THE THEORY OF STATIONARY POINT PROCESSES

BY

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Table of contents

Abstract

An axiomatic formulation is presented for point processes which may be interpreted as ordered sequences of points randomly located on the real line. Such concepts as forward recurrence times and number of points in intervals are defined and related in set-theoretic

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terms, and conditions established under which these random variables are finite valued. Several types of stationarity are defined, and it is shown that these (each requiring a kind of statistical uniformity over the entire real axis) are equivalent to one another. Stationarity does *not* imply that the intervals between points are either independently or identically distributed.

Convexity and absolute continuity properties are found for the forward recurrence times of the stationary point process (s.p.p.). The moments of the number of points in an interval are described in terms of these distributions, which appear in series whose convergences are necessary and sufficient conditions for the finiteness of the moments. Local and global properties of the moments are related, and it is shown that any existent moment is an absolutely continuous function of the interval length. The distribution functions of the forward recurrences are related to the statistics of the point sequence and the interval times. Moment properties are also determined in terms of the latter. An ergodic theorem relates the behavior of individual realizations of the number of points to their statistical averages.

Several classes of point processes are described, and stationarity verified where applicable, using the most convenient of the (equivalent) criteria for each case. The preceding theory is applied to the problem of calculating moments and other process statistics.

1.0. Introduction and Summary

A stationary point process, like a recurrent or renewal process, may be interpreted as an ordered sequence of points randomly located on the real line. The stationary point process (hereafter abbreviated s.p.p.) generalizes certain aspects of renewal processes; in particular, the intervals between points on the line need be neither independently nor identically distributed. On the other hand, the s.p.p. is required to retain a certain statistical uniformity over the entire open line, so that it more properly becomes a generalization of the equilibrium renewal process (see Cox [3]).

The s.p.p. is not only of interest for its own sake, but also leads to applications for which the renewal process is an inadequate model. Included are the examination of randomly timed modulations of random processes in communication theory, analyses of zero crossings of stochastic processes, and those problems in queue arrivals, traffic flow, etc., whose current behavior depends on past history.

Perhaps the most direct motivation for this study is a paper by McFadden [10], whose results are admittedly heuristic. Some of the properties to be demonstrated here have been stated by McFadden for processes of similar nature, and by others for renewal processes (see Smith [14], Takacs [15], or Cox [3] for a summary). An earlier paper by Wold [16] suggests the same stationary conditions later utilized by McFadden, but fails to develop the properties consequent to the definition.

McFadden's work unfortunately incurs gaps that make mandatory the choice of a different structure than his for a rigorous treatment. He creates a process consisting of randomly located points t_n whose indices are "floating"; the index $n=1$ refers always to the point immediately to the right of various "arbitrarily chosen" numbers t. Aside from the question of the meaning of "arbitrarily chosen", we find that ω-sets such as $[\omega: t_n(\omega) \leqslant x]$ cannot be specified as probability sets in the accepted sense. Moreover, McFadden erroneously draws the conclusion(1) (of which he makes frequent use) that the intervals between points necessarily constitute a stationary random process [10] [11].

Our program is as follows. We shall define point processes axiomatically, and establish forward and backward recurrence times as well as numbers of points in intervals in terms of the probability space induced by the process $\{t_n\}$. Stationarity is then introduced, using forward recurrence times. It is shown that this implies a similar property for the backward recurrence times. Further, McFadden's definition [10], as well as an apparently weaker version, are proved equivalent to the preceding stationarity notions.

In the second chapter, we derive convexity and absolute continuity properties for the forward recurrence distributions time of the s.p.p. The moments of the number of points in an interval are described in terms of these distributions, which appear in series whose convergence is a necessary and sufficient condition for the finiteness of the moments. Local and global properties of the moments are related; any existent moment is an absolutely continuous function of the interval length. The distribution functions of forward recurrence times are used to obtain interval statistics, and certain results on statistics of the t_n. The final portion of this chapter presents an ergodic theorem that illustrates the rich structure of s.p.p.

In the fourth and final chapter, several classes of point processes are analyzed. Stationarity is verified for these, and more specialized results obtained.

2.0. Stationary properties for point processes

A stochastic point process can be intuitively described in terms of randomly located points on the real axis. Given such a process, one considers such random variables as $N(t, x)$, the number of points falling in the interval $(t, t+x]$, and $L_n(t)$, the time required

(1) This was first pointed out to us by Prof. W. L. Root, who provided a counterexample that constitutes the basis for the generalization to be presented in Section 4.2.

for the nth point after t to occur. To be meaningful, however, the process must be enunciated in strictly mathematical terms that translate these intuitive concepts into a rigorous structure. Here, the process is viewed as an ordered, non-decreasing sequence of random variables, $\{t_n\}$, properly defined on a probability space. When this is done, $N(t, x)$ and $L_n(t)$ are specified in set-theoretic language consistent with their desired intuitive interpretation.

To be a fruitful object of study, however, $\{t_n\}$ must be endowed with certain additional properties. For example, in renewal theory only positive indices are considered, and $\{t_{n+1}-t_n\}$ is assumed mutually independent and identically distributed (at least for $n=1, 2, \ldots$) (see again Cox [3]). To us, these assumptions seem unnecessarily restrictive, and lead to an unacceptable model for many problems, including that of random sampling of random processes, which initially motivated this study.

McFadden [10] has suggested by his work that the following intuitively appealing assumption should be made: $\{t_n\}$ is a stationary point process (s.p.p.) if the multivariate distribution functions of $N(t_j, x_j)$, $j=1, 2, \ldots, N$ remains invariant if, with any number h, t_j is replaced by t_j+h. With only this hypothesis, and occasionally the added assumption that the expected number of points in an interval is finite, we shall be able to obtain all the results which follow.

Although our work makes little use of interval statistics [those of $N(t, x)$ and $L_n(t)$ generally being more convenient], it is advantageous to define the process by beginning with (the intuitive equivalent of) intervals. That is, we let $\{\tau_n\}$, $n=0, \pm 1, \pm 2, \ldots$ be a discrete parameter stochastic process associated with a probability space $(\Omega, \mathfrak{F}, P)$. We require that each τ_n be finite-valued and non-negative with probability one.

Definition 2.0.1. If $\{\tau_n\}$ is as specified above, and

$$t_n = \begin{cases} \sum_0^n \tau_k & \text{if } n \geqslant 0 \\ \\ \tau_0 - \sum_n^{-1} \tau_k & \text{if } n \leqslant -1, \end{cases} \tag{2.0.1}$$

$\{t_n\}$ is called a *stochastic point process.*

It is clear from the definition that, with probability 1, $\{t_n\}$ is an ordered nondecreasing sequence, each of whose members is finite-valued. There is no loss of generality in supposing that these properties of $\{t_n\}$ hold for every $\omega \in \Omega$, and we shall so assume henceforth. We observe also that the σ-sub-fields induced on $(\Omega, \mathfrak{F}, P)$ by $\{t_n\}$ and $\{\tau_n\}$ are identical; we may take $\mathfrak{F}$ to be that σ-sub-field itself, and treat $(\Omega, \mathfrak{F}, P)$ as the basic probability space underlying both $\{t_n\}$ and $\{\tau_n\}$.

All sets and random variables to appear in this paper will be expressed in terms of countable set operations on the "basic building block" sets $B_n(t)$, where n is any integer, and t any real number. We define

$$B_n(t) = \{\omega: t_n(\omega) \leqslant t\}. \tag{2.0.2}$$

It is evident that each $B_n(t)$ is ω-measurable, and that these sets satisfy both $B_{n+1}(t) \subset B_n(t)$ and (for $s \leqslant t$) $B_n(s) \subset B_n(t)$.

2.1. Forward recurrence times and points in an interval

For each t, we wish to define a new (discrete parameter) random process $L_k(t)$ for positive integer k. $L_k(t)$ is to be interpreted as the length of time required for the kth point after t to occur. For this reason, $L_k(t)$ is called the kth forward recurrence time after t (compare Cox [3], p. 27 and elsewhere). To be precise, we let

$$E_n(t, x) = \bigcup_m [B^*_{m+1}(t) \cap B_{m+n}(t+x)], \quad x \geqslant 0,\ n \geqslant 1, \tag{2.1.1}$$

where B_k has been defined by (2.0.2) and * denotes complement. Clearly, $E_n(t, x)$ is the measurable set carrying the intuitive meaning "at least n of the t_j fall in $(t, t+x]$." We now take as the definition of $L_n(t)$

Definition 2.1.1. $L_0(t) = 0$ for all $\omega \in \Omega$. For $n \geqslant 1$, $L_n(t)$ is the random variable satisfying

$$\{\omega: L_n(t) \leqslant x\} = E_n(t, x), \tag{2.1.2}$$

with $L_n(t) = \infty$ on $[\Omega - \lim_{x\to\infty} E_n(t, x)]$.

The sets on the right of (2.1.2) are empty for $x \leqslant 0$ and non-decreasing in x, so the definition makes sense. The interpretation of $L_n(t)$ as an nth recurrence time becomes more persuasive when we recall that (2.1.2) implies $L_n(t) = \inf_{\omega \in E_n(t,x)} x$. The set on which $L_n(t) = \infty$ appears to be bothersome, but we shall find that the stationarity condition renders the set void.

For future reference, we give alternate expressions for $E_n(t, x)$, as these are more useful in certain derivations.

LEMMA 2.1.1. *For $x \geqslant 0$, $E_n(t, x)$ may be represented by*

$$E_n(t, x) = \{\bigcup_m [B_m(t) \cap B^*_{m+1}(t) \cap B_{m+n}(t+x)]\} \cup [(\bigcap_k B^*_k(t)) \cap (\bigcup_k B_k(t+x))], \tag{2.1.3}$$

in which $\{\,\}$ *and* $[\,]$ *are disjoint, and the indicated union in* $\{\,\}$ *is itself disjoint over m. Another expression for $E_n(t, x)$ is*

$$E_n(t, x) = \{\bigcap_m [B^*_m(t) \cup B_{m+n}(t+x)]\} \cap [\bigcup_k B_k(t+x)] \cap [\bigcup_k B^*_k(t)]. \tag{2.1.4}$$

Verification of these equalities is achieved by the usual method of showing that each side of (2.1.3) and (2.1.4) contains the other. That (2.1.3) is a disjoint union as claimed follows from $B_{m+k}(t) \cap B^*_{m+1}(t) = \emptyset$ for $k=1, 2, \ldots$. Although (2.1.3) appears to be more complicated than (2.1.1), $[\,]=\emptyset$ if $\{t_n\}$ is an s.p.p. (see Theorems 2.1.1 and 2.2.1), so that $E_n(t, x)$ is simply represented as a disjoint union. The last form of $E_n(t, x)$, (2.1.4), is of interest chiefly because it enables one to use DeMorgan's relations to obtain

$$E^*_n(t, x) = \{\bigcup_m [B_m(t) \cap B^*_{m+n}(t+x)]\} \cup [\bigcap_k B^*_k(t+x)] \cup [\bigcap_k B_k(t)]. \tag{2.1.5}$$

Let us next define sets having the intuitive significance "exactly n of the t_k occur in $(t, t+x]$". We denote these sets by $A_n(t, x)$, and define them for $n \geq 0$ by

$$A_n(t, x) = E_n(t, x) \cap E^*_{n+1}(t, x). \tag{2.1.6}$$

To complete the definition of the A_n, we adopt the convention $E_0(t, x) = \Omega$.

