Acta Math., 174 (1995), 1–84

Completeness of translates in weighted spaces on the half-line

by

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1. Introduction

Imagine a one-dimensional monochromatic film, infinitely extended along a straight line, and a one-point signal emitter attached to an infinite rail running parallel to the film, which sends light signals to the film. An emitted signal is recorded on the film, and we may think of the result as a real-valued Borel measurable function on the line. The recording process is assumed reversible, in the sense that if a signal is received, and afterwards the opposite signal is received, the net result is zero. We may move the emitter freely along the rail, and there is a volume dial on the emitter, which permits us to vary the amplitude of the signal, and even reverse it. Suppose the emitter is equipped with a single signal. A natural question is what kind of images can be obtained if the emitter is placed in several positions along the rail and the signal, modified by adjusting the volume dial, is emitted from each of these positions. An interesting subquestion is that of determining which signals may be used to approximate every conceivable image.

When we translate this model to a mathematical setting, we need to define the distance between recorded images. The usual way would be to take the square root of the integral along the film of the square modulus of the difference of the two images, that is, the L^2 metric. Should the sensitivity of the film not be homogeneous, a weight function can be used to express the degree of inhomogeneity.

The above described model is a physical interpretation of translation invariance in function spaces on the real line. This area was initiated in the early 1930's by Norbert Wiener, Arne Beurling, Izrail Gel'fand, and Laurent Schwartz, and a multitude of beautiful papers were produced by them and their followers between 1930 and, say, 1960.

(1) Address before July 1, 1992: POMI, St. Petersburg, Russia.

(2) Partially supported by the Swedish Natural Science Research Council (NFR), and by the Wallenberg Prize from the Swedish Mathematical Society.

A number of difficult problems remained. In this paper, we solve one of them. To give the flavor of results obtained in the time period 1930–1960, we describe the contents of Theorems A and B stated below. They are concerned with the spaces of (equivalence classes of) square Lebesgue integrable complex-valued functions on the real line $\mathbf{R}$, $L^2(\mathbf{R})$, and on the half-line $\mathbf{R}_+=[0,+\infty[$, $L^2(\mathbf{R}_+)$. First, we need some terminology. The translation operator $T_x\colon L^2(\mathbf{R})\to L^2(\mathbf{R})$ associated with the real number x is defined by the formula

$$T_x f(t)=f(t-x), \quad t\in\mathbf{R}.$$

We consider $L^2(\mathbf{R}_+)$ to be the norm closed subspace of $L^2(\mathbf{R})$ of functions equal to 0 on the negative half-axis $\mathbf{R}_-=]-\infty,0[$. The right translation operators T_x, $x\in\mathbf{R}_+$, then act on the space $L^2(\mathbf{R}_+)$, which makes it natural to study the closed subspaces of $L^2(\mathbf{R}_+)$ that are invariant with respect to all of them. The Fourier transform of an $L^2(\mathbf{R})$ function f is given by the formula

$$\mathfrak{F}f(x)=\int_{-\infty}^{+\infty} e^{-itx} f(t)\,dt, \quad x\in\mathbf{R}, \tag{1.1}$$

with the usual convention of how to interpret this integral in case it is not absolutely convergent; by the Plancherel theorem, $\mathfrak{F}f\in L^2(\mathbf{R})$. The Fourier image of $L^2(\mathbf{R}_+)$ is known as $H^2(\mathbf{C}_-)$, which can also be described as the subspace of $L^2(\mathbf{R})$ consisting of those functions whose harmonic extensions to the lower half-plane $\mathbf{C}_-$ via the Poisson integral formula are holomorphic. One frequently thinks of the elements of the space $H^2(\mathbf{C}_-)$ as being holomorphic functions on $\mathbf{C}_-$ rather than square integrable functions on $\mathbf{R}$.

THEOREM A (Wiener [30], Ditkin [7]). *Every closed translation invariant subspace of $L^2(\mathbf{R})$ is determined by a Lebesgue measurable set $E\subset\mathbf{R}$, in the sense that the subspace coincides with the set of all functions whose Fourier transforms vanish on E.*

An inner function on $\mathbf{C}_-$ is a bounded analytic function on $\mathbf{C}_-$ whose (nontangential) boundary values on $\mathbf{R}$ have modulus 1 almost everywhere. The inner functions on $\mathbf{C}_-$ are isometric multipliers on $H^2(\mathbf{C}_-)$.

THEOREM B (Beurling [3], Lax [23]). *Every closed right translation invariant subspace of $L^2(\mathbf{R}_+)$ is either the $\{0\}$ subspace, or determined by an inner function q on the lower half-plane $\mathbf{C}_-$, in the sense that the subspace coincides with the set of all functions whose Fourier transforms belong to $qH^2(\mathbf{C}_-)$.*

Theorem A incorporates, among other things, the L^2 analog of Wiener's classical theorem on the completeness of translates of a given collection of functions in the space

$L^1(\mathbf{R})$. Theorem B, or perhaps more accurately, the corresponding statement for the unit disk, has been vital to the development of operator theory. Theorem B easily answers the question of when the right translates of a given collection of functions in $L^2(\mathbf{R}_+)$ span a dense subspace of $L^2(\mathbf{R}_+)$. The L^1 analog of this question was solved by Bertil Nyman in his 1950 thesis [25]. We note that for $f\in L^1(\mathbf{R}_+)$, its Fourier transform $\mathfrak{F}f$, as given by the formula

$$\mathfrak{F}f(z)=\int_0^{+\infty} e^{-itz}f(t)\,dt, \quad x\in\overline{\mathbf{C}}_-, \tag{1.2}$$

is continuous and bounded throughout $\overline{\mathbf{C}}_-$, holomorphic in the interior $\mathbf{C}_-$, and vanishes at infinity.

THEOREM C (Nyman). *Let $\mathfrak{S}$ be a collection of functions in $L^1(\mathbf{R}_+)$. Then the right translates T_xf, with $0\leqslant x$ and $f\in\mathfrak{S}$, span a dense subspace of $L^1(\mathbf{R}_+)$ if and only if*

(a) *for each $z\in\overline{\mathbf{C}}_-$, there exists an $f\in\mathfrak{S}$ with $\mathfrak{F}f(z)\neq 0$, and*

(b) *there is no interval $[0,\varepsilon]$, $0<\varepsilon$, such that all functions in $\mathfrak{S}$ vanish (almost everywhere) on it.*

Let us say that $\omega:\mathbf{R}_+=[0,+\infty[\,\to\,]0,+\infty[$ is a weight function on $\mathbf{R}_+$ if ω is continuous, the function $\log\omega$ is concave on $\mathbf{R}_+$, and

$$\log\omega(t)=o(t), \quad \text{as } t\to+\infty.$$

With this definition, every weight function ω has $t\mapsto\omega(t+x)/\omega(t)$ bounded on $\mathbf{R}_+$, for each $x\in\mathbf{R}_+$. In fact (Proposition 4.3), one can show that

$$\omega(0)\omega(s+t)\leqslant\omega(s)\omega(t), \quad s,t\in\mathbf{R}_+. \tag{1.3}$$

The above property implies that right translation is a bounded operation on the weighted L^p spaces $L^p(\mathbf{R}_+,\omega)$ which we are about to introduce. For weight functions ω on $\mathbf{R}_+$, and for a real parameter p with $1\leqslant p<+\infty$, the space $L^p(\mathbf{R}_+,\omega)$ consists of all (equivalence classes of) complex-valued Lebesgue measurable functions f on $\mathbf{R}_+$ for which

$$\|f\|_{L^p(\omega)}=\left(\int_0^{+\infty}|f(t)|^p\omega(t)^p\,dt\right)^{1/p}<+\infty;$$

here, $\|\cdot\|_{L^p(\omega)}$ defines a norm on $L^p(\mathbf{R}_+,\omega)$ which makes it a Banach space, and for $p=2$, a Hilbert space. Observe that since $0<\omega(0)\leqslant\omega(t)$ on $\mathbf{R}_+$, $L^p(\mathbf{R}_+,\omega)$ is a subspace of $L^p(\mathbf{R}_+)$, and as a matter of fact, the imbedding $L^p(\mathbf{R}_+,\omega)\to L^p(\mathbf{R}_+)$ is continuous.

The space $L^p(\mathbf{R}_+,\omega)$ is a subspace of $L^1(\mathbf{R}_+)$ automatically for $p=1$, and for $1<p<+\infty$, this is so, provided that

$$\int_0^{+\infty} \omega(t)^{-p'}\,dt < +\infty, \tag{1.4}$$

where p' is the *conjugate exponent* to p $(1/p+1/p'=1)$, by using Hölder's inequality. The reason why this question is of interest is that the condition that $L^p(\mathbf{R}_+,\omega)$ be contained in $L^1(\mathbf{R}_+)$ assures us that the Fourier image of a function in $L^p(\mathbf{R}_+,\omega)$, as defined by (1.2), is continuous and bounded on $\overline{\mathbf{C}}_-$, and vanishes at infinity. It is clear without any additional condition that $\mathfrak{F}f$ is holomorphic in the interior $\mathbf{C}_-$ for $f\in L^p(\mathbf{R}_+,\omega)$.

A key interest of Beurling was the notion of quasianalyticity, in the many shapes this concept took following the initial work of Arnaud Denjoy and Torsten Carleman. According to Beurling's classification, a weight function ω on $\mathbf{R}_+$ is said to be non-quasianalytic if

$$\int_0^{+\infty} \frac{\log\omega(t)}{1+t^2}\,dt < +\infty,$$

and quasianalytic if the above integral diverges. The relevance of the concept of quasianalyticity is better understood in the context of weighted L^p spaces on the whole real line. Let $\widetilde{\omega}$ be the symmetric extension to $\mathbf{R}$ of the weight function ω, so that $\widetilde{\omega}(t)=\omega(|t|)$ for all $t\in\mathbf{R}$, and consider the space $L^p(\mathbf{R},\widetilde{\omega})$ of all (equivalence classes of) complex-valued Lebesgue measurable functions f on $\mathbf{R}$ for which

$$\|f\|_{L^p(\widetilde{\omega})} = \left(\int_{-\infty}^{+\infty} |f(t)|^p\widetilde{\omega}(t)^p\,dt\right)^{1/p} < +\infty.$$

The Fourier transform on this space is given by formula (1.1).

THEOREM D (Paley–Wiener [26], $1\leqslant p<+\infty$). *Let ω be a weight function on $\mathbf{R}_+$ satisfying* (1.4) *if $1<p<+\infty$, with symmetric extension $\widetilde{\omega}$ to all of $\mathbf{R}$. Then the space $L^p(\mathbf{R},\widetilde{\omega})$ contains a nonzero element having Fourier transform with compact support if and only if ω is non-quasianalytic.*

For a simple proof of the above result in the case $p=1$, which easily carries over to general p, we refer to Garth Dales' and Walter Hayman's paper [6, p. 143]. Theorem D is the key element in the standard proof of the following extension of Wiener's completeness theorem, Theorem E (a). We note that, by (1.3), the spaces $L^1(\mathbf{R}_+,\omega)$ and $L^1(\mathbf{R},\widetilde{\omega})$ are Banach algebras when supplied with convolution multiplication, and that the same holds true for general p, $1<p<+\infty$, if we add some slight regularity conditions on ω. Let us say that a collection of functions $\mathfrak{S}$ in $L^1(\mathbf{R})$ has the *Wiener property* if for each $x\in\mathbf{R}$ there exists an $f\in\mathfrak{S}$ with $\mathfrak{F}f(x)\neq0$. Moreover, let us say that a collection of functions $\mathfrak{S}$ in $L^p(\mathbf{R},\widetilde{\omega})$ is *translation complete* in $L^p(\mathbf{R},\widetilde{\omega})$ if the translates T_xf, with $x\in\mathbf{R}$ and $f\in\mathfrak{S}$, span a dense subspace of $L^p(\mathbf{R},\widetilde{\omega})$.

THEOREM E ($1\leqslant p<+\infty$). *Fix a weight function ω on $\mathbf{R}_+$, meeting condition* (1.4) *for $1<p<+\infty$.*

(a) (Beurling [2], $p=1$) *If ω is non-quasianalytic, then each collection $\mathfrak{S}$ in $L^1(\mathbf{R},\tilde{\omega})$ which has the Wiener property is translation complete in $L^1(\mathbf{R},\tilde{\omega})$.*

(b) (Domar [9]) *If ω is quasianalytic, then there exists a collection $\mathfrak{S}$ in $L^p(\mathbf{R},\tilde{\omega})$ with the Wiener property which is not translation complete in $L^p(\mathbf{R},\tilde{\omega})$.*

There is another way to prove Theorem E (a) which does not explicitly use Theorem D, but instead employs a function-theoretic device, known as the 'log-log' theorem; see for instance [17, pp. 142–143]. The log-log theorem seems not to be well-known to a wide audience, so we present here a version of it. Evseĭ Dyn'kin showed in [12] that it should be thought of as a dual formulation of Theorem D. An account of who did what pertaining to the log-log theorem can be found in [10].

