WEIGHTED POLYNOMIAL APPROXIMATION ON ARITHMETIC PROGRESSIONS OF INTERVALS OR POINTS

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Introduction and definitions

The classical Bernstein problem on weighted polynomial approximation is as follows: Given a continuous function $W(x) \geqslant 1$ on $(-\infty, \infty)$ such that, for every $n \geqslant 0$,

$$\frac{|x|^n}{W(x)} \to 0 \quad \text{as} \quad x \to \pm\infty;$$

determine whether or not every continuous function $f(x)$ satisfying

$$\frac{f(x)}{W(x)} \to 0, \quad x \to \pm\infty$$

can be approximated uniformly by polynomials with respect to the weight W, that is, whether or not, corresponding to every such f, there exist polynomials P making

$$\sup_{-\infty < x < \infty} \frac{|f(x) - P(x)|}{W(x)}$$

arbitrarily small.

In this problem, whose solution is known, it is approximation over the whole real line that is in question. The present study is concerned with the similar problem that arises when the real line is replaced by certain unbounded subsets thereof, namely those obtained when a fixed segment is translated to and fro through all integral multiples of a fixed distance, or even by discrete subsets, like the set of integers.

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We will consider approximation by finite trigonometric sums as well as by polynomials. At this point, it is convenient to introduce some special notations, which will be followed in the rest of this paper.

If $0 \leqslant \varrho \leqslant \frac{1}{2}$, E_ϱ denotes the set $\bigcup_{n=-\infty}^{\infty} [n-\varrho, n+\varrho]$. Thus, E_ϱ is the real line, $\mathbf{R}$, if $\varrho = \frac{1}{2}$, and for $\varrho = 0$, E_ϱ reduces to $\mathbf{Z}$, the set of integers.

Let $W(x) \geqslant 1$ be a continuous function defined on E_ϱ, having the property that $W(x) \to \infty$ as $x \to \pm\infty$ in E_ϱ. Then $\mathcal{C}_W(E_\varrho)$ will denote the set of functions f, defined and continuous on E_ϱ, fulfilling the condition:

$$\frac{f(x)}{W(x)} \to 0 \quad \text{as} \quad x \to \pm\infty \quad \text{in} \quad E_\varrho.$$

Writing

$$\|f\|_{W, E_\varrho} = \sup_{x \in E_\varrho} \frac{|f(x)|}{W(x)}$$

for $f \in \mathcal{C}_W(E_\varrho)$ makes the latter into a Banach space, with norm $\| \ \|_{W, E_\varrho}$.

For $A > 0$, $\mathcal{C}_W(E_\varrho, A)$ is the closure (with respect to $\| \ \|_{W, E_\varrho}$) in $\mathcal{C}_W(E_\varrho)$ of the set of finite sums of the form $\sum_{-A \leqslant \lambda \leqslant A} a_\lambda e^{i\lambda x}$. Also, provided that $W(x)$ has the supplementary property:

$$\frac{|x|^n}{W(x)} \to 0 \quad \text{as} \quad x \to \pm\infty \quad \text{in} \quad E_\varrho \quad \text{for all} \quad n \geqslant 0,$$

we define $\mathcal{C}_W(E_\varrho, 0)$ as the closure, in $\mathcal{C}_W(E_\varrho)$, of the set of polynomials.

In terms of these notations, the classical Bernstein problem can be restated as follows:

If $W(x)$ has the aforementioned supplementary property, under what additional conditions on $W(x)$ does $\mathcal{C}_W(\mathbf{R}, 0) = \mathcal{C}_W(\mathbf{R})$?

The solution ([1], [2]) is as follows:

$\mathcal{C}_W(\mathbf{R}, 0) = \mathcal{C}_W(\mathbf{R})$ *if and only if there exist polynomials P, satisfying $\|P\|_{W,\mathbf{R}} \leqslant 1$, that make the integral*

$$\int_{-\infty}^{\infty} \frac{\log |P(x)|}{1+x^2}\, dx$$

arbitrarily large.

The condition on $W(x)$ provided by this result is not a very explicit one, but it does lead immediately to the important corollary, due to T. Hall:

$$\textit{If} \int_{-\infty}^{\infty} \frac{\log W(x)}{1+x^2}\, dx < \infty, \quad \textit{then} \quad \mathcal{C}_W(\mathbf{R}, 0) \neq \mathcal{C}_W(\mathbf{R}).$$

The purpose of this paper is to investigate what happens to these and related results when $\mathbf{R}$ is replaced by E_ϱ with $0 \leqslant \varrho < \frac{1}{2}$. If $0 < \varrho < \frac{1}{2}$, it turns out that the only change in them consists in the replacement of integrals over $\mathbf{R}$ by integrals over E_ϱ, the integrands themselves remaining unmodified. For the case $\varrho=0$, i.e., that of weighted approximation on the *integers*, we have not obtained a full solution to the problem, but only an analogue of the above corollary (1). It is rather remarkable that the most obvious adaptation of that result is actually valid in this case, namely:

$$\textit{If} \quad \sum_{-\infty}^{\infty} \frac{\log W(n)}{1+n^2} < \infty, \quad \textit{then} \quad \mathsf{C}_W(\mathbf{Z},0) \neq \mathsf{C}_W(\mathbf{Z}).$$

This statement reminds one of Beurling and Malliavin's multiplication theorems, set forth in [5] (see Theorems I and II of that paper). Indeed, part of our proof bears a superficial resemblance to the reasoning in [5], insofar as the same harmonic function (see formula (18) and the beginning of § 8 in Part II below) figures in both developments, and its potential theoretic properties are used, albeit in quite different ways. The connection of our results relating to $\mathbf{Z}$ with those of Beurling and Malliavin is nevertheless not clear, and it does not seem possible to obtain ours from theirs without much labor, if at all.

The examination of the case involving E_ϱ with $0<\varrho<\frac{1}{2}$ is carried out in Part I of the present study. Part II is devoted to the case when $\varrho=0$, i.e., when E_ϱ reduces to $\mathbf{Z}$. The method used here is different from that of Part I, so that the two parts of the paper can be read independently.

I am indebted to Professor Carleson, editor of these *Acta*, for valuable criticism of Part I, § 1, thanks to which the exposition of that section was shortened and simplified considerably.

Part I. Weighted approximation on E_ϱ, $0<\varrho<\frac{1}{2}$

The solution of the classical Bernstein problem, for the case $E_\varrho=\mathbf{R}$, is based to a large extent on the elementary majoration

$$H(z) \leqslant \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{|y|\,H(t)}{(x-t)^2+y^2}\,dt,$$

valid for all continuous subharmonic functions $H(z)$ of sufficiently slow growth. The main step in our solution of the problem for the case $0<\varrho<\frac{1}{2}$ consists in the derivation of a similar estimate, expressed in terms of the values of $H(t)$ on E_ϱ, instead of on the whole real line.

1. Harmonic measure and harmonic majoration in the complement of E_ϱ

We denote by D_ϱ the complement of E_ϱ. D_ϱ is an open subset of the complex plane, of infinite connectivity.

(1) *Note added in proof:* We have since extended the work of the present paper so as to obtain the complete solution for this case also ($\varrho=0$). This appears in *Comptes Rendus*, t. 262, no. 20, Ser. A, pp. 1100–1102 (1966).

Lemma. *Let $V(z)$ be real and bounded above in the complex plane, continuous on a neighborhood of each component of E_ϱ, and subharmonic in D_ϱ. Then, for every z, $V(z) \leqslant \sup_{t \in E_\varrho} V(t)$.*

This lemma follows easily from an elementary Phragmén–Lindelöf argument; one may use

$$\log \left| \frac{z}{\varrho} + \sqrt{\frac{z^2}{\varrho^2} - 1} \right|,$$

with proper determination of the radical, as the Phragmén–Lindelöf function.

Definition. *We denote by $\omega(z)$ a function having the following properties:*

i) *$\omega(z)$ is continuous in the complex plane, and harmonic in D_ϱ.*

ii) $0 \leqslant \omega(z) \leqslant 1$.

iii) $\omega(t) \equiv 1, \ -\varrho \leqslant t \leqslant \varrho$.

iv) $\omega(t) \equiv 0, \ n - \varrho \leqslant t \leqslant n + \varrho$, *for* $n = \pm 1, \ \pm 2, \ldots$.

In our case, the existence of $\omega(t)$ is guaranteed by fairly simple general considerations. According to the lemma, there can only be one function $\omega(z)$.

The function $\omega(z)$ is the harmonic measure of the component $[-\varrho, \varrho]$ of E_ϱ relative to D_ϱ, as seen from the point z. It is clear that the harmonic measure of the component $[n-\varrho, n+\varrho]$ of E_ϱ is given by $\omega(z-n)$, so that, if $H(z)$ is bounded and continuous in the complex plane, harmonic in D_ϱ, and assumes, for each n, the constant value h_n on the component $[n-\varrho, n+\varrho]$ of E_ϱ, we will have

$$H(z) = \sum_{-\infty}^{\infty} h_n \omega(z-n) \tag{1}$$

by the lemma.

We wish to estimate $\omega(z)$ from above. According to a natural extension of an idea due to Keldysh and Sedov (see [4], pp. 284–288), it would be possible to express

$$\frac{\partial \omega(z)}{\partial x} \quad \text{as} \quad \Re \left\{ \frac{\varphi(z)}{\sqrt{\sin^2 \pi z - \sin^2 \pi \varrho}} \right\},$$

where $\varphi(z)$ is a certain entire function, having a simple zero in each of the intervals $(n+\varrho, n+1-\varrho)$, $n \neq -1, 0$, and no other zeros. One can use the periodicity of the set E_ϱ to establish an integral equation for φ which leads to a complicated but explicit formula for it, and hence to the determination of $\omega(z)$.

As we are only interested in estimating $\omega(z)$, we shall proceed somewhat differently. Our idea is to approximate $\omega(z)$ by replacing the function $\varphi(z)$ in the above expression by the elementary one $(\cos \pi z)/(z^2 - \frac{1}{4})$ whose zeros are close to those of $\varphi(z)$.

THEOREM 1. $$\omega(z) \leqslant C_\varrho \frac{|y|+1}{x^2+(|y|+1)^2},$$

where C_ϱ depends only on ϱ.

Proof. Using the branch of $(\sin^2 \pi z - \sin^2 \pi\varrho)^{\frac{1}{2}}$ which is single valued in D_ϱ and positive on the interval $(\varrho, 1-\varrho)$, we write, for each complex z,

$$\Omega(z) = -\Re \int_0^\infty \frac{\cos \pi(z+t)\, dt}{[(z+t)^2 - \frac{1}{4}]\sqrt{\sin^2 \pi(z+t) - \sin^2 \pi\varrho}}. \tag{2}$$

It is not hard to see that $\Omega(z)$ is continuous and bounded in the complex plane, and constant on each of the segments $[n-\varrho, n+\varrho]$, since $(\sin^2 \pi z - \sin^2 \pi\varrho)^{\frac{1}{2}}$ has imaginary boundary values on both sides of those segments. By its very form (differentiate (2) with respect to x), $\Omega(z)$ is seen to be harmonic in D_ϱ.

The function $\Omega(x)$ is clearly even, and we proceed to estimate it for $x \geqslant 0$. If the integer n is $\geqslant 0$, the quantity $(\cos \pi x)/(\sin^2 \pi x - \sin^2 \pi\varrho)^{\frac{1}{2}}$ is positive on $(n+\varrho, n+\frac{1}{2})$ and negative on $(n+\frac{1}{2}, n+1-\varrho)$, and its integrals over these two intervals are equal to C and $-C$ respectively, where C is a certain positive constant whose exact value we do not need to know (in fact, $C = \pi^{-1} \arg\cosh (1/\sin \pi\varrho)$). Because of this, (2) yields, by the second mean value theorem,

$$-\frac{C}{n^2 - \frac{1}{4}} \leqslant \Omega(n+\varrho) = \Omega(n) \leqslant 0, \quad n = 1, 2, \ldots. \tag{3}$$

And the same argument shows, quite generally, that

$$|\Omega(x)| \leqslant \frac{O(1)}{x^2+1} \tag{4}$$

for $x \geqslant 0$, hence for all real x, since $\Omega(x)$ is even. Another use of (2) shows us that

$$\Omega(0) - \Omega(1) = \Omega(\varrho) - \Omega(1-\varrho) = -\int_\varrho^{1-\varrho} \frac{\cos \pi x\, dx}{(x^2 - \frac{1}{4})\sqrt{\sin^2 \pi x - \sin^2 \pi\varrho}} > 4C + \tfrac{4}{3}C$$

which, with (3), yields $$\Omega(0) = 4C\gamma_\varrho, \tag{5}$$

where γ_ϱ is a constant >1 depending on ϱ.

Applying formula (1) to the function $\Omega(z)$, we obtain

$$\Omega(z) = \sum_{-\infty}^{\infty} \Omega(n)\, \omega(z-n).$$

Since $\omega(z) \geqslant 0$, we can substitute estimates (3) and (5) into this last relation, getting

$$\omega(z) - \sum_{-\infty}^{\infty} A_n \omega(z-n) \leqslant \frac{\Omega(z)}{\Omega(0)}, \tag{6}$$

where

$$A_n = \begin{cases} 0, & n=0 \\ \dfrac{1}{\gamma_\varrho(4n^2-1)}, & n \neq 0. \end{cases} \tag{7}$$

We shall use (6) to estimate $\omega(x)$ for real x. The elementary formula

$$\sum_{-\infty}^{\infty} \frac{e^{in\vartheta}}{1-4n^2} = \frac{\pi}{2} \sin \frac{|\vartheta|}{2}, \quad |\vartheta| \leqslant \pi \tag{8}$$

shows first of all that $\sum_{-\infty}^{\infty} |A_n| = 1/\gamma_\varrho < 1$, so that $\omega(z)$ can be expressed in terms of the *left member*, $\psi(z)$, of (6) by

$$\omega(z) = \psi(z) + \sum_{-\infty}^{\infty} B_n \psi(z-n), \tag{9}$$

where the B_n are related to the A_n through the equality

$$1 + \sum_{-\infty}^{\infty} B_n e^{in\vartheta} = \frac{1}{1-\sum_{-\infty}^{\infty} A_n e^{in\vartheta}}. \tag{10}$$

From (7) we have $A_n \geqslant 0$, from which it is easy to see, by expanding the right side of (10) in powers of $\sum_{-\infty}^{\infty} A_n e^{in\vartheta}$, that the B_n are all $\geqslant 0$. Because of this, we can replace $\psi(z)$ in (9) by the right-hand member of (6), yielding

$$\omega(z) \leqslant \frac{\Omega(z)}{\Omega(0)} + \sum_{-\infty}^{\infty} B_n \frac{\Omega(z-n)}{\Omega(0)}. \tag{11}$$

Now

$$B_n \leqslant \frac{O(1)}{n^2+1}. \tag{12}$$

Indeed, from (7) and (10), we have, by formula (8),

$$B_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{\cos n\vartheta \, d\vartheta}{F(\vartheta)} \quad \text{for} \quad n \neq 0, \tag{13}$$

where

$$F(\vartheta) = \frac{\gamma_\varrho - 1}{\gamma_\varrho} + \frac{\pi}{2\gamma_\varrho} \sin \frac{|\vartheta|}{2}.$$

Since $\gamma_\varrho > 1$, $d(1/F(\vartheta))/d\vartheta$ is of bounded variation on $[-\pi, \pi]$, so that the integral in (13) can be integrated by parts twice and thereby proven to be $O(1/n^2)$.

For real x, (12), (11), and (4) yield

$$\omega(x) \leqslant \sum_{-\infty}^{\infty} \frac{O(1)}{(x-n)^2+1} \cdot \frac{O(1)}{n^2+1} \leqslant \frac{C_\varrho}{x^2+1}, \tag{14}$$

where C_ϱ depends only on ϱ.

The function $\omega(z)$ is continuous and bounded in $\Im z \geqslant 0$, and harmonic in $\Im z > 0$, so for such z we have by Poisson's formula

$$\omega(z) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{y}{(x-t)^2+y^2}\, \omega(t)\, dt.$$

Substitution of (14) into this yields $\omega(z) \leqslant C_\varrho (y+1)^2/(x^2+(y+1)^2)$ for $y>0$, and for $y<0$ a similar argument applies, completing the proof of the inequality affirmed by the theorem.

Theorem 1 will figure in our applications through the use of a

Corollary. *Let $H(z)$ be subharmonic and bounded above in D_ϱ, and continuous in a neighborhood of E_ϱ, and suppose that for each integer n there is a number $h_n \geqslant 0$ such that*

$$H(x) \leqslant h_n \text{ for } n-\varrho \leqslant x \leqslant n+\varrho.$$

Then
$$H(i) \leqslant K_\varrho \sum_{-\infty}^{\infty} \frac{h_n}{n^2+1}$$
with a constant K_ϱ depending only on ϱ.

Proof. Combine the above theorem with the lemma given at the beginning of this section.

2. Application to weighted approximation on E_ϱ

The special notations used below have already been explained in the introduction to this paper.

Theorem 2. *Let $A>0$. Then $\mathcal{C}_W(E_\varrho, A) = \mathcal{C}_W(E_\varrho)$ if and only if the integral*

$$\int_{E_\varrho} \frac{\log |f(x)|}{1+x^2}\, dx$$

is unbounded above when f ranges over the set of finite sums of the form

$$f(x) = \sum_{-A \leqslant \lambda \leqslant A} a_\lambda e^{i\lambda x}$$

satisfying $\|f\|_{W,E_\varrho} \leqslant 1$.

Proof. Suppose first of all that the integral in question does remain bounded above, say by K, when f ranges over the set of finite sums of the given form satisfying $\|f\|_{W,E_\varrho} \leqslant 1$.

Then we will prove that $\mathcal{C}_W(E_\varrho, A)$ consists only of entire functions, hence cannot be equal to $\mathcal{C}_W(E_\varrho)$.

We remark first of all that $\mathcal{C}_W(E_\varrho, A)$ is generated by the finite sums of the form

$$\sum_{-A\leqslant\lambda\leqslant A} a_\lambda e^{i\lambda x},$$

where, in each particular sum, all the λ belong to some arithmetic progression (depending on the sum). This is true because with such sums we can come arbitrarily close in $\| \ \|_{W,E_\varrho}$ norm to any other one of the same form, but not necessarily having its λ in arithmetic progression. That fact is in turn an easy consequence of the conditions

$$W(x)\geqslant 1, \quad x\in E_\varrho,$$
$$W(x)\to\infty \text{ for } x\to\pm\infty \text{ in } E_\varrho.$$

In order to prove that $\mathcal{C}_W(A, E_\varrho)$ consists only of entire functions, it is thus sufficient to show that the set of finite sums

$$f(z)=\sum_{-A\leqslant\lambda\leqslant A} a_\lambda e^{i\lambda z}$$

with $\|f\|_{W,E_\varrho}\leqslant 1$, the λ in each sum being in arithmetic progression, constitutes a normal family (in the complex plane). It is even enough to show this for the smaller set made up of all such sums which are real on the real axis, for any other can be written as the sum of two, one real and one purely imaginary on the real axis.

