynomial P_{l-1} has one and only one zero on every interval (b_i, a_{i+1}), $1\leqslant i\leqslant l-1$, and hence we can write with some $\tau_i\in(b_i, a_{i+1})$

$$P_{l-1}(x)=\prod_{i=1}^{l-1}(x-\tau_i).$$

Consider the function

$$H(z)=\frac{\sqrt{\prod_{j=1}^{l}(a-a_j)(a-b_j)}}{\sqrt{\prod_{j=1}^{l}(z-a_j)(z-b_j)}}\cdot\frac{P_{l-1}(z)}{P_{l-1}(a)}$$

on the Riemann sphere cut along Σ, where we take that branch of the square root which is positive for positive z. Let $\alpha,\beta\in[a_i,b_i]$. There is a branch of $\log((z-\beta)/(z-\alpha))$ that is analytic outside $[a_i,b_i]$. Thus,

$$\frac{1}{2\pi i}\oint_{\Sigma}\frac{H(z)}{z-a}\log\frac{z-\beta}{z-\alpha}\,dz=H(a)\log\frac{a-\beta}{a-\alpha}=\log\frac{a-\beta}{a-\alpha}.$$

Take here real parts. Since $H(z)=\mp|H(x)|i$ for $z=x\pm i0$, $x\in\Sigma$, we get

$$-\frac{1}{\pi}\int_{\Sigma}\frac{|H(x)|}{x-a}\log\frac{|x-\beta|}{|x-\alpha|}\,dx=\log\frac{|a-\beta|}{|a-\alpha|},$$

and thus the function

$$V(\alpha)=-\log|a-\alpha|+\frac{1}{\pi}\int_{\Sigma}\frac{|H(x)|}{|x-a|}\log|x-\alpha|\,dx$$

is constant on each interval $[a_i,b_i]$. Similarly, if $\alpha=b_i$, $\beta=a_{i+1}$, then by cutting the sphere along $\Sigma\cup[b_i,a_{i+1}]$ we get as before

$$\frac{1}{2\pi i}\oint_{\Sigma\cup[b_i,a_{i+1}]}\frac{H(z)}{z-a}\log\frac{z-\beta}{z-\alpha}\,dz=H(a)\log\frac{a-\beta}{a-\alpha}=\log\frac{a-\beta}{a-\alpha}.$$

Take again real parts, and notice that on (b_i,a_{i+1}) the function $H(z)$ is real, and for $z=x\pm i0$, $x\in(b_i,a_{i+1})$ we have

$$\log\frac{z-\beta}{z-\alpha}=\log\frac{|x-\beta|}{|x-\alpha|}\pm i\pi.$$

From these we obtain

$$\frac{1}{\pi}\int_{\Sigma}\frac{|H(x)|}{|x-a|}\log\frac{|x-\beta|}{|x-\alpha|}\,dx-\log\frac{|a-\beta|}{|a-\alpha|}+\int_{b_i}^{a_{i+1}}\frac{H(x)}{x-a}\,dx=0.$$

Here the last term is zero by the choice of the polynomial P_{l-1}, and we can conclude that $V(\alpha)$ is constant on all of Σ. Furthermore,

$$\frac{1}{2\pi i}\oint_{\Sigma}\frac{H(z)}{z-a}\,dz=H(a)=1,$$

so by taking real parts we obtain

$$\frac{1}{\pi}\int_{\Sigma}\frac{|H(x)|}{|x-a|}\,dx=1.$$

Thus, the measure given by the density function

$$\frac{1}{\pi}\frac{|H(x)|}{|x-a|}$$

is the balayage measure $\hat{\delta}_a$, and the lemma is verified. □

When we form the balayage onto Σ of an arbitrary measure ν with support on $\mathbf{R}$ and with $\nu(\Sigma)=0$, then the density is obtained by integrating the density in (2.6) with respect to $d\nu(a)$.

It follows from the lemma that as $a\searrow b_k$, the numbers $\tau_i=\tau_i(a)$, $i\neq k$, converge to some τ_i' that supply the solution of (2.7) for $a=b_k$, and again $\tau_i'\in(b_i,a_{i+1})$. In this case

$$\frac{\prod_1^l|(a-a_j)(a-b_j)|^{1/2}}{|P_{l-1}(a)|}=(1+o(1))C_k\sqrt{a-b_k} \tag{2.10}$$

with some positive C_k, where $o(1)$ denotes a quantity that tends to zero as $a\searrow b_k$.

Using these facts we can calculate the Jacobian of the mapping

$$M(x_1,\dots,x_{l-1}).$$

Since each of the sets $\Sigma(x_1,\dots,x_{l-1})$ is just like Σ, it is enough to do that at the origin. First let $1\leqslant k\leqslant l-1$, $1\leqslant i\leqslant l$ including $i=l$, but first let $i\neq k$, and we calculate the partial derivative

$$\frac{\partial\mu_{\Sigma(x_1,\dots,x_{l-1})}([a_i,b_i+x_i])}{\partial x_k}(0,\dots,0)$$

