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The topology of the space of rational curves on a toric variety

by

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1. Introduction

A toric variety is, roughly speaking, a complex algebraic variety which is the (partial) compactification of an algebraic torus $T^{\mathbf{C}}=(\mathbf{C}^*)^r$. It admits (by definition) an action of $T^{\mathbf{C}}$ such that, for some point $*\in X$, the orbit of $*$ is an embedded copy of $T^{\mathbf{C}}$. The most significant property of a toric variety is the fact that it is characterized entirely by a combinatorial object, namely its *fan*, which is a collection of convex cones in $\mathbf{R}^r$. As a general reference for the theory of toric varieties we use [Od1], together with the recent lecture notes [Fu].

In this paper we shall study the space of rational curves on a compact toric variety X. We shall obtain a configuration space description of the space $\mathrm{Hol}(S^2,X)$ of all holomorphic (equivalently, algebraic) maps from the Riemann sphere $S^2=\mathbf{C}\cup\infty$ to X. Our main application of this concerns fixed components $\mathrm{Hol}_D^*(S^2,X)$ of $\mathrm{Hol}^*(S^2,X)$, where the symbol D will be explained later, and where the asterisk indicates that the maps are required to satisfy the condition $f(\infty)=*$. If $\mathrm{Map}_D^*(S^2,X)$ denotes the corresponding space of continuous maps, we shall show that *the inclusion*

$$\mathrm{Hol}_D^*(S^2,X)\to\mathrm{Map}_D^*(S^2,X)$$

induces isomorphisms of homotopy groups up to some dimension $n(D)$, and we shall give a procedure for computing $n(D)$.

A theorem of this type was proved in the case $X=\mathbf{C}P^n$ by Segal ([Se]), and indeed that theorem provided the motivation for the present work. Our main idea is that the result of Segal may be interpreted as a result about configurations of distinct points in $\mathbf{C}$ which have labels in a certain *partial monoid.* We shall show that $\mathrm{Hol}_D^*(S^2,X)$ may be identified with a space $Q_D^X(\mathbf{C})$ of configurations of distinct points in $\mathbf{C}$ which have labels in a partial monoid M_X, where M_X is derived from the fan of X. Then we shall extend Segal's method so that it applies to this situation.

A feature of the method is the idea that the functor $U \mapsto \pi_i Q_D^X(U)$ resembles a homology theory. This functor has some similarities with the Lawson homology functor introduced in [La], in the sense that both are generalizations of the Dold–Thom functor $U \mapsto \pi_i \operatorname{Sp}^d(U)$, where $\operatorname{Sp}^d(U)$ is the dth symmetric product of U. This latter space can be considered as (a subspace of) the space of configurations of distinct points in U which have labels in the partial monoid $\{1, 2, ..., d\}$. (The label of a point, here, is simply its multiplicity.) The Dold–Thom functor resembles the ordinary homology functor, in the sense that $\pi_i \operatorname{Sp}^d(U) \cong \widetilde{H}_i U$, for i less than some dimension $n(d, U)$ ([DT]). It is important to note that the labelled configurations here are topologized so that when two distinct points "collide", their labels are added; if the addition is not defined, then the collision is prohibited.

A second aspect concerns a well known problem inspired by Morse theory. To explain this, we note that the above theorem (with $n(D)$ non-trivial!) is definitely not valid for arbitrary compact complex analytic spaces, or even complex manifolds. For example, there are no non-constant holomorphic maps from S^2 to the Hopf surface $S^1 \times S^3$, or indeed to any abelian variety. Nevertheless, there is reason to believe that a theorem of the above type might hold for compact Kähler manifolds, because in this case the holomorphic maps (in a fixed component of smooth maps) are precisely the absolute minima of the energy functional. A suitable extension of Morse theory would then explain such a result (although there would be no guarantee that $n(D)$ would be non-zero; for example it is known that there exist Kähler manifolds with very few rational curves). Our results confirm this Morse theory principle for smooth toric varieties, and they provide some evidence that it extends even to certain *singular* varieties.

This paper is arranged as follows. After a brief review of toric varieties in §2, we proceed to describe the correspondence between holomorphic maps and labelled configurations in §3, in the case of a projective toric variety (singular or not). The proof of the main theorem in the case of a non-singular projective toric variety (Theorem 4.1) is given in §4. It falls into three parts. First, we show that the homotopy groups of $Q_D^X(\mathbf{C})$ "stabilize" as $D \to \infty$. This can be reduced to the corresponding fact for the symmetric product. The method we use here is based on [GKY], [Gu1] as the method used in [Se] for the case $X = \mathbf{C}P^n$ does not seem to extend to the case of general X. Second, we show (using the homology-like properties of $\pi_i Q_D^X(\mathbf{C})$) that $Q_\infty^X(\mathbf{C})$ is actually homotopy equivalent to a component of $\operatorname{Map}(S^2, X)$. This idea, due originally to Gromov and Segal, has been used several times in the past, e.g. in [Mc], [Se], [Gu1], [Gu2]. Third, we show that this homotopy equivalence actually arises from the inclusion map of the theorem. In §5 we discuss the considerably more difficult case of singular projective toric varieties. We are not able to give a general result here which covers all cases, but we shall establish the result in a basic situation (Theorems 5.1 and 5.2) and then illustrate by examples how our method can in principle be applied to the remaining cases. We also sketch how the results may be generalized to non-projective toric varieties. For technical reasons we study only compact toric varieties in this paper, although it seems likely that a similar method also works in the non-compact case (see [GKY] for an example).

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2. Toric varieties

We shall summarize some of the basic properties of toric varieties, from [Od1]. Let X be an irreducible normal algebraic variety. One says that X is a *toric variety* if it has an algebraic action of an algebraic torus $T^{\mathbf{C}}=(\mathbf{C}^*)^r$, such that the orbit $T^{\mathbf{C}}\cdot *$ of some point $*\in X$ is dense in X and isomorphic to $T^{\mathbf{C}}$.

A toric variety is characterized up to isomorphism by its *fan*, which is a finite collection Δ of strongly convex rational polyhedral cones in $\mathbf{R}^r$ such that every face of an element of Δ belongs to Δ, and the intersection of any two elements of Δ is a face of each. (A strongly convex rational polyhedral cone in $\mathbf{R}^r$ is a subset of $\mathbf{R}^r$ of the form $\{\sum_{i=1}^s a_i n_i \,|\, a_i \geqslant 0\}$, where $\{n_1,\ldots,n_s\}\subseteq \mathbf{Z}^r$, which does not contain any line.) Given a fan Δ, an associated toric variety may be constructed abstractly as a union of affine varieties U_σ, $\sigma\in\Delta$.

It is possible to give a concrete description of *projective* toric varieties, as follows. Let $m_1,\ldots,m_N\in\mathbf{Z}^r$, such that the elements m_i-m_j generate $\mathbf{Z}^r$. Consider the action of $(\mathbf{C}^*)^r$ on $\mathbf{C}P^N$ given by the formula $u\cdot[z_0;\ldots;z_N]=[z_0;u^{m_1}z_1;\ldots;u^{m_N}z_N]$, where $u=(u_1,\ldots,u_r)$, $m_i=((m_i)_1,\ldots,(m_i)_r)$, and $u^{m_i}=u_1^{(m_i)_1}\ldots u_r^{(m_i)_r}$. Then the closure of the orbit of $[1;\ldots;1]$ is a toric variety. This gives rise to a second characterization of a toric variety embedded in projective space, namely that it is defined by equations of the form "monomial in $z_0,\ldots,z_N$ = monomial in $z_0,\ldots,z_N$". A third explicit description will be mentioned at the end of §5.

From the general construction, it follows that there is a one-to-one correspondence between $T^{\mathbf{C}}$-orbits of codimension i in X and cones of dimension i in Δ. The closure of the $T^{\mathbf{C}}$-orbit corresponding to a cone σ is a toric subvariety; it is the union of the orbits corresponding to those $\tau \in \Delta$ such that σ is a face of τ. Not surprisingly, geometrical and topological properties of X are reflected in the fan Δ. For example, X is non-singular if and only if, for each $\sigma \in \Delta$, the generators $n_1, \ldots, n_s$ can be extended to a generating set for $\mathbf{Z}^r$ ([Od1, Theorem 1.10]). The variety X is compact if and only if $\bigcup_{\sigma\in\Delta} \sigma = \mathbf{R}^r$ ([Od1, Theorem 1.11]). *From now on we shall assume that X is a compact toric variety.*

We shall be particularly concerned with the topology of X, and with line bundles over X. It is known that the fundamental group of any toric variety X is isomorphic to the quotient of $\mathbf{Z}^r$ by the subgroup generated by $\bigcup_{\sigma\in\Delta} \sigma \cap \mathbf{Z}^r$ ([Od1, Proposition 1.9]). From this and the compactness criterion, $\pi_1 X = 0$. To get further information, we need to introduce some more notation. Let $\sigma_1, \ldots, \sigma_u$ be the one-dimensional cones in Δ. We have $\sigma_i \cap \mathbf{Z}^r = \mathbf{Z} v_i$ for some $v_i \in \mathbf{Z}^r$. Let $X_1, \ldots, X_u$ be the closures of the corresponding (codimension one) $T^{\mathbf{C}}$-orbits in X. Equivariant line bundles on X correspond to invariant Cartier divisors on X. If X is compact, these correspond to "Δ-linear support functions", i.e. functions $h: \Delta \to \mathbf{R}$ which are linear on each cone σ and $\mathbf{Z}$-valued on $\Delta \cap \mathbf{Z}^r$ ([Od1, Proposition 2.4]). Let $\mathrm{SF}(\Delta)$ denote the group of Δ-linear support functions. For $h \in \mathrm{SF}(\Delta)$, a divisor of the corresponding line bundle is given by $\sum_{i=1}^u h(v_i) X_i$. Hence we obtain an inclusion $\mathrm{SF}(\Delta) \to \bigoplus_{i=1}^u \mathbf{Z}\sigma_i$, $h \mapsto \sum_{i=1}^u h(v_i)\sigma_i$. From now on, we shall *identify* $\mathrm{SF}(\Delta)$ with a subgroup of $\bigoplus_{i=1}^u \mathbf{Z}\sigma_i$. It represents the subgroup consisting of invariant Cartier divisors of the group $\bigoplus_{i=1}^u \mathbf{Z}\sigma_i$ of invariant Weil divisors. The inclusion is an isomorphism if X is non-singular ([Od1, Proposition 2.1]).

We have another natural inclusion $\iota: \mathbf{Z}^r \to \mathrm{SF}(\Delta)$, $m \mapsto \langle m, \cdot \rangle$, and the quotient group is isomorphic to the Picard group $\mathrm{Pic}(X) \cong H^1(X, \mathcal{O}_X^*)$, if X is compact ([Od1, Corollary 2.5]). Moreover, in this case, it is known that $H^i(X, \mathcal{O}_X) = 0$ for $i \geqslant 1$ ([Od1, Corollary 2.8]), so we have $H^2 X \cong H^1(X, \mathcal{O}_X^*) \cong \mathrm{Pic}(X) \cong \mathrm{SF}(\Delta)/\mathbf{Z}^r$. Since we consider maps $S^2 \to X$ in this paper, we shall need a description of the group $\pi_2 X$. If $H_2 X$ is torsion free, then $H_2 X \cong (H^2 X)^*$, and so we have

$$\pi_2 X \cong H_2 X \cong (H^2 X)^* \cong (\mathrm{SF}(\Delta)/\mathbf{Z}^r)^* \cong \mathrm{Ker}\, \iota^*,$$

where $\iota^*: \mathrm{SF}(\Delta)^* \to (\mathbf{Z}^r)^*$ is the dual of ι. If X is non-singular then $H_2 X$ is torsion free, because the integral cohomology of X is torsion free, by a theorem of Jurkiewicz–Danilov ([Od1, p. 134]).

Example 2.1: *Complex projective space, $X = \mathbf{C}P^n$.* Let $E_1, \ldots, E_n$ be the standard orthonormal basis of $\mathbf{R}^n$. Let $\varrho_0, \ldots, \varrho_n$ be the one-dimensional cones spanned by $E_0 = -\sum_{i=1}^n E_i, E_1, \ldots, E_n$ (respectively). We obtain a fan in $\mathbf{R}^n$ by taking the cones spanned by all proper subsets of $\{\varrho_0, \ldots, \varrho_n\}$ (together with the zero-dimensional cone given by the origin). The associated toric variety is isomorphic to $\mathbf{C}P^n$. In terms of the usual homogeneous coordinates for $\mathbf{C}P^n$, the algebraic torus $T^{\mathbf{C}} = (\mathbf{C}^*)^n$ acts by the formula

$(v_1, ..., v_n) \cdot [z_0; ...; z_n] = [z_0; v_1 z_1; ...; v_n z_n]$. Let $* = [1; ...; 1]$. The closures of the codimension one $T^{\mathbf{C}}$-orbits are the hyperplanes $P_0, ..., P_n$, where P_i is defined by the condition $z_i = 0$.