We see that all we need to do is give a bound, depending only on z and not on f, for $|f(z)|$, where f is any finite sum of the form

$$f(x)=\sum_{-A\leqslant\lambda\leqslant A} a_\lambda e^{i\lambda x}$$

having the λ in arithmetic progression, such that $\|f\|_{W,E_\varrho}\leqslant 1$ and $f(x)$ is real on the real axis. Let f be such a sum. Then the function

$$g(x)=1+(f(x))^2$$

is $\geqslant 1$ on the real axis, and can be expressed as a finite sum

$$\sum_{-2A\leqslant\mu\leqslant 2A} b_\mu e^{i\mu x},$$

where the μ belong also to some arithmetic progression. It follows by a theorem of Fejèr and Riesz ([6], p. 117) that we can write $g(x)=|h(x)|^2$, $-\infty<x<\infty$, with a finite sum h of the form

$$h(x) = \sum_{-A \leqslant \nu \leqslant A} c_\nu e^{i\nu x},$$

having the property that all the zeros of $h(z)$ lie in $\Im z < 0$ (see [6], p. 118).

Now $|h(x)| = (1+(f(x))^2)^{\frac{1}{2}} \leqslant 1 + |f(x)|$, $-\infty < x < \infty$, so since $\|f\|_{W,E_\varrho} \leqslant 1$ and $W(x) \geqslant 1$ for $x \in E_\varrho$, $\|\frac{1}{2}h\|_{W,E_\varrho} \leqslant 1$. In view of the supposition made at the beginning of this proof, this implies

$$\int_{E_\varrho} \frac{\log|\frac{1}{2}h(x)|}{1+x^2}\,dx < K,$$

i.e.

$$\int_{E_\varrho} \frac{\log|h(x)|}{1+x^2}\,dx < K + \pi \log 2,$$

where the constant K does not depend on the choice of f.

The function

$$U(z) = \frac{1}{\varrho}\int_{-\varrho/2}^{\varrho/2} \log|h(z+t)|\,dt$$

is subharmonic in the complex plane, hence surely in $D_{\varrho/2}$. Since all the zeros of $h(z)$ lie in $\Im z < 0$, $U(z)$ is continuous in a neighborhood of E_ϱ.

The function $h(z)$ is, by its form, bounded on the real axis, and of exponential type A. So by the Phragmèn–Lindelöf theorem, $\log|h(z)| \leqslant O(1) + A|y|$, and the function

$$V(z) = U(z) - \frac{A}{\pi}\log\left|\frac{\sin \pi z}{\sin \frac{1}{2}\pi\varrho} + \sqrt{\frac{\sin^2 \pi z}{\sin^2 \frac{1}{2}\pi\varrho} - 1}\right|$$

(same determination of the radical as in the proof of Theorem 1, § 1) is bounded above in $D_{\varrho/2}$. $V(z)$ is clearly subharmonic in $D_{\varrho/2}$ and continuous in a neighborhood of $E_{\varrho/2}$, and for $x \in E_{\varrho/2}$, $V(x) = U(x)$. Denote by h_n the maximum of $U(x) = V(x)$ on the component $[n - \varrho/2, n + \varrho/2]$ of $E_{\varrho/2}$, then, since $|h(x)| \geqslant 1$ for x real, we have $h_n \geqslant 0$, and by the corollary of § 1,

$$V(i) \leqslant \bar{K}_{\varrho/2} \sum_{-\infty}^{\infty} \frac{h_n}{1+n^2}.$$

Using again the fact that $|h(x)| \geqslant 1$ for x real, we see from the definition of $U(x)$ that

$$h_n \leqslant \frac{1}{\varrho}\int_{n-\varrho}^{n+\varrho} \log|h(t)|\,dt,$$

so

$$V(i) \leqslant \frac{\bar{K}_{\varrho/2}}{\varrho}\sum_{-\infty}^{\infty} \frac{\int_{n-\varrho}^{n+\varrho}\log|h(t)|\,dt}{1+n^2} \leqslant \frac{\bar{K}_{\varrho/2}}{\varrho(1-\varrho)^2}\int_{E_\varrho} \frac{\log|h(x)|}{1+x^2}\,dx$$

which is, as we have seen,

$$\leqslant \frac{K_{\varrho/2}}{\varrho(1-\varrho)^2}(K+\pi\log 2)=C,$$

where C is completely independent of the choice of f.

In terms of $U(i)$, this last inequality yields

$$U(i) \leqslant C + A\left(1 - \frac{1}{\pi}\log\sin\frac{\pi\varrho}{2}\right),$$

that is,
$$\frac{1}{\varrho}\int_{-\varrho/2}^{\varrho/2}\log|h(i+t)|\,dt \leqslant M,$$

with a number M independent of the choice of f.

Now $h(z)$ is bounded on the real axis and of exponential type A, and has no zeros in $\Im z \geqslant 0$. So Poisson's formula may be applied ([7], p. 92) to yield, for t real,

$$\log|h(i+t)| = A + \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\log|h(x)|\,dx}{(x-t)^2+1}.$$

Integrating both sides of this relation with respect to t over $[-\varrho/2, \varrho/2]$, and using once more the fact that $\log|h(x)| \geqslant 0$, $-\infty < x < \infty$, we see from the previous inequality that

$$\int_{-\infty}^{\infty}\frac{\log|h(x)|}{x^2+1}\,dx \leqslant M',$$

where M' depends only on M and ϱ, and is hence completely independent of f. Substituting $|h(x)| = (1+(f(x))^2)^{\frac{1}{2}}$, this yields finally

$$\int_{-\infty}^{\infty}\frac{\log^+|f(x)|}{x^2+1}\,dx \leqslant M'.$$

Since $f(z)$ is, by its form, bounded on the real axis and of exponential type A, it is a consequence ([7], p. 93) of Poisson's formula that

$$\log|f(z)| \leqslant A|y| + \frac{|y|}{\pi}\int_{\infty-}^{\infty}\frac{\log^+|f(t)|}{(x-t)^2+y^2}\,dt.$$

It is possible, with the help of the Phragmèn–Lindelöf theorem, to derive, from these last two inequalities, a relation of the form $\log|f(z)| \leqslant \alpha(A+M')(|z|+1)$, where α is a purely numerical constant (see, for instance, [3]). This estimate holds for any finite sum f of the form described above, as long as it is real on the real axis and satisfies the condition $\|f\|_{W,E_\varrho} \leqslant 1$. Since α, A and M' are independent of f, we have found the bound we needed, and hence proved that $\mathcal{C}_W(E_\varrho, A) \neq \mathcal{C}_W(E_\varrho)$ under the supposition made at the beginning of this demonstration.

In the other direction, the theorem is immediate. Suppose, indeed, that we have a sequence of finite sums $f_n(x)$, each of the form $\sum_{-A \leq \lambda \leq A} a_\lambda e^{i\lambda x}$, such that $\|f_n\|_{W, E_\varrho} \leq 1$, but

$$\int_{E_\varrho} \frac{\log |f_n(x)|}{1+x^2}\, dx \to \infty, \quad n \to \infty.$$

Then,
$$\int_{E_\varrho} \frac{\log^+ |f_n(x)|}{1+x^2}\, dx \to \infty, \quad n \to \infty,$$

whence, a fortiori,
$$\int_{-\infty}^{\infty} \frac{\log^+ |f_n(x)|}{1+x^2}\, dx \to \infty \quad \text{as} \quad n \to \infty.$$

Since $\|f_n\|_{W, E_\varrho} \leq 1$, the previous formula implies that $\mathcal{C}_W(E_\varrho) = \mathcal{C}_W(E_\varrho, A)$, according to a theorem of Akhiezer (see [1], [2], and [3]).

Theorem 2 is completely proved.

COROLLARY. *If* $\displaystyle\int_{E_\varrho} \frac{\log W(x)}{1+x^2}\, dx < \infty$, *then* $C_W(E_\varrho, A) \neq C_W(E_\varrho)$.

Proof. Clear.

THEOREM 3. *Let* $W(x)$ *have the property that* $|x|^n / W(x) \to 0$ *as* $x \to \pm\infty$ *in* E_ϱ, *for all* $n \geq 0$. *Then, in order that* $\mathcal{C}_W(E_\varrho, 0) = \mathcal{C}_W(E_\varrho)$, *it is necessary and sufficient that the integral*

$$\int_{E_\varrho} \frac{\log |P(x)|}{1+x^2}\, dx$$

be unbounded above as $P(x)$ *ranges over the set of polynomials with* $\|P\|_{W, E_\varrho} \leq 1$.

Proof. Let us suppose that the integral in question is bounded above, by K say, for all polynomials P satisfying $\|P\|_{W, E_\varrho} \leq 1$. Then we shall prove that the polynomials satisfying this inequality form a normal family in the complex plane. In that case $\mathcal{C}_W(E_\varrho, 0)$ can only consist of analytic functions, and hence cannot equal $\mathcal{C}_W(E_\varrho)$. To see this, it is enough to show that the set of polynomials P which are real on the real axis and satisfy $\|P\|_{W, E_\varrho} \leq 1$ form such a normal family, for any polynomial is the sum of two, one real and one purely imaginary on $(-\infty, \infty)$.

Having made this preliminary reduction, the proof proceeds very much as in the case of Theorem 2, save that here $|P(x)|$ is not bounded on the real axis, as $|f(x)|$ was in the proof of that theorem.

To get around this difficulty, we first show that there is a constant L_ϱ, depending only on ϱ, such that

$$\int_{-\infty}^{\infty} \frac{\log [1+(P(x))^2]}{1+x^2}\,dx \leqslant L_\varrho \int_{E_\varrho} \frac{\log [1+(P(x))^2]}{1+x^2}\,dx$$

for any polynomial P, real on $(-\infty, \infty)$. Let P be such a polynomial; say it is of degree N. Given $\eta>0$, consider the function

$$g(z)=1+(P(z))^2\left(\frac{\sin \eta z}{\eta z}\right)^{2N}.$$

$g(z)$ is an entire function of exponential type $2N\eta$; it is real and bounded on the real axis, and satisfies there the inequality $g(x)\geqslant 1$. It follows from these facts, by an extension of the theorem of Fejèr and Riesz used earlier (see [7], p. 125), that there is an entire function $h(z)$ of exponential type $N\eta$, having all its zeros in $\Im z<0$, and satisfying $|h(x)|^2=g(x)$ for $-\infty<x<\infty$. On account of this, $|h(x)|$ is bounded and $\geqslant 1$ on the real axis, and we may treat the present function $h(z)$ just as the one denoted by the same letter was handled in the proof of Theorem 5, provided we replace A by $N\eta$ (the type of $h(z)$) in the definition of $V(z)$ given there. That reasoning leads to a relation which, in the present case, takes the form:

$$N\eta+\frac{1}{\pi}\int_{-\infty}^{\infty}\left(\frac{1}{\varrho}\int_{-\varrho/2}^{\varrho/2}\frac{dt}{(x-t)^2+1}\right)\log|h(x)|\,dx$$
$$\leqslant \frac{K_{\varrho/2}}{\varrho(1-\varrho^2)}\int_{E_\varrho}\frac{\log|h(x)|}{x^2+1}\,dx+N\eta\left(1-\frac{\log \sin \frac{1}{2}\pi\varrho}{\pi}\right).$$

Since $\log|h(x)|\geqslant 0$, $-\infty<x<\infty$, we can argue as in the proof of Theorem 2 to deduce

$$\int_{-\infty}^{\infty}\frac{\log|h(x)|}{1+x^2}\,dx \leqslant L_\varrho\left[\int_{E_\varrho}\frac{\log|h(x)|}{x^2+1}dx-N\eta\log\sin\tfrac{1}{2}\pi\varrho\right]$$

with a constant L_ϱ depending only on ϱ.

In terms of $P(x)$, the last formula can be written

$$\int_{-\infty}^{\infty}\frac{\log\left[1+(P(x))^2\left(\frac{\sin\eta x}{\eta x}\right)^{2N}\right]}{1+x^2}\,dx$$
$$\leqslant L_\varrho\int_{E_\varrho}\frac{\log\left[1+(P(x))^2\left(\frac{\sin\eta x}{\eta x}\right)^{2N}\right]}{1+x^2}\,dx-L_\varrho N\eta\log\sin\tfrac{1}{2}\pi\varrho,$$

and on making $\eta\to 0$, we get

$$\int_{-\infty}^{\infty} \frac{\log [1+(P(x))^2]}{1+x^2}\, dx \leqslant L_\varrho \int_{E_\varrho} \frac{\log [1+(P(x))^2]}{1+x^2}\, dx,$$

using Lebesgue's dominated convergence theorem.

The rest of this first part of the proof is now rapidly completed. If the polynomial $P(x)$ is real on the real axis, we can find a polynomial Q so that $|Q(x)|^2 = 1+(P(x))^2$, $-\infty < x < \infty$. Since $W(x) \geqslant 1$ for $x \in E_\varrho$, the inequality $\|P\|_{W, E_\varrho} \leqslant 1$ implies $\|\frac{1}{2}Q\|_{W, E_\varrho} \leqslant 1$, and from this we get

$$\int_{E_\varrho} \frac{\log |\frac{1}{2}Q(x)|}{1+x^2}\, dx \leqslant K,$$

a constant independent of P, according to the supposition made at the beginning of this demonstration. In terms of P, this last relation implies

$$\int_{E_\varrho} \frac{\log [1+(P(x))^2]}{1+x^2}\, dx \leqslant 2K + \pi \log 4,$$

and we can now apply the inequality proven above to conclude that

$$\int_{-\infty}^{\infty} \frac{\log^+ |P(x)|}{1+x^2}\, dx \leqslant L_\varrho (K + \pi \log 2),$$

a fixed constant, whenever the polynomial P is real on the real axis and satisfies $\|P\|_{W, E_\varrho} \leqslant 1$.

The set of polynomials P satisfying the relation just proven is a normal family in the complex plane, as may be seen by the methods mentioned in the proof of Theorem 2. The first part of the present demonstration is now complete.

The second part consists in showing that $\mathcal{C}_W(E_\varrho, 0) = \mathcal{C}_W(E_\varrho)$ if there exists a sequence of polynomials P_n such that $\|P_n\|_{W, E_\varrho} \leqslant 1$ whilst

$$\int_{E_\varrho} \frac{\log |P_n(x)|}{1+x^2}\, dx \to \infty, \quad n \to \infty.$$

This parallels exactly the reasoning at the end of the proof of Theorem 2.

Theorem 3 is now established.

Corollary. *Let* $|x|^n / W(x) \to 0$, $x \to \pm\infty$ *in* E_ϱ, *for all* $n \geqslant 0$. *If*

$$\int_{E_\varrho} \frac{\log W(x)}{1+x^2}\, dx < \infty,$$

then $\mathcal{C}_W(E_\varrho, 0) \neq \mathcal{C}_W(E_\varrho)$.

Part II. Weighted approximation on the integers

We are interested in seeing whether or not the corollary to Theorem 3 at the end of Part I can be extended to the case $\varrho=0$, when E_ϱ reduces to $\mathbf{Z}$. If, in the relation used in proving Theorem 3,

$$\int_{-\infty}^{\infty}\frac{\log[1+(P(x))^2]}{1+x^2}\,dx\leqslant L_\varrho\int_{E_\varrho}\frac{\log[1+(P(x))^2]}{1+x^2}\,dx,$$

valid for polynomials P real on the real axis, one tries to make $\varrho\to 0$, it is found, after calculation of the order of magnitude of L_ϱ, that $\varrho L_\varrho\to\infty$ as $\varrho\to 0$. It is therefore not possible to apply the results of Part I so as to obtain an estimate of the form

$$\int_{-\infty}^{\infty}\frac{\log[1+(P(x))^2]}{1+x^2}\,dx\leqslant K\sum_{-\infty}^{\infty}\frac{\log[1+(P(m))^2]}{1+m^2}.$$

Indeed, such an inequality is not valid without qualification. To see this, one may take

$$P_N(x)=(1-x^2)^{[N/\log N]}\prod_2^N\left(1-\frac{x^2}{n^2}\right),$$

and one easily finds that

$$\sum_{-\infty}^{\infty}\frac{\log^+|P_N(m)|}{1+m^2}<10$$

for all sufficiently large N, even though $P_N(i)\to\infty$ as $N\to\infty$.

It would thus appear that the corollary to Theorem 3 could not be adapted to the limiting case $E_\varrho=\mathbf{Z}$. However, if one tries to refine the above example so as to have smaller and smaller positive upper bounds on the given sum in place of the number 10, it seems to be impossible to proceed beyond a certain point without forcing boundedness of the sequence $|P_N(i)|$. This suggests that there might be a uniform majoration for $|P(z)|$, applicable to polynomials P for which $\sum_{-\infty}^{\infty}(\log^+|P(m)|)/(1+m^2)$ is sufficiently small. The main work of the following sections consists in the establishment of such a result for polynomials P of a certain special form.

During the remainder of Part II, $P(x)$ will denote a polynomial of the form

$$P(x)=\prod_k\left(1-\frac{x^2}{x_k^2}\right),$$

where the x_k are real and positive, and $n(t)$ will indicate the number of points x_k in the interval $[0,t]$. (The usefulness of the function $n(t)$ in determining the size of $|P(z)|$ for complex z is well known.) We shall prove:

To any $\varepsilon > 0$ *corresponds a* $\delta > 0$ *such that, for any polynomial* P *of the given form,*

$$\sum_1^\infty \frac{\log^+|P(m)|}{m^2} < \delta \quad \textit{implies that} \quad \frac{n(t)}{t} < \varepsilon \quad \textit{for all} \quad t > 0.$$

1. Estimation of sums from below by integrals

Direct calculation of the second derivative shows that, for $x > 0$, $\log|P(x)|$ is concave (downward) on any interval free of points x_k.

