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On the Dirichlet problem for degenerate Monge–Ampère equations

by

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1. Introduction

In this paper, we resolve an outstanding problem concerning the global regularity of solutions of the Dirichlet problem for the *degenerate* Monge–Ampère equation in Euclidean n-space, $\mathbf{R}^n$, $n\geqslant 2$.

THEOREM 1.1. *Let Ω be a uniformly convex domain in $\mathbf{R}^n$ with boundary $\partial\Omega\in C^{3,1}$, $\varphi\in C^{3,1}(\bar\Omega)$ and let f be a non-negative function in Ω such that $f^{1/(n-1)}\in C^{1,1}(\bar\Omega)$. Then there exists a unique convex solution $u\in C^{1,1}(\bar\Omega)$ of the Dirichlet problem*

$$\begin{aligned} \det D^2u &= f \quad \text{in } \Omega,\\ u&=\varphi \quad \text{on } \partial\Omega. \end{aligned} \tag{1.1}$$

Consequently, any generalized solution of (1.1) *in $C^0(\bar\Omega)$ must belong to $C^{1,1}(\bar\Omega)$. Moreover, the solution u satisfies an estimate*

$$\|u\|_{C^{1,1}(\bar\Omega)}\leqslant C, \tag{1.2}$$

where C is a constant depending on Ω and the norms of the functions φ and $\tilde f=f^{1/(n-1)}$ in the spaces $C^{3,1}(\bar\Omega)$ and $C^{1,1}(\bar\Omega)$ respectively.

A function $u\in C^0(\bar\Omega)$ is a generalized solution of equation (1.1), in the sense of Aleksandrov, if u is convex in Ω with subgradient (or normal) mapping χ having density f with respect to Lebesgue measure in $\mathbf{R}^n$. The regularity assertion in Theorem 1.1 is an immediate consequence of the existence assertion and the uniqueness of the generalized solution (see, for example, [19]). In turn, the existence assertion follows from the a priori estimate (1.2) applied to smooth solutions of approximating non-degenerate problems, whose solvability is guaranteed by the fundamental second-derivative Hölder estimates of Calabi [4], Caffarelli, Nirenberg and Spruck [2], and Krylov [14], [15]. The Monge–Ampère equation (1.1) is referred to as non-degenerate when the inhomogeneous term f is positive (or equivalently when it is uniformly elliptic with respect to a convex classical solution). For non-degenerate equations, the global second-derivative estimate (1.2) was established by Ivochkina [11], and again with a different proof in [2], with constant C depending also on $\inf_\Omega f > 0$.

The global regularity problem for degenerate Monge–Ampère equations (that is, when the non-negative function f is allowed to vanish somewhere in $\bar{\Omega}$) has been studied by various authors (see [3], [7], [10], [15]–[18], [20], [23], [24]), with the strongest result to date due to Krylov [16]–[18] who established Theorem 1.1 in the case when $f^{1/n} \in C^{1,1}(\bar{\Omega})$, with the constant C in (1.2) depending on the norm of $f^{1/n}$ (rather than $f^{1/(n-1)}$). We remark that the techniques of this paper are completely different to those of [16]–[18]. When the equation is completely degenerate, that is, the function f is identically zero, Theorem 1.1 was proved by Caffarelli, Nirenberg and Spruck [3], following the corresponding interior regularity result of Trudinger and Urbas [24].

The assumption that $f^{1/n} \in C^{1,1}(\bar{\Omega})$ appears natural as then the estimation of second derivatives is readily reduced to boundary estimation, by applying the Aleksandrov–Bakelman maximum principle [5] to the twice differentiated equation as in [1], [23]. Recently, Guan [7] observed that by following an argument analogous to that of Pogorelov [5], [19], this reduction to boundary estimation can be achieved by only assuming that $f^{1/(n-1)} \in C^{1,1}(\bar{\Omega})$, as in Theorem 1.1. However, he has to suppose that either $\varphi \equiv 0$ or f is positive near the boundary, to derive an a priori estimate for second derivatives on the boundary by the usual methods.

The significance of Theorem 1.1 is not so much that it is an improvement of earlier results but that it is *optimal*. Indeed, an example of Wang [25] shows that if the function $\tilde{f} = f^{1/(n-1)} \notin C^{1,1}(\bar{\Omega})$, then $C^{1,1}$-regularity is false in general. The assumption $\varphi \in C^{3,1}(\bar{\Omega})$ is also optimal for global regularity [3]. In the non-degenerate case, Wang [26] shows that $\varphi \in C^3(\bar{\Omega})$ suffices for global $C^{1,1}$-regularity while if only $\varphi \in C^{2,1}(\bar{\Omega})$, the solution may even fail to lie in the Sobolev space $W^{2,p}(\Omega)$ for sufficiently large p. We also note here that for degenerate Monge–Ampère equations, $C^{1,1}$-regularity is the best that can be expected. This is readily seen by letting $B = B_1(0)$ be the unit ball in $\mathbf{R}^2$, and defining u on B by

$$u(x,y) = \left[\max\left\{\left(x^2 - \tfrac{1}{2}\right)^+, \left(y^2 - \tfrac{1}{2}\right)^+\right\}\right]^2. \tag{1.3}$$

We then have $\det D^2 u = 0$ in B, with u analytic on ∂B. In the two-dimensional case, better regularity is possible if the Hessian matrix $D^2 u$ has at least one positive eigenvalue [8].