Example 2.2: *The Hirzebruch surface* $X = \Sigma_k$ ([Od1, p. 9, Example (iii)]). Consider the fan in $\mathbf{R}^2$ given by the four two-dimensional cones (and all their faces) spanned by the four vectors $v_1 = (1, 0)$, $v_2 = (0, 1)$, $v_3 = (-1, k)$, $v_4 = (0, -1)$. Thus, the one-dimensional cones in this fan are $\sigma_i = \mathbf{R}_{\geqslant 0} v_i$ for $i = 1, 2, 3, 4$, and we have $\sigma_i \cap \mathbf{Z}^2 = \mathbf{Z} v_i$. The construction produces a variety isomorphic to Σ_k. The classes $[\sigma_1], [\sigma_2], [\sigma_3], [\sigma_4]$ in $\bigoplus_{i=1}^4 \mathbf{Z}\sigma_i / \mathbf{Z}^2$ satisfy the relations $[\sigma_1] = [\sigma_3]$, $[\sigma_4] = [\sigma_2] + k[\sigma_3]$ (corresponding to the generators $(1, 0)$, $(0, 1)$ of $\mathbf{Z}^2$). Hence $H^2 \Sigma_k \cong \mathbf{Z} \oplus \mathbf{Z}$. To describe this variety directly, let us consider the embedding

$$\Sigma_k = \{([x_0; x_1; x_2], [y_1; y_2]) \mid x_1 y_1^k = x_2 y_2^k\} \subseteq \mathbf{C}P^1 \times \mathbf{C}P^2.$$

The algebraic torus $T^{\mathbf{C}} = (\mathbf{C}^*)^2$ acts on Σ_k as follows:

$$(v_1, v_2) \cdot ([x_0; x_1; x_2], [y_1; y_2]) = ([v_1 x_0; v_2^k x_1; x_2], [y_1; v_2 y_2]).$$

Let $* = ([1; 1; 1], [1; 1])$. The closures of the codimension one $T^{\mathbf{C}}$-orbits are the four embedded copies of $\mathbf{C}P^1$ defined by $X_1 = \{x_2 = 0, y_1 = 0\}$, $X_2 = \{x_1 = 0, x_2 = 0\}$, $X_3 = \{x_1 = 0, y_2 = 0\}$, $X_4 = \{x_0 = 0\}$. The natural projection $\Sigma_k \to \mathbf{C}P^1$ exhibits Σ_k as $\mathbf{P}(\mathcal{O}(0) \oplus \mathcal{O}(-k))$, which is the bundle obtained from $\mathcal{O}(-k)$ by fibre-wise one point compactification. The 0-section and ∞-section are given by X_2 and X_4, and the fibres over $[0; 1], [1; 0]$ are given by X_1, X_3.

Example 2.3: *The weighted projective spaces* $X = P(a_0, ..., a_n)$ ([Fu, §2.2]). The weighted projective space $P(a_0, ..., a_n)$ is defined to be the quotient of $\mathbf{C}P^n$ by the action of the finite group $(\mathbf{Z}/a_0\mathbf{Z}) \times ... \times (\mathbf{Z}/a_n\mathbf{Z})$ given by

$$(\omega_0, ..., \omega_n).[z_0; ...; z_n] = [\omega_0 z_0; ...; \omega_n z_n],$$

where ω_i is a primitive a_ith root of unity. Without loss of generality we may assume $a_0 = 1$. In this case, a suitable fan is generated by the vectors $-\sum_{i=1}^n a_i E_i, E_1, ..., E_n$, and we have $H^2 P(a_0, ..., a_n) \cong \mathbf{Z}$. These varieties may have singularities.

Example 2.4: *Compact non-singular toric surfaces.* These are classified in [Fu, §2.6], and [Od1, §§1.6, 1.7]. They are obtained from $\mathbf{C}P^2$ or Σ_k by blowing up a finite number of fixed points of the torus action.

Example 2.5: *The closure of an algebraic torus orbit in a (generalized) flag manifold.* A Lie-theoretic description of the fan is given in [FH]; see also [Da] and [Od2], and the references given in [Od2] to closely related work of the Gelfand school. It was pointed out in [Od2] that the normality of these varieties remains to be verified. This omission has recently been rectified in [Da], in the case of generic orbits. These varieties may have singularities. As a concrete example, we mention the famous "tetrahedral complex", which is a singular three-dimensional subvariety X of the Grassmannian $\mathrm{Gr}_2(\mathbf{C}^4)$. For historical remarks,(1) including a description of the role played by this variety in the early development of Lie theory, we refer to §4.2 of [GM]. In mundane terms, if $\mathrm{Gr}_2(\mathbf{C}^4)$ is realized as the subvariety of $\mathbf{C}P^5$ given by the usual Plücker equation $z_0z_1-z_2z_3+z_4z_5=0$, then X is given by the equations $z_0z_1=\alpha z_2z_3=\beta z_4z_5$, where α,β are fixed complex numbers such that $1-\alpha^{-1}+\beta^{-1}=0$ and $\alpha,\beta\neq 0,1,\infty$. If, on the other hand, $\mathrm{Gr}_2(\mathbf{C}^4)$ is considered as a generalized flag manifold of the group $\mathrm{SL}_4(\mathbf{C})$, and if $(\mathbf{C}^*)^3$ is considered in the usual way to be a maximal torus of $\mathrm{SL}_4(\mathbf{C})$, which therefore acts naturally on $\mathrm{Gr}_2(\mathbf{C}^4)$, then X occurs as the closure of a generic orbit. It follows that the fan of X can be obtained from the results of §4 of [FH]. After some re-normalization, it is the fan in $\mathbf{R}^3$ with six three-dimensional cones (and all their faces) spanned by the vectors $(\pm 1,\pm 1,\pm 1)$. The lattice is taken to be that which is generated by $(\pm 1,\pm 1,\pm 1)$, however, rather than $\mathbf{Z}^3$. One has $H^2X\cong\mathbf{Z}$.

3. The configuration space for projective toric varieties

As in §2, let $\sigma_1,\dots,\sigma_u$ be the one-dimensional cones in the fan Δ, and let $X_1,\dots,X_u$ be the closures of the codimension one $T^{\mathbf{C}}$-orbits in X. Thus, $X_1\cup\dots\cup X_u$ is the complement of the "big orbit" $T^{\mathbf{C}}\cdot *$. We shall assume as usual that H_2X is torsion free, in order to make use of the description of π_2X which was given in §2.

If f is a holomorphic map such that $f(\infty)=*$, then $f(S^2)$ is not contained in any of the subvarieties X_i, and so $f(S^2)\cap X_i$ must be a finite (possibly empty) set of points. We associate to f the finite set of distinct points $\{z\in\mathbf{C}\mid f(z)\notin T^{\mathbf{C}}\cdot *\}$, and to each such point z we associate—provisionally—a "label" $l_z=((l_z)_1,\dots,(l_z)_u)$, where $(l_z)_i$ is the non-negative integer given by the (suitably defined) intersection number of f and X_i at z.

It turns out to be more natural to regard the label l_z as an element of $\mathrm{SF}(\Delta)^*$, i.e. $\mathrm{Hom}(\mathrm{SF}(\Delta),\mathbf{Z})$. Therefore, the provisional definition of the labelled configuration associated to f will be replaced by the following construction. Let $Q(\mathbf{C};\mathrm{SF}(\Delta)^*)$ be the space of configurations of distinct points in $\mathbf{C}$, where the points have labels in the group $\mathrm{SF}(\Delta)^*$. An element of $Q(\mathbf{C};\mathrm{SF}(\Delta)^*)$ may be written in the form $\{(z,l_z)\}_{z\in I}$, where I is a finite subset of $\mathbf{C}$ and $\{l_z\}_{z\in I}\subseteq\mathrm{SF}(\Delta)^*$. There is a natural topology on $Q(\mathbf{C};\mathrm{SF}(\Delta)^*)$

(1) See also T. Hawkins, The birth of Lie's theory of groups, *Math. Intelligencer*, 16 (1994), 6–17.

which permits two distinct points in a configuration to "coalesce", whereupon their labels are added. Thus, $Q(\mathbf{C};\mathrm{SF}(\Delta)^*)$ consists of a collection of disconnected components $Q_D(\mathbf{C};\mathrm{SF}(\Delta)^*)$, indexed by elements $D=\sum_{z\in I} l_z$ of $\mathrm{SF}(\Delta)^*$. Each component is contractible as all particles may be moved to the origin.

Definition. To a holomorphic map $f\in\mathrm{Hol}^*(S^2,X)$ we associate a configuration in $Q(\mathbf{C};\mathrm{SF}(\Delta)^*)$ by means of the map

$$\alpha^X:\mathrm{Hol}^*(S^2,X)\to Q(\mathbf{C};\mathrm{SF}(\Delta)^*),\quad \alpha^X(f)=\{(z,l_z)\mid f(z)\notin T^{\mathbf{C}}\cdot *\},$$

where, for any (Cartier) divisor $\tau\in\mathrm{SF}(\Delta)$, $l_z(\tau)$ is the multiplicity of z in the divisor $f^{-1}(\tau)$.

Our main observation will be that the map α^X is a homeomorphism to its image, and that the image has a simple characterization. To obtain this characterization, we observe that the configuration obtained from a map f must satisfy two kinds of properties.

First, geometry forces the following conditions on the label l_z of a point z:

(X) If $\tau\geqslant 0$, then $l_z(\tau)\geqslant 0$. If $\tau_{i_1}\cap\ldots\cap\tau_{i_j}=\varnothing$, then $l_z(\tau_{i_1})\ldots l_z(\tau_{i_j})=0$ (i.e. $l_z(\tau_{i_1})$, $\ldots, l_z(\tau_{i_j})$ cannot all be non-zero).

If X is non-singular, so that $\mathrm{SF}(\Delta)^*\cong\bigoplus_{i=1}^u \mathbf{Z}\sigma_i$, then condition (X) says that at least one of the (non-negative) integers $l_z(\sigma_{i_1}),\ldots,l_z(\sigma_{i_j})$ must be zero whenever $\sigma_{i_1},\ldots,\sigma_{i_j}$ do not lie in a single cone.

Second, we may interpret topologically the integer $\sum_z l_z(\tau)$ as the class $f^*[\tau]\in H^2S^2$. Since the image of the inclusion $\iota:\mathbf{Z}^r\to\mathrm{SF}(\Delta)$ is zero in $\mathrm{SF}(\Delta)/\mathbf{Z}^r\cong H^2X$, we have:

(D) The vector $D=\sum_z l_z\in\mathrm{SF}(\Delta)^*$ is in the kernel of the map $\iota^*:\mathrm{SF}(\Delta)^*\to(\mathbf{Z}^r)^*$.

It follows from the identification $\pi_2X\cong\mathrm{Ker}\,\iota^*$ (when H_2X is torsion free; see §2) that we may regard D as the homotopy class of f. We shall write $\mathrm{Map}_D(S^2,X)$ for this component of the space of continuous maps, and $\mathrm{Hol}_D(S^2,X)$ for its subset consisting of holomorphic maps.

Condition (X) is a local condition, in the sense that it is purely "label-theoretic". It depends only on the toric variety X. Condition (D), on the other hand, is a global condition, which depends on f. We shall show that (X) and (D) are the *only* conditions on the configuration associated to f.

Definition. $Q_D^X(\mathbf{C})$ denotes the space of configurations of distinct points in $\mathbf{C}$ with labels in $\mathrm{SF}(\Delta)^*$ such that conditions (X) and (D) are satisfied (for a fixed $D\in\mathrm{SF}(\Delta)^*$).

It should be noted that $Q_D^X(\mathbf{C})$ is in general a topologically non-trivial subspace of the contractible space $Q_D(\mathbf{C};\mathrm{SF}(\Delta)^*)$, because condition (X) prevents certain types of collisions.

