SPECIAL FUNCTIONS ON LOCALLY COMPACT FIELDS

BY

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§ 1. Introduction

In this paper we establish a variety of facts relating to analysis on a locally compact, totally disconnected, non-discrete field, referred to hereafter as K. These fields, which include the $\mathfrak{p}$-adic number fields and their finite algebraic extensions, have been studied in some detail in relation to algebraic number theory and, more recently, in the subject of group representations ([2], [3], [5], [7]).

In these studies certain "special functions" arise as Fourier transforms of additive or multiplicative characters (and combinations of them). The usual approach has been to truncate the characters so as to produce L^1 functions on the additive structure (K^+, dx), or the multiplicative structure (K^*, d^*x) of K and then work with the transforms of the truncated characters.

In [3], however, the non-truncated characters are used in an essential way. It is our purpose to examine the transforms of the non-truncated characters in somewhat more detail than that to be found in [3]. The "special functions" that arise here are the gamma, beta, and Bessel functions. These functions coincide with those introduced in [3]. We also treat the Hankel transform which is not mentioned explicitly in [3]. The gamma, beta, and Bessel functions are first introduced as complex valued functions on appropriate domains and these functions are then related to various distributions on (K^+, dx), (K^*, d^*x) and $(\hat{K}^*, d\pi)$, the group of unitary (multiplicative) characters on K^*.

This paper will be followed with applications to the representations of $SL(2, K)$ by the former author, and to the study of potential spaces and Lipschitz spaces on the finite dimensional vector spaces over K by the latter author.

In § 2 basic harmonic analysis on K^+ and K^* is treated, mostly without proof, but in a form required in the later sections. The major portion of the results stated are either well-known or may be found in [3]. We supply the proof of a few of the less well-known results.

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In § 3 we present the gamma and beta functions. We show that these functions can be computed explicitly. The gamma function turns out to be equal to the invariant factor $\varrho(\pi)$ (π a multiplicative character) defined by Tate (see [4], Ch. VII, § 3).

In § 4 the Bessel function is computed in terms of gamma functions and "trigonometric polynomials". The properties of the Bessel function as a distribution are also detailed. We note that Saito [6] has also defined a Bessel function for $\mathfrak{p}$-adic fields with certain restrictions. This arises in a somewhat different context from the Bessel function that we discuss here. These two Bessel functions are not the same, but there seems to be a connection between them which is somewhat analogous to the connection between Bessel functions of different kinds in the classical case.

In § 5 the Hankel transform is treated and its properties as an operator on $L^p(K^+)$, $1 \leqslant p \leqslant 2$, are developed. In particular, it is shown to be a unitary operator on $L^2(K^+)$.

§ 2. Preliminaries

Let K be a locally compact, totally disconnected, non-discrete field. Such fields have been completely classified (see [1], p. 159). If K has characteristic zero, then K is either a $\mathfrak{p}$-adic field for some rational prime p or a finite algebraic extension of such a field. If K has characteristic $p \neq 0$, then K is isomorphic to a field of formal power series over a finite field of characteristic p.

Let K^+ be the additive group and K^* the multiplicative group of K, and let dx be a Haar measure on K^+. There is a natural (non-archimedean) norm on K satisfying the relations: $d(ax) = |a|dx$; $|x+y| \leqslant \max[|x|, |y|]$; and $|x+y| = \max[|x|, |y|]$ if $|x| \neq |y|$. A Haar measure on K^* is given by $d^*x = |x|^{-1}dx$.

The set $O = \{x : |x| \leqslant 1\}$ is a subring of K, the ring of integers. We normalize dx so that O has measure 1. The set $\mathfrak{P} = \{x : |x| < 1\}$ is the unique maximal ideal in O ($\mathfrak{P}$ is also principal). $O/\mathfrak{P}$ is a finite field and contains q elements, q a prime power. Let $\mathfrak{p}$ be a generator ot $\mathfrak{P}$. Then $|\mathfrak{p}| = q^{-1}$, and, for all $x \in K$ either $|x| = 0$ (when and only when $x = 0$) or $|x| = q^n$ for some integer n. It follows that $\mathfrak{P}^n$ has measure q^{-n}, $n \geqslant 1$, and the measure of $\{x : |x| = q^n\}$ is q^n/q' $(1/q' = 1 - 1/q)$ for all integers n. By an abuse of standard notation we define $\mathfrak{P}^n = \{x : |x| \leqslant q^{-n}\}$ for all integers n. The collection $\{\mathfrak{P}^n\}_{n=0}^{\infty}$ is a neighborhood basis for the identity in K^+.

There is a non-trivial character χ on K^+ which is trivial on $O = \mathfrak{P}^0$ but is non-trivial on $\mathfrak{P}^{-1}$. Every unitary character on K^+ has the form $\chi_u(x) = \chi(ux)$ for some $u \in K$. The mapping $u \to \chi_u$ is a topological isomorphism of K^+ onto $\hat{K}^+$, so we identify K^+ and its dual. The Fourier transform on K^+ is initially defined on the complex-valued functions in $L^1(K^+)$ as $\hat{f}(u) = \int_K f(x)\chi(ux)\,dx$.

Let S be the set of functions on K which have compact support and are constant on the cosets (in K^+) of $\mathfrak{P}^n$ for some n. S is a linear space of continuous functions on K and a topology is induced in S if we define a null sequence to be $\{\varphi_k\}$ where the φ_k all vanish outside a fixed compact set, are constant on the cosets of a fixed $\mathfrak{P}^n$, and tend uniformly to zero. In this topology, S is complete. S is called the space of testing functions on K^+. S', the topological dual of S, with the weak topology is called the space of distributions on K^+. S' is also a linear complete space. The action of f in S' on an element φ in S is denoted (f, φ). The Fourier transform is a topological isomorphism of S onto itself. The following important relations are used in an essential way in the sequel.

LEMMA (A_1). *Suppose that $\varphi \in S$. φ is supported on $\mathfrak{P}^n$ and is constant on the cosets (in K^+) of $\mathfrak{P}^m$ if and only if $\hat{\varphi}$ is supported on $\mathfrak{P}^{-m}$ and is constant on the cosets (in K^+) of $\mathfrak{P}^{-n}$.*

Proof. If φ is supported on $\mathfrak{P}^n$ and we take any $h \in K$ such that $|h| \leqslant q^n$, then

$$\hat{\varphi}(u+h) = \int_{|x| \leqslant q^{-n}} \chi(xu+xh)\, \varphi(x)\, dx = \int_{|x| \leqslant q^{-n}} \chi(xu)\, \varphi(x)\, dx = \hat{\varphi}(u),$$

since $|xh| \leqslant 1$ for x in the given range. Hence $\hat{\varphi}$ is constant on the cosets of $\mathfrak{P}^{-n}$. On the other hand, if φ is constant on the cosets of $\mathfrak{P}^m$, then

$$\hat{\varphi}(u) = \int_K \chi(xu)\, \varphi(x)\, dx = \int_K \chi(xu)\, \varphi(x+h)\, dx$$

for any $h \in K$ such that $|h| \leqslant q^{-m}$. Changing variables we get

$$\hat{\varphi}(u) = \chi(-hu) \int_K \chi(xu) \varphi(x)\, dx = \chi(-hu)\hat{\varphi}(u).$$

If $|u| > q^m$, then there is an element h in $\mathfrak{P}^m$ so that $|uh| > 1$ and $\chi(uh) \neq 1$. Hence $\hat{\varphi}(u) = 0$. Q.E.D.

For every $f \in S'$, the Fourier transform of f is in S' and is defined by $(\hat{f}, \varphi) = (f, \hat{\varphi})$. The Fourier transform is a topological isomorphism of S'.

We define Φ_n to be the characteristic function of $\mathfrak{P}^n$. $\Phi_n \in S$ and $\hat{\Phi}_n = q^{-n}\Phi_{-n}$, so that Φ_0 is an eigenfunction of the Fourier transform on $S \subset L^2(K^+)$. We note that $\{q^n \Phi_n\}_{n=0}^{\infty}$ is an approximation to the identity on (K^+, dx) which is non-negative; $q^n\Phi_n \in S$ for all n. It is easy to see that S is dense in $L^p(K^+, dx) = L^p(K^+)$, $1 \leqslant p < \infty$, and also in the continuous functions that vanish at infinity. Locally integrable functions and measures on K^+ are identified with the elements in S' which they induce. For example, if f is locally integrable we set $(f, \varphi) = \int_K f(x)\varphi(x)\, dx$, $\varphi \in S$. We single out one measure, the "delta function" δ, which has mass one at the origin, $(\delta, \varphi) = \varphi(0)$, $\varphi \in S$.

For locally integrable f we define $[f]_n$ to be the function which is equal to f on the set $\{x: q^{-n} \leqslant |x| \leqslant q^n\}$ and zero otherwise. The principal value integral of f is defined by

$$P\int_K f(x)\,dx = \lim_{n\to\infty} \int_K [f]_n(x)\,dx.$$

Plancherel's theorem on K^+ takes the following form: If $f \in L^2(K^+)$, then $\{\int_K [f]_n(x)\chi(ux)\,dx\}$ converges in the L^2-norm (as $n \to \infty$) to $\hat{f}$, $\hat{f} \in L^2$ and $\|\hat{f}\|_2 = \|f\|_2$. If $P\int_K [f]_n(x)\chi(ux)\,dx$ exists for a.e. $u \in K$, then $\hat{f}(u) = P\int_K f(x)\chi(ux)\,dx$ for a.e. $u \in K$.

Let $\mathcal{O}^* = \mathcal{O} - \mathfrak{P}$ be the group of units in $\mathcal{O}$. There is an element ε in $\mathcal{O}^*$ of order $q-1$. The set $\{0, 1, \varepsilon, \varepsilon^2, \ldots, \varepsilon^{q-2}\}$ is a complete set of coset representatives for $\mathcal{O}/\mathfrak{P}$. The set $A = \{x: |1-x| < 1\}$ is a compact subgroup of K^*, and every x in K^* can be written uniquely in the form $x = \mathfrak{p}^n \varepsilon^k a$, where $|x| = q^{-n}$, $0 \leqslant k \leqslant q-2$ and $a \in A$. Thus K^* is the direct product of three groups, $K^* = \mathbf{Z} \times \mathbf{Z}_{q-1} \times A$, where $\mathbf{Z}$ here denotes the infinite cyclic group of elements $\{\mathfrak{p}^n\}_{n=-\infty}^{\infty}$ and $\mathbf{Z}_{q-1}$ denotes the finite cyclic group of elements $\{\varepsilon^k, 0 \leqslant k \leqslant q-2\}$. Define $A_0 = \mathcal{O}^*$, $A_1 = A = 1 + \mathfrak{P}$, $A_n = 1 + \mathfrak{P}^n$, $n \geqslant 1$. The collection $\{A_n\}_{n=0}^{\infty}$ is a neighborhood basis for the identity in K^*.

Let $\pi \in \hat{K}^*$ be a (multiplicative) unitary character on K^*. Since π is continuous we see that π is trivial on some A_n. If π is trivial on $A_0 = \mathcal{O}^*$, we say that π is unramified or has ramification degree 0. If π is trivial on A_n, but not on A_{n-1} ($n \geqslant 1$), we say that π is ramified and has ramification degree n.

The direct product representation for K^* shows that $\hat{K}^*$ is the direct product of the circle group (unitary characters depending only on the norm of x, $\pi(x) = |x|^{i\alpha}$, $-\pi/\ln q < \alpha \leqslant \pi/\ln q$), the cyclic group of order $q-1$ (unitary characters depending only on ε^k, $\pi(\varepsilon)$ a $(q-1)$st root of unity) and an infinite discrete group $\hat{A}$, the dual of A (unitary characters determined by their values on A). There are only a finite number of characters of A of each ramification degree, so that $\hat{A}$ is countable. Writing $\pi \in \hat{K}^*$ as $\pi = \pi^* |x|^{i\alpha}$, α real, π^* a character on $\mathcal{O}^* = \mathbf{Z}_{q-1} \times A$, we see that $\hat{K}^*$ can be viewed as a countable discrete collection of circles, each circle T_{π^*} indexed by a character π^* on $\mathcal{O}^*$. Haar measure, $d\pi$, on $\hat{K}^*$ is then given by integrating over each circle T_{π^*} with respect to the usual measure $d\alpha$, and then summing over the countable collection $\{T_{\pi^*}, \pi^* \in \hat{\mathcal{O}}^*\}$.

If $f \in L^1(\hat{K}^*, d\pi)$ and we choose a suitable normalizing factor (to be determined below), then

$$\int_{\hat{K}^*} f(\pi)\,d\pi = \sum_{\pi^*} \frac{1}{a} \int_{\pi \in T_{\pi^*}} f(\pi)\,d\alpha = \sum_{\pi^*} \frac{1}{a} \int_{-\pi/\ln q}^{\pi/\ln q} f(\pi^* |x|^{i\alpha})\,d\alpha.$$

The collection of test functions, S^*, on K^* is the set of complex valued functions on K^* with compact support ($\varphi \in S^*$ implies $\varphi(x) = 0$ if $|x| < q^{-n}$ or $|x| > q^n$ for some integer n)

and which are constant on the cosets (in K^*) of some A_m. A topology is induced in S^* by defining a null sequence to be $\{\varphi_k\}$ where the φ_k all vanish outside a fixed compact set in K^*, are constant on the cosets of a fixed A_n and tend uniformly to zero. The topological dual, $S^{*\prime}$, of S^* is called the space of distributions on K^*.

The following lemmas give the exact relationship between the space S of test functions on K^+ and the space S^* of test functions on K^*.

LEMMA (A_2). *Suppose $\varphi \in S$ and φ is constant on the cosets of $\mathfrak{P}^n$. Then φ is constant on the cosets of A_{n-k} in $\mathfrak{P}^k$ for all integers $k \leqslant n$. In particular, if $\varphi(0)=0$, then $\varphi \in S^*$.*

Proof. Assume that $aA_{n-k}=bA_{n-k}$, $|a|=|b|=q^{-k}$, $k \leqslant n$. Then $ab^{-1} \in A_{n-k}$, so that $1-ab^{-1} \in \mathfrak{P}^{n-k}$. Using the fact that $|a-b|=|b|\,|1-ab^{-1}|$, we see that $a-b \in \mathfrak{P}^n$ so that $\varphi(a)=\varphi(b)$. Furthermore, if φ is supported on $\mathfrak{P}^{-m}$, $-m \leqslant n$, then φ is constant on the cosets of A_{n+m}. Hence, if $\varphi(0)=0$, $\varphi \in S^*$. Q.E.D.