LEMMA 2.1.2. *The A_n are disjoint (for different indices), and each A_n can be expressed as the disjoint union*

$$A_n(t, x) = \bigcup_m [B_m(t) \cap B^*_{m+1}(t) \cap B_{m+n}(t+x) \cap B^*_{m+n+1}(t+x)]. \tag{2.1.7}$$

Proof. From (2.1.1), $E_{n+1} \subset E_n$, so $E_{n+j} \cap E^*_{n+1} = \emptyset$ for $j=1, 2, \ldots$; hence, the A_n given by (2.1.5) are disjoint as claimed. The rest of the lemma is proved by substituting (2.1.3) and (2.1.5) into (2.1.6), and eliminating intersections of disjoint terms from the multiple unions.

We have earlier spoken of a random variable $N(t, x)$ as the "number of points in $(t, t+x]$." This notion is given formal meaning by

Definition 2.1.2. $N(t, x)$ is the random variable satisfying

$$\{\omega: N(t, x) = n\} = A_n(t, x) \tag{2.1.8}$$

for $n = 0, 1, 2, \ldots$, and

$$\{\omega: N(t, x) = \infty\} = \Omega - \bigcup_0^\infty A_n(t, x). \tag{2.1.9}$$

Most of our work will deal with finite-valued $N(t, x)$, in fact with processes that guarantee $N(t, x)$ to be finite for all t, x. As we would expect, the finiteness of $N(t, x)$ is intimately related to the location of the limit points of $\{t_n\}$.

Since $\{t_n\}$ is non-decreasing, $\{t_n\}$ has exactly two limit points if we admit $\pm\infty$. The ω-set such that the upper limit point falls on $(t, t+x]$ is denoted by $C_u(t, x)$, and is specified by

$$C_u(t, x) = [\bigcup B^*_n(t)] \cap [\bigcap B_k(t+x)]. \tag{2.1.10}$$

We also define
$$C_l(t, x) = [\bigcap B_n^*(t)] \cap [\bigcup B_k(t+x)] \tag{2.1.11}$$

which is associated with the lower limit point in the sense that the set (2.1.11) contains $\{\omega: t \leqslant \lim_{n\to-\infty} t_n < t+x\}$ and is in turn contained in $\{\omega: t \leqslant \lim_{n\to-\infty} t_n \leqslant t+x\}$. Finally, the ω-set "both limit points lie in $[t, t+x]$" is the intersection of C_u and C_l, viz.,

$$C(t, x) = C_l(t, x) \cap C_u(t, x) = [\bigcap B_n^*(t)] \cap [\bigcap B_k(t+x)]. \tag{2.1.12}$$

THEOREM 2.1.1. *The following are equivalent*

(i) $N(t. x)$ *is finite valued* (*probability* 1);

(ii) $\lim_{n\to\infty} P[E_n(t, x)] = 0;$ (2.1.13)

(iii) $P[C_l(t, x)] = P[C_u(t, x)] = 0.$ (2.1.14)

If any of the above hold, (2.1.3) *simplifies to*

$$E_n(t, x) = \bigcup_m [B_m(t) \cap B_{m+1}^*(t) \cap B_{m+n}(t+x)] \tag{2.1.15}$$

which agrees with (2.1.3) *to a zero probability equivalence.*

Proof. We begin by noting that $\bigcup_0^n A_k(t, x) = E_{n+1}^*(t, x)$, which follows from (2.1.6) and $E_n^* \subset E_{n+1}^*$ by induction. Now

$$P[N(t, x) = \infty] = P[\{\bigcup_0^\infty A_n(t, x)\}^*] = P[\lim\ E_n(t, x)]; \tag{2.1.16}$$

since measures and monotone limits commute, we already have the asserted equivalence of (i) and (ii).

Next, if (ii) holds, $P[\bigcap_1^\infty E_n(t, x)] = 0$. Applying this form of (ii) to (2.1.3), and noting that each term in the disjoint union must have zero probability, we obtain

$$P\{[\bigcap B_k^*(t)] \cap [\bigcup B_n(t+x)]\} = 0 \tag{2.1.17}$$

and
$$P\{[B_m(t) \cap B_{m+1}^*(t)] \bigcap_{-\infty}^{\infty} B_n(t+x)\} = 0 \tag{2.1.18}$$

for every m. The first equation merely states that $P(C_l) = 0$. For the proof of $P(C_u) = 0$, it is sufficient to show $P(C_u - C) = 0$. To this end, we write the easily proved set relations

$$\begin{aligned} C_u(t, x) - C(t, x) &= [\bigcup_n B^*(t) - \bigcap_n B_n^*(t)] \bigcap_k B_k(t+x) \\ &= \bigcup_m \{[B_m(t) \cap B_{m+1}^*(t)] \bigcap_k B_k(t+x)\}. \end{aligned} \tag{2.1.19}$$

Hence, (2.1.18) implies $P(C_u - C) = 0$. Conversely, the validity of (iii) yields (2.1.17) from $P(C_u) = 0$ and (2.1.18) from $P(C_u - C) = 0$; this means $P[\bigcap_1^\infty E_n] = 0$, whence (ii) follows. Finally, (2.1.15) is obtained by applying (2.1.17) to (2.1.3).

The equality

$$N(s_0, s_M - s_0) = \sum_{j=0}^{M-1} N(s_j, s_{j+1} - s_j), \tag{2.1.20}$$

valid for any $s_0 \leqslant s_1 \leqslant \ldots \leqslant s_M$, will find several applications later in the paper. Although it is intuitively evident that the number of points in an interval is equal to the sum of points over its subintervals, our development demands a rigorous proof.

LEMMA 2.1.3. (2.1.20) *holds for each* $\omega \in \Omega$.

Proof. If we can show that $N(t, y) = N(t, x) + N(t + x, y - x)$, $0 \leqslant x \leqslant y$, the rest follows by finite induction. On each set $A_n(t, y)$, $n = 1, 2, \ldots$, the preceding equation may be inferred from

$$A_n(t, y) = \bigcup_{k=0}^{n} [A_k(t, x) \cap A_{n-k}(t + x, y - x)]. \tag{2.1.21}$$

It is convenient to express the A_j by the disjoint unions of Lemma 2.1.2, and to use the fact that $[\bigcup F_n] \cap [\bigcup G_n] = \bigcup [F_n \cap G_n]$ whenever $F_n \cap G_m = \emptyset$ for $m \neq n$. A separate but simpler computation is applicable to $A_0(t, y)$. Once (2.1.21) is proved for $n = 0, 1, 2, \ldots$ we may show that the sum property holds even if $N(t, y)$ is infinite. By taking unions over n in (2.1.21), and then transforming indices,

$$\bigcup_0^\infty A_n(t, y) = [\bigcup_{k=0}^\infty A_k(t, x)] \cap [\bigcup_{j=0}^\infty A_j(t + x, \; y - x)]. \tag{2.1.22}$$

If we take complements in (2.1.22), we shall have

$$\{\omega\colon N(t, y) = \infty\} = \{\omega\colon N(t, x) = \infty\} \cup \{\omega\colon N(t + x, y - x) = \infty\},$$

which extends the proof of the lemma to infinite numbers of points.

The discussion on the finiteness of $N(t, x)$ has a counterpart in that of $L_n(t)$: the set on which $L_n(t, x) = \infty$ is related to the set on which the upper limit point is less than t.

THEOREM 2.1.2. *For* $n = 1, 2, \ldots$

$$\lim_{x \to \infty} E_n(t, x) = \bigcup_m B_m^*(t). \tag{2.1.23}$$

Proof. If for some k $\omega \in B_k^*(t)$, $t < t_{k+1} \leqslant t_{k+n} < \infty$ for this ω. Then $\omega \in B_{k+1}^*(t) \cap B_{k+n}(t + x)$ for any $x \geqslant [t_{k+n}(\omega) - t]$, which is to say $\omega \in \lim_{x \to \infty} E_n(t, x)$ from (2.1.1). Conversely, $\omega \in E_n(t, x)$

for some x implies $\omega \in B_m^*(t)$ for some m, again by (2.1.1). Then $\omega \in \bigcup B_m^*(t)$, as we wished to prove.

COROLLARY 2.1.1. $\lim_{x\to\infty} E_n(t, x)$ *is the same set for each* $n=1, 2, \ldots$.

Proof. The right side of (2.1.23) is the same for each such n.

COROLLARY 2.1.2. $L_n(t)$ *is (for any* $n=0, 1, 2, \ldots$*) a finite-valued random variable iff* $P[\bigcup_m B_m^*(t)]=1$.

Proof. The result follows directly from (2.1.23) and Definition 2.1.1.

2.2. Backward recurrence times and stationarity

Although we have deduced some elementary properties of $\{t_n\}$ on the basis of Definition 2.0.1 alone, much stronger properties accrue from some sort of stationarity assumption. McFadden [10] and others [16] have introduced such a notion: a process is stationary if all multivariate distributions of $N(t_1, x_1)$, $N(t_2, x_2)$, ..., $N(t_n, x_n)$ are invariant under all shifts $t_j \to t_j + h$. This definition is both intuitively appealing and analytically fruitful. Our definition, oriented as it is toward forward recurrence statistics, seems somewhat different from the above, but will actually turn out to be equivalent.

Definition 2.2.1. The process $\{t_n\}$ is said to be *forward* [*backward*] *stationary* if for each positive [negative] integer set $k_1, k_2, \ldots, k_n$, and $x_1, x_2, \ldots, x_n$, and any h,

$$P[\bigcap_1^n E_{k_j}(t, x_j)] = P[\bigcap_1^n E_{k_j}(t+h, x_j)], \tag{2.2.1}$$

i.e., if the multivariate distribution for each set $L_{k_1}(t), L_{k_2}(t), \ldots, L_{k_n}(t)$ is invariant under translation.

In the above, backwards recurrence times $L_{-n}(t)$, $n \geqslant 1$, are consistently specified by Definition 2.1.1, extended to all n, with the understanding that

$$E_{-n}(t, x) = \bigcup_m [B_{m+1}^*(t-x) \cap B_{m+n}(t)], \quad n=1, 2, \ldots. \tag{2.2.2}$$

Thus, $L_{-n}(t)$ becomes the time interval between the nth point before t and t itself. If we combine (2.1.1) with (2.2.2) we obtain for $n \geqslant 1$

$$\{\omega: L_{-n}(t) \leqslant x\} = \{\omega: L_n(t-x) \leqslant x\}; \tag{2.2.3}$$

thus, L_n and L_{-n} have the same distribution under assumptions of forward [backward] stationarity. This property fails to extend to multivariate distributions. However, forward [backward] stationarity will be shown to imply a stronger form of stationarity, which is described by

Definition 2.2.2. The process $\{t_n\}$ is *strongly stationary* if for each integer set $k_1, k_2, \ldots, k_n$ (not necessarily of the same sign), each $t_1, t_2, \ldots, t_n$, each $x_1, x_2, \ldots, x_n$, and any h

$$P[\bigcap_1^n E_{k_j}(t_j, x_j)] = P[\bigcap_1^n E_{k_j}(t_j + h, x_j)]. \tag{2.2.4}$$

Note that Definition 2.2.1 refers only to a single point of origin t, and to k_j of the same sign; it is therefore immediate that a strongly stationary process is both forward and backward stationary. Stationarity may alternatively be defined by the statistics of numbers of points in intervals.