PROPOSITION 3.1. *Let X be a projective toric variety, such that H_2X is torsion free. Then the map $\alpha^X: \mathrm{Hol}^*_D(S^2, X) \to Q^X_D(\mathbf{C})$ is a homeomorphism.*

Proof. Let $\theta: X \to \mathbf{C}P^N$ be an equivariant embedding with $\theta(*)=[1;\dots;1]$. The fact that θ is equivariant means that it is induced by a *map of fans* $\phi: \Delta^X \to \Delta^{\mathbf{C}P^N}$ ([Od1, §1.5]), i.e. a $\mathbf{Z}$-linear homomorphism $\phi: \mathbf{Z}^r \to \mathbf{Z}^N$ whose $\mathbf{R}$-linear extension carries each cone of Δ^X into some cone of $\Delta^{\mathbf{C}P^N}$. Here, Δ^X is the fan of X, and $\Delta^{\mathbf{C}P^N}$ is the fan of $\mathbf{C}P^N$. Let $e_1, \dots, e_r$ and $E_1, \dots, E_N$ be the standard orthonormal bases of $\mathbf{R}^r$ and $\mathbf{R}^N$, respectively. We denote the one-dimensional cones of Δ^X by $\sigma_1, \dots, \sigma_u$, and those of $\Delta^{\mathbf{C}P^N}$ by $\varrho_0, \dots, \varrho_N$, as usual.

We have an inclusion map $\theta': \mathrm{Hol}^*_D(S^2, X) \to \mathrm{Hol}^*_d(S^2, \mathbf{C}P^N)$, where $\mathrm{Hol}^*_d(S^2, \mathbf{C}P^N)$ denotes the space of holomorphic maps f of some degree d (depending on D) such that $f(\infty)=[1;\dots;1]$. We also have a map $\theta'': Q^X_D(\mathbf{C}) \to Q^{\mathbf{C}P^N}_d(\mathbf{C})$, induced by $T^*: \mathrm{SF}(\Delta^X)^* \to \mathrm{SF}(\Delta^{\mathbf{C}P^N})^*$, where $T: \mathrm{SF}(\Delta^{\mathbf{C}P^N}) \to \mathrm{SF}(\Delta^X)$ is given by composition with ϕ. The following diagram is commutative:

$$\begin{array}{ccc}
\mathrm{Hol}^*_D(S^2, X) & \xrightarrow{\theta'} & \mathrm{Hol}^*_d(S^2, \mathbf{C}P^N) \\
\Big\downarrow{\scriptstyle \alpha^X} & & \Big\downarrow{\scriptstyle \alpha^{\mathbf{C}P^N}} \\
Q(\mathbf{C}; \mathrm{SF}(\Delta^X)^*) & \xrightarrow{\theta''} & Q(\mathbf{C}; \mathrm{SF}(\Delta^{\mathbf{C}P^N})^*).
\end{array}$$

Note that the map $\alpha^{\mathbf{C}P^N}$ gives the well known homeomorphism between $\mathrm{Hol}^*_d(S^2, \mathbf{C}P^N)$ and the space of $(N+1)$-tuples of coprime monic polynomials of degree d.

Next we claim that $\theta''(Q^X_D(\mathbf{C})) \subseteq Q^{\mathbf{C}P^N}_d(\mathbf{C})$. If $\{(z_\alpha, l_{z_\alpha})\}_\alpha \in Q^X_D(\mathbf{C})$, then we must check that the configuration $\{(z_\alpha, T^*(l_{z_\alpha}))\}_\alpha$ satisfies the two conditions "$(\mathbf{C}P^N)$" and "(d)". The first of these is clear from geometrical considerations. The second amounts to the condition that $\sum_\alpha T^*(l_{z_\alpha})$ belongs to the kernel of the map $\mathrm{SF}(\Delta^{\mathbf{C}P^N})^* \to (\mathbf{Z}^N)^*$. That this is true follows from the fact that $\sum_\alpha l_{z_\alpha}$ belongs to the kernel of the map $\mathrm{SF}(\Delta^X)^* \to (\mathbf{Z}^r)^*$.

Since θ' and $\alpha^{\mathbf{C}P^N}$ are injective, it follows from the above diagram that α^X is injective. To show that α^X maps surjectively onto $Q^X_D(\mathbf{C})$, we must show that a holomorphic map $f: S^2 \to \mathbf{C}P^N$, which has been constructed from a configuration in the image of θ'', actually factors through the embedding $\theta: X \to \mathbf{C}P^N$. To do this, we shall need to describe the maps θ', θ'' more explicitly. We begin with θ'. The embedding $\theta: X \to \mathbf{C}P^N$ (and hence the map θ') is determined by the restriction $(\mathbf{C}^*)^r \to (\mathbf{C}^*)^N$ of θ to the corresponding tori ([Od1, Theorem 1.13]). This is given by $(z_1, \dots, z_r) \mapsto (z^{m_1}, \dots, z^{m_N})$, where $\phi(x) = \sum_{i=1}^N \langle m_i, x \rangle E_i$, and where z^{m_i} means $z_1^{(m_i)_1} \dots z_r^{(m_i)_r}$. Thus, X may be described explicitly as the closure in $\mathbf{C}P^N$ of the set of elements of the form $[1; z^{m_1}; \dots; z^{m_N}]$,

$z\in(\mathbf{C}^*)^r$. Now we turn to θ''. Let $\{(z_\alpha, l_{z_\alpha})\}_\alpha$ be an element of $Q_D^X(\mathbf{C})$; its image under θ'' is $\{(z_\alpha, T^*(l_{z_\alpha}))\}_\alpha$. By the remarks above, this configuration lies in $Q_d^{\mathbf{C}P^N}(\mathbf{C})$, so it corresponds to an $(N+1)$-tuple $(p_0, ..., p_N)$ of coprime monic polynomials of degree d. The exponent of $z-z_\alpha$ in $p_i(z)$ is $T^*(l_{z_\alpha})(\varrho_i)=l_{z_\alpha}(T(\varrho_i))$. To find the explicit form of p_i, we have to compute $T(\varrho_i)$. By definition we have $T(\varrho_i)=\sum_{j=1}^u\langle\varrho_i, \phi(v_j)\rangle\sigma_j$. Now, one may verify by direct calculation that $\langle\varrho_i-\varrho_0, \sum_{k=1}^N x_k E_k\rangle=x_i$, so we obtain

$$T(\varrho_i)-T(\varrho_0)=\sum_{j=1}^u\langle\varrho_i-\varrho_0, \phi(v_j)\rangle\sigma_j=\sum_{j=1}^u\langle m_i, v_j\rangle\sigma_j.$$

Hence the exponent of $z-z_\alpha$ in $p_i(z)p_0(z)^{-1}$ is $l_{z_\alpha}(\sum_{j=1}^u\langle m_i, v_j\rangle\sigma_j)=\sum_{k=1}^r(m_i)_k a_k^\alpha$, where $a_k^\alpha=l_{z_\alpha}(\sum_{j=1}^u(v_j)_k\sigma_j)$. Observe that $\sum_{j=1}^u(v_j)_k\sigma_j$ belongs to $\mathrm{SF}(\Delta)$, since its value on v_i is just $(v_i)_k$. Hence a_k^α is an integer, and we have

$$p_i(z)p_0(z)^{-1}=\prod_\alpha(z-z_\alpha)^{\sum_{k=1}^r(m_i)_k a_k^\alpha}=q_1(z)^{(m_i)_1}\dots q_r(z)^{(m_i)_r}=q^{m_i},$$

where $q_k(z)$ denotes the rational function $\prod_\alpha(z-z_\alpha)^{a_k^\alpha}$. This completes our explicit determination of the map θ''. It follows immediately from this and the earlier description of θ' that the map represented by $(p_0, ..., p_N)$ factors through X. Hence α^X maps surjectively onto $Q_D^X(\mathbf{C})$, as required. We have now shown that α^X is bijective. It is a homeomorphism because it is a restriction of $\alpha^{\mathbf{C}P^N}$, which is a homeomorphism. □

Example 3.2: *Complex projective space* $\mathbf{C}P^n$. With the notation of Example 2.1, we have $\mathrm{SF}(\Delta)\cong\bigoplus_{i=0}^n\mathbf{Z}\varrho_i\cong\mathrm{SF}(\Delta)^*$. The map ι^* is given by

$$\iota^*\colon\sum_{i=0}^n x_i\varrho_i\mapsto(x_1-x_0, ..., x_n-x_0).$$

For $D=\sum_{i=0}^n d\varrho_i\in\operatorname{Ker}\iota^*$, $Q_D^X(\mathbf{C})$ consists of all configurations such that the labels $l_z=\sum_{i=0}^n x_i\varrho_i$ satisfy the conditions

(X) $x_0, ..., x_n\geqslant 0$ and $x_0\dots x_n=0$,

(D) $\sum_z x_0=\dots=\sum_z x_n$ $(=d)$.

(Explanation: A map $f\in\mathrm{Hol}_d^*(S^2, \mathbf{C}P^n)$ may be identified explicitly with an $(n+1)$-tuple $(p_0, ..., p_n)$ of monic polynomials of degree d with no common factor. The divisor $f^{-1}(P_i)$ is given by the roots of p_i. Thus, the labelled configuration associated to f is the set of distinct roots z of $p_0\dots p_n$, where the label $l_z=\sum_{i=0}^n x_i\varrho_i$ of z indicates that z is a root of p_i of multiplicity x_i.)

Example 3.3: *The Hirzebruch surface* Σ_k. From Example 2.2,

$$\mathrm{SF}(\Delta)\cong\bigoplus_{i=1}^{4}\mathbf{Z}\sigma_i\cong\mathrm{SF}(\Delta)^*,$$

and the map ι^* is given by

$$\iota^*:\sum_{i=1}^{4}x_i\sigma_i\mapsto(x_1-x_3,x_2+kx_3-x_4).$$

Conditions (X) and (D) are:

(X) $x_1,\dots,x_4\geqslant 0$ and $x_1x_3=0$, $x_2x_4=0$,

(D) $\sum_z x_1=\sum_z x_3$, $\sum_z x_2+k\sum_z x_3=\sum_z x_4$.

(Explanation: From the embedding $\Sigma_k\subseteq\mathbf{C}P^1\times\mathbf{C}P^2$ of Example 2.2, we see that a map $f\in\mathrm{Hol}^*(S^2,\Sigma_k)$ may be identified explicitly with a 5-tuple of monic polynomials $((p_4,p_2p_3^k,p_2p_1^k),(p_1,p_3))$, such that p_1,p_3 are coprime and p_2,p_4 are coprime. The divisor $f^{-1}(X_i)$ is given by the roots of p_i. Thus, the labelled configuration associated to f is the set of distinct roots z of $p_1p_2p_3p_4$, where the label $l_z=\sum_{i=1}^{4}x_i\sigma_i$ of z indicates that z is a root of p_i of multiplicity x_i.)

Example 3.4: *The "quadric cone"* $z_2^2=z_1z_3$ *in* $\mathbf{C}P^3$. (The space of rational curves on this variety was considered in detail in [Gu1].) Consider the fan in $\mathbf{R}^2$ given by the three two-dimensional cones (and all their faces) spanned by the vectors $v_1=(1,0),v_2=(-1,2),v_3=(0,-1)$. It can be shown that this fan arises from the quadric cone X in $\mathbf{C}P^3$ which is defined by the equation $z_2^2=z_1z_3$. Indeed, this is an example of a weighted projective space (see Example 2.3), namely $P(1,1,2)$. The torus $(\mathbf{C}^*)^2$ acts on X by $(u,v)\cdot[z_0;z_1;z_2;z_3]=[z_0;uvz_1;uz_2;uv^{-1}z_3]$. We have

$$\mathrm{SF}(\Delta)\cong\{h_1\sigma_1+h_2\sigma_2+h_3\sigma_3\mid h_1,h_2,h_3\in\mathbf{Z},\ h_1+h_2\in 2\mathbf{Z}\},$$
$$\mathrm{SF}(\Delta)^*\cong\{x_1\sigma_1+x_2\sigma_2+x_3\sigma_3\mid x_1,x_2\in\tfrac{1}{2}\mathbf{Z},\ x_3,x_1+x_2\in\mathbf{Z}\},$$

where we have identified $\mathrm{SF}(\Delta)^*$ in an obvious way with a lattice in $\mathbf{R}\sigma_1\oplus\mathbf{R}\sigma_2\oplus\mathbf{R}\sigma_3$. The map ι^* is given by

$$\iota^*:x_1\sigma_1+x_2\sigma_2+x_3\sigma_3\mapsto(x_1-x_2,2x_2-x_3).$$

Thus, for $D=d\sigma_1+d\sigma_2+2d\sigma_3$, we see that $Q_D^X(\mathbf{C})$ consists of all configurations such that the labels $l_z=x_1\sigma_1+x_2\sigma_2+x_3\sigma_3$ satisfy the conditions

(X) $x_1,x_2,x_3\geqslant 0$ and $x_1x_2x_3=0$,

(D) $2\sum_z x_1=2\sum_z x_2=\sum_z x_3$ $(=2d)$.