LEMMA (M_1). *Suppose $\varphi \in S^*$ and φ is constant on the cosets of A_n, $n \geqslant 0$. Then φ is constant on the cosets of $\mathfrak{P}^{n+k}$ in $K-\mathfrak{P}^{k+1}$ for all integers k. In particular, $\varphi \in S$.*

Proof. The first part is essentially the reversal of the corresponding proof in Lemma (A_2). If $\varphi(x)\equiv 0$ on $\mathfrak{P}^{m+1}$, then φ is constant on the cosets of $\mathfrak{P}^{m+n}$ so that $\varphi \in S$. Q.E.D.

The collection of test functions $\hat{S}^*$ on $\hat{K}^*$ is the set of complex valued functions on $\hat{K}^*$ with compact support (vanishing off a finite number of circles), and on each circle is a trigonometric polynomial (that is, $\varphi(\pi)=\varphi(\pi^*|x|^{i\alpha})=\sum_{\nu=-m}^{m} a_\nu(\pi^*)q^{i\nu\alpha}$). A topology is induced in $\hat{S}^*$ by defining a null sequence to be $\{\varphi_k\}$ where the φ_k all vanish outside a fixed compact set in $\hat{K}^*$, the degrees of the restrictions of φ_k to T_{π^*} have a common bound for each circle T_{π^*} and tend uniformly to zero. The topological dual, $\hat{S}^{*\prime}$, of $\hat{S}^*$ is called the space of distributions on $\hat{K}^*$. $\hat{S}^*$ and $\hat{S}^{*\prime}$ are complete linear spaces. As usual we identify the distributions induced by locally integrable functions and measures on K^* and $\hat{K}^*$ with the respective functions or measures.

Let Ψ_n be the characteristic function of A_n, Λ_n the characteristic function of those circles in $\hat{K}^*$ corresponding to the characters π^* which are reamified of degree less than or equal to n. We denote this set of characters by A_n'.

The Fourier transforms on K^* and $\hat{K}^*$ are denoted Mellin transforms in this paper. They are initially defined on $L^1(K^*, d^*x)$ and $L^1(\hat{K}^*, d\pi)$. If $f \in L^1$, the Mellin transform of f is denoted by $\tilde{f}$. If $f \in L^1(K^*, d^*x)$,

$$\tilde{f}(\pi)=\int_{K^*} f(x)\,\pi(x)\,d^*x.$$

If $f \in L^1(\hat{K}^*, d\pi)$, $$\check{f}(x) = \int_{\hat{K}^*} f(\pi)\, \pi^{-1}(x)\, d\pi.$$

It is easily checked that the Mellin transform is a topological isomorphism of S^* onto $\hat{S}^*$ and of $\hat{S}^*$ onto S^*.

These Mellin transforms extend to $L^2(K^*)$ and $L^2(\hat{K}^*)$ by the usual Plancherel argument. We may select the normalizing factor, a, so that $\|f\|_{2,d^*x} = \|\tilde{f}\|_{2,d\pi}$ for $f \in L^2(K^*)$ and $\|g\|_{2,d\pi} = \|\check{g}\|_{2,d^*x}$ for $g \in L^2(\hat{K}^*)$.

The following facts are used in an essential way in the sequel.

LEMMA (M_2). *$\varphi \in S^*$ and φ is supported on the set $\{x : q^{-n} \leqslant |x| \leqslant q^n\}$ and is constant on the cosets (in K^*) of A_m if and only if $\tilde{\varphi} \in \hat{S}^*$, $\tilde{\varphi}$ is supported on A'_m and the restriction of $\tilde{\varphi}$ to each circle T_{π^*} is of degree bounded by n.*

Proof. Suppose $\varphi \in S^*$ and φ is constant on the cosets of A_m. Take $\pi \in \hat{K}^*$ such that the ramification degree of π is greater than m. Then

$$\tilde{\varphi}(\pi) = \int_{q^{-n} \leqslant |x| \leqslant q^n} \varphi(x)\, \pi(x)\, d^*x = \sum_s \varphi(\beta_s) \int_{\beta_s A_m} \pi(x)\, d^*x.$$

Since π is not trivial on A_m, the integrals in the sum are all zero. Hence $\tilde{\varphi}(\pi) = 0$ if $\pi \notin A'_m$. Furthermore, we see that, if $\pi(x) = \pi^*(x)|x|^{i\alpha}$, then

$$\tilde{\varphi}(\pi) = \int_{q^{-n} \leqslant |x| \leqslant q^n} \varphi(x)\, \pi^*(x)\, |x|^{i\alpha} d^*x = \sum_{k=-n}^{n} \left(\int_{|x| = q^k} \varphi(x)\, \pi^*(x)\, d^*x \right) q^{ik\alpha}.$$

If we restrict $\tilde{\varphi}$ to a fixed circle T_{π^*}, then we get trigonometric polynomials of degree not greater than n.

Now suppose $\tilde{\varphi} \in \hat{S}^*$, $\tilde{\varphi}$ restricted to T_{π^*} has the form

$$\tilde{\varphi}(\pi^* |x|^{i\alpha}) = \sum_{\nu=-k}^{k} a_\nu(\pi^*)\, q^{i\nu\alpha}.$$

Suppose further that $\tilde{\varphi}$ is supported on A'_m and that the degrees of the trigonometric polynomials above are bounded uniformly by n. We then have

$$\varphi(x) = \int_{\hat{K}^*} \tilde{\varphi}(\pi)\, \pi^{-1}(x)\, d\pi = \sum_{\pi^* \in A'_m} \frac{1}{a} \sum_{\nu=-k(\pi^*)}^{k(\pi^*)} a_\nu(\pi^*)\, \pi^{*-1}(x) \int_{-\pi/\ln q}^{\pi/\ln q} q^{i\nu\alpha} |x|^{-i\alpha} d\alpha.$$

It is clear that, if $|x| < q^{-n}$ or $|x| > q^n$, each of the integrals in the last sum is zero. Hence φ is supported on the set $\{x : q^{-n} \leqslant |x| \leqslant q^n\}$. Finally, if x and y are in the same coset of A_m in K^*, then $\pi(x) = \pi(y)$ for all $\pi \in A'_m$. Substituting into the formula above, we see that $\varphi(x) = \varphi(y)$. Q.E.D.

We see easily that $q'\tilde{\Psi}_0=\Lambda_0$, $q^n\tilde{\Psi}_n=\Lambda_n$, $n\geqslant 1$. This shows that the measure of A_0' is q', and that the measure of A_n', $n\geqslant 1$, is q^n (using Plancherel). We conclude that the normalizing constant a is given by $a=q'\ln q/2\pi$. It then follows that the collection $\{q'\Psi_0, q^n\Psi_n\}_{n=1}^{\infty}$ is a non-negative approximation to the identity with each of its terms in S^*. Also, it follows that the number of characters on O^* ramified of degree 0 is 1, of degree 1 is $(q-q')/q'=q-2$, and of degree $n\geqslant 2$ is $(q^n-q^{n-1})/q'=q^{n-2}(q-1)^2$.

We now note that if $f\in L^1(K^*)$, $g\in L^1(\hat{K}^*)$, then $\int_{K^*}f(x)\tilde{g}(x)d^*x=\int_{\hat{K}^*}\hat{f}(\pi)g(\pi)d\pi$. If f is a distribution on $K^*(\hat{K}^*)$, we denote the action on a test function φ by $\langle f,\varphi\rangle$. We extend the Mellin transform to any distribution on $K^*(\hat{K}^*)$ by defining for any $f\in S^{*\prime}(\hat{S}^{*\prime})$ a corresponding $\hat{f}\in\hat{S}^{*\prime}(S^{*\prime})$ which satisfies $\langle\hat{f},\varphi\rangle=\langle f,\tilde{\varphi}\rangle$ for $\varphi\in\hat{S}^*(S^*)$. The Mellin transform is a topological isomorphism of $S^{*\prime}(\hat{S}^{*\prime})$ onto $\hat{S}^{*\prime}(S^{*\prime})$.

Let
$$D_n(x)=\tfrac{1}{2}+\sum_{\nu=1}^{n}\cos\nu x=\frac{\sin(n+1/2)x}{2\sin(x/2)},$$
the Dirichlet kernel. Then let
$$d_n(\pi)=d_n(\pi^*|x|^{i\alpha})=\begin{cases}\dfrac{2}{q'}D_n(\alpha\ln q), & \pi^*\equiv 1\\ 0 & ,\ \pi^*\not\equiv 1.\end{cases}$$
$$d_n(|x|^{i\alpha})=\frac{1}{q'}\sum_{\nu=-n}^{n}q^{i\nu\alpha}.$$

Each $d_n\in\hat{S}^*$, and $\{d_n\}_{n=0}^{\infty}$ is an approximation to the identity but is not non-negative.

Let
$$K_n(x)=\frac{1}{(n+1)}\sum_{\nu=0}^{n}D_\nu(x)=\frac{2}{2n+1}\left[\frac{\sin(n+1)x/2}{2\sin(x/2)}\right]^2=\tfrac{1}{2}\sum_{\nu=-n}^{n}\left(1-\frac{|\nu|}{n+1}\right)e^{i\nu x},$$
the Fejer kernel. Then let
$$k_n(\pi)=k_n(\pi^*|x|^{i\alpha})=\begin{cases}\dfrac{2}{q'}K_n(\alpha\ln q), & \pi^*\equiv 1\\ 0 & ,\ \pi^*\not\equiv 1.\end{cases}$$
$$k_n(|x|^{i\alpha})=\frac{1}{q'}\sum_{\nu=-n}^{n}\left(1-\frac{|\nu|}{n+1}\right)q^{i\nu\alpha}.$$

Each $k_n\geqslant 0$, $k_n\in\hat{S}^*$ for all n and $\{k_n\}_{n=0}^{\infty}$ is an approximation to the identity.

Let Θ be the characteristic function of O^*. We calculate and find that $\tilde{d}_n$ is the characteristic function of the set $\{x: q^{-n}\leqslant|x|\leqslant q^n\}$ so that
$$\tilde{d}_n(x)=\sum_{\nu=-n}^{n}\Theta(\mathfrak{p}^\nu x).$$

Similarly,
$$\tilde{k}_n(x)=\sum_{\nu=-n}^{n}\left(1-\frac{|\nu|}{n+1}\right)\Theta(\mathfrak{p}^\nu x).$$

From our definition of the principal value integral we see that

$$P\int_K f(x)\,dx = \lim_{n\to\infty}\int_K f(x)\,\tilde{d}_n(x)\,dx = \lim_{n\to\infty}\langle(|x|f)^\sim, d_n\rangle.$$

By an obvious analogy we define $(C, 1)$ (Cesaro) summability of integrals on K by

$$(C,1)\int_K f(x)\,dx = \lim_{n\to\infty}\int_K f(x)\,\tilde{k}_n(x)\,dx = \lim_{n\to\infty}\frac{1}{n+1}\sum_{\nu=0}^{n}\int_{q^{-\nu}\leqslant|x|\leqslant q^{\nu}} f(x)\,dx$$
$$= \lim_{n\to\infty}\langle(|x|f)^\sim, k_n\rangle.$$

If π is any (multiplicative) character on K^*, not necessarily unitary, we may write $\pi(x)=\pi_1(x)|x|^{\beta}$ where π_1 is a unitary character and β is a complex number. We may also write $\pi(x)=\pi^*(x)|x|^{\gamma}$ where π^* is a character on O^*. In this case, π is unitary if and only if $\mathrm{Re}(\gamma)=0$. Given two characters $\pi_1(x)=\pi_1^*(x)|x|^{\gamma_1}$ and $\pi_2(x)=\pi_2^*(x)|x|^{\gamma_2}$, we see that $\pi_1=\pi_2$ if and only if $\pi_1^*=\pi_2^*$ and $\gamma_1-\gamma_2=(2k\pi i)\ln q$ for some integer k. In the remainder of this paper we restrict ourselves to the range $-\pi/\ln q<\mathrm{Im}(\gamma)\leqslant\pi/\ln q$. Thus, when we consider functions analytic in the parameter γ we restrict ourselves to this strip. All the functions we consider may be extended to entire or meromorphic functions on the complex plane by periodicity.

§ 3. Gamma functions and beta functions

We define the gamma function, $\Gamma(\pi)=\Gamma(\pi^*|x|^{\alpha})=\Gamma_{\pi^*}(\alpha)$, for all characters π (not necessarily unitary) on K^* except $\pi\equiv 1$.

Definition i) If π is ramified, $\Gamma(\pi)=\Gamma_{\pi^*}(\alpha)=P\int_K\chi(x)\pi(x)|x|^{-1}dx$.
ii) If $\mathrm{Re}(\alpha)>0$, $\Gamma(|x|^{\alpha})=\Gamma_1(\alpha)=P\int_K\chi(x)|x|^{\alpha-1}dx$.
iii) If $\mathrm{Re}(\alpha)=0$, $\alpha\neq 0$, $\Gamma_1(\alpha)=(C,1)\int_K\chi(x)|x|^{\alpha-1}dx$.
iv) If $\mathrm{Re}(\alpha)<0$, $\Gamma_1(\alpha)$ is the analytic continuation of Γ_1 into the left half-plane.

In our first theorem below we show that this definition makes sense by explicitly evaluating the integrals in the definition. The crucial details in the computation of the gamma function are contained in the following lemmas.

LEMMA 1. *Let π be ramified of degree $h\geqslant 1$. Then, if $|u|\neq q^h$, $\int_{|x|=1}\chi(ux)\pi(x)\,dx=0$.*

Proof. Let $f(x)=\pi(x)$ if $|x|=1$, $f(x)=0$ otherwise. Then, by Lemma (M_1) $f\in S$ and f is constant on the cosets (in K^+) of $\mathfrak{P}^h$. Hence, by Lemma (A_1),

$$\hat{f}(u)=\int_{|x|=1}\chi(ux)\,\pi(x)\,dx=0, \text{ if } |u|>q^h.$$

Now let $g(x)=\chi(ux)$, $|x|=1$, $g(x)=0$ otherwise. Suppose $|u|=q^k$. Then, if $k\geq 1$, it follows from Lemma (A_2) that $g\in S^*$ and g is constant on the cosets (in K^*) of A_k. It follows from Lemma (M_2) that

$$\tilde{g}(\pi)=\int_{|x|=1}\chi(ux)\,\pi(x)\,dx=0 \text{ if } 1\leq k<h.$$

Finally, if $k<1$, the result is immediate. Q.E.D.