Definition 2.2.3. The process $\{t_n\}$ is *interval stationary* if for each set of non-negative integers $k_1, k_2, \ldots, k_n$, each sets $t_1, t_2, \ldots, t_n$ and $x_1, x_2, \ldots, x_n$ and each real h

$$P[\bigcap_1^n A_{k_j}(t_j, x_j)] = P[\bigcap_1^n A_{k_j}(t_j + h, x_j)]. \tag{2.2.5}$$

The above definition considers any arbitrary (finite) collection of finite intervals. If we restrict these to be consecutive, we have

Definition 2.2.4. The process $\{t_n\}$ is *consecutive interval stationary* if (2.2.5) is satisfied whenever

$$t_k + x_k = t_{k+1}, \quad k = 1, 2, \ldots, n-1. \tag{2.2.6}$$

Evidently, interval stationarity implies consecutive interval stationarity. We shall see, however, that each implies the other, both being necessary and sufficient conditions for strong stationarity.

To prove the results at which we have already hinted, certain preliminary work is required, viz.

THEOREM 2.2.1. *If $\{t_n\}$ is forward stationary, there is a set Λ with $P(\Lambda) = 0$ and such that*

$$C_u(t, x) \subset \Lambda \quad and \quad C_l(t, x) \subset \Lambda \tag{2.2.7}$$

for all t, x.

Remark. This theorem is of interest in its own right, since it permits the simplification of (2.1.3) to (2.1.15). Moreover, Λ is the same for all t, x, so that (without loss of generality) we may take $E_n(t, x)$ to be given by (2.1.15) for all $\omega \in \Omega$.

Proof. It suffices to show that $P[C_l(-n, 2n)]=0$ for each positive integer n, and likewise for $C_u(-n, 2n)$. For C_l this is so because by (2.1.11) $C_l(t, x) \subset C_l(-n, 2n)$ whenever $-n \leqslant t$ and $t+x \leqslant n$; then every $C_l(t, x)$ is contained in a denumerable union of sets of null probability. An identical argument applies to C_u, and the same null set Λ (union of null set for C_l and C_u) taken for both.

If $P[C_l(-n, 2n)]$ were not zero, there exists an integer m such that $P[C_l(-n, 2n)]>m^{-1}$. To show that this supposition leads to a contradiction, consider that

$$E_1^*(-n-2mn, 2mn) \cap E_1(-n-2mn, 2[m+1]n)$$

contains $C_l(-n, 2n)$; the latter is demonstrated by representing E_1 and E_1^* respectively by (2.1.3) and (2.1.5), eliminating terms having empty intersection, and comparing with (2.1.11). Hence

$$P[E_1^*(-n+2jn-2mn, 2mn) \cap E_1(-n+2jn-2mn, 2[m+1]n)]>m^{-1}, \tag{2.2.8}$$

in which the left-hand probability is the same for any j by the forward stationarity of $\{t_n\}$. Further, we argue that the sets on the left side of (2.2.8) are disjoint for $j=0, 1, \ldots, m-1$. We use the set identity to be proved as Lemma 2.3.2 (with $A_0=E_1^*$). Then we need only show that $E_1^*(-n+2kn-2mn, 2mn) \cap E_1(-n+2jn, 2n)=\emptyset$ whenever $0 \leqslant j<k \leqslant m-1$. To see that this intersection is in fact disjoint, observe that

$$E_1^*(-n+2kn-2mn, 2mn) \subset E_1^*(-n, 2kn)$$

while $E_1(-n+2jn, 2n) \subset E_1(-n, 2kn)$. The sets in (2.2.8) being disjoint as claimed, we have

$$P\{\bigcup_{j=0}^{m-1} [E_1^*(-n+2jn-2mn, 2mn) \cap E_1(-n+2jn-2mn, 2[m+1]n)\}>1; \tag{2.2.9}$$

this is the desired contradiction.

To finish the proof, consider $C_u - C$ as specified by (2.1.19). If we write

$$E(-n, 2n)=[\bigcup \{B_m(-n) \cap B_{m+1}^*(-n)\}] \bigcap B_k(n),$$

we need to prove that $P[E(-n, 2n)]=0$. Now $E(-n, 2n)$ differs from the monotone limit of $E_j(-n, 2n)$ by $C_l(-n, 2n)$, which is contained in a set of probability zero. Under the assumption of forward stationarity,

$$P[E(-n, 2n)]=\lim_j P[E_j(-n, 2n)]=\lim_j P[E_j(-n+h, 2n)]=P[E(-n+h, 2n)].$$

Furthermore, if $h \geqslant 2n$, $E(-n, 2n) \cap E(-n+h, 2n)=\emptyset$ because $E(-n, 2n) \subset B_p(n)$ while $E(-n+h, 2n) \subset B_p^*(-n+h)$ for some index p.

Let us assume that there exists an integer r such that $P[E(-n, 2n)] > r^{-1}$. From the arguments of the preceding paragraph

$$P[\bigcup_{j=0}^{r-1} E(-n+2jn, 2n)] = \sum_{j=0}^{r-1} P[E(-n+2jn, 2n)] = rP[E(-n, 2n)] > ; \qquad (2.2.10)$$

hence $P[E(-n, 2n)]$ must be zero, and the proof is complete.

COROLLARY 2.2.1. *If $\{t_n\}$ is forward stationary, $L_n(t)$ and $N(t, x)$ are finite-valued except on a set of measure zero that does not depend on n, t, or x. More precisely, let Λ be the set specified in the theorem. Then for any $x \geq 0$ and any t, $\bigcup_{k=0}^{\infty} A_k(t, x) \supset \Omega - \Lambda$; and for $n = 0, 1, 2, \ldots, \lim_{y\to\infty} E_n(t, y) \supset \Omega - \Lambda$.*

Proof. From the first statement in the proof of Theorem 2.1.1 $[\bigcup A_k(t, x)]^* = E(t, x)$ which we have just shown to be a subset of Λ.

By (2.1.23), the second statement is equivalent to $\bigcap B_m(t) \subset \Lambda$. Now

$$C_u(t-x, x) = [\bigcup B_n^*(t-x)] \cap [\bigcap B_k(t)] \subset \Lambda$$

for all t and x. Taking (monotone) limits on this expression yields

$$[\lim_{v\to-\infty} \bigcup_n B_n^*(v)] \cap [\bigcap_k B_k(t)] \subset \Lambda.$$

But $\lim_{v\to-\infty} B_0^*(v) = \Omega$ because t_0 is non-negative, whence $\bigcap_k B_k(t) \subset \Lambda$.

2.3. Equivalent stationarity conditions

In this section, we prove that each of several stationarity conditions implies the others. To render the principal theorem more transparent, we mention two set relations that are again intuitively obvious, but require some manipulations for rigorous demonstration.

LEMMA 2.3.1. *Let $0 = k_0 \leq k_1 \leq k_2 \leq \ldots \leq k_n$ be a set of integers, and $0 = x_0 \leq x_1 \leq x_2 \leq \ldots \leq x_n$ a set of reals. Define $m_j = k_j - k_{j-1}$ and $y_j = x_j - x_{j-1}$. Then*

$$\bigcap_{j=1}^{n} A_{k_j}(t, x_j) = \bigcap_{j=1}^{n} A_{m_j}(t + x_{j-1}, y_j). \qquad (2.3.1)$$

Proof. For $n = 2$, $A_{k_2}(t, x_2)$ is expressed as in (2.1.21). We then intersect both sides with $A_{k_1}(t, x_1)$ for the desired result. From what has been proved for $n = 2$, it is possible to proceed by induction to complete the verification of the lemma.

LEMMA 2.3.2.
$$A_r(t, x) \cap E_k(t+x, y) = A_r(t, x) \cap E_{k+r}(t, x+y). \tag{2.3.2}$$

Proof. For $k=0$, the result is obvious from $E_0=\Omega$ and $E_r \supset A_r$. For positive k, we verify instead the equality with E_{k+r} and E_k replaced by their complements. Now $A_r(t, x) \cap A_j(t+x, y) = A_r(t, x) \cap A_{j+r}(t, x+y)$ from the preceding lemma. If we take the union of both sides on j from zero to $k-1$ the desired equality is attained in view of $A_r(t, x) \cap A_j(t, x+y) = \emptyset$ for $j<r$.

The principal result comparing the various stationarity definitions introduced in Section 2.2 is the subject of the theorem below. By virtue of this theorem, the name "stationary point process" (hereafter abbreviated s.p.p.) can be indifferently applied to a process $\{t_n\}$ satisfying any of the conditions named.

THEOREM 2.3.1. *The following statements are equivalent*: $\{t_n\}$ *is*

(i) *forward stationary,*
(ii) *consecutive interval stationary,*
(iii) *interval stationary,*
(iv) *strongly stationary,*
(v) *backwards stationary.*

Proof. If (i) is true, it follows from the complementation of sets and corresponding probabilities that
$$P[\bigcap_1^n F_{k_j}(t, x_j)] = P[\bigcap_1^n F_{k_j}(t+h, x_j)], \tag{2.3.3}$$
where each F_i may be chosen as either E_i or E_i^*. In particular,
$$P[\bigcap_1^n E_{k_j}(t, x_j) \cap E^*_{k_j+1}(t, x_j)]$$
does not depend on t. This fact, together with (2.1.6) and Lemma 2.3.1 implies (ii). Conversely, let (ii) hold; then $P[\bigcap_1^n A_{k_j}(t, x_j)]$ does not depend on t from (2.3.1). This property is retained if we sum over each k_j from zero to m_j. By the disjointness of the A's, and because $\bigcup_0^m A_j = E^*_{m+1}$, we have that $P[\bigcap_1^n E^*_{m_j}(t, x_j)]$ does not depend on t. This means that (i) is valid, so we have shown that (ii) implies (i).

We prove (iv) from (i). From Lemma 2.3.2, $A_r(t, t_j-t) \cap E_{k_j+r}(t, t_j-t+x_j) = A_r(t, t_j-t) \cap E_{k_j}(t_j, x_j)$. By (i) and (2.3.3), the probability of the intersection of any such sets remain the same if t, t_j is replaced by $t+h, t_j+h$. Thus also
$$\sum_{=0}^{\infty} P[\bigcap_{j=1}^n A_r(t, t_j-t) \cap E_{k_j}(t, x_j)] = \sum_{r=0}^{\infty} P[\bigcap_{j=1}^n A_r(t+h, t_j-t) \cap E_{k_j}(t_j+h, x_j)]. \tag{2.3.4}$$

Because of Corollary 2.2.1 and the disjointness of the A_r, (2.3.4) yields (2.2.4), but only for positive k_j. Since, however, $E_{-k}(t, x) = E_k(t-x, x)$, (2.2.4) holds for any combination of integer k_j if only it holds for positive k_j. From (iv) we obtain (iii), using combinations of the E_{k_j} and their complements.