Example 3.5: *The weighted projective space* $P(1,2,3)$ (see Example 2.3). This is a del Pezzo surface. A suitable fan is the one generated by the vectors $v_1=(1,0), v_2=(0,1), v_3=(-2,-3)$. It can be shown (see Appendix 2) that

$$\mathrm{SF}(\Delta)\cong\{h_1\sigma_1+h_2\sigma_2+h_3\sigma_3 \mid h_1,h_2,h_3\in\mathbf{Z},\ h_2+h_3\in 2\mathbf{Z},\ h_1+2h_3, 2h_1+h_3\in 3\mathbf{Z}\},$$

$$\mathrm{SF}(\Delta)^*\cong\{x_1\sigma_1+x_2\sigma_2+x_3\sigma_3 \mid x_1\in\tfrac{1}{3}\mathbf{Z},\ x_2\in\tfrac{1}{2}\mathbf{Z},\ x_3\in\tfrac{1}{6}\mathbf{Z},\\ x_1+4x_3, 2x_1+2x_3, x_2+3x_3, x_1+x_2+x_3\in\mathbf{Z}\},$$

and that the map ι^* is given by

$$\iota^*: x_1\sigma_1+x_2\sigma_2+x_3\sigma_3\mapsto(x_1-2x_3, x_2-3x_3).$$

For $D=2d\sigma_1+3d\sigma_2+d\sigma_3$, we see that $Q_D^X(\mathbf{C})$ consists of all configurations such that the labels $l_z=x_1\sigma_1+x_2\sigma_2+x_3\sigma_3$ satisfy the conditions

(X) $x_1,x_2,x_3\geqslant 0$ and $x_1x_2x_3=0$,

(D) $\sum_z x_1=2d$, $\sum_z x_2=3d$, $\sum_z x_3=d$.

Example 3.6: *The tetrahedral complex* (see Example 2.5). We shall use the following notation:

$$v_{12}=(1,1,-1),\quad v_{13}=(1,-1,1),\quad v_{23}=(-1,1,1),\quad v_{123}=(1,1,1),$$
$$v'_{12}=(-1,-1,1),\quad v'_{13}=(-1,1,-1),\quad v'_{23}=(1,-1,-1),\quad v'_{123}=(-1,-1,-1),$$

and we shall write $\sigma_*=\mathbf{R}_{\geqslant 0}v_*$, $\sigma'_*=\mathbf{R}_{\geqslant 0}v'_*$, where $*$ ranges over the subscripts 12, 13, 23, 123. One obtains the identification

$$\mathrm{SF}(\Delta)\cong\left\{\sum h_*\sigma_*+\sum h'_*\sigma'_*\ \middle|\ h_*,h'_*\in\mathbf{Z},(H)\right\},$$

where (H) denotes the system of equations

$$h_{12}+h_{13}=h_{123}+h'_{23},\quad h'_{12}+h'_{13}=h'_{123}+h_{23},$$
$$h_{12}+h_{23}=h_{123}+h'_{13},\quad h'_{12}+h'_{23}=h'_{123}+h_{13},$$
$$h_{13}+h_{23}=h_{123}+h'_{12},\quad h'_{13}+h'_{23}=h'_{123}+h_{12}.$$

To identify the dual group $\mathrm{SF}(\Delta)^*$, let us denote by V the subspace of $W=(\bigoplus\mathbf{R}\sigma_*)\oplus(\bigoplus\mathbf{R}\sigma'_*)$ defined by the equations (H). Then $\mathrm{SF}(\Delta)^*$ may be identified with a lattice in the vector space W^*/V° (where $V^\circ=\{f\in W^*\mid f(V)=0\}$), namely the lattice of functionals which take integer values on $\mathrm{SF}(\Delta)$. Since V° is generated by the six elements $\sigma_{12}+\sigma_{13}-\sigma_{123}-\sigma'_{23}$ etc., one may represent any element of W^*/V° by a unique element $x=$

$\sum x_* \sigma_* \in W$. This defines an element of $\mathrm{SF}(\Delta)^*$ if and only if it takes integer values on $\mathrm{SF}(\Delta)$. It is easy to check that this is so if and only if $x_* \in \mathbf{Z}$ for all $*$. Thus we arrive at the identification

$$\mathrm{SF}(\Delta)^* \cong \bigoplus \mathbf{Z}\sigma_*.$$

Conditions (X) and (D) are as follows:

(X) The integers x_{12}, x_{13}, x_{23}, $x_{12}+x_{13}+x_{123}$, $x_{12}+x_{23}+x_{123}$, $x_{13}+x_{23}+x_{123}$ are non-negative, but not simultaneously positive,

(D) $\sum_z x_{12} = \sum_z x_{13} = \sum_z x_{23} = -\sum_z x_{123} = d$, where $D = d\sigma_{12} + d\sigma_{13} + d\sigma_{23} + d\sigma_{123}$.

4. The theorem for non-singular projective toric varieties

If X is a non-singular toric variety, then we have $\mathrm{SF}(\Delta)^* \cong \bigoplus_{i=1}^u \mathbf{Z}\sigma_i$, and so the homotopy class of a map $S^2 \to X$ is given by an element $D = \sum_{i=1}^u d_i \sigma_i$ of the kernel of the homomorphism $\iota^* : \bigoplus_{i=1}^u \mathbf{Z}\sigma_i \to \mathbf{Z}^r$, $\sigma_i \mapsto v_i$, where each d_i is a non-negative integer. (Recall from §2 that, in the non-singular case, $\pi_2 X$ may be identified with $\mathrm{Ker}\,\iota^*$.) From the identification $\mathrm{Hol}_D^*(S^2, X) \cong Q_D^X(\mathbf{C})$ of Proposition 3.1, we have the following consequences:

(i) $\mathrm{Hol}_D^*(S^2, X)$ is connected,

(ii) the fundamental group of $\mathrm{Hol}_D^*(S^2, X)$ is free abelian of finite rank.

The first of these follows from the fact that the space $Q_D^X(\mathbf{C})$ may be obtained from the affine space of u-tuples of monic polynomials of degrees $d_1, \ldots, d_u$, by removing a finite collection of complex hypersurfaces. The second is proved in the Appendix of [GKY].

THEOREM 4.1. *Let X be a non-singular projective toric variety. Then the inclusion*

$$\mathrm{Hol}_D^*(S^2, X) \to \mathrm{Map}_D^*(S^2, X)$$

is a homotopy equivalence up to dimension d, where $d = \min\{d_1, \ldots, d_u\}$ (i.e. this map induces isomorphisms on homotopy groups in dimensions less than d, and an epimorphism in dimension d).

There are two approaches to proving this theorem, depending on how one views the space $Q^X(\mathbf{C})$. The first way, which was the original motivation for this paper, is to view $Q^X(\mathbf{C})$ as a subspace of $Q^{\mathbf{C}P^N}(\mathbf{C})$ via the inclusion map θ'' (see the proof of Proposition 3.1). It is the subspace consisting of configurations of points whose labels satisfy the additional condition that they belong to the image of the linear transformation T^*. This condition arises because of the fact (mentioned earlier) that an embedding of X in $\mathbf{C}P^N$ may be chosen such that the equations of X are all of the form “monomial = monomial”.

It suggests that the theorem might be proved by interpreting Segal's proof for $\mathbf{C}P^N$ in terms of labelled configurations, and then imposing the additional linear condition on the labels. While this is essentially valid, there are some technical difficulties, and it is more convenient (and perhaps more natural) to work with the space $Q^X(\mathbf{C})$ directly, without using a particular projective embedding of X. This is what we shall do.

Let $Q^X_{d_1,\ldots,d_u}(U)$ be the space of configurations of distinct points z in $U(\subseteq\mathbf{C})$ with labels $l_z \in \mathrm{SF}(\Delta)^*$ which satisfy condition (X), with $\sum_z l_z(\sigma_i)=d_i$ for all i (where $d_i \geqslant 0$ for all i). Let V be an open subset of $\mathbf{C}$ with $U\subset V$. One may define a natural inclusion $j: Q^X_{d_1,\ldots,d_i,\ldots,d_u}(U)\to Q^X_{d_1,\ldots,d_i+1,\ldots,d_u}(V)$ by adjoining to each configuration a fixed point in $V-U$ with the label σ_i.

PROPOSITION 4.2. *The inclusion* $j: Q^X_{d_1,\ldots,d_i,\ldots,d_u}(U)\to Q^X_{d_1,\ldots,d_i+1,\ldots,d_u}(V)$ *is a homotopy equivalence up to dimension* d_i.

Proof. This result is proved in [GKY]. The idea of the proof is to reduce it to the well known fact that the inclusion $\mathrm{Sp}^d(U)\to\mathrm{Sp}^{d+1}(V)$ of symmetric products is a homotopy equivalence up to dimension d. □

The space $Q^X_D(\mathbf{C})$ of the previous section is of this form. Let $D=(d_1,\ldots,d_u)$ and $D'=(d'_1,\ldots,d'_u)$ be multi-degrees with $d_i\leqslant d'_i$ for all i (we write $D\leqslant D'$). By adjoining a fixed labelled configuration in $V-U$ we obtain an inclusion $j: Q^X_D(U)\to Q^X_{D'}(V)$. Evidently we have:

COROLLARY 4.3. *The inclusion* $j: Q^X_D(U)\to Q^X_{D'}(V)$ *is a homotopy equivalence up to dimension* $d=\min\{d_1,\ldots,d_u\}$. □

We shall introduce a stabilized space using the idea of [Se]. Let $\xi=\{(z_i,l_i)\,|\,i=1,2,\ldots\}$ be a sequence of points of $Q^X(\mathbf{C})$ with $z_i\to\infty$.

Definition. $\widehat{Q}^{\xi}_{k_1,\ldots,k_u}(\mathbf{C})$ is the set of sequences $\{(w_i,m_i)\,|\,i=1,2,\ldots\}$ of points of $Q^X(\mathbf{C})$, which agree with ξ except possibly for a finite number of terms, such that $\sum_i(l_i-m_i)_j=k_j$ for $j=1,\ldots,u$. We shall write $\widehat{Q}^{\xi}_0(\mathbf{C})$ for $\widehat{Q}^{\xi}_{0,\ldots,0}(\mathbf{C})$.

Let $D_1<D_2<D_3<\ldots$ be a sequence of multi-degrees. We may choose open discs $U_1\subset U_2\subset U_3\subset\ldots$ in $\mathbf{C}$ and labelled configurations in each U_i-U_{i-1} so as to obtain a sequence of inclusions

$$Q^X_{D_1}(U_1)\to Q^X_{D_2}(U_2)\to Q^X_{D_3}(U_3)\to\ldots.$$

The choice of labelled configurations defines a sequence ξ such that $\widehat{Q}^{\xi}_0$ is $\bigcup_{i\geqslant 1}Q^X_{D_i}(U_i)$. (Since $Q^X_{D_i}(U_i)$ is homeomorphic to $Q^X_{D_i}(\mathbf{C})$ and hence to $\mathrm{Hol}^*_{D_i}(S^2,X)$, this construction may be taken as the definition of the limit "$\lim_{D\to\infty}\mathrm{Hol}^*_D(S^2,X)$" of §1.)