(we set $x_l=0$). Since $i\neq k$, this is the limit of the quotient

$$\frac{\mu_{\Sigma(0,\dots,0,x_k,0,\dots,0)}([a_i,b_i])-\mu_\Sigma([a_i,b_i])}{x_k}$$

as x_k tends to 0 through positive values. For positive x_k we have

$$\Sigma\subset\Sigma(0,\dots,0,x_k,0,\dots,0),$$

so μ_Σ is the balayage of $\mu_{\Sigma(0,\dots,0,x_k,0,\dots,0)}$ onto Σ (see [16, Theorem IV.1.6 (e)]). Therefore, the numerator of the preceding ratio is nothing else than the measure of $[a_i,b_i]$ with respect to the measure that we obtain by taking the balayage of

$$\mu_{\Sigma(0,\dots,0,x_k,0,\dots,0)}|_{(b_k,b_k+x_k)}$$

onto Σ. Thus, in view of formula (2.6) and (2.10) the preceding ratio equals

$$(1+o(1))\frac{C_k}{x_k}\int_{a_i}^{b_i}\int_{b_k}^{b_k+x_k}\frac{\sqrt{a-b_k}\,\prod_{1\leqslant j\leqslant l,j\neq k}|t-\tau_j'|}{\pi\prod_1^l|(t-a_j)(t-b_j)|^{1/2}}\frac{1}{|t-a|}\,d\mu_{\Sigma(0,\dots,0,x_k,0,\dots,0)}(a)\,dt.$$

Here for $b_k<a<b_k+x_k$ we have by (2.4)

$$d\mu_{\Sigma(0,\dots,0,x_k,0,\dots,0)}(a)=(1+o(1))\frac{D_k}{\sqrt{b_k+x_k-a}}\,da$$

with some positive constant D_k. Therefore, the previous double integral can be written with some positive constant $E_{k,i}$ as

$$\begin{aligned}(1+o(1))E_{k,i}\int_{b_k}^{b_k+x_k}\frac{\sqrt{a-b_k}}{\sqrt{b_k+x_k-a}}\,da&=x_k(1+o(1))E_{k,i}\int_0^1\sqrt{\frac{u}{1-u}}\,du\\&=x_k(1+o(1))E_{k,i}\cdot\tfrac{1}{2}\pi.\end{aligned}$$

Thus, for $k\neq i$,

$$\frac{\partial\mu_{\Sigma(x_1,\dots,x_{l-1})}([a_i,b_i+x_i])}{\partial x_k}(0,\dots,0)=\tfrac{1}{2}\pi E_{k,i}.$$

However,

$$\sum_{1\leqslant i\leqslant l}\mu_{\Sigma(x_1,\dots,x_{l-1})}([a_i,b_i+x_i])=1,$$

and therefore for the case $k=i$ we obtain

$$\frac{\partial\mu_{\Sigma(x_1,\dots,x_{l-1})}([a_k,b_k+x_k])}{\partial x_k}(0,\dots,0)=-\sum_{i\neq k}\tfrac{1}{2}\pi E_{k,i}.$$

This already proves that the Jacobian of $V(x_1,\dots,x_{l-1})$ is diagonally dominant, since the preceding sum is the kth diagonal element, but the sum of the kth row without this diagonal element is

$$-\sum_{\substack{1\leqslant i\leqslant l-1\\ i\neq k}}\tfrac{1}{2}\pi E_{k,i},$$

which is smaller in absolute value than the absolute value of the kth diagonal element by $\frac{1}{2}\pi E_{k,l}$, and this is a positive number. □

3. The Bernstein inequality on several intervals

Let $\|\cdot\|_E$ denote the supremum norm on the set E. The Bernstein inequality

$$|P_n'(x)| \leqslant \frac{n}{\sqrt{1-x^2}} \|P_n\|_{[-1,1]}, \quad x\in[-1,1], \tag{3.1}$$

and the Markoff inequality

$$\|P_n'\|_{[-1,1]} \leqslant n^2 \|P_n\|_{[-1,1]} \tag{3.2}$$

for polynomials P_n of degree at most n play a fundamental role in several branches of mathematics (see e.g. [19] and [5]).

In the second part of the paper we are going to deal with the analogues of the aforementioned classical inequalities for sets E consisting of several intervals. First we show that the Bernstein inequality (actually, the sharper Szegő inequality) holds with replacing $n/\sqrt{1-x^2}$ by π times the equilibrium measure of the set E, namely we prove the following generalization of (3.1).

THEOREM 3.1. *Let E be a set consisting of a finite number of intervals, and let ω_E be the density of the equilibrium measure of E. Then for any n and any polynomial P_n of degree at most n we have*

$$\left(\frac{|P_n'(x)|}{\pi\omega_E(x)}\right)^2 + n^2 P_n^2(x) \leqslant n^2 \|P_n\|_E^2, \quad x\in E. \tag{3.3}$$

As a corollary we obtain

$$|P_n'(x)| \leqslant \pi\omega_E(x) n \|P_n\|_E, \quad x\in E, \tag{3.4}$$

which is the extension of (3.1) to several intervals.

If $E=[-1,1]$, then (3.3) becomes

$$\left(\sqrt{1-x^2}\,|P_n'(x)|\right)^2 + n^2 P_n^2(x) \leqslant n^2 \|P_n\|_{[-1,1]}^2, \quad x\in[-1,1],$$

which is a well-known generalization of the Bernstein inequality due to Szegő [18].

For possible later reference let us state here the following corollary, which is the same theorem but for general compact sets.