THEOREM F (Carleman, Levinson, Sjöberg, Wolf, Beurling, Domar). *Let $M\colon]0,1]\to [e,+\infty[$ be a continuous decreasing function, and suppose f is a holomorphic function in the strip*

$$\Sigma_{(-1,1)}=\{z\in\mathbf{C}:\ -1<\operatorname{Im} z<1\},$$

which there obeys the growth control

$$|f(z)|\leqslant M(|\operatorname{Im} z|),\quad z\in\Sigma_{(-1,1)}. \tag{1.5}$$

If the function M satisfies

$$\int_0^1 \log\log M(t)\,dt<+\infty,$$

then the function f must be bounded throughout $\Sigma_{(-1,1)}$. If, on the other hand, the above integral diverges, then there exists an f, satisfying (1.5), *which is unbounded on $\Sigma_{(-1,1)}$.*

In a paper from 1964, Vladimir Gurariĭ and Boris Levin ([16], see also [14]) extended Bertil Nyman's theorem (Theorem C) to the context of $L^1(\mathbf{R}_+,\omega)$, where ω is non-quasianalytic. The result is described below.

THEOREM G (Gurariĭ–Levin). *Let ω be a non-quasianalytic weight function on $\mathbf{R}_+$, and suppose $\mathfrak{S}$ is a collection of functions in $L^1(\mathbf{R}_+,\omega)$. Then the right translates $T_x f$, with $0\leqslant x$ and $f\in\mathfrak{S}$, span a dense subspace of $L^1(\mathbf{R}_+,\omega)$ if and only if*

(a) *for each $z\in\overline{\mathbf{C}}_-$, there exists an $f\in\mathfrak{S}$ with $\mathfrak{F}f(z)\neq 0$, and*

(b) *there is no interval $[0,\varepsilon]$, $0<\varepsilon$, such that all functions in $\mathfrak{S}$ vanish (almost everywhere) on it.*

The main technical vehicle for proving Theorem G, the way Gurariĭ and Levin did it, is the log-log theorem (Theorem F). It comes in naturally at a particular stage of the proof, where a certain function

$$M_\omega(x)=\int_0^{+\infty} e^{-xt}\omega(t)\,dt, \quad 0<x<+\infty,$$

appears, and one uses heavily the fact that the integrals

$$\int_0^{+\infty} \frac{\log\omega(t)}{1+t^2}\,dt$$

and

$$\int_0^1 \log\log M_\omega(t)\,dt$$

converge simultaneously. So, for a while, it seemed reasonable to suppose that the natural extension of Theorem G to quasianalytic ω should be false, just as Beurling's theorem (Theorem E (a)) failed for quasianalytic weights ω. But the apparent need of the log-log theorem to control the growth of analytic functions was an illusion, as shown by the theorem below, the proof of which constitutes the bulk of this paper. We note that the regularity condition imposed on ω entails that $L^p(\mathbf{R}_+,\omega)\subset L^1(\mathbf{R}_+)$.

The function $\theta(p)$ appearing in the theorem is defined as follows: $\theta(p)=3-1/p$ for $1<p<2$ and $2<p<+\infty$, $\theta(2)=\frac{1}{2}$, and $\theta(1)=3$.

MAIN THEOREM ($1\leqslant p<+\infty$). *Let ω be a weight function on $\mathbf{R}_+$, such that $\log\omega(t)-(\theta(p)+\varepsilon)\log(1+t)$ is concave, for some fixed ε, $0<\varepsilon$. Let $\mathfrak{S}$ be a collection of functions in $L^p(\mathbf{R}_+,\omega)$. Then the right translates $T_x f$, with $0\leqslant x$ and $f\in\mathfrak{S}$, span a dense subspace of $L^p(\mathbf{R}_+,\omega)$ if and only if*

(a) *for each $z\in\overline{\mathbf{C}}_-$, there exists an $f\in\mathfrak{S}$ with $\mathfrak{F}f(z)\neq 0$, and*

(b) *there is no interval $[0,\delta]$, $0<\delta$, such that all functions in $\mathfrak{S}$ vanish (almost everywhere) on it.*

The smaller you can get $\theta(p)$, the less regularity is required of ω; thus, the sharpest result is obtained for $p=2$.

One of the basic ingredients in the proof of the Main Theorem is the fact that we are able to successfully model the Fourier image of $L^p(\mathbf{R}_+,\omega)$ as a fairly concrete space of asymptotically holomorphic functions (see §§ 2 and 3). The term asymptotically holomorphic function is used here to mean a function of a complex variable whose $\bar\partial$ derivative is controlled (by some kind of weight function), and frequently quite small, near a prescribed set. The concept of asymptotically holomorphic functions originates with Dyn'kin [11] (see also [12]). We should like to mention at this point that in the works of Sergeĭ Bernshteĭn and Beurling one can trace ideas closely linked to the notion of asymptotic holomorphicity.

In [15], Gurariĭ conjectured that our Main Theorem should hold for $p=1$.

The organization of the paper is as follows. In §2, we assume $1<p<+\infty$, and identify the Fourier image of $L^p(\mathbf{R}_+,\omega)$ with the space $Q^p(\mathbf{C}_-,\sigma)$ of Cauchy transforms of elements of a space of densities $\mathfrak{L}^p(\Sigma,\sigma)$. We explain the details in the special case $p=2$. The elements of the space $\mathfrak{L}^2(\Sigma,\sigma)$ are Borel measurable complex-valued functions g in the strip $\Sigma=\mathbf{R}\times]0,1[$, subject to the norm condition

$$\|f\|_{\mathfrak{L}^2(\Sigma,\sigma)}=\left(\frac{1}{\pi}\int_\Sigma |g(x+iy)|^2\sigma(y)^2\,dx\,dy\right)^{1/2}<+\infty,$$

where $\sigma\colon]0,1]\to]0,+\infty[$ is continuous, and σ^{-2} is in $L^1(]0,1[)$. The Cauchy transform of $g\in\mathfrak{L}^2(\Sigma,\sigma)$ is the function

$$\mathfrak{C}g(\zeta)=\frac{1}{\pi}\int_\Sigma \frac{g(z)}{\zeta-z}\,dS(z),\quad \zeta\in\mathbf{C}_-.$$

The space $Q^2(\mathbf{C}_-,\sigma)$ of Cauchy transforms then coincides with the Fourier image of $L^2(\mathbf{R}_+,\omega)$, provided the weights ω and σ are related by the identity

$$\omega(t)=\omega_{\sigma,2}(t)=\left(2\int_0^1 \frac{e^{-2ty}}{\sigma(y)^2}\,dy\right)^{-1/2},\quad t\in[0,+\infty[.$$

In §3 an analogous model is developed for $p=1$. It should be mentioned that Dyn'kin, in his unpublished 1972 Leningrad thesis, found a related isometry construction for weighted l^2 sequences on the positive integers. In §4, we study what classes of weights ω correspond to certain given classes of weights σ. In §5, we reformulate the Main Theorem, discuss the analogies with our previous paper [5], and derive an important corollary describing all closed right translation invariant subspaces of $L^p(\mathbf{R}_+,\omega)$ whose Fourier transforms have no common zeros. The topic of §6 is to reformulate translation invariance for closed subspaces as invariance with respect to convolution with cut-off exponentials e_λ, with λ ranging over the upper half-plane $\mathbf{C}_+$. In §7 we introduce the concept of multipliers on space of densities. In §§8 and 9 we study how conditions on the weight σ influence properties of the space $Q^p(\mathbf{C}_-,\sigma)$, such as when it is a Banach algebra under pointwise multiplication of functions.

§10 is devoted to the topic of what we call *the resolvent transform*, which has its roots in early work by Carleman, Gel'fand, and Beurling. The resolvent transform $\mathfrak{R}\phi$ of an element ϕ of the dual space $L^{p'}(\mathbf{R}_+,\omega^{-1})$ is essentially the usual Laplace transform. We assume ϕ annihilates all right translates of $\mathfrak{S}$, and study when $\mathfrak{R}\phi$ extends to an entire function. The plot is to show in later sections that $\mathfrak{R}\phi$ satisfies estimates that together with the fact that it is entire force it to vanish identically. Initially, we carry out our manipulations in the operator algebra on a quotient space. Later on (more precisely, in Proposition 10.8), a convolution algebra assumption is made on $L^p(\mathbf{R}_+,\omega)$. The resolvent transform method was used by Gurariĭ and Levin in their proof of Theorem G. It is sometimes possible to get resolvent transforms of dual elements to extend analytically although the underlying space lacks a Banach algebra structure (see [19]).

We next describe a procedure for estimating the resolvent transform, which we call the *holomorphization process* (§§ 12 and 13). Given a function $f\in Q^p(\mathbf{C}_-,\sigma)$, which does not vanish identically, one constructs another function $g\in Q^p(\mathbf{C}_-,\sigma)$ such that fg extends analytically to a region slightly bigger than $\mathbf{C}_-$. This is done by solving a particular $\bar\partial$ equation. The method resembles to some extent what Alexander Volberg [29] did for asymptotically holomorphic functions in his proof of the celebrated result that the logarithm of an L^1 function on the unit circle $\mathbf{T}$ is summable, provided its negative Fourier coefficients decrease quasianalytically rapidly. The proof of the Main Theorem is then brought to a conclusion in §14.

A number of technical results concerning moment problems are contained in Appendices A and B.

At the time when Gurariĭ and Levin wrote their paper, it was not clear that the resolvent transform technique works also for quasianalytic ω, in the sense that it turns the completeness problem into a question involving entire functions. This point was later (in 1975) clarified by Yngve Domar [8]. Still, even after Domar's contribution, people were not able to stretch the validity of Theorem G beyond the border of quasianalyticity. In retrospect, we can say that the reason why Gurariĭ and Levin stop there is that they use too little of the information available about the size of the resolvent transform.

2. The isometry construction: $1<p<+\infty$

Throughout this section, we fix a p, $1<p<+\infty$, and write $p'=p/(p-1)$. Let σ be positive and continuous on $]0,1]$, and satisfy

$$\int_0^1 \frac{dt}{\sigma(t)^{p'}}<+\infty. \tag{2.1}$$

We associate with σ the weight function $\omega_{\sigma,p}$,

$$\omega_{\sigma,p}(t)=\left(2\int_0^1 \frac{e^{-p'ty}}{\sigma(y)^{p'}}\,dy\right)^{-1/p'},\quad t\in[0,+\infty[. \tag{2.2}$$

By Proposition 4.1 (§4 is independent of this one, as it is based on results from Appendices A and B), $\log \omega_{\sigma,p}$ is concave and increasing on $[0,+\infty[$, and has

$$\log \omega_{\sigma,p}(t) = o(t), \quad \text{as } t \to +\infty. \tag{2.3}$$

Let Σ be the strip

$$\Sigma = \{x+iy \in \mathbf{C} : 0 < y < 1\},$$

and write dS for area measure on $\mathbf{C}$,

$$dS(z) = dx\,dy, \quad z = x+iy.$$

The space $L^p(\Sigma, \sigma)$ of *pre-densities* consists of all (equivalence classes of) Borel measurable complex-valued functions g on Σ meeting the integrability condition

$$\|g\|_{L^p(\Sigma,\sigma)} = \left(2\int_\Sigma |g(t+iy)|^p \sigma(y)^p \, dS(t+iy)\right)^{1/p} < +\infty.$$

For $p=2$, this is a Hilbert space. The *density space* $\mathfrak{L}^p(\Sigma,\sigma)$ is the image of $L^p(\Sigma,\sigma)$ under the operation of taking the Fourier transform in the variable t. More precisely, if g_y denotes the function $g_y(t)=g(t+iy)$, then

$$\tilde{g}(x+iy) = \mathfrak{F} g_y(x), \quad x+iy \in \Sigma, \tag{2.4}$$

is a general element of $\mathfrak{L}^p(\Sigma,\sigma)$. For almost all y, $0<y<1$, g_y belongs to $L^p(\mathbf{R})$, so that, for $1<p\leqslant 2$, $\mathfrak{F} g_y$ makes sense as a function in $L^{p'}(\mathbf{R})$, by the Hausdorff–Young theorem. The norm on $\mathfrak{L}^p(\Sigma,\sigma)$ is the one that makes the mapping $g \mapsto \tilde{g}$ an isometry. The Plancherel theorem states that for L^2 functions φ on the real line, the norm identity

$$\int_{-\infty}^{+\infty} |\mathfrak{F}\varphi(x)|^2 \, dx = 2\pi \int_{-\infty}^{+\infty} |\varphi(t)|^2 \, dt$$

holds, so for $p=2$ we may use this to rewrite the norm on $\mathfrak{L}^p(\Sigma,\sigma)$, and the result is

$$\|\tilde{g}\|_{\mathfrak{L}^2(\Sigma,\sigma)} = \left(\frac{1}{\pi}\int_\Sigma |\tilde{g}(z)|^2 \sigma(\operatorname{Im} z)^2 \, dS(z)\right)^{1/2}. \tag{2.5}$$