LEMMA 2.
$$\int_{|x|=q^k}\chi(x)\,dx=\begin{cases} q^k/q', & k\leq 0,\\ -1, & k=1,\\ 0, & k>1,\end{cases}$$

Proof. $\chi(x)=1$ if $|x|\leq 1$ so the cases for $k\leq 0$ are immediate. χ is a non-trivial character on the compact groups $\mathfrak{P}^{-k}$, $k\geq 1$. Thus $\int_{\mathfrak{P}^{-k}}\chi(x)\,dx=0$, $k\geq 1$. The result follows by setting

$$\int_{|x|=q^k}\chi(x)\,dx=\int_{\mathfrak{P}^{-k}}\chi(x)\,dx-\int_{\mathfrak{P}^{-k+1}}\chi(x)\,dx,\quad k\geq 1.\quad \text{Q.E.D.}$$

There is also a multiplicative analogue to Lemma 2.

LEMMA 3. *If π is ramified of degree $h\geq 1$, then*

$$\int_{A_k-A_{k+1}}\pi(x)\,dx=\begin{cases} 0, & 0\leq k<h-1\\ -q^{-h}, & k=h-1\\ q^{-k}/q', & k\geq h.\end{cases}$$

Proof. The proof follows from the facts that

$$\int_{A_k}\pi(x)\,dx=0 \text{ if } 0\leq k<h, \text{ and } \int_{A_k}\pi(x)\,dx=\int_{\mathfrak{P}^k}dx=q^{-k} \text{ if } k\geq h.\quad \text{Q.E.D.}$$

THEOREM 1.

i) *If π is ramified of degree $h\geq 1$, $\Gamma(\pi)=\Gamma_{\pi^*}(\alpha)=C_{\pi^*}q^{h(\alpha-\frac{1}{2})}$, where $|C_{\pi^*}|=1$, $C_{\pi^{*-1}}C_{\pi^*}=\pi^*(-1)$.*

ii) $\Gamma_1(\alpha)=(1-q^{\alpha-1})/(1-q^{-\alpha})$, $\alpha\neq 0$. *$\Gamma_1(\alpha)$ has a simple pole at $\alpha=0$ with residue $1/q'\ln q$. $1/\Gamma_1(\alpha)$ has a simple pole at $\alpha=1$ with residue $-1/q'\ln q$. $\alpha=0$ is the only singularity of $\Gamma_1(\alpha)$, $\alpha=1$ the only zero.*

iii) *For all π,*

$\Gamma_{\pi^*}(\alpha)=\pi^*(-1)\,\overline{\Gamma_{\pi^{*-1}}(\bar{\alpha})}$, $\Gamma_{\pi^*}(\alpha)\,\Gamma_{\pi^{*-1}}(1-\alpha)=\pi^*(-1)$, *and so* $\Gamma_{\pi^*}(\alpha)\,\overline{\Gamma_{\pi^*}(1-\bar{\alpha})}=1$

with the obvious interpretation at $\pi(x)=|x|$.

Proof. If π is ramified of degree $h \geqslant 1$, then Lemma 1 implies that

$$\Gamma_{\pi^*}(\alpha) = \int_{|x|=q^h} \chi(x)\,\pi(x)\,|x|^{-1}\,dx = q^{h(\alpha-\frac{1}{2})}\int_{|x|=q^h} \chi(x)\,\pi^*(x)\,|x|^{-\frac{1}{2}}\,dx$$
$$= \Gamma_{\pi^*}(\tfrac{1}{2})\,q^{h(\alpha-\frac{1}{2})} = C_{\pi^*}q^{h(\alpha-\frac{1}{2})}, \text{ where } C_{\pi^*} = \Gamma_{\pi^*}(\tfrac{1}{2}).$$

Now suppose that π is unramified, $\pi(x) = |x|^\alpha$. From Lemma 2 we see that

$$\int_K [\chi(x)\,|x|^{\alpha-1}]_n\,dx = -q^{\alpha-1} + 1/q' \sum_{k=0}^{n} q^{-k\alpha}.$$

If $\operatorname{Re}(\alpha) > 0$, the limit is $(1-q^{\alpha-1})/(1-q^{-\alpha})$. If $\operatorname{Re}(\alpha)=0$, $\alpha \neq 0$, the $(C, 1)$ limit is also $(1-q^{\alpha-1})/(1-q^{-\alpha})$. Continuing the function into the left half plane we see that $\Gamma_1(\alpha)$ and $1/\Gamma_1(\alpha)$ are meromorphic functions as stated. The calculation of the residues is left to the reader.

Part iii) for unramified characters is immediate from the expression for $\Gamma_1(\alpha)$, $\alpha \neq 0$. Now suppose that π is ramified of degree $h \geqslant 1$. The fact that

$$\Gamma_{\pi^*}(\alpha) = \pi^*(-1)\,\overline{\Gamma_{\pi^{*-1}}(\bar{\alpha})}$$

is easily seen by changing variables in the defining integral. Let $f(x) = \pi(x)\,|x|^{-1}$, if $|x| = 1$, $f(x) = 0$ otherwise. Then $f \in S$, and, using Lemma 1, we see that $\hat{f}(u) = \Gamma(\pi)\pi^{-1}(u)$ if $|u| = q^h$ and zero otherwise. Using Lemma 1 again, we see that

$$\Gamma_{\pi^*}(\alpha)\,\Gamma_{\pi^{*-1}}(1-\alpha) = \Gamma(\pi)\int_{|u|=q^h} \pi^{-1}(u)\,|u|\,\chi(u)\,|u|^{-1}\,du$$
$$= \int_{|u|=q^h} \Gamma(\pi)\,\pi^{-1}(u)\,\chi(u)\,du = \int_K \hat{f}(u)\,\chi(u)\,du = f(-1) = \pi(-1)$$

since Fourier inversion holds on S. The relations for C_{π^*} in i) are an immediate consequence of iii). Q.E.D.

For any two multiplicative characters, $\pi = \pi^*|x|^\alpha$, $\lambda = \lambda^*|x|^\beta$, which are ramified, the constants C_{π^*}, C_{λ^*} and $C_{\pi^*\lambda^*}$ are related in a simple fashion.

THEOREM 2. *Suppose that π^* and λ^* are characters on O^* which are ramified of degree $h_1 \geqslant 1$, $h_2 \geqslant 1$ respectively. Let h_3 be the ramification degree of $\pi^*\lambda^*$. Then*

(a) $$C_{\pi^*}C_{\lambda^*} = C_{\pi^*\lambda^*}q^{(2h_1-h_2)/2}\int_{|x|=q^{h_2-h_1}} \pi^*(1-x)\,\lambda^*(x)\,dx, \quad h_1 > h_2.$$

(b) $$C_{\pi^*}C_{\lambda^*} = C_{\pi^*\lambda^*}q^{(3h_3-2h_1)/2}\int_{|x|=q^{h_1-h_3}} \pi^*(x)\,\lambda^*(1-x)\,dx, \quad h_1 = h_2,\ 0 < h_3 < h_1.$$

(c) $$C_{\pi^*}C_{\lambda^*} = C_{\pi^*\lambda^*}q^{h_1/2}\int_{\substack{|x|=1\\|1-x|=1}} \pi^*(x)\,\lambda^*(1-x)\,dx, \quad h_1 = h_2 = h_3.$$

If $h_3 = 0$, then $\lambda^* = \pi^{*-1}$ and $C_{\pi^*}C_{\lambda^*} = \pi^*(-1)$ by Theorem 1, i).

Proof. If $h_1 > h_2$, then $\pi^* \lambda^*$ is ramified of degree h_1. We then have, from Lemma 1 and Theorem 1,

$$\begin{aligned}
C_{\pi^*\lambda^*} & \int_{|x|=q^{h_2-h_1}} \pi^*(1-x)\,\lambda^*(x)\,dx \\
&= \left(\int_{|u|=q^{h_1}} \pi^*\lambda^*(u)\,\chi(u)\,|u|^{-\frac{1}{2}}\,du\right)\left(\int_{|x|=q^{h_2-h_1}} \pi^*(1-x)\,\lambda^*(x)\,dx\right) \\
&= q^{-3h_1/2} \int_{|u|=q^{h_1}} \chi(u) \int_{|x|=q^{h_2}} \pi^*(u-x)\,\lambda^*(x)\,dx\,du \\
&= q^{-h_1} q^{h_2/2} \int_{|x|=q^{h_2}} \lambda^*(x)\,\chi(x)\,|x|^{-\frac{1}{2}}\,dx\Big) \left(\int_{|u|=q^{h_1}} \pi^*(u)\,\chi(u)\,|u|^{-\frac{1}{2}}\,du\right) \\
&= q^{-h_1+h_2/2} C_{\pi^*} C_{\lambda^*}.
\end{aligned}$$

This proves (a). The other relations are proved in exactly the same fashion. For (c) we observe that

$$\int_{\substack{||x|=1 \\ |1-x|=1}} \pi^*(x)\,\lambda^*(1-x)\,dx = \int_{|x|=1} \pi^*(x)\,\lambda^*(1-x)\,dx.$$

We now proceed to show that the Γ-function arises in a natural way as a factor in the Fourier transform of a certain distribution. This is the original definition of the Γ-function given in [3]. If $\pi(x) = \pi^*(x)|x|^\alpha$, Re $(\alpha) \geqslant 0$, $\alpha \neq 0$, then $\pi(x)|x|^{-1}$ induces an element of S' (which we denote by $\pi|x|^{-1}$) as follows. For $\varphi \in S$,

$$(\pi|x|^{-1}, \varphi) = (C, 1) \int_K \pi(x)\,|x|^{-1} \varphi(x)\,dx \left(= \int_K \pi(x)\,|x|^{-1} \varphi(x)\,dx,\ \mathrm{Re}\,(\alpha) > 0\right). \tag{3.1}$$

If π is ramified and φ is constant on $\mathfrak{P}^n$ and supported on $\mathfrak{P}^{-n}$, $n \geqslant 1$, we may use Lemma 1 (with $|u| \leqslant 1$) to obtain

$$(\pi|x|^{-1}, \varphi) = \int_{q^{-n} < |x| \leqslant q^n} \pi^*(x)\,\varphi(x)\,|x|^{\alpha-1}\,dx. \tag{3.2}$$

If π is unramified, then

$$\begin{aligned}
(|x|^{\alpha-1}, \varphi) &= \varphi(0)\,(C, 1) \int_{|x| \leqslant q^{-n}} |x|^{\alpha-1}\,dx + \int_{q^{-n} < |x| \leqslant q^n} \varphi(x)\,|x|^{\alpha-1}\,dx \\
&= \varphi(0)\,q^{-n\alpha}/q'(1-q^{-\alpha}) + \int_{q^{-n} < |x| \leqslant q^n} \varphi(x)\,|x|^{\alpha-1}\,dx.
\end{aligned} \tag{3.3}$$

In either case we see that, if $\varphi \in S$, $(\pi^*|x|^{\alpha-1}, \varphi)$ is an analytic function of α, Re $(\alpha) \geqslant 0$, $\alpha \neq 0$ if $\pi^* \equiv 1$. This analytic function may be extended to an entire function of α if π is ramified, and to a meromorphic function with a single pole at $\alpha = 0$ if π is unramified. In the latter case the residue at $\alpha = 0$ is $\varphi(0)/q' \ln q$. We see from Theorem 1, ii), that

$1/\Gamma_1(\alpha)(\pi^*|x|^{\alpha-1}, \varphi)$ has a removable singularity at $\alpha=0$, and is analytic at $\alpha=0$ if we define its value there to be $\varphi(0)=(\delta, \varphi)$. It is therefore natural to make the following definition.

$$1/\Gamma_1(0)|x|^{-1}=\delta, \text{ as a distribution.} \tag{3.4}$$

THEOREM 3. *If* $\pi \not\equiv 1$, $\pi|x|^{-1} \in S'$. $1/\Gamma_1(0)|x|^{-1} \in S'$.

Proof. We may assume $\pi \not\equiv 1$. We shall show that, if $\{\varphi_k\}$ is a null sequence in S, then $(\pi|x|^{-1}, \varphi_k) \to 0$ as $k \to \infty$. For some fixed n, $(\pi|x|^{-1}, \varphi_k)$ is given by (3.2) or (3.3) with φ replaced by φ_k. Since φ_k tends uniformly to zero as $k \to \infty$, it follows that $(\pi|x|^{-1}, \varphi_k) \to 0$ as $k \to \infty$. Q.E.D.

THEOREM 4. $(\pi|x|^{-1})^\wedge = \Gamma(\pi)\pi^{-1}$.