LEMMA. *Suppose* $0 < a < m$, *and that* $b > m$ *is determined so as to satisfy*

$$\frac{\log \frac{b}{a}}{1 - \frac{a}{b}} = \frac{m}{a}. \tag{1}$$

Then, if there are no points x_k *in* $[a, b]$,

$$\int_a^b \frac{\log|P(x)|}{x^2}\,dx \leqslant \log|P(m)| \int_a^b \frac{dx}{x^2}.$$

Proof. If b satisfies (1), then

$$\int_a^b \frac{dx}{x} = m \int_a^b \frac{dx}{x^2}. \tag{2}$$

Suppose, without loss of generality, that $|P(a)|$ is the smaller of the two quantities $|P(a)|$ and $|P(b)|$; then,

$$\int_a^b \frac{\log|P(x)|}{x^2}\,dx = \int_a^b \frac{\log|P(a)|}{x^2}\,dx + \int_a^b \frac{\log|P(x)| - \log|P(a)|}{x^2}\,dx.$$

Denote by M the value of $d \log|P(x)|/dx$ for $x = m$. Then, since there are no points x_k in $[a, b]$, $\log|P(x)|$ is concave downward for $a \leqslant x \leqslant b$, so (let the reader draw a figure),

$$0 \leqslant \log|P(x)| - \log|P(a)| \leqslant M(x - m) + \log|P(m)| - \log|P(a)|,$$

$a \leqslant x \leqslant b$. From this we have

$$\int_a^b \frac{\log|P(x)| - \log|P(a)|}{x^2}\,dx \leqslant \int_a^b \frac{\log|P(m)| - \log|P(a)|}{x^2}\,dx + M\left\{\int_a^b \frac{dx}{x} - m\int_a^b \frac{dx}{x^2}\right\}.$$

According to (2), the last term on the right vanishes, and, adding $\log|P(a)| \int_a^b x^{-2}dx$ to both sides of the resulting inequality, we have the lemma.

LEMMA. *If $m \geqslant 7$ and $m-1 \leqslant a \leqslant m$, the solution $b \geqslant m$ of* (1) *satisfies $b < m+2$.*

Proof. If we write $a/b = \varrho$, then $0 < \varrho \leqslant 1$, and (1) takes the form $(\log 1/\varrho)/(1-\varrho) = m/a$ which yields, on expanding the numerator on the right in powers of $1-\varrho$,

$$1-\varrho \leqslant \frac{2(m-a)}{a}.$$

Strict inequality holds here unless $\varrho = 1$, in which case $a = m = b$, and the lemma is trivially true. So, assuming strict inequality, we get

$$\varrho = \frac{a}{b} > \frac{3a-2m}{a},$$

whence, since $m-1 \leqslant a \leqslant m$,

$$b - m < \frac{(m-a)(2m-a)}{3a-2m} \leqslant \frac{m+1}{m-3} \leqslant 2$$

if $m \geqslant 7$, completing the proof.

THEOREM 4. *Let $6 \leqslant a < b$. Then there is a number b^*, $b \leqslant b^* < b+3$ such that, provided there are no zeros x_k of $P(x)$ in $[a, b^*]$,*

$$\int_a^{b^*} \frac{\log |P(x)|}{x^2}\,dx \leqslant 5 \sum_{a<m<b^*} \frac{\log^+ |P(m)|}{m^2}, \tag{3}$$

the sum on the right being taken over the integers m satisfying $a < m < b^*$.

DEFINITION. *During the rest of this paper, we will say that b^* is well disposed with respect to a.*

Proof of Theorem 4. Let the integer m_1 be such that $m_1 - 1 \leqslant a < m_1$; then $m_1 \geqslant 7$, and according to the previous lemma, we can find an a_1, $m_1 < a_1 < m_1 + 2$, such that

$$\frac{\log \dfrac{a_1}{a}}{1 - \dfrac{a}{a_1}} = \frac{m_1}{a}.$$

During the remainder of this proof, we shall write a_0 for a. Then, since $m_1 - 1 \leqslant a_0 < m_1$, $b > a_0$, and $m_1 < a_1 < m_1 + 2$, we cannot have $a_1 \geqslant b+3$.

In case $a_1 \geqslant b$, we take $b^* = a_1$, then $b \leqslant b^* < b+3$, and if there are no points x_k in $[a_0, b^*]$, we get

$$\int_{a_0}^{b^*} \frac{\log |P(x)|}{x^2} dx \leqslant \log |P(m_1)| \int_{a_0}^{a_1} \frac{dx}{x^2}, \tag{4}$$

according to the first lemma in this paragraph. Since $a_1 - a_0 < 3$ and $m_1/a_0 < 7/6$, we have $\int_{a_0}^{a_1} x^{-2} dx < 5/m_1^2$, so that (4) certainly implies (3), which is thus proven in case $a_1 \geqslant b$.

Suppose now that $a_1 < b$. We take m_2 as the integer satisfying $m_2 - 1 \leqslant a_1 < m_2$, and observe that $m_2 > m_1$ since $a_1 > m_1$. There is an a_2, $m_2 < a_2 < m_2 + 2$ satisfying

$$(\log a_2/a_1)/(1 - a_1/a_2) = m_2/a_1,$$

and since, in the present case, $a_1 < b$, we cannot have $a_2 \geqslant b + 3$. If there are no points x_k on $[a_1, a_2]$, we have by the first lemma,

$$\int_{a_1}^{a_2} \frac{\log |P(x)|}{x^2} dx \leqslant \log |P(m_2)| \int_{a_1}^{a_2} \frac{dx}{x^2},$$

where, as in the previous step, $\int_{a_1}^{a_2} x^{-2} dx < 5/m_2^2$. If now $a_2 \geqslant b$, we put $b^* = a_2$, then $b \leqslant b^* < b + 3$. If not, we continue in this fashion, until we first reach an a_l with $b \leqslant a_l < b + 3$. We then put $b^* = a_l$. If there are no points x_k in $[a, b^*] = [a_0, a_l]$, we can write, for $j = 1, \ldots, l$:

$$\int_{a_{j-1}}^{a_j} \frac{\log |P(x)|}{x^2} dx \leqslant \log |P(m_j)| \int_{a_{j-1}}^{a_j} \frac{dx}{x^2}. \tag{5}$$

Here, the integers m_j satisfy $a_{j-1} < m_j < a_j$, so that, in particular,

$$a = a_0 < m_1 < m_2 < \ldots < m_l < a_l = b^*.$$

Also,
$$\int_{a_{j-1}}^{a_j} \frac{dx}{x^2} < \frac{5}{m_j^2} \text{ for each } j,$$

as we saw in the case of the first two steps. Adding both sides of (5) for $j = 1, \ldots, l$, we get, since $a = a_0$, $b^* = a_l$,

$$\int_a^{b^*} \frac{\log |P(x)|}{x^2} dx \leqslant \sum_{j=1}^{l} \log |P(m_j)| \int_{a_{j-1}}^{a_j} \frac{dx}{x^2} \leqslant 5 \sum_{j=1}^{l} \frac{\log^+ |P(m_j)|}{m_j^2} \leqslant 5 \sum_{a<m<b^*} \frac{\log^+ |P(m)|}{m^2},$$

establishing (3) and proving the theorem.

THEOREM 5. *Let $10 \leqslant a' < b$. There is an a, $a' - 3 < a \leqslant a'$, such that b is well disposed with respect to a; that is,*

$$\int_a^b \frac{\log |P(x)|}{x^2} dx \leqslant 5 \sum_{a<m<b} \frac{\log^+ |P(m)|}{m^2}$$

provided that there are no points x_k in $[a, b]$.

The proof of Theorem 5 is very similar to that of Theorem 4, and we omit it.

2. Inclusion of the zeros of $P(x)$ in certain intervals

We are going to carry out a series of geometrical constructions on the graph of $n(t)$ vs. t. Recall that, for $t>0$, $n(t)$ is the number of points x_k (zeros of $P(x)$) in the interval $[0, t]$. At this point, we extend the definition of $n(t)$ to the whole real line by putting $n(t)\equiv 0$ for $t\leqslant 0$. The function $n(t)$ is non-decreasing on $(-\infty, \infty)$, identically zero on some open interval including $(-\infty, 0]$, and constant for all sufficiently large values of t. The graph of $n(t)$ vs. t consists of horizontal portions separated by jumps, and at each jump $n(t)$ increases by an integral multiple of one.

In the constructions that follow, we shall include in the graph of $n(t)$ vs. t its *vertical portions*, i.e., if $n(t)$ has a jump discontinuity at t_0, the vertical line segment joining $(t_0, n(t_0-))$ to $(t_0, n(t_0+))$ is considered as forming part of that graph.

Our constructions are arranged in three steps.

First step. Construction of the Bernstein intervals

We begin by taking a number $p>0$; beyond the requirement that p be *small* ($p<1/20$, say), its choice is unrestricted. Once p is chosen, however, it is to remain fixed throughout the series of steps that follow.

Denote by O the set of points t_0, $-\infty<t_0<\infty$, having the property that a straight line of slope p through the point $(t_0, n(t_0))$ cuts or touches the graph of $n(t)$ vs. t *only once*. O is open, and its complement in $\mathbf{R}$ is the union of a finite number of closed intervals, B_0, B_1, B_2, ..., called the Bernstein intervals of the polynomial P associated with the slope p. (Together, these intervals make up what V. Bernstein called the neighborhood set of the points x_k—see [8], p. 259. His construction of the intervals is different from ours.) The formation and disposition of the Bernstein intervals is shown in Figure 1.

From the figure, we see that all the zeros x_k of $P(x)$ (i.e., the points of discontinuity of $n(t)$) are contained in the union of the B_k. Moreover, *if for any* B_k, *we write* $B_k=[a, b]$, *we have:*

That part of the graph of $n(t)$ vs. t *corresponding to the values* $a\leqslant t\leqslant b$ *lies between the two parallel lines of slope* p *passing through the points* $(a, n(a))$ *and* $(b, n(b))$.

There is yet another important property of the intervals B_k which is not so apparent. Henceforth, if I is a closed interval, say $I=[\alpha, \beta]$, we write $n(I)$ for $n(\beta+)-n(\alpha-)$, and $|I|$ for the *length* of I. Then we have:

LEMMA. *For each* B_k, $\dfrac{n(B_k)}{p|B_k|}\geqslant\frac{1}{2}$.

Proof. We begin by making the geometrically evident observation that a line of slope

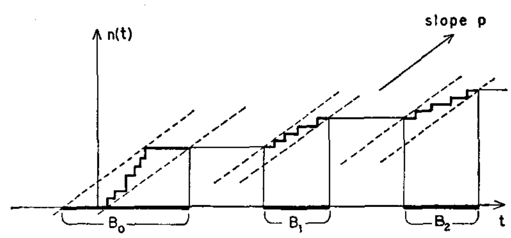

Fig. 1.

p which cuts (or touches) the graph of $n(t)$ vs. t *more than once* must come into contact with some vertical portion of that graph (let the reader make a diagram).

Take any interval B_k, denote it by $[a, b]$, and denote that portion of the graph of $n(t)$ vs. t corresponding to the values $a \leqslant t \leqslant b$ by G. We indicate by L and M the lines of slope p passing through the points $(a, n(a))$ and $(b, n(b))$ respectively. According to the definition of the intervals B_k, any line N of slope p which lies between L and M must cut (or touch) the graph of $n(t)$ vs. t at least twice. N must therefore come into contact with some vertical portion of that graph, indeed, it must come into contact with some vertical portion of G, for it can never touch any part of the graph that does not lie over $[a, b]$ (see Figure 1).

The lemma will thus be proved if we show that the inequality

$$\frac{n(B_k)}{p|B_k|} < \tfrac{1}{2}$$

implies the existence of a line N of slope p, lying between L and M, that does not come into contact with any vertical portion of G.

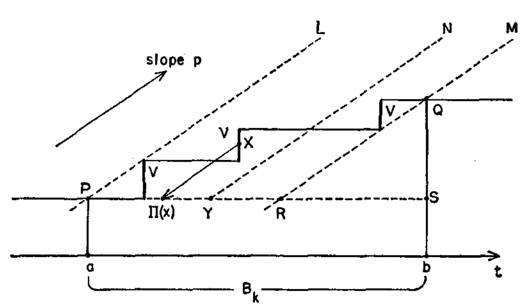

Fig. 2.

In Figure 2, $|B_k| = \overline{PS}$ and $n(B_k) = \overline{QS}$. We have to prove that if $\overline{QS} < \tfrac{1}{2}p \cdot \overline{PS}$, there is a line N of slope p, lying between L and M, which does not touch any of the vertical portions of G. Denote the union of the vertical portions of G by V, and for $X \in V$ let $\Pi(X)$

denote the result of projecting X downwards in a direction of slope p onto the segment PR. The result, $\Pi(V)$, of applying Π to all the points of V is a certain closed subset of PR, and if we use $|\ |$ to denote linear Lebesgue measure, we clearly have

$$p|\Pi(V)| \leqslant |V|,$$

by definition of the projection Π. Since $p\cdot\overline{RS}=\overline{QS}$, we have $p\cdot\overline{PR}>\overline{QS}$ if $\frac{1}{2}p\cdot\overline{PS}>\overline{QS}$. Also, $|V|=\overline{QS}$, so the inequality $\frac{1}{2}p\cdot\overline{PS}>\overline{QS}$ implies $|\Pi(V)|<\overline{PR}$. There is thus a point $Y\in\overline{PR}$ such that $Y\notin\Pi(V)$; if then N is the line of slope p passing through Y, N cannot touch V, and since N lies between L and M, we are done.

Second step. Modification of the collection of Bernstein intervals

The Bernstein intervals B_k constructed in the preceding step are inconvenient in that the ratios $n(B_k)/p|B_k|$ may vary. The purpose of the present construction is to remedy this defect.

For $k=0, 1, 2, \ldots$, we denote B_k by $[a_k, b_k]$, and assume the indices k so ordered that $b_k<a_{k+1}$. We also indicate the smallest of the positive zeros x_k of $P(x)$ by α_0; α_0 is the first point of discontinuity of $n(t)$, and $a_0<\alpha_0<b_0$. Recall that we have assumed $0<p<1/20$.

We are going to construct a finite set of intervals $I_k=[\alpha_k, \beta_k]$, $k=0, 1, \ldots$, having the following properties:

i) *All the points x_k are contained in the union of the I_k.*

ii) $n(I_k)/p|I_k|=\frac{1}{2}$, $k=0, 1, \ldots$.

iii) *For $\alpha_0\leqslant t\leqslant\beta_0$,*

$$n(\beta_0)-n(t)\leqslant\frac{p}{1-3p}(\beta_0-t),$$

and for $\alpha_k\leqslant t\leqslant\beta_k$ with $k\geqslant 1$,

$$n(t)-n(\alpha_k)\leqslant\frac{p}{1-3p}(t-\alpha_k),$$

$$n(\beta_k)-n(t)\leqslant\frac{p}{1-3p}(\beta_k-t).$$

iv) *For $k\geqslant 1$, α_k is well disposed (see § 1) with respect to β_{k-1}.*

We begin by constructing I_0. For $\tau\geqslant b_0$, let Λ_τ be the line of slope p passing through the point $(\tau, n(\tau))$, and write $J_\tau=[\alpha_0, \tau]$. Since α_0 was taken as the smallest of the x_k, we have, for $\tau=b_0$,

$$\frac{n(J_\tau)}{p|J_\tau|}>\frac{n(B_0)}{p|B_0|},\quad\text{which is }\geqslant\tfrac{1}{2}$$

by the lemma proved in the preceding step. For $\tau \in [b_0, a_1)$, $n(J_\tau)$ continues to have the constant value $n(B_0)$, so the ratio $n(J_\tau)/p|J_\tau|$ is a decreasing function of τ on $[b_0, a_1)$.

Suppose that $n(J_\tau)/p|J_\tau| = \frac{1}{2}$ for some $\tau \in [b_0, a_1)$. Then we take β_0 as that value of τ, and put $I_0 = [\alpha_0, \beta_0]$. Property ii) certainly holds for I_0, and property iii) does also. Indeed, if $b_0 \leq \tau < a_1$, it is evident, from the construction of the intervals B_k, that the portion of the graph of $n(t)$ vs. t corresponding to the values $0 \leq t \leq \tau$ lies entirely to the left of the line Λ_τ (look at Figure 1). That is, for such τ we have

$$n(\tau) - n(t) \leq p(\tau - t), \quad 0 \leq t \leq \tau,$$

whence, a fortiori,

$$n(\tau) - n(t) \leq \frac{p}{1-3p}(\tau - t), \quad 0 \leq t \leq \tau,$$

(since $p < 1/20$, $1-3p > 0$), and property iii) holds.

It may happen, however, that $n(J_\tau)/p|J_\tau|$ remains $> \frac{1}{2}$ for $b_0 \leq \tau < a_1$. In that case, we will still have $n(J_\tau)/p|J_\tau| \geq \frac{1}{2}$ for $\tau = b_1$. This is true because $n(B_1)/p|B_1| \geq \frac{1}{2}$ by the lemma of the preceding step, and

$$n([\alpha_0, b_1]) = n(a_1 -) - n(\alpha_0 -) + n(B_1),$$

$$b_1 - \alpha_0 = a_1 - \alpha_0 + |B_1|.$$

In the present situation, $n(J_\tau)/p|J_\tau| \geq \frac{1}{2}$ for $\tau = b_1$, and decreases on the interval $[b_1, a_2)$. Also, when τ belongs to $[b_1, a_2)$, the part of the graph of $n(t)$ vs. t corresponding to the values $0 \leq t \leq \tau$ lies entirely to the left of the line Λ_τ, just as in the discussion of the previous case. So if $n(J_\tau)/p|J_\tau| = \frac{1}{2}$ for some $\tau \in [b_1, a_2)$, we take β_0 as that value of τ, and $I_0 = [\alpha_0, \beta_0]$ has properties ii) and iii).

If yet $n(J_\tau)/p|J_\tau|$ remains $> \frac{1}{2}$ for $b_1 \leq \tau < a_2$, we will have $n(J_\tau)/p|J_\tau| \geq \frac{1}{2}$ for $\tau = b_2$, by the argument already used, and we then repeat the above procedure, looking for β_0 in the interval $[b_2, a_3)$. The process continues in this way until we either get a β_0 between two successive intervals B_k, B_{k+1} (perhaps coinciding with the right endpoint of B_k), or we have passed through the half-open interval separating the *last two* B_k, without $n(J_\tau)/p|J_\tau|$ ever having gotten $\leq \frac{1}{2}$. If this second eventuality occurs, suppose $B_l = [a_l, b_l]$ is the last B_k; then, as before, $n(J_\tau)/p|J_\tau| \geq \frac{1}{2}$ for $\tau = b_l$. Here, since $n(t)$ remains constant for $t \geq b_l$, we can either take $\beta_0 = b_l$, or, if necessary, simply increase τ until $n(J_\tau)/p|J_\tau|$ has diminished to $\frac{1}{2}$, and use the resulting value of τ as β_0. In this situation, there is only one interval I_k, namely $I_0 = [\alpha_0, \beta_0]$, and the construction is finished, since property i) obviously now holds, ii) and iii) hold by the above reasoning, and iv) is vacuously true.