For $2<p<+\infty$, it is necessary to interpret (2.4) as expressing a tempered distribution on $\mathbf{C}$ supported on the closed strip $\bar{\Sigma}$, so that for test functions φ on $\mathbf{C}$,

$$\langle \varphi, \tilde{g} \rangle = \int_\Sigma \mathfrak{F}\varphi_y(t) g(t+iy) \, dS(t+iy), \tag{2.6}$$

where $\varphi_y(t)=\varphi(t+iy)$, as expected. By Hölder's inequality,

$$\begin{aligned}|\langle\varphi,\tilde g\rangle| &\leqslant \int_\Sigma |\mathfrak{F}\varphi_y(t)g(t+iy)|\,dS(t+iy)\\ &\leqslant \biggl(\int_\Sigma |\mathfrak{F}\varphi_y(t)|^{p'}\sigma(y)^{-p'}\,dS(t+iy)\biggr)^{1/p'}\\ &\qquad\times\biggl(\int_\Sigma |g(t+iy)|^p\sigma(y)^p\,dS(t+iy)\biggr)^{1/p},\end{aligned}\tag{2.7}$$

(2.6) makes sense for all φ in $\mathfrak{L}^{p'}(\Sigma,\sigma^{-1})$, this space being defined analogously (here, $\sigma^{-1}=1/\sigma$). The first factor on the right hand side of (2.7) is estimated as follows,

$$\int_\Sigma |\mathfrak{F}\varphi_y(t)|^{p'}\sigma(y)^{-p'}\,dS(t+iy) \leqslant \sup_{0<y<1}\|\mathfrak{F}\varphi_y\|^{p'}_{L^{p'}(\mathbf{R})}\int_0^1\frac{dy}{\sigma(y)^{p'}};$$

thus, by (2.1), the integral on the right hand side of (2.6) is summable for test functions φ in $\mathcal{S}(\mathbf{C})$, the space of C^∞ test functions on $\mathbf{C}$ which, along with their partial derivatives, decay more rapidly than $|z|^{-n}$ near infinity, for all $n=1,2,3,\ldots$. The above distributional interpretation of (2.4) for $2<p<+\infty$ extends to the general case $1<p<+\infty$, and for $1<p\leqslant 2$ it coincides with our earlier interpretation of it as a Borel measurable (and, in fact, locally integrable on Σ) function. The Cauchy kernel is the function

$$\mathcal{C}(z,\zeta)=\pi^{-1}(\zeta-z)^{-1},\quad z,\zeta\in\mathbf{C},\ z\neq\zeta;$$

we write $\mathcal{C}_y(x,\zeta)=\mathcal{C}(x+iy,\zeta)$. The Fourier transform of the function $\mathcal{C}_y(x,\zeta)$ with respect to the x variable is

$$\mathcal{K}_y(t,\zeta)=\begin{cases}-2i\exp(-t(y+i\zeta))H(-t), & y<\operatorname{Im}\zeta,\\ 2i\exp(-t(y+i\zeta))H(t), & \operatorname{Im}\zeta<y,\end{cases}\tag{2.8}$$

where H is the Heaviside function,

$$H(t)=\begin{cases}1, & 0\leqslant t,\\ 0, & t<0.\end{cases}$$

The *Cauchy transform* of a density $\tilde g$ is the holomorphic function in the lower half-plane

$$\begin{aligned}\mathfrak{C}\tilde g(\zeta)&=\langle\mathcal{C}(\,\cdot\,,\zeta),\tilde g\rangle=\int_\Sigma\mathcal{K}_y(t,\zeta)g(t+iy)\,dS(t+iy)\\ &=2i\int_{\mathbf{R}_+\times]0,1[}\exp(-t(y+i\zeta))g(t+iy)\,dS(t+iy),\quad \zeta\in\mathbf{C}_-,\end{aligned}\tag{2.9}$$

which satisfies the inequality

$$\begin{aligned}|\mathfrak{C}\tilde{g}(\zeta)| &\leqslant 2\int_{\mathbf{R}_+\times]0,1[}|\exp(-t(y+i\zeta))g(t+iy)|\,dS(t+iy)\\ &\leqslant \biggl(2\int_{\mathbf{R}_+\times]0,1[}|g(t+iy)|^p\sigma(y)^p\,dS(t+iy)\biggr)^{1/p}\\ &\quad\times\biggl(2\int_{\mathbf{R}_+\times]0,1[}\exp(-tp'(y-\operatorname{Im}\zeta))\,\frac{dS(t+iy)}{\sigma(y)^{p'}}\biggr)^{1/p'}\\ &\leqslant \|\tilde{g}\|_{\mathfrak{L}^p(\Sigma,\sigma)}\biggl(\frac{2}{p'}\int_0^1\frac{dy}{(y-\operatorname{Im}\zeta)\sigma(y)^{p'}}\biggr)^{1/p'}<+\infty,\quad \zeta\in\mathbf{C}_-.\end{aligned}\tag{2.10}$$

If the density $\tilde{g}$ is a genuine function on Σ, as is the case for $1<p\leqslant 2$, its Cauchy transform may be computed as follows (one needs to be careful with the convergence of the integral for $p\neq 2$),

$$\mathfrak{C}\tilde{g}(\zeta)=\frac{1}{\pi}\int_\Sigma\frac{\tilde{g}(z)}{\zeta-z}\,dS(z),\quad \zeta\in\mathbf{C}_-.\tag{2.11}$$

We write $Q^p(\mathbf{C}_-,\sigma)$ for the space $\mathfrak{C}\mathfrak{L}^p(\Sigma,\sigma)$ of Cauchy transforms of functions in $\mathfrak{L}^p(\Sigma,\sigma)$. It is a Banach space of holomorphic functions in the lower half-plane $\mathbf{C}_-$, when equipped with the norm

$$\|h\|_{Q^p(\mathbf{C}_-,\sigma)}=\inf\{\|g\|_{\mathfrak{L}^p(\Sigma,\sigma)}: g\in\mathfrak{L}^p(\Sigma,\sigma),\ \mathfrak{C}g=h\}.$$

THEOREM 2.1. *The Fourier transform* $\mathfrak{F}$ *maps* $L^p(\mathbf{R}_+,\omega_{\sigma,p})$ *isometrically onto* $Q^p(\mathbf{C}_-,\sigma)$.

For the proof, we introduce the operators $\mathfrak{D}^p$ and $\mathcal{E}$, as follows. Given an $f\in L^p(\mathbf{R}_+,\omega_{\sigma,p})$, associate to it the *pre-density function*

$$g[f,p](t+iy)=-i\sigma(y)^{-p'}f(t)\omega_{\sigma,p}(t)^{p'}\exp(-typ'/p),\quad t+iy\in\Sigma,\tag{2.12}$$

(we agree that $f(t)=0$ for $t<0$, so that the right hand side vanishes for $t<0$) and the *density function*

$$\begin{aligned}\mathfrak{D}^pf(x+iy)&=\tilde{g}[f,p](x+iy)\\ &=-i\sigma(y)^{-p'}\int_0^{+\infty}f(t)\omega_{\sigma,p}(t)^{p'}\exp(-t(ix+yp'/p))\,dt,\quad x+iy\in\Sigma.\end{aligned}\tag{2.13}$$

For $0<y<+\infty$ and $0\leqslant\alpha<+\infty$,

$$\begin{aligned}\int_0^{+\infty}t^\alpha|f(t)|\,\omega_{\sigma,p}(t)^{p'}\exp(-typ'/p)\,dt &\leqslant\biggl(\int_0^{+\infty}|f(t)|^p\omega_{\sigma,p}(t)^p\,dt\biggr)^{1/p}\\ &\quad\times\biggl(\int_0^{+\infty}t^{\alpha p'}\omega_{\sigma,p}(t)^q\exp(-tyq)\,dt\biggr)^{1/p'}\\ &=\|f\|_{L^p(\mathbf{R}_+,\omega_{\sigma,p})}\biggl(\int_0^{+\infty}t^{\alpha p'}\omega_{\sigma,p}(t)^q\exp(-tyq)\,dt\biggr)^{1/p'}\end{aligned}$$

holds, where $q=p'(p'-1)=(p')^2/p$, so that with the notation,

$$\tilde{\sigma}(p,\alpha,y)=\int_0^{+\infty} t^{\alpha p'}\omega_{\sigma,p}(t)^q \exp(-tyq)\,dt, \tag{2.14}$$

this results in an estimate of the nth partial derivate with respect to x of the density function $\mathfrak{D}^p f(x+iy)$,

$$|\partial_x^n \mathfrak{D}^p f(z)| \leqslant \frac{\tilde{\sigma}(p,n,y)^{1/p'}}{\sigma(y)^{p'}} \|f\|_{L^p(\mathbf{R}_+,\omega_{\sigma,p})}, \quad z=x+iy\in\Sigma. \tag{2.15}$$

The function $\tilde{\sigma}(p,\alpha,y)$ is finite for $0<y$, by (2.3). By the same token, $\mathfrak{D}^p f(z)$ is continuous on the half-open strip $\check{\Sigma}$,

$$\check{\Sigma}=\{x+iy\in\mathbf{C}:\, 0<y\leqslant 1\},$$

and of class C^∞ in the x variable. In fact, if we write

$$\mathfrak{A}^p f(z) = -i\int_0^{+\infty} \exp(itz)f(t)\omega_{\sigma,p}(t)^{p'}\,dt, \quad z\in\mathbf{C}_+,$$

which represents a holomorphic function in the upper half-plane, the expression for the density $\mathfrak{D}^p f$ becomes

$$\mathfrak{D}^p f(x+iy)=\sigma(y)^{-p'}\mathfrak{A}^p f(-(x-iyp'/p)), \quad x+iy\in\Sigma.$$

LEMMA 2.2. *The operator $\mathfrak{D}^p$ maps $L^p(\mathbf{R}_+,\omega_{\sigma,p})$ isometrically into $\mathfrak{L}^p(\Sigma,\sigma)$, that is,*

$$\int_0^{+\infty} |f(t)|^p\omega_{\sigma,p}(t)^p\,dt = 2\int_\Sigma |g[f,p](t+iy)|^p\sigma(y)^p\,dS(t+iy)$$

holds for all $f\in L^p(\mathbf{R}_+,\omega_{\sigma,p})$.

Proof. By the definition (2.2) of the weight $\omega_{\sigma,p}$, we have

$$\begin{aligned}
2\int_\Sigma & |g[f,p](t+iy)|^p\sigma(y)^p\,dS(t+iy) \\
&= 2\int_\Sigma \sigma(y)^p\sigma(y)^{-pp'}|f(t)|^p\omega_{\sigma,p}(t)^{pp'}\exp(-p'ty)\,dS(t+iy) \\
&= \int_0^{+\infty} |f(t)|^p\omega_{\sigma,p}(t)^p\left(\omega_{\sigma,p}(t)^{p'}2\int_0^1 \frac{\exp(-p'ty)}{\sigma(y)^{p'}}\,dy\right)dt \\
&= \|f\|^p_{L^p(\mathbf{R}_+,\omega_{\sigma,p})}.
\end{aligned}$$

The proof is complete. □

Densities of the type $\mathfrak{D}^p f$ for some $f\in L^p(\mathbf{R}_+,\omega_{\sigma,p})$ will be called *canonical densities*, and by Lemma 2.2, they form a closed subspace of $\mathfrak{L}^p(\Sigma,\sigma)$.

LEMMA 2.3. *The operators $\mathfrak{C}\mathfrak{D}^p$ and $\mathfrak{F}$ coincide on $L^p(\mathbf{R}_+,\omega_{\sigma,p})$, that is, $\mathfrak{C}\mathfrak{D}^p f(z)=\mathfrak{F}f(z)$ holds on $\mathbf{C}_-$ for all $f\in L^p(\mathbf{R}_+,\omega_{\sigma,p})$.*

Proof. A computation based on the definition of $\mathfrak{D}^p f$ and the expression (2.9) for the Cauchy transform reveals that

$$\begin{aligned}\mathfrak{C}\mathfrak{D}^p f(\zeta)&=\mathfrak{C}\tilde{g}[f,p](\zeta)=2i\int_{\mathbf{R}_+\times]0,1[}\exp(-t(y+i\zeta))g[f,p](t+iy)\,dS(t+iy)\\&=2\int_0^{+\infty}\int_0^1\exp(-t(y+i\zeta))\sigma(y)^{-p'}f(t)\omega_{\sigma,p}(t)^{p'}\exp(-typ'/p)\,dy\,dt\\&=\int_0^{+\infty}\int_0^1\exp(-it\zeta)\omega_{\sigma,p}(t)^{p'}f(t)\biggl(2\int_0^1\frac{e^{-typ'}}{\sigma(y)^{p'}}\,dy\biggr)\,dt\\&=\int_0^{+\infty}\exp(-it\zeta)f(t)\,dt=\mathfrak{F}f(\zeta),\quad \zeta\in\mathbf{C}_-,\end{aligned}$$

which completes the proof. □

Given an element $\tilde{g}$ of $\mathfrak{L}^p(\Sigma,\sigma)$, its *extraction* is the function

$$\mathcal{E}\tilde{g}(t)=2i\int_0^1\exp(-ty)g(t+iy)\,dy,\quad t\in[0,+\infty[,\tag{2.16}$$

where g is the pre-density associated with $\tilde{g}$. For $1<p\leqslant 2$, $\tilde{g}$ is a function, and the extraction may be written as

$$\mathcal{E}\tilde{g}(t)=\frac{i}{\pi}\int_\Sigma\exp(itz)\tilde{g}(z)\,dS(z),\quad t\in[0,+\infty[,$$

where one has to be a little careful with convergence of the integral. By Hölder's inequality,

$$\begin{aligned}|\mathcal{E}\tilde{g}(t)|&\leqslant 2\int_0^1 e^{-ty}|g(t+iy)|\,dy\\&\leqslant\biggl(2\int_0^1\frac{e^{-typ'}}{\sigma(y)^{p'}}\,dy\biggr)^{1/p'}\biggl(2\int_0^1|g(t+iy)|^p\sigma(y)^p\,dy\biggr)^{1/p},\quad t\in[0,+\infty[,\end{aligned}$$

and if we use the explicit formula (2.2) giving $\omega_{\sigma,p}$, we arrive at the norm control

$$\begin{aligned}\int_0^{+\infty}|\mathcal{E}\tilde{g}(t)|^p\omega_{\sigma,p}(t)^p\,dt&\leqslant 2\int_0^{+\infty}\int_0^1|g(t+iy)|^p\sigma(y)^p\,dy\,dt\\&\leqslant 2\int_0^1\int_{-\infty}^{\infty}|g(t+iy)|^p\sigma(y)^p\,dt\,dy=\|\tilde{g}\|^p_{\mathfrak{L}^p(\Sigma,\sigma)}.\end{aligned}$$

We formulate this result as a lemma.