The rest of this paper is devoted to the proof of the second-derivative estimate (1.2). In the following section, we reduce that estimation to estimation on the boundary, as in [7] (Lemma 2.1). The estimation of the second derivatives of the solution of (1.1) on the boundary $\partial\Omega$ is carried out in the ensuing sections. As in the non-degenerate case [2], [11], the estimation of the double normal derivatives can be achieved, from that of the other second derivatives, through the equation itself. However, in the degenerate case, certain precise forms of these preliminary estimates are absolutely vital. In §3, we provide the necessary estimate from below for the cofactor of the double normal derivative (Lemma 3.1), while in §§4 and 5, we derive crucial *linear* estimates for the mixed tangential-normal second derivatives with respect to the square roots of the corresponding tangential derivatives. Our techniques rest strongly on the behaviour of the equation (1.1) with respect to affine transformations and on the convexity of the solutions.

Finally, in §6, we complete the proof of estimate (1.2) and remark on the extension to more general inhomogeneous terms. The notation in this paper is standard with Lipschitz spaces and their norms being defined as, for example, in [5]. Almost always our constants C will depend on the same quantities as in the statement of Theorem 1.1, but to avoid too much repetition, this is not always indicated.

2. Global second-derivative bounds

As mentioned in the introduction, it suffices to prove the estimate (1.2) for smooth solutions of non-degenerate problems. The estimation of the solution itself and its gradient is well known from standard barrier considerations [5]. In this section we follow [7] to reduce the estimation of second derivatives of solutions of (1.1) to their estimation on the boundary.

LEMMA 2.1. *Let $u\in C^4(\Omega)\cap C^2(\bar\Omega)$ satisfy equation (1.1) in Ω with $f>0$ in Ω. Then*

$$\sup_\Omega |D^2u| \leqslant C+\sup_{\partial\Omega}|D^2u|, \tag{2.1}$$

where C is a constant depending on $|\tilde f|_{1,1}$ and Ω.

Proof. Since the estimate (2.1) is already proved in [7], we present here a proof for completeness.

Let $w(x)=\Delta u(x)+\frac12 M|x|^2$. Suppose that w attains its maximum at x_0. If $x_0\in\partial\Omega$, we are through. So to prove (2.1) we need only to consider the case $x_0\in\Omega$. By rotating the coordinates we may suppose that D^2u is diagonal at x_0. Then at x_0 we have

$$0=w_k(x_0)=\sum_{i=1}^n u_{iik}+Mx_k,$$

$$0\geqslant w_{kk}(x_0)=\sum_{i=1}^n u_{iikk}+M.$$

Differentiating the equation $\log\det(D^2u)=\log f$ with respect to x_k gives

$$u^{ij}u_{ijkk}=\frac{f_{kk}}{f}-\frac{f_k^2}{f^2}+\sum u^{ir}u^{js}u_{ijk}u_{rsk}\geqslant\left\{\frac{f_{kk}}{f}-\frac{n-2}{n-1}\cdot\frac{f_k^2}{f^2}\right\}-\frac{1}{n-1}\cdot\frac{f_k^2}{f^2}=:\mathrm{I}+\mathrm{II},$$

where (u^{ij}) is the inverse of (u_{ij}), which is also diagonal at x_0. Since $\tilde f=f^{1/(n-1)}\in C^{1,1}(\bar\Omega)$, we have

$$\mathrm{I}=\frac{f_{kk}}{f}-\frac{n-2}{n-1}\cdot\frac{f_k^2}{f^2}\geqslant -Cf^{-1/(n-1)},$$

$$|\nabla\tilde f|\leqslant C\sqrt{\tilde f},$$

and hence also

$$\mathrm{II}\geqslant -Cf^{-1/(n-1)},$$

where C depends on $|\tilde f|_{1,1}$ and Ω. Therefore we obtain

$$u^{ii}u_{iikk}\geqslant -Cf^{-1/(n-1)}.$$

Consequently

$$0\geqslant u^{ii}w_{ii}(x_0)=u^{ii}u_{iikk}+Mu^{ii}\geqslant M\sum\frac{1}{u_{ii}}-Cf^{-1/(n-1)}.$$

Observing that

$$\sum\frac{1}{u_{ii}}\geqslant\left[\frac{1}{u_{22}\ldots u_{nn}}\right]^{1/(n-1)}\geqslant f^{-1/(n-1)}$$

if $u_{11}=\max u_{ii}\geqslant 1$, we reach a contradiction if M is large enough. □

3. Tangential derivatives

In this section we establish upper and lower bounds for the tangential second-order derivatives of solutions on the boundary. For any point $x_0\in\partial\Omega$, without loss of generality we may suppose that x_0 is the origin and that the x_n-axis is the inner normal there. By transforming the coordinates $x'=(x_1,\ldots,x_{n-1})$ we may suppose that in a neighbourhood $\mathcal{N}$ of the origin, $\partial\Omega$ is represented by

$$x_n=\varrho(x')=\tfrac12|x'|^2+\text{cubic of }x'+O(|x'|^4) \tag{3.1}$$

as $x'\to 0$, where $x'=(x_1,\ldots,x_{n-1})$. Subtracting a linear function we may suppose that

$$\nabla u(0)=0 \quad\text{and}\quad \inf_\Omega u(x)=u(0)=0, \tag{3.2}$$

while, by rotation of the coordinates x', we may further suppose that $u_{ij}(0)=0$ for $i\neq j$, $i,j=1,...,n-1$. By Taylor expansion,

$$\varphi(x',\varrho(x'))=\frac{1}{2}\sum_{i=1}^{n-1}b_ix_i^2+R(x')+O(|x'|^4), \tag{3.3}$$