Proof. For $\pi \equiv 1$ and $\pi(x)=|x|$, the result follows from (3.4), Theorem 1, iii), and the fact that $(1/\Gamma_1(0)|x|^{-1})^\wedge = \hat{\delta} = 1$. Now assume that $\pi \not\equiv 1$, $\pi(x) \neq |x|$. We saw above, using (3.2) and (3.3), that, for $\varphi \in S$, $((\pi|x|^{-1})^\wedge, \varphi) = (\pi|x|^{-1}, \hat{\varphi}) = (\pi^*|x|^{\alpha-1}, \hat{\varphi})$ is a meromorphic function of α with at most a pole at $\alpha=0$ (if $\pi^*=1$). Thus it suffices to assume that $0<\operatorname{Re}(\alpha)<1$. Take $\varphi \in S$ such that φ is constant on the cosets of $\mathfrak{P}^n$ and is supported on $\mathfrak{P}^{-n}$, $n \geq 1$. Then, by Lemma (A_1), $\hat{\varphi}$ is also constant on the cosets of $\mathfrak{P}^n$ and is supported on $\mathfrak{P}^{-n}$. Then, using (3.2) and (3.3), we have

$$(\Gamma(\pi)\pi^{-1}, \varphi) = \Gamma(\pi)\int_{|x|\leq q^n} \varphi(x)\pi^{-1}(x)\,dx$$
$$= \begin{cases} \varphi(0)q^{-n(1-\alpha)}/q'(1-q^{-\alpha}) + \Gamma(\pi)\displaystyle\int_{q^{-n}<|x|\leq q^n} \varphi(x)\pi^{-1}(x)\,dx, & \pi \text{ unramified,} \\ \Gamma(\pi)\displaystyle\int_{q^{-n}<|x|\leq q^n} \varphi(x)\pi^{-1}(x)\,dx, & \pi \text{ ramified.} \end{cases}$$

If π is unramified,

$$(\pi|u|^{-1}, \hat{\varphi}) = \int_{|u|\leq q^n} \pi(u)|u|^{-1}\hat{\varphi}(u)\,du = \int_{|x|\leq q^{-n}} \varphi(0)\int_{|u|\leq q^n} \pi(u)|u|^{-1}\,du\,dx$$
$$+ \int_{q^{-n}<|x|\leq q^n} \varphi(x)\int_{|u|\leq q^n} \pi(u)|u|^{-1}\chi(ux)\,du\,dx.$$

A simple computation shows that

$$\int_{|x|\leq q^{-n}} \varphi(0)\int_{|u|\leq q^n} \pi(u)|u|^{-1}\,du\,dx = \varphi(0)q^{-n(1-\alpha)}/q'(1-q^{-\alpha}).$$

In the second integral, we let $y=ux$, and this gives

$$\int_{q^{-n}<|x|\leqslant q^n} \varphi(x)\,\pi^{-1}(x) \int_{|y|\leqslant q^n|x|} \chi(y)\,\pi(y)\,|y|^{-1}\,dy\,dx.$$

Since $q^n|x| \geqslant q$ for $|x| > q^{-n}$, the theorem is proved for unramified characters.

Now suppose π is ramified of degree $h \geqslant 1$. From Lemma (A_2) we know that $\varphi(x)$ is constant on the cosets of A_{n-k} in $\mathfrak{P}^k$, $k \leqslant n$. Furthermore, φ is constant on the cosets of A_{2n} throughout K^*. If $h > 2n$, we can decompose $\int_{|x|=q^k} \varphi(x)\pi^{-1}(x)\,dx$, $-n < k \leqslant n$, into

$$\sum_s \varphi(\beta_s) \int_{\beta_s A_{2n}} \pi^{-1}(x)\,dx$$

and each of the integrals in the sum is zero. If $h \leqslant 2n$, and $q^{-n} < q^{-k} < q^{-n+h}$, then $n-k < h$ and we have

$$\int_{|x|=q^{-k}} \varphi(x)\,\pi^{-1}(x)\,dx = \sum_s \varphi(\beta_s) \int_{\beta_s A_{n-k}} \pi^{-1}(x)\,dx.$$

Again each of the integrals in the sum is zero. Summarizing, we have

$$\Gamma(\pi) \int_{q^{-n}<|x|\leqslant q^n} \varphi(x)\,\pi^{-1}(x)\,dx = \begin{cases} 0, & h > 2n \\ \Gamma(\pi) \displaystyle\int_{q^{-n+h}\leqslant|x|\leqslant q^n} \varphi(x)\,\pi^{-1}(x)\,dx, & h \leqslant 2n. \end{cases}$$

On the other hand, from (3.2) we have

$$\begin{aligned}
(\pi|u|^{-1}, \hat{\varphi}) &= \int_{q^{-n}<|u|\leqslant q^n} \pi(u)\,|u|^{-1}\hat{\varphi}(u)\,du \\
&= \int_{q^{-n}<|u|\leqslant q^n} \pi(u)\,|u|^{-1} \int_{|x|\leqslant q^n} \varphi(x)\,\chi(ux)\,dx\,du \\
&= \int_{q^{-n}<|u|\leqslant q^n} \pi(u)\,|u|^{-1} \int_{q^{-n}<|x|\leqslant q^n} \varphi(x)\,\chi(ux)\,dx\,du \\
&= \int_{q^{-n}<|x|\leqslant q^n} \varphi(x)\,\pi^{-1}(x) \int_{q^{-n}|x|<|y|\leqslant q^n|x|} \chi(y)\,\pi(y)\,|y|^{-1}\,dy \\
&= \begin{cases} 0, & h > 2n \\ \Gamma(\pi) \displaystyle\int_{q^{-n+h}\leqslant|x|\leqslant q^n} \varphi(x)\,\pi^{-1}(x)\,dx, & h \leqslant 2n. \quad \text{Q.E.D.} \end{cases}
\end{aligned}$$

Remark. Theorem 4 shows that our definition of the gamma function is equivalent to that in [3], § 2.5. This theorem also shows that our gamma function is precisely the invariant factor $\varrho(\pi)$ defined by Tate (see [4], Ch. VII, § 3). Tate considers the class of continuous functions f in $L_1(K^+)$ with the property that $\hat{f}$ is continuous and in $L_1(K^+)$, and also that $f|x|^\sigma$, $\hat{f}|x|^\sigma$ are in $L_1(K^*)$ for all $\sigma>0$. This class of functions is strictly larger than S as may be seen from the example $f(x)=0$, $|x|\leqslant 1$; $f(x)=e^{-|x|}$, $|x|>1$. For any character π, $0<\operatorname{Re}(\pi)<1$, and f as above, the local zeta function is defined by

$$\zeta(f,\pi)=\int_K f(x)\,\pi(x)\,|x|^{-1}\,dx.$$

If $f\in S$, Theorem 3 shows that $\zeta(\cdot,\pi)\in S'$. Defining $\hat{\pi}=\pi^{-1}|x|$, Tate proves the local functional equation

$$\zeta(f,\pi)\,\zeta(\hat{g},\hat{\pi})=\zeta(\hat{f},\hat{\pi})\,\zeta(g,\pi).$$

In our case this follows directly (for f, $g\in S$) from Theorem 4 and Plancherel's formula. The factor $\varrho(\pi)$ is defined by the formula $\zeta(f,\pi)=\varrho(\pi)\,\zeta(\hat{f},\hat{\pi})$. Theorem 4 shows that $\varrho(\pi)=\Gamma(\pi)$. In this setting, the results of Theorem 1 are contained in Tate's local computations ([4], Ch. VII, § 4).

Tate's approach suggests another method of proving Theorem 4. We first state the definition of homogeneous distributions (cf. [3], p. 45).

Definition. $f\in S'$ is said to be *homogeneous of degree* π (π a multiplicative character on K^*) if, for all $t\in K^*$, $f_t=\pi(t)f$, where $(f_t,\varphi)=(f,|t|^{-1}\varphi_{1/t})$, $\varphi\in S$; $\varphi_s(x)=\varphi(sx)$, $s\in K^*$.

LEMMA 4. $(\pi|x|^{-1})^\wedge$ *is homogeneous of degree* π^{-1}, $\pi\not\equiv 1$, $\pi(x)\neq|x|$.

LEMMA 5. $f\in S'$ *is homogeneous of degree* π, $\pi(x)\neq|x|^{-1}$, *if and only if* $f=c\pi$ *for some constant c.* $f=c\delta$, *for some constant c, if and only if* $(f,\varphi)=0$ *for all* $\varphi\in S$ *such that* $\varphi(0)=0$.

For proofs of Lemma 4 and Lemma 5 see [3], § 2.

Remark. The missing cases in Lemma 4 are filled in by means of (3.4). It follows from the last two lemmas that $(\pi|x|^{-1})^\wedge=A_\pi\pi^{-1}$. If we take $\varphi\in S$ such that $\varphi(0)=0$, then a simple computation shows (with the help of the Lebesque dominated convergence theorem) that $A_\pi=\Gamma(\pi)$. The proof of Theorem 4 given above is more in keeping with the spirit of proving results by direct computation providing the detail does not become too cumbersome.

Definition. Let π and λ be multiplicative characters, $\pi=\pi^*|x|^\alpha$, $\lambda=\lambda^*|x|^\beta$. Then the *beta function* $B(\pi,\lambda)$ is defined by

$$B(\pi,\lambda)=\frac{\Gamma(\pi)\,\Gamma(\lambda)}{\Gamma(\pi\lambda)}=\frac{\Gamma_{\pi^*}(\alpha)\,\Gamma_{\lambda^*}(\beta)}{\Gamma_{\pi^*\lambda^*}(\alpha+\beta)},$$

where the various gamma functions are defined.

Remark. $B(\pi,\lambda)$ can be considered as a meromorphic function in two complex variables α, β for fixed π^*, λ^*. In a number of cases $B(\pi,\lambda)$ is constant as a function of one or both of two variables. For example, if π is unramified and λ is ramified of degree $h \geqslant 1$, then $B(\pi, \lambda) = \Gamma_1(\alpha) q^{-h\alpha}$. If π is ramified of degree $h_1 \geqslant 1$ and λ is ramified of degree $h_2 \geqslant 1$, $h_1 < h_2$, then

$$B(\pi, \lambda) = (C_{\pi^*} C_{\lambda^*} / C_{\pi^*\lambda^*})\, q^{h_1(\alpha - \frac{1}{2}) - h_2 \alpha}.$$

Finally, if π, λ and $\pi\lambda$ are all ramified of degree $h \geqslant 1$,

$$B(\pi, \lambda) = (C_{\pi^*} C_{\lambda^*} / C_{\pi^*\lambda^*})\, q^{-h/2}.$$

It is clear that $B(\pi, \lambda) = B(\lambda, \pi)$ for all π, λ.

Our aim is to obtain an integral representation of the beta function under suitable conditions. This is the method used to define the beta function in [3]. Let π and λ be as in the definition of the beta function with

$$0 < \operatorname{Re}(\alpha),\ \operatorname{Re}(\beta),\ \operatorname{Re}(\alpha + \beta) < 1. \tag{3.5}$$

If $u \in K^*$, then the integral defining

$$k(u) = (\pi |x|^{-1} \ast \lambda |x|^{-1})(u) = \int_K \pi(x) |x|^{-1} \lambda(u - x) |u - x|^{-1} dx \tag{3.6}$$

converges absolutely. Setting

$$b(\pi, \lambda) = \int_K \pi(x) |x|^{-1} \lambda(1 - x) |1 - x|^{-1} dx, \tag{3.7}$$

and changing variables in (3.6), we see that $k(u) = \pi\lambda(u) |u|^{-1} b(\pi, \lambda)$. Theorem 4 implies that $\hat{k} = b(\pi, \lambda) \Gamma(\pi\lambda)(\pi\lambda)^{-1}$. We wish to show that $b(\pi, \lambda) = B(\pi, \lambda)$ which will follow from the relation $\hat{k} = \Gamma(\pi)\Gamma(\lambda)(\pi\lambda)^{-1}$. This is the substance of the following lemma.

LEMMA 6. *If* $\pi = \pi^* |x|^\alpha$, $\lambda = \lambda^* |x|^\beta$ *satisfy* (3.5), *then* $(\pi |x|^{-1} \ast \lambda |x|^{-1})^\wedge = (\pi |x|^{-1})^\wedge (\lambda |x|^{-1})^\wedge = \Gamma(\pi)\Gamma(\lambda)(\pi\lambda)^{-1}$, *and* $b(\pi, \lambda)\Gamma(\pi\lambda) = \Gamma(\pi)\Gamma(\lambda)$.

Proof. For all $n \geqslant 1$, it is easy to see that $([\pi |x|^{-1}]_n \ast \lambda |x|^{-1})(u)$ represents an element of S' since it is dominated by $|u|^{\operatorname{Re}(\alpha+\beta)-1} b(|x|^{\operatorname{Re}(\alpha)-1}, |x|^{\operatorname{Re}(\beta)-1})$, a locally integrable function. Fix $\varphi \in S$, $\varphi(0) = 0$, so that φ is supported on the set $\{x : q^{-k} < |x| \leqslant q^k\}$. We then have

$$\begin{aligned}
(([\pi |x|^{-1}]_n \ast \lambda |x|^{-1})^\wedge, \varphi) &= ([\pi |x|^{-1}]_n \ast \lambda |x|^{-1}, \hat{\varphi}) \\
&= \int_{|u| \leqslant q^k} \hat{\varphi}(u) \int_{q^{-n} \leqslant |x| \leqslant q^n} \pi(x) |x|^{-1} \lambda(u - x) |u - x|^{-1} dx\, du \\
&= \int_{q^{-n} \leqslant |x| \leqslant q^n} \pi(x) |x|^{-1} \int_{|u| \leqslant q^k} \lambda(u - x) |u - x|^{-1} \hat{\varphi}(u)\, du\, dx,
\end{aligned}$$

by Fubini's theorem. We observe that

$$\int_{|u|\leqslant q^k}\lambda(u-x)\,|u-x|^{-1}\hat{\varphi}(u)\,du=(\tau_x\lambda|u|^{-1},\hat{\varphi})=((\tau_x\lambda|u|^{-1})^{\wedge},\varphi)$$
$$=\int_{|u|\leqslant q^k}\chi(ux)\,\lambda^{-1}(u)\,\Gamma(\lambda)\,\varphi(u)\,du,$$

by Theorem 4. We then use the fact that $\pi(x)\,|x|^{-1}\chi(ux)\,\lambda^{-1}(u)\,\varphi(u)$ is absolutely integrable on $(q^{-k}<|u|\leqslant q^k)\times(q^{-n}\leqslant|x|\leqslant q^n)$, apply Fubini's theorem and obtain

$$(([\pi|x|^{-1}]_n*\lambda|x|^{-1})^{\wedge},\varphi)=\int_{q^{-k}<|u|\leqslant q^k}\Gamma(\lambda)\,\lambda^{-1}(u)\,\varphi(u)\int_{q^{-n}\leqslant|x|\leqslant q^n}\pi(x)\,|x|^{-1}\chi(ux)\,dx\,du$$
$$=\int_{q^{-k}<|u|\leqslant q^k}\Gamma(\lambda)\,(\pi\lambda)^{-1}(u)\,\varphi(u)\int_{q^{-n}/|u|\leqslant|x|\leqslant q^n/|u|}\pi(x)\,|x|^{-1}\chi(x)\,dx\,du.$$

$(\pi\lambda)^{-1}(u)\,\varphi(u)\in L^1(K^+,du)$ and the inner integral converges uniformly to $\Gamma(\pi)$ on the support of φ as $n\to\infty$, so that

$$\lim_{n\to\infty}(([\pi|x|^{-1}]_n*\lambda|x|^{-1})^{\wedge},\varphi)=((\pi\lambda)^{-1}\,\Gamma(\pi)\,\Gamma(\lambda),\varphi),\quad\text{for }\varphi\in S,\ \varphi(0)=0.$$

Lemma 5 implies that

$$\lim_{n\to\infty}([\pi|x|^{-1}]_n*\lambda|x|^{-1})^{\wedge}=(\pi\lambda)^{-1}\,\Gamma(\pi)\,\Gamma(\lambda)+c_1\delta,$$

in the sense of convergence in S', for some constant c_1.