Let A be any subspace of $S^2=\mathbf{C}\cup\infty$, and let B be a closed subspace of A. Let $Q^X(A,B)$ denote $Q^X(A)/\sim$, where $Q^X(A)$ is the space of configurations of distinct points in A with labels in $\mathrm{SF}(\Delta)^*$ satisfying condition (X), and where two labelled configurations are defined to be equivalent if they agree on the complement of B in A. (This is a connected space, if A is connected.) If Σ is a labelled configuration, then $\Sigma\cap B_z$ defines an element of $Q^X(B_z,\partial B_z)$, where B_z is the closed unit disc with centre z. We may identify $Q^X(B_z,\partial B_z)$ canonically with $Q^X(S^2,\infty)$ and so we obtain a map

$$\mathbf{C}\times Q^X_D(\mathbf{C})\to \big(Q^X(B_z,\partial B_z)\to\big)\, Q^X(S^2,\infty),\quad (z,\Sigma)\mapsto\Sigma\cap B_z.$$

This extends to a continuous map $S^2\times Q^X_D(\mathbf{C})\to Q^X(S^2,\infty)$ with $(\infty,\Sigma)\mapsto\varnothing$. The adjoint map

$$S_D\colon Q^X_D(\mathbf{C})\to\Omega^2 Q^X(S^2,\infty)$$

will be called the scanning map. This is a generalization of a construction introduced in [Se] for the case $X=\mathbf{C}P^n$. As in [Se] we obtain a stabilized map

$$\widehat{S}\colon \widehat{Q}^\xi_0(\mathbf{C})\to\Omega^2_0 Q^X(S^2,\infty),$$

where Ω^2_0 denotes the component of Ω^2 which contains the constant maps.

PROPOSITION 4.4. *$\widehat{S}$ is a homotopy equivalence.*

Proof. This is entirely analogous to the proof in §3 of [Se] for the case $\mathbf{C}P^n$. Another treatment of the same argument was given in [Gu1] (in the case of the quadric cone) and in [Gu2] (in the case of $\mathbf{C}P^n$). □

Next we shall examine the relation between S_D and the inclusion map

$$I_D\colon \mathrm{Hol}^*_D(S^2,X)\to\mathrm{Map}^*_D(S^2,X).$$

PROPOSITION 4.5. *The maps S_D, I_D may be identified with each other, up to homotopy.*

Proof. Let $F^X=\{f\colon U\to X\mid f=g|_U,\ g\in\mathrm{Hol}(S^2,X)\}$, where U is the open unit disc in $\mathbf{C}$. The evaluation map $e\colon F^X\to X$, $f\mapsto f(0)$ is a homotopy equivalence. Let $\widetilde{F}^X=\{f\in F^X\mid f(U)\cap(T^{\mathbf{C}}\cdot *)\neq\varnothing\}$. Then $\widetilde{F}^X$ is obtained from F^X by removing those maps with image in the complement of $T^{\mathbf{C}}\cdot *$ (a subspace of F^X of infinite codimension), and so the evaluation map $e\colon\widetilde{F}^X\to X$ is also a homotopy equivalence.

The action of $T^{\mathbf{C}}$ on $\widetilde{F}^X$ is (by construction) free, in contrast to the action of $T^{\mathbf{C}}$ on X. (Thus, $\widetilde{F}^X/T^{\mathbf{C}}$ is the homotopy quotient $X//T^{\mathbf{C}}$.) Let $p\colon\widetilde{F}^X\to\widetilde{F}^X/T^{\mathbf{C}}$ be the natural map. There is a map $u\colon\widetilde{F}^X/T^{\mathbf{C}}\to Q^X(\overline{U},\partial\overline{U})$, defined by sending the labelled configuration Σ which represents an element $[f]\in\widetilde{F}^X/T^{\mathbf{C}}$ to the labelled configuration $\Sigma\cap\bar{U}$. It may be shown by an elementary argument as in [Se, Proposition 4.8] that u is a homotopy equivalence.

The discussion so far may be summarized in the following diagram:

$$\begin{tikzcd} \widetilde{F}^X \arrow[r, "e"] \arrow[d, "p"'] & X \\ \widetilde{F}^X/T^C \arrow[r, "u"] & Q^X(\bar{U},\partial\bar{U}). \end{tikzcd}$$

Consider next the diagram below

$$\begin{tikzcd} Q^X_D(\mathbf{C}) \arrow[r, "s_D"] \arrow[d, "="'] & \mathrm{Map}(\mathbf{C},\widetilde{F}^X) \arrow[r] \arrow[d] & \mathrm{Map}(\mathbf{C},X) \\ Q^X_D(\mathbf{C}) \arrow[r, "\bar{s}_D"] & \mathrm{Map}(\mathbf{C},\widetilde{F}^X/T^{\mathbf{C}}) \arrow[r] & \mathrm{Map}(\mathbf{C},Q^X(\bar{U},\partial\bar{U})), \end{tikzcd}$$

where $s_D(f)(z)$ is the map $w\mapsto f(w+z)$ in $\widetilde{F}^X$, for $f\in\mathrm{Hol}^*_D(S^2,X)\cong Q^X_D(\mathbf{C})$. (The right hand side of the diagram is induced from the previous diagram, and $\bar{s}_D$ is the map induced by s_D.)

Observe that $\mathrm{Map}(\mathbf{C},\cdot\,)$ can be replaced by $\mathrm{Map}^*(S^2,\cdot\,)$, i.e. all the relevant maps extend from $\mathbf{C}$ to $\mathbf{C}\cup\infty=S^2$ (as based maps). Thus we obtain the following commutative diagram, where the suffix D denotes the appropriate component:

$$\begin{tikzcd} Q^X_D(\mathbf{C}) \arrow[r, "s_D"] \arrow[d, "="'] & \mathrm{Map}^*_D(S^2,\widetilde{F}^X) \arrow[r, "\Omega^2 e"] \arrow[d, "\Omega^2 p"'] & \mathrm{Map}^*_D(S^2,X) \\ Q^X_D(\mathbf{C}) \arrow[r, "\bar{s}_D"] & \mathrm{Map}^*_D(S^2,\widetilde{F}^X/T^{\mathbf{C}}) \arrow[r, "\Omega^2 u"] & \mathrm{Map}^*_D(S^2,Q^X(\bar{U},\partial\bar{U})). \end{tikzcd}$$

The top row is the inclusion map I_D, and the bottom row is the scanning map S_D. The proof of the proposition is completed by noting that $\Omega^2 e,\Omega^2 u$ are homotopy equivalences (because e,u are), and that $\Omega^2 p$ is a homotopy equivalence because p is a fibration with fibre $(\mathbf{C}^*)^r$. $\square$

Corollary 4.3, Proposition 4.4 and Proposition 4.5 constitute a proof of Theorem 4.1.

5. Sketch of the theorem in the general case

In this section we shall sketch how Theorem 4.1 may be extended to arbitrary compact toric varieties. As it seems hard to give a single general statement, we shall first obtain a result under certain special assumptions, and then explain very briefly how to proceed when the assumptions are not satisfied. There are two independent parts to the result. First, we must show:

(I) The inclusion $\mathrm{Hol}_D^*(S^2,X)\to\mathrm{Map}_D^*(S^2,X)$ induces a homotopy equivalence in the limit $D\to\infty$.

Then we must find an integer $n(D)$ such that:

(II) The inclusion $\mathrm{Hol}_D^*(S^2,X)\to\mathrm{Hol}_{D'}^*(S^2,X)$ is a homotopy equivalence up to dimension $n(D)$.

Part (I) can be carried out by making technical modifications to the argument of §4, as we shall show in Theorem 5.1. On the other hand, part (II) needs a new idea, which we give in Theorem 5.2.

To carry out (I), we need to define an appropriate stabilization procedure. Let $\mathrm{SF}(\Delta)^*_{\geqslant 0}$ denote the non-negative elements of $\mathrm{SF}(\Delta)^*$, i.e. those which take non-negative values on positive divisors. Let $D_1, D_2, D_3, \ldots$ be a sequence in $\mathrm{SF}(\Delta)^*_{\geqslant 0}$, such that each $D_i-D_{i-1}=l_i$ is a valid label; we write $D_1<D_2<D_3 \ldots$. As in the non-singular case, we may choose open discs $U_1\subset U_2\subset U_3\subset\ldots$ in $\mathbf{C}$ and a sequence $\xi=\{(z_i,l_i)\,|\,i=1,2,\ldots\}$ of labelled points (with $z_i\in U_i-U_{i-1}$) so as to obtain a sequence of inclusions

$$Q^X_{D_1}(U_1)\to Q^X_{D_2}(U_2)\to Q^X_{D_3}(U_3)\to\ldots .$$

We obtain a stabilized space $\widehat{Q}^\xi_0(\mathbf{C})=\bigcup_{i\geqslant 0}Q^X_{D_i}(U_i)$ in the usual way. The inclusion $Q^X_{D_i}(U_i)\to Q^X_{D_{i+1}}(U_{i+1})$ may be regarded (up to homotopy) as a map $\mathrm{Hol}^*_{D_i}(S^2,X)\to\mathrm{Hol}^*_{D_{i+1}}(S^2,X)$, when $D_i, D_{i+1}\in\mathrm{Ker}\,\iota^*$. We then have:

THEOREM 5.1. *Let X be a projective toric variety, such that H_2X is torsion free, and such that the configuration spaces $Q^X_D(\mathbf{C})$ are (non-empty and) connected for all $D\in\mathrm{SF}(\Delta)^*_{\geqslant 0}$. Then the inclusion*

$$\lim_{D\to\infty}\mathrm{Hol}_D^*(S^2,X)\to\lim_{D\to\infty}\mathrm{Map}_D^*(S^2,X)\simeq\mathrm{Map}_0^*(S^2,X)$$

is a homotopy equivalence.

Proof. This is similar to the proof of Propositions 4.4 and 4.5, so we shall just point out the new features. For the proof of Proposition 4.4 one needs to know

(a) $Q^X_D(\mathbf{C})$ is (non-empty and) connected for all $D\in\mathrm{SF}(\Delta)^*_{\geqslant 0}$,

(b) the components of $\mathrm{Map}^*(S^2,Q^X(S^2,\infty))$ are indexed by $\mathrm{SF}(\Delta)^*$, i.e.

$$\pi_2 Q^X(S^2,\infty)\cong\mathrm{SF}(\Delta)^*,$$

and

(c) $\pi_1 Q^X_D(\mathbf{C})$ is abelian for D sufficiently large (see the proof of Lemma 3.4 of [Se]). These three points are easy to establish when X is smooth, but are not immediately obvious when X is singular. We have included part (a) as a hypothesis; in Appendix 1 to this paper we give a method to determine when this hypothesis is satisfied. Part (b) follows from the existence of an isomorphism $\mathrm{SF}(\Delta)\to H^2(X//T^{\mathbf{C}})$, which is given by assigning to a $T^{\mathbf{C}}$-equivariant line bundle on X the first Chern class of the associated line bundle on $X//T^{\mathbf{C}}$. The truth of part (c) is explained in Appendix 1. Finally, Proposition 4.5 is proved in exactly the same way, when X is singular. $\square$

Our method for (II) depends on the fact that it is possible to choose a *toric resolution* $\theta\colon\widehat{X}\to X$. We shall recall briefly this procedure (see [Fu, §2.7], and [Od1, §1.5]). First, assume that the fan Δ of X is simplicial, i.e. for any cone $\sigma=\{\sum_{i=1}^k a_i v_i \mid a_i\geqslant 0\}$ in Δ, where $\sigma\cap\mathbf{Z}^r=\mathbf{Z}v_i$, the vectors $v_1,\dots,v_k\in\mathbf{Z}^r$ are linearly independent. Let σ be a maximal cone in Δ. The multiplicity of σ is defined to be the index of $\bigoplus_{i=1}^k \mathbf{Z}v_i$ in $\mathbf{Z}^r$. By the criterion for singularity, X is singular if and only if there is a maximal cone σ in Δ of multiplicity greater than one. In such a case, there is some $v=\sum_{i=1}^k c_i v_i\in\sigma\cap\mathbf{Z}^r$ such that $0\leqslant c_i<1$ for all i. Let $\hat{\Delta}$ be the fan obtained by sub-dividing Δ in the obvious way, i.e. by replacing σ by the joins of $\mathbf{R}_{\geqslant 0}v$ with all the faces of σ. This is a fan corresponding to a toric variety $\widehat{X}$ which is "less singular" than X. This process of "inserting a ray" may be repeated finitely many times, to obtain a non-singular variety $\widehat{X}$ and an equivariant map $\theta\colon\widehat{X}\to X$ which is a resolution of X. Finally, if the fan Δ is not simplicial, it is easy to see that Δ may be made simplicial by inserting suitable rays.

The configuration space for X is related to the configuration space for $\widehat{X}$ by the (set theoretic) formula

$$Q_D^X(\mathbf{C})=\bigcup_{\widehat{D}\in\theta_*^{-1}(D)} Q_{\widehat{D}}^{\widehat{X}}(\mathbf{C}),$$

where $\theta_*\colon\pi_2\widehat{X}\to\pi_2 X$ is the homomorphism induced by $\theta\colon\widehat{X}\to X$. The right hand side of this formula inherits the topology of the left hand side; in Appendix 2 we shall give a more concrete description of this topology. The idea of our method is to use the fact that a result of the required type is known for the spaces $Q_{\widehat{D}}^{\widehat{X}}(\mathbf{C})$ (Proposition 4.2, Corollary 4.3).