THEOREM 3.2. *Let $E\subset\mathbf{R}$ be a compact set with nonempty interior, and let ω_E be the density of the equilibrium measure of E on that interior. Then for any n and any polynomial P_n of degree at most n we have*

$$\left(\frac{|P_n'(x)|}{\pi\omega_E(x)}\right)^2 + n^2 P_n^2(x) \leqslant n^2 \|P_n\|_E^2, \quad x\in \mathrm{Int}(E). \tag{3.5}$$

This immediately follows from Theorem 3.1. In fact, we can approximate any compact set by sets E^* consisting of finitely many intervals in such a manner that in the interior of E the densities $\omega_{E^*}(x)$ converge to $\omega_E(x)$. Since Theorem 3.1 holds with universal constant independent of the sets E^*, its validity will be preserved from the sets E^* to the set E. We shall not give more details, for they are fairly standard. Of course, the equilibrium measure μ_E is absolutely continuous in the interior of E, so ω_E is meaningful there.

As an immediate corollary we obtain (3.4) for an arbitrary compact set E and for $x \in \mathrm{Int}(E)$. Next we show that (3.4) is sharp for any compact E:

THEOREM 3.3. *Let $\varepsilon>0$. For each x lying in the interior of E, and for every large n, there is a polynomial $P_n \not\equiv 0$ of degree at most n such that*

$$|P_n'(x)| \geqslant (1-\varepsilon)\pi\omega_E(x)n\|P_n\|_E. \tag{3.6}$$

Furtheremore, if E is the polynomial inverse image of an interval, then the equality sign holds in (3.4) *for some appropriate polynomials for infinitely many degrees and for a set of points that becomes dense as the degree tends to infinity.*

Proof of Theorem 3.1. The outline of the proof is the following: First we show the validity of (3.4) for a special class of sets and for a special class of polynomials. Then, by approximation, we obtain (3.4) with an additional $(1+o(1))$-factor on the right, where $o(1)$ may depend on x and the degree n, but tends to zero as n tends to infinity. Finally, from here we obtain the full (3.3) by a transformation.

First we prove (3.4) for a family of polynomials on sets that are polynomial inverse images of intervals. As in the first part of this paper let T_N be a polynomial of degree $N \geqslant 2$ with real and simple zeros $X_1<X_2<...<X_N$, let $Y_1<Y_2<...<Y_{N-1}$ be the zeros of T_N', and assume that $|T_N(Y_j)| \geqslant 1$, $j=1,...,N-1$. We denote by $E_1,...,E_N$ those closed intervals for which $T_N(E_i)=[-1,1]$, $X_i \in E_i$ and for $1 \leqslant i \leqslant N-1$ the set $E_i \cap E_{i+1}$ contains at most one point. In the first part of the proof we assume that

$$E = \bigcup_{i=1}^{N} E_i = T_N^{-1}([-1,1]),$$

i.e. E is the polynomial inverse image of $[-1,1]$ via T_N. Note that here E_i are not necessarily the intervals that E consists of, because several E_i may combine to form the subintervals of E.

We denote by $T_{N,i}^{-1}$ that branch of T_N^{-1} that maps $[-1,1]$ into E_i, and if ν_0 is a measure on $[-1,1]$, we set

$$\nu(A) := \frac{1}{N}\nu_0(T_N(A)) \quad \text{for } A \subset E_i. \tag{3.7}$$

It is known that if ν_0 is the equilibrium measure of $[-1,1]$, then the measure ν obtained this way is the equilibrium measure for E, see [6, Theorem 11]. Thus, in the present case,

$$\omega_E(x)=\frac{|T_N'(x)|}{\pi N\sqrt{1-T_N^2(x)}},\quad x\in E. \tag{3.8}$$

Suppose now that $P_n(x)$ is of the form $R_m(T_N(x))$ with some polynomial R_m of degree m. Then $n=mN$, and by applying the Bernstein inequality to R_m, we obtain

$$|P_n'(x)|=|R_m'(T_N(x))|\cdot|T_N'(x)|\leqslant\frac{m}{\sqrt{1-T_N^2(x)}}\|R_m\|_{[-1,1]}|T_N'(x)|,$$

which, in view of (3.8) and $\|R_m\|_{[-1,1]}=\|P_n\|_E$, is (3.4).

Next let E be an arbitrary set consisting of a finite number of intervals. We show that for any x_0 lying in the interior of E we have

$$|P_n'(x_0)|\leqslant(1+o(1))\pi\omega_E(x_0)n\|P_n\|_E, \tag{3.9}$$

where $o(1)$ denotes a term that is independent of P_n, and tends to zero as $n\to\infty$.

Let $\varepsilon>0$ be arbitrary. Then by Theorem 2.1 there are polynomial inverse image sets E^* consisting of the same number of intervals as E such that the corresponding endpoints of the subintervals of E and E^* are as close as we wish. Therefore, we can choose $E^*\subset\operatorname{Int}(E)$ so that

$$\omega_{E^*}(x_0)\leqslant(1+\varepsilon)\omega_E(x_0) \tag{3.10}$$

is satisfied. Let, as before, $E^*(x)=T_N^{-1}([-1,1])$, and let $E_i^*=T_{N,i}^{-1}([-1,1])$, $i=1,...,N$, be the N inverse image intervals of $[-1,1]$ under the N branches of T_N^{-1}. Since any translate of E^* is the polynomial inverse image of $[-1,1]$ via a translate of T_N, we may assume without loss of generality, that x_0 is not an endpoint of any of the intervals E_i^*, i.e. x_0 is lying in the interior of $E_{i_0}^*$ for some i_0.