That (iv) implies both (i) and (v) is obvious, so the theorem is complete if we can prove any of the other conditions from (v). That (ii) follows from (v) is easily shown by methods like those used above, and we omit that proof.

3.0. Distribution functions, moments, and sample averages of the s.p.p.

In renewal theory, the distribution function of the interval between renewals determines the process, and is therefore basic to its study. Although s.p.p. could be investigated in a similar manner, it is not convenient to do so. In the first place, the interval process $\{\tau_n\}$ need not be a stationary process; secondly, the recurrence statistics do not induce interval statistics uniquely. The statistics of $\{\tau_n\}$ are likewise unpromising objects of study (cf. Section 3.5).

Our approach to s.p.p. will be through the forward recurrence times. Their distribution functions have interesting convexity and differentiability properties. The moments of $N(t, x)$ also find convenient expression in terms of recurrence time statistics, and these facilitate the calculation of global and asymptotic properties of the moments of $N(t, x)$.

We shall use the following notation, which always refers to an s.p.p.

$$G_n(x) = P[L_n(t) \leqslant x], \tag{3.0.1}$$

that is, G_n is the distribution function for the nth forward recurrence time. It will be convenient to denote the sum of the first n G_k by S_n, i.e.,

$$S_n(x) = \sum_{k=1}^{n} G_k(x). \tag{3.0.2}$$

Particular derivatives (suitably chosen from the equivalence class of Radon–Nikodym derivates) of S_n and G_n will be denoted by s_n and g_n, respectively.

3.1. Convexity and absolute continuity

There are many properties of S_n (and, a fortiori, G_1) that depend only on the assumption that $\{t_n\}$ is an s.p.p. These follow from the fact that S_n is concave (upward convex).

LEMMA 3.1.1. *For each n, S_n is a concave function on* $[0, \infty)$.

Proof. Consider the set equality

$$E_m(t, x+h) - E_m(t, x) = \bigcup_{k=0}^{m-1} \{[A_k(t, x)] \cap [E_{m-k}(t+x, h)]\}, \quad h > 0. \tag{3.1.1}$$

Since the sets in the indicated union are disjoint, we obtain the probability on the right as a sum. If we then sum over m, and interchange (finite) summations, we have

$$S_n(x+h) - S_n(x) = \sum_{k=1}^{n} \sum_{j=k}^{n} P\{[A_{j-k}(t, x)] \cap [E_k(t+x, h)]\}. \tag{3.1.2}$$

But for $z \geqslant y \geqslant 0$, there is the set inequality

$$\bigcup_{j=k}^{n} [A_{j-k}(t, z) \cap E_k(t+z, h)] \subset \bigcup_{j=k}^{n} [A_{j-k}(t+(z-y), y) \cap E_k(t+z, h)] \tag{3.1.3}$$

from which (by stationarity and because the A's are disjoint)

$$\sum_{j=k}^{n} P[A_{j-k}(t, z) \cap E_k(t+z, h)] \leqslant \sum_{j=k}^{n} P[A_{j-k}(t, y) \cap E_k(t+y, h)]. \tag{3.1.4}$$

If we sum (3.1.4) over k as in (3.1.2), the result is $S_n(z+h) - S_n(z) \leqslant S_n(y+h) - S_n(y)$. This corresponds to the usual notion of concavity; take $y = x_1$, $z = (x_1 + x_2)/2$, $h = (x_2 - x_1)/2$ with $0 \leqslant x_1 \leqslant x_2$.

Thus, S_n is for each n a monotone bounded concave function on $[0, \infty)$, and its properties are precisely those derived in Section 3.18 of Hardy, Littlewood, Polya [7]. In particular, the right derivate $s_n^+(x) = \lim_{h \to 0+} [S_n(x+h) - S_n(x)/h]$ exists for all $x > 0$, and if $0 < x < y < z$, $|S_n(z) - S_n(y)| \leqslant s_n^+(x)|z-y|$. That S_n meets this Lipschitz condition implies that S_n is absolutely continuous on any interval $[\delta, \infty)$, $\delta > 0$, whence

$$S_n(x) = S_n(\delta) + \int_\delta^x s_n(u)\, du \tag{3.1.5}$$

in which s_n is any Radon–Nikodym derivative of S_n. But $s_n = s_n^+$ almost everywhere, and we shall always mean s_n^+ when we speak of s_n as the derivative of S_n. Then also s_n is monotone non-increasing.

We may take $\delta \to 0+$ in (3.1.5), obtaining (since S_n is continuous from the right)

THEOREM 3.1.1. *S_n is absolutely continuous on any $[\delta, \infty)$, $\delta > 0$, and its derivative s_n may be taken to be monotone non-increasing. Further*

$$S_n(x) = S_n(0+) + \int_0^x s_n(u)\, du. \tag{3.1.6}$$

Although $s_n(0+)$ need not be finite, $s_n(x) < \infty$ for any $x > 0$. Then $G_n = S_n - S_{n-1}$, $n \geqslant 2$, is absolutely continuous, and we may take $g_n = s_n - s_{n-1}$ for $x > 0$. This formulation assures that $g_n \geqslant 0$, and that $g_n(0+)$ makes sense whenever $s_n(0+) < \infty$. For future use, we state a relation between sums of $g_n(0+)$ and derivates of the G_n and S_n, viz.

LEMMA 3.1.2. *Let $s_n(0+) < \infty$ for each n. Then $g_n(0+) = \lim_{h \to 0+} [G_n(h) - G_n(0+)/h]$ is well-defined and finite, and*

$$\sum_1^\infty g_n(0+) = \lim_{h \to 0+} [\sum_1^\infty g_n(h)] = \lim_{h \to 0+} \sum_1^\infty [G_n(h) - G_n(0)/h], \tag{3.1.7}$$

where the limits are not required to be finite.

Proof. From Section 3.18 of [7], $[S_n(x+h) - S_n(x)]/h$ is non-decreasing in x, h, and n as $x \searrow$, $h \searrow$, $n \nearrow$. Limits of these variables can then be freely interchanged. Moreover, the limits are finite for each n, so that these quotients and their limits can be expressed in terms of partial sums of G_n and g_n.

Remark. If $\lim_{x \to 0+} \Sigma[G_n(x)/x] = \beta$, $\beta < \infty$, $S_n(0+) = 0$ and $G_n(0+) = 0$ for each n. By an argument again based on interchanging monotone limits, $s_n(0+) \leqslant \beta$ for all n. Then from the lemma, $\lim_{h \to 0+} \Sigma g_n(h) = \Sigma g_n(0+) = \beta$.

Added in proof: It was left unsettled whether $G_1(0+) > 0$ is possible. The answer is negative, and a fortiori, $S_n(0+) = 0$ and $G_n(0+) = 0$ for each n. Consequently, Theorem 3.1.1 and Lemma 3.1.2 (as well as the remark immediately following) may be simplified by the omission of these terms, and Corollary 3.3.1 becomes vacuous.

We show by contradiction that $G_1(0+) = \alpha > 0$ is impossible. Indeed, $G_1(0+) = \alpha$ implies $P[E_1(t, x)] \geqslant \alpha$ for each $x > 0$, and hence $P[\lim_{x \to 0+} E_1(t, x)] \geqslant \alpha$ because measures are continuous from above ([6], p. 39). Now $E_1(t, n^{-1}) \supset \lim_{x \to 0+} E_1(t, x)$ for each positive integer n, so that $P[\bigcap_{n=1}^\infty E_1(t, n^{-1})] \geqslant \alpha$. On the other hand, the limit point properties of the realizations $\{t_n(\omega)\}$ of the s.p.p. permit us to conclude that $\bigcap_{n=1}^\infty E_1(t, n^{-1}) \subset C_l(t, x)$. But $P[C_l(t, x)] = 0$ by Theorem 2.2.1, and the desired contradiction is attained.

3.2. Existence and global properties of moments

The number of points in time intervals is of equal interest with recurrence times, and should receive equal attention. The G_n introduced earlier turn out to provide the ideal tools for the study of $N(t, x)$ also. We call $p(n, x) = P[A_n(t, x)]$; this is simply the probability that n occurrences fall in the interval $(t, t+x]$. It is then easy to deduce from the earlier set identities that $G_n(x) = \Sigma_{k=n}^\infty p(k, x)$ and the equivalent expression $p(n, x) = G_n(x) - G_{n+1}(x)$. Here all probabilities are zero for $x < 0$, and $G_0(x)$ is interpreted as unity.

Our first theorem relates local moment properties of $N(t, x)$ to global ones.

THEOREM 3.2.1. *Suppose $E\{[N(t,h)]^k\}$ is finite for fixed $k\geqslant 1$ and some $h>0$. Then $E\{[N(t,x)]^k\}$ is finite for all positive x, and in fact*

$$E\{[N(t,x)]^k\}=O(x^k) \text{ as } x\to\infty.$$

Proof. We may suppose from the hypothesis that $E\{[N(t,h)]^k\}=M<\infty$. First, we recall that $N(t,x)$ is non-decreasing in x for every ω whence $E\{[N(t,x)]^k\}\leqslant E\{[N(t,mh)]^k\}$ where we choose $m=[x/h]+1$. Next, we write the identity $N(t,mh)=\sum_0^{m-1}N(t+kh,h)$ from Lemma 2.1.3. Combining these, and using the Minkowski inequality and stationarity, we obtain $(E\{[N(t,x)]^k\})^{1/k}\leqslant m(E\{[N(t,h)]^k\})^{1/k}$. We now take both sides to the k power, and note that $m^k=O(x^k)$.

The next result characterizes finiteness of moments in terms of forward recurrence distributions, and even provides an explicit evaluation.

THEOREM 3.2.2. *For each $k\geqslant 1$,*

(i) $E\{[N(t,x)]^k\}$

(ii) $\sum_n n^k[G_n(x)-G_{n+1}(x)]$

(iii) $\sum_n [n^k-(n-1)^k]\,G_n(x)$

are equal, whether finite or infinite.

Proof. $E\{[N(t,x)]^k\}=\Sigma n^k p(n,x)$, so that (ii) and the series for (i) are equal term-by-term. Moreover, these series (and all others of the theorem) are composed of non-negative summands, so that they converge, if only to infinity.

To relate (ii) and (iii), consider their respective partial sums U_n and W_n. For these

$$W_n=n^kG_{n+1}+U_n; \tag{3.2.1}$$

since $U_n\leqslant W_n$, the proof is completed by showing that

$$\lim_{n\to\infty} n^k G_{n+1}(x)=0 \tag{3.2.2}$$

follows from the finiteness of (ii). To this end, we use the identity

$$n^k[G_n-G_{n+p+1}]+\{\sum_{n+1}^{n+p}[j^k-(j-1)^k]\,G_j-[(n+p)^k-n^k]\,G_{n+p+1}\}=U_{n+p}-U_n. \tag{3.2.3}$$

The term in braces is non-negative because $G_{n+p+1}\leqslant G_j$ for $j\leqslant n+p$. Because U_n converges, there is for each $\delta>0$ an n_0 (not dependent on p) such that $n>n_0$ implies $0\leqslant n^k[G_n(x)-G_{n+p+1}(x)]<\delta$. Then (3.2.2) follows by taking $p\to\infty$, provided that $G_p\to 0$. But the latter is precisely (2.1.13), which is a consequence of stationarity (cf. Theorem 2.2.1).