We shall begin by considering the special case where $\Delta,\hat{\Delta}$ are simplicial and $\hat{\Delta}$ is obtained from Δ by inserting a single vector $v'=\sum_{i=1}^k c_i v_i$ (with $0\leqslant c_i<k$) into a k-dimensional cone σ spanned by vectors $v_1,\dots,v_k$. Let $v_{k+1},\dots,v_u$ be the generating vectors of the remaining one-dimensional cones of Δ. We have $\mathrm{SF}(\hat{\Delta})^*\cong\mathbf{Z}\sigma'\oplus(\bigoplus_{i=1}^u\mathbf{Z}\sigma_i)$ and we may identify $\mathrm{SF}(\Delta)^*$ with a subspace of $\bigoplus_{i=1}^u\mathbf{R}\sigma_i$. The map $T^*\colon\mathrm{SF}(\hat{\Delta})^*\to\mathrm{SF}(\Delta)^*$ is given by $T^*(x\sigma'+\sum_{i=1}^u x_i\sigma_i)=\sum_{i=1}^k(x_i+c_i x)\sigma_i+\sum_{i=k+1}^u x_i\sigma_i$. Let $D=\sum_{i=1}^u e_i\sigma_i\in$

$\operatorname{Ker}\iota^*$. From the form of T^*, we see that

$$\theta_*^{-1}(D)=\left\{d\sigma'+\sum_{i=1}^{k}(e_i-c_id)\sigma_i+\sum_{i=k+1}^{u}e_i\sigma_i\in\operatorname{Ker}\hat{\iota}^*\;\middle|\;0\leqslant d\leqslant\min\{e_i/c_i\mid c_i\neq 0\}\right\}.$$

The possible values of d here are of the form $d_0, d_0+l, \dots, d_0+bl, \dots, d_0+ml$, for some non-negative integers d_0, m. We denote by $\widehat{D}_b$ the element of $\theta_*^{-1}(D)$ corresponding to $d=d_0+bl$.

From now on we shall denote simply by Q_D^X the spaces $Q_D^X(\mathbf{C})$ or $Q_D^X(U)$. For $D, D'\in \operatorname{Ker}\iota$, we write $D'\geqslant D$ if $D'=D+\sum_{i=1}^{k}ac_il\sigma_i+\sum_{i=k+1}^{u}a_i\sigma_i\in\operatorname{Ker}\iota^*$, where $a, a_{k+1}, \dots, a_u$ are non-negative. We then have a stabilization map $s\colon Q_D^X\to Q_{D'}^X$.

THEOREM 5.2. *Let X be a projective toric variety which admits a resolution $\widehat{X}\to X$ of the above form. Let $D=\sum_{i=1}^{u}e_i\sigma_i$ and $D'\geqslant D$. Then the stabilization map $s\colon Q_D^X\to Q_{D'}^X$ is a homotopy equivalence up to dimension $n(D)=\min\{e, e_{k+1}, \dots, e_u\}$, where*

$$e=\max\{\min\{d_0+jl, e_1-c_1(d_0+jl), \dots, e_k-c_k(d_0+jl)\}\mid j=0,1,\dots,m\}.$$

Moreover, $\lim_{D\to\infty}n(D)=\infty$.

Later we shall give an example to show that the hypothesis on the resolution is not a serious restriction (Example 5.4).

Proof. Let $Q_{D;j}^X=\bigcup_{b=j}^{m}Q_{\widehat{D}_b}^{\widehat{X}}$. The stabilization map $s\colon Q_D^X\to Q_{D'}^X$ induces a stabilization map $s_j\colon Q_{D;j}^X\to Q_{D';j+a}^X$ for each j. We claim that

($*$) the stabilization map s_j is a homology equivalence up to dimension $n(D;j)=\min\{d_0+jl, e_{k+1}, \dots, e_u\}$, and

($**$) the inclusion $Q_{D;j}^X\to Q_{D;0}^X=Q_D^X$ is a homology equivalence up to dimension $m(D;j)=\min\{e_1-c_1(d_0+jl), \dots, e_k-c_k(d_0+jl)\}$.

These statements imply that s is a homology equivalence up to dimension

$$\min\{m(D;j), m(D', j+a), n(D;j)\}=\min\{m(D;j), n(D;j)\}.$$

By choosing j so as to maximize this number, we obtain the stated value of $n(D)$, and it is easy to verify that $\lim_{D\to\infty}n(D)=\infty$. To prove the theorem, therefore, we must prove ($*$) and ($**$), and then show that "homology" can be replaced by "homotopy".

Statements ($*$) and ($**$) are analogous to statements (3) and (4) in the proof of Proposition 3.2 of [Gu1], so we shall just summarize their proofs.

Proof of ($*$). We have

$$\theta_*^{-1}(D')=\{\widehat{D}_0', \widehat{D}_1', \dots, \widehat{D}_{m+a}'\},$$

and (for $b=0,1,...,m$)

$$\widehat{D}'_{b+a}=\widehat{D}'_b+al\sigma'+\sum_{i=k+1}^{u}a_i\sigma_i.$$

For each $b=j,...,m$, we have a stabilization map $\hat{s}_b: Q^{\widehat{X}}_{\widehat{D}_b}\to Q^{\widehat{X}}_{\widehat{D}'_{b+a}}$, defined by adding a point of $U'-U$ with label $\widehat{D}'_{b+a}-\widehat{D}_b$. The stabilization map $s_j: Q^X_{D;j}\to Q^X_{D';j+a}$ is defined by adding the same point with label $D'-D=\sum_{b=j}^m T^*(\widehat{D}'_{b+a}-\widehat{D}_b)$. These are compatible, in the sense that the following diagram is commutative:

$$\begin{tikzcd}
Q^X_{D;j} \arrow[r, "="] \arrow[d, "s_j"'] & \bigcup_{b=j}^m Q^{\widehat{X}}_{\widehat{D}_b} \arrow[d, "\bigcup_{b=j}^m \hat{s}_b"] \\
Q^X_{D';j+a} \arrow[r, "="] & \bigcup_{b=j}^m Q^{\widehat{X}}_{\widehat{D}'_{b+a}}
\end{tikzcd}$$

By Corollary 4.3, each map $\hat{s}_b$ is a homology equivalence up to dimension $\min\{d_0+bl, e_{k+1},...,e_u\}$. By the Mayer–Vietoris argument used in [GKY, Theorem 2.5], it follows that $\bigcup_{b=j}^m \hat{s}_b$ (and hence s_j) is a homology equivalence up to dimension $\min\{d_0+jl, e_{k+1},...,e_u\}$.

Proof of $(**)$. It suffices to prove that the inclusion $Q^X_{D;j}\to Q^X_{D;j-1}$ is a homology equivalence up to dimension $m(D;j)$. To prove this, we shall use an identification

$$Q^X_{D;j-1}\simeq Q^X_{D;j}\bigcup_f P^{\widehat{X}}_{\widehat{D}_{j-1}},$$

where $P^{\widehat{X}}_{\widehat{D}_{j-1}}$ denotes the configuration space defined in exactly the same way as $Q^{\widehat{X}}_{\widehat{D}_{j-1}}$ except that condition ($\widehat{\mathrm{X}}$) is relaxed: for any point z with label l, the integers $\{l(\sigma_i)|c_i\neq 0\}$ are allowed to be simultaneously zero. We define $P^{\widehat{X};1}_{\widehat{D}_{j-1}}$ to be the closed subspace of $P^{\widehat{X}}_{\widehat{D}_{j-1}}$ consisting of configurations for which the label of at least one point satisfies the condition $l(\sigma_i)\geqslant 1$, for all i such that $c_i\neq 0$. (It follows then that $l(\sigma_i)\geqslant lc_i$.) We have $Q^{\widehat{X}}_{\widehat{D}_j}=P^{\widehat{X}}_{\widehat{D}_{j-1}}-P^{\widehat{X};1}_{\widehat{D}_{j-1}}$, and the attaching map f is the natural map

$$f: P^{\widehat{X};1}_{\widehat{D}_{j-1}}\to Q^X_{D;j}.$$

Hence, the lemma is equivalent to the assertion that the inclusion

$$p_{j-1}: P^{\widehat{X};1}_{\widehat{D}_{j-1}}\to P^{\widehat{X};0}_{\widehat{D}_{j-1}}=P^{\widehat{X}}_{\widehat{D}_{j-1}}$$

is a homology equivalence up to dimension $m(D;j)$. To prove this assertion, we use the stabilization map $s: P^{\widehat{X};0}_E\to P^{\widehat{X};1}_{E+F}$, where $E\in\mathrm{SF}(\hat{\Delta})^*$ and $F=\sum_{i=1}^k c_il\sigma_i$. Consider the composition

$$P^{\widehat{X};0}_{\widehat{D}_{j-1}-F}\xrightarrow{s}P^{\widehat{X};1}_{\widehat{D}_{j-1}}\xrightarrow{p_{j-1}}P^{\widehat{X};0}_{\widehat{D}_{j-1}}.$$

This is homotopic to a stabilization map of the type of Proposition 4.2, hence is a homology equivalence up to dimension

$$\min\{e_1-c_1(d_0+(j-1)l)-c_1l,\ldots,e_k-c_k(d_0+(j-1)l)-c_kl\}.$$

From this we conclude that the map p_{j-1} induces surjections in homology up to dimension $m(D;j)$. Next consider the composition

$$P^{\widehat{X};1}_{\widehat{D}_{j-1}} \xrightarrow{p_{j-1}} P^{\widehat{X};0}_{\widehat{D}_{j-1}} \xrightarrow{s} P^{\widehat{X};1}_{\widehat{D}_{j-1}+F}.$$

This is homotopic to the stabilization map $P^{\widehat{X};1}_{\widehat{D}_{j-1}} \to P^{\widehat{X};1}_{\widehat{D}_{j-1}+F}$. By the method of [Gu1, Proposition 3.2], it may be deduced from Proposition 4.2 that this is a homology equivalence up to dimension $\min\{e_1-c_1(d_0+(j-1)l)-c_1l,\ldots,e_k-c_k(d_0+(j-1)l)-c_kl\}$. From this we conclude that the map p_{j-1} induces injections in homology up to dimension $m(D;j)-1$. Thus, p_{j-1} is a homology equivalence up to dimension $m(D;j)$, as required. This completes the proof of $(**)$.

To pass from homology to homotopy, we make use of the fact (see [HH]) that a map induces isomorphisms of homotopy groups if and only if it induces (a) isomorphisms of homology groups with arbitrary local coefficients, and (b) an isomorphism of fundamental groups. The stabilization map $Q^X_D \to Q^X_{D'}$ satisfies (a), because the above argument for homology with integer coefficients extends word for word to the case of arbitrary local coefficients: the basic ingredients were the Mayer–Vietoris exact sequence and the exact sequence of a pair, together with Proposition 4.2. For (b), we combine the homology statement with the fact that $\pi_1 Q^X_D$ is abelian (see Appendix 1). □

Example 5.3: *The "quadric cone" $z_2^2=z_1z_3$ in $\mathbf{C}P^3$* (see Example 3.4). Here, $\pi_2 X \cong \mathbf{Z}$. The variety has one singular point, $[1;0;0;0]$, which corresponds to the cone σ spanned by v_1, v_2. The multiplicity of σ is 2. A toric resolution may be obtained by "inserting" the vector $v'=\frac{1}{2}v_1+\frac{1}{2}v_2=(0,1)$. The corresponding variety $\widehat{X}$ is the Hirzebruch surface Σ_2 (cf. Example 2.2).

Let $D=g\sigma_1+g\sigma_2+2g\sigma_3 \in \operatorname{Ker}\iota^*$, with $g\in\frac{1}{2}\mathbf{Z}$. For simplicity, let us assume that $g\in\mathbf{Z}$ (the case where $g-\frac{1}{2}\in\mathbf{Z}$ is similar). Then $\theta_*^{-1}(D)=\{\widehat{D}_0,\widehat{D}_1,\ldots,\widehat{D}_g\}$, where $\widehat{D}_b=(g-b)\sigma_1+2b\sigma'+(g-b)\sigma_2+2g\sigma_3$. Here we have $c_1=c_2=\frac{1}{2}$, $l=2$, $d_0=0$, $m=g$. We have $D'=D+a\sigma_1+a\sigma_2+2a\sigma_3$. By Theorem 5.2, the stabilization map $Q^X_D \to Q^X_{D'}$ is a homotopy equivalence up to dimension $\min\{e,2g\}$, where

$$e=\max\{\min\{2j,g-j\}\mid j=0,1,\ldots,g\}=\left[\tfrac{2}{3}g\right].$$

We conclude that $n(D)=\left[\frac{2}{3}g\right]$ in this case.