LEMMA 2.4. *For every $\tilde{g}\in\mathfrak{L}^p(\Sigma,\sigma)$, we have $\mathcal{E}\tilde{g}\in L^p(\mathbf{R}_+,\omega_{\sigma,p})$, and the mapping $\mathcal{E}:\mathfrak{L}^p(\Sigma,\sigma)\to L^p(\mathbf{R}_+,\omega_{\sigma,p})$ is a contraction: $\|\mathcal{E}\tilde{g}\|_{L^p(\mathbf{R}_+,\omega_{\sigma,p})}\leqslant\|\tilde{g}\|_{\mathfrak{L}^p(\Sigma,\sigma)}$.*

The relevance of the extraction operator is due to the following fact.

LEMMA 2.5. *The operators $\mathfrak{F}\mathcal{E}$ and $\mathfrak{C}$ coincide on $\mathfrak{L}^p(\Sigma,\sigma)$, that is, $\mathfrak{F}\mathcal{E}\tilde{g}(z)=\mathfrak{C}\tilde{g}(z)$ holds on $\mathbf{C}_-$ for all $\tilde{g}\in\mathfrak{L}^p(\Sigma,\sigma)$.*

Proof. Let g be the pre-density associated with $\tilde{g}$. Then, by the definitions (2.9) and (2.16) of $\mathfrak{C}$ and $\mathcal{E}$,

$$\begin{aligned}\mathfrak{F}\mathcal{E}\tilde{g}(\zeta)&=\int_0^{+\infty}\exp(-it\zeta)\mathcal{E}\tilde{g}(t)\,dt\\&=2i\int_0^{+\infty}\int_0^1\exp(-t(y+i\zeta))g(t+iy)\,dy\,dt=\mathfrak{C}\tilde{g}(\zeta),\quad \zeta\in\mathbf{C}_-,\end{aligned}$$

as asserted. □

Proof of Theorem 2.1. We will use the results of Lemmas 2.2, 2.3, and 2.4, freely, without particular reference. The mapping $\mathfrak{C}:\mathfrak{L}^p(\Sigma,\sigma)\to Q^p(\mathbf{C}_-,\sigma)$ is norm contractive by definition. Hence $\mathfrak{F}=\mathfrak{C}\mathfrak{D}^p$ is contractive $L^p(\mathbf{R}_+,\omega_{\sigma,p})\to Q^p(\mathbf{C}_-,\sigma)$. Take an $h\in Q^p(\mathbf{C}_-,\sigma)$; by definition, this means that there is a $\tilde{g}\in\mathfrak{L}^p(\Sigma,\sigma)$ with $\mathfrak{C}\tilde{g}=h$. If we put $f=\mathcal{E}\tilde{g}\in L^p(\mathbf{R}_+,\omega_{\sigma,p})$ and $\tilde{g}_0=\mathfrak{D}^pf\in\mathfrak{L}^p(\Sigma,\sigma)$, then $\mathfrak{F}f=\mathfrak{F}\mathcal{E}\tilde{g}=\mathfrak{C}\tilde{g}=h$ and $\mathfrak{C}\tilde{g}_0=\mathfrak{C}\mathfrak{D}^p\mathfrak{F}\mathcal{E}\tilde{g}=\mathfrak{F}\mathcal{E}\tilde{g}=h$. In particular, h belongs to the Fourier image of $L^p(\mathbf{R}_+,\omega_{\sigma,p})$. Moreover, $\|\tilde{g}_0\|_{\mathfrak{L}^p(\Sigma,\sigma)}\leqslant\|\tilde{g}\|_{\mathfrak{L}^p(\Sigma,\sigma)}$, which implies that $\|h\|_{Q^p(\mathbf{C}_-,\sigma)}=\|\tilde{g}_0\|_{\mathfrak{L}^p(\Sigma,\sigma)}$. Note that $f=\mathcal{E}\tilde{g}_0$ since both sides have the same Fourier transform, and thus

$$\|f\|_{L^p(\mathbf{R}_+,\omega_{\sigma,p})}\leqslant\|\tilde{g}_0\|_{\mathfrak{L}^p(\Sigma,\sigma)}=\|h\|_{Q^p(\mathbf{C}_-,\sigma)}=\|\mathfrak{F}f\|_{Q^p(\mathbf{C}_-,\sigma)}.$$

The proof is complete. □

The global Cauchy transform $\mathfrak{C}_*\tilde{g}$ of a density $\tilde{g}$ is the extension of $\mathfrak{C}\tilde{g}$ to all of $\mathbf{C}$, which for $p=2$ (and for $1<p<2$ also, if there are no convergence problems) takes the form

$$\mathfrak{C}_*\tilde{g}(\zeta)=\frac{1}{\pi}\int_\Sigma\frac{\tilde{g}(z)}{\zeta-z}\,dS(z),\quad \zeta\in\mathbf{C},$$

wherever the integrand is summable (this happens area-almost everywhere). For general p, $1<p<+\infty$, this corresponds to putting

$$\begin{aligned}\mathfrak{C}_*\tilde{g}(\zeta)&=\int_\Sigma\mathcal{K}_y(t,\zeta)g(t+iy)\,dS(t+iy)\\&=2i\int_{U(\zeta)}\exp(-t(i\zeta+y))g(t+iy)\,dS(t+iy)\\&\quad-2i\int_{V(\zeta)}\exp(-t(i\zeta+y))g(t+iy)\,dS(t+iy),\quad \zeta\in\mathbf{C},\end{aligned}\tag{2.17}$$

where $U(\zeta)=\mathbf{R}_+\times(]\operatorname{Im}\zeta,1]\cap]0,1[)$ and $V(\zeta)=]-\infty,0[\times(]0,\operatorname{Im}\zeta]\cap]0,1[)$. To properly interpret $\mathfrak{C}_*\tilde{g}$ as a tempered distribution, we need to know that it is a well-defined function area-almost everywhere, and that it belongs to a reasonable space of locally integrable functions on $\mathbf{C}$. For $1\leqslant q<+\infty$, let us agree to say that a Lebesgue area measurable function f on $\mathbf{C}$ is in $L^q_\infty(\mathbf{C})$ provided that

$$\sup_{z\in\mathbf{C}}\int_{\mathbf{D}}|f(z+\zeta)|^q\,dS(\zeta)<+\infty,$$

where $\mathbf{D}$ is the unit disk, with the usual agreement to identify functions that coincide with the exception of a set of area measure 0.

LEMMA 2.6. *For $\tilde{g}\in\mathfrak{L}^p(\Sigma,\sigma)$, we have $\mathfrak{C}_*\tilde{g}\in L^1_\infty(\mathbf{C})$.*

Proof. By (2.8) and Hölder's inequality,

$$\begin{aligned}\int_{-\infty}^{+\infty}|\mathcal{K}_y(t,\zeta)g(t+iy)|\,dt &\leqslant\left(\int_{-\infty}^{+\infty}|\mathcal{K}_y(t,\zeta)|^{p'}\,dt\right)^{1/p'}\left(\int_{-\infty}^{+\infty}|g(t+iy)|^p\,dt\right)^{1/p}\\ &=2(p'|y-\operatorname{Im}\zeta|)^{-1/p'}\|g_y\|_{L^p(\mathbf{R})},\end{aligned}$$

so that, by (2.17),

$$|\mathfrak{C}_*\tilde{g}(\zeta)|\leqslant\int_\Sigma|\mathcal{K}_y(t,\zeta)g(t+iy)|\,dS(t+iy)\leqslant 2\int_0^1(p'|y-\operatorname{Im}\zeta|)^{-1/p'}\|g_y\|_{L^p(\mathbf{R})}\,dy.$$

If R is a square with side length 1, then

$$\int_R|y-\operatorname{Im}\zeta|^{-1/p'}\,dS(\zeta)\leqslant 2p,$$

and consequently,

$$\begin{aligned}\int_R|\mathfrak{C}_*\tilde{g}(\zeta)|\,dS(\zeta)&\leqslant C(p)\int_0^1\|g_y\|_{L^p(\mathbf{R})}\,dy\\ &\leqslant C(p)\left(\int_0^1\frac{dy}{\sigma(y)^{p'}}\right)^{1/p'}\left(\int_0^1\|g_y\|^p_{L^p(\mathbf{R})}\sigma(y)^p\,dy\right)^{1/p}\\ &=C(p)\|\tilde{g}\|_{\mathfrak{L}^p(\Sigma,\sigma)}\left(\int_0^1\frac{dy}{\sigma(y)^{p'}}\right)^{1/p'},\end{aligned}$$

where $C(p)=4p(p')^{-1/p'}$. The assertion is now immediate, in view of (2.1). □

So far, we have only defined the global Cauchy transform $\mathfrak{C}_*$ on spaces of functions or distributions supported on the closure of Σ. There is, however, no reason for doing so in general; for instance, if φ is a test function in $\mathcal{S}(\mathbf{C})$, we may define

$$\mathfrak{C}_*\varphi(z)=\frac{1}{\pi}\int_{\mathbf{C}}\frac{\varphi(\zeta)}{z-\zeta}\,dS(\zeta),\quad z\in\mathbf{C},$$

and observe that $\bar\partial\mathfrak{C}_*\varphi(z)=\mathfrak{C}_*\bar\partial\varphi(z)=\varphi(z)$ holds on $\mathbf{C}$. In particular,

$$\varphi_y(x)=\varphi(x+iy)=-\int_{\mathbf{C}} \mathcal{C}_y(x,\zeta)\bar\partial\varphi(\zeta)\,dS(\zeta),$$

so that

$$\mathfrak{F}\varphi_y(t)=-\int_{\mathbf{C}} \mathcal{K}_y(t,\zeta)\bar\partial\varphi(\zeta)\,dS(\zeta). \tag{2.18}$$

PROPOSITION 2.7. *If $\tilde g\in\mathfrak{L}^p(\Sigma,\sigma)$, then $f=\mathfrak{C}_*\tilde g\in L^1_\infty(\mathbf{C})$, and $\bar\partial f=\tilde g$, in the sense of distribution theory. On the other hand, if $f\in L^1_\infty(\mathbf{C})$, and its distributional derivative $\bar\partial f$ belongs to the space $\mathfrak{L}^p(\Sigma,\sigma)$, then there exists a constant $\beta(f)$ such that $f=\beta(f)+\mathfrak{C}_*\bar\partial f$ holds area-almost everywhere on $\mathbf{C}$; in particular, the restriction to $\mathbf{C}_-$ of $f-\beta(f)$ belongs to $Q^p(\mathbf{C}_-,\sigma)$.*

Note. One should think of $\beta(f)$ as the value of f at infinity.

Proof. That $f=\mathfrak{C}_*\tilde g\in L^1_\infty(\mathbf{C})$ for $\tilde g\in\mathfrak{L}^p(\Sigma,\sigma)$ was demonstrated in Lemma 2.6. We proceed to check that $\bar\partial f=\tilde g$. For test functions φ on $\mathbf{C}$, we have, by (2.18) and Fubini's theorem,

$$\begin{aligned}
\langle\varphi,\bar\partial\mathfrak{C}_*\tilde g\rangle&=-\langle\bar\partial\varphi,\mathfrak{C}_*\tilde g\rangle=-\int_{\mathbf{C}}\mathfrak{C}_*\tilde g(\zeta)\bar\partial\varphi(\zeta)\,dS(\zeta)\\
&=-\int_{\mathbf{C}}\int_\Sigma \mathcal{K}_y(t,\zeta)g(t+iy)\,dS(t+iy)\,\bar\partial\varphi(\zeta)\,dS(\zeta)\\
&=-\int_\Sigma\int_{\mathbf{C}} \mathcal{K}_y(t,\zeta)\,\bar\partial\varphi(\zeta)\,dS(\zeta)\,g(t+iy)\,dS(t+iy)\\
&=\int_\Sigma \mathfrak{F}\varphi_y(t)g(t+iy)\,dS(t+iy)=\langle\varphi,\tilde g\rangle,
\end{aligned}$$

as claimed.