where $R(x')$ denotes the cubic term. Then $b_i=u_{ii}(0)>0$ for $i=1,...,n-1$. We may further suppose

$$0<b_1\leqslant b_2\leqslant...\leqslant b_{n-1}.$$

Differentiating the equality $u(x',\varrho(x'))=\varphi(x',\varrho(x'))$, we get

$$b_i=\varphi_{ii}(0)+\varphi_n(0),\quad i=1,...,n-1, \tag{3.4}$$

which implies an upper bound for b_i (in terms of ∇u). To proceed further, we first observe the following obvious fact. Suppose that $n=2$, u is non-negative and

$$u=\varphi=\alpha x_1^2+\beta x_1^3+R_4\quad\text{on } \partial\Omega,$$

where $\alpha\geqslant 0$, $\beta\in\mathbf{R}$ and $|R_4|\leqslant A|x_1|^4$. Then

$$|\beta|\leqslant(1+A)\sqrt{\alpha}, \tag{3.5}$$

since $u\geqslant 0$ at the points $x_1=\pm\sqrt{\alpha}$. A lower bound for the tangential derivatives b_i is given by the following lemma.

LEMMA 3.1. *There exists a positive constant γ_0 depending on Ω, $|\varphi|_{3,1}$ and $|\tilde{f}|_{1,1}$ such that*

$$\prod_{i=1}^{n-1}b_i\geqslant\gamma_0 f(0). \tag{3.6}$$

Proof. To prove (3.6), we make a dilation $x\to y=T(x)$ defined by

$$y_i=M_ix_i,\quad i=1,...,n,$$

where

$$\begin{cases} M_i=b_i^{1/2}M, & i=1,...,n-1,\\ M_n=M=1/b_1. \end{cases}$$

Let

$$v(y)=M^2u(x).$$

Then

$$\det(D_y^2 v) = g(y) =: \frac{f(y_i/M_i)M^{2n}}{\prod_{i=1}^n M_i^2} \quad \text{in } \widetilde{\Omega} = T(\Omega). \tag{3.7}$$

By (3.1), the boundary $\partial\widetilde{\Omega}$ can be represented in a neighbourhood $\widetilde{\mathcal{N}}$ $(=T(\mathcal{N}))$ of the origin by

$$y_n = \tilde{\varrho}(y') = \frac{1}{2}\sum_{i=1}^{n-1} d_i y_i^2 + O(|y'|^3) \tag{3.8}$$

with

$$d_i = \frac{M_n}{M_i^2} = \frac{b_1}{b_i} \leqslant 1, \quad i = 1, ..., n-1. \tag{3.9}$$

Moreover,

$$|D_{y'}^k \tilde{\varrho}| \leqslant |D_{x'}^k \varrho| \leqslant C$$

for $k=3,4$, where C depends on $\partial\Omega$. On $\widetilde{\mathcal{N}} \cap \partial\widetilde{\Omega}$ we have

$$v = \widetilde{\varphi}(y') = \frac{1}{2}\sum_{i=1}^{n-1} \tilde{b}_i y_i^2 + R(y') + O(|y'|^4), \quad (y', \tilde{\varrho}(y')) \in \partial\widetilde{\Omega}, \tag{3.10}$$

where

$$\tilde{b}_i = \frac{M^2}{M_i^2} b_i = 1 \tag{3.11}$$

and $R(y')$ is the cubic term. Noticing that

$$|D_{y_i}^4 \widetilde{\varphi}(y)| = \frac{M^2}{M_i^4} |D_{x_i}^4 \varphi(x)| \leqslant C,$$

we see that all fourth-order derivatives of $\widetilde{\varphi}$ are bounded. Hence in (3.10), the third term $O(|y'|^4) \leqslant C|y'|^4$ for some constant C depending only on $\partial\Omega$ and $|\varphi|_{3,1}$. Applying (3.5) to every direction in the tangent plane $\{y_n=0\}$ of $\partial\widetilde{\Omega}$, we conclude that the coefficients of R are uniformly bounded. Hence near the origin both $\partial\widetilde{\Omega}$ and $\widetilde{\varphi}$ are $C^{3,1}$-smooth and their $C^{3,1}$-norms are independent of the upper bound of M. Without loss of generality we may assume $\widetilde{\Omega} \cap \{y_n < 1\} \subset \widetilde{\mathcal{N}}$. Let

$$\omega = \{y \in \widetilde{\Omega} \mid y_n < 1, |y_i| < 1, i = 2, ..., n-1\}.$$

Then ω is bounded in terms of $\partial\Omega$ and $|\varphi|_{3,1}$ since $d_1=1$ in (3.8). By the convexity of v, and noticing that $v = \widetilde{\varphi} \leqslant C$ on $\partial\omega \cap \partial\widetilde{\Omega}$, we have

$$v \leqslant C \quad \text{in } \omega, \tag{3.12}$$

where C depends on Ω and $|\varphi|_{3,1}$. We claim that

$$\sup\{g(y) \mid y \in \omega\} \leqslant C \tag{3.13}$$

for some constant C depending on $|\tilde{f}|_{1,1}$, $|\varphi|_{3,1}$ and Ω, where by (3.7),

$$g(y)=\frac{f(y_i/M_i)}{\prod_{i=1}^{n-1} b_i}. \tag{3.14}$$

From (3.13) and (3.14) we obtain (3.6) with $\gamma_0=1/C$. To prove (3.13), we compute

$$\frac{\partial}{\partial y_i} g^{1/(n-1)}(y)=\frac{\partial f^{1/(n-1)}/\partial y_i}{\left[\prod_{i=1}^{n-1} b_i\right]^{1/(n-1)}}=\frac{\partial f^{1/(n-1)}/\partial x_i}{M_i\left[\prod_{i=1}^{n-1} b_i\right]^{1/(n-1)}}.$$