For any projective toric variety X, the *method* of Theorem 5.2 may be used after factoring a resolution $\widehat{X}\to X$ into maps of the above form. Rather than attempt to give a general formula for $n(D)$, however, we shall just illustrate the method in the following particular but non-trivial case.

Example 5.4: *The weighted projective space* $P(1,2,3)$ (see Examples 2.3, 3.5). Here, $\pi_2 X\cong\mathbf{Z}$. The cone spanned by v_2, v_3 has multiplicity 2, and the cone spanned by v_1, v_3 has multiplicity 3. By inserting $v'=\frac{1}{2}v_2+\frac{1}{2}v_3=(-1,-1)$ we can resolve the singular point represented by the first cone. By inserting $v''=\frac{2}{3}v_1+\frac{1}{3}v_3=(0,-1)$, we replace the second cone by two maximal cones, one of which represents a singular point, namely that spanned by v'', v_3. It has multiplicity 2 (<3). We resolve this singularity by inserting $v'''=\frac{1}{2}v''+\frac{1}{2}v_3=(-1,-2)$.

To use the method of Theorem 5.2, we need to factor our resolution $\widehat{X}\to X$ into a sequence of three simple resolutions. Let us re-number the vectors defining the fan $\hat{\Delta}$ of the resolution $\widehat{X}$ as follows: $v_1=(1,0)$, $v_2=(0,1)$, $v_3=(-1,-1)$, $v_4=(-2,-3)$, $v_5=(-1,-2)$, $v_6=(0,-1)$. We shall use the sequence of resolutions

$$\widehat{X}=X_{356}\to X_{36}\to X_3\to X,$$

where X_3 denotes the toric variety whose fan is obtained from the fan of X by inserting v_3, and so on.

First step: $X_{356}\to X_{36}$. We shall (temporarily) write $\widehat{X}=X_{356}$, $X=X_{36}$. Let $D=\sum_{i\neq 5} e_i\sigma_i\in \mathrm{SF}(\Delta)^*$. Here $e_4, e_6\in\frac{1}{2}\mathbf{Z}$ and $e_1, e_2, e_3, e_4+e_6\in\mathbf{Z}$. The map $T^*\colon \mathrm{SF}(\hat{\Delta})^*\to \mathrm{SF}(\Delta)^*$ is given by

$$T^*\left(\sum_{i=1}^{6} x_i\sigma_i\right)=x_1\sigma_1+x_2\sigma_2+x_3\sigma_3+\left(x_4+\tfrac{1}{2}x_5\right)\sigma_4+\left(x_6+\tfrac{1}{2}x_5\right)\sigma_6.$$

We have $\theta_*^{-1}(D)=\{\widehat{D}_0,\widehat{D}_1,\dots,\widehat{D}_m\}$, where $\widehat{D}_b=\sum_{i=1}^{6} d_i\sigma_i$, and $e_1=d_1$, $e_2=d_2$, $e_3=d_3$, $e_4=d_4+\frac{1}{2}d_5$, $e_6=d_6+\frac{1}{2}d_5$. Let us assume that $e_4, e_6\in\mathbf{Z}$. Then we may write $d_5=2b$, hence $\widehat{D}_b=e_1\sigma_1+e_2\sigma_2+e_3\sigma_3+(e_4-b)\sigma_4+2b\sigma_5+(e_6-b)\sigma_6$ for $b=0,1,\dots,m$. We have $c_4=c_6=\frac{1}{2}$, $l=2$, $d_0=0$, $m=\min\{e_4,e_6\}$. This is very similar to the situation of Example 5.3. By Theorem 5.2, the stabilization map $Q_D^X\to Q_{D'}^X$ is a homotopy equivalence up to dimension $\min\{\left[\frac{2}{3}e_4\right],\left[\frac{2}{3}e_6\right],e_1,e_2,e_3\}$.

Second step: $X_{36}\to X_3$. We write $\widehat{X}=X_{36}$, $X=X_3$. Let $D=\sum_{i=1}^{4} e_i\sigma_i\in\mathrm{SF}(\Delta)^*$. Here $e_1, e_4\in\frac{1}{3}\mathbf{Z}$ and $e_2, e_3, e_1+e_4\in\mathbf{Z}$. The map $T^*\colon\mathrm{SF}(\hat{\Delta})^*\to\mathrm{SF}(\Delta)^*$ is given by

$$T^*\left(\sum_{i\neq 5} x_i\sigma_i\right)=\left(x_1+\tfrac{2}{3}x_6\right)\sigma_1+x_2\sigma_2+x_3\sigma_3+\left(x_4+\tfrac{1}{3}x_6\right)\sigma_4.$$

Let us assume that $e_1, e_4 \in \mathbf{Z}$. Then we have $\theta_*^{-1}(D)=\{\widehat{D}_0, \widehat{D}_1, ..., \widehat{D}_m\}$, where $\widehat{D}_b=(e_1-2b)\sigma_1+e_2\sigma_2+e_3\sigma_3+(e_4-b)\sigma_4+3b\sigma_6$. In this situation, $c_1=\frac{2}{3}$, $c_4=\frac{1}{3}$, $l=3$, $d_0=0$, $m=\min\{[\frac{1}{2}e_1], e_4\}$. We may now apply (the method of) Theorem 5.2. A slight strengthening of the result of the first step is needed here, namely that the "individual" stabilization map $s_i: Q_D^{\widehat{X}} \to Q_{D+\sigma_i}^{\widehat{X}}$ is a homotopy equivalence up to dimension e_i for $i=1,2,3$, and up to dimension $[\frac{2}{3}e_i]$ for $i=4,6$. (We omit the proof of this result.) We find that the stabilization map $Q_D^X \to Q_{D'}^X$ is a homotopy equivalence up to dimension $\min\{e, e_2, e_3\}$, where

$$e=\max\{\min\{\tfrac{2}{3}(3j), e_1-2j, [\tfrac{2}{3}(e_4-j)]\} \mid j=0,1,...,m\}=\min\{[\tfrac{1}{2}e_1], [\tfrac{1}{2}e_4]\}.$$

Third step: $X_3 \to X$. We write $\widehat{X}=X_3$. Let $D=e_1\sigma_1+e_2\sigma_2+e_4\sigma_4 \in \mathrm{SF}(\Delta)^*$. The conditions on e_1, e_2, e_4 were given earlier. The map $T^*: \mathrm{SF}(\hat{\Delta})^* \to \mathrm{SF}(\Delta)^*$ is given by

$$T^*\left(\sum_{i=1}^{4} x_i\sigma_i\right)=x_1\sigma_1+\left(x_2+\tfrac{1}{2}x_3\right)\sigma_2+\left(x_4+\tfrac{1}{2}x_3\right)\sigma_4.$$

Let us assume that $e_2, e_4 \in \mathbf{Z}$. Then we have $\theta_*^{-1}(D)=\{\widehat{D}_0, \widehat{D}_1, ..., \widehat{D}_m\}$, where $\widehat{D}_b=e_1\sigma_1+(e_2-b)\sigma_2+2b\sigma_3+(e_4-b)\sigma_4$. This time we have $c_2=c_4=\frac{1}{2}$, $l=2$, $d_0=0$, $m=\min\{e_2, e_4\}$. By the above method we find that the stabilization map $Q_D^X \to Q_{D'}^X$ is a homotopy equivalence up to dimension $\min\{e, [\frac{1}{2}e_1]\}$, where

$$e=\max\{\min\{2j, e_2-j, [\tfrac{1}{2}(e_4-j)]\} \mid j=0,1,...,m\}=\min\{[\tfrac{2}{3}e_2], [\tfrac{2}{5}e_4]\}.$$

In conclusion, we have shown that for $X=P(1,2,3)$ the inclusion $\mathrm{Hol}_D^*(S^2, X) \to \mathrm{Map}_D^*(S^2, X)$ is a homotopy equivalence up to dimension

$$n(D)=\min\{[\tfrac{1}{2}e_1], [\tfrac{2}{3}e_2], [\tfrac{2}{5}e_4]\}, \quad \text{where } D=e_1\sigma_1+e_2\sigma_2+e_4\sigma_4.$$

Thus, if $D=2d\sigma_1+3d\sigma_2+d\sigma_4$, then $n(D)=[\frac{2}{5}d]$.

Finally, we shall indicate how to proceed in the case of a (compact) toric variety which is not covered by our methods up to this point. There are two problems to deal with, namely (i) the description of the connected components of $\mathrm{Hol}(S^2, X)$ when H_2X has torsion, and (ii) the extension of all our previous results to the case of a non-projective toric variety.

Regarding problem (i), let us consider a (singular, projective, compact) toric variety X, and let us choose $D \in \mathrm{Ker}\,\iota^*$. There is a surjection $\delta: H_2X(\cong \pi_2X) \to \mathrm{Ker}\,\iota^*(\cong (H^2X)^*)$, whose kernel is the torsion subgroup of H_2X. We define

$$\mathrm{Hol}_D^*(S^2, X)=\{f \in \mathrm{Hol}^*(S^2, X) \mid \delta[f]=D\},$$

where $[f]\in\pi_2 X$ is the homotopy class of f. The space $Q_D^X(\mathbf{C})$ is defined in the usual way, and we have $\mathrm{Hol}_D^*(S^2,X)\cong Q_D^X(\mathbf{C})$. The problem now is to describe the connected components of $Q_D^X(\mathbf{C})$. Although this may be done in any particular case by the method of Appendix 1, we are unable to give a general statement. Therefore, we shall content ourselves by giving the following example.

Example 5.5: *The tetrahedral complex* (see Examples 2.5, 3.6). In this case, the group H_2X has been computed by M. McConnell (in a private communication) to be $\mathbf{Z}\oplus(\mathbf{Z}/2)^{11}$. For $D=d\sigma_{12}+d\sigma_{13}+d\sigma_{23}+d\sigma_{123}\in\mathrm{Ker}\,\iota^*$, the method described in Appendix 1 shows that Q_D^X has eight components, if $d\geqslant 3$. Four of the components correspond to holomorphic maps whose image lies in one of four copies of $\mathbf{C}P^1$, each of which is given by conditions of the form $z_i=z_j=z_k$ with $i\in\{0,1\}$, $j\in\{2,3\}$, $k\in\{4,5\}$. The other four components consist of "full" holomorphic maps.

In the terminology of Appendix 1, the "simple" labels here are:

$$\begin{aligned}
&l_{12}=(1,1,0,-1), && l_{13}=(1,0,1,-1), && l_{23}=(0,1,1,-1), && l_{123}=(1,1,1,-2),\\
&l'_{12}=(0,0,1,0), && l'_{13}=(0,1,0,0), && l'_{23}=(1,0,0,0), && l'_{123}=(0,0,0,1).
\end{aligned}$$

To define a stabilization procedure, let us consider the sequence of labels $l_1,l_2,l_3,\ldots$, where $l_1,\ldots,l_8$ are the above simple labels and $l_i=l_{i-8}$ for $i>8$. Let $z_1,z_2,z_3,\ldots$ be a sequence of points such that $z_i\in U_i-U_{i-1}$, as above. With $D_i=l_1+\ldots+l_i$, we define $\underline{Q}^X_{D_i}(U_i)$ to be the component of $Q^X_{D_i}(U_i)$ which contains the configuration $\{(z_j,l_j)\,|\,j=1,\ldots,i\}$. Then for $i=8j$, the space $\underline{Q}^X_{D_i}(U_i)$ may be identified up to homotopy with a distinguished component $\underline{\mathrm{Hol}}^*_{4j}(S^2,X)$ of $\mathrm{Hol}^*_{4j}(S^2,X)$. Let $\underline{\mathrm{Map}}^*_{4j}(S^2,X)$ denote the component of $\mathrm{Map}^*(S^2,X)$ containing the image of $\underline{\mathrm{Hol}}^*_{4j}(S^2,X)$. The method of the proof of Theorem 5.1 then gives a homotopy equivalence $\lim_{j\to\infty}\underline{\mathrm{Hol}}^*_{4j}(S^2,X)\to\lim_{j\to\infty}\underline{\mathrm{Map}}^*_{4j}(S^2,X)$. Thus, we obtain a *modified* version of our main theorem in this case.