We turn to check the third assertion, which states that if $f\in L^1_\infty(\mathbf{C})$, and its distributional derivative $\bar\partial f$ belongs to the space $\mathfrak{L}^p(\Sigma,\sigma)$, then $f-\mathfrak{C}_*\bar\partial f$ equals a constant almost everywhere (dS) on $\mathbf{C}$. By what we have done so far, it is clear that the function $\varphi=\mathfrak{C}_*\bar\partial f$ belongs to $L^1_\infty(\mathbf{C})$, and has $\bar\partial\varphi=\bar\partial f$. The difference $\varphi-f$ is then an entire function in $L^1_\infty(\mathbf{C})$, which of course must be constant, by the mean value property and Liouville's theorem. If we denote this constant by $\beta(f)$, the claim is verified. □

For $\tilde g\in\mathfrak{L}^p(\Sigma,\sigma)$, the restriction of $\mathfrak{C}_*\tilde g$ to $\mathbf{C}_-$ coincides with the earlier introduced Cauchy transform $\mathfrak{C}\tilde g$. Thus, according to Lemma 2.3, the operator $\mathfrak{F}^p_*=\mathfrak{C}_*\mathfrak{D}^p$ supplies a *canonical generalized Fourier transform.* We shall obtain an explicit expression for $\mathfrak{F}^p_*f$. For $0<y<1$, write

$$q_{\sigma,p}(t,y)=2\omega_{\sigma,p}(t)^{p'}\int_y^1\frac{e^{-p'ut}}{\sigma(u)^{p'}}\,du,\quad t\in\mathbf{R}_+,$$

and put $q_{\sigma,p}(t,y)=1$ for $y\leqslant 0$, and $q_{\sigma,p}(t,y)=0$ for $1\leqslant y$. The function $q_{\sigma,p}$ is continuous, and has $0\leqslant q_{\sigma,p}(t,y)\leqslant 1$ for all $(t,y)\in\mathbf{R}_+\times\mathbf{R}$. Moreover, for $0<\varepsilon\leqslant y$ and $1<\theta<p'$, we have the crude estimate

$$q_{\sigma,p}(t,y)=O(e^{-\theta ty}),\quad t\to+\infty, \tag{2.19}$$

uniformly in y, if ε and θ are fixed.

PROPOSITION 2.8 ($1<p<+\infty$). *In terms of the function $q_{\sigma,p}$, the canonical generalized Fourier transform on $L^p(\mathbf{R}_+,\omega_{\sigma,p})$ is given by the formula*

$$\mathfrak{F}^p_* f(z)=\mathfrak{C}_*\mathfrak{D}^p f(z)=\int_0^{+\infty} e^{-itz} f(t) q_{\sigma,p}(t,\operatorname{Im} z)\,dt,\quad z\in\mathbf{C}\backslash\mathbf{R}.$$

Proof. Let $\varrho(t)=1/\sigma(t)$ for $0<t<1$, and set $\varrho(t)=0$ elsewhere on $\mathbf{R}$. If we write $z=x+iy$, we have for $1<p<+\infty$, in the sense of distribution theory,

$$\begin{aligned}\bar\partial_z(e^{-itz}q_{\sigma,p}(t,y))&=e^{-itz}\bar\partial_z q_{\sigma,p}(t,y)=\tfrac12 ie^{-itz}\frac{\partial}{\partial y}q_{\sigma,p}(t,y)\\&=-ie^{-itz}\omega_{\sigma,p}(t)^{p'}\varrho(y)^{p'}e^{-typ'}\\&=-ie^{-itx}\omega_{\sigma,p}(t)^{p'}\varrho(y)^{p'}e^{-typ'/p},\quad z\in\mathbf{C}.\end{aligned}\tag{2.20}$$

For an $f\in L^p(\mathbf{R}_+,\omega_{\sigma,p})$, let $\mathfrak{F}^p_* f$ denote the function given by the expression involving $q_{\sigma,p}$, so that what we need to check is that $\mathfrak{F}^p_* f=\mathfrak{C}_*\mathfrak{D}^p f$. The function $\mathfrak{D}^p f$ is, as an element of $\mathfrak{L}^p(\Sigma,\sigma)$, a tempered distribution on $\mathbf{C}$ with support contained in $\bar\Sigma$, and $\mathfrak{C}_*\mathfrak{D}^p f\in L^1_\infty(\mathbf{C})$, by Proposition 2.6. The density $\mathfrak{D}^p f$ is furthermore a locally bounded function in $\mathbf{C}_+$ (it is declared to vanish off Σ), making $\mathfrak{C}_*\mathfrak{D}^p f$ continuous on $\mathbf{C}_+$, due to local elliptic regularity. Since $\mathfrak{C}_*\mathfrak{D}^p f$ is automatically holomorphic in $\mathbf{C}_-$, we see that it is continuous on $\mathbf{C}\backslash\mathbf{R}$. Now suppose temporarily that f has compact support, so that $\mathfrak{F}^p_* f\in L^\infty(\mathbf{C})\cap C(\mathbf{C})$, by the properties of $q_{\sigma,p}$. Using the identity (2.20), summability, and Fubini's theorem, one quickly verifies that, in the sense of distributions,

$$\bar\partial_z\mathfrak{F}^p_* f(z)=\mathfrak{D}^p f(z),\quad z\in\mathbf{C},$$

holds. By Proposition 2.7, $\mathfrak{F}^p_* f=\mathfrak{C}_*\mathfrak{D}^p f$, as claimed. In fact, this identity holds pointwise on $\mathbf{C}\backslash\mathbf{R}$, because there, at least, both functions are continuous. To get the identity for general $f\in L^p(\mathbf{R}_+,\omega_{\sigma,p})$, we use an approximation argument. So, let $\{f_n\}_n$ be a sequence of compactly supported functions in $L^p(\mathbf{R}_+,\omega_{\sigma,p})$, converging in norm to an arbitrary $f_\infty\in L^p(\mathbf{R}_+,\omega_{\sigma,p})$. Estimate (2.19) assures that $\mathfrak{F}^p_* f_n\to\mathfrak{F}^p_* f_\infty$ as $n\to+\infty$, uniformly on compact subsets of $\mathbf{C}_+$. On the other hand, by general Fourier analysis, it is clear that we have uniform convergence on compact subsets of $\mathbf{C}_-$ as well. The same of course happens for the operator $\mathfrak{C}_*\mathfrak{D}^p$, which finishes off the proof. □

The dual space of bounded linear functionals on $L^p(\mathbf{R}_+, \omega_{\sigma,p})$ may be identified with $L^{p'}(\mathbf{R}_+, \omega_{\sigma,p}^{-1})$, the space of functions ϕ on $\mathbf{R}_+$ satisfying

$$\|\phi\|_{L^{p'}(\mathbf{R}_+,\omega_{\sigma,p}^{-1})} = \left(\int_0^{+\infty} |\phi(t)|^{p'} \omega_{\sigma,p}(t)^{-p'}\, dt\right)^{1/p'} < +\infty,$$

with the dual action

$$\langle f, \phi\rangle = \int_0^{+\infty} f(t)\phi(t)\, dt, \quad f \in L^p(\mathbf{R}_+, \omega_{\sigma,p}),\ \phi \in L^{p'}(\mathbf{R}_+, \omega_{\sigma,p}^{-1}).$$

The *resolvent transform* of a $\phi \in L^{p'}(\mathbf{R}_+, \omega_{\sigma,p}^{-1})$ is the function

$$\mathfrak{R}\phi(z) = -i\int_0^{+\infty} \exp(itz)\phi(t)\, dt, \quad z \in \mathbf{C}_+, \tag{2.21}$$

which is holomorphic in the upper half-plane $\mathbf{C}_+$. It is a transformation of Fourier–Laplace type, and will prove invaluable for the proof of the Main Theorem.

For $z \in \mathbf{C}\setminus\mathbf{R}$, let

$$\phi_z^p(t) = \exp(-itz) q_{\sigma,p}(t, \operatorname{Im} z), \quad t \in \mathbf{R}_+,$$

so that $\langle f, \phi_z^p\rangle = \mathfrak{F}_*^p f(z)$ for $f \in L^p(\mathbf{R}_+, \omega_{\sigma,p})$. One then has

$$\|\phi_z^p\|_{L^{p'}(\mathbf{R}_+,\omega_{\sigma,p}^{-1})} = \left(\int_0^{+\infty} e^{typ'} \omega_{\sigma,p}(t)^{-p'} q_{\sigma,p}(t,y)^{p'}\, dt\right)^{1/p'}, \quad z = x+iy,$$

and this is thus the norm of the point evaluation functional. In view of this observation, the following is immediate.

PROPOSITION 2.9. *For $f \in L^p(\mathbf{R}_+, \omega_{\sigma,p})$ of norm $\leqslant 1$,*

$$\begin{aligned}|\mathfrak{F}_*^p f(z)| &\leqslant \|\mathfrak{F}_*^p f(\cdot + iy)\|_{\mathfrak{F}L^1(\mathbf{R})} \leqslant \|\phi_z^p\|_{L^{p'}(\mathbf{R}_+,\omega_{\sigma,p}^{-1})} \\ &= \sup\{|\mathfrak{F}_*^p h(z)| : h \in L^p(\mathbf{R}_+, \omega_{\sigma,p}),\ \|h\|_{L^p(\mathbf{R}_+,\omega_{\sigma,p})} \leqslant 1\}, \quad z = x+iy \in \mathbf{C}\setminus\mathbf{R}.\end{aligned}$$

Remark 2.10. The norm on $\mathfrak{F}L^1(\mathbf{R})$ is the one that makes the Fourier transform $\mathfrak{F}$ an isometry $L^1(\mathbf{R}) \to \mathfrak{F}L^1(\mathbf{R})$.

The relations between the various mappings occurring in this section are illustrated in the following two commutative diagrams; $\mathcal{H}^p(\Sigma, \sigma)$ is the closed subspace of *canonical* *densities* in $\mathfrak{L}^p(\Sigma,\sigma)$, which is the image of $L^p(\mathbf{R}_+,\omega_{\sigma,p})$ under $\mathfrak{D}^p$, R is the restriction operator $f\mapsto f|_{\mathbf{C}_-}$, P is the projection $\mathfrak{L}^p(\Sigma,\sigma)\rightarrow\mathcal{H}^p(\Sigma,\sigma)$ which makes the diagram commute, and $=$ is used to indicate the identity mapping, provided it commutes both ways. For $p=2$, P is the orthogonal projection $\mathfrak{L}^2(\Sigma,\sigma)\rightarrow\mathcal{H}^2(\Sigma,\sigma)$.

$$\begin{tikzcd}
L^p(\mathbf{R}_+,\omega_{\sigma,p}) \arrow[d, "\mathfrak{D}^p"'] & \mathfrak{L}^p(\Sigma,\sigma) \arrow[l, "\mathcal{E}"'] \arrow[d, equal] & \mathfrak{C}_*\mathfrak{L}^p(\Sigma,\sigma) \arrow[l, "\bar{\delta}"'] \arrow[d, equal] \\
\mathcal{H}^p(\Sigma,\sigma) \arrow[d, "\mathfrak{C}"'] & \mathfrak{L}^p(\Sigma,\sigma) \arrow[l, "P"'] \arrow[r, "\mathfrak{C}_*"] \arrow[d, "\mathfrak{C}"'] & \mathfrak{C}_*\mathfrak{L}^p(\Sigma,\sigma) \arrow[d, "R"'] \\
Q^p(\mathbf{C}_-,\sigma) \arrow[r, equal] & Q^p(\mathbf{C}_-,\sigma) \arrow[r, equal] & Q^p(\mathbf{C}_-,\sigma)
\end{tikzcd}$$

$$\begin{tikzcd}
\mathcal{H}^p(\Sigma,\sigma) \arrow[r, equal] \arrow[d, "\mathcal{E}"'] & \mathcal{H}^p(\Sigma,\sigma) \arrow[r, equal] & \mathcal{H}^p(\Sigma,\sigma) \arrow[r, "\mathfrak{C}"] \arrow[d, "\mathfrak{C}_*"'] & Q^p(\mathbf{C}_-,\sigma) \arrow[d, equal] \\
L^p(\mathbf{R}_+,\omega_{\sigma,p}) \arrow[r, equal] & L^p(\mathbf{R}_+,\omega_{\sigma,p}) \arrow[u, "\mathfrak{D}^p"'] \arrow[r, "\mathfrak{F}^p_*"] \arrow[d, "\mathfrak{F}"'] & \mathfrak{C}_*\mathcal{H}^p(\Sigma,\sigma) \arrow[r, "R"] \arrow[d, "R"'] & Q^p(\mathbf{C}_-,\sigma) \arrow[d, equal] \\
Q^p(\mathbf{C}_-,\sigma) \arrow[r, equal] \arrow[u, "\mathfrak{F}^{-1}"] & Q^p(\mathbf{C}_-,\sigma) \arrow[r, equal] & Q^p(\mathbf{C}_-,\sigma) \arrow[r, equal] & Q^p(\mathbf{C}_-,\sigma)
\end{tikzcd}$$

3. The isometry construction: $p=1$

Let σ be a positive strictly decreasing C^2 function on $]0,1]$, with limit $\sigma(y)\rightarrow+\infty$ as $0<y\rightarrow0$. Suppose, furthermore, that its logarithm $y\mapsto\log\sigma(y)$ is strictly convex, in the strong sense that its second derivative is positive throughout $]0,1]$. Put

$$\omega_{\sigma,1}(t)=\inf\{\exp(ty)\sigma(y):0<y\leqslant1\}, \tag{3.1}$$

which then has a concave logarithm $\log\omega_{\sigma,1}$, and has the limit $\omega_{\sigma,1}(t)\rightarrow+\infty$ as $t\rightarrow+\infty$. By the general properties of the Legendre transform, we may recover σ from $\omega_{\sigma,1}$,

$$\sigma(y)=\sup\{\exp(-ty)\omega_{\sigma,1}(t):t\in\mathbf{R}_+\},\quad 0<y\leqslant1. \tag{3.2}$$

We assume, moreover, that $\sigma(1)=1$ and $\sigma'(1)=0$; as a consequence, $\omega_{\sigma,1}(0)=1$. The function $y_\sigma(t)$ is defined by the equality

$$\omega_{\sigma,1}(t)=\exp(ty_\sigma(t))\sigma(y_\sigma(t)); \tag{3.3}$$

one then computes that $y_\sigma(t)=(\log\omega_{\sigma,1})'(t)$, which is a decreasing function on $\mathbf{R}_+$. The inverse function of $y_\sigma(t)$ is denoted by $t_\sigma(y)$, and one finds that $t_\sigma(y)=-\sigma'(y)/\sigma(y)$.