Since $\tilde{f}=f^{1/(n-1)}\in C^{1,1}$ and f is non-negative, we have, as in the proof of Lemma 2.1,

$$|\nabla \tilde{f}|\leqslant C\sqrt{\tilde{f}}\,,$$

where C depends on Ω and $|\tilde{f}|_{1,1}$. We then obtain

$$\left|\frac{\partial}{\partial y_i} g^{1/(n-1)}(y)\right|\leqslant \frac{C f^{1/2(n-1)}}{M_i\left[\prod_{i=1}^{n-1} b_i\right]^{1/(n-1)}}=\frac{C g^{1/2(n-1)}}{M_i\left[\prod_{i=1}^{n-1} b_i\right]^{1/2(n-1)}}.$$

Observing that

$$M_i^{2(n-1)}\prod_{i=1}^{n-1} b_i\geqslant M_1^{2(n-1)}\prod_{i=1}^{n-1} b_i\geqslant M_1^{2(n-1)} b_1^{n-1}=1,$$

we obtain

$$\left|\frac{\partial}{\partial y_i} g^{1/2(n-1)}(y)\right|\leqslant C. \tag{3.15}$$

Hence if $\sup_\omega g$ is large enough, so is $\inf_\omega g$, which implies by (3.12) and the comparison principle that $\inf_\omega v<0$. On the other hand, by (3.2) we have $v\geqslant 0$. This contradiction shows that g is uniformly bounded in ω, in accordance with (3.13). Hence (3.6) holds. □

4. Mixed tangential-normal derivatives

In this and the following section we prove an estimate for the mixed tangential-normal second derivatives of solutions of the Dirichlet problem (1.1) in terms of the corresponding tangential second derivatives. We first prove a preliminary estimate in terms of the *largest* tangential second derivative. Using the coordinate system (3.1), introduced in the preceding section, we formulate this estimate as follows.

LEMMA 4.1. *Letting γ and τ denote respectively the unit inner normal to $\partial\Omega$ and any unit tangential vector to $\partial\Omega$, we have for any $x\in\mathcal{N}\cap\partial\Omega$, with $|x|\leqslant\frac{1}{2}\sqrt{b_{n-1}}$, the estimate*

$$|u_{\gamma\tau}(x)|\leqslant C\sqrt{b_{n-1}}\,, \tag{4.1}$$

where C is a positive constant depending on Ω, $|\varphi|_{3,1}$ and $|\tilde{f}|_{1,1}$, and b_{n-1} is defined by (3.3).

Proof. Let $M=1/b_{n-1}$. We make a dilation $x\to y=T(x)$ by defining

$$\begin{cases} y'=\sqrt{M}\,x', \\ y_n=Mx_n. \end{cases}$$

Let $v(y)=M^2u(x)$. Then v satisfies

$$\det(D_y^2 v)=g(y)=:M^{n-1}f(T^{-1}(y)) \quad \text{in } \widetilde{\Omega}=T(\Omega). \tag{4.2}$$

In the neighbourhood $\widetilde{\mathcal{N}}=T(\mathcal{N})$, $\partial\widetilde{\Omega}$ is represented by

$$y_n=\tfrac{1}{2}|y'|^2+O(|y'|^3), \tag{4.3}$$

and on the boundary $\partial\widetilde{\Omega}\cap\widetilde{\mathcal{N}}$,

$$v(y)=\widetilde{\varphi}(y')=\frac{1}{2}\sum_{i=1}^{n-1}\tilde{b}_i y_i^2+R(y')+O(|y|^4), \quad y=(y',\tilde{\varrho}(y'))\in\partial\widetilde{\Omega}, \tag{4.4}$$

where

$$\tilde{b}_i=\frac{b_i}{b_{n-1}}\leqslant 1.$$

As in the preceding section, we see that near the origin, both $\partial\widetilde{\Omega}$ and $\widetilde{\varphi}$ are $C^{3,1}$-smooth with $C^{3,1}$-norms independent of M. Similarly to (3.13) we also infer that the function g is uniformly bounded on the domain

$$\omega=\{y\in\widetilde{\Omega}\mid y_n<1\},$$

which is also bounded independently of M, by virtue of (4.3). By (4.2) and the uniform convexity of $\widetilde{\mathcal{N}}\cap\partial\widetilde{\Omega}$ implied by (4.3), we then obtain, by the usual barrier considerations for the Monge–Ampère equation ([5, Theorem 17.21]), a normal derivative bound

$$|v_{\widetilde{\gamma}}(y)|\leqslant C \tag{4.5}$$

for $y\in\partial\widetilde{\Omega}$ satisfying $y_n\leqslant\frac{7}{8}$, where $\tilde{\gamma}$ is the unit inner normal to $\partial\widetilde{\Omega}$ and C depends as usual on Ω, $|\psi|_{3,1}$ and $|\tilde{f}|_{1,1}$. From (4.5) we have

$$|Dv|\leqslant C \tag{4.6}$$

on $\partial\widetilde{\Omega}\cap\{y_n\leqslant\frac{3}{4}\}$, which extends to $\widetilde{\Omega}\cap\{y_n\leqslant\frac{3}{4}\}$ by the convexity of v. By differentiating equation (4.2) with respect to a tangential vector field as in [5], [24], or with respect to a linear vector field with skew-symmetric Jacobian matrix as in [2], [12], we obtain, again from the uniform convexity of $\widetilde{\mathcal{N}}\cap\partial\widetilde{\Omega}$, the mixed tangential-normal second-derivative estimates

$$|v_{\tilde{\gamma}\tilde{\tau}}|\leqslant C \tag{4.7}$$

on $\partial\widetilde{\Omega}\cap\{y_n<\frac{1}{2}\}$, where $\tilde{\tau}$ is a unit tangent vector to $\partial\widetilde{\Omega}$ and C depends on Ω, $|\varphi|_{3,1}$ and $|\tilde{f}|_{1,1}$. Since a more complicated version of this argument will be needed later, in the absence of uniform convexity of $\partial\widetilde{\Omega}$, we omit the details here. Pulling back to our original coordinates, we obtain (4.1). □