Regarding problem (ii), the main question is whether Proposition 3.1 can be proved in the case of a non-projective toric variety. It suffices to consider the non-singular case, because singular varieties may be dealt with by using a resolution. A proof may be obtained from the following construction of a toric variety X from its fan Δ, described in [Col], [Au]. Let

$$Z=\left\{\sum x_i\sigma_i\in\bigoplus_{i=1}^{u}\mathbf{C}\sigma_i\;\middle|\;\prod_{i\in I_\sigma^c}x_i=0\text{ for all }\sigma\in\Delta\right\},$$

where $I_\sigma=\{i\,|\,\sigma_i\subseteq\sigma\}$ and I_σ^c is the complement of I_σ in $\{1,\ldots,u\}$. Let G be the kernel of the map

$$(\mathbf{C}^*)^u\cong\bigoplus_{i=1}^{u}\mathbf{C}\sigma_i\Big/\bigoplus_{i=1}^{u}\mathbf{Z}\sigma_i\to\mathbf{C}^r/\mathbf{Z}^r$$

induced by $\iota\otimes\mathbf{C}$. This is an algebraic group, which acts naturally on $\bigoplus_{i=1}^{u}\mathbf{C}\sigma_i$.

THEOREM 5.6 ([Co1], [Au]). *Let X be a simplicial toric variety. The action of G preserves $\bigoplus_{i=1}^{u}\mathbf{C}\sigma_i - Z$, and the quotient space is isomorphic to X. If X is non-singular (hence, in particular, simplicial), the action of G is free.* □

This is analogous to the usual description of $\mathbf{C}P^n$ as the quotient

$$(\mathbf{C}^{n+1}-\{0\})/(\mathbf{C}^*)^n,$$

which is a special case.

We may now give an alternative proof of Proposition 3.1, in the non-singular case. The idea of the proof is that, just as a (based) holomomorphic map $S^2\to\mathbf{C}P^n$ may be represented by an $(n+1)$-tuple of monic polynomials which have no common factor, a (based) holomorphic map $S^2\to X$ may be represented by a u-tuple of monic polynomials $(p_1,\ldots,p_u)$ such that $(p_1(z),\ldots,p_u(z))\notin Z$ for all $z\in\mathbf{C}$. Let $I_z=\{i\,|\,p_i(z)=0\}$. Now, $(p_1(z),\ldots,p_u(z))\notin Z$ if and only if there is a cone $\sigma\in\Delta$ such that $I_z\cap(I_\sigma)^c=\varnothing$, i.e. $I_z\subseteq I_\sigma$. So the condition on $p_1,\ldots,p_u$ is that, if $p_{i_1},\ldots,p_{i_j}$ have a common factor, then $\sigma_{i_1},\ldots,\sigma_{i_j}$ belong to a single cone of the fan. This is precisely condition (X) of §3. A complete proof, in a more general context, has been given recently by Cox in [Co2].

Appendix 1: $\pi_0 Q_D^X$ and $\pi_1 Q_D^X$

For a toric variety X we have given a correspondence between (based) holomorphic maps $S^2\to X$ and configurations of points with labels in $\mathrm{SF}(\Delta)^*$. These labels satisfy condition (X) of §3. The set of all such labels, being a subset of the monoid $\mathrm{SF}(\Delta)^*_{\geqslant 0}$, has the structure of a partial monoid, which we shall denote by M_X. In this appendix we shall indicate briefly how the algebraic structure of M_X determines $\pi_i\,\mathrm{Hol}_D^*(S^2,X)$ (i.e. $\pi_i Q_D^X(\mathbf{C})$) for $i=0,1$.

Let $l_1,\ldots,l_p$ be the simple elements of M_X (an element is said to be simple if it cannot be written in the form m_1+m_2, with m_1 and m_2 both non-zero). To investigate whether Q_D^X is connected, we note first that any element of Q_D^X may be moved continuously to a configuration of points whose labels are all simple. If that configuration contains certain points $z_1,\ldots,z_q$ with labels $l_{i_1},\ldots,l_{i_q}$, such that the sum $l_{i_1}+\ldots+l_{i_q}$ is defined (in M_X), then it may be moved continuously to a configuration in which $z_1,\ldots,z_q$ are replaced by a single point z (with label $l_{i_1}+\ldots+l_{i_q}$). Our aim now is to repeat this reduction process until we arrive at a canonical configuration; if this is possible, then we will have shown that Q_D^X is path-connected.

Let us suppose that there exist *linearly independent* labels $m_1, ..., m_r \in M_X$, such that the above reduction process eventually leads to a configuration of the form $\{(z_i, k_i m_i) \mid i=1,...,r\}$, where $k_1,...,k_r$ are non-negative integers. Since $D=\sum k_i m_i$ (and $m_1,...,m_r$ are linearly independent), the integers $k_1,...,k_r$ are determined by D. Therefore, we have succeeded in moving to a canonical configuration, and so Q_D^X is connected.

In the examples occurring in this paper it is straightforward to determine whether $m_1,...,m_r$ exist. For example, in the case of the quadric cone (Examples 3.4, 5.3), the simple labels are $\sigma_1, \frac{1}{2}(\sigma_1+\sigma_2), \sigma_2, \sigma_3$. If $D=d_1\sigma_1+d_2\sigma_2+d_3\sigma_3$, a suitable choice of $m_1,...,m_r$ would be $\sigma_1, \frac{1}{2}(\sigma_1+\sigma_2), \sigma_3$ (if $d_1 \geqslant d_2$), or $\frac{1}{2}(\sigma_1+\sigma_2), \sigma_2, \sigma_3$ (if $d_1 \leqslant d_2$).

The simple labels are also the main ingredient in the computation of the fundamental group of Q_D^X. By a slight generalization of the argument used in the Appendix of [GKY], it follows that the fundamental group is abelian. Moreover, there is one generator for each pair of simple labels l_i, l_j such that the sum l_i+l_j is *not* defined (in M_X). The order of such a generator is the least positive integer n such that $n(l_i+l_j) \in M_X$.

Appendix 2: Representation of holomorphic maps by polynomials

In the case of a non-singular toric variety X, Proposition 3.1 (and Theorem 5.6) gives a description of any $f \in \mathrm{Hol}_D^*(S^2, X)$ as a sequence $(p_1,...,p_u)$ of monic polynomials, where

(X) $p_{i_1},...,p_{i_j}$ are coprime if $X_{i_1} \cap ... \cap X_{i_j} = \varnothing$,

(D) $\deg p_i = d_i$, where $D=\sum_{i=1}^u d_i \sigma_i$.

The roots of the polynomial p_i represent the divisor $f^{-1}(\sigma_i)$. This description is canonical; it does not depend on any embedding in projective space. Such an embedding merely converts the above polynomial description into a more complicated one, as we have seen in Example 3.3.

In the case of a singular variety X, the divisors $\sigma_1,...,\sigma_u$ are not necessarily Cartier divisors, so we cannot expect the same procedure to work. Instead, let us *choose* a generating set $\tau_1 ..., \tau_v$ for the positive divisors in $\mathrm{SF}(\Delta)$, and then define monic polynomials $q_1,...,q_v$ by taking the roots of q_i to represent the divisor $f^{-1}(\tau_i)$. This is the same as $\hat{f}^{-1}(T(\tau_i))$, where $T\colon \mathrm{SF}(\Delta) \to \mathrm{SF}(\hat{\Delta})$ is the map induced by a toric resolution $\widehat{X} \to X$, and where $\hat{f}\colon S^2 \to \widehat{X}$ corresponds to $f\colon S^2 \to X$. If $T(\tau_i)=\sum_{j=1}^{\hat{u}} b_{ij}\hat{\sigma}_j$, then we obtain

$$(q_1,...,q_v) = (p_1^{b_{11}} ... p_{\hat{u}}^{b_{1\hat{u}}}, ..., p_1^{b_{v1}} ... p_{\hat{u}}^{b_{v\hat{u}}}) = (p^{b_1}, ..., p^{b_v}).$$

The proof of Proposition 3.1 shows that elements of $\mathrm{Hol}_D^*(S^2, X)$ may be identified with v-tuples $(p^{b_1},...,p^{b_v})$, where the polynomials $p_1,...,p_{\hat{u}}$ satisfy

(X) $p_{i_1},...,p_{i_j}$ are coprime if $\widehat{X}_{i_1} \cap ... \cap \widehat{X}_{i_j} = \varnothing$,

(D) $\deg p^{b_i} = D(\tau_i)$, $i=1,...,v$.

This illustrates how $\mathrm{Hol}_D^*(S^2, X)$ is topologized as the union of those $\mathrm{Hol}_{\widehat{D}}^*(S^2, \widehat{X})$ for which $\theta_*(\widehat{D})=D$: each $\mathrm{Hol}_{\widehat{D}}^*(S^2, \widehat{X})$ has its usual topology, but a collection of roots of polynomials $p_{i_1}, \ldots, p_{i_j}$ may coalesce to give a root of another polynomial p_{i_k}, where p_{i_k} is associated with a ray which sub-divides the cone associated to $p_{i_1}, \ldots, p_{i_j}$.

This kind of polynomial description of elements of $\mathrm{Hol}_D^*(S^2, X)$ also arises if we consider a suitable embedding of X in projective space.

Example A2.1: *The quadric cone* (see Examples 3.4, 5.3). Let us choose the generators $\sigma_4, 2\sigma_1, \sigma_1+\sigma_3, 2\sigma_3$ of $\mathrm{SF}(\Delta)$. We have $T(\sigma_4)=\sigma_4$, $T(2\sigma_1)=2\sigma_1+\sigma_2$, $T(\sigma_1+\sigma_3)=\sigma_1+\sigma_2+\sigma_3$, $T(2\sigma_3)=2\sigma_3+\sigma_2$, so $f \in \mathrm{Hol}_D^*(S^2, X)$ is represented by a 4-tuple of polynomials

$$(q_1, q_2, q_3, q_4) = (p_4, p_1^2 p_2, p_1 p_2 p_3, p_2 p_3^2),$$

where

(X) p_1, p_3 are coprime, p_2, p_4 are coprime,

(D) $\deg p_4 = \deg p_1^2 p_2 = \deg p_1 p_2 p_3 = \deg p_2 p_3^2 = 2g$.

This is in fact the polynomial representation which arises from the given embedding in $\mathbf{C}P^3$. In other words, as one readily verifies, a 4-tuple (q_1, q_2, q_3, q_4) of coprime monic polynomials of degree $2g$ satisfies the equation $q_3^2 = q_2 q_4$ if and only if it is of the above form.

Example A2.2: *The weighted projective space* $P(1,2,3)$ (see Examples 3.5, 5.4). Let us choose the generators $6\sigma_4, \sigma_1+4\sigma_4, 2\sigma_1+2\sigma_4, 3\sigma_1, \sigma_2+3\sigma_4, 2\sigma_2, \sigma_1+\sigma_2+\sigma_4$ of $\mathrm{SF}(\Delta)$. (The reason for this choice will become clear in a moment.) Applying the map T, we find that any $f \in \mathrm{Hol}_D^*(S^2, X)$ is represented by a 7-tuple of polynomials

$$(q_1, q_2, q_3, q_4, q_5, q_6, q_7)$$

of the form

$$(p_3^3 p_4^6 p_5^4 p_6^2,\ p_1 p_3^2 p_4^4 p_5^3 p_6^2,\ p_1^2 p_3 p_4^2 p_5^2 p_6^2,\ p_1^3 p_5 p_6^2,\ p_2 p_3^2 p_4^3 p_5^2 p_6,\ p_2^2 p_3,\ p_1 p_2 p_3 p_4 p_5 p_6),$$

where $p_1, p_2, p_3, p_4, p_5, p_6$ satisfy the conditions

(X) p_i, p_j are coprime except possibly when $|i-j|=1$ or $\{i,j\}=\{1,6\}$,

(D) $\deg q_i = 6g$, $i=1, \ldots, 7$.

In [Ha, Example 10.27], it is shown that $P(1,2,3)$ may be embedded in $\mathbf{C}P^6$ via the equations $z_0 z_2 = z_1^2$, $z_2 z_5 = z_6^2$, $z_1 z_3 = z_2^2$, $z_1 z_5 = z_4 z_6$. Our polynomial representation was chosen to be compatible with this embedding.

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