The functions y_σ and t_σ are of class C^1; y_σ is strictly increasing on $\mathbf{R}_+$, and t_σ is strictly decreasing on $]0,1]$. By assumption, $\sigma(1)=1$ and $\sigma'(1)=0$, so the image of $]0,1]$ under t_σ is $\mathbf{R}_+$, and accordingly, the image of $\mathbf{R}_+$ under y_σ is $]0,1]$.

Let $\mathcal{M}(\mathbf{R})$ be the set of all finite complex-valued Borel measures on $\mathbf{R}$. Given $\mu\in\mathcal{M}(\mathbf{R})$, we write $\|\mu\|_{\mathcal{M}(\mathbf{R})}$ for the total variation of μ.

If ω is a weight function on $\mathbf{R}_+$ (in the sense used in the introduction), we consider the space $\mathcal{M}(\mathbf{R}_+,\omega)$ of all Borel measures μ supported on $\mathbf{R}_+=[0,+\infty[$, subject to

$$\|\mu\|_{\mathcal{M}(\mathbf{R}_+,\omega)}=\int_{\mathbf{R}_+}\omega(t)\,d|\mu|(t)<+\infty.$$

It is a Banach space with the above norm, which contains $L^1(\mathbf{R}_+,\omega)$ as a closed subspace in a canonical fashion: an L^1 function f is mapped onto the measure $f\,dt$.

Let $\mathcal{M}(\check\Sigma)$ be the set of all finite Borel measures on the half-open strip $\check\Sigma=\mathbf{R}\times]0,1]$, normed appropriately:

$$\|\mu\|_{\mathcal{M}(\check\Sigma)}=\int_{\check\Sigma}d|\mu|(z).$$

Every measure μ in $\mathcal{M}(\check\Sigma)$ may be decomposed (see, for instance, [24, pp. 595–618])

$$d\mu(t+iy)=d\mu_y(t)\,d\nu(y),\quad t+iy\in\check\Sigma, \tag{3.4}$$

where ν is the finite positive Borel measure on $]0,1]$ obtained by setting $\nu(E)=|\mu|(\mathbf{R}\times E)$, and the mapping $y\mapsto\mu_y$ is Borel measurable and well-defined almost everywhere $(d\nu)$ as a mapping from $]0,1]$ into the closed unit ball of $\mathcal{M}(\mathbf{R})$. If μ is a probability measure, μ_y is the conditional distribution of t for fixed y. The space $\mathcal{M}(\check\Sigma,\sigma)$ of *pre-densities* is the subspace of $\mathcal{M}(\check\Sigma)$ consisting of those Borel measures μ on $\check\Sigma$ with finite norm,

$$\|\mu\|_{\mathcal{M}(\check\Sigma,\sigma)}=2\int_{\check\Sigma}\sigma(y)\,d|\mu|(t+iy)<+\infty.$$

The *density space* $\mathfrak{L}^1(\check\Sigma,\sigma)$ is defined to be the image of $\mathcal{M}(\check\Sigma,\sigma)$ by the Fourier transform in the variable t. More precisely, if $\mu\in\mathcal{M}(\check\Sigma,\sigma)$ and μ has the decomposition (3.4), then

$$d\tilde\mu(x+iy)=\mathfrak{F}\mu_y(x)\,dx\,d\nu(y),\quad x+iy\in\check\Sigma, \tag{3.5}$$

is a typical element of $\mathfrak{L}^1(\check\Sigma,\sigma)$, where

$$\mathfrak{F}\mu_y(x)=\int_{\mathbf{R}}\exp(-itx)\,d\mu_y(t),\quad x+iy\in\check\Sigma,$$

which is bounded and continuous in the x variable. The elements of $\mathfrak{L}^1(\check\Sigma,\sigma)$ are thus Borel measures on $\check\Sigma$. The norm in $\mathfrak{L}^1(\check\Sigma,\sigma)$ is defined to be the one induced by the pre-density space $\mathcal{M}(\check{\Sigma},\sigma)$: $\|\tilde{\mu}\|_{\mathfrak{L}^1(\check{\Sigma},\sigma)}=\|\mu\|_{\mathcal{M}(\check{\Sigma},\sigma)}$. Since they are Borel measures on $\check{\Sigma}$, the elements of $\mathfrak{L}^1(\check{\Sigma},\sigma)$ are naturally distributions on $\mathbf{C}$; as in §2, this is compatible with defining the action of $\tilde{\mu}\in\mathfrak{L}^1(\check{\Sigma},\sigma)$ on a test function $\varphi\in\mathcal{S}(\mathbf{C})$ to be

$$\langle\varphi,\tilde{\mu}\rangle=\int_{\check{\Sigma}}\mathfrak{F}\varphi_y(t)\,d\mu(t+iy).$$

Again as in §2, the right hand side expression makes sense for a larger class of φ than those in $\mathcal{S}(\mathbf{C})$. The Cauchy transform is defined on $\mathfrak{L}^1(\check{\Sigma},\sigma)$ by a formula analogous to (2.9),

$$\begin{aligned}\mathfrak{C}\tilde{\mu}(\zeta)&=\langle\mathcal{C}(\,\cdot\,,\zeta),\tilde{\mu}\rangle=\int_{\check{\Sigma}}\mathcal{K}_y(t,\zeta)\,d\mu(t+iy)\\&=2i\int_{[0,+\infty]\times]0,1]}\exp(-t(y+i\zeta))\,d\mu(t+iy),\quad \zeta\in\mathbf{C}_-.\end{aligned}\tag{3.6}$$

This definition involves a choice of how the Borel measure μ_y acts on the Heaviside function H. The Cauchy image $\mathfrak{C}\tilde{\mu}$ of $\tilde{\mu}$ is a holomorphic function in the lower half-plane $\mathbf{C}_-$, because the integral defining it converges absolutely for all $\zeta\in\mathbf{C}_-$, since

$$\begin{aligned}|\mathfrak{C}\tilde{\mu}(\zeta)|&\leqslant 2\int_{[0,+\infty[\times]0,1]}\exp(-t(y-\operatorname{Im}\zeta))\,d|\mu|(t+iy)=2\int_{]0,1]}\frac{d\nu(y)}{y-\operatorname{Im}\zeta}\\&\leqslant\frac{\|\mu\|_{\mathcal{M}(\check{\Sigma},\sigma)}}{\inf\{(y-\operatorname{Im}\zeta)\sigma(y):0<y\leqslant 1\}}.\end{aligned}$$

Write $Q^1(\mathbf{C}_-,\sigma)$ for the image $\mathfrak{C}\mathfrak{L}^1(\check{\Sigma},\sigma)$ of $\mathfrak{L}^1(\check{\Sigma},\sigma)$ under the Cauchy transform $\mathfrak{C}$. It is a Banach space of holomorphic functions in the lower half-plane $\mathbf{C}_-$, when supplied with the norm

$$\|h\|_{Q^1(\mathbf{C}_-,\sigma)}=\inf\{\|g\|_{\mathfrak{L}^1(\check{\Sigma},\sigma)}:g\in\mathfrak{L}^1(\check{\Sigma},\sigma),\ \mathfrak{C}g=h\}.$$

THEOREM 3.1. *The Fourier transform* $\mathfrak{F}$ *maps* $\mathcal{M}(\mathbf{R}_+,\omega_{\sigma,1})$ *isometrically onto* $Q^1(\mathbf{C}_-,\sigma)$.

As in §2, the first step is to introduce the operators $\mathfrak{D}^1$ and $\mathcal{E}$. Given a measure $\xi\in\mathcal{M}(\mathbf{R}_+,\omega_{\sigma,1})$, let $\xi\circ t_\sigma$ be the measure on $]0,1]$ defined by

$$\xi\circ t_\sigma(E)=\xi(t_\sigma(E)),$$

for Borel sets E. Consider the pre-density measure

$$d\mu[\xi](t+iy)=\frac{1}{2i}\exp(yt_\sigma(y))\,d\delta_0(t-t_\sigma(y))\,d(\xi\circ t_\sigma)(y),\quad t+iy\in\check{\Sigma},\tag{3.7}$$

and the associated density $\mathfrak{D}^1\xi = \tilde{\mu}[\xi]$,

$$d\tilde{\mu}[\xi](x+iy) = \frac{1}{2i}\exp(yt_\sigma(y))\exp(-it_\sigma(y)x)\,dx\,d(\xi\circ t_\sigma)(y), \quad x+iy\in\check{\Sigma}.$$

The measure $\mu[\xi]$ has norm

$$\begin{aligned}
\|\mu[\xi]\|_{\mathcal{M}(\check{\Sigma},\sigma)} &= \int_{\check{\Sigma}} \sigma(y)\exp(yt_\sigma(y))\,d\delta_0(t-t_\sigma(y))\,d(|\xi|\circ t_\sigma)(y)\\
&= \int_{]0,1]} \sigma(y)\exp(yt_\sigma(y))\,d(|\xi|\circ t_\sigma)(y)\\
&= \int_{\mathbf{R}_+} \sigma(y_\sigma(t))\exp(ty_\sigma(y))\,d|\xi|(t)\\
&= \int_{\mathbf{R}_+} \omega_{\sigma,1}(t)\,d|\xi|(t) = \|\xi\|_{\mathcal{M}(\mathbf{R}_+,\omega_{\sigma,1})};
\end{aligned}$$

thus $\xi\mapsto\mu[\xi]$ is an isometry $\mathcal{M}(\mathbf{R}_+,\omega_{\sigma,1})\to\mathcal{M}(\check{\Sigma},\sigma)$, and $\mathfrak{D}^1$ is an isometry $\mathcal{M}(\mathbf{R}_+,\omega_{\sigma,1})\to\mathfrak{L}^1(\check{\Sigma},\sigma)$. By (3.6), the Cauchy transform of $\mathfrak{D}^1\xi$ is

$$\begin{aligned}
\mathfrak{C}\tilde{\mu}[\xi](\zeta) &= 2i\int_{[0,+\infty[\times]0,1]} \exp(-t(y+i\zeta))\,d\mu[\xi](t+iy)\\
&= \int_{[0,+\infty[\times]0,1]} \exp(-t(y+i\zeta)+yt_\sigma(y))\,d\delta(t-t_\sigma(y))\,d(\xi\circ t_\sigma)(y)\\
&= \int_{]0,1]} \exp(-it_\sigma(y)\zeta)\,d(\xi\circ t_\sigma)(y)\\
&= \int_{\mathbf{R}_+} \exp(-it\zeta)\,d\xi(t) = \mathfrak{F}\xi(\zeta), \quad \zeta\in\mathbf{C}_-.
\end{aligned}$$

Given a pre-density $\mu\in\mathcal{M}(\check{\Sigma},\sigma)$, let $\xi[\mu]$ be the Borel measure on $\mathbf{R}_+$ which assigns the mass

$$\xi[\mu](E) = 2i\int_{E\times]0,1]} \exp(-ty)\,d\mu_y(t)\,d\nu(y) \tag{3.8}$$

to a Borel set E. The extraction of the density $\tilde{\mu}\in\mathfrak{L}^1(\check{\Sigma},\sigma)$ is then $\mathcal{E}\tilde{\mu}=\xi[\mu]$, which is an element of $\mathcal{M}(\mathbf{R}_+,\omega_{\sigma,1})$ (use (3.2)):

$$\begin{aligned}
\|\xi[\mu]\|_{\mathcal{M}(\mathbf{R}_+,\omega_{\sigma,1})} &= \int_{\mathbf{R}_+} \omega_{\sigma,1}(t)\,d|\xi|[\mu](t)\\
&\leqslant 2\int_{\mathbf{R}_+\times]0,1]} \exp(-ty)\omega_{\sigma,1}(t)\,d|\mu|_y(t)\,d\nu(y)\\
&\leqslant 2\int_{\mathbf{R}_+\times]0,1]} \sigma(y)\,d|\mu|_y(t)\,d\nu(y) = \|\mu\|_{\mathcal{M}(\check{\Sigma},\sigma)}.
\end{aligned}$$

By inspection, the Fourier transform of $\xi[\mu]$ coincides with $\mathfrak{C}\tilde{\mu}$,

$$\begin{aligned}\mathfrak{F}\xi[\mu](\zeta) &= \int_{\mathbf{R}_+} \exp(-it\zeta)\, d\xi[\mu](t) \\ &= 2i\int_{\mathbf{R}_+} \exp(-it\zeta - ty)\, d\mu_y(t)\, d\nu(y) = \mathfrak{C}\tilde{\mu}(\zeta), \quad \zeta\in\mathbf{C}_-.\end{aligned}$$

Let us gather these observations in a proposition.