If, on the other hand, the construction gives us a $\beta_0 \in [b_k, a_{k+1})$ where there is still a Bernstein interval B_{k+1}, we have to construct $I_1 = [\alpha_1, \beta_1]$. To do this, we must first choose α_1 so that property iv) is ensured. Observe first of all that $n(t)$ increases by at least one at each of its jumps, so that, by construction of I_0, $p(\beta_0 - \alpha_0) = 2n(I_0) \geqslant 2$, whence $a_{k+1} > \beta_0 > 2/p > 40$, because $0 < p < 1/20$. Theorem 4 of § 1 thus applies to yield the existence of an α_1, $a_{k+1} \leqslant \alpha_1 < a_{k+1} + 3$ which is *well disposed* with respect to β_0.

Although α_1 may lie to the right of a_{k+1}, I claim that $n(\alpha_1) = n(\beta_0 -)$, and besides

$$n(t) - n(\alpha_1) \leqslant \frac{p}{1-3p}(t - \alpha_1) \quad \text{for} \quad t \geqslant \alpha_1.$$

This follows directly from the facts that $n(t)$ increases by at least one at each jump, and that $1/p > 3$, as is evident from Figure 3:

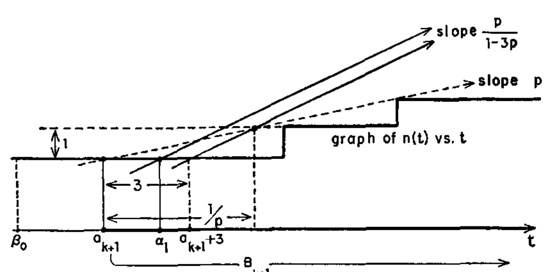

Fig. 3.

We see that this choice of α_1 guarantees not only property iv), but also i) and iii), insofar as their validity depends on the position of α_1.

We must now choose $\beta_1 > \alpha_1$ in such a way as to continue to ensure the properties in question. This step is very much like the determination of β_0. For $\tau \geqslant b_{k+1}$, we write $J'_\tau = [\alpha_1, \tau]$, and take Λ_τ as before. Then, since $n(b_{k+1}) - n(\alpha_1) = n(B_{k+1})$, we certainly have, for $\tau = b_{k+1}$,

$$\frac{n(J'_\tau)}{p|J'_\tau|} \geqslant \frac{n(B_{k+1})}{p|B_{k+1}|} \geqslant \tfrac{1}{2},$$

by the lemma already used in the construction of I_0. We may therefore proceed just as above to find a τ, lying either in the half open interval separating two successive Bernstein intervals, or beyond all of them, such that $n(J'_\tau)/p|J'_\tau| = \frac{1}{2}$. For this τ, the part of the graph of $n(t)$ vs. t corresponding to the values $t \leqslant \tau$ lies entirely to the left of Λ_τ, whence, a fortiori,

$$n(\tau) - n(t) \leqslant \frac{p}{1-3p}(\tau - t), \quad 0 \leqslant t \leqslant \tau.$$

We then take β_1 equal to the τ just found, put $I_1=[\alpha_1, \beta_1]$, and see at once that properties ii) and iii) hold for I_0 and I_1.

If $I_0 \cup I_1$ does not already include all the Bernstein intervals B_k, β_1 must lie between two of them, and we can proceed to get an α_2 just as α_1 was found above. Then we can construct an I_2. Since there are only a finite number of B_k, the process will eventually stop, and we will have a finite number of intervals I_k having properties ii)–iv). Property i) will now also hold, since, when we finish, the union of the I_k includes that of the B_k.

Third step. Replacement of the first few intervals I_k by a single one, if $n(t)/t$ is not always $\leqslant p/(1-3p)$

We now introduce a new parameter, η, which will continue to intervene until the last sections of this paper, when a decision will finally be made concerning the value to be assigned to it. Until then, we require only that $0<\eta<2/3$, but η is considered as fixed, once chosen, throughout the following discussions. From time to time we will state various intermediate results whose validity depends on η's having been taken sufficiently small to begin with; the final determination of η will come about when we combine those results.

In Figure 4 we show the intervals $I_k=[\alpha_k, \beta_k]$ constructed in the preceding step.

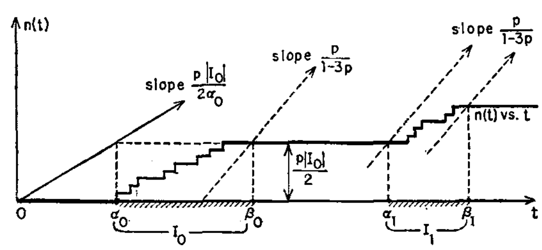

Fig. 4.

LEMMA. *If* $0<\eta<\frac{2}{3}$, $\sup_t \dfrac{n(t)}{t} > \dfrac{p}{1-3p}$ *implies* $\dfrac{|I_0|}{\beta_0} > \eta$.

Proof. Figure 4 shows that

$$\sup_t \frac{n(t)}{t} \leqslant \max\left\{\frac{p}{1-3p}, \frac{p|I_0|}{2\alpha_0}\right\} = \max\left\{\frac{p}{1-3p}, \frac{p}{2}\frac{|I_0|}{\beta_0}\left(1-\frac{|I_0|}{\beta_0}\right)^{-1}\right\},$$

and this last expression equals

$$\frac{p}{1-3p} \text{ if } \frac{|I_0|}{\beta_0} \leqslant \eta < \tfrac{2}{3}. \quad \text{Q.E.D.}$$

Our ultimate purpose, in all that is being done, is to show that $\sup_t n(t)/t$ is small if $\sum_1^\infty m^{-2}\log^+|P(m)|$ is. The way we are going to go about this is to assume that $\sup_t n(t)/t$ is *not* small, and arrive at a positive lower bound for $\sum_1^\infty m^{-2}\log^+|P(m)|$. The constructions will therefore continue under the assumption that $\sup_t n(t)/t > p/(1-3p)$, which, by the above lemma, implies that $|I_0|/\beta_0 > \eta$.

Suppose $|I_0|/\beta_0 > \eta$, where $0 < \eta < 2/3$. We will then replace the first few intervals I_k by a single one, according to the procedure that now follows.

Let $\omega_I(t)$ be the continuous and piecewise linear function, defined on $[0, \infty)$, that has slope 1 on each of the intervals I_k, and slope *zero* elsewhere—we put $\omega_I(0) = 0$. The ratio $\omega_I(t)/t$ is continuous, and tends to zero as $t \to \infty$, since there are only a finite number of intervals I_k. Besides this, $\omega_I(t)/t$ *increases* on the interior of each I_k. For if t belongs to the interior of an I_k,

$$\frac{d}{dt}\left(\frac{\omega_I(t)}{t}\right) = \frac{1}{t} - \frac{\omega_I(t)}{t^2} > 0,$$

since clearly $\omega_I(t)/t < 1$ for $t > 0$.

The assumption $|I_0|/\beta_0 > \eta$ means that $\omega_I(\beta_0)/\beta_0 > \eta$. In view of the above remarks, there is a *largest* t, which we will call d, for which $\omega_I(t)/t = \eta$, and d cannot belong to the interior, or be the left endpoint, of any of the intervals I_k.

Since $d > \beta_0$, there is a *last* interval I_k, say I_l, lying entirely to the left of d. If I_l is also the last of all the intervals I_k, we define $d_0 = d$, $c_0 = (1-\eta)d$, and put $J_0 = [c_0, d_0]$. In this case all the points x_k (discontinuities of $n(t)$) lie to the *left* of d_0. Otherwise, I_l is not the last of all the I_k, and there is an $I_{l+1} = [\alpha_{l+1}, \beta_{l+1}]$; according to what we have said about d, $d < \alpha_{l+1}$. Now surely $d > \beta_0$, and as we saw in the second step, where the I_k were constructed, $\beta_0 > 2/p > 10$, since $p < 1/20$. So we may apply Theorem 5 of § 1 to conclude that there is a d_0, $d - 3 < d_0 \leqslant d$, such that α_{l+1} is *well disposed with respect to* d_0. We will then put $c_0 = d_0 - d\eta$ and define intervals J_k by

$$J_0 = [c_0, d_0],$$
$$J_1 = I_{l+1},$$
$$J_2 = I_{l+2},$$
$$\cdots$$

We will also relabel the endpoints of the J_k with $k \geqslant 1$, putting $\alpha_{l+1} = c_1$, $\beta_{l+1} = d_1$, etc.; thus, $J_k = [c_k, d_k]$.

Now the point d_0, although it may lie to the left of β_l, still lies to the right of all the points x_k in the interval I_l. Besides this, we can say that the part of the graph of $n(t)$ vs. t corresponding to the values $t \leqslant d_0$ lies entirely to the left of a line of slope $p/(1-3p)$ through the point $(d_0, n(d_0))$. Indeed, since $d \geqslant \beta_l$, $d_0 > \beta_l - 3$. We also know, from the construction of the intervals I_k, that the part of the graph of $n(t)$ vs. t corresponding to the values $t \leqslant \beta_l$ lies entirely to the left of a line of slope p (and not only of slope $p/(1-3p)$) through the point $(\beta_l, n(\beta_l))$. The two statements just made concerning d_0 can therefore be verified from a diagram similar to Figure 3.

We also have $\eta \leqslant |J_0|/d_0 \leqslant 2\eta/(2-3p)$. For

$$\frac{|J_0|}{d_0} = \frac{|J_0|}{d} \cdot \frac{d}{d_0} \leqslant \frac{|J_0|}{d} \cdot \frac{d}{d-3} < \eta \cdot \frac{2}{2-3p},$$

since by definition of c_0, $|J_0|/d = \eta$, and since $d > \beta_0 > 2/p$. In particular, on account of our permanent assumption that $p < 1/20$, we certainly have $\eta \leqslant |J_0|/d_0 \leqslant 40\,\eta/37$, and $c_0 = d_0 - |J_0| > 0$ since $0 < \eta < 2/3$.

The ratio $n(d_0)/p|J_0|$ is equal to $\frac{1}{2}$. For since d_0 and d both lie strictly between all the points x_k in I_l and the interval I_{l+1} (or beyond the last I_k, if J_0 is the only J_k), $n(d_0) = n(d) = \sum_0^l n(I_k) = \frac{1}{2}p\sum_0^l |I_k|$ by property ii) of the I_k. According to the definition of $\omega_I(t)$, this last expression equals, by choice of d, $\frac{1}{2}p\omega_I(d) = \frac{1}{2}p\eta d = \frac{1}{2}p|J_0|$.

Let us define $\omega_J(t)$ as the continuous and piecewise linear function on $[0, \infty)$, taking the value 0 at the origin, which has slope 1 on every interval J_k, and slope zero elsewhere. Then there is an η', $\eta \leqslant \eta' \leqslant 2\eta$, such that

$$\frac{\omega_J(t)}{t} \leqslant \eta' \text{ for all } t > 0,$$

whilst $\omega_J(d_0)/d_0 = \eta'$. Indeed, $\omega_J(t) \equiv 0$, $0 \leqslant t \leqslant c_0$, and on $[c_0, d_0] = J_0$, $\omega_J(t)/t$ increases to $\omega_J(d_0)/d_0 = |J_0|/d_0$ which lies, as we have seen, between η and 2η. For $d_0 \leqslant t \leqslant d$, $\omega_J(t)/t = |J_0|/t$ decreases, and for $t \geqslant d$, $\omega_J(t) = \omega_I(t)$, so that $\omega_J(t)/t \leqslant \eta$, $t \geqslant d$, by choice of d.

The purpose of this entire section has been to construct the intervals J_k, and it is with them that we shall work during the remainder of the paper. It is best to summarize all that we have done in

THEOREM 6. *Suppose $p < 1/20$, and $\sup_t n(t)/t > p/(1-3p)$. Given η, $0 < \eta < \frac{2}{3}$, we can construct a finite set of intervals $J_k = [c_k, d_k]$, $k \geqslant 0$, with $0 < c_0$ and $d_{k-1} < c_k$, $k \geqslant 1$, that have the following properties:*

i) *All the points x_k (discontinuities of $n(t)$) lie in $(0, d_0) \cup \bigcup_{k \geqslant 1} J_k$.*

ii) $$\frac{n(d_0)}{p|J_0|} = \frac{1}{2}, \quad \frac{n(J_k)}{p|J_k|} = \frac{1}{2}, \quad k \geqslant 1.$$

iii) *For* $0 \leqslant t \leqslant d_0$,

$$n(d_0) - n(t) \leqslant \frac{p}{1-3p}(d_0 - t);$$

For $c_k \leqslant t \leqslant d_k$, $k \geqslant 1$,

$$\begin{cases} n(t) - n(c_k) \leqslant \dfrac{p}{1-3p}(t - c_k) \\ n(d_k) - n(t) \leqslant \dfrac{p}{1-3p}(d_k - t). \end{cases}$$

iv) *For* $k \geqslant 1$, c_k *is well disposed with respect to* d_{k-1}.

v) *If, for* $t \geqslant 0$, $\omega_J(t) = |\bigcup_{k\geqslant 0} J_k \cap [0, t]|$, *there is a number* $\eta', \eta \leqslant \eta' \leqslant 2\eta$, *such that*

$$\frac{\omega_J(t)}{t} \leqslant \eta' \quad \textit{for all} \quad t > 0,$$

whilst

$$\frac{\omega_J(d_0)}{d_0} = = \eta'.$$

3. Replacement of the distribution $n(t)$ by a continuous one

Throughout this section, we assume that $\sup n(t)/t > p/(1-3p)$. Taking an η, $0 < \eta < \frac{2}{3}$, we can then construct the intervals $J_k = [c_k, d_k]$, $k = 0, 1, \ldots$, having the properties listed in Theorem 6 of the previous section.

Notation. Suppose J_l is the last J_k. Then we write

$$O = (d_0, c_1) \cup (d_1, c_2) \cup \ldots \cup (d_{l-1}, c_l) \cup (d_l, \infty).$$

(This is *not* the same O as that used at the beginning of § 2!)

LEMMA. *If* $P(x)$ *is the polynomial* $\prod_k (1 - x^2/x_k^2)$,

$$\int_O \frac{\log|P(x)|}{x^2} dx \leqslant 5 \sum_{m=1}^{\infty} \frac{\log^+|P(m)|}{m^2}.$$

Proof. By property i) affirmed in Theorem 6 of § 2, there are no x_k in O. By property iv), c_k is well disposed with respect to d_{k-1} for $k = 1, \ldots, l$, hence, by Theorem 4 of § 1,

$$\int_{d_{k-1}}^{c_k} \frac{\log|P(x)|}{x^2} dx \leqslant 5 \sum_{d_{k-1} < m < c_k} \frac{\log^+|P(m)|}{m^2}$$

(in the sum, m takes integral values), for $k = 1, \ldots, l$. Also, since all the x_k are less than

d_l, $\log|P(x)|$ is concave downward and increasing on $[d_l, \infty)$, and from this it follows easily (e.g., by the reasoning of § 1) that

$$\int_{d_l}^{\infty} \frac{\log|P(x)|}{x^2}\,dx \leqslant 5 \sum_{d_l<m<\infty} \frac{\log^+|P(x)|}{m^2}.$$

Adding all these inequalities,

$$\int_{O} \frac{\log|P(x)|}{x^2}\,dx \leqslant 5 \sum_{m\in O} \frac{\log^+|P(m)|}{m^2},$$

which implies the lemma.

Notation. Let $\mu(t)$ be piecewise linear (perhaps with jump discontinuities) and increasing on $[0, \infty)$, zero for all t sufficiently close to 0, and constant for all sufficiently large t. Then we write

$$V_\mu(x) = \int_0^\infty \log\left|1 - \frac{x^2}{t^2}\right| d\mu(t). \tag{6}$$

Remark. Since $P(x) = \prod_k (1 - x^2/x_k^2)$, we have, by definition of $n(t)$,

$$\log|P(x)| = V_n(x).$$

LEMMA.

$$V_\mu(x) = -x\int_0^\infty \log\left|\frac{x+t}{x-t}\right| d\left(\frac{\mu(t)}{t}\right). \tag{7}$$

This formula is known (see [9], p. 137), but for the reader's convenience, we give a quick proof.

It is enough to check (7) for the case where $\mu(t)$ is continuous at x, for both sides of (7) obviously equal $-\infty$ if $\mu(x-) < \mu(x+)$. We may also suppose $x > 0$.

We have

$$-\int_0^x \log\left|\frac{x+t}{x-t}\right| d\left(\frac{\mu(t)}{t}\right) = \int_0^x \frac{1}{t^2}\log\left|\frac{x+t}{x-t}\right| \mu(t)\,dt - \int_0^x \frac{1}{t}\log\left|\frac{x+t}{x-t}\right| d\mu(t).$$

The first integral on the right is integrated by parts, using the formula

$$\frac{\partial}{\partial t}\left(\frac{1}{t}\log\left|\frac{x+t}{x-t}\right| + \frac{1}{x}\log\left|1 - \frac{x^2}{t^2}\right|\right) = -\frac{1}{t^2}\log\left|\frac{x+t}{x-t}\right|,$$

valid for $t > 0$, $t \neq x$, and we find, in view of the above assumptions on the point x, and the fact that $\mu(t)$ vanishes identically near 0:

$$-\int_0^x \log\left|\frac{x+t}{x-t}\right| d\left(\frac{\mu(t)}{t}\right) = -\frac{2\mu(x)\log 2}{x} + \frac{1}{x}\int_0^x \log\left|1-\frac{x^2}{t^2}\right| d\mu(t). \tag{8}$$

In the same way, we get

$$-\int_x^\infty \log\left|\frac{x+t}{x-t}\right| d\left(\frac{\mu(t)}{t}\right) = \frac{2\mu(x)\log 2}{x} + \frac{1}{x}\int_x^\infty \log\left|1-\frac{x^2}{t^2}\right| d\mu(t), \tag{9}$$

and (7) follows on adding (8) and (9).

Notation. We write $J=\bigcup_{k\geqslant 0} J_k$, and $\Omega=(0,\infty)\sim J$.