We remark here, for purposes of illustration, that Lemmas 3.1 and 4.1 are already sufficient to ensure a second-derivative estimate in the two-dimensional case. To see this, we write equation (1.1) at the origin in the form

$$u_{11}(0)u_{22}(0)=u_{12}^2(0)+f(0), \tag{4.8}$$

so that

$$0\leqslant u_{22}(0)=\frac{u_{12}^2(0)}{b_1}+\frac{f(0)}{b_1}\leqslant C,$$

by Lemmas 3.1 and 4.1.

5. Mixed tangential-normal derivatives, continued

We proceed from Lemma 4.1, by means of induction, to obtain the following refinement.

LEMMA 5.1. *For $i=1,...,n-1$, we have the estimate*

$$|u_{in}(0)|\leqslant C_0\sqrt{b_i}\,, \tag{5.1}$$

where C_0 is a constant depending on Ω, $|\varphi|_{3,1}$ and $|\tilde{f}|_{1,1}$, and $b_i=u_{ii}(0)$ is defined by (3.3).

Proof. For $i=1,...,n-1$, let us denote by $\tau_i=\tau_i(x)$ the tangential direction of $\partial\Omega$ at $x\in\mathcal{N}\cap\partial\Omega$, which lies in the two-dimensional plane parallel to the x_i- and x_n-axes and passes through the point x. Our induction hypothesis is that for some $k=1,...,n-2$ and $i=k+1,...,n-1$, there exists a constant $\theta_i>0$, depending on Ω, $|\varphi|_{3,1}$ and $|\tilde{f}|_{1,1}$, such that for $x\in\mathcal{N}\cap\partial\Omega$, with $|x|\leqslant\theta_i\sqrt{b_i}$, we have the estimates

$$|u_{\gamma\tau_j}(x)|\leqslant C\sqrt{b_i} \tag{5.2}$$

for $j=1,...,i$, where C is a constant depending on Ω, $|\varphi|_{3,1}$ and $|\tilde{f}|_{1,1}$. When $k=n-2$, (5.2) is exactly (4.1) with $\theta_{n-1}=\frac{1}{2}$. We shall prove that there exists a constant θ_k, also depending on Ω, $|\varphi|_{3,1}$ and $|\tilde{f}|_{1,1}$, such that for $x\in\mathcal{N}\cap\partial\Omega$, with $|x|\leqslant\theta_k\sqrt{b_k}$, we have

$$|u_{\gamma\tau_j}(x)|\leqslant C\sqrt{b_k} \tag{5.3}$$

for $j=1,...,k$, where C is a constant depending on Ω, $|\varphi|_{3,1}$ and $|\tilde{f}|_{1,1}$. The estimate (5.1) then follows from (4.1) and (5.3) by induction.

To prove (5.3), we introduce the dilation $x\to y=T(x)$ defined by

$$y_i=M_ix_i,\quad i=1,...,n,$$

where

$$M_i=\begin{cases}\sqrt{M}, & i=1,...,k,\\ \sqrt{b_i}\,M, & i=k+1,...,n-1,\\ M, & i=n,\end{cases}$$

and

$$M=\frac{1}{b_k}.$$

We may suppose that $b_k\leqslant\frac{1}{2}b_{k+1}$, otherwise (5.3) follows immediately from (5.2). Let $v(y)=M^2u(x)$. Then v satisfies

$$\det(D_y^2v)=g(y)=:\frac{f(y_i/M_i)\,M^{2n}}{\prod_{i=1}^n M_i^2}\quad\text{in } \widetilde{\Omega}=T(\Omega).$$

Near the origin $\partial\widetilde{\Omega}$ is represented by

$$y_n=\tilde{\varrho}(y')=\frac{1}{2}\sum_{i=1}^{n-1}d_iy_i^2+O(|y'|^3), \tag{5.4}$$

where

$$d_i=\begin{cases}1, & i\leqslant k,\\ b_k/b_i\ (\leqslant 1), & i\geqslant k+1.\end{cases}$$

After the transformation we have, for $\widetilde{\mathcal{N}}=T(\mathcal{N})$,

$$v(y)=\tilde{\varphi}(y')=\frac{1}{2}\sum_{i=1}^{n-1}\tilde{b}_i y_i^2+R(y')+O(|y'|^4), \quad y=(y',\tilde{\varrho}(y'))\in\widetilde{\mathcal{N}}\cap\partial\widetilde{\Omega}, \tag{5.5}$$

where

$$\tilde{b}_i=\begin{cases}1, & i\geqslant k,\\ b_i/b_k\ (\leqslant 1), & i<k.\end{cases}$$

As above we see that near the origin, both $\tilde{\varphi}$ and $\partial\widetilde{\Omega}$ are $C^{3,1}$-smooth and their $C^{3,1}$-norms are independent of M.