A result of the above theorem is that $n^k G_n \to 0$ with n if the kth moment is to be finite. However this condition is not sufficient, as is seen from the fact that $\Sigma n^{k-1} G_n < \infty$ is both necessary and sufficient. It is also easy to show that moments of all orders exist iff, for some $x > 0$, $n^k G_n(x)$ is bounded for each $k = 1, 2, \ldots$ as $n \to \infty$. Finally, we note that each term of (ii) and (iii) is increasing in x, so that convergence for any $x > 0$ implies uniform convergence on every $[0, x_0]$. The latter also shows that any existent moment is continuous in x, but we shall obtain stronger properties for these moments later.

3.3. First and second moments

It is instructive to consider certain relations between the lower moments and the G_n. Indeed, the existence of the first moment already implies that $G_n(0+) = 0$ for all n, and that the g_n are bounded and have a convergent sum. Knowing this, one returns to the higher moments and elicits properties of more general type.

The first theorem of this section specifies completely the functional form of $E[N(t, x)]$.

THEOREM 3.3.1. *If $N(t, x)$ has a finite first moment,*

$$E[N(t, x)] = \beta x \tag{3.3.1}$$

for all $x \geq 0$, β being some non-negative constant.

Proof. From stationarity, $E[N(t, x)]$ is a function only of x, say $f(x)$. Now take $x, y \geq 0$, and write $N(t, x+y) = N(t, x) + N(t+x, y)$ which follows from Lemma 2.1.3. Taking the expectation of both sides of (3.3.2) yields the functional equation $f(x+y) = f(x) + f(y)$. Since f is bounded on any subset of the positive real axis, there is only one solution (cf. [7], p. 96), namely $f(x) = \beta x$. Here β is non-negative because $N(t, x)$ is non-negative for every $\omega \in \Omega$.

The theorem which follows asserts another condition from which the existence of the first moment of $N(t, x)$ can be deduced, and the parameter β calculated.

THEOREM 3.3.2. *The conditions*

$$E[N(t, x)] < \infty \tag{3.3.2}$$

and

$$\lim_{x \to 0+} \sum_1^\infty [G_n(x)/x] < \infty \tag{3.3.3}$$

each imply the other. In either event, $G_n(0+) = 0$, $n = 1, 2, \ldots,$ and

$$\sum_1^\infty g_n(0+) = \beta, \tag{3.3.4}$$

where β is the parameter appearing in (3.3.1).

Proof. If we combine Theorem 3.2.2 (iii) with (3.3.1) we have for each $x>0$ that $\Sigma[G_n(x)/x]=\beta$; the conclusions of the theorem then follow from Lemma 3.1.2 and the subsequent remark.

Conversely, let (3.3.3) be equal to $\alpha<\infty$. An interchange of (monotone) limits shows that $G_n(0+)=0$, and that (again by the concavity of S_n and an interchange of monotone limits) $\Sigma[G_n(x)/x]\leqslant\alpha<\infty$ for all $x>0$. Therefore, $\Sigma G_n(x)<\infty$, and so (3.3.2) is applicable by virtue of Theorem 3.2.2 (iii). The proof is completed by retracing the steps of the preceding paragraph.

The intrinsic interest of the above theorem is enhanced by one of its implications, which is useful in proving the absolute continuity of any existent moments (Theorem 3.3.3).

COROLLARY 3.3.1. *Under the conditions of the theorem, the S_n and G_n are absolutely continuous, and in fact*

$$S_n(x)=\int_0^x s_n(u)\,du. \tag{3.3.5}$$

Proof. Since $S_n(0+)=0$ for each n, (3.1.6) becomes (3.3.5).

COROLLARY 3.3.2. *Let $N(t, x)$ have a finite first moment. Then for every $x>0$*

$$\sum g_n(x)=\beta. \tag{3.3.6}$$

Proof. $\Sigma g_n=\lim_{n\to\infty}s_n$ is non-increasing in x, and consequently from (3.3.4) $\Sigma g_n(x)\leqslant \Sigma g_n(0+)=\beta$. Suppose now that at some x_0 we have $\Sigma g_n(x_0)=\alpha$ where $\alpha<\beta$. Then $\Sigma g_n(x)\leqslant\alpha$ for all $x\geqslant x_0$. But for any such x

$$\sum G_n(x)=\sum\int_0^x g_n(u)\,du=\int_0^x[\sum g_n(u)]\,du<\beta x, \tag{3.3.7}$$

the interchange of summation and integration being legitimate by the bounded convergence theorem. As (3.3.7) contradicts $\Sigma G_n(x)=\beta x$, the corollary is proved.

THEOREM 3.3.3. *If $E\{[N(t,x)]^k\}<\infty$ for some $x>0$, the indicated expectation is an absolutely continuous function of x over the reals.*

Proof. From Theorem 3.2.1 the kth moment exists for all x and we may write this moment as in Theorem 3.2.2. (iii). Using (3.3.5), together with $G_n=S_n-S_{n-1}$ and $g_n=s_n-s_{n-1}$, puts the kth moment into the form

$$E\{[N[t,x)]^k\}=\sum_n\int_0^x[n^k-(n-1)^k]\,g_n(u)\,du. \tag{3.3.8}$$

Each term of the integrand is non-negative, so that interchange of summation and integration is admissible, i.e.,

$$E\{[N(t,x)]^k\} = \int_0^x \sum_n [n^k - (n-1)^k]\, g_n(u)\, du. \tag{3.3.9}$$

Since the integral of the non-negative sum is finite for every x, the desired conclusion follows.

COROLLARY 3.3.3. $\Sigma[n^k-(n-1)^k]g_n(x)$ *converges absolutely for almost all* x.

Proof. Because of (3.3.9) the sum cannot be infinite on a set of positive measure. Since each term of the sum is non-negative, convergence is absolute.

3.4. Interval statistics and the computation of moments

In renewal theory, the starting point is the common distribution of the (mutually independent) intervals. We, on the other hand, have obtained our results from recurrence distributions G_n; it would indeed be difficult to use τ_n statistics, since these need be neither stationary nor independent.

This section treats the relationships between interval and forward occurrence statistics. The F_n, defined by (3.4.1) in terms of the s_n, correspond to distributions of n successive intervals, given that any one of the t_n occurs at the initial time. For equilibrium renewal processes, precisely the same equation relates forward recurrence times to interval distributions (Cox [3], p. 33). Moreover, Khinchin ([3], Chapter 3) has obtained functions like our F_n for onesided s.p.p. whose intervals need be neither independent nor identically distributed. His results in relating the F_n to conditional distributions are roughly equivalent to ours, and his methods more similar than those of the other authors cited [3], [10]. McFadden's work [10] should also be mentioned, although his interpretation of the F_n as unconditioned distributions is valid only in a highly restricted context.

The F_n which appear below will be defined as functions of s_n. Only later, after we explore the moment properties of $N(t, x)$ relative to the F_n, will we interpret the significance of the F_n as conditional interval distributions.

Let $\{F_n\}$, $n=0, 1, \ldots$ be functions on the reals, with $F_n(x)=0$ for $x<0$, and

$$F_n(x) = 1-\beta^{-1}s_n(x) \quad \text{for} \quad x \geqslant 0. \tag{3.4.1}$$

Here s_0 is taken to be zero (as usual). We assume throughout this section that $E[N(t,x)]<\infty$, so that β is well-defined as in Section 3.3. Since the s_n are each non-increasing, the F_n

are non-decreasing. Furthermore, the s_n are integrable ($\int s_n = n$), so that $s_n(x) \to 0$ as $x \to \infty$. Therefore, $\lim_{x\to\infty} F_n(x) = 1$ (all n), and the F_n are distribution functions. We also have $F_n(x) \leqslant F_m(x)$ for $n \geqslant m$ and $\lim_{n\to\infty} F_n(x) = 0$ (all x); the former is true because $s_n \nearrow$ with n, and the latter by Corollary 3.3.2.

In the theory of renewal processes, $H(x) = \Sigma F_n(x)$ is called the renewal function, where F_n is the distribution for the length of n adjacent intervals. For the equilibrium renewal process, not only the first, but also the second moment of $N(t, x)$ appears in terms of H (cf. Cox [3], eq. 4.5.6). Precisely the same formula applies to the s.p.p.; this assertion is a special case of

THEOREM 3.4.1. *Let $k > 1$ be an integer. If either*

$$\text{(i)}\quad E\{[N(t, x)]^k\} < \infty \tag{3.4.2}$$

or

$$\text{(ii)}\quad \sum_1^\infty [(n+1)^k - 2n^k + (n-1)^k]\, F_n(x) < \infty \tag{3.4.3}$$

both are finite. Moreover, $\lim_{n\to\infty} n^{k-1} F_n(x) = 0$ *and*

$$E\{[N(t, x)]^k\} = \beta\left\{x + \int_0^x \sum_{n=1}^\infty [(n+1)^k - 2n^k + (n-1)^k]\, F_n(u)\, du\right\}. \tag{3.4.4}$$

The derivative $d(E\{[N(t, x)]^k\})/dx$ is almost everywhere equal to a non-decreasing function, and finally

$$E\{[N(t, x)]^k\} = O(x) \quad \text{as } x \to 0. \tag{3.4.5}$$

Part of the proof of the theorem hinges upon the following lemma, which follows from an argument already used to prove Theorem 3.2.2.

LEMMA 3.4.1. *Let $\{a_n\}$, $\{b_n\}$ be sequences of non-negative numbers with $a_n \nearrow$ and $b_n \searrow 0$. Then*

$$\sum_1^\infty (b_j - b_{j+1})\, a_j = a_0 b_1 + \sum_1^\infty (a_j - a_{j-1})\, b_j \tag{3.4.6}$$

even if the values in (3.4.6) *are not finite. If either side is finite, we also have* $\lim_{n\to\infty} a_n b_n = 0$.

Proof. In view of (i), $E\{[N(t, x)]^k\}$ is furnished by (3.3.9). Substituting $g_n = \beta[F_{n-1} - F_n]$ from (3.4.1) into this equation, and employing Lemma 3.4.1 with b_n as F_{n-1}, $a_0 = 0$, $a_n = n^k - (n-1)^k$ for $n \geqslant 1$, yields (3.4.4) as well as $\lim_{n\to\infty} n^{k-1} F_n = 0$. Next, we deduce from

$$\sum_1^\infty [n^k - (n-1)^k]\, g_n(x) = \beta\{1 + \sum_1^\infty [(n+1)^k - 2n^k + (n-1)^k] F_n(x)\} \tag{3.4.7}$$

and Corollary 3.3.2 that (3.4.3) holds a.e. (almost everywhere). But in fact, the summands on the right of (3.4.7) are non-negative and non-decreasing in x; hence (3.4.3) is everywhere true, and (3.4.7) is non-decreasing, being bounded on each interval $[0, x_0]$. The latter property implies (3.4.5), and we also obtain as one version of $d(E\{[N(t, x)]^k\})/dx$

$$\frac{d}{dx}(E\{[N(t, x)]^k\}) = \beta\{1 + \sum_1^{\infty}[(n+1)^k - 2n^k + (n-1)^k] F_n(x)\} \tag{3.4.8}$$

which is non-decreasing as required.