THEOREM 7. *Under the assumptions of this section, we have*

$$\sum_1^\infty \frac{\log^+|P(m)|}{m^2} \geqslant \frac{1}{5}\int_\Omega \frac{V_\mu(x)}{x^2}\,dx,$$

where $\mu(t)$ is a certain increasing function that can be described as follows:

i) *$\mu(t)$ is piecewise linear, continuous, and increasing on $[0,\infty)$.*

ii) *Outside J, the slope of $\mu(t)$ is zero.*

iii) *For each $k\geqslant 0$ there are points γ_k and δ_k in $J_k=[c_k,d_k]$, with $c_k<\gamma_k<\delta_k<d_k$ for $k\geqslant 1$ and $c_0=\gamma_0<\delta_0<d_0$, satisfying*

$$\frac{\gamma_k-c_k+d_k-\delta_k}{d_k-c_k}=\frac{1-3p}{2},\quad k\geqslant 0,$$

and such that $\mu(t)$ has slope $p/(1-3p)$ on each of the intervals $[c_k,\gamma_k]$, $[\delta_k,d_k]$, and zero slope elsewhere in J_k.

In order that the reader may easily see the behavior of $\mu(t)$, we show its graph in Figure 5:

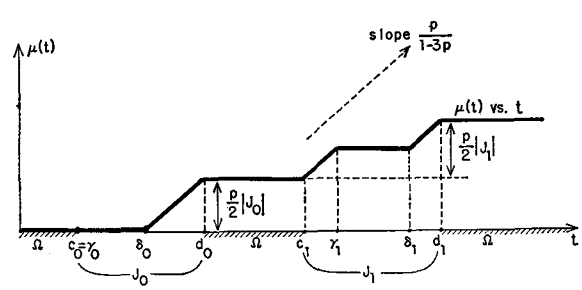

Fig. 5.

Proof of Theorem 7. Let $\mu(t)$ be any function having properties i), ii) and iii). We observe first of all that $\mu(t)$ is completely specified by the values of the numbers γ_k, δ_k for $k \geq 1$. According to the first lemma of this section, we will be done if we show how to assign values to γ_k and δ_k for $k \geq 1$, compatible with the constraint in iii), so that, for the resulting function $\mu(t)$,

$$\int_O \frac{\log |P(x)|}{x^2}\, dx \geq \int_\Omega \frac{V_\mu(x)}{x^2}\, dx.$$

Property iii) implies that $\mu(d_0) = \frac{1}{2}p|J_0|$, $\mu(J_k) = \frac{1}{2}p|J_k|$, $k \geq 1$ ($\mu(J_k)$ denotes the *increase* of $\mu(t)$ on J_k), so by Theorem 6 of § 2 we must have $\mu(d_0) = n(d_0)$, $\mu(J_k) = n(J_k)$, $k \geq 1$. In other words, $\mu(t)$ and $n(t)$ agree on the closure of O, whence, by (6) and the remark following,

$$\log |P(x)| - V_\mu(x) = \int_0^{d_0} \log \left| 1 - \frac{x^2}{t^2} \right| (dn(t) - d\mu(t)) + \sum_{k \geq 1} \int_{c_k}^{d_k} \log \left| 1 - \frac{x^2}{t^2} \right| (dn(t) - d\mu(t)). \tag{10}$$

By Theorem 6 of § 2 and property iii), $n(t) \geq \mu(t)$ for $0 \leq t \leq d_0$ with equality for $t = d_0$. This implies that the first integral on the right in (10) is non-negative for $x > d_0$, because $\log (x^2/t^2 - 1)$ is a decreasing function of t for $0 < t < x$. Therefore,

$$\int_O \frac{dx}{x^2} \int_0^{d_0} \log \left| 1 - \frac{x^2}{t^2} \right| (dn(t) - d\mu(t)) \geq 0 \tag{11}$$

in view of the definition of O.

We now show that for $k \geq 1$, γ_k and δ_k can be chosen, compatible with the constraint in iii), in such a way that

$$\int_O \frac{dx}{x^2} \int_{c_k}^{d_k} \log \left| 1 - \frac{x^2}{t^2} \right| (dn(t) - d\mu(t)) \geq 0. \tag{12}$$

If we apply the second lemma of this section, first with $\mu(t)$, and then with the function

$$\mu_k(t) = \begin{cases} \mu(t), & t \notin [c_k, d_k] \\ n(t), & c_k \leq t \leq d_k \end{cases}$$

in place of $\mu(t)$, we find

$$\int_{c_k}^{d_k} \log \left| 1 - \frac{x^2}{t^2} \right| (dn(t) - d\mu(t)) = x \int_{c_k}^{d_k} \log \left| \frac{x+t}{x-t} \right| \left(d\left(\frac{\mu(t)}{t}\right) - d\left(\frac{n(t)}{t}\right) \right),$$

and this, on substitution into the left side of (12), yields the expression

$$\int_{c_k}^{d_k} F(t)\left(d\left(\frac{\mu(t)}{t}\right) - d\left(\frac{n(t)}{t}\right)\right),$$

where
$$F(t) = \int_O \log\left|\frac{x+t}{x-t}\right| \frac{dx}{x}. \tag{13}$$

Integrating by parts, we see that the left-hand side of (12) is equal to

$$\int_{c_k}^{d_k} F'(t)\,\frac{n(t)-\mu(t)}{t}\,dt. \tag{14}$$

Successive differentiation of (13) yields, for $c_k < t < d_k$,

$$F'(t) = \int_O \frac{2\,dx}{x^2 - t^2},$$

$$F''(t) = \int_O \frac{4\,t dx}{(x^2 - t^2)^2}.$$

From the first of these relations, we see that

$$F'(t) \to -\infty \text{ as } t \to c_k \text{ in } (c_k, d_k)$$

$$F'(t) \to \infty \text{ as } t \to d_k \text{ in } (c_k, d_k),$$

since $O \supset (d_{k-1}, c_k) \cup (d_k, c_{k+1})$. The second relation shows that $F'(t)$ is strictly increasing on (c_k, d_k), and it follows that $F'(t)$ has exactly one zero, t_k, in (c_k, d_k), and that $F'(t) < 0$ for $c_k < t < t_k$, whilst $F'(t) > 0$ for $t_k < t < d_k$.

Starting with the point t_k, the numbers γ_k and δ_k are found by the construction of Figure 6.

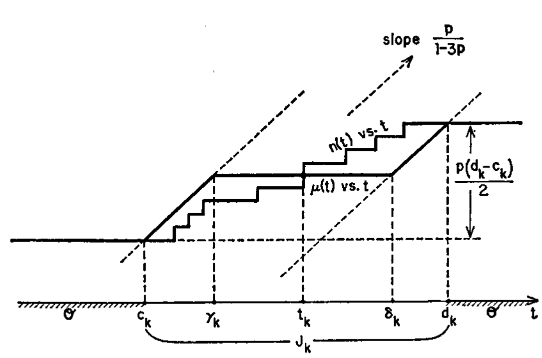

Fig. 6. Showing case where t_k is a point of discontinuity of $n(t)$.

It is evident in Figure 6 that this choice of γ_k and δ_k is compatible with the constraint in iii), and leads to a specification of $\mu(t)$ on J_k satisfying

$$n(t)-\mu(t)\leqslant 0, \quad c_k\leqslant t\leqslant t_k$$

$$n(t)-\mu(t)\geqslant 0, \quad t_k\leqslant t\leqslant d_k.$$

In view of the behaviour of $F'(t)$ on (c_k, d_k), these inequalities imply that the integral (14) is $\geqslant 0$, that is, (12) holds.

This procedure may be used to specify $\mu(t)$ on all the intervals J_k, $k\geqslant 1$, after which

$$\int_{O}\frac{\log|P(x)|}{x^2}\,dx\geqslant\int_{O}\frac{V_\mu(x)}{x^2}\,dx$$

will follow from (10), by adding the inequalities (11) and (12).

We observe finally that $\Omega=(0, c_0)\cup O$, and that $V_\mu(x)$ is obviously $\leqslant 0$ for $0<x<c_0$. Hence

$$\int_{O}\frac{V_\mu(x)}{x^2}\,dx\geqslant\int_{\Omega}\frac{V_\mu(x)}{x^2}\,dx,$$

which, with the previous inequality, yields

$$\int_{O}\frac{\log|P(x)|}{x^2}\,dx\geqslant\int_{\Omega}\frac{V_\mu(x)}{x^2}\,dx,$$

proving the theorem.

Remark. Thanks to the results of this section we have reduced the problem of finding a positive lower bound for

$$\sum_1^\infty\frac{\log^+|P(m)|}{m^2},$$

under the assumption that $\sup_t n(t)/t>p/(1-3p)$, to the purely analytical one of finding a positive lower bound for

$$\int_{(0,\infty)\sim J}\frac{dx}{x^2}\int_0^\infty\log\left|1-\frac{x^2}{t^2}\right|d\mu(t),$$

where $\mu(t)$ has the very special form shown in Figure 5.

4. Auxiliary formulas

We proceed to develop certain relations. Throughout this section, $\nu(t)$ and $\omega(t)$ will denote continuous piecewise linear increasing functions on $[0, \infty)$, vanishing on a neighborhood of the origin, and constant for all sufficiently large t.

LEMMA.

$$\int_0^\infty \int_0^\infty \log\left|1-\frac{x^2}{t^2}\right| d\nu(t)\, \frac{dx-d\omega(x)}{x^2} = \int_0^\infty \int_0^\infty \log\left|\frac{x+t}{x-t}\right| d\left(\frac{\nu(t)}{t}\right) \frac{d\omega(x)}{x}.$$

Proof. By the second lemma of § 3,

$$\frac{1}{x}\int_0^\infty \log\left|1-\frac{x^2}{t^2}\right| d\nu(t) = -\int_0^\infty \log\left|\frac{x+t}{x-t}\right| d\left(\frac{\nu(t)}{t}\right)$$

for $x>0$. Also, if $t>0$,

$$\int_0^\infty \log\left|\frac{x+t}{x-t}\right| \frac{dx}{x} = \int_0^\infty \log\left|\frac{x+1}{x-1}\right| \frac{dx}{x} = \frac{\pi^2}{2}.$$

This last, coupled with the obvious relation $\int_0^\infty d(\nu(t)/t)=0$ and the preceding formula, yields the desired result.

LEMMA.

$$\int_0^\infty \int_0^\infty \log\left|1-\frac{x^2}{t^2}\right| d\nu(t)\, \frac{dx-d\nu(x)}{x^2} = \int_0^\infty \int_0^\infty \log\left|\frac{x+t}{x-t}\right| d\left(\frac{\nu(t)}{t}\right) d\left(\frac{\nu(x)}{x}\right).$$

Proof. By the preceding lemma, the expression on the left is equal to

$$\int_0^\infty \int_0^\infty \log\left|\frac{x+t}{x-t}\right| d\left(\frac{\nu(t)}{t}\right) \frac{d\nu(x)}{x},$$

so since

$$d\left(\frac{\nu(x)}{x}\right) = \frac{d\nu(x)}{x} - \frac{\nu(x)}{x^2}\,dx,$$

we will be done if we show

$$\int_0^\infty \int_0^\infty \log\left|\frac{x+t}{x-t}\right| d\left(\frac{\nu(t)}{t}\right) \frac{\nu(x)}{x^2}\,dx = 0. \tag{15}$$

Let us write $\varrho(t)=\nu(t)/t$, then the expression of the left in (15) becomes

$$\int_0^\infty \int_0^\infty \log\left|\frac{x+t}{x-t}\right| \frac{\varrho(x)}{x}\,dx\,d\varrho(t). \tag{16}$$

The function $\varrho(t)$ is identically zero for all sufficiently small $t>0$, and is $O(1/t)$ at ∞. Moreover, since $\varrho(x)$ is bounded,

$$\int_0^\infty \log\left|\frac{x+t}{x-t}\right| \frac{\varrho(x)}{x}\,dx \leqslant \text{const.} \int_0^\infty \log\left|\frac{x+t}{x-t}\right| \frac{dx}{x}$$

which is constant for $t>0$. If, therefore, in (16), we integrate by parts with respect to t, the integrated term vanishes, yielding for (16) the value

$$-\int_0^\infty \frac{d}{dt}\left\{\int_0^\infty \log\left|\frac{x+t}{x-t}\right| \frac{\varrho(x)}{x}\,dx\right\}\varrho(t)\,dt. \tag{17}$$

Making the substitution $x=t\xi$ and differentiating under the integral sign (which is easily justified by Lebesgue's dominated convergence theorem for functions $\varrho(x)=\nu(x)/x$, when $\nu(x)$ has the properties imposed on it at the beginning of this section), we find, for $t>0$,

$$\frac{d}{dt}\int_0^\infty \log\left|\frac{x+t}{x-t}\right|\frac{\varrho(x)}{x}\,dx = \frac{d}{dt}\int_0^\infty \log\left|\frac{\xi+1}{\xi-1}\right|\frac{\varrho(t\xi)}{\xi}\,d\xi$$

$$= \int_0^\infty \log\left|\frac{\xi+1}{\xi-1}\right|\varrho'(t\xi)\,d\xi = \frac{1}{t}\int_0^\infty \log\left|\frac{x+t}{x-t}\right| d\varrho(x).$$

Substituting this result in (17), we find that the expression (16) is equal to its negative, and hence must vanish. This proves (15).

Notation. Let $\varrho(t)$ and $\sigma(t)$ be any real absolutely continuous functions on $[0, \infty)$, having there piecewise continuous and bounded derivatives which are of the form const.$/t^2$ for all sufficiently large t, and vanishing on a neighborhood of the origin. (This will always be the case if $\varrho(t)$ and $\sigma(t)$ are of the form $\nu(t)/t$, where $\nu(t)$ has the properties stated at the beginning of this section.) Then we write

$$E(d\varrho(t), d\sigma(t)) = \int_0^\infty\int_0^\infty \log\left|\frac{x+t}{x-t}\right| d\varrho(x)\,d\sigma(t).$$

The bilinear form $E(\ ,\ )$ turns out to be positive definite. This result belongs to the elements of potential theory ($E(d\varrho(t), d\varrho(t))$ is simply the energy of the Green potential generated in the right half plane by the mass distribution $d\varrho(t)$), and is a direct consequence of the following

LEMMA. *Let, for* $\Re z>0$,

$$u(z) = \int_0^\infty \log\left|\frac{z+t}{z-t}\right| d\varrho(t). \tag{18}$$

Then
$$E(d\varrho(t), d\varrho(t)) = \frac{1}{\pi}\int_0^\infty\int_0^\infty \{(u_x(z))^2 + (u_y(z))^2\}\,dx\,dy. \tag{19}$$

Proof. Let us denote $\lim_{\eta\to 0+} u_y(x+i\eta)$ by $u_y(x+)$, and let us show first of all that

$$E(d\varrho(t), d\varrho(t)) = -\frac{1}{\pi}\int_0^\infty u_y(x+)\,u(x)\,dx. \tag{20}$$

From (18) we have, for $y>0$,

$$u_y(z) = \int_0^\infty \frac{\partial}{\partial y}\log\left|\frac{z+t}{z-t}\right| d\varrho(t) = \int_0^\infty \left\{\frac{y}{(x+t)^2+y^2} - \frac{y}{(x-t)^2+y^2}\right\} d\varrho(t),$$

so $u_y(x+) = -\pi\varrho'(x)$ for $x>0$ by Poisson's formula. Substituting this into the right side of (20), and using (18) to express $u(x)$ therein, we get the integral which was used to define $E(d\varrho(t), d\varrho(t))$.

To complete the proof of the lemma, we must show that the right side of (20) equals the right side of (19). From (18), we see that $u(z)$ is harmonic in the quadrant $\Re z>0$, $\Im z>0$, and that $u(iy)\equiv 0$ for $y>0$. Therefore, if $R>0$,

$$-\int_0^R u(x)\,u_y(x+)\,dx + \int_0^{\pi/2} u(Re^{i\varphi})\frac{\partial u(Re^{i\varphi})}{\partial R}\,R\,d\varphi = \iint\limits_{\substack{x^2+y^2<R^2\\ x>0,\, y>0}} \{(u_x(z))^2 + (u_y(z))^2\}\,dx\,dy \tag{21}$$

by Green's theorem (the possible discontinuities of u's partial derivatives at the boundary of the sector cause no trouble here).

I claim that the second integral on the left in (21) tends to 0 as $R\to\infty$. In view of the restrictions on $\varrho(t)$, we can write $u(z) = u_1(z) + u_2(z)$, with

$$u_1(z) = \int_0^M \log\left|\frac{z+t}{z-t}\right| d\varrho(t), \tag{22}$$

$$u_2(z) = C\int_M^\infty \log\left|\frac{z+t}{z-t}\right| \frac{dt}{t^2}, \tag{23}$$

$M>0$ and C being certain constants. If $|z|>M$, we can, to evaluate (22), first expand $\log((z+t)/(z-t))$ in powers of t/z, then use the real part of the resulting expression; in this way we find that

$$|u_1(Re^{i\varphi})| \leqslant O\left(\frac{1}{R}\right), \quad \left|\frac{\partial u_1(Re^{i\varphi})}{\partial R}\right| \leqslant O\left(\frac{1}{R^2}\right)$$

uniformly for $0<\varphi<\frac{1}{2}\pi$, as $R\to\infty$. Making the change of variable $t=R\tau$ in (23) we see easily that $|u_2(Re^{i\varphi})| \leqslant O((\log R)/R)$ uniformly for $0<\varphi<\frac{1}{2}\pi$ as $R\to\infty$. Finally, we differentiate the right side of (23) under the integral sign with respect to R and calculate the resulting expression explicitly. This yields:

$$\left|\frac{\partial u_2(Re^{i\varphi})}{\partial R}\right| \leqslant O\left(\frac{1}{R}\right) \text{ uniformly for } 0<\varphi<\tfrac{1}{2}\pi \text{ as } R\to\infty.$$

Putting together these estimates, we get:

$$\left|u(Re^{i\varphi})\,\frac{\partial u(Re^{i\varphi})}{\partial R}\right| \leqslant O\left(\frac{\log R}{R^2}\right) \text{ uniformly for } 0<\varphi<\tfrac{1}{2}\pi \quad\text{as}\quad R\to\infty,$$

and this shows that the second integral on the left in (21) tends to 0 as $R\to\infty$. Making $R\to\infty$ in (21) thus proves the equality of the right-hand members of (20) and (19), and the lemma is established.

LEMMA. $$E(d\varrho(t),\, d\sigma(t)) \leqslant \sqrt{E(d\varrho(t),\, d\varrho(t))\cdot E(d\sigma(t),\, d\sigma(t))}.$$

This follows from the positivity of the bilinear form $E(\ ,\ )$ (a direct consequence of the preceding lemma) by Schwarz' inequality.