On the other hand, if $\varphi\in S$, $\varphi(0)=0$,

$$([\pi|x|^{-1}]_n*\lambda|x|^{-1},\varphi)=\int_{q^{-k}<|u|\leqslant q^k}\varphi(u)\int_{q^{-n}\leqslant|x|\leqslant q^n}\pi(x)\,|x|^{-1}\lambda(u-x)\,|u-x|^{-1}\,dx\,du$$
$$=\int_{q^{-k}\leqslant|u|\leqslant q^k}\varphi(u)\,(\pi\lambda)\,(u)\,|u|^{-1}\int_{q^{-n}/|u|\leqslant|x|\leqslant q^n/|u|}\pi(x)\,|x|^{-1}\lambda(1-x)\,|1-x|^{-1}\,dx\,du$$
$$\to\int_{q^{-k}<|u|\leqslant q^k}\varphi(u)\,(\pi\lambda)\,(u)\,|u|^{-1}b(\pi,\lambda)\,du=(b(\pi,\lambda)\,\pi\lambda\,|u|^{-1},\varphi)\quad\text{as }n\to\infty,$$

by arguments similar to those above. Thus

$$\lim_{n\to\infty}([\pi|x|^{-1}]_n*\lambda|x|^{-1})=b(\pi,\lambda)\,\pi\lambda\,|u|^{-1}+c_2\delta$$

for some constant c_2, and so

$$\lim_{n\to\infty}([\pi|x|^{-1}]_n*\lambda|x|^{-1})^{\wedge}=b(\pi,\lambda)\,\Gamma(\pi\lambda)\,(\pi\lambda)^{-1}+c_2=\Gamma(\pi)\,\Gamma(\lambda)\,(\pi\lambda)^{-1}+c_1\delta.$$

A homogeneity argument shows that $c_1=c_2=0$, $b(\pi,\lambda)\,\Gamma(\pi\lambda)=\Gamma(\pi)\,\Gamma(\lambda)$. Therefore

$$(\pi|x|^{-1}*\lambda|x|^{-1})^{\wedge}=b(\pi,\lambda)\,\Gamma(\pi\lambda)\,(\pi\lambda)^{-1}=(\Gamma(\pi)\,\pi^{-1})\,(\Gamma(\lambda)\,|\lambda|^{-1})=(\pi|x|^{-1})^{\wedge}\,(\lambda|x|^{-1})^{\wedge}.$$

Q.E.D.

THEOREM 5. *Let* $\pi=\pi^*|x|^\alpha$, $\lambda=\lambda^*|x|^\beta$. *Then*

i) *if* $0<\mathrm{Re}(\alpha)$, $\mathrm{Re}(\beta)$, $\mathrm{Re}(\alpha+\beta)<1$,

$$B(\pi,\lambda)=\int_K \pi(x)\,|x|^{-1}\,\lambda(1-x)\,|1-x|^{-1}\,dx.$$

Furthermore, $[\pi|x|^{-1}]_n * \lambda|x|^{-1} \to B(\pi,\lambda)\pi\lambda|x|^{-1}$ *in* S' *as* $n\to\infty$.

ii) (a) *If* π, λ *and* $\pi\lambda$ *are ramified,*

$$B(\pi,\lambda)=P\int_K \pi(x)\,|x|^{-1}\lambda(1-x)\,|1-x|^{-1}\,dx,$$

for all $\alpha,\beta\in\mathbf{C}$.

(b) *If* π, λ *are ramified and* $\pi\lambda$ *is unramified,*

$$B(\pi,\lambda)=P\int_K \pi(x)\,|x|^{-1}\lambda(1-x)\,|1-x|^{-1}\,dx$$

for $\mathrm{Re}(\alpha+\beta)<1$.

(c) *If* π *is ramified and* λ *is unramified, or* λ *is ramified and* π *is unramified, then*

$$B(\pi,\lambda)=P\int_K \pi(x)\,|x|^{-1}\lambda(1-x)\,|1-x|^{-1}\,dx$$

for $\mathrm{Re}\,(\beta)>0$ *and* $\alpha\in\mathbf{C}$, *or* $\mathrm{Re}(\alpha)>0$ *and* $\beta\in\mathbf{C}$ *respectively.*

Proof. i) is an immediate consequence of Lemma 6. For ii) we consider

$$\begin{aligned}\int_K [\pi]_n(x)\,|x|^{-1}\,[\lambda]_n(1-x)\,|1-x|^{-1}\,dx &= \int_{O^*-A} \pi^*(x)\,\lambda^*(1-x)\,dx \\ &+ \int_{1<|x|\leqslant q^n} \pi^*(x)\,\lambda^*(1-x)\,|x|^{\alpha+\beta-2}\,dx + \int_{q^{-n}\leqslant|x|<1} \pi^*(x)\,\lambda^*(1-x)\,|x|^{\alpha-1}\,dx \\ &+ \int_{q^{-n}\leqslant|x|<1} \pi^*(1-x)\,\lambda^*(x)\,|x|^{\beta-1}\,dx = I_1+I_2+I_3+I_4.\end{aligned}$$

I_1 converges absolutely and is independent of α and β. Let h_2 be the degree of ramification of λ. Then $\lambda^*(1-x)=\lambda^*(-1)\lambda^*(x)\lambda^*(1-1/x)=\lambda^*(-1)\lambda^*(x)$ if $|x|\geqslant q^{h_2}$. Therefore

$$\lim_{n\to\infty} I_2=\int_{1<|x|<q^{h_2}} \pi^*(x)\,\lambda^*(1-x)\,|x|^{\alpha+\beta-2}\,dx$$

if $\pi\lambda$ is ramified (using Lemma 1 with $|u|\leqslant 1$). Similarly, we see that $\lambda^*(1-x)=1$ if $|x|\leqslant q^{-h_2}$ and $\pi^*(1-x)=1$ if $|x|\leqslant q^{-h_1}$ where h_1 is the degree of ramification of π. Therefore

$$\lim_{n\to\infty} I_3 = \int_{q^{-h_2}<|x|<1} \pi^*(x)\,\lambda^*(1-x)\,|x|^{\alpha-1}\,dx,$$

$$\lim_{n\to\infty} I_4 = \int_{q^{-h_1}<|x|<1} \lambda^*(x)\,\pi^*(1-x)\,|x|^{\beta-1}\,dx$$

if π and λ respectively are ramified. Thus, if the conditions from part ii) on the "size" of α and β are satisfied along with the conditions of ramification on π and λ, the principal value integral converges to an absolutely integrable function which is analytic in α and β in their appropriate ranges. Using part i) and an analytic continuation argument, we have the desired result.

Remark. In part ii) of the last theorem, we can extend to the boundary cases ($\mathrm{Re}(\alpha+\beta)=1$, $\alpha+\beta\neq 1$; $\mathrm{Re}(\alpha)=0$, $\alpha\neq 0$; $\mathrm{Re}(\beta)=0$, $\beta\neq 0$) by use of $(C, 1)$ convergence at the respective singularities as we did in Theorem 1 for the gamma function. The verification is obvious and is left to the reader.

COROLLARY. *Let convolution be defined by the integral as in Lemma 6. Then* $[1/\Gamma_1(\alpha)]\,|x|^{\alpha-1} * [1/\Gamma_1(\beta)]\,|x|^{\beta-1} = [1/\Gamma_1(\alpha+\beta)]\,|x|^{\alpha+\beta-1}$, $0<\mathrm{Re}\,(\alpha)$, $\mathrm{Re}\,(\beta)$, $\mathrm{Re}\,(\alpha+\beta)<1$.

Proof. An immediate consequence of Lemma 6.

The results in part ii) of Theorem 5 concerning the existence of the principal value integrals do not reveal the full story about the convolution of two multiplicative characters on the additive structure of K.

THEOREM 6. *Suppose that* $\pi=\pi^*|x|^\alpha$, $\lambda=\lambda^*|x|^\beta$ *are multiplicative characters satisfying* (3.5). *Suppose that* π *is ramified of degree* $h_1\geqslant 0$, λ *is ramified of degree* $h_2\geqslant 0$ *and* $\pi\lambda$ *is ramified of degree* h_3. *Then, if* $k(u)=(\pi|x|^{-1} * \lambda|x|^{-1})(u)$ *as in* (3.6), *we have*

i) $$k(u) = |u|^{\alpha-1}\int_{|x|=|u|} \pi^*(x)\,\lambda^*(u-x)\,|u-x|^{\beta-1}\,dx,\quad h_1>h_2\geqslant 0;$$

ii) $$k(u) = |u|^{\alpha-1}\int_{|x|=|u|q^{h_2-h_1}} \pi^*(u-x)\,|x|^{\beta-1}\lambda^*(x)\,dx,\quad h_1>h_2\geqslant 1;$$

iii) $$k(u) = \int_{|x|=|u|q^{h_1-h_3}} \pi^*(x)\,\lambda^*(u-x)\,|x|^{\alpha+\beta-2}\,dx,\quad h_1=h_2>h_3\geqslant 1;$$

iv) $$k(u) = \int_{\substack{|x|=|u|\\ |x-u|=|u|}} \pi^*(x)\,\lambda^*(u-x)\,|x|^{\alpha+\beta-2}\,dx,\quad h_1=h_2=h_3\geqslant 1;$$

v) $$k(u) = \int_{|x|\geqslant|u|q^{h_1-1}} \pi^*(x)\,\lambda^*(u-x)\,|x|^{\alpha+\beta-2}\,dx,\quad h_1=h_2>1,\ h_3=0;$$

vi) $$k(u) = \int_{\substack{|x|\geqslant|u|\\ |x-u|=|x|}} \pi^*(x)\,\lambda^*(u-x)\,|x|^{\alpha+\beta-2}\,dx,\quad h_1=h_2=1,\ h_3=0.$$

Proof. We give only the proof of i). The remaining statements can be proved in a similar fashion using the techniques developed in this section.

We first write

$$k(u) = |u|^{\beta-1}\int_{|x|<|u|} \pi^*(x)\,\lambda^*(u-x)\,|x|^{\alpha-1}\,dx$$
$$+\int_{|x|>|u|} \pi^*(x)\,\lambda^*(u-x)\,|x|^{\alpha+\beta-2}\,dx + |u|^{\alpha-1}\int_{|x|=|u|} \pi^*(x)\,\lambda^*(u-x)\,|u-x|^{\beta-1}\,dx.$$

If λ is unramified ($h_2=0$), the result follows from Lemma 1. Now suppose $h_2>0$, and take $q^k<|u|$. Define $f_1(x)=\pi^*(x)$ if $|x|=q^k$, $f_1(x)=0$ otherwise; $f_2(x)=\lambda^*(x)$ if $|x|=|u|$, $f_2(x)=0$ otherwise. Then $(f_1 * f_2)(u)=\int_{|x|=q^k}\pi^*(x)\lambda^*(u-x)\,dx$. From Lemma 1 we see that

$$\hat{f}_1(v)=\int_{|x|=q^k} \chi(vx)\,\pi^*(x)\,dx=0 \ \text{ unless } \ |v|=q^{h_1-k},$$

and

$$\hat{f}_2(v)=\int_{|x|=|u|} \chi(vx)\,\lambda^*(x)\,dx=0 \ \text{ unless } \ |v|=q^{h_2}|u|^{-1}.$$

It follows $\hat{f}_1(v)\hat{f}_2(v)=(f_1 * f_2)^\wedge(v)=0$, and hence $(f_1 * f_2)(u)=0$. This shows that the first integral in the above decomposition of $k(u)$ is equal to zero. If $q^k>|u|$, we take $f_1(x)$ as above and define $f_2(x)=\lambda^*(x)$ if $|x|=q^k$, $f_2(x)=0$ otherwise. Then $\hat{f}_1(v)=0$ unless $|v|=q^{h_1-k}$, $\hat{f}_2(v)=0$ unless $|v|=q^{h_2-k}$. We conclude that the second integral above is zero and the lemma is proved. Q.E.D.

Remark. By using Theorem 1, Theorem 2, Theorem 6 and Lemma 3, one can prove by direct (rather laborious) computation that $k(1)=B(\pi,\lambda)$ for all characters π, λ satisfying (3.5).

We now show that the gamma function may also be considered (up to a δ-function) as a Mellin transform for unitary characters π. The additive character χ is bounded on K^* and so represents an element of $S^{*\prime}$. It follows that $\tilde{\chi}\in\hat{S}^{*\prime}$. We define (3.8) $\tilde{\chi}=\Gamma^*$. The structure of $\hat{K}^*$ is that of a countable discrete collection of circles $\{T_{\pi^*}\}$ indexed by the characters π^* on O^*. Hence we may view Γ^* as a collection $\Gamma^*\sim\{\Gamma^*_{\pi^*}\}$ where each $\Gamma^*_{\pi^*}$ is a distribution on T_{π^*}, a copy of the circle. Observe that $P\Gamma_1(i\alpha)$ and $\Gamma_{\pi^*}(i\alpha)$, $\pi^*\not\equiv 1$, where $\alpha\in\mathbf{R}$, induce distributions on the circle. So the ordinary gamma function, restricted to unitary characters, may be considered as a distribution on $\hat{K}^*$, $\Gamma\sim\{P\Gamma_1(i\alpha),\ \Gamma_{\pi^*}(i\alpha),\ \pi^*\not\equiv 1\}$.

Remark. By $Pf(i\alpha)$ we mean the distribution (if it is one) defined by

$$\langle Pf,\varphi\rangle=\lim_{\varepsilon\to 0+} q'\ln q/2\pi\int_{\varepsilon\leqslant|\alpha|\leqslant\pi/\ln q} f(i\alpha)\,\varphi(\alpha)\,d\alpha,$$

where φ is a trigonometric polynomial on the circle, $\varphi(\alpha)=\sum_{\nu=-k}^{k} a_\nu q^{i\nu\alpha}$.