Let P_n be an arbitrary polynomial of degree n, and consider the polynomial

$$P_n^*(x)=(1-\alpha(x-x_0)^2)^{[\sqrt{n}]}P_n(x), \tag{3.11}$$

where $\alpha>0$ is fixed so that $1-\alpha(x-x_0)^2>0$ on E. Clearly, P_n^* has degree at most $n+2\sqrt{n}$, $\|P_n^*\|_E\leqslant\|P_n\|_E$, $P_n^*(x_0)=P_n(x_0)$, $(P_n^*)'(x_0)=P_n'(x_0)$, and there is a $0<\beta<1$ such that

$$|P_n^*(x)|\leqslant\beta^{\sqrt{n}}\|P_n\|_E,\quad|(P_n^*)'(x)|\leqslant\beta^{\sqrt{n}}\|P_n\|_E \tag{3.12}$$

uniformly for $x\in E\setminus E_{i_0}$. In fact, for the last relations just observe that the factor $1-\alpha(x-x_0)^2$ is nonnegative and stricly less than one on $E\setminus E_{i_0}$. For $x\in E^*$ consider the sum

$$S(x)=\sum_{i=1}^N P_n^*(T_{N,i}^{-1}(T_N(x))). \tag{3.13}$$

We claim that this is a polynomial of degree at most $(n+2\sqrt{n})/N$ of $T_N(x)$, i.e. $S(x)=S_n(T_N(x))$ for some polynomial S_n of degree at most $(n+2\sqrt{n})/N$. To this end let $x_i=T_{N,i}^{-1}(T_N(x))$, $i=1,...,N$. Then

$$S(x)=S(x_1,...,x_N)=\sum_{i=1}^{N}P_n^*(x_i)$$

is a symmetric polynomial of the variables $x_1,...,x_N$, and hence it is a polynomial of the elementary symmetric polynomials

$$S_j(x_1,...,x_N)=\sum_{1\leqslant k_1<k_2<...<k_j\leqslant N}x_{k_1}x_{k_2}...x_{k_j},\quad 1\leqslant j\leqslant N.$$

However, $x_1,x_2,...,x_N$ are the zeros in t of the polynomial equation $T_N(t)=T_N(x)$, and so if $T_N(x)=d_Nx^N+...+d_0$, then it follows that

$$S_j(x_1,...,x_k)=(-1)^j\frac{d_{N-j}}{d_N}$$

if $1\leqslant j<N$, while

$$S_N(x_1,...,x_k)=(-1)^N\frac{d_0-T_N(x)}{d_N},$$

from which the claim that S is a polynomial of $T_N(x)$ follows. On comparing the degree of the homogeneous parts of these polynomials, we can see that the degree of

$$S_n(u):=S(T_{N,i_0}^{-1}(u))$$

is at most $\deg(P_n^*)/N\leqslant(n+2\sqrt{n})/N$ in u.

From the properties (3.12) and (3.13) it is also clear that

$$\|S\|_{E^*}\leqslant\left(1+N\beta^{\sqrt{n}}\right)\|P_n\|_E,\quad |S'(x_0)-P_n'(x_0)|\leqslant N\beta^{\sqrt{n}}\|P_n\|_E.$$

Now S is already of the type for which we have verified (3.4) above, so if we apply to S the inequality (3.4) at $x=x_0$, and if we use (3.10) and the preceding estimates, we obtain (3.9):

$$\begin{aligned}|P_n'(x_0)|&\leqslant|S'(x_0)|+N\beta^{\sqrt{n}}\|P_n\|_E\\&\leqslant\left(n+2\sqrt{n}\right)\pi\omega_{E^*}(x_0)\|S\|_{E^*}+N\beta^{\sqrt{n}}\|P_n\|_E\\&\leqslant\left(n+2\sqrt{n}\right)(1+\varepsilon)\pi\omega_E(x_0)\left(1+N\beta^{\sqrt{n}}\right)\|P_n\|_E+N\beta^{\sqrt{n}}\|P_n\|_E\\&=(1+o(1))n\pi\omega_E(x_0)\|P_n\|_E,\end{aligned}$$

since $\varepsilon>0$ was arbitrary.

Finally, we verify (3.3). Let P_n be any polynomial, and x_0 be any point in the interior of E. Without loss of generality we may assume that $\|P_n\|_E=1$. Let $\mathcal{T}_m(z)=\cos(m\arccos z)$ be the classical Chebyshev polynomials, and for some $0<\alpha_m<1$ and $0\leqslant\varepsilon_m<1-\alpha_m$ consider the polynomials

$$R_{mn}(x)=\mathcal{T}_m(\alpha_m P_n(x)+\varepsilon_m),$$

where $\alpha_m<1$ and $0\leqslant\varepsilon_m<1-\alpha_m$ are chosen so that $\alpha_m P_n(x_0)+\varepsilon_m$ is one of the zeros of $\mathcal{T}_m$. Since the distance of neighbouring zeros of $\mathcal{T}_m$ is smaller than $10/m$, we can do this with $\alpha_m=1-10/m$ and with some $0\leqslant\varepsilon_m<10/m$, and then $\alpha_m\to1$ and $\varepsilon_m\to0$ as $m\to\infty$. Now apply (3.9) to R_{mn}. It follows that

$$|R'_{mn}(x_0)|\leqslant(1+o(1))\pi\omega_E(x_0)mn\|R_{mn}\|_E,$$

where the term $o(1)$ tends to zero as $m\to\infty$. Here, on the right, $\|R_{mn}\|_E\leqslant1$, and on the left we have

$$|R'_{mn}(x_0)|=|\mathcal{T}'_m(\alpha_m P_n(x_0)+\varepsilon_m)|\cdot|P'_n(x_0)|\alpha_m.$$