To complete the proof, we show that (ii) implies (i). Since the finiteness of (ii) for some $x > 0$ requires its boundedness on $[0, x]$, the left side of (3.4.7) will likewise be bounded on the same interval. From this fact, together with the monotone convergence theorem, we have

$$\sum_{n=1}^{\infty}[n^k - (n-1)^k]\, G_n(x) = \int_0^x \sum_{n=1}^{\infty}[n^k - (n-1)^k]\, g_n(u)\, du < \infty. \tag{3.4.9}$$

An application of Theorem 3.2.2 (iii) then verifies (3.4.2).

Having examined moment properties of the F_n, we now turn to the meaning of these distribution functions themselves. In order to provide an intuitively appealing result, we shall make an assumption that means roughly "the probability that there are two or more points in a sufficiently small interval is negligible when compared with the probability of one (or no) point".

LEMMA 3.4.2. *The following are equivalent:*

$$P[E_1(t, h) - A_1(t, h)] = o(h) \quad as \ h \to 0+ \tag{3.4.10}$$

$$P[E_2(t, h)] = o(h) \qquad as \ h \to 0+ \tag{3.4.11}$$

$$\lim_{h \to 0+}[G_n(h)/h] = 0 \quad n = 2, 3, \dots \tag{3.4.12}$$

$$\lim_{h \to 0+}[G_1(h)/h] = \beta. \tag{3.4.13}$$

Proof. Since $E_1 - A_1 = E_2$, (3.4.11) is a consequence of (3.4.10), and conversely. In turn, (3.4.11) and (3.4.12) are the same for $n = 2$; for $n \geqslant 3$, (3.4.12) is true by virtue of $G_{n+1} \leqslant G_n$. Let (3.4.12) hold, and consider $\lim_{h \to 0+} \Sigma[G_n(h)/h] = \beta$, which then becomes (3.4.13) by the interchange of monotone limits. On the other hand, in the event (3.4.13) is true, the interchange yields (3.4.12) because each summand $\lim_{h \to 0+}[G_n(h)/h]$ is non-negative.

The hypothesis of the lemma will be assumed for the remainder of this section. Consider next the set equality

$$E_{-1}(t,\delta)\cap E_n^*(t,x)=\bigcup_{k=1}^{n}E_{-k}(t+x,x+\delta)\cap A_{k-1}(t,x) \tag{3.4.14}$$

in which the right side consists of disjoint sets. If we define

$$G_{n,-1}(x,u)=P[E_n(t,x)\cap E_{-1}(t,u)] \tag{3.4.15}$$

we shall have from (3.4.14)

$$G_1(\delta)-G_{n,-1}(x,\delta)=\sum_{k=1}^{n}\{P[E_k(t-\delta,x+\delta)]-P[E_k(t-\delta,x+\delta)\cap\{E_{k-1}^*(t,x)\cup E_k(t,x)\}]\}. \tag{3.4.16}$$

For $n=1$, the second probability on the right side becomes merely $G_1(x)$, and so $G_1(\delta)-G_{n,-1}(x,\delta)=G_1(x+\delta)-G_1(x)$. If $n\geqslant 2$, $E_0^*=\emptyset$ and $E_k(t,x)\subset E_k(t-\delta,x+\delta)$ permit this term to be evaluated as

$$\sum_{k=2}^{n}P[E_k(t-\delta,x+\delta)\cap E_{k-1}^*(t,x)]+S_n(x). \tag{3.4.17}$$

Now each intersection term is dominated by $E_2(t-\delta,\delta)$ [use $E_{k-1}^*=\bigcup_0^{k-2}A_j$ and Lemma 2.3.2], so that the sum in (3.4.17) can be no greater than $(n-1)G_2(\delta)=o(\delta)$. Thus the final form of (3.4.16) is

$$G_1(\delta)-G_{n,-1}(x,\delta)=S_n(x+\delta)-S_n(x)+o(\delta), \tag{3.4.18}$$

which holds also for $n=1$ with $o(\delta)\equiv 0$. If we divide by δ and take $\delta\to 0+$ in (3.4.18), the limits exist for every term except possibly $G_{n,-1}$. Then the latter limit must be well-defined also, and therefore

$$F_n(x)=\beta^{-1}\{\lim_{h\to 0+}[G_{n,-1}(x,h)/h]\}, \tag{3.4.19}$$

with the aid of (3.4.13) and (3.4.1). By using inequalities similar to those of Lemma 3.1.1, we verify that $G_{n,-1}(x,u)$ is concave in u for each fixed x. Indeed, the arguments of Section 3.1 are applicable to $G_{n,-1}$ without change, and so $\partial G_{n,-1}(x,u)/\partial u$ may be defined (for each fixed x) as a non-decreasing function with $u\searrow$, $u>0$. Moreover, monotonicity of the limiting operation shows as in Lemma 3.1.2 that

$$\left[\frac{\partial G_{n,-1}(x,u)}{\partial u}\right]_{u=0+}=\lim_{h\to 0+}\left\{\left[\frac{G_{n,-1}(x,h)}{h}\right]\right\}. \tag{3.4.20}$$

Here we note that $G_{n,-1}(x,0+)=0$, as is deduced from $G_{n,-1}(x,u)\leqslant G_1(u)$.

An intuitive meaning may be abstracted from (3.4.19). We use the fact $g_1(h)=\beta h+o(h)$ to obtain the rigorous result

$$F_n(x)=\lim_{h\to 0+}[G_{n,-1}(x,h)/g_1(h)] \tag{3.4.21}$$

having the intuitive meaning "probability of at least n points in $(t, t+x]$, given that there was a point at t". This theme is carried forward by relating it to the corresponding conditional probability distribution in the wide sense (cf. Doob [16], Section 1.9). To this end, let $\mathcal{B}$ be the Borel field generated by the random variables $L_{-1}(t)$, with t fixed. The conditional probability $P[E_n(t, x)\,|\,\mathcal{B}]$ is constant (up to a probability 1 equivalence) on the elementary sets of $\mathcal{B}$, that is on the sets of constancy $L_{-1}(t)=a$. On each such set, then, we can take the conditional probability of $E_n(t, x)$ in the wide sense relative to $\mathcal{B}$ as

$$P[E_n(t, x)\,|\,L_{-1}(t)=a]=\frac{\left.\dfrac{\partial G_{n,-1}(x, u)}{\partial u}\right|_{u=a}}{g_1(a)}. \tag{3.4.22}$$

We have already pointed out that the derivative on the right is well-defined. The other requirements for a conditional distribution can be verified directly, but we shall not do this. Such conditional distributions constitute an equivalence class (up to zero probability sets), but (3.4.22) is in a sense the "natural" definition, which we may suppose represents the conditional distribution for all $a \geqslant 0$. The interpretation attached to (3.4.21) is then actually valid in the sense that $F_n(x)=P[E_n(t, x)\,|\,L_{-1}(t)=0+]$. More generally, we may develop an alternative expression to (3.4.22) as follows: In (3.4.18) with $n=1$, let δ be successively $u+h$ and u, subtract the second from the first, divide the equation by h, and take the limit $h \to 0+$. Substituting the resulting form of $\partial G_{1,-1}(x, u)/\partial u$ in (3.4.22) yields(1)

$$P[E_1(t, x)\,|\,L_{-1}(t)=a]=1-[g_1(x+a)/g_1(a)]; \tag{3.4.23}$$

here again the right side becomes $F_1(x)$ if $a=0+$, and $n=1$ [recall $g_1(0+)=\beta$]. The preceding discussion is summarized in

THEOREM 3.4.2. *The F_n are related to the recurrence times through equation* (3.4.19). *The derivative of $G_{n,-1}(x, \cdot)$ exists and* (3.4.20) *permits the F_n to be expressed in terms of this derivative. Moreover, F_n is one version of the conditional probability distribution of E_n in the wide sense relative to $\mathcal{B}$, and evaluated on the set of constancy $L_{-1}(t)=0+$. For the case $n=1$, this conditional probability is also given by* (3.4.23).

3.5. Distribution of the t_n

It would be convenient if the recurrence and (unconditioned) interval statistics were directly related, so that one set of statistics could be calculated from the other. However,

(1) We had conjectured that $\partial G_{n,-1}(x, u)/\partial u = g_1(u) - s_n(x+u)$, whence $P[E_n(t, x)\,|\,L_{-1}(t)=a] = 1-[s_n(s+a)/g_1(a)]$. Mr. P. M. Lee showed by means of an example that the conjecture is false.

the probability structure of the t_n—or even their univariate distributions—cannot be uniquely inferred from that of the recurrence statistics. Consequently, we shall have little to say on the distribution of the t_n, although we can determine their general character. The first result along these lines is

THEOREM 3.5.1. *If $E[N(t, x)] < \infty$, the probability distribution functions of the t_n are uniformly absolutely continuous.*

Proof. We show that H_n, the distribution function of t_n, meets a uniform Lipschitz condition. In fact, $\{\omega: u < t_n \leqslant v\} = B_n^*(u) \cap B_n(v) \subset E_1(u, v-u)$ by (2.1.1). Therefore, since $G_1(x) \leqslant \beta x$ for $x \geqslant 0$

$$0 \leqslant H_n(v) - H_n(u) \leqslant G_1(v-u) \leqslant \beta(v-u). \tag{3.5.1}$$

The next theorem demonstrates the essential lack of uniqueness of $\{t_n\}$ for specified recurrence statistics.

THEOREM 3.5.2. *Let $\{t_n\}$ be a stationary point process. Then there exists another point process $\{t_n'\}$ with the same statistics on $\{L_n(t)\}$ (and hence also stationary).*

Proof. For the new s.p.p., we define

$$\tau_n' = \tau_n + \delta_{0n} T, \tag{3.5.2}$$

in which δ refers to the Kronecker delta, and T is any finite-valued random variable independent of $\{\tau_n\}$. To be precise, take $(\Omega^1, \mathfrak{F}^1, P^1)$ to be the probability space on which $\{\tau_n\}$ (and thus also $\{t_n\}$) is defined, and similarly $(\Omega^2, \mathfrak{F}^2, P^2)$ as the probability space for T. For the new process, specify a probability space $(\Omega, \mathfrak{F}, P)$, in which $\Omega = \Omega^1 \times \Omega^2$, $\mathfrak{F}$ is the σ-algebra which is the completion of the extension of $\mathfrak{F}^1 \times \mathfrak{F}^2$, and P is the extension of the product of P^1 and P^2 to $\mathfrak{F}$; see Halmos [9], Section 49.