LEMMA. $$E(d\varrho(t),\, d\varrho(t)) = \int_0^\infty\int_0^\infty \left\{\frac{\varrho(x)-\varrho(y)}{x-y}\right\}^2 \frac{x^2+y^2}{(x+y)^2}\,dx\,dy.$$

Proof. Let us write $x=e^{\xi}$, $t=e^{\tau}$, $\varrho(e^{\xi})=R(\xi)$, $K(\xi)=\log\left|\coth\tfrac{1}{2}\xi\right|$. Then

$$E(d\varrho(t),\, d\varrho(t)) = \int_0^\infty\int_0^\infty \log\left|\frac{x+t}{x-t}\right| d\varrho(x)\,d\varrho(\tau) = \int_{-\infty}^\infty\int_{-\infty}^\infty K(\xi-\eta)\,dR(\xi)\,dR(\tau). \quad (24)$$

Under the conditions of the present lemma, a formula of Beurling (see § 4 of [10]—I am grateful to J. P. Kahane for having called my attention to this paper of Beurling) applies and says that

$$\int_{-\infty}^\infty\int_{-\infty}^\infty K(\xi-\eta)\,dR(\xi)\,dR(\tau) = \int_0^\infty K''(\tau)\int_{-\infty}^\infty [R(\xi+\tau)-R(\xi)]^2\,d\xi\,d\tau. \quad (25)$$

For the reader's convenience, we go through a quick verification of (25). Start with the Fourier transforms

$$\hat{K}(\lambda) = \int_{-\infty}^\infty e^{i\lambda\xi}K(\xi)\,d\xi,$$

$$r(\lambda) = \int_{-\infty}^\infty e^{i\lambda\xi}\,dR(\xi).$$

If we use the Fourier inversion formula to express $K(\xi-\tau)$ in terms of $\hat{K}(\lambda)$, multiply both sides of the resulting expression by $dR(\xi)dR(\tau)$, and integrate we find, after changing the order of integration (which is not hard to justify here), that the left side of (25) is equal to $(2\pi)^{-1}\int_{-\infty}^{\infty}\hat{K}(\lambda)|r(\lambda)|^2 d\lambda$. In terms of $r(\lambda)$, the Fourier inversion formula yields

$$R(\xi+\tau)-R(\xi)=\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{-i\xi\lambda}\,\frac{1-e^{-i\tau\lambda}}{i\lambda}\,r(\lambda)\,d\lambda,$$

whence, by Plancherel's theorem

$$\int_{-\infty}^{\infty}(R(\xi+\tau)-R(\xi))^2\,d\xi=\frac{2}{\pi}\int_{-\infty}^{\infty}\frac{\sin^2\left(\frac{1}{2}\lambda\tau\right)}{\lambda^2}\,|r(\lambda)|^2\,d\lambda. \tag{26}$$

If we integrate $\int_0^\infty K''(\tau)(1-\cos\lambda\tau)\lambda^{-2}d\tau$ by parts twice, the integrated term vanishes each time, and we end with the expression $\int_0^\infty K(\tau)\cos\lambda\tau\,d\tau=\frac{1}{2}\hat{K}(\lambda)$, since our function $K(\tau)$ is even. Multiplying both sides of (26) by $K''(\tau)$ and using this fact we find, after integrating with respect to τ, that the right side of (25) also has the value $(2\pi)^{-1}\int_{-\infty}^{\infty}\hat{K}(\lambda)|r(\lambda)|^2\,d\lambda$, and (25) is verified.

We return to the proof of the lemma. In terms of the variable t,

$$K''(\tau)=\frac{\cosh\tau}{\sinh^2\tau}=\frac{2t(t^2+1)}{(t^2-1)^2},$$

so, substituting in (25), going back to the variables x and t, and using (24), we get, after changing the order of integration,

$$E(d\varrho(t),d\varrho(t))=2\int_0^\infty\int_1^\infty\left\{\frac{\varrho(xt)-\varrho(x)}{t-1}\right\}^2\frac{t^2+1}{(t+1)^2}\,dt\,\frac{dx}{x}.$$

On writing $tx=y$, this last expression becomes

$$2\int_0^\infty\int_x^\infty\left\{\frac{\varrho(y)-\varrho(x)}{y-x}\right\}^2\frac{x^2+y^2}{(x+y)^2}\,dy\,dx=\int_0^\infty\int_0^\infty\left\{\frac{\varrho(y)-\varrho(x)}{y-x}\right\}^2\frac{x^2+y^2}{(x+y)^2}\,dy\,dx,$$

and the lemma is proved.

5. Estimation of $E(d(\mu(t)/t),\, d(\mu(t)/t))$ from below

We again suppose that $\sup_t n(t)/t>p/(1-3p)$, and maintain this assumption throughout the present section. Choosing η, $0<\eta<\frac{2}{3}$, we can then construct the intervals $J_k=[c_k,d_k]$, $k=0,1,\ldots$ whose properties are given by Theorem 6, § 2.

At the end of § 3, it was remarked that the present study leads to the problem of estimating

$$\int_{(0,\infty)\sim J} \frac{dx}{x^2} \int_0^\infty \log \left| 1 - \frac{x^2}{t^2} \right| d\mu(t)$$

from below, $\mu(t)$ being the function described in the statement of Theorem 7, § 3, and sketched in Figure 5. As a first step towards the solution of this problem, we shall estimate a similar integral, in which $J = \bigcup_{k\geq 0} J_k$ is replaced by the set on which $\mu(t)$ is strictly increasing.

The reader should now recall the meaning of the numbers γ_k, δ_k, $k \geq 0$, defined in the statement of Theorem 7, § 3. During the remainder of this paper, we use the following

Notation.
$$F = \bigcup_{k\geq 0} [\delta_k, d_k] \cup \bigcup_{k\geq 1} [c_k, \gamma_k],$$

$\nu(t) = ((1-3p)/p)\mu(t)$, where $\mu(t)$ is the function defined in the statement of Theorem 7, § 3.

F is simply the set on which $\mu(t)$ is strictly increasing (look at Figure 5). $\nu(t)$ is continuous and piecewise linear, and is such that $\nu'(t) \equiv 1$ on the interior of F, whilst $\nu'(t) \equiv 0$ elsewhere.

LEMMA.
$$\int_{(0,\infty)\sim F} \left\{ \int_0^\infty \log \left| 1 - \frac{x^2}{t^2} \right| d\mu(t) \right\} \frac{dx}{x^2} = \frac{p}{1-3p} E\left(d\left(\frac{\nu(t)}{t}\right), d\left(\frac{\nu(t)}{t}\right) \right). \tag{27}$$

Proof. In (27), the integral on the left is equal to

$$\frac{p}{1-3p} \int_0^\infty \int_0^\infty \log \left| 1 - \frac{x^2}{t^2} \right| d\nu(t) \frac{dx - d\nu(x)}{x^2},$$

in view of the remarks just preceding the statement of this lemma. If we now apply the second lemma of § 4 and recall the definition following it, we see that the expression just written has the same value as the right side of (27).

Concerning the right side of (27) we now have

THEOREM 8. *If the parameter $\eta > 0$ used in the construction of the intervals J_k is chosen sufficiently small,*

$$E\left(d\left(\frac{\nu(t)}{t}\right), d\left(\frac{\nu(t)}{t}\right) \right) \geq \left(\frac{3}{2} - \log 2 - K\eta \right) \sum_{k\geq 0} \left\{ \left(\frac{\gamma_k - c_k}{\gamma_k} \right)^2 + \left(\frac{d_k - \delta_k}{d_k} \right)^2 \right\},$$

where K is a purely numerical constant, independent of p or the configuration of the J_k, and γ_k, δ_k are defined in the statement of Theorem 7, § 3.

Remark. We record, for future reference, the numerical value $3/2 - \log 2 = 0.80685 \ldots$.

Proof of Theorem 8. By the fifth lemma of § 4 we have:

$$E\left(d\left(\frac{\nu(t)}{t}\right), d\left(\frac{\nu(t)}{t}\right)\right)=\int_0^\infty\int_0^\infty\left(\frac{\frac{\nu(x)}{x}-\frac{\nu(y)}{y}}{x-y}\right)^2\frac{x^2+y^2}{(x+y)^2}\,dx\,dy$$

$$\geqslant\tfrac{1}{2}\sum_{k\geqslant 0}\int_{J_k}\int_{J_k}\left(\frac{\frac{\nu(x)}{x}-\frac{\nu(y)}{y}}{x-y}\right)^2 dx\,dy \tag{28}$$

Given an interval $J_k=[c_k, d_k]$, we define $\gamma_k'=c_k+2(\gamma_k-c_k)$, $\delta_k'=d_k-2(d_k-\delta_k)$, and observe that, since $(\gamma_k-c_k+d_k-\delta_k)/(d_k-c_k)=\frac{1}{2}(1-3p)<\frac{1}{2}$, we have

$$c_k\leqslant\gamma_k\leqslant\gamma_k'<\delta_k'<\delta_k<d_k.$$

Therefore, for each k,

$$\int_{J_k}\int_{J_k}\left(\frac{\frac{\nu(x)}{x}-\frac{\nu(y)}{y}}{x-y}\right)^2 dx\,dy\geqslant\left\{\int_{c_k}^{\gamma_k'}\int_{c_k}^{\gamma_k'}+\int_{\delta_k'}^{d_k}\int_{\delta_k'}^{d_k}\right\}\left(\frac{\frac{\nu(x)}{x}-\frac{\nu(y)}{y}}{x-y}\right)^2 dx\,dy. \tag{29}$$

We carry out the estimation of the second integral on the right in (29)—the first one is handled similarly.

First of all,

$$\int_{\delta_k'}^{d_k}\int_{\delta_k'}^{d_k}\left(\frac{\frac{\nu(x)}{x}-\frac{\nu(y)}{y}}{x-y}\right)^2 dx\,dy\geqslant\left\{\int_{\delta_k}^{d_k}\int_{\delta_k}^{d_k}+\int_{\delta_k'}^{\delta_k}\int_{\delta_k}^{d_k}+\int_{\delta_k}^{d_k}\int_{\delta_k'}^{\delta_k}\right\}\left(\frac{\frac{\nu(x)}{x}-\frac{\nu(y)}{y}}{x-y}\right)^2 dx\,dy. \tag{30}$$

Of the three double integrals on the right in (30), the first one is the easiest to evaluate. Since $\nu'(t)\equiv 1$ on (δ_k, d_k),

$$\frac{\nu(x)}{x}=1+\frac{\nu(\delta_k)-\delta_k}{x},\quad \delta_k\leqslant x\leqslant d_k.$$

Using this, a simple direct calculation shows that the first double integral on the right in (30) has the value

$$\left(1-\frac{\nu(\delta_k)}{\delta_k}\right)^2\left(\frac{d_k-\delta_k}{d_k}\right)^2.$$

From the definitions of $\nu(t)$ and F we have clearly $\nu(t)=|[0,t]\cap F|$. But $F\subset J$, and the J_k were so constructed that $|[0,t]\cap J|\leqslant 2\eta t$ (Theorem 6, § 2).

Therefore, for all $t>0$,

$$\frac{\nu(t)}{t}\leqslant 2\eta \tag{31}$$

which, with the preceding result, yields

$$\int_{\delta_k}^{d_k}\int_{\delta_k}^{d_k}\left(\frac{\dfrac{\nu(x)}{x}-\dfrac{\nu(y)}{y}}{x-y}\right)^2 dx\,dy \geqslant (1-2\eta)^2\left(\frac{d_k-\delta_k}{d_k}\right)^2. \tag{32}$$

We turn now to the second double integral figuring in the right hand member of (30). Let us, to simplify the notation, make the change of variables $x=\delta_k+s$, $y=\delta_k-t$, and denote $d_k-\delta_k=\delta_k-\delta_k'$ by Δ. Since $\nu(y)\equiv\nu(\delta_k)$ for $\delta_k'\leqslant y\leqslant\delta_k$ (look at Figure 5, keeping in mind that $\delta_k'>\gamma_k$), the integral in question can be written as

$$\int_0^\Delta\int_0^\Delta\left\{\frac{\dfrac{\nu(\delta_k)+s}{\delta_k+s}-\dfrac{\nu(\delta_k)}{\delta_k-t}}{s+t}\right\}^2 ds\,dt$$

in terms of s, t and Δ. This last expression simplifies to

$$\int_0^\Delta\int_0^\Delta\left\{\frac{s}{(\delta_k+s)(t+s)}-\frac{\nu(\delta_k)}{(\delta_k-t)(\delta_k+s)}\right\}^2 ds\,dt$$
$$\geqslant\frac{1}{d_k^2}\int_0^\Delta\int_0^\Delta\left(\frac{s}{s+t}\right)^2 dt\,ds-2\,\frac{\nu(\delta_k)}{\delta_k}\cdot\frac{1}{\delta_k'\,\delta_k}\int_0^\Delta\int_0^\Delta\frac{s}{s+t}\,ds\,dt\geqslant\frac{\Delta^2}{d_k^2}(1-\log 2)-\frac{4\eta\Delta^2}{\delta_k'\,\delta_k},$$

using again (31). Now $\Delta=d_k-\delta_k=\nu(d_k)-\nu(\delta_k)\leqslant\nu(d_k)\leqslant 2\eta d_k$ by (31), so since $\delta_k=d_k-\Delta$, $\delta_k'=d_k-2\Delta$,

$$\frac{4\eta\Delta^2}{\delta_k'\,\delta_k}\leqslant\frac{4\eta}{(1-2\eta)(1-4\eta)}\,\frac{\Delta^2}{d_k^2}.$$

Substituting this into the previous expression, and replacing Δ by $d_k-\delta_k$, we get

$$\int_{\delta_k}^{\delta_k}\int_{\delta_k}^{d_k}\left(\frac{\dfrac{\nu(x)}{x}-\dfrac{\nu(y)}{y}}{x-y}\right)^2 dx\,dy\geqslant\left(1-\log 2-\frac{4\eta}{(1-2\eta)(1-4\eta)}\right)\left(\frac{d_k-\delta_k}{d_k}\right)^2. \tag{33}$$

For the third integral on the right in (30) we have the same estimate, and using (33) and (32) therein, we get finally

$$\int_{\delta_k'}^{d_k}\int_{\delta_k'}^{d_k}\left(\frac{\dfrac{\nu(x)}{x}-\dfrac{\nu(y)}{y}}{x-y}\right)^2 dx\,dy\geqslant(3-2\log 2-15\eta)\left(\frac{d_k-\delta_k}{d_k}\right)^2, \tag{34}$$

as long as $\eta>0$ is sufficiently small.

In the right-hand side of (29), we estimate the second double integral by (34), and apply a similar inequality to the first double integral. The result, when substituted into (28), yields the relation asserted by the theorem. We are done.

6. Estimations on a certain auxiliary harmonic function

Given l and $A>0$, we denote by $R(l, Al)$ the rectangle with vertices at the points $0, l, l+iAl$, and iAl.

Notation. With a number $\varrho \geqslant 1$, take any function $V(z)$ harmonic inside $R(l, Al)$ and having the following boundary data:

$$V_y(x)=\begin{cases} \varrho, & 0<x<\dfrac{l}{1+\varrho} \\ -1, & \dfrac{l}{1+\varrho}<x<l \end{cases} \tag{35}$$

$$V_y \equiv 0 \text{ along the } \textit{top} \text{ of } R(l, Al), \tag{36}$$

$$V_x \equiv 0 \text{ along the } \textit{vertical sides} \text{ of } R(l, Al). \tag{37}$$

Then we write
$$X_l(\varrho, A)=\iint_{R(l,Al)} (V_x(z))^2\,dx\,dy$$
$$Y_l(\varrho, A)=\iint_{R(l,Al)} (V_y(z))^2\,dx\,dy$$
$$Y_l^+(\varrho, A)=\iint_{R^+(l,Al)} (V_y(z))^2\,dx\,dy,$$

where $R^+(l, Al)=\{z\in R(l, Al)\,|\,V_y(z)>0\}$. (Since any two harmonic functions $V(z)$ satisfying (35), (36) and (37) must differ by a constant, the quantities $X_l(\varrho, A)$, etc. depend only on l, ϱ, and A.)

LEMMA. *Given $\varepsilon>0$, there exists an $A>0$ such that, for all $\varrho\geqslant 1$ sufficiently close to 1 and all $l>0$,*

$$\pi X_l(\varrho, A)\leqslant(\lambda+\varepsilon)l^2,\quad \pi Y_l(\varrho, A)\leqslant(\lambda+\varepsilon)l^2,\quad \pi Y_l^+(\varrho, A)\leqslant\tfrac{1}{2}(\lambda+\varepsilon)l^2,$$

where
$$\lambda=\frac{4}{\pi^2}\sum_1^\infty \frac{\sin^2\frac{1}{2}\pi n}{n^3}. \tag{38}$$

Remark. For future reference, we record the numerical estimate

$$\lambda=\frac{4}{\pi^2}\left(1+\frac{1}{27}+\frac{1}{125}+\dots\right)<\frac{4}{\pi^2}\times 1.052412<0.4268.$$

Proof. We first evaluate $X_\pi(1, A)$, $Y_\pi(1, A)$ and $Y_\pi^+(1, A)$. I am indebted to C. Schubert for having suggested to me the method followed in the calculation of these quantities.

Since our function $V(z)$ is harmonic in $R(\pi, A\pi)$, and is determined only to within an additive constant, we may take it as being given by an expansion of the form:

$$V(z) = \sum_1^\infty (A_n e^{-ny} + B_n e^{ny}) \cos nx. \tag{39}$$

If $V(z)$ is given by (39), the boundary condition (37) is automatically satisfied. From (39) we also find

$$V_y(z) = -\sum_1^\infty n(A_n e^{-ny} - B_n e^{ny}) \cos nx, \tag{40}$$

which, with the condition (36) yields (here $l = \pi$):

$$A_n e^{-n\pi A} - B_n e^{n\pi A} = 0. \tag{41}$$

Relations (40) and (35) yield

$$-n(A_n - B_n) = \frac{2}{\pi}\int_0^\pi V_y(x) \cos nx\, dx = \frac{4 \sin \frac{1}{2}\pi n}{\pi n}, \tag{42}$$

since in the present case $\varrho = 1$ and $l = \pi$. Solving (41) and (42) for the A_n and B_n, and substituting back into (39), we get

$$V(z) = -\frac{4}{\pi}\sum_1^\infty \frac{\sin \frac{1}{2}\pi n}{n^2 \sinh \pi n A} \cosh n(\pi A - y) \cos nx.$$

From this we have easily, by direct calculation (using Parseval's relation to effect the inner integration in each of the following double integrals),

$$X_\pi(1, A) = \int_0^{\pi A}\int_0^\pi \{V_x(x+iy)\}^2\, dx\, dy = \frac{4}{\pi}\sum_1^\infty \frac{\sin^2 \frac{1}{2}\pi n}{n^3}\left[\frac{\cosh \pi n A}{\sinh \pi n A} + \frac{\pi n A}{\sinh^2 \pi n A}\right], \tag{43}$$

$$Y_\pi(1, A) = \int_0^{\pi A}\int_0^\pi \{V_y(x+iy)\}^2\, dx\, dy = \frac{4}{\pi}\sum_1^\infty \frac{\sin^2 \frac{1}{2}\pi n}{n^3}\left[\frac{\cosh \pi n A}{\sinh \pi n A} - \frac{\pi n A}{\sinh^2 \pi n A}\right]. \tag{44}$$

In the present case ($\varrho = 1$), it is evident by symmetry from (35), (36) and (37) that $Y_\pi^+(1, A) = \frac{1}{2} Y_\pi(1, A)$.