Let

$$\omega=\{y\in\widetilde{\Omega}\mid y_n<\beta^2,\ |y_i|<\beta,\ i=k+1,\dots,n-1\}, \tag{5.6}$$

where β will be chosen small such that the third- and high-order terms in (5.4) and (5.5) do no harm to the following estimation. As before we may assume that $\omega\subset\widetilde{\mathcal{N}}$ and, by (5.4), that ω is bounded independently of M. By the convexity of v we have

$$v\leqslant C \quad \text{in } \omega.$$

Similarly to (3.13) we have

$$\sup\{g(y)\mid y\in\omega\}\leqslant C. \tag{5.7}$$

To prove (5.3) it is crucial to establish a bound for the normal derivative of v near the origin. The main difficulty is that we cannot control the convexity of $\partial\widetilde{\Omega}$ near the origin. We construct a lower barrier $\underline{v}$ by setting

$$\underline{v}(y)=\tfrac{1}{2}\sigma|y'|^2+\tfrac{1}{2}Ky_n^2-K^2y_n, \tag{5.8}$$

where $\sigma>0$ small and $K>1$ large will be chosen so that

$$\det(D^2\underline{v})=\sigma^{n-1}K\geqslant\sup_{\omega}g(y). \tag{5.9}$$

We claim that $\underline{v}\leqslant v$ on $\partial\omega$ (with appropriate choice of β, σ and K). For later application we will prove the stronger inequality

$$\underline{v}\leqslant\tfrac{1}{2}v \quad \text{on } \partial\omega. \tag{5.10}$$

To prove (5.10) we first consider the piece $\partial_1\omega:=\partial\omega\cap\partial\widetilde{\Omega}$. For $y\in\partial_1\omega$ we have, by (5.5),

$$v(y)=\frac{1}{2}\sum_{i=1}^{n-1}\tilde{b}_i y_i^2+O(|y'|^3)\geqslant\frac{1}{4}|\tilde{y}|^2-C|\hat{y}|^2$$

provided β is small, where $\hat{y}=(y_1,...,y_k)$ and $\tilde{y}=(y_{k+1},...,y_{n-1})$. By (5.4) we have

$$\underline{v}(y)\leqslant\tfrac{1}{2}\sigma|y'|^2-\tfrac{1}{2}K^2y_n\leqslant\tfrac{1}{2}\sigma|\tilde{y}|^2-\tfrac{1}{4}K^2|\hat{y}|^2.$$

Hence (5.10) holds on $\partial_1\omega$. On $\partial_2\omega:=\partial\omega\cap\{y_n=\beta^2\}$ we have $v\geqslant 0$. For $\sigma>0$ small and $K>1$ large, we have $\underline{v}\leqslant-\frac{1}{2}K^2\beta^2$, so that (5.10) also holds on $\partial_2\omega$.

Finally we consider the piece $\partial_3\omega:=\partial\omega\cap\{|y_i|=\beta \text{ for some } i=k+1,...,n-1\}$. We only consider the piece $\partial_3'\omega:=\partial\omega\cap\{y_{n-1}=\beta\}$ since other pieces of $\partial_3\omega$ can be handled in the same way. First we prove that

$$v(y)\geqslant\tfrac{1}{4}\beta^2\quad\text{on } \partial_3'\omega\cap\{y_n<\varepsilon_0\beta^2\}\tag{5.11}$$

provided ε_0 is small enough. If $\partial_3'\omega\subset\{y_n\geqslant\varepsilon_0\beta^2\}$, we have nothing to prove, so we may suppose $\partial_3'\omega\cap\{y_n<\varepsilon_0\beta^2\}\neq\varnothing$. To prove (5.11) we first fix a point $p=(\hat{p},\tilde{p},p_n)\in\partial_3'\omega$, where $\hat{p}=(p_1,...,p_k)\neq 0$, $\tilde{p}=(p_{k+1},...,p_{n-1})$, $p_n<\varepsilon_0\beta^2$. For $\delta\geqslant 0$ sufficiently small, we then fix a further point $p^*=(0,\tilde{p},p_n+\delta)$ so that the straight line through p and p^* meets $\partial_3'\omega$ in a point $\bar{p}$ satisfying

$$\tfrac{1}{2}|p-p^*|\leqslant|\bar{p}-p^*|\leqslant|\bar{p}-p|.\tag{5.12}$$

In view of the convexity of $\widetilde{\Omega}$ and the representation (5.4), we may accomplish (5.12) by taking

$$\delta=|\nabla_{\hat{y}}\tilde{\varrho}(0,\tilde{p})|\cdot|\hat{p}|\leqslant O(|\beta|^3)$$

and β sufficiently small. Now let $p^0=(0,\tilde{p},\tilde{\varrho}(0,\tilde{p}))$ be the projection of p^* on $\partial\widetilde{\Omega}$. We claim that

$$|v_{\tilde{\gamma}}(p^0)|\leqslant C,\tag{5.13}$$

where $\tilde{\gamma}$ is the unit inner normal at p^0. Indeed, for any $y=(0,\tilde{y},\tilde{\varrho}(0,\tilde{y}))\in\partial\widetilde{\Omega}\cap B_{\theta_{k+1}}(0)$, with θ_{k+1} as given in (5.2), we have (for $x\in T^{-1}(y)$)

$$|x_i|=\left|\frac{y_i}{M_i}\right|\leqslant\theta_{k+1}\frac{b_k}{b_i^{1/2}},\quad i=k+1,...,n-1.$$