Since at the zeros z of $\mathcal{T}_m$ we have $\mathcal{T}'_m(z)=m/\sqrt{1-z^2}$, it follows that

$$\frac{m}{\sqrt{1-(\alpha_m P_n(x_0)+\varepsilon_m)^2}}|P'_n(x_0)|\alpha_m\leqslant(1+o(1))\pi\omega_E(x_0)mn,$$

where the term $o(1)$ tends to zero as $m\to\infty$. On dividing here by m and letting m tend to infinity we obtain

$$\frac{|P'_n(x_0)|}{\sqrt{1-P_n^2(x_0)}}\leqslant\pi\omega_E(x_0)n,$$

and this is the inequality (3.3) at the point x_0 because in our case $\|P_n\|_E=1$. □

Proof of Theorem 3.3. If $E=T_N^{-1}([-1,1])$ is the polynomial inverse image of $[-1,1]$, then, in view of (3.8), we have equality in (3.4) for $P_N=T_N$ at those points x for which $T_N(x)=0$. We can repeat this argument with each $\mathcal{T}_m(T_N)$ instead of T_N, where $\mathcal{T}_m$ are the classical Chebyshev polynomials, for obviously $\mathcal{T}_m(T_N)$ also have the properties set forth in the preceding proof, namely that $E=(\mathcal{T}_m(T_N))^{-1}([-1,1])$ (cf. [10, Remark 2.2 (c)]), and as $m\to\infty$, the zeros of $\mathcal{T}_m(T_N)$ become denser in E. This proves the last statement in Theorem 3.3.

If E and $\varepsilon>0$ are arbitrary and $x_0\in\operatorname{Int}E$, then select a polynomial inverse image set $E^*=T_N^{-1}([-1,1])$ such that $E\subset E^*$ and $\omega_{E^*}(x)\geqslant(1-\frac{1}{2}\varepsilon)\omega_E(x)$ are satisfied. Consider the polynomials

$$R_{mN}(x)=\mathcal{T}_m(\alpha_m T_N(x)+\varepsilon_m),$$

where $\alpha_m<1$ and $0\leqslant\varepsilon_m<1-\alpha_m$ are chosen so that $\alpha_m T_N(x_0)+\varepsilon_m$ is one of the zeros of $\mathcal{T}_m$. As in the previous proof we can assume that $\alpha_m\to1$, $\varepsilon_m\to0$ as $m\to\infty$, and then from

$$|R'_{mN}(x_0)|=|\mathcal{T}'_m(\alpha_m T_N(x_0)+\varepsilon_m)|\cdot|T'_N(x_0)|\alpha_m,$$

from (3.8) and from the fact that at the zeros z of $\mathcal{T}_m$ we have $\mathcal{T}'_m(z)=m/\sqrt{1-z^2}$, it follows that

$$|R'_{mN}(x_0)|=\frac{m}{\sqrt{1-(\alpha_m T_N(x_0)+\varepsilon_m)^2}}|T'_N(x_0)|\alpha_m\geqslant(1-o(1))\pi\omega_{E^*}(x_0)mN,$$

where the term $o(1)$ tends to zero as $m\to\infty$. Now taking into account $\omega_{E^*}(x)\geqslant(1-2/\varepsilon)\omega_E(x)$ and the inequality $\|R_{mN}\|_E\leqslant\|R_{mN}\|_{E^*}\leqslant1$, we can conclude the validity of (3.6). □

4. The Markoff inequality on several intervals

Next we consider the extension of the Markoff inequality (3.2) to sets consisting of several intervals. If we apply the original form of the Markoff inequality on each subinterval of E, then we obtain that

$$\|P'_n\|_E\leqslant Cn^2\|P_n\|_E \tag{4.1}$$

with some constant C, but this produces only a rough estimate on the best possible constant. Our aim is to determine the asymptotically best constant in the inequality (4.1).

Actually, we shall be interested in several Markoff inequalities, one-one around each endpoint of E. In fact, by Theorem 3.1 we have

$$|P'_n(x)|\leqslant C_K n\|P_n\|_E,\quad x\in K, \tag{4.2}$$

uniformly on compact subsets K of the interior of E, and this shows that inside E the Bernstein–Markoff factor is $O(n)$. However, around the endpoints of E this factor is of the order $O(n^2)$, and the best constant may depend on which endpoint we are considering.

We shall prove that the analogue of (3.2) around any endpoint of the set holds with an asymptotically best constant that depends on the endpoint in question. We shall determine these best constants, and it turns out that they are also connected with the equilibrium measure μ_E of the set E.