A consequence of (3.5.2) is that

$$B_n'(t) = B_n(t-T), \tag{3.5.3}$$

where B_n' is defined in an obvious manner, and B_n is a set in the product space that resumes its previous connotation for every section determined by a $\omega^2 \in \Omega^2$. Because $E_n[E_n']$ is an intersection of $B_n[B_n']$ sets, the relation (3.5.3) implies $E_n'(t, x) = E_n(t-T, x)$ in the same sense as (3.5.3). We have therefore

$$P[\bigcap_1^n E_{k_j}'(t_j, x_j)] = P^2\{P^1[\bigcap_1^n E_{k_j}(t_j - T, x_j)]\}. \tag{3.5.4}$$

in which Fubini's theorem provides assurance that the integration can be undertaken in the order indicated on the right–hand side. But for each $\omega^2 \in \Omega^2$ (always excepting a set of zero product measure)

$$P^1[\bigcap_1^n E_{k_j}(t_j - T, x_j)] = P^1[\bigcap_1^n E_{k_j}(t_j, x_j)] \tag{3.5.5}$$

by the stationarity of $\{t_n\}$. The right side of (3.5.5) is no longer a function of ω^2, so that it remains unaffected by the application of P^2. Thus we have shown that $P[\bigcap_1^n E'_{k_j}(t_j, x_j)] = P^1[\bigcap_1^n E_{k_j}(t_j, x_j)]$, which is the conclusion of the theorem.

In addition to the theorems of this section, there are a number of negative results that deserve mention. For instance, the multivariate distributions of the $L_n(t)$ fail to determine (uniquely) those of the τ_n. If it is assumed that the τ_n are mutually independent, it need not follow that they are also identically distributed (cf. Section 4.2). Alternatively, the τ_n, $n \neq 0$, may be identically distributed without being independent (cf. Section 4.5). Finally, we note that the phenomenon asserted by Theorem 3.5.2 may occur in cases where $\{t_n\}$ and $\{t'_n\}$ differ by other than a simple shift (cf. Section 4.3).

3.6. An ergodic theorem

In renewal theory, an argument based on the strong law of large numbers implies that $\lim_{x\to\infty}[N(t, x)/x] = \beta$, except on a set of zero probability independent of t. The same need not be true for an s.p.p. with finite mean; in fact, the limit of $N(t, x)/x$ may well be a random variable, which means that the field of invariant sets generated by the shift transformation is non-trivial (see Section 4.3 for examples). As would be expected, there is an ergodic theorem that takes into account the circumstances just described.

THEOREM 3.6.1. *Let $E[N(t, x)] < \infty$, and let $\mathcal{B}$ be the field of invariant sets under the transformation T, defined by $T^k[N(0, 1)] = N(k, 1)$. If $E[N(0, 1) \mid \mathcal{B}]$ is any (fixed) version of the indicated conditional expectation, there is an ω-set Λ (depending on neither a nor t), with $P(\Lambda) = 0$, such that when $a \geq 0$,*

$$\lim_{x\to\infty} [N[t, ax)/x] = aE[N[0, 1) \mid \mathcal{B}], \quad \text{all } \omega \notin \Lambda. \tag{3.6.1}$$

Proof. By Theorem 2.3.1 (ii), $N(n, 1)$, $n = 0, 1, 2, \ldots$, is a stationary process whose mean (by assumption) is finite. The Birkoff ergodic theorem (cf. Doob [4]) then states that there is a set Λ, $P(\Lambda) = 0$, such that

$$N(0, n)/n = n^{-1} \sum_{j=1}^{n} N(j-1, 1) \to E[N(0, 1) \mid \mathcal{B}] \tag{3.6.2}$$

whenever $\omega \notin \Lambda$. To show that the same result applies to $N(t, n)/n$, $t \geq 0$, consider $N(t, n) =$

$N(0, n) - N(0, t) + N(n, t)$, which follows from Lemma 2.1.3 if $N(0, t) < \infty$. The latter is true except on a set of zero probability that does not depend on t (Corollary 2.2.1); hence we even have $[N(0, t)/n] \to 0$ as $n \to \infty$, except on this fixed set. Moreover, $[N(n, t)/n] \to 0$ as $n \to \infty$ except on a null probability set that applies to all t. To prove this claim, it suffices (on account of the monotonicity of N for all ω—see the remark following definition 2.0.1) to show

$$P[\{\overline{\lim_{n\to\infty}}\, N(n, k)/n\} \geqslant m^{-1}] = 0 \tag{3.6.3}$$

for each positive integer m and k. To this end, define

$$C(n, m, k) = \{\omega: N(n, k) \geqslant m^{-1} n\}. \tag{3.6.4}$$

$C(n, m, k) \subset \{\omega: L_{[nm^{-1}]}(n) \leqslant k\}$, so that from stationarity and Theorem 3.2.2 (iii)

$$\sum_n P[C(n, m, k)] \leqslant m \sum_n G_n(k) < \infty \tag{3.6.5}$$

whence an application of the Borel–Cantelli lemma (cf. Doob [4]) verifies (3.6.3). For $t < 0$, the proof that

$$\lim_{n\to\infty} [N(t, n)/n] = E[N(0, 1) \,|\, \mathcal{B}], \quad \omega \notin \Lambda, \tag{3.6.6}$$

is entirely analogous, and will be omitted. The Λ appearing in (3.6.6), although perhaps a larger zero probability set than that denoted by the same symbol in (3.6.1), fails to depend on t.

If $a > 0$, $N(t, [ax])/[ax]$ and $N(t, [ax]+1)/[ax]$ both have (as $x \to \infty$) the limit property ascribed to $N(t, n)/n$ in (3.6.6). But $N(t, [ax]) \leqslant N(t, ax) \leqslant N(t, [ax]+1)$ for all ω (see remark after Definition 2.0.1). Therefore, this limit property extends to $N(t, ax)/[ax]$ and, since $ax/[ax] \to 1$, to $N(t, ax)/ax$. This completes the proof.

Since $\{t_n\}$ has no finite limits points, it is conjectured that sample averages may be well-defined even if $E[N(t, x)] = \infty$. Indeed, Section 4.3 suggests the construction of a process for which $N(t, x)/x$ converges (uniformly!) to a finite-valued random variable, but whose mean is unbounded.

4.0. Classes and examples of stationary point processes

In this chapter, we study a number of classes of point processes, and examine their stationarity properties. Because most of these processes were motivated by applications, their analysis is doubly rewarding, particularly when explicit formulas are derived. The results obtained earlier are often adaptable to this purpose, furnishing relations that facilitate computation.

Several of the proofs that certain processes are stationary are exceedingly tedious, and are therefore presented only in outline form. Fortunately, Theorem 2.3.1 suggests that stationarity may be verified in any one of several forms, so that a difficult proof can often be replaced by an easier one.

4.1. Poisson processes

Among renewal processes, the Poisson process is the best known; it provides a plausible probability model for many phenomena in reliability, queuing, insurance, etc., while possessing attractive mathematical and computational properties. It might be expected that there is an s.p.p. corresponding to the equilibrium Poisson renewal process, but this is unfortunately not the case. Such a process, with independent, identically distributed intervals and τ_0 independent of all these intervals, simply cannot exist according to Theorem 4.6.1. However, there is an s.p.p. subject to the following conditions (compare Parzen [12], p. 118 for the axioms of the Poisson renewal process):

(i) The numbers of points in disjoint intervals are mutually independent.

(ii) (3.4.13) is satisfied.

From (i), $G_{n-1}(x, h) = P[\{N(t, x) \geqslant n\} \cap \{N(t-h, h) \geqslant 1\}] = G_n(x) G_1(h)$, so that $F_n = G_n$ because of (3.4.19) and (ii). This relationship, together with (3.4.1), yields (for positive argument), the sequence of differential equations

$$\beta^{-1} S_n' + S_n = 1 + S_{n-1}, \quad S_n(0+) = 0 \tag{4.1.1}$$

which may be solved recursively. We write the solutions as

$$f_n(x) = g_n(x) = \beta(\beta x)^{n-1} e^{-\beta x}(n-1)!, \tag{4.1.2}$$

which corresponds to the classical result.

In place of the Poisson renewal process requirement that the intervals be mutually independent and identically distributed (precluded for an s.p.p. by Theorem 4.6.2), we substitute the weaker condition

(iii) The τ_k are mutually independent, $k \neq 0$.

We shall construct a process meeting (i), (ii), and (iii); further, the intervals will be identically distributed with the exception of τ_{-1}. Indeed, each τ_n, $n \neq -1$, has probability density f_1 as given in (4.1.2). Let τ be a new random variable having this same density, with τ independent of τ_n, $n \neq 1$. Then τ_{-1} shall be specified by $\tau_{-1} = \tau_0 + \tau$.

For the process just described $t_n = \Sigma_0^n \tau_k$ for $n \geqslant 0$ and $t_n = -\tau - \Sigma_n^{-2} \tau_k$ for $n \leqslant -1$. The process on the positive half axis is thus a Poisson renewal process, and the one on the negative half axis is a reflection of such a process. Further, the processes on the two half axes are independent, and have identical statistics for numbers of points in intervals.

To prove consecutive interval stationarity it remains to consider $N(t, x)$ such that $(t, x+t]$ contains the origin. Now $N(t, x) = N(t, -t) + N(0, x+t)$ for this case, so that $\{N(t, -t), N(0, x+t), N(t_1, x_1), N(t_2, x_2), \ldots\}$ is a collection of mutually independent random variables if the $(t_j, x_j + t_j]$ are not only all disjoint, but also disjoint from $(t, t+x]$. Then $\{N(t, x), N(t_1, x_1), N(t_2, x_2) \ldots\}$ also constitute a mutually independent set. In view of this, consecutive interval stationarity requires only that the statistics of $N(t, x)$ be the same as that for any other interval of length x. But the latter follows from

$$P[N(t, x) = n] = \sum_{k=0}^{n} P[N(t, -t) = k]\, P[N(0, x+t) = n-k]$$
$$= \sum_{k=0}^{n} (-\beta t)^k e^{+\beta t} (\beta[x+t])^{n-k} e^{-\beta[x+t]}/k!\,(n-k)! = (\beta x)^n e^{-\beta x}/n! \qquad (4.1.3)$$

which is just what it should be.

One can generalize the Poisson s.p.p. by relaxing (ii) and (iii). For example, one lets $\tau_{-3}, \tau_{-1}, \tau_0, \tau_2, \ldots$ remain as before, and takes the other τ_k as zero with probability one, thus assuring $t_{2n} = t_{2n+1}$. More complicated variations involve τ_k that are either zero or exponentially distributed, the choice being subject to some (stationary) probability law; if the choices for successive τ_k are mutually independent, we retain (iii) but not (ii).

4.2. Periodic processes

The class of s.p.p. includes some whose intervals are all determinate, and which manifest a periodically recurrent pattern. We shall describe periodic processes, show them to be stationary, and calculate some of their statistical parameters. It will be assumed throughout that the period in question is unity; this is done for convenience, as changes to an arbitrary period are easy to make.