Our last theorem of this section shows that, up to a delta function, Γ and Γ^* are the same distributions.

THEOREM 7. i) $\Gamma_1^* = P\Gamma_1(i\alpha) + (\pi/q' \ln q)\,\delta$.

ii) $\Gamma_{\pi^*}^* = \Gamma_{\pi^*}(i\alpha), \quad \pi^* \neq 1$.

Proof. Suppose that $\varphi \in \hat{S}^*$, so that $\tilde{\varphi} \in S^*$. Without loss of generality we may assume that φ is supported on T_{π^*} for each of the cases above. Then

$$\langle \Gamma_{\pi^*}^*, \varphi \rangle = \langle \tilde{\chi}, \varphi \rangle = \langle \chi, \tilde{\varphi} \rangle = \int_K \chi(x)\,\tilde{\varphi}(x)\,|x|^{-1}\,dx$$

$$= \lim_{n\to\infty} \int_K [\chi]_n(x)\,\tilde{\varphi}(x)\,|x|^{-1}\,dx = \lim_{n\to\infty} \langle [\chi]_n, \tilde{\varphi} \rangle = \lim_{n\to\infty} \langle [\chi]_n^\sim, \varphi \rangle.$$

Now $[\chi]_n$ is compactly supported on K^* and is constant on the cosets of A_n. Thus

$$[\chi]_n \in S^* \quad \text{and} \quad [\chi]_n^\sim(\pi) = \int_{q^{-n} \leq |x| \leq q^n} \chi(x)\,\pi(x)\,|x|^{-1}\,dx.$$

If π^* is ramified of degree $h \geq 1$, then $[\chi]_n^\sim(\pi) = \Gamma(\pi)$, $n \geq h$, and we obtain $\langle \Gamma_{\pi^*}^*, \varphi \rangle = \lim_{n\to\infty} \langle [\chi]_n^\sim, \varphi \rangle = \langle \Gamma_{\pi^*}, \varphi \rangle$. Now assume that π^* is unramified so that φ is supported on T_1. Then, from Lemma 2, we see that

$$[\chi]_n^\sim(|x|^{i\alpha}) = \int_{q^{-n} \leq |x| \leq q^n} \chi(x)\,|x|^{i\alpha - 1}\,dx = -q^{i\alpha - 1} + 1/q' \sum_{k=0}^{n} q^{-ki\alpha}$$

$$= -q^{i\alpha-1} + 1/2q' + 1/q'\,[D_n(\alpha \ln q) - i\tilde{D}_n(\alpha \ln q)],$$

where D_n and $\tilde{D}_n$ are the Dirichlet and conjugate Dirichlet kernels respectively ([8], v. I, p. 49).

It is well known that, as distributions, $D_n(\alpha \ln q) \to (\pi/\ln q)\delta$, and $\tilde{D}_n(\alpha \ln q) \to P(1/2 \cot(\alpha \ln q/2))$. An elementary calculation shows that $-q^{i\alpha-1} + 1/2q'[1 - i \cot(\alpha \ln q/2)] = \Gamma_1(i\alpha)$. Hence $[\chi]_n^\sim(|x|^{i\alpha}) \to P\Gamma_1(i\alpha) + (\pi/q' \ln q)\delta$ and $\Gamma_1^* = P\Gamma_1(i\alpha) + (\pi/q' \ln q)\delta$. Q.E.D.

§ 4. Bessel functions

The Bessel function is a complex valued function defined for each $\pi \in \hat{K}^*$ and u, v in K^*.

Definition. The *Bessel function* (of order π), denoted $J_\pi(u, v)$ is the value of the principal value integral.

$$P\int_K \chi(ux + v/x)\,\pi(x)\,|x|^{-1}\,dx. \tag{4.1}$$

In Theorems 8 and 9 below we establish that (4.1) exists for all characters $\pi \in \hat{K}^*$ and $u, v \in K^*$. Using this fact we may find several elementary properties of the Bessel function by changing variables in (4.1).

LEMMA 7. i) $J_\pi(u, v) = J_{\pi^{-1}}(v, u)$.

ii) $\pi(u) J_\pi(u, v) = \pi(v) J_\pi(v, u)$.

iii) $J_\pi(u, v) = \overline{J_{\pi^{-1}}(-u, -v)} = \pi(-1)\overline{J_{\pi^{-1}}(u, v)}$.

iv) *If* $\pi(-1) = 1$, $J_\pi(u, u)$ *is real valued.*

If $\pi(-1) = -1$, $J_\pi(u, u)$ *is pure imaginary valued.*

For k a positive integer, $\pi \in \hat{K}^*$ and $v \in K^*$, we set

$$F_\pi(k, v) = \int_{|x| = q^k} \chi(x)\, \chi(v/x)\, \pi(x)\, |x|^{-1}\, dx. \tag{4.2}$$

LEMMA 8. *Suppose that* $|v| = q^m$ *and* $1 \leqslant k < m$. *Then*

i) *for* π *unramified,* $F_\pi(k, v) \neq 0$ *if and only if* m *is even and* $k = m/2$.

ii) *For* π *ramified of degree* $h \geqslant 1$, $F_\pi(k, v) \neq 0$ *if and only if one of the following is valid.*

a) m *is even,* $m \geqslant 2h$ *and* $k = m/2$.

b) m *is even,* $m < 2h < 2m$ *and* $k = m/2$, h *or* $m - h$.

c) m *is odd,* $m < 2h < 2m$ *and* $k = h$ *or* $m - h$.

Proof. Set $f_1(x) = \pi(x)\chi(x)$ if $|x| = q^k$, $f_1(x) = 0$ otherwise; $f_2(x) = \chi(x)$ if $|x| = q^{m-k}$, $f_2(x) = 0$ otherwise. Then $F_\pi(k, v) = (f_1 * f_2)(v)$, where convolution is taken with respect to the multiplicative structure (K^*, d^*x). The corresponding Mellin transforms are

$$\tilde{f}_1(\pi') = \int_{|x| = q^k} \pi'\pi(x)\, \chi(x)\, |x|^{-1}\, dx \quad \text{and} \quad \tilde{f}_2(\pi') = \int_{|x| = q^{m-k}} \pi'(x)\, \chi(x)\, |x|^{-1}\, dx.$$

Since $(f_1 * f_2)^\sim = \tilde{f}_1 \tilde{f}_2$ (f_1 and f_2 are in S^*), we see that $F_\pi(k, v) \neq 0$ if and only if $\tilde{f}_1(\pi')\tilde{f}_2(\pi') \neq 0$ for some π'. From our calculations for the gamma function (Lemma 1, Lemma 2, Theorem 1) we see that $\tilde{f}_1(\pi') \neq 0$ if and only if $\pi'\pi$ is unramified and $k = 1$ or $\pi'\pi$ is ramified of degree k. Similarly $\tilde{f}_2(\pi') \neq 0$ if and only if π' is unramified and $k = m - 1$ or π' is ramified of degree $m - k$. A straightforward check of the possibilities for π' gives our result. These possibilities are:

π unramified, π' and $\pi\pi'$ have the same ramification degree.

π ramified of degree $h \geqslant 1$ and

i) π' unramified, $\pi\pi'$ ramified of degree h.

ii) π' ramified of degree h, $\pi\pi'$ unramified.

iii) π' ramified of degree h, $\pi\pi'$ ramified of degree s, $1 \leqslant s \leqslant h$.

iv) π' ramified of degree $t+h$, $t \geqslant 1$; $\pi\pi'$ ramified of degree $s = \max[t, h]$.

The details are left to the reader. Q.E.D.

LEMMA 9. $|F_\pi(k, v)| \leqslant 1/q'$.

Proof.
$$|F_\pi(k,v)| \leqslant \int_{|x|=q^k} |x|^{-1}\,dx = 1/q'. \quad \text{Q.E.D.}$$

We now give explicit formulas for $J_\pi(u, v)$ in terms of gamma functions and the functions $F_\pi(k, v)$.

THEOREM 8. *If $\pi \in \hat{K}^*$, π unramified, $\pi \not\equiv 1$, and $u, v \in K^*$,*

$$J_\pi(u,v) = \begin{cases} \pi(v)\,\Gamma(\pi^{-1}) + \pi^{-1}(u)\,\Gamma(\pi), & |uv| \leqslant q, \\ \pi^{-1}(u)\,F_\pi(m/2, uv), & |uv| = q^m,\ m > 1,\ m \text{ even}, \\ 0, & |uv| = q^m,\ m > 1,\ m \text{ odd}. \end{cases}$$

If $\pi \equiv 1$, the only change is that $J_1(u, v) = (m+1)/q' - 2/q = 1/q'\ [\ln(1/|uv|)/\ln q + 1] - 2/q$ for $|uv| = q^{-m} \leqslant q$.

Remark. $J_1(1, v)$ is the natural analogue of the usual Bessel function of order zero.

THEOREM 9. *If $\pi \in \hat{K}^*$, π ramified of degree $h \geqslant 1$ and $u, v \in K^*$,*

$$J_\pi(u,v) = \begin{cases} \pi(v)\,\Gamma(\pi^{-1}) + \pi^{-1}(u)\,\Gamma(\pi), & |uv| \leqslant q^h, \\ \pi^{-1}(u)\,F_\pi(m/2, uv), & |uv| = q^m,\ m \geqslant 2h,\ m \text{ even}, \\ 0, & |uv| = q^m,\ m > 2h,\ m \text{ odd}, \\ \pi^{-1}(u)\,[F_\pi(h, uv) + F_\pi(m-h, uv) + F_\pi(m/2, uv)], & \\ \qquad\qquad |uv| = q^m,\ m < 2h < 2m,\ m \text{ even}, & \\ \pi^{-1}(u)\,[F_\pi(h, uv) + F_\pi(m-h, uv)], & |uv| = q^m,\ m < 2h < 2m,\ m \text{ odd}. \end{cases}$$

Proof of Theorem 8. Set $\pi(x) = |x|^\alpha$. For $|uv| = q^{-m} \leqslant q$ we write

$$J_\pi(u,v) = \pi^{-1}(u)\,P\int_{|x|\leqslant 1} \chi(uv/x)\,|x|^{\alpha-1}\,dx + \pi^{-1}(u)\,P\int_{|x|>1} \chi(x)\,|x|^{\alpha-1}\,dx.$$

Lemma 2 implies that

$$\pi^{-1}(u)\,P\int_{|x|>1} \chi(x)\,|x|^{\alpha-1}\,dx = -\pi^{-1}(u)\,q^{\alpha-1}.$$

From this same Lemma, we also see that

$$\pi^{-1}(u)\,P\int_{|x|\leqslant 1} \chi(uv/x)\,|x|^{\alpha-1}\,dx = \pi(v)\int_{q^{-m}\leqslant |x| \leqslant q} \chi(x)\,|x|^{-\alpha-1}\,dx = \pi(v)\left\{1/q' \sum_{k=0}^{m} q^{k\alpha} - q^{-\alpha-1}\right\}.$$

If $\alpha \neq 0$, this last term can be written

$$\pi(v)\{\Gamma(\pi^{-1})+\pi^{-1}(uv)\,q^{\alpha-1}(1-q)/(1-q^{\alpha})\}=\pi(v)\,\Gamma(\pi^{-1})+\pi^{-1}(u)\,(1-1/q)/(1-q^{-\alpha}).$$

The result for $|uv| \leqslant q$ and all α now follows immediately. If $|uv| > q$, we write

$$J_\pi(u,v)=\pi^{-1}(u)\,P\int_{|x|\leqslant 1}\chi(uv/x)\,|x|^{\alpha-1}\,dx$$
$$+\pi^{-1}(u)\,P\int_{|x|\geqslant|uv|}\chi(x)\,|x|^{\alpha-1}\,dx+\pi^{-1}(u)\int_{1<|x|<|uv|}\chi(x)\,\chi(uv/x)\,|x|^{\alpha-1}\,dx.$$

Lemma 2 shows that the first two terms are zero. The result then follows from Lemma 8. Q.E.D.

Remark. The proof of Theorem 8 shows that, for π unramified,

$$\pi(v)\,\Gamma(\pi^{-1})+\pi^{-1}(u)\,\Gamma(\pi)\to 1/q'[-\ln|uv|/\ln q+1]-2/q \text{ as } \pi\to 1.$$

Proof of Theorem 9. We break up the integral exactly as in the proof of Theorem 8. A direct application of the results of Lemma 1, Lemma 8 and the definition of the gamma function gives the result.

COROLLARY. i) *For fixed* $\pi\in\hat{K}^*$, $\pi\not\equiv 1$, $J_\pi(u,v)$ *is bounded as a function of* $u, v\in K^*$. $J_1(u,v)$ *is bounded for* uv *bounded away from zero.* $J_1(u,v)\simeq -\ln|uv|/q'\ln q$ *as* $|uv|\to 0$.

ii) *For fixed* $u, v\in K^*$, $J_\pi(u,v)$ *is bounded as a function of* $\pi\in\hat{K}^*$.

Proof. i) If $\pi\not\equiv 1$, this follows from Lemma 9 and the expression for the gamma function in Theorem 1. For $\pi\equiv 1$, we use the representation for $J_1(u,v)$ given in Theorem 8. This representation yields both the boundedness of $J_1(u,v)$ for $|uv|$ bounded away from zero, and the asymptotic formula. We observe here that $J_1(u,v)$ is a locally integrable function of u on (K^+, du) for fixed $v\in K^*$.

ii) For π ramified, this is immediate from Lemma 9 and the expression for $\Gamma(\pi)$. For π unramified and $|uv|>q$, we again use Lemma 9. If $|uv|=q^{-m}\leqslant q$, the proof of Theorem 8 shows that $|J_\pi(u,v)|\leqslant(m+1)/q'+2/q$ for all unramified π. Q.E.D.

In the next three theorems we show that $J_\pi(u,v)$ can be regarded as the Mellin transform of a distribution on K^*, the Fourier transform of a distribution on K^+, and the Mellin transform of a distribution on $\hat{K}^*$.