Thus, let $E=\bigcup_{i=1}^{l}[a_{2i-1},a_{2i}]$, where the intervals $[a_{2i-1},a_{2i}]$ are disjoint. Let a_j be an endpoint of E, and let E^j be that part of E that lies closer to a_j than to any other endpoint, i.e. we set

$$E^j=\{x\in E:|x-a_j|<|x-a_i|,\,i\neq j\}.$$

In this situation we shall be interested in the best possible constant M_j such that the inequality

$$\|P_n'\|_{E^j} \leqslant (1+o(1)) M_j n^2 \|P_n\|_E \tag{4.3}$$

holds for all polynomials P_n of degree at most n, where $o(1)$ denotes a term that tends to zero as $n \to \infty$. Then asymptotically the best Markoff factor in (4.1) is just the maximum of these M_j. Thus, in this respect we are speaking about $2l$ Markoff constants, one is related to each endpoint.

By (2.4)–(2.8) the equilibrium density ω_E is of the form

$$\omega_E(x) = \frac{\prod_{i=1}^{l-1} |x-\tau_i|}{\pi \sqrt{\prod_{i=1}^{2l} |x-a_i|}}, \quad x \in E, \tag{4.4}$$

where the τ_i are the unique numbers satisfying

$$\int_{a_{2j}}^{a_{2j+1}} \frac{\prod_{i=1}^{l-1} (x-\tau_i)}{\pi \sqrt{\prod_{i=1}^{2l} |x-a_i|}}\, dx = 0$$

for $j=1, ..., l-1$. We also know that we have exactly one τ_i (say τ_j) in each of the contiguous intervals (a_{2j}, a_{2j+1}), $j=1, ..., l-1$.

THEOREM 4.1. *With the above notations and with*

$$M_j = 2 \frac{\prod_{i=1}^{l-1} (a_j-\tau_i)^2}{\prod_{i \neq j} |a_j - a_i|} \tag{4.5}$$

we have for each $1 \leqslant j \leqslant 2l$

$$\|P_n'\|_{E^j} \leqslant (1+o(1)) M_j n^2 \|P_n\|_E, \tag{4.6}$$

and this is asymptotically the best possible, for there is a sequence $\{P_n\}$ *of polynomials of corresponding degree at most* $n=1,2,...$ *such that*

$$|P_n'(a_j)| \geqslant (1+o(1)) M_j n^2 \|P_n\|_E. \tag{4.7}$$

Before giving the proof we consider an example. Let $E=[-b,-a] \cup [a,b]$ with $0<a<b$. In this case $l=2$ and (by symmetry) $\tau_1=0$, and we have $M_j = b/(b^2-a^2)$ if $a_j = \pm b$, and $M_j = a/(b^2-a^2)$ if $a_j = \pm a$. Thus, we have the inequalities

$$\|P_n'\|_{[-b,-b+(b-a)/2] \cup [a+(b-a)/2, b]} \leqslant (1+o(1)) \frac{b}{b^2-a^2} n^2 \|P_n\|_E$$

and

$$\|P_n'\|_{[-b+(b-a)/2,-a] \cup [a, a+(b-a)/2]} \leqslant (1+o(1)) \frac{a}{b^2-a^2} n^2 \|P_n\|_E.$$

Furthermore, the constants on the right cannot be replaced by any smaller one. Combining these two inequalities we obtain

$$\|P_n'\|_E \leqslant (1+o(1))\frac{b}{b^2-a^2}n^2\|P_n\|_E,$$

which is a result of P. Borwein [4].

Theorem 4.1 implies

$$\|P_n'\|_E \leqslant (1+o(1))\big(\max_{1\leqslant j\leqslant 2l} M_j\big)n^2\|P_n\|_E. \tag{4.8}$$

Here the maximum cannot be attained for $j=2$ or $j=2l-1$ (because, as elementary consideration shows, $M_2<M_1$ and $M_{2l-1}<M_{2l}$), but otherwise (depending on the structure of the l intervals $[a_{2j-1},a_{2j}]$) the maximum can occur at any other j. One can also show (see [3]) that in (4.8) the factor $1+o(1)$ cannot be dropped.

Proof of Theorem 4.1. We start the proof by computing the derivative of a polynomial at an endpoint of the associated polynomial inverse image of $[-1,1]$. Thus, let us suppose that E is the polynomial inverse image of $[-1,1]$ under the mapping $x\to T_N(x)$, where the polynomial T_N satisfies the properties set forth in the preceding sections. It is known [10, Theorem 2.3] that if we set

$$H(x)=\prod_{i=1}^{2l}(x-a_i), \quad r(x)=\prod_{i=1}^{l-1}(x-\tau_i),$$

then there is a polynomial U_{N-l} of degree $N-l$ such that

$$T_N^2(x)-H(x)U_{N-l}^2(x)=1,$$

and this U_{N-l} satisfies the equation

$$T_N'(x)=NU_{N-l}(x)r(x) \tag{4.9}$$

(see [10, Remark 2.6 (b), p. 194], which appears with a slight error, namely the factor N is missing on the right). On differentiating the first equality, setting $x=a_j$ and making use of the fact that $T_N(a_j)=\pm1$ we obtain

$$\pm 2T_N'(a_j)=H'(a_j)U_{N-l}^2(a_j),$$

while (4.9) yields

$$T_N'(a_j)=NU_{N-l}(a_j)r(a_j).$$

If we express from here $U_{N-l}(a_j)$, and substitute the resulting equality into the previous formula, then we obtain

$$|T'_N(a_j)| = N^2 \cdot 2 \frac{\prod_{i=1}^{l-1}(a_j-\tau_i)^2}{\prod_{i\neq j}|a_j-a_i|} = N^2 M_j. \tag{4.10}$$

Next we verify (4.6) for the special case when $E=T_N^{-1}([-1,1])$ is the polynomial inverse image of $[-1,1]$, and P_n is of the form $P_n(x)=R_m(T_N(x))$.