Hence by (5.2),

$$|u_\gamma(x)|\leqslant u_\gamma(0)+C\sum_{i=k+1}^{n-1}\sup\{|x_i|\cdot|u_{\gamma\tau_i}(x)|\mid|x_i|\leqslant\theta_{k+1}b_k/b_i^{1/2}\}\leqslant Cb_k,$$

and since $u_\gamma(0)=0$ by (3.2), we obtain

$$|\nabla u(x)|\leqslant Cb_k\quad\text{at } x=T^{-1}(y).$$

By the definition of T,

$$|v_{y_n}(y)| = M|u_{x_n}(x)| \leqslant CMb_k = C.$$

Noticing that $v=\tilde{\varphi}\in C^3$ on $\partial\tilde{\Omega}$, we obtain (5.13). From (5.13) and the convexity of v, we have

$$v(p^*) \geqslant v(p^0) - C|p_n^* - p_n^0|, \tag{5.14}$$

while, from (5.5),

$$v(p^0) = \frac{1}{2}\sum_{i=k+1}^{n-1} |p_i^0|^2 + O(|p^0|^3)$$

and

$$v(\bar{p}) = \frac{1}{2}\sum_{i=1}^{n-1} \tilde{b}_i|\bar{p}_i|^2 + O(|\bar{p}|^3) = v(p^0) + \frac{1}{2}\sum_{i=1}^{k} \tilde{b}_i|\bar{p}_i|^2 + O(|\bar{p}|^3).$$

Noticing that $p_{n-1}^0 = \beta$, we have $v(p^0) \geqslant \frac{1}{2}\beta^2 - O(\beta^3) \geqslant \frac{3}{8}\beta^2$ if β is small enough. Since $d_i=1$ in (5.4) when $i\leqslant k$, we see that

$$\sum_{i=1}^{k} \tilde{b}_i|\bar{p}_i|^2 \leqslant C(|\bar{p}_n - p_n^0| + \beta^3) \leqslant C(\bar{p}_n + \beta^3).$$

Hence we may first fix β small and then choose ε_0 small so that $\bar{p}_n \leqslant 2\varepsilon_0\beta^2$ and

$$v(\bar{p}) \leqslant \tfrac{9}{8} v(p^*),$$

whence by (5.14), $v(p^*) \geqslant \frac{1}{3}\beta^2$. Hence by (5.12) and the convexity of v, we have

$$v(p) \geqslant 3v(p^*) - 2v(\bar{p}) \geqslant \tfrac{3}{4} v(p^*) \geqslant \tfrac{1}{4}\beta^2,$$

and (5.11) is proved.

From (5.11) we have

$$v(y) \geqslant \tfrac{1}{8}\beta^2 - Cy_n \quad \text{on } \partial_3'\omega,$$

where C depends on β and ε_0. On the other hand,

$$\underline{v}(y) \leqslant C\sigma - K^2 y_n.$$

Hence $\underline{v} < \frac{1}{2}v$ on $\partial_3'\omega$ if σ is small and K large enough. This completes the proof of inequality (5.10).

The next step in our proof is to adapt the barrier $\underline{v}$ to obtain a normal derivative bound near the origin. For any point $y_0\in\partial_1\omega$, let $\xi=(\xi_1,...,\xi_n)=A(y-y_0)$ be an orthogonal basis at y_0 so that ξ_n coincides with the inner normal, where A is some orthogonal matrix. Instead of (5.8), we now set

$$\underline{v}(y)=\tfrac{1}{2}\sigma|\xi'|^2+\tfrac{1}{2}K\xi_n^2-K^2\xi_n+l(\xi'),$$

where $\sigma>0$ small and $K>1$ large will be chosen so that (5.9) holds, $\xi'=(\xi_1,...,\xi_{n-1})$, and l is a linear function such that $|l(\xi')-v(y)|=o(|\xi'|)$ as $\xi'\to 0$. Since both $\widetilde{\varphi}$ and $\partial\widetilde{\Omega}$ are $C^{3,1}$-smooth near the origin, arguing as above, and by virtue of (5.11), we see that $\underline{v}\leqslant\frac{1}{2}v$ on $\partial\omega$ if $|y_0|$ and $\sigma\,(>0)$ are sufficiently small and $K>1$ is sufficiently large. By (5.9) and the comparison principle, it follows that $\underline{v}(y)\leqslant v(y)$ on ω. We therefore obtain, by the convexity of v,

$$|v_{\tilde{\gamma}}(y)|\leqslant C \quad \text{for } y\in\partial\widetilde{\Omega} \text{ near the origin.} \tag{5.15}$$

By choosing a new β we can then ensure that the estimate (5.15) holds for all $y\in\partial_1\omega=\partial\omega\cap\partial\widetilde{\Omega}$, where ω is given by (5.6), and subsequently, from the convexity of v,

$$|Dv|\leqslant C \quad \text{in } \omega. \tag{5.16}$$

We can now complete the proof of (5.1) by standard arguments [2], [5]. For convenience we follow that in [2]. Let

$$\mathcal{L}w=v^{ij}w_{ij},$$

where $\{v^{ij}\}$ denotes the inverse of the Hessian D^2v, and $T=(T_1,...,T_k)$, where