Formulas (43), (44), and the last one show that given $\varepsilon > 0$, we can choose $A > 0$ sufficiently large so that

$$X_\pi(1, A) < \pi(\lambda + \tfrac{1}{2}\varepsilon), \tag{45}$$

$$Y_\pi(1, A) < \pi(\lambda + \tfrac{1}{2}\varepsilon), \tag{46}$$

$$Y_\pi^+(1, A) < \tfrac{1}{2}\pi(\lambda + \tfrac{1}{2}\varepsilon), \tag{47}$$

the number λ being given by (38).

Once we fix an A such that (45), (46) and (47) are satisfied, it is clear that the quantities $X_\pi(\varrho, A)$, $Y_\pi(\varrho, A)$ and $Y^+_\pi(\varrho, A)$ will be continuous functions of ϱ, at least for ϱ near 1, so that

$$X_\pi(\varrho, A) < \pi(\lambda+\varepsilon)$$
$$Y_\pi(\varrho, A) < \pi(\lambda+\varepsilon)$$
$$Y^+_\pi(\varrho, A) < \tfrac{1}{2}\pi(\lambda+\varepsilon)$$

for all $\varrho \geq 1$ and sufficiently close to 1.

The lemma follows from these last inequalities, since $X_l(\varrho, A) = (l/\pi)^2 X_\pi(\varrho, A)$, $Y_l(\varrho, A) = (l/\pi)^2 Y_\pi(\varrho, A)$, and $Y^+_l(\varrho, A) = (l/\pi)^2 Y^+_\pi(\varrho, A)$, as may be seen by an obvious homothety argument.

7. An approximation to $\displaystyle\int_{(0,\infty)\sim J}\int_0^\infty \log\left|1-\frac{x^2}{t^2}\right| d\nu(t)\,\frac{dx}{x^2}$

We again make the assumption that $\sup_t n(t)/t > p/(1-3p)$ so that, for an arbitrary η, $0<\eta<\frac{2}{3}$, we can construct the intervals $J_k = [c_k, d_k]$, $k \geq 0$ (Theorem 6, § 2), and Theorem 7 of § 3 holds. As in § 5, we consider the set

$$F = \bigcup_{k\geq 0}[\delta_k, d_k] \cup \bigcup_{k\geq 1}[c_k, \gamma_k]$$

(the reader will recall the meaning of the numbers γ_k, δ_k if he glances at Figure 5 of § 3), and the function $\nu(t)$ (the graph of $\nu(t)$ looks just like that of $\mu(t)$ shown in Figure 5, save that its slanting portions have slope 1 instead of $p/(1-3p)$). As usual, we write $J = \bigcup_{k\geq 0} J_k$.

Notation. In each interval $J_k = [c_k, d_k]$ we take $g_k \in [\gamma_k, \delta_k)$ as the point such that

$$g_k - c_k = \frac{2}{1-3p}(\gamma_k - c_k), \quad d_k - g_k = \frac{2}{1-3p}(d_k - \delta_k).$$

(This is possible, since $(\gamma_k - c_k + d_k - \delta_k)/(d_k - c_k) = \frac{1}{2}(1-3p)$.)

We remind the reader at this point that $g_0 = c_0$ since $\gamma_0 = c_0$. For $k \geq 1$, we have, on the other hand, $c_k < \gamma_k < g_k < \delta_k < d_k$.

THEOREM 9. *There is a purely numerical constant C, independent of p or the configuration of the J_k such that, whenever the parameter η used in the construction of the latter is sufficiently small,*

$$\int_{(0,\infty)\sim J}\int_0^\infty \log\left|1-\frac{x^2}{t^2}\right| d\nu(t)\,\frac{dx}{x^2} \geq \sum_{k\geq 0}\left\{\frac{1}{c_k}\int_{c_k}^{g_k} + \frac{1}{d_k}\int_{g_k}^{d_k}\right\} u(x)\,dx - C\sqrt{\eta}\,E, \tag{48}$$

$$\sum_{k\geqslant 0}\left\{\frac{1}{c_k}\int_{c_k}^{\gamma_k}+\frac{1}{d_k}\int_{\delta_k}^{d_k}\right\}u(x)\,dx \geqslant \int_{(0,\infty)\sim F}\int_0^\infty \log\left|1-\frac{x^2}{t^2}\right|d\nu(t)\,\frac{dx}{x^2}-C\sqrt{\eta}\,E, \tag{49}$$

where

$$u(x)=\int_0^\infty \log\left|\frac{x+t}{x-t}\right| d\left(\frac{\nu(t)}{t}\right), \tag{50}$$

and

$$E=E\left(d\left(\frac{\nu(t)}{t}\right),\ d\left(\frac{\nu(t)}{t}\right)\right).$$

Remark, Of course,

$$\int_{(0,\infty)\sim F}\int_0^\infty \log\left|1-\frac{x^2}{t^2}\right|d\nu(t)\,\frac{dx}{x^2}=E\left(d\left(\frac{\nu(t)}{t}\right),\ d\left(\frac{\nu(t)}{t}\right)\right),$$

by the lemma of § 5.

Proof. We show only (48), (49) being proved in exactly the same way.

According to (50) and the first lemma of § 4, the left-hand member of (48) equals

$$\int_{(0,\infty)\sim J} u(x)\,\frac{dx}{x}.$$

If we subtract the sum on the right in (48) from the left side of (48), we therefore obtain a result equal to

$$\int_0^\infty u(x)\,\varphi(x)\,dx, \tag{51}$$

where $\varphi(x)\equiv 0$ for $x\notin J$, and if $x\in J_k=[c_k, d_k]$,

$$\varphi(x)=\begin{cases}\dfrac{1}{x}-\dfrac{1}{c_k}, & c_k\leqslant x<g_k\\[2mm] \dfrac{1}{x}-\dfrac{1}{d_k}, & g_k\leqslant x\leqslant d_k.\end{cases}$$

Inequality (48) will be proved if we show that the expression (51) is $\geqslant -C\eta^{\frac{1}{2}}E$ provided η is sufficiently small. Using (50) again, the integral (51) is seen to equal $E(\varphi(t)\,dt,\ d(\nu(t)/t))$, whence, by the fourth lemma of § 4,

$$\int_0^\infty u(x)\,\varphi(x)\,dx\geqslant -\sqrt{E\cdot E(\varphi(t)\,dt,\ \varphi(t)\,dt)}. \tag{52}$$

We proceed to estimate $E(\varphi(t)\,dt,\ \varphi(t)\,dt)$. We have, if $c_k\leqslant x<g_k$, $|\varphi(x)|\leqslant(g_k-c_k)/xc_k$, and if $g_k\leqslant x\leqslant d_k$, $|\varphi(x)|\leqslant(d_k-g_k)/x^2$. Since $|J\cap[0,t]|\leqslant 2\eta t$ (Theorem 6, § 2), we see easily, as in the proof of Theorem 8, § 5, that

$$|\varphi(x)| \leqslant \frac{2\eta}{(1-2\eta)x}, \tag{53}$$

and moreover, for each k,

$$|\varphi(x)| \leqslant \begin{cases} \dfrac{g_k - c_k}{(1-2\eta)^2 \gamma_k^2}, & c_k \leqslant x < g_k \\ \dfrac{d_k - g_k}{(1-2\eta)^2 d_k^2}, & g_k \leqslant x \leqslant d_k, \end{cases} \tag{54}$$

whilst $\varphi(x) \equiv 0$ for $x \notin J$.

From (53),

$$\left| \int_0^\infty \log \left| \frac{x+t}{x-t} \right| \varphi(t)\, dt \right| \leqslant \frac{2\eta}{1-2\eta} \int_0^\infty \log \left| \frac{x+t}{x-t} \right| \frac{dt}{t} \leqslant \frac{\eta\pi^2}{1-2\eta}, \quad x > 0. \tag{55}$$

Also from (54),

$$\int_0^\infty |\varphi(x)|\, dx = \int_J |\varphi(x)|\, dx \leqslant \sum_{k\geqslant 0} \frac{1}{(1-2\eta)^2} \left[\left(\frac{g_k - c_k}{\gamma_k} \right)^2 + \left(\frac{d_k - g_k}{d_k} \right)^2 \right]$$

$$= \frac{4}{(1-3p)^2 (1-2\eta)^2} \sum_{k\geqslant 0} \left[\left(\frac{\gamma_k - c_k}{\gamma_k} \right)^2 + \left(\frac{d_k - \delta_k}{d_k} \right)^2 \right].$$

We now apply Theorem 8, § 5, according to which this last sum is $\leqslant$ say

$$2E\left(d\left(\frac{\nu(t)}{t} \right),\ d\left(\frac{\nu(t)}{t} \right) \right) = 2E,$$

if $\eta > 0$ is small enough. Since we are supposing throughout this paper that $p < 1/20$, this yields

$$\int_0^\infty |\varphi(x)|\, dx < 16\, E \tag{56}$$

for all sufficiently small η.

From (55) and (56) we find

$$E(\varphi(t)\, dt, \varphi(t)\, dt) = \int_0^\infty \left\{ \int_0^\infty \log \left| \frac{x+t}{x-t} \right| \varphi(t)\, dt \right\} \varphi(x)\, dx \leqslant 25\, \pi^2 E \eta$$

if $\eta > 0$ is sufficiently small. This result, substituted into (52) yields

$$\int_0^\infty u(x)\, \varphi(x)\, dx \geqslant -5\pi \sqrt{\eta}\, E,$$

whenever $\eta > 0$ is sufficiently small, and as we have seen, this is enough to prove (48).

Since (49) can be proved in the same way, we are done.

8. Final estimation of $\int_{(0,\infty)\sim J}\int_0^\infty \log\left|1-\frac{x^2}{t^2}\right| d\nu(t)\,\frac{dx}{x^2}$ from below

We continue with the notations used in the preceding section. In order to carry out the estimation mentioned in the heading to this section, it is, in view of Theorem 9, § 7 and Theorem 8, § 5, of interest to us to compare the terms of the sum in (48) with those of the one in (49). This will be done with the help of the results found in § 6.

Notation. During the remainder of this paper, we denote by ϱ the quantity $(1+3p)/(1-3p)$. We note that $\varrho>1$, and is very near one if p is sufficiently small.

We also assume that the function u, used in the preceding section, is defined for complex values of its argument according to the formula

$$u(z)=\int_0^\infty \log\left|\frac{z+t}{z-t}\right| d\left(\frac{\nu(t)}{t}\right).$$

The function $u(z)$ is, of course, harmonic in the upper half plane, and continuous up to the real axis, since $\nu(t)$ is piecewise linear, continuous, and vanishes for all values of t sufficiently close to the origin.

We proceed to estimate the difference

$$\int_{\gamma_k}^{g_k} u(x)\,dx-\varrho\int_{c_k}^{\gamma_k} u(x)\,dx \tag{57}$$

from below. We may obviously suppose $k>0$ here, since $c_0=\gamma_0=g_0$. So, letting $g_k-c_k=l$, we denote by $R(l, Al)$ a rectangle of height Al having for base the segment $[c_k, g_k]$ of the real axis. (This differs only by a translation along the real axis from the rectangle denoted by the same symbol in § 6.)

According to the lemma of § 6 and the remark thereto, we can choose A sufficiently large so that the quantities $X_l(\varrho, A)$, etc., defined there satisfy

$$\left.\begin{aligned}\pi X_l\,(\varrho, A)&\leqslant 0.427\,l^2\\ \pi Y_l\,(\varrho, A)&\leqslant 0.427\,l^2\\ \pi Y_l^{+}(\varrho, A)&\leqslant 0.214\,l^2\end{aligned}\right\} \tag{58}$$

for all $\varrho>1$ sufficiently close to 1, and all l. We then take such an A, and fix it – this value of A will be adhered to throughout the remainder of this section.

Let $V(z)$ be a function harmonic inside $R(l, Al)$ and having the following boundary data:

$V_y(x) = \varrho$ for $c_k \leqslant x < \gamma_k$

$V_y(x) = -1$ for $\gamma_k \leqslant x \leqslant g_k$

$V_y \equiv 0$ along the *top* of $R(l, Al)$

$V_x \equiv 0$ along the *vertical sides* of $R(l, Al)$.

We note that $\gamma_k - c_k = (1+\varrho)^{-1}(g_k - c_k) = l/(1+\varrho)$ according to the definition of the number g_k (§ 7), since $(g_k - c_k)/(\gamma_k - c_k) = 2/(1-3p)$ and $\varrho = (1+3p)/(1-3p)$. Our present function $V(z)$ thus differs from the one denoted by the same letter in § 7 only in that its domain of definition has been translated along the x-axis.

Expression (57) is, in terms of V, obviously equal to

$$-\int_{c_k}^{g_k} u(x) V_y(x)\, dx = \int_{\partial R(l,\, Al)} u(z) \frac{\partial V(z)}{\partial n_z} |dz|, \tag{59}$$

where n_z denotes the outward normal to $\partial R(l, Al)$ at the point z. If we apply Green's theorem to the right member of (59) we find for it the value

$$\iint_{R(l,\, Al)} (u_x(z) V_x(z) + u_y(z) V_y(z))\, dx\, dy, \tag{60}$$

since $V_{xx} + V_{yy} \equiv 0$ inside $R(l, Al)$. (The discontinuities of the partial derivatives of u and V at the boundary of $R(l, Al)$ give no trouble here.) Our problem has thus reduced to the one of finding a lower bound for (60).

We must now resist the temptation to apply Schwarz' inequality directly to (60), as the estimate thus obtained would be too crude for our purposes.

We begin by breaking up the integral $\iint_{R(l,\, Al)} u_y(z) V_y(z) dx\, dy$ into two, the first taken over

$$R^+(l, Al) = \{z \in R(l, Al) \mid V_y(z) > 0\}$$

and the second over

$$R^-(l, Al) = R(l, Al) \sim R^+(l, Al).$$

From the formula for $u(z)$ we have, for $y > 0$:

$$u_y(z) = \int_0^\infty \left\{ \frac{y}{(x+t)^2 + y^2} - \frac{y}{(x-t)^2 + y^2} \right\} d\left(\frac{\nu(t)}{t}\right). \tag{61}$$

This shows that $u_y(z)$ is bounded in the upper half plane since $|d(\nu(t)/t)/dt| \leqslant$ const., as is clear from the definition of $\nu(t)$. From (61) we also have $u_y(iy) \equiv 0$, $y \geqslant 0$, and, as in the proof of the third lemma, § 4,

$$\lim_{\eta\to 0+} u_y(x+i\eta) = -\pi \frac{d}{dx}\left(\frac{\nu(x)}{x}\right), \quad x>0.$$

But since $\nu(x)$ is increasing.

$$-\frac{d}{dx}\left(\frac{\nu(x)}{x}\right) \leqslant \frac{\nu(x)}{x^2} \leqslant \frac{2\eta}{x} \quad \text{(see (31), § 5)},$$

so that

$$\lim_{\eta\to 0+} u_y(x+i\eta) \leqslant \frac{2\pi\eta}{x}, \quad x>0.$$

This, coupled with the facts that $u_y(iy)\equiv 0$, $y\geqslant 0$, and that $u_y(z)$ is harmonic and bounded in the first quadrant, yields, by the principle of maximum,

$$u_y(z) \leqslant \frac{2\pi\eta x}{x^2+y^2} \quad \text{for } x>0, y>0. \tag{62}$$

From (62) we have clearly

$$u_y(z) \leqslant \frac{2\pi\eta}{c_k} \quad \text{for } z \text{ in } R(l, Al),$$

whence, since $V_y(z)\leqslant 0$ in $R^-(l, Al)$,

$$\begin{aligned}\iint_{R^-(l,\,Al)} u_y(z)V_y(z)\,dx\,dy &\geqslant -\frac{2\pi\eta}{c_k}\iint_{R^-(l,\,Al)} V_y(z)dx\,dy \\ &\geqslant -\frac{2\pi\eta}{c_k}\left[Al^2\iint_{R(l,\,Al)}(V_y(z))^2\,dx\,dy\right]^{\frac{1}{2}}\end{aligned}$$

by Schwarz' inequality. Here, $R(l, Al)$ and $V(z)$ differ only by a translation along the x-axis from the objects denoted by the same symbols in § 6, so this last relation can be rewritten as

$$\iint_{R^-(l,\,Al)} u_y(z)V_y(z)\,dx\,dy \geqslant -2\pi\sqrt{A}\,\eta\frac{l}{c_k}\sqrt{Y_l(\varrho, A)}. \tag{63}$$

Direct application of Schwarz' inequality now yields

$$\iint_{R^+(l,\,Al)} u_y(z)V_y(z)\,dx\,dy \geqslant -\sqrt{\iint_{R(l,\,Al)}(u_y(z))^2dx\,dy\,Y_l^+(\varrho, A)}, \tag{64}$$

$$\iint_{R(l,\,Al)} u_x(z)V_x(z)\,dx\,dy \geqslant -\sqrt{\iint_{R(l,\,Al)}(u_x'(z))^2\,dx\,dy\,X_l(\varrho, A)}, \tag{65}$$

in terms of the quantities $Y_l^+(\varrho, A)$, $X_l(\varrho, A)$ defined in § 6. Adding (63), (64) and (65) now gives us

$$\iint_{R(l,\,Al)}[u_x(z)V_x(z)+u_y(z)V_y(z)]\,dx\,dy$$

$$\geqslant -\sqrt{X_l(\varrho,A)}\sqrt{\iint_{R(l,\,Al)}(u_x)^2\,dx\,dy}-\sqrt{Y_l^+(\varrho,A)}\sqrt{\iint_{R(l,\,Al)}(u_y)^2\,dx\,dy}$$

$$-2\pi\sqrt{A}\,\eta\frac{l}{c_k}\sqrt{Y_l(\varrho,A)}$$

which, in turn, is $\geqslant$

$$-\sqrt{[X_l(\varrho,A)+Y_l^+(\varrho,A)]\iint_{R(l,\,Al)}[u_x^2+u_y^2]\,dx\,dy}-2\pi\sqrt{A}\,\eta\frac{l}{c_k}\sqrt{Y_l(\varrho,A)}, \qquad (66)$$

by Cauchy's inequality. Substituting into (66) the numerical estimates (58), valid for all $\varrho>1$ sufficiently close to 1, we see that

$$\iint_{R(l,\,Al)}[u_x(z)V_x(z)+u_y(z)V_y(z)]\,dx\,dy$$

$$\geqslant -\left[0.641\,l^2\cdot\frac{1}{\pi}\iint_{R(l,\,Al)}[(u_x(z))^2+(u_y(z))^2]\,dx\,dy\right]^{\frac{1}{2}}-A'\eta\frac{l^2}{c_k} \qquad (67)$$

provided that $p>0$ is sufficiently small, A' being some fixed constant.