Let $\varepsilon>0$ be arbitrary. Because of (4.2), for every $\eta>0$ we have

$$\|P'_n\|_{E^j\setminus[a_j-\eta,a_j+\eta]} \leqslant M_j n^2 \|P_n\|_E$$

for sufficiently large n, and therefore it is enough to consider the η-neighborhood of a_j. We can choose this η so small that for $x\in[a_j-\eta,a_j+\eta]$ we have $|T'_N(x)|\leqslant(1+\varepsilon)|T'_N(a_j)|=(1+\varepsilon)M_jN^2$. Then for $x\in[a_j-\eta,a_j+\eta]\cap E$ we obtain from the classical Markoff inequality applied to the polynomial R_m that

$$|P'_n(x)| = |R'_m(T_n(x))|\cdot|T'_N(x)| \leqslant m^2\|R_m\|_{[-1,1]}(1+\varepsilon)M_jN^2 \leqslant (1+\varepsilon)M_jn^2\|P_n\|_E,$$

where we used that the norm of R_m over $[-1,1]$ coincides with the norm of P_n over E, and that $n=Nm$.

Finally, let E be an arbitrary set consisting of a finite number of intervals. By Theorem 2.1 we can choose a polynomial inverse image set $E^*=T_N^{-1}([-1,1])$ consisting of l intervals that lies arbitrarily close to E. Furthermore, in selecting E^* we can also achieve that a_j is an endpoint of E^*, and $E^*\subset E$ (see the remarks made after Theorem 2.1). It is also true that the numbers τ_i are C^∞-functions of the endpoints a_j (see §2), hence we can assume that if M_j^* is the number M_j from (4.5) computed for E^*, then $M_j^*\leqslant(1+\varepsilon)M_j$, where $\varepsilon>0$ is any given number. Now we can prove (4.6) for arbitrary P_n via polynomials like P_n^* and $S(x)$ from (3.11) and (3.13) from the preceding section just as was done there. In fact, let $E_i^*=T_{N,i}^{-1}([-1,1])$, $i=1,...,N$, be the inverse images under the N branches of T_N^{-1}. Assume that $a_j\in E_{j_0}^*$, and let $\eta>0$ be so small that $[a_j-\eta,a_j+\eta]\cap E\subset E_{j_0}^*$ and this set does not contain the other endpoint of $E_{j_0}^*$. There are polynomials (see [16, Theorem VI.3.6]) $L_{\sqrt{n}}$ of degree at most $\sqrt{n}$ such that with some constants $0<\beta<1$ and C we have

$$\begin{aligned} &0\leqslant 1-L_{\sqrt{n}}(x)\leqslant C\beta^{\sqrt{n}} && \text{for } x\in[a_j-\eta,a_j+\eta],\\ &0\leqslant L_{\sqrt{n}}(x)\leqslant C\beta^{\sqrt{n}} && \text{for } x\in \bigcup_{1\leqslant i\leqslant N,\, i\neq j_0} E_i^*, \end{aligned}$$

and otherwise $0\leqslant L_{\sqrt{n}}(x)\leqslant 1$ on E^*. Instead of (3.11) consider the polynomial

$$P_n^*(x) = L_{\sqrt{n}}(x)P_n(x), \tag{4.11}$$

which has similar properties as the P_n^* from (3.11), namely P_n^* has degree at most $n+\sqrt{n}$, $\|P_n^*\|_{E^*}\leqslant\|P_n\|_{E^*}$, $P_n^*(x)=\big(1+O(\beta^{\sqrt{n}})\big)P_n(x)$ for $x\in[a_j-\eta, a_j+\eta]$,

$$|(P_n^*)'(x)-P_n'(x)|=O\big(n^2\beta^{\sqrt{n}}\big)\|P_n\|_{E^*}\quad\text{for } x\in[a_j-\eta,a_j+\eta],$$

and

$$|P_n^*(x)|=O\big(\beta^{\sqrt{n}}\big)\|P_n\|_{E^*},\quad |(P_n^*)'(x)|=O\big(n^2\beta^{\sqrt{n}}\big)\|P_n\|_{E^*}$$

uniformly for $x\in E^*\setminus E_{i_0}^*$, where we have also used the classical Markoff inequality (3.2) in the estimates of the derivatives. These show that if we define

$$S(x)=\sum_{i=1}^N P_n^*(T_{N,i}^{-1}(T_N(x)))$$

as in (3.13), then exactly as after (3.13) we can conclude that this is a polynomial of T_N, and

$$\|S\|_{E^*}\leqslant\big(1+O(\beta^{\sqrt{n}})\big)\|P_n\|_{E^*},\quad |S'(x)-P_n'(x)|\leqslant O\big(n^2\beta^{\sqrt{n}}\big)\|P_n\|_{E^*}$$

for all $x\in E^*\cap[a_j-\eta,a_j+\eta]$. Now S is already of the type for which we have verified (4.6) above, so if we apply to S the inequality (4.6) and also take into account (4.2), then we can conclude (4.6):