THEOREM 10. *Let* f *be the distribution induced by* $\chi(ux+v/x)$ *on* (K^*, d^*x), $u, v\in K^*$. *Then* $\hat{f}$ *is the distribution induced by* $J_\pi(u,v)$ *on* $(\hat{K}^*, d\pi)$.

Proof. For fixed $u, v\in K^*$, $\chi(ux+v/x)$ is bounded on (K^*, d^*x) and $J_\pi(u,v)$ is bounded on $(\hat{K}^*, d\pi)$ by the corollary above. For $\varphi\in\hat{S}^*$, we wish to show that

$$\int_{K^*} \chi(ux+v/x)\,\tilde{\varphi}(x)\,|x|^{-1}\,dx = \int_{\hat{K}^*} J_\pi(u,v)\,\varphi(\pi)\,d\pi.$$

If u and v are fixed in K^* and the ramification degree of π is bounded above by h_0, then there is a fixed compact set A in K^* depending only on $|uv|$ and h_0 such that $J_\pi(u,v) = \int_A \chi(ux+v/x)\pi(x)|x|^{-1}dx$. Since $\tilde{\varphi}\in S^*$, the set A may be chosen so that it contains the support of $\tilde{\varphi}$. Also observe that $\varphi\in\hat{S}^*$ implies that $\varphi(\pi)=0$ if the ramification degree of π is large enough, say greater than h_0. Letting A'_{h_0} denote the compact set (in $\hat{K}^*$) of π such that the ramification degree of $\pi \leqslant h_0$, we see that

$$\int_{\hat{K}^*} J_\pi(u,v)\,\varphi(\pi)\,d\pi = \int_{A'_{h_0}} \varphi(\pi)\int_A \chi(ux+v/x)\,\pi(x)\,|x|^{-1}\,dx\,d\pi$$
$$= \int_A \chi(ux+v/x)\,\tilde{\varphi}(x)\,|x|^{-1}\,dx = \int_{K^*} \chi(ux+v/x)\,\tilde{\varphi}(x)\,|x|^{-1}\,dx,$$

by Fubini's theorem. Q.E.D.

THEOREM 11. *$f=P(\chi(v/x)\pi(x)|x|^{-1})$, $v\in K^*$, $\pi\in\hat{K}^*$, is a distribution on (K^+, dx). $\hat{f}$ is the distribution $J_\pi(u,v)$ on (K^+, du).*

Proof. Suppose $\varphi\in S$, φ supported on $\mathfrak{P}^{-k}$ and constant on the cosets (in K^+) of $\mathfrak{P}^k$, $k\geqslant 1$. Let h be the maximum of 1 and the ramification degree of π. Let l be such that $q^l=\max\,[q^h/|v|,\,q^k]$. If $n>l$, then

$$(*)\qquad \int_K [\chi(v/x)\,\pi(x)\,|x|^{-1}]_n\,\varphi(x)\,dx$$
$$=\varphi(0)\int_{q^{-n}\leqslant|x|<q^{-l}} \chi(v/x)\,\pi(x)\,|x|^{-1}\,dx + \int_{q^{-l}\leqslant|x|\leqslant q^k} \varphi(x)\,\chi(v/x)\,\pi(x)\,|x|^{-1}\,dx.$$

By means of Lemma 1 or Lemma 2 we see that the first summand is zero since $|v/x|>q^h$ when $q^{-n}\leqslant|x|<q^{-l}$. It follows that (*) is constant if $n>l$, so that

$$(f,\varphi) = \int_{q^{-l}\leqslant|x|\leqslant q^k} \varphi(x)\,\chi(v/x)\,\pi(x)\,|x|^{-1}\,dx,$$

where k and l depend on φ, π and v. If $\{\varphi_s\}$ is a null sequence in S, there are fixed integers k and l such that, for all s,

$$(f,\varphi_s) = \int_{q^{-l}\leqslant|x|\leqslant q^k} \varphi_s(x)\,\chi(v/x)\,\pi(x)\,|x|^{-1}\,dx.$$

Since φ_s tends uniformly to zero as $s\to\infty$, $(f,\varphi_s)\to 0$ as $x\to\infty$. This shows $f\in S'$. Note that $[\chi(v/x)\pi(x)|x|^{-1}]_n\in S$ for all n. Then, for n large enough,

$$(\hat{f}, \varphi) = (f, \hat{\varphi}) = \int_K [\chi(v/x)\,\pi(x)\,|x|^{-1}]_n \hat{\varphi}(x)\,dx$$
$$= \int_K [\chi(v/x)\,\pi(x)\,|x|^{-1}]_n^{\wedge}(u)\,\varphi(u)\,du = \int_K \varphi(u) \int_{q^{-n}\leqslant|x|\leqslant q^n} \chi(ux+v/x)\,\pi(x)\,|x|^{-1}\,dx\,du.$$

If $\pi \not\equiv 1$, the inner integral converges boundedly to $J_\pi(u, v)$ as $n \to \infty$, so that $(\hat{f}, \varphi) = \int_K J_\pi(u, v)\varphi(u)\,du$, $\hat{f} = J_\pi(u, v)$ on (K^+, dx).

Now suppose $\pi \equiv 1$. Without loss of generality we may choose k so that $|v| \leqslant q^{k+1}$. We can write

$$(\hat{f}, \varphi) = \varphi(0) \int_{|u|\leqslant q^{-k}} \int_{q^{-n}\leqslant|x|\leqslant q^n} \chi(ux+v/x)\,|x|^{-1}\,dx\,du$$
$$+ \int_{q^{-k}<|u|\leqslant q^k} \varphi(u) \int_{q^{-n}\leqslant|x|\leqslant q^n} \chi(ux+v/x)\,|x|^{-1}\,dx\,du.$$

Since $|u|$ is bounded away from zero in the second integral, this term converges to

$$\int_{q^{-k}<|u|\leqslant q^k} \varphi(u)\,J_1(u, v)\,du \quad \text{as} \quad n \to \infty.$$

Thus, we must prove that

$$\int_{|u|\leqslant q^{-k}} \int_{q^{-n}\leqslant|x|\leqslant q^n} \chi(ux+v/x)\,|x|^{-1}\,dx \to \int_{|u|\leqslant q^{-k}} J_1(u, v)\,du \quad \text{as} \quad n \to \infty.$$

We first observe that, in this case, $|uv| \leqslant q$. If we take n large enough, we may write

$$\int_{|u|\leqslant q^{-k}} \int_{q^{-n}\leqslant|x|\leqslant q^n} \chi(ux)\,\chi(v/x)\,|x|^{-1}\,dx\,du$$
$$= \int_{|u|\leqslant q^{-k}} \left[\int_{q^{-n}\leqslant|x|\leqslant q^k} \chi(v/x)\,|x|^{-1}\,dx + \int_{q^k<|x|\leqslant q^n} \chi(ux)\,|x|^{-1}\,dx\right] du$$
$$= \int_{|u|\leqslant q^{-k}} \int_{q^{-k}|v|\leqslant|x|\leqslant q} \chi(x)\,|x|^{-1}\,dx\,du + \int_{|u|\leqslant q^{-n}} \int_{q^k<|x|\leqslant q^n} |x|^{-1}\,dx\,du$$
$$+ \int_{q^{-n}<|u|\leqslant q^{-k}} \int_{q^k<|x|\leqslant q|u|^{-1}} \chi(ux)\,|x|^{-1}\,dx\,du.$$

Now

$$\int_{|u|\leqslant q^{-n}} \int_{q^k<|x|\leqslant q^n} |x|^{-1}\,dx\,du = \frac{n-k}{q'} \sum_{j=n}^{\infty} q^{-j} = \frac{n-k}{(q')^2}\,q^{-n} \to 0 \quad \text{as} \quad n \to \infty.$$

Hence, as $n \to \infty$,

$$\int_{|u|\leqslant q^{-k}}\int_{q^{-n}\leqslant|x|\leqslant q^n}\chi(ux)\,\chi(v/x)\,|x|^{-1}\,dx$$

$$\to\int_{|u|\leqslant q^{-k}}\left\{\int_{q^{-k}|v|\leqslant|x|\leqslant q}\chi(x)\,|x|^{-1}\,dx+\int_{q^k<|x|\leqslant q|u|^{-1}}\chi(ux)\,|x|^{-1}\,dx\right\}du$$

$$=\int_{|u|\leqslant q^{-k}}\left\{\int_{q^{-k}|v|\leqslant|x|\leqslant q}\chi(x)\,|x|^{-1}\,dx+\int_{q^k|u|<|x|\leqslant q}\chi(x)\,|x|^{-1}\,dx\right\}du.$$

The inner integrals may be summed directly as in the proof of Theorem 8. If $|v|=q^j$, $u=|q|^{-l}$, we get

$$\left(\frac{k-j+1}{q'}-q^{-1}\right)+\left(\frac{l-k}{q'}-q^{-1}\right)=\frac{l-j+1}{q'}-\frac{2}{q}.\quad \text{Q.E.D.}$$

THEOREM 12. *Let f be the distribution on (K^*, d^*v) induced by the locally integrable function $J_\pi(u, v)$. Then $\tilde{f}$ is the distribution on $(K^*, d\pi_1)$ given by $\tilde{f}=(\pi\pi_1)^{-1}(u)\Gamma^*(\cdot)\Gamma^*(\pi\cdot)$.*

Proof. A change of variables shows that the distribution

$$(J_\pi(u,\cdot))^\sim=(\pi\pi_1)^{-1}(u)\,(J_\pi(1,\cdot))^\sim.$$

So it will suffice to show that $(J_\pi(1,\cdot))^\sim=\Gamma^*(\cdot)\Gamma^*(\pi\cdot)$. We now let f be the distribution induced by $J_\pi(1, v)$. We will construct a sequence in S^* that converges to f in $S^{*\prime}$. Let

$$g_n(v)=\int_K[\chi(x)\,\chi(v/x)\,\pi(x)\,|x|^{-1}]_n\,dx=\int_K\chi(vx)\,[\chi(1/x)\,\pi^{-1}(x)\,|x|^{-1}]_n\,dx.$$

Since $[\chi(1/x)\pi^{-1}(x)|x|^{-1}]_n$ has compact support and is constant on the cosets of $\mathfrak{P}^{l_n}$, $l_n=\max[2n, h+n]$, then $g_n\in S$ and $g_n(v)=0$ when $|v|>q^{l_n}$, where h is the degree of ramification of π. If $\varphi\in S^*$, it can be checked that $\langle f,\varphi\rangle=\langle g_n,\varphi\rangle$ if φ is supported on the set $\{x:q^{-k}\leqslant|x|\leqslant q^k\}$ and $n\geqslant k+1$. Let $f_n=[g_n]_{n+1}$. Clearly $f_n\in S^*$ for all n and $\langle f,\varphi\rangle=\langle f_n,\varphi\rangle$ for $n\geqslant k+1$.

Choose $\varphi\in\hat{S}^*$. Then $\langle\tilde{f},\varphi\rangle=\langle f,\tilde{\varphi}\rangle=\langle f_n,\tilde{\varphi}\rangle=\langle\tilde{f}_n,\varphi\rangle$ for n large enough. Thus it will suffice to determine the limiting value on $\hat{S}^{*\prime}$ of

$$(*)\qquad \tilde{f}_n(\pi_1)=\int_K f_n(v)\,\pi_1(v)\,|v|^{-1}\,dv$$

$$=\int_{q^{-(n+1)}\leqslant|v|\leqslant q^{n+1}}\pi_1(v)\,|v|^{-1}\int_{q^{-n}\leqslant|x|\leqslant q^n}\chi(x)\,\chi(v/x)\,\pi(x)\,|x|^{-1}\,dx\,dv$$

$$=\int_{q^{-n}\leqslant|x|\leqslant q^n}\chi(x)\,\pi(x)\,|x|^{-1}\int_{q^{-(n+1)}\leqslant|v|\leqslant q^{n+1}}\chi(v/x)\,\pi_1(v)\,|v|^{-1}\,dv\,dx$$

$$=\int_{q^{-n}\leqslant|x|\leqslant q^n}\chi(x)\,\pi\pi_1(x)\,|x|^{-1}\int_{q^{-(n+1)}|x|\leqslant|v|\leqslant q^{n+1}/|x|}\chi(v)\,\pi_1(v)\,|v|^{-1}\,dv\,dx.$$

Suppose π_1, $\pi\pi_1$ are ramified of degrees h_1, $h_2 \geqslant 1$. When $n+1 \geqslant \max[h_2, |h_1-h_2|]$ then $\hat{f}_n(\pi_1)=\Gamma(\pi\pi_1)\Gamma(\pi_1)$ as we see from Lemma 1. It follows that if φ is supported on a circle where π_1 and $\pi\pi_1$ are both ramified $\langle \hat{f}, \varphi\rangle = \langle \Gamma^*(\cdot)\Gamma^*(\pi\cdot), \varphi\rangle$.

The remainder of the proof is concerned with evaluating $\langle \hat{f}, \varphi\rangle$ when φ is supported on a circle where either or both of π_1 and $\pi\pi_1$ are unramified.

Define $S_{-1}(\alpha) = -q^{i\alpha-1}$, $S_n(\alpha) = -q^{i\alpha-1} + 1/q'(1+q^{-i\alpha}+\dots+q^{-in\alpha})$.

If π is ramified of degree $h \geqslant 1$ and π_1 is such that π_1 and $\pi\pi_1$ are not both ramified then either i) π_1 is unramified and $\pi\pi_1$ is ramified of degree h or ii) $\pi\pi_1$ is unramified and π_1 is ramified of degree h.

i) Set $\pi_1 = |x|^{i\alpha}$.
$(*) = \Gamma(\pi\pi_1)S_{n+h}(\alpha)$, $n \geqslant h$. The argument of Theorem 7 shows that $S_{n+h}(\alpha) \to \Gamma_1^*$ as $n \to \infty$ so that if φ is supported on T_1,

$$\langle \hat{f}, \varphi\rangle = \lim_{n\to\infty} \langle \Gamma(\pi\cdot)S_{n+h}(\cdot), \varphi\rangle = \langle \Gamma^*(\pi\cdot)\Gamma^*(\cdot), \varphi\rangle.$$

ii) $\pi\pi_1 = |x|^{i\alpha}$. In $(*)$ the inner integral is non-zero only if x satisfies the condition $q^{-(n+h+1)} \leqslant |x| \leqslant q^{n+1-h}$ in which case it is $\Gamma(\pi_1)$. Thus if $n \geqslant h$

$$(*) = \Gamma(\pi_1)\int_{q^{-n}\leqslant|x|\leqslant q} \chi(x)\pi\pi_1(x)|x|^{-1}dx = \Gamma(\pi_1)S_n(\alpha).$$

As in case i) we see that if φ is supported on the circle where $\pi\pi_1$ is ramified, then $\langle \hat{f}, \varphi\rangle = \langle \Gamma^*(\cdot)\Gamma^*(\pi\cdot), \varphi\rangle$.