We now change the name for $R(l, Al)$ to R_k; R_k is thus a rectangle having the segment $[c_k, g_k]$ as its base, and having height $A(g_k-c_k)$, where A is the fixed number chosen after relations (58). We recall that $l=g_k-c_k=(1+\varrho)(\gamma_k-c_k)$, and that $c_k\geqslant(1-2\eta)\gamma_k$, since $|J\cap[0,t]|\leqslant 2\eta t$. Taking account of these relations, and remembering that expression (60) equals the right member of (59), whose left member is the same as (57), we get, by (67):

$$\frac{1}{c_k}\int_{c_k}^{g_k}u(x)\,dx-\frac{1+\varrho}{c_k}\int_{c_k}^{\gamma_k}u(x)\,dx$$

$$\geqslant -(1+\varrho)\left[0.65\left(\frac{\gamma_k-c_k}{\gamma_k}\right)^2\cdot\frac{1}{\pi}\iint_{R_k}[(u_x(z))^2+(u_y(z))^2]\,dx\,dy\right]^{\frac{1}{2}}-K\eta\left(\frac{\gamma_k-c_k}{\gamma_k}\right)^2, \qquad (68)$$

provided that $\eta>0$ and $p>0$ are sufficiently small, where K is some fixed numerical constant.

We are now prepared to prove

THEOREM 10. *There is a positive numerical constant L such that, for all $p>0$ and $\eta>0$ sufficiently small*

$$\int_{(0,\infty)\sim J}\int_0^\infty\log\left|1-\frac{x^2}{t^2}\right|d\nu(t)\frac{dx}{x^2}\geqslant LE\left(d\left(\frac{\nu(t)}{t}\right),\ d\left(\frac{\nu(t)}{t}\right)\right). \qquad (69)$$

Proof. For each k, denote by S_k a rectangle whose base is the segment $[g_k, d_k]$, and whose height is equal to $A(d_k - g_k)$. By following the reasoning used above, we can prove that

$$\frac{1}{d_k}\int_{g_k}^{d_k} u(x)\,dx - \frac{1+\varrho}{d_k}\int_{\delta_k}^{d_k} u(x)\,dx$$
$$\geqslant -(1+\varrho)\left[0.65\left(\frac{d_k-\delta_k}{d_k}\right)^2\cdot\frac{1}{\pi}\iint_{S_k}(u_x^2+u_y^2)\,dx\,dy\right]^{\frac{1}{2}} - K\eta\left(\frac{d_k-\delta_k}{d_k}\right)^2 \tag{70}$$

for $p>0$ and $\eta>0$ sufficiently small, this relation being entirely analogous to (68).

We now combine (68) and (70) with the inequalities (48), (49) of Theorem 9, § 7 and find, taking into account the remark thereto, that

$$\int_{(0,\infty)\sim J}\int_0^\infty \log\left|1-\frac{x^2}{t^2}\right| d\nu(t)\,\frac{dx}{x^2} \geqslant (1+\varrho)\,E\left(d\left(\frac{\nu(t)}{t}\right),\ d\left(\frac{\nu(t)}{t}\right)\right)$$
$$-(2+\varrho)\,C\sqrt{\eta}\,E\left(d\left(\frac{\nu(t)}{t}\right),\ d\left(\frac{\nu(t)}{t}\right)\right) - K\eta\sum_{k\geqslant 0}\left[\left(\frac{\gamma_k-c_k}{\gamma_k}\right)^2+\left(\frac{d_k-\delta_k}{d_k}\right)^2\right]$$
$$-(1+\varrho)\sum_{k\geqslant 0}\left[0.65\left(\frac{\gamma_k-c_k}{\gamma_k}\right)^2\cdot\frac{1}{\pi}\iint_{R_k}(u_x^2+u_y^2)\,dx\,dy\right]^{\frac{1}{2}}$$
$$-(1+\varrho)\sum_{k\geqslant 0}\left[0.65\left(\frac{d_k-\delta_k}{d_k}\right)^2\cdot\frac{1}{\pi}\iint_{S_k}(u_x^2+u_y^2)\,dx\,dy\right]^{\frac{1}{2}}, \tag{71}$$

whenever $p>0$ and $\eta>0$ are sufficiently small. According to Cauchy's inequality, the last two terms on the right in (71) are together bounded below by

$$-(1+\varrho)\left[0.65\sum_{k\geqslant 0}\left\{\left(\frac{\gamma_k-c_k}{\gamma_k}\right)^2+\left(\frac{d_k-\delta_k}{d_k}\right)^2\right\}\right]^{\frac{1}{2}}\left[\frac{1}{\pi}\sum_{k\geqslant 0}\left\{\iint_{R_k}+\iint_{S_k}\right\}(u_k^2+u_y^2)\,dx\,dy\right]^{\frac{1}{2}}. \tag{72}$$

If $\eta>0$ is sufficiently small, the expression inside the first factor in brackets of (72) has, according to Theorem 8, § 5 and the remark thereto, the majorant:

$$\frac{0.65}{0.80}\,E\left(d\left(\frac{\nu(t)}{t}\right),\ d\left(\frac{\nu(t)}{t}\right)\right).$$

Also, since the rectangles R_k and S_k are obviously all disjoint, and lie in the first quadrant,

$$\frac{1}{\pi}\sum_{k\geqslant 0}\left\{\iint_{R_k}+\iint_{S_k}\right\}(u_x^2+u_y^2)\,dx\,dy \leqslant \frac{1}{\pi}\int_0^\infty\int_0^\infty (u_x^2+u_y^2)\,dx\,dy,$$

which is, however, equal to $E(d(\nu(t)/t),\ d(\nu(t)/t))$ by the third lemma of § 4.

Putting these estimates into (72) and then going back to (71) (of which the third term on the right can also be estimated by Theorem 8, § 5), we see that

$$\int_{(0,\infty)\sim J}\int_0^\infty \log\left|1-\frac{x^2}{t^2}\right| d\nu(t)\,\frac{dx}{x^2}$$

$$\geqslant \left[(1+\varrho)\left(1-\sqrt{\frac{65}{80}}\right)-(2+\varrho)\,C\sqrt{\eta}-2K\eta\right] E\left(d\left(\frac{\nu(t)}{t}\right),\; d\left(\frac{\nu(t)}{t}\right)\right)$$

for all $p>0$ and $\eta>0$ sufficiently small. Since K and C are numerical constants and $\varrho\to 1$ as $p\to 0$, this proves the theorem.

9. Return to $\sum_1^\infty \frac{\log^+|P(m)|}{m^2}$

According to Theorem 10 of the preceding section, inequality (69) holds whenever $p>0$ and $\eta>0$ are sufficiently small. We can therefore take a positive value of $\eta(<\frac{2}{3})$ such that (69) holds for all sufficiently small $p>0$. We choose such a value of η and *fix* it; η is henceforth to be considered as a *positive numerical constant.*

Returning to the notation of the first sections of this second part of the paper, we let $P(x)$ be any polynomial of the form $\prod_k(1-x^2/x_k^2)$ with the x_k real and positive, and denote by $n(t)$ the number of points x_k in $[0, t]$. We then have:

THEOREM 11. *There is a positive numerical constant D, independent of the choice of $P(x)$, such that, for any $p>0$ sufficiently small,*

$$\sup_t \frac{n(t)}{t} > \frac{p}{1-3p}$$

implies

$$\sum_1^\infty \frac{\log^+|P(m)|}{m^2} > Dp.$$

Proof. Suppose $\sup_t n(t)/t > p/(1-3p)$ with $0<p<1/20$. Then, according to Theorem 6, § 2, we can construct the intervals J_k, $k\geqslant 0$, using our fixed $\eta>0$.

By Theorem 7 of § 3 we have then

$$\sum_1^\infty \frac{\log^+|P(m)|}{m^2} \geqslant \frac{1}{5}\int_\Omega \frac{V_\mu(x)}{x^2}\,dx, \tag{73}$$

where $\mu(t)$ is the function shown in Figure 5,

$$V_\mu(x) = \int_0^\infty \log\left|1-\frac{x^2}{t^2}\right| d\mu(t),$$

and

$$\Omega = (0,\infty)\sim \bigcup_{k\geqslant 0} J_k = (0,\infty)\sim J.$$

The function $\nu(t)$ was defined as $((1-3p)/p)\mu(t)$ at the beginning of § 5. The right-hand side of (73) is thus equal to $p/(5-15p)$ times the left hand member of (69), and Theorem 10, § 8 yields

$$\sum_1^\infty \frac{\log^+|P(m)|}{m^2} \geqslant \frac{pL}{5-15p} E\left(d\left(\frac{\nu(t)}{t}\right),\ d\left(\frac{\nu(t)}{t}\right)\right) \tag{74}$$

for all sufficiently small $p>0$, in view of our choice of the constant η.

To estimate the right side of (74), we apply Theorem 8 of § 5, according to which

$$E\left(d\left(\frac{\nu(t)}{t}\right),\ d\left(\frac{\nu(t)}{t}\right)\right) \geqslant \frac{1}{2}\left(\frac{d_0-\delta_0}{d_0}\right)^2. \tag{75}$$

Now from Theorem 7, § 3, $d_0-\delta_0=\frac{1}{2}(1-3p)(d_0-c_0)$, and from Theorem 6, § 2, $d_0-c_0\geqslant\eta d_0$. Therefore, by (75),

$$E\left(d\left(\frac{\nu(t)}{t}\right),\ d\left(\frac{\nu(t)}{t}\right)\right) \geqslant \frac{1}{8}(1-3p)^2\eta^2,$$

and this, substituted into (74), proves the theorem, since L and η are constants >0.

THEOREM 12. *Let $Q(z)$ be any polynomial of the form $\prod_k(1-z^2/z_k^2)$ with the z_k not necessarily real. There is a numerical constant C, independent of the choice of $Q(z)$, such that*

$$\frac{\log|Q(z)|}{|z|} \leqslant C\sum_1^\infty \frac{\log^+|Q(m)|}{m^2} \tag{76}$$

for all complex z, provided that the sum on the right is less than some constant $\alpha>0$, independent of the choice of Q.

Proof. Let $P(x)=\prod_k(1-x^2/|z_k|^2)$, and denote by $n(t)$ the number of points $|z_k|$ in the interval $[0, t]$.

We have, on the one hand,

$$\sum_1^\infty \frac{\log^+|P(m)|}{m^2} \leqslant \sum_1^\infty \frac{\log^+|Q(m)|}{m^2},$$

and, on the other,

$$\log|Q(z)| \leqslant \sum_k \log\left[1+\frac{|z|^2}{|z_k|^2}\right] = \int_0^\infty \log\left(1+\frac{|z|^2}{t^2}\right) dn(t)$$

$$= 2|z|\int_0^\infty \frac{|z|}{t^2+|z|^2}\,\frac{n(t)}{t}\,dt \leqslant \pi|z|\sup_t \frac{n(t)}{t}.$$

These two inequalities, together with Theorem 11, yield the desired result.

Remark 1. The set of integers **Z** thus behaves somewhat as if it had positive logarithmic capacity, even though it is of capacity zero.

Remark 2. Theorem 12 is not dependent on the special arithmetic character of **Z**. An analoguous result is valid if the set $\{1, 2, 3, ...\}$ over which the sum in (76) is taken is replaced by any positive increasing sequence $\{\lambda_m\}$ having the property that $\lambda_{m+1}-\lambda_m \leqslant h$ for some constant h. Indeed, the work of § 1 is readily adapted to this more general situation, and the reasoning of §§ 2–8 then goes through practically without change.

10. Weighted polynomial approximation on Z

Let $W(m) \geqslant 1$ be defined on **Z**, and have the property that

$$\frac{|m|^k}{W(m)} \to 0 \text{ as } m \to \pm\infty \text{ for every } k > 0.$$

In terms of the spaces $\mathcal{C}_W(\mathbf{Z})$ and $\mathcal{C}_W(\mathbf{Z}, 0)$ defined in the introduction to this paper we now have:

THEOREM 13. *If* $\sum_{-\infty}^{\infty} \frac{\log W(m)}{1+m^2}$ *converges, then* $C_W(Z, 0) \neq C_W(Z)$.

Proof. I claim that if $F(m)=0$ for $m \neq 0$ and $F(0)=1$, there do not exist polynomials P which make $\|F-P\|_W$ arbitrarily small.

For suppose, on the contrary, that there is a sequence $\{P_r\}$ of polynomials such that $\|F-P_r\|_W \to 0$, $r \to \infty$. Then clearly $P_r(0) \underset{r}{\to} 1$, so, since F is even, if we put

$$Q_r(x) = \frac{P_r(x)+P_r(-x)}{2P_r(0)},$$

we will have

$$\|F-Q_r\|_{W_1} \to 0, \; r \to \infty,$$

that is,

$$\sup_m \frac{|F(m)-Q_r(m)|}{W_1(m)} \to 0, \quad r \to \infty, \tag{77}$$

where $W_1(m) = \sup(W(m), W(-m))$.

In view of the hypothesis,

$$\sum_1^{\infty} \frac{\log W_1(m)}{m^2} < \infty,$$

so, given any $\varepsilon > 0$ we can find a number N_ε such that

$$\sum_{N_\varepsilon}^{\infty} \frac{\log W_1(m)}{m^2} < \frac{\varepsilon}{2}. \tag{78}$$

Since (77) implies that $\sup_{m>0} |Q_r(m)|/W_1(m) < 1$ for all sufficiently large r, we have by (78),

$$\sum_{N_\varepsilon}^{\infty} \frac{\log^+ |Q_r(m)|}{m^2} < \frac{\varepsilon}{2} \tag{79}$$

for all sufficiently large r. On the other hand, $Q_r(m) \to 0$, $r \to \infty$ for each $m \neq 0$, by the definition of F. Therefore

$$\sum_{1}^{N_\varepsilon - 1} \frac{\log^+ |Q_r(m)|}{m^2} \to 0, \quad r \to \infty,$$

which, when combined with (79), yields

$$\sum_{1}^{\infty} \frac{\log^+ |Q_r(m)|}{m^2} \to 0, \quad r \to \infty, \tag{80}$$

since $\varepsilon > 0$ is arbitrary.

Now $Q_r(z)$, being even and *one* at the origin is, for each r, of the form

$$\prod_k \left(1 - \frac{z^2}{z_{k,r}^2}\right).$$

So (80) enables us to apply Theorem 12, § 9, according to which

$$\sup_z \frac{\log |Q_r(z)|}{|z|} \to 0, \quad r \to \infty.$$

From this we see that the functions $Q_r(z)$ form a normal family in the complex plane, and a subsequence of them tends to an entire function $Q(z)$ of *zero exponential type*. Clearly, if $m \in \mathbf{Z}$, $Q(m) = F(m)$ is 1 for $m = 0$ and 0 for $m \neq 0$. But this is impossible, by Jensen's theorem, since $Q(z)$ is of zero exponential type.

We have reached a contradiction, and the theorem is proved.

Remark. An analogous result holds for weighted polynomial approximation on any set of the form $\{\lambda_m | m \in \mathbf{Z}\}$, where $\lambda_0 = 0$, $\lambda_{-m} = -\lambda_m$, and the sequence $\{\lambda_m\}$ is increasing, provided there exists an h such that

$$\lambda_{m+1} - \lambda_m \leqslant h \text{ for all } m.$$

(See Remark 2 of the preceding section.)

As an application of Theorem 13, we give

THEOREM 14. *Let $M(m) \geqslant 1$, $m \in \mathbf{Z}$. If*

$$\sum_{-\infty}^{\infty} \frac{\log M(m)}{1+m^2} < \infty, \tag{81}$$

there is a periodic infinitely differentiable function $f(\vartheta)$ of the form

$$f(\vartheta) = \sum_{-\infty}^{\infty} \alpha_m e^{im\vartheta}$$

such that $\alpha_0 > 0$, $|\alpha_m| \leqslant 1/M(m)$, whilst $f^{(k)}(\pm\pi) = 0$, $k = 0, 1, 2, \ldots$.

Proof. In order to have a function $W(m) \geqslant 1$ such that $|m|^k/W(m) \to 0$, $m \to \pm\infty$, for all $k > 0$, we put $W(m) = \sup\{M(m), e^{|m|^{\frac{1}{2}}}\}$. If (81) holds, we will also have

$$\sum_{-\infty}^{\infty} \frac{\log W(m)}{1+m^2} < \infty,$$

so that the preceding theorem may be applied. If $F(m)$ is as in the proof of that theorem, $F \notin \mathcal{C}_W(\mathbf{Z}, 0)$, and the Hahn–Banach theorem tells us that there is a two-way sequence $\{\gamma_m\}$ with $\sum_{-\infty}^{\infty} |\gamma_m| < \infty$, such that

$$\sum_{-\infty}^{\infty} \frac{F(m)}{W(m)} \gamma_m > 0, \tag{82}$$

while
$$\sum_{-\infty}^{\infty} \frac{Q(m)}{W(m)} \gamma_m = 0 \text{ for all } Q \in \mathcal{C}_W(\mathbf{Z}, 0). \tag{83}$$

By (82), $\gamma_0/W(0) > 0$, and by (83) $\sum_{-\infty}^{\infty} m^k \gamma_m / W(m) = 0$, $k = 0, 1, 2, \ldots$. So, since $W(m) \geqslant M(m)$, we can put $\alpha_m = C(-1)^m \gamma_m / W(m)$ with $C > 0$ sufficiently small, and we will have the desired result.

Remark. Results like that of Theorem 14, but in which $M(m)$ is subjected to additional restrictive conditions on the regularity of its behaviour, have long been known (see, for instance [11], p. 78 and p. 80). In the establishment of these results, the extra properties that $M(m)$ was assumed to enjoy have always played an essential role.

We now see that the condition (81) is by itself already enough to ensure the existence of a function $f(\vartheta)$ having the properties stated in the theorem, and that no assumptions whatever need be made concerning the regularity of $M(m)$.

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