$$T_i=\partial_i+(y_i\partial_n-y_n\partial_i), \quad i=1,...,k.$$

Applying the operator T to both sides of the equation

$$F(D^2v):=\log\det(D^2v)=\log g,$$

we obtain

$$\mathcal{L}(Tv)=T(\log g),$$

where $\mathcal{L}=v^{ij}\partial_i\partial_j$. Setting

$$w(y)=\pm T(v-\widetilde{\varphi})(y)+B|y|^2,$$

we have, since $d_i=1$ for $i\leqslant k$ in (5.4),

$$|w(y)|\leqslant C_B(|y|^2+y_n) \quad \text{on } \partial\omega. \tag{5.17}$$

Similarly to (3.15) we have

$$|\nabla g^{1/2(n-1)}(y)| \leqslant C,$$

so that

$$|\nabla g| \leqslant C g^{(2n-3)/2(n-1)}(y).$$

Hence

$$\begin{aligned}\mathcal{L}w &\geqslant B\sum v^{ii} - C\Big(g^{-1/2(n-1)} + \sum v^{ii}\Big)\\ &\geqslant \tfrac{1}{2}B\sum v^{ii} - Cg^{-1/2(n-1)}\\ &\geqslant \tfrac{1}{2}Bg^{-1/n} - Cg^{-1/2(n-1)} \geqslant 0\end{aligned}$$

provided B is large enough. Now set

$$\widetilde{w}(y) = A(\underline{v} - v - y_n) + w(y),$$

where $\underline{v}$ is given by (5.8), and $A>1$ is a sufficiently large constant to be chosen later. By (5.9) and the concavity of F, we have

$$\mathcal{L}(\underline{v} - v) \geqslant F(D^2\underline{v}) - F(D^2 v) \geqslant 0.$$

Consequently

$$\mathcal{L}\widetilde{w} \geqslant 0, \tag{5.18}$$

which implies that the function $\widetilde{w}$ attains the maximum on the boundary of ω.

We claim that $\widetilde{w}(y) \leqslant 0$ on $\partial\omega$ for sufficiently large A. This is because, by (5.10) and (5.17),

$$\widetilde{w}(y) \leqslant C_B(|y'|^2 + y_n) - \tfrac{1}{2}A(v(y) + y_n).$$

Using (5.4) and (5.5), we then choose A large enough so that $\widetilde{w}(y) \leqslant 0$ on $\partial\omega$.

Noticing that $\widetilde{w}(0)=0$, we have therefore $(\partial\widetilde{w}/\partial y_n)(0) \leqslant 0$. Namely, $|v_{in}(0)| \leqslant C$, $i=1, ..., k$. Similarly we have $|v_{\tilde{\gamma}\tilde{\tau}_i}(y)| \leqslant C$, $i=1, ..., k$, for $y \in \partial\omega \cap \partial\widetilde{\Omega}$ near the origin, where $\tilde{\tau}_i$ is a tangential direction of $\partial\widetilde{\Omega}$ at y which lies in the plane parallel to the y_i- and y_n-axes and passes through the point y. Pulling back to the x-coordinates we obtain (5.1). □

6. Concluding remarks

To complete the proof of the second-derivative estimate (1.2) in Theorem 1.1, we write (as in the case $n=2$ in (4.8)) equation (1.1) at the origin in the form

$$\left(\prod_{i=1}^{n-1} b_i\right) u_{nn}(0) = \sum_{j=1}^{n-1}\left(\prod_{i\neq j} b_i\right) u_{jn}^2(0) + f(0), \tag{6.1}$$

so that

$$0 \leqslant u_{nn}(0) = \sum_{j=1}^{n-1} \frac{u_{jn}^2(0)}{b_j} + \frac{f(0)}{\prod_{i=1}^{n-1} b_i} \leqslant C \tag{6.2}$$

by Lemmas 3.1 and 5.1. Combining (3.4), (4.1) or (5.1) and (6.2), we thus obtain

$$|D^2 u| \leqslant C \tag{6.3}$$

on $\partial\Omega$, and subsequently in Ω by (2.1), where in both (6.2) and (6.3) the constant C depends on Ω, $|\tilde{f}|_{1,1}$ and $|\varphi|_{3,1}$. The estimate (1.1) is thus established for the non-degenerate case $f>0$ on $\bar{\Omega}$. To get the full generality of Theorem 1.1, we need to solve approximating Dirichlet problems (using [2] or [5]), with f replaced by $f+\varepsilon$, for $\varepsilon>0$ constant, and deduce the existence assertion of Theorem 1.1 by sending ε to zero.

The conditions on the function f can be weakened somewhat. In particular, in the derivation of the estimate (6.3) on the boundary $\partial\Omega$, we have only used the condition $f^{1/2(n-1)} \in C^{0,1}(\bar{\Omega})$. For the full second-derivative estimate in Ω, we only need to assume additionally that $\Delta f^{1/(n-1)}$ is bounded from below (in the sense of distributions) in the proof of Lemma 2.1 [7]. Moreover, by employing the Aleksandrov–Bakelman estimate [5] instead, we can replace the latter condition by $\Delta f^{1/n}$ being bounded from below by a function in $L^n(\Omega)$. For application to problems in differential geometry (as, for example, in [8]), which provided the motivation for Guan's approach in [7], it is desirable to impose no restriction on the non-negative function f apart from smoothness, as in the cases $n=2,3$ in [7].

Finally we remark that the results of this work carry over to more general Monge–Ampère equations of the form

$$\det D^2 u = f(x,u)\psi(Du), \tag{6.4}$$

where f is monotone increasing in u and ψ convex. In particular, we may consider the equation of prescribed Gauss curvature

$$\det D^2 u = f(x)(1+|Du|^2)^{(n+2)/2} \tag{6.5}$$

with again $f^{1/(n-1)} \in C^{1,1}(\bar{\Omega})$, in the presence of barriers, so that Theorem 1.1 would continue to hold if

$$\int_\Omega f < \omega_n \tag{6.6}$$

and, for example, $f=0$ on $\partial\Omega$ [23].

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