Gathering results, we see that if π is ramified of degree $h>1$, then

$$\langle \hat{f}, \varphi\rangle = \langle \Gamma^*(\cdot)\Gamma^*(\pi\cdot), \varphi\rangle$$

for all $\varphi \in \hat{S}^*$.

To complete the proof we need to consider the case when π is unramified so that $\pi\pi_1$ and π_1 are unramified on T_1. Let $\pi_1 = |x|^{i\alpha}$, $\pi\pi_1 = |x|^{i(\alpha+\beta)}$. Then

$$(**) \qquad \hat{f}_n(\pi_1) = \int_{q^{-n}\leqslant|x|\leqslant q} \chi(x)\pi\pi_1(x)S_{n+k+1}(\alpha)|x|^{-1}dx, \quad |x| = q^k$$
$$= -q^{i(\alpha+\beta)-1}S_{n+2}(\alpha) + 1/q'\sum_{k=0}^{n} q^{-ik(\alpha+\beta)}S_{n-k+1}(\alpha).$$

It is easy to see that, when applied to φ supported on T_1, $\Gamma^*(\pi\cdot)\Gamma^*(\cdot)$ depends only on a finite number of terms in the formal product

$$\left(-q^{i\alpha-1}+1/q'\sum_{k=0}^{\infty}q^{-ik\alpha}\right)\left(-q^{i(\alpha+\beta)-1}+1/q'\sum_{k=0}^{\infty}q^{-ik(\alpha+\beta)}\right),$$

and if n is large enough any fixed set of finite terms of that product is contained in (**). Thus, if φ is supported on T_1 and π is unramified, $\langle \hat{f}, \varphi \rangle = \langle \Gamma^*(\cdot)\Gamma^*(\pi \cdot), \varphi \rangle$.

We have shown that for any π and any φ, $\langle \hat{f}, \varphi \rangle = \langle \Gamma^*(\cdot)\Gamma^*(\pi \cdot), \varphi \rangle$, so $\hat{f} = \Gamma^*(\cdot)\Gamma^*(\pi \cdot)$. Q.E.D.

§ 5. Hankel transforms

Definition. For $\pi \in \hat{K}^*$ and $\varphi \in S$, we define the *Hankel transform* (*of order* π), $H_\pi \varphi$, by

$$H_\pi \varphi(v) = \int_K \varphi(u) J_\pi(u, v)\, du, \quad v \in K^*. \tag{5.1}$$

Remark. H_π is well-defined on S since $J_\pi(u, v)$ is locally integrable on (K^+, du). If $\pi \not\equiv 1$, $J_\pi(u, v)$ is bounded so that H_π can be directly defined on all of $L^1(K^+)$. In particular, if $f \in L^1(K^+)$ and we let $A_\pi = \|J_\pi(\cdot, v)\|_\infty$, then

$$H_\pi f(v) = \int_K f(u) J_\pi(u, v)\, du \in L^\infty(K^+),$$

and $\|H_\pi f\|_\infty \leqslant A_\pi \|f\|_1$. We first establish a basic representation lemma for $H_\pi \varphi$, $\varphi \in S$.

LEMMA 10. *If* $\varphi \in S$, $H_\pi \varphi(v) = (\pi^{-1}(x)|x|^{-1}\hat{\varphi}(1/x))\hat{\ }(v)$. $H_\pi \varphi \in L^2(K^+)$ *and* $\|H_\pi \varphi\|_2 = \|\varphi\|_2$.

Proof. Let $f = P(\chi(v/x)\pi(x)|x|^{-1})$. Theorem 11 shows that $f \in S'$ and that $\hat{f} = J_\pi(\cdot, v)$. Therefore

$$\begin{aligned} H_\pi \varphi(v) = (J_\pi(\cdot, v), \varphi) = (f, \hat{\varphi}) &= P\int_K \chi(v/x)\, \pi(x)\, |x|^{-1} \hat{\varphi}(x)\, dx \\ &= P\int_K \pi^{-1}(x)\, |x|^{-1} \hat{\varphi}(1/x)\, \chi(vx)\, dx. \end{aligned}$$

Since $\hat{\varphi}(x) \in L^2(K^+)$, we see that $\pi^{-1}(x)|x|^{-1}\hat{\varphi}(1/x) \in L^2(K^+)$, and that

$$\|\hat{\varphi}\|_2 = \|\pi^{-1}(\cdot)|\cdot|^{-1}\hat{\varphi}(1/\cdot)\|_2.$$

Since $\hat{\varphi}(1/x) = 0$ for $|x|$ small, and $\hat{\varphi}(1/x)$ is constant for $|x|$ large, it follows from Lemma 1 and Lemma 2 that $P\int_K \pi^{-1}(x)|x|^{-1}\hat{\varphi}(1/x)\chi(vx)\,dx$ converges for $v \in K^*$. An application of Plancherel's theorem shows that this last integral converges to $(\pi^{-1}(x)|x|^{-1}\hat{\varphi}(1/x))\hat{\ }(v)$, and that $(\pi^{-1}(x)|x|^{-1}\hat{\varphi}(1/x))\hat{\ } \in L^2(K^+)$. Finally, since $\|\varphi\|_2 = \|\hat{\varphi}\|_2$, we have $\|H\varphi\|_2 = \|\varphi\|_2$. Q.E.D.

COROLLARY. *If* $\hat{\varphi} \in S^*$ (*equivalently* $\varphi \in S$, $\int_K \varphi\, dx = 0$), *then* $H_\pi \varphi \in S$.

Proof. $\hat{\varphi}(x) \in S^*$ if and only if $\hat{\varphi}(1/x) \in S^*$. Therefore $\pi^{-1}(x)|x|^{-1}\hat{\varphi}(1/x) \in S^* \subset S$, so that $H_\pi \varphi = (\pi^{-1}(x)|x|^{-1}\hat{\varphi}(1/x))\hat{\ } \in S$. Q.E.D.

THEOREM 13. *If* $\pi \not\equiv 1$, H_π *is a bounded linear map from* $L^1(K^+)$ *into* $L^\infty(K^+)$ *with norm* A_π. $H_\pi f(v)$, $f \in L^1(K^+)$, *is continuous for* $v \neq 0$. $H_\pi f(v) = \Gamma(\pi)\int_K \pi^{-1}(u) f(u)\,du + \Gamma(\pi^{-1})\,\pi(v)\int_K f(u)\,du + o(1)$ *as* $|v| \to 0$, $H_\pi f(v) \to 0$ *as* $|v| \to \infty$.

Proof. The remark above proves the first statement of the theorem. Since $J_\pi(u, v)$ is a continuous function of $v \neq 0$ for all $u \neq 0$, and $|J_\pi(u, v) f(u)| \leqslant A_\pi |f(u)|$, an application of the Lebesgue dominated convergence theorem shows that $H_\pi f(v)$ is continuous for $v \neq 0$.

To determine the behavior of $H_\pi f$ near zero we note that, if $|u| \leqslant q^h/|v|$ (h the ramification degree of π), $J_\pi(u, v) = \Gamma(\pi)\pi^{-1}(u) + \Gamma(\pi^{-1})\pi(v)$, by Theorems 8 and 9. Then

$$H_\pi f(v) = \Gamma(\pi)\int_{|u| \leqslant q^h/|v|} \pi^{-1}(u) f(u)\,du + \Gamma(\pi^{-1})\,\pi(v)\int_{|u| \leqslant q^h/|v|} f(u)\,du + \int_{|u| > q^h/|v|} O(1) f(u)\,du$$

$$= \Gamma(\pi)\int_K \pi^{-1}(u) f(u)\,du + \Gamma(\pi^{-1})\,\pi(v)\int_K f(u)\,du + o(1), \quad |v| \to 0.$$

To show that $H_\pi f(v) \to 0$ as $|v| \to \infty$, it will suffice to assume $f \in \mathcal{S}$, since $\mathcal{S}$ is dense in $L^1(K^+)$. Then, from the proof of Lemma 10, we have

$$H_\pi f(v) = \int_{q^{-n} \leqslant |x| < q^n} \pi^{-1}(x)\,|x|^{-1} \hat{f}(1/x)\,\chi(vx)\,dx + f(0)\,P\int_{|x| \geqslant q^n} \pi^{-1}(x)\,|x|^{-1}\chi(vx)\,dx,$$

where f is constant on $\mathfrak{P}^n$ and supported on $\mathfrak{P}^{-n}$, $n \geqslant 1$. It follows from Lemma 1 or Lemma 2, that the second integral is zero for $|v| > q^{-n+h+1}$, h the ramification degree of π. Now observe that $\hat{f}(1/x)$ is constant on the cosets of $\mathfrak{P}^{3n}$ in the range $q^{-n} \leqslant |x| < q^n$. Also, if π is ramified of degree $h \geqslant 1$, it follows from Lemma (M_1) that $\pi^{-1}(x)$ is constant on the cosets of $\mathfrak{P}^{h+n}$ in the range $q^{-n} \leqslant |x| < q^n$. In any case, setting $k = \max[3n, h+n]$, we may write

$$\int_{q^{-n} \leqslant |x| < q^n} \pi^{-1}(x)\,|x|^{-1}\hat{f}(1/x)\,\chi(vx)\,dx = \sum_s \pi^{-1}(a_s)\,\hat{f}(1/a_s)\int_{a_s + \mathfrak{P}^k} |x|^{-1}\chi(vx)\,dx,$$

where a_s runs through a complete set of coset representatives of $\mathfrak{P}^k$ in the given range. Lemma 2 shows that each of the integrals in the sum is zero for $|v|$ large enough. Hence $H_\pi f$ has compact support and, *a fortiori*, $H_\pi f(v) \to 0$ as $|v| \to \infty$. Q.E.D.

The operator H_π defined by (5.1) for $\varphi \in \mathcal{S}$ may be extended to an operator on all of $L^2(K^+)$ as follows. By Lemma 10 we see that H_π is a linear isometry on $\mathcal{S}$, considered as a subspace of L^2. Since $\mathcal{S}$ is dense in $L^2(K^+)$, H_π may be extended to $L^2(K^+)$ as a linear isometry.

THEOREM 14. *H_π is a unitary map on $L^2(K^+)$. If $f, g \in L^2(K^+)$, then*

(a) $$\int_K H_\pi f(x)\,g(x)\,dx = \int_K f(x)\,H_{\pi^{-1}} g(x)\,dx,$$

$$\text{(b)} \qquad \int_K H_\pi f(x)\,\overline{g(x)}\,dx = \int_K f(x)\,\pi(-1)\,\overline{H_\pi g(x)}\,dx.$$

This shows that $\pi(-1)H_\pi$ is the adjoint of H_π, and hence its inverse.

Proof. The remark before the statement shows that H_π is an isometry. If we show (a), then (b) follows by a change of variables. If (a) holds and H_π is not onto, there is a $g\in L^2(K^+)$, $g\not\equiv 0$, such that

$$0=\int_K H_\pi f(x)\,g(x)\,dx=\int_K f(x)\,H_{\pi^{-1}}g(x)\,dx, \quad \text{for all } f\in L^2(K^+).$$

This shows that $H_{\pi^{-1}}g\equiv 0$. But $H_{\pi^{-1}}$ is an isometry so that $g\equiv 0$. The proof will be complete if we show (a). Formally

$$\begin{aligned}\int_K H_\pi f(v)\,g(v)\,dv &= \int_K (\pi^{-1}(x)\,|x|^{-1}\hat{f}(1/x))\hat{\ }(v)\,g(v)\,dv\\ &= \int_K \pi^{-1}(x)\,|x|^{-1}\hat{f}(1/x)\,\hat{g}(x)\,dx = \int_K \hat{f}(x)\,\pi(x)\,|x|^{-1}\hat{g}(1/x)\,dx\\ &= \int_K f(v)\,(\pi(x)\,|x|^{-1}g(1/x))\hat{\ }(v)\,dv = \int_K f(v)\,H_{\pi^{-1}}g(v)\,dv.\end{aligned}$$

These relations all hold if f, g, $H_\pi f$, $H_{\pi^{-1}}g$ are in S which follows from the corollary to Lemma 10, if we take $\hat{f}$, $\hat{g}\in S^*$. Since the set of all such f and g is dense in L^2, the result extends to all f, $g\in L^2(K^+)$. Q.E.D.

THEOREM 15. *If $\pi\not\equiv 1$ and $\varphi\in S$, then $H_\pi\varphi\in L^{p'}$, $1\leqslant p\leqslant 2$, $1/p+1/p'=1$. We have*

$$(*) \qquad \|H_\pi\varphi\|_{p'}\leqslant A_\pi^{2/p-1}\|\varphi\|_p.$$

H_π can be extended to a linear operator on all L^p, maintaining (*). For any $f\in L^p(K^+)$, $1\leqslant p\leqslant 2$, $H_\pi f$ is equal almost everywhere to F_1+F_2 where F_1 is bounded and continuous on K^* and $F_2\in L^2(K^+)$.

Proof. If $\pi\neq 1$, Theorem 13 shows that $\|H_\pi\varphi\|_\infty\leqslant A_\pi\|\varphi\|_1$. Theorem 14 shows that $\|H_\pi\varphi\|_2=\|\varphi\|_2$. An application of the Riesz–Thorin interpolation theorem ([8], v. II, p. 95) shows that (*) holds for $\varphi\in S$ and that the operator extends to $L^p(K^+)$ maintaining (*).

To obtain the decomposition $H_\pi f=F_1+F_2$, we need only write $f=f_1+f_2$, $f_1\in L^1$, $f_2\in L^2$, and set $H_\pi f_i=F_i$, $i=1,2$. A standard argument shows that $H_\pi f_1+H_\pi f_2$ agrees almost everywhere with $H_\pi f$ defined by extending H_π from its values on S. Q.E.D.

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