PROPOSITION 3.2. *The following assertions are valid.*

(a) *The mapping $\mathfrak{D}^1$ is an isometry $\mathcal{M}(\mathbf{R}_+,\omega_{\sigma,1})\to\mathfrak{L}^1(\check{\Sigma},\sigma)$.*

(b) *For each $\xi\in\mathcal{M}(\mathbf{R}_+,\omega_{\sigma,1})$, $\mathfrak{C}\mathfrak{D}^1\xi(z)=\mathfrak{F}\xi(z)$ holds on $\mathbf{C}_-$.*

(c) *The mapping $\mathcal{E}$ is a norm contraction $\mathfrak{L}^1(\check{\Sigma},\sigma)\to\mathcal{M}(\mathbf{R}_+,\omega_{\sigma,1})$.*

(d) *$\mathfrak{F}\mathcal{E}=\mathfrak{C}$ as mappings $\mathfrak{L}^1(\check{\Sigma},\sigma)\to Q^1(\mathbf{C}_-,\sigma)$.*

Proof of Theorem 3.1. Mimic that of Theorem 2.1. □

We need to extend the Cauchy transform to a global one, denoted $\mathfrak{C}_*$. As was the case with the usual Cauchy transform, there will be some degree of arbitrariness in our choice, due to the fact that we are dealing with measures, not functions. For $\mu\in\mathcal{M}(\check{\Sigma},\sigma)$, put

$$\begin{aligned}\mathfrak{C}_*\tilde{\mu}(\zeta) = 2i&\int_{U(\zeta)} \exp(-t(i\zeta+y))\, d\mu(t+iy) \\ &-2i\int_{V(\zeta)} \exp(-t(i\zeta+y))\, d\mu(t+iy), \quad \zeta\in\mathbf{C},\end{aligned} \tag{3.9}$$

where $U(\zeta)=\mathbf{R}_+\times(]\operatorname{Im}\zeta,1]\cap]0,1])$ and $V(\zeta)=]-\infty,0[\times(]0,\operatorname{Im}\zeta]\cap]0,1])$. It is clear that this defines a bounded Borel measurable function on $\mathbf{C}$, and that in fact

$$\|\mathfrak{C}_*\tilde{\mu}\|_{L^\infty(\mathbf{C})} \leqslant \|\mu\|_{\mathcal{M}(\check{\Sigma},\sigma)} \tag{3.10}$$

holds.

PROPOSITION 3.3. *If $\tilde{\mu}\in\mathfrak{L}^1(\check{\Sigma},\sigma)$, so that $f=\mathfrak{C}_*\tilde{\mu}\in L^\infty(\mathbf{C})$, then $\bar{\partial}f=\tilde{\mu}$, in the sense of distribution theory. On the other hand, if $f\in L^1_\infty(\mathbf{C})$, and its distributional derivative $\bar{\partial}f$ belongs to the space $\mathfrak{L}^1(\check{\Sigma},\sigma)$, then there exists a constant $\beta(f)$ such that $f=\beta(f)+\mathfrak{C}_*\bar{\partial}f$ holds area-almost everywhere on $\mathbf{C}$; in particular, the restriction of f to $\mathbf{C}_-$ belongs to $Q^1(\mathbf{C}_-,\sigma)$.*

Proof. Analogous to that of Proposition 2.7. □

By Proposition 3.2, the operator $\mathfrak{F}^1_*=\mathfrak{C}_*\mathfrak{D}^1$ supplies a *canonical generalized Fourier transform.* It is given by the formula stated in the proposition below.

PROPOSITION 3.4. *The canonical generalized Fourier transform on* $\mathcal{M}(\mathbf{R}_+,\omega_{\sigma,1})$ *is given by the formula*

$$\mathfrak{F}^1_*\xi(z)=\mathfrak{C}_*\mathfrak{D}^1\xi(z)=\int_{[0,t_\sigma(y)[} e^{-itz}\,d\xi(t), \quad z=x+iy\in\mathbf{C},$$

where we agree that $t_\sigma(y)=+\infty$ *for* $y\leqslant 0$, *and* $t_\sigma(y)=0$ *for* $1\leqslant y<+\infty$. *It is thus a bounded Borel measurable function on* $\mathbf{C}$. *If the measure* ξ *in* $\mathcal{M}(\mathbf{R}_+,\omega_{\sigma,1})$ *is absolutely continuous with respect to Lebesgue measure, then* $\mathfrak{F}_*\xi$ *is continuous on* $\mathbf{C}$.

Note. The interval $[0,0[$ is the empty set, and an integral over the empty set is 0. Hence $\mathfrak{F}^1_*\xi(z)=0$ for $1\leqslant \operatorname{Im} z$.

Proof of Proposition 3.4. Since $0\leqslant t_\sigma(y)$ always holds, the measure $\mu[\xi]$ in (3.7) places no mass on the set $V(\zeta)$ occurring in the definition (3.9) of $\mathfrak{C}_*\tilde\mu(\zeta)$, and thus

$$\begin{aligned}\mathfrak{C}_*\tilde\mu[\xi](\zeta)&=2i\int_{U(\zeta)}\exp(-t(i\zeta+y))\,d\mu[\xi](t+iy)\\ &=\int_{]\operatorname{Im}\zeta,1]\cap]0,1]}\exp(-it_\sigma(y)\zeta)\,d(\xi\circ t_\sigma)(y)=\int_{[0,t_\sigma(y)[}e^{-it\zeta}\,d\xi(t).\end{aligned}$$

The boundedness and continuity mentioned follow from direct inspection. □

The space $L^1(\mathbf{R}_+,\omega_{\sigma,1})$ will be identified with the closed subspace of $\mathcal{M}(\mathbf{R}_+,\omega_{\sigma,1})$ of measures that are absolutely continuous with respect to linear Lebesgue measure. The dual space of bounded linear functionals on $L^1(\mathbf{R}_+,\omega_{\sigma,1})$ may be identified with $L^\infty(\mathbf{R}_+,\omega_{\sigma,p}^{-1})$, the space of functions ϕ on $\mathbf{R}_+$ satisfying

$$\|\phi\|_{L^\infty(\mathbf{R}_+,\omega_{\sigma,1}^{-1})}=\operatorname{ess\,sup}\{|\phi(t)|/\omega_{\sigma,1}(t):t\in\mathbf{R}_+\}<+\infty,$$

with the dual action

$$\langle f,\phi\rangle=\int_0^{+\infty}f(t)\phi(t)\,dt,\quad f\in L^1(\mathbf{R}_+,\omega_{\sigma,1}),\ \phi\in L^\infty(\mathbf{R}_+,\omega_{\sigma,p}^{-1}).$$

The *resolvent transform* of a $\phi\in L^\infty(\mathbf{R}_+,\omega_{\sigma,p}^{-1})$ is the function

$$\mathfrak{R}\phi(z)=-i\int_0^{+\infty}\exp(itz)\phi(t)\,dt,\quad z\in\mathbf{C}_+, \tag{3.11}$$

which is holomorphic in the upper half-plane $\mathbf{C}_+$.

For $z\in\mathbf{C}$, let

$$\phi^1_z(t)=\exp(-itz)q_{\sigma,1}(t,\operatorname{Im} z),\quad t\in\mathbf{R}_+,$$

where $q_{\sigma,1}(\cdot\,,y)$ is the characteristic function of the interval $[0,t_\sigma(y)[$, so that $\langle f,\phi^1_z\rangle=\mathfrak{F}^1_*f(z)$ holds for $f\in L^1(\mathbf{R}_+,\omega_{\sigma,1})$. One then has, with $z=x+iy$, that $\phi^1_z=0$ for $1\leqslant y<+\infty$, and

$$\|\phi^1_z\|_{L^\infty(\mathbf{R}_+,\omega_{\sigma,1}^{-1})}=\operatorname{ess\,sup}\{e^{ty}/\omega_{\sigma,1}(t):0\leqslant t<t_\sigma(y)\}=1,\quad -\infty<y<1,$$

so the norm of the point evaluation functional is either 0 or 1. Just as in the previous section, the following is immediate.

PROPOSITION 3.5. *For $f \in L^1(\mathbf{R}_+, \omega_{\sigma,1})$ of norm $\leqslant 1$,*

$$|\mathfrak{F}^1_* f(z)| \leqslant \|\mathfrak{F}^1_* f(\cdot + iy)\|_{\mathfrak{F}L^1(\mathbf{R})} \leqslant 1, \quad z = x+iy \in \mathbf{C}.$$

4. The class of weight functions of type $\omega_{\sigma,p}$

Fix a p, $1 \leqslant p < +\infty$, and write $p' = p/(p-1)$, with the usual convention that $p' = \infty$ if $p=1$. The continuous function $\sigma:]0,1] \to]0,+\infty[$ satisfies (2.1) if $1<p<+\infty$, and for $p=1$, it is assumed to be strictly decreasing and have a strictly convex logarithm, as in §3. It is then of interest to know which weights ω on $\mathbf{R}_+$ are of the form $\omega_{\sigma,p}$, but since this question has a complicated answer, and we actually are only interested in the spaces $L^p(\mathbf{R}_+, \omega)$, we shall be equally happy to know for which ω we have $\omega \asymp \omega_{\sigma,p}$. Here, we use the notation $f \asymp g$, and say in words that f and g are *comparable* on $\mathbf{R}_+$, if f and g are two functions on $\mathbf{R}_+$ with values in $[0, +\infty[$, which satisfy

$$C_1 f(t) \leqslant g(t) \leqslant C_2 f(t), \quad t \in \mathbf{R}_+,$$

for some constants C_1, C_2, $0 < C_1, C_2 < +\infty$. One easily convinces oneself that given two different weight functions ω_1 and ω_2 on $\mathbf{R}_+$, the spaces $L^p(\mathbf{R}_+, \omega_1)$ and $L^p(\mathbf{R}_+, \omega_2)$ are the same (and the associated norms equivalent) if and only if $\omega_1 \asymp \omega_2$.

We first treat the case $1<p<+\infty$. Let $\mathfrak{W}$ denote the collection of all continuous (weight) functions $\omega: \mathbf{R}_+ \to]0,+\infty[$ which are increasing, have limit $\omega(t) \to +\infty$ as $t \to +\infty$, and possess a logarithm $\log \omega$ which is concave, and satisfies

$$\log \omega(t) = o(t), \quad \text{as } t \to +\infty.$$

PROPOSITION 4.1 ($1<p<+\infty$). *If σ is as above, then $\omega_{\sigma,p}$ belongs to $\mathfrak{W}$. If, on the other hand, ω is in $\mathfrak{W}$, then a σ can be found such that $\omega_{\sigma,p} \asymp \omega$.*

Proof. Let us start with having a σ, and try to prove $\omega_{\sigma,p} \in \mathfrak{W}$. Since, with the notation of Appendix B, and $d\mu(x) = dx/\sigma(x)^{p'}$,

$$\omega_{\sigma,p}(t) = (2F_{d\mu}(p't))^{-1/p'}, \quad t \in \mathbf{R}_+, \tag{4.1}$$

holds, and $F_{d\mu}$ belongs to $\mathfrak{V}$ in view of Proposition B.1, the assertion that $\omega_{\sigma,p}$ is in $\mathfrak{W}$ is immediate from the definitions of the classes $\mathfrak{V}$ and $\mathfrak{W}$.