$$\begin{aligned}\|P_n'\|_{E^j}&\leqslant\max\big(O(n)\|P_n\|_{E^*},\|S'\|_{E^*\cap[a_j-\eta,a_j+\eta]}+O\big(n^2\beta^{\sqrt{n}}\|P_n\|_{E^*}\big)\big)\\&\leqslant(1+o(1))M_j^*(\deg(S))^2\|S\|_{E^*}+O\big((n+n^2\beta^{\sqrt{n}})\|P_n\|_{E^*}\big)\\&\leqslant(1+o(1))n^2M_j^*\|P_n\|_{E^*}\leqslant(1+o(1))(1+\varepsilon)n^2M_j\|P_n\|_E,\end{aligned}$$

where in the last step we used that $E^*\subset E$ and $M_j^*\leqslant(1+\varepsilon)M_j$. Since here $\varepsilon>0$ is arbitrary, (4.6) follows.

The proof of (4.7) follows from the above considerations. In fact, as before, select a polynomial inverse image set $E^*=T_N^{-1}([-1,1])$ consisting of l intervals that lies arbitrarily close to E for which a_j is an endpoint, and for which M_j^* is close to M_j, but now we select E^* so that it contains E. If $\mathcal{T}_m$ are the classical Chebyshev polynomials, then using that $T_N(a_j)=\pm1$, and $|\mathcal{T}_m'(\pm1)|=m^2$, we get from (4.10) for $m=[n/N]$

$$|(\mathcal{T}_m(T_N))'(a_j)|=m^2N^2M_j^*,$$

and since here $n^2/m^2N^2\to1$ as $n\to\infty$, and M_j^* is as close to M_j as we wish, (4.7) follows. □

References

[1] ACHYESER, N. [AKHIEZER, N. I.], Über einige Funktionen, welche in zwei gegebenen Intervallen am wenigsten von Null abweichen. *Bull. Acad. Sci. URSS Sér. Math.* (*7*) [*Izv. Akad. Nauk SSSR Sér. Math.* (*7*)], 1932:9, 1163–1202.

[2] APTEKAREV, A. I., Asymptotic properties of polynomials orthogonal on a system of contours and periodic motions of Toda lattices. *Mat. Sb.* (*N.S.*), 125 (167) (1984), 231–258 (Russian); English translation in *Math. USSR-Sb.*, 53 (1986), 223–260.

[3] BENKŐ, D. & TOTIK, V., Sets with interior extremal points for the Markoff inequality. *J. Approx. Theory*, 110 (2001), 261–265.

[4] BORWEIN, P., Markov's and Bernstein's inequalities on disjoint intervals. *Canad. J. Math.*, 33 (1981), 201–209.

[5] DEVORE, R. A. & LORENTZ, G. G., *Constructive Approximation.* Grundlehren Math. Wiss., 303. Springer-Verlag, Berlin, 1993.

[6] GERONIMO, J. S. & VAN ASSCHE, W., Orthogonal polynomials on several intervals via a polynomial mapping. *Trans. Amer. Math. Soc.*, 308 (1988), 559–581.

[7] LANDKOF, N. S., *Foundations of Modern Potential Theory.* Grundlehren Math. Wiss., 180. Springer-Verlag, New York, 1972.

[8] PEHERSTORFER, F., On Bernstein–Szegő orthogonal polynomials on several intervals, I. *SIAM J. Math. Anal.*, 21 (1990), 461–482.

[9] — On Bernstein–Szegő orthogonal polynomials on several intervals, II. Orthogonal polynomials with periodic recurrence coefficients. *J. Approx. Theory*, 64 (1991), 123–161.

[10] — Orthogonal and extremal polynomials on several intervals. *J. Comput. Appl. Math.*, 48 (1993), 187–205.

[11] — Elliptic orthogonal and extremal polynomials. *J. London Math. Soc.* (*3*), 70 (1995), 605–624.

[12] — Approximation of several intervals by an inverse polynomial mapping. To appear in *J. Approx. Theory.*

[13] PEHERSTORFER, F. & SCHIEFERMAYR, K., Description of extremal polynomials on several invervals and their computation, I. *Acta Math. Hungar.*, 83 (1999), 27–58.

[14] PEHERSTORFER, F. & STEINBAUER, R., On polynomials orthogonal on several intervals. *Ann. Numer. Math.*, 2 (1995), 353–370.

[15] RANSFORD, T., *Potential Theory in the Complex Plane.* London Math. Soc. Stud. Texts, 28. Cambridge Univ. Press, Cambridge, 1995.

[16] SAFF, E. B. & TOTIK, V., *Logarithmic Potentials with External Fields.* Grundlehren Math. Wiss., 316. Springer-Verlag, Berlin, 1997.

[17] STAHL, H. & TOTIK, V., *General Orthogonal Polynomials.* Encyclopedia Math. Appl., 43. Cambridge Univ. Press, Cambridge, 1992.

[18] SZEGŐ, G., Über einen Satz des Herrn Serge Bernstein. *Schr. Königsberg. Gel. Ges.*, 5 (1928/29), 59–70.

[19] TIMAN, A. F., *Theory of Approximation of Functions of a Real Variable.* Dover, New York, 1994.

[20] TSUJI, M., *Potential Theory in Modern Function Theory.* Maruzen, Tokyo, 1959.

[21] WIDOM, H., Extremal polynomials associated with a system of curves in the complex plane. *Adv. Math.*, 3 (1969), 127–232.

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