Suppose that we have non-negative numbers $\tau_1, \tau_2, \ldots, \tau_N$ with the additional property

$$\sum_{1}^{N} \tau_k = 1. \qquad (4.2.1)$$

We then take $\tau_{k+N} = \tau_k$ to complete the definition of $\{\tau_n\}$ for positive indices. For the negative indices $k = -1, -2, \ldots, -N$ let $\tau_k = \sum_{j=0}^{-k-1} \tau_{N-j}$, the definition again being completed by $\tau_{k+N} = \tau_k$. The only truly random element in the process enters through τ_0. We identify Ω with $[0, 1]$, $\mathcal{F}$ with the Lebesgue measurable sets on this interval, and P with Lebesgue measure. If $\{t_n\}$ is obtained from $\{\tau_n\}$ in the manner prescribed by Definition 2.0.1, the possibility that $\{t_n\}$ is an s.p.p. hinges on the specification of τ_0. We shall choose $\tau_0(\omega) = \omega$, which is the simplest choice of the random variable τ_0 that ensures the stationarity of $\{t_n\}$.

To prove that $\{t_n\}$ is forward stationary, we introduce $I_{kj}(t, x_j)$, the function unity on $E_{kj}(t, x_j)$ and zero otherwise. We need to show that $E[\prod_{j=1}^n I_{kj}(t, x_j)]$ does not depend on t. If we can establish that each $I_{kj}(t, x_j)$ can be expressed as a function of $t-\omega$ periodic in its argument (with period unity), the same will be true of any product of characteristic functions of this type. For fixed but arbitrary k_j, x_j we call the product function f; then

$$E[\prod_{j=1}^{n} I_{kj}(t, x_j)] = \int_0^1 f(t-\omega)\, d\omega = \int_{t-1}^{t} f(t-\omega)\, d\omega = \int_0^1 f(\omega)\, d\omega. \tag{4.2.2}$$

Here we have used the fact that $P(d\omega) = d\omega$ on $[0, 1]$, and that the change in limits is valid because of the periodicity of f.

To complete the argument, we return to $I_k(t, x)$, and prove that this random variable is a periodic function of $t-\omega$. Let $\{t_n'\}$ be the non-random quantities $t_n' = t_n - \tau_0(\omega)$; these satisfy the relations $t'_{n+jN} = t'_n + j$ for all integer j. Substitution in (2.1.1) leads to

$$E_k(t, x) = \bigcup_m \{[t'_{m+1} > t - \tau_0(\omega)] \cap [t'_{m+k} \leqslant t - \tau_0(\omega) + x]\}. \tag{4.2.3}$$

Since $\tau_0(\omega) = \omega$, it is seen that, for fixed k and x, $\omega \in E_k(t, x)$ iff $t-\omega$ satisfies the inequalities stated in (4.2.3). Thus $I_k(t, x)$ is a function of $t-\omega$. The periodicity of this function is a consequence of $t'_{n+jN} = t'_n + j$. Indeed, replacing t'_{m+1} by t'_{m+jN+1} and t'_{m+k} by t'_{m+jN+k} (any integer j) does not alter the right side of (4.2.3) for the translation of indices is immaterial in view of the indicated union over all indices. Finally, we observe that $\{\omega: t'_{m+jN+1} > t - \omega\} = \{\omega: t'_{m+1} > (t-j) - \omega\}$ (and likewise for the other set), so that $E_k(t, x) = E_k(t-j, x)$; this completes the proof.

For $N \geqslant 2$, the τ_j, $j = 1, 2, \ldots, N$ can be chosen to have different lengths, so that the τ_n, $n \neq 0$, are not identically distributed. Since such a process is also interval stationary (cf. Theorem 2.3.1), it is stationary by McFadden's definition [10] but—contrary to McFadden's assertion—$\{\tau_n\}$ does not constitute a stationary stochastic process.

We shall now compute some of the statistics of the periodic s.p.p. Let $\tau_{k_1}, \tau_{k_2}, \ldots, \tau_{k_N}$ be a rearrangement of τ_j, $j = 1, 2, \ldots, N$ such that $0 < \tau_{k_1} \leqslant \ldots \leqslant \tau_{k_N}$, with the understanding that $\tau_{k_0} = 0$. By a simple calculation

$$G_1(x) = \sum_0^n \tau_{k_j} + (N-n)\, x, \quad \tau_{k_n} \leqslant x < \tau_{k_{n+1}}, \quad n = 0, 1, \ldots, N-1 \tag{4.2.4}$$

and $G_2(x) = o(x)$. As is expected, one obtains from this $E[N(t, x)] = Nx$ by application of (3.3.4). We may also calculate F_1, which is

$$F_1(x) = n/N \quad \text{for} \quad \tau_{k_n} \leqslant x < \tau_{k_{n+1}} \tag{4.2.5}$$

from (4.2.4) and (3.4.1); the same result is attained by heuristic reasoning.

The higher order distributions F_2, F_3, ... and G_2, G_3, ... depend on the order in which the intervals of different lengths appear. In general, it is easier to obtain the F's, using the intuitive concept of counting "starting from an occurrence". It is, however, possible to draw some conclusions on the F_n and the higher moments of $N(t, x)$ without explicit computation. Moments of all orders exist, and we may obtain upper and lower bounds for these moments. All but (at most) N of the F_n are either zero or one, and all but (at most) $([x]+1)N$ of the $F_n(x)$ must be zero. For those whose values lie in (0, 1) the inequality $0 \leqslant F_n \leqslant F_1$ is useful if $x < 1$. In any case, this information may be used in (3.4.4) to secure the aforesaid bounds.

4.3. Compound processes

The s.p.p. we have described thus far possess finite moments of all orders, and are ergodic in the sense of Section 3.6. Here, we shall sketch the structure of a class of processes which may—by proper choice of a parameter set—have any desired moment and ergodicity characteristics.

Suppose that $(\Omega^i, \mathcal{J}^i, P^i)$ is a sequence of s.p.p. for $i = 1, 2, \ldots$. Our compound process is then characterized by $\Omega = \bigcup \Omega^i$ (where the right-hand side is regarded a disjoint union), $\mathcal{J}$ consisting of all sets of the type $A = \bigcup A^i$ with $A^i \in \mathcal{J}^i$, and the probability measure specified by $P(A) = \Sigma p_i P^i(A_i)$. It is assumed, of course, that the $p_i \geqslant 0$, and that $\Sigma p_i = 1$. We omit the proof that $(\Omega, \mathcal{J}, P)$ is a probability space which generates a point process, and indeed an s.p.p. The individual s.p.p. and the p_i can be chosen to assure arbitrary moment properties of $N(t, x)$, subject only to the usual inequalities and stationarity properties. As a specific example, we take $p_i = 2^{-i}$, and let the ith process be periodic, with uniform spacing $t_n^i - t_{n-1}^i = 2^{-i}$. Then $N(t, x)/x$ converges uniformly to 2^i for some i, but $E[N(t, x)] = \infty$.

4.4. The generalized skip process

The so-called skip process is generated from an ordinary s.p.p. by expunging certain of the points. An analogous process appears in renewal theory in connection with the study of paralyzable counters (cf. Parzen [12]), in which formulas similar to ours have been derived. Such problems may occur in queuing theory, where some of the queued customers do not, after all, require servicing. Our generalization will admit not only the deletion of points (as suggested above), but also the creation of several points where only one existed before.

It is assumed that $\{t_n'\}$ is an s.p.p. associated with a probability space $(\Omega^1, \mathcal{J}^1, P^1)$. There is also a space $(\Omega^2, \mathcal{J}^2, P^2)$ which defines a stationary process $\{y_n\}$, with the y_n

assuming only non-negative integer values. The product space $\Omega^1 \times \Omega^2$ is generated, $\mathcal{F}$ being the completion of the extension of $\mathcal{F}^1 \times \mathcal{F}^2$, with probability measure generated by $P(A^1 \times A^2) = P(A^1)P(A^2)$, $A^1 \in \mathcal{F}^1$, $A^2 \in \mathcal{F}^2$. Hence $\{t_n'\}$ and $\{y_n\}$ are statistically independent. With this terminology, we define the generalized skip process $\{t_n\}$ by

$$t_n = \min_{\sum_0^k y_j > n} t_k' \tag{4.4.1}$$

for $n \geqslant 0$, and similarly for $n < 0$. This is clearly a point process, but one admitting the possibility of certain pathologies. To preclude these, we require that $P[y_n > 0$ for only a finite number of indices$] = 0$.

We assert that the relation

$$N(t, x) = \sum y_k I(t_k') \tag{4.4.2}$$

is consistent with both (4.4.1) and Definition 2.1.2. Here $I(t_k')$ is (for fixed x and t) the random variable which is unity if $t_k' \in (t, t+x]$ and zero otherwise. Suppose then that $\omega \in [N(t, x) = n]$, where $N(t, x)$ is specified by (4.4.2). For this ω, there is (since $\{t_n'\}$ has no finite limit points) a largest index m such that $t_m' \leqslant t$, and in fact $t_m' \leqslant t < t_{m+1}' \leqslant \ldots \leqslant t_{m+r}' \leqslant t + x < t_{m+r+1}'$. Suppose $\sum_0^m y_j = s+1$; then $t_s \leqslant t_m' \leqslant t$. Also, if y_{m+j} is the first non-zero y_n whose index exceeds m, $\sum_0^{m+j} y_j = s+p+1$, $j \leqslant r$, where $y_{m+j} = p$, say. Thus $t_{s+1} = t_{m+j}' > t$, or $\omega \in B_{s+1}^*(t) \cap B_s(t)$. Since $\sum_{m+1}^{m+r} y_j = n$, $\sum_0^{m+r} y_j = n+s+1$ and so $t_{s+n} \leqslant t_{m+r}' \leqslant t+x$. A similar argument demonstrates that $t_{s+n+1} > t+x$. In other words,

$$\{\omega : N(t, x) = n\} \subset \bigcup_s [B_s(t) \cap B_{s+1}^*(t) \cap B_{s+n}(t+x) \cap B_{s+n+1}^*(t+x)], \tag{4.4.3}$$

where the left side refers to the $N(t, x)$ furnished by (4.4.2).

Conversely, if ω belongs to the right side of (4.4.3), ω belongs to one and only one term of the (disjoint) union. Then there is an m such that $\sum_0^m y_j \leqslant s+1$, $\sum_0^{m+1} y_j \geqslant s+2$, whence $t_{m+1}' = t_{s+1} > t$, and $y_j = 0$ or $t_j' \leqslant t$ if $j \leqslant m$. Actually, $\sum_0^m y_j = s+1$, for otherwise $t_s = t_{s+1}$, which is impossible. The same argument leads to $\sum_0^{m+r} y_j = s+n+1$, $\sum_0^{m+r+1} y_j \geqslant s+n+2$ for some r, with $t_{m+r}' \leqslant t$ and, for $j \geqslant m+r+1$, $t_j' > t$ or $y_j = 0$. Then

$$\sum y_k I(t_k') = \sum_{m+1}^{m+r} y_k = \Big(\sum_0^{m+r} - \sum_0^m\Big) y_j = n, \tag{4.4.4}$$

and the proof is finished for $n \geqslant 1$ and $t \geqslant 0$. The cases of $n = 0$ and/or negative t are treated analogously.

For the skip process, it seems more convenient to prove successive interval stationarity than any of the other criteria of Theorem 2.3.1. The successive interval statistics will