We proceed to the case when we have an ω, and seek a σ. Let the function F be defined via

$$\omega(t) = (2F(p't))^{-1/p'}, \quad t \in \mathbf{R}_+, \tag{4.2}$$

and use Proposition B.1 to produce a $\mu\in\mathfrak{P}$ of the form $d\mu(t)=\varphi(t)\,dt$, where φ is a continuous positive real-valued function on $]0,1]$, for which $F_{d\mu}\asymp F$. The choice $\sigma(t)=\varphi(t)^{-1/p'}$ then does the trick. $\square$

For a fixed real parameter s, $0\leqslant s<+\infty$, let $\mathfrak{W}_s$ denote the set of all weights ω in $\mathfrak{W}$ with the additional property that the function $t\mapsto(1+t)^{-s}\omega(t)$ is in $\mathfrak{W}$. In particular, $\mathfrak{W}_0$ coincides with $\mathfrak{W}$. Recall from Appendix B the definition of the sets $\mathfrak{P}_s$, with $0<s<+\infty$, of fractional integrals of positive Borel measures in $\mathfrak{P}$.

PROPOSITION 4.2 ($1<p<+\infty$). *Fix a real parameter s, $0<s<+\infty$. If σ is such that $\sigma^{-p'}$ is in $\mathfrak{P}_{sp'}$, then there exists a weight $\widetilde{\omega}_{\sigma,p}$ in $\mathfrak{W}_s$ such that $\widetilde{\omega}_{\sigma,p}\asymp\omega_{\sigma,p}$. On the other hand, if an element ω of $\mathfrak{W}_s$ is given, then there exists a σ with $\sigma^{-p'}\in\mathfrak{P}_{sp'}$ with the property that $\omega_{\sigma,p}\asymp\omega$.*

Proof. If we start out with σ, we put $\varrho=\sigma^{-p'}$, which is assumed in $\mathfrak{P}_{sp'}$, and note that

$$\omega_{\sigma,p}(t)=(2F_\varrho(p't))^{-1/p'},\quad t\in\mathbf{R}_+, \tag{4.3}$$

which is (4.1) specialized to the present situation. Let $\widetilde{F}_\varrho$ be as in Corollary B.4, and define $\widetilde{\omega}_{\sigma,p}$ by the identity

$$\widetilde{\omega}_{\sigma,p}(t)=(2\widetilde{F}_\varrho(p't))^{-1/p'},\quad t\in\mathbf{R}_+.$$

Corollary B.4 tells us that $\widetilde{F}_\varrho\asymp F_\varrho$, hence $\widetilde{\omega}_{\sigma,p}\asymp\omega_{\sigma,p}$, and that $\widetilde{F}_\varrho\in\mathfrak{V}_{sp'}$, hence $\widetilde{\omega}_{\sigma,p}\in\mathfrak{W}_s$.

We next deal with the case when we begin with an ω, and look for a σ. Let F be related to ω via (4.2), so that $\omega\in\mathfrak{W}_s$ entails $F\in\mathfrak{V}_{sp'}$. Proposition B.5 provides us with a function $\varrho\in\mathfrak{P}_{sp'}$ such that $F_\varrho\asymp F$, so if we put $\sigma=\varrho^{-1/p'}$, we get $\omega_{\sigma,p}\asymp\omega$. This completes the proof. $\square$

It is time to formulate an assertion about weights in $\mathfrak{W}$, which we mentioned without proof in the introduction.

PROPOSITION 4.3. *Suppose $\omega\in\mathfrak{W}$; this is so if ω is of the type $\omega_{\sigma,p}$, by Proposition 4.1. We then have the inequality*

$$\omega(0)\omega(t+x)\leqslant\omega(t)\omega(x),\quad t,x\in\mathbf{R}_+.$$

Proof. By definition, $\nu(t)=\log\omega(t)$ is increasing and concave on $\mathbf{R}_+$. In particular, $\nu'(t)$ decreases with t, so that

$$\nu(t+x)-\nu(t)=\int_0^x\nu'(t+s)\,ds\leqslant\int_0^x\nu'(s)\,ds=\nu(x)-\nu(0),$$

from which the claimed inequality

$$\omega(0)\omega(t+x) \leqslant \omega(t)\omega(x), \quad t, x \in \mathbf{R}_+,$$

is immediate. □

Recall the notation

$$\tilde{\sigma}(p,\alpha,y) = \int_0^{+\infty} t^{\alpha p'} \omega_{\sigma,p}(t)^q \exp(-tyq)\,dt, \tag{4.4}$$

from §2, where $q=p'(p'-1)=(p')^2/p$. As a function of y, $\tilde{\sigma}(p,\alpha,y)$ is strictly decreasing, and has $0<\tilde{\sigma}(p,\alpha,y)<+\infty$ for $y\in\,]0,+\infty[$, because $\omega_{\sigma,p}\in\mathfrak{W}$ (Proposition 4.1).

PROPOSITION 4.4 ($1<p<+\infty$). *Given s and α, subject to the restraints $1/p'<s<+\infty$ and $0\leqslant\alpha<+\infty$, there exists a constant $C=C(s,\sigma,p,\alpha)$, $0<C<+\infty$, such that if $\sigma^{-p'}$ is in $\mathfrak{P}_{sp'}$, then*

$$\tilde{\sigma}(p,\alpha,y)^{1/p'} \leqslant C\sigma(y)^q, \quad 0<y\leqslant 1,$$

where $q=(sp'/p+\alpha+1/p')/(s-1/p')$.

Proof. With the notation of Appendix B,

$$\omega_{\sigma,p}(t) = (2F_{\sigma^{-p'}}(p't))^{-1/p'}, \quad 0\leqslant t<+\infty,$$

so we see that by substituting $u=p't$,

$$\tilde{\sigma}(p,\alpha,y) = 2^{-p'/p}(p')^{-\alpha p'-1}\int_0^{+\infty} u^{\alpha p'} \exp(-uyp'/p) F_{\sigma^{-p'}}(u)^{-p'/p}\,du, \quad 0<y<+\infty.$$

The assertion is now an immediate consequence of Proposition B.8. □

Consider the related functions ($0<r<+\infty$)

$$\begin{cases} \hat{\sigma}(p,y) = \int_0^{+\infty} \exp(-pty)\omega_{\sigma,p}(t)^p\,dt, & 0<y<+\infty, \\ \check{\sigma}(p,r,y) = \int_0^{+\infty} \exp(-tyrp')\omega_{\sigma,p}(t)^{rp'}\,dt, & 0<y<+\infty, \end{cases} \tag{4.5}$$

which, as functions of y, are strictly decreasing, and have values in $]0,+\infty[$ for $0<y<+\infty$, because $\omega_{\sigma,p}\in\mathfrak{W}$ (Proposition 4.1). They have similar estimates.

PROPOSITION 4.5 ($1<p<+\infty$). *If s has $1/p'<s<+\infty$, there exist two constants $C=C(s,\sigma,p)$ and $K=K(s,\sigma,p,r)$, $0<C,K<+\infty$, such that if $\sigma^{-p'}$ is in $\mathfrak{P}_{sp'}$, then*

$$\hat{\sigma}(p,y)^{1/p} \leqslant C\sigma(y)^{\alpha}, \quad 0<y\leqslant 1,$$
$$\check{\sigma}(p,r,y)^{1/p'} \leqslant K\sigma(y)^{\beta}, \quad 0<y\leqslant 1,$$

with $\alpha=(s+1/p)/(s-1/p')$ *and* $\beta=(rs+1/p')/(s-1/p')$.

Proof. Analogous to that of Proposition 4.4. $\square$

We turn to the case $p=1$. If we do not make any additional regularity assumptions on σ, the relationship between σ and $\omega_{\sigma,1}$ is sufficiently well understood in §3. To treat regularity conditions on the weights σ and $\omega_{\sigma,1}$, we proceed as follows. Start with an $\omega\in\mathfrak{W}_s$, where s has $0<s$, replace it with $\widetilde{\omega}\in\mathfrak{W}_s$ in C^2, having $\widetilde{\omega}\asymp\omega$, $\widetilde{\omega}(0)=1$, and $\widetilde{\omega}'(0)=1$, and put

$$\sigma(y)=\sup\{e^{-ty}\widetilde{\omega}(t):t\in\mathbf{R}_+\},\quad 0<y\leqslant 1. \tag{4.6}$$

This σ then has all the properties requested of it in §3; we mention here that in particular, $\sigma(1)=1$, and $\sigma'(1)=0$. By the definition (4.7) of $\omega_{\sigma,1}$, and standard properties of the Legendre transform, $\omega_{\sigma,1}(t)=\widetilde{\omega}(t)$ holds on all of $\mathbf{R}_+$. To study the functions $y_\sigma(t)$ and $t_\sigma(y)$ considered in §3 more closely, introduce temporarily the notation $\alpha(t)=\log\omega_{\sigma,1}(t)$ and $\beta(y)=\log\sigma(y)$, and note that these functions are connected via

$$\begin{cases}\alpha(t)=\inf\{\beta(y)+ty:0<y\leqslant 1\}=\beta(y_\sigma(t))+ty_\sigma(t),\\ \beta(y)=\sup\{\alpha(t)-yt:t\in\mathbf{R}_+\}=\alpha(t_\sigma(y))-yt_\sigma(y).\end{cases}\tag{4.7}$$

By differential calculus,

$$\begin{cases}y_\sigma(t)=\alpha'(t),\\ t_\sigma(y)=-\beta'(y),\end{cases}$$

so that since the functions $y_\sigma(t)$ and $t_\sigma(y)$ are inverse to each other by definition, it follows that

$$\begin{cases}\alpha'(-\beta'(y))=y,\\ \beta'(\alpha'(t))=-t.\end{cases}$$

Differentiating once more, we get

$$\beta''(y)=-(\alpha''(-\beta'(y)))^{-1}.\tag{4.8}$$

LEMMA 4.6. *In the above context, the following estimates hold*:

(a) $0\leqslant t_\sigma(y)=-\beta'(y)\leqslant e\cdot\sigma(y)^{1/s}$, $y\in]0,1]$,

(b) $-(e^2/s)\sigma(y)^{2/s}\leqslant t'_\sigma(y)=-\beta''(y)\leqslant 0$, $y\in]0,1]$.

Proof. Since $\omega_{\sigma,1}=\widetilde{\omega}\in\mathfrak{W}_s$, we have $\alpha(t)=s\log(1+t)+\alpha_0(t)$, where α_0 is concave, and has $\alpha_0(t)=o(t)$ as $t\to+\infty$. Moreover, since $\alpha(0)=0$, $\alpha_0(0)=0$, and thus, by concavity,

$$0\leqslant\int_0^t(\alpha_0'(\tau)-\alpha_0'(t))\,d\tau=\alpha_0(t)-t\alpha_0'(t),\quad t\in\mathbf{R}_+.$$

We conclude that

$$s(\log(1+t)-t/(1+t))\leqslant\alpha(t)-t\alpha'(t),\quad t\in\mathbf{R}_+,$$

and since $\beta(y_\sigma(t))=\beta(\alpha'(t))=\alpha(t)-t\alpha'(t)$, that

$$e^{-1}(1+t)\leqslant\exp(\beta(y_\sigma(t))/s),$$

or, equivalently,

$$1+t_\sigma(y)\leqslant e\cdot\exp(\beta(y)/s). \tag{4.9}$$

Assertion (a) is immediate. Furthermore, assertion (b) follows from (4.8), (4.9), and the inequality $\alpha''(t)\leqslant-s(1+t)^{-2}$. □

PROPOSITION 4.7. *If $\omega_{\sigma,1}\in\mathfrak{W}_s$ for some s, $0<s<+\infty$, then*

$$\hat\sigma(1,y)=\int_0^{+\infty}e^{-ty}\omega_{\sigma,1}(t)\,dt\leqslant e^3\Gamma(s)\sigma(y)^{1+1/s},\quad 0<y\leqslant1.$$

Proof. The assumption on $\omega_{\sigma,1}$ should be thought of as

$$\alpha''(t)\leqslant-s(1+t)^{-2},\quad t\in\mathbf{R}_+. \tag{4.10}$$

The integrand is $\exp(-ty+\alpha(t))$, and by (4.7), it attains its maximum value $\sigma(y)=\exp(\beta(y))$ at $t=t_\sigma(y)$. By (4.10), the function $t\mapsto-ty+\alpha(t)$ is strictly concave, and one easily derives the estimate

$$-ty+\alpha(t)\leqslant\beta(y)+s\left(\log(1+t)-\log(1+t_\sigma(y))-\frac{t-t_\sigma(y)}{1+t_\sigma(y)}\right),\quad t\in\mathbf{R}_+.$$

When the integrand is replaced with the exponential of the right hand expression, one arrives at

$$\int_0^{+\infty}\exp(-ty+\alpha(t))\,dt\leqslant e^2\Gamma(s)(1+t_\sigma(y))\sigma(y),$$

whence the assertion follows by invoking (4.9). □

We now formulate three equivalent ways of expressing quasianalyticity. A proof is essentially contained in Beurling's paper [4].

PROPOSITION 4.8 ($1\leqslant p<+\infty$). *Suppose σ is unbounded and decreasing. Fix a real number δ, $0<\delta<1$, with the property that $e\leqslant\min\{\sigma(\delta),\hat\sigma(p,\delta)\}$. Then the following three conditions are equivalent:*

(a) $\int_0^\delta\log\log\sigma(t)\,dt<+\infty$,

(b) $\int_0^\delta\log\log\hat\sigma(p,t)\,dt<+\infty$,

(c) $\int_0^{+\infty}(\log\omega_{\sigma,p}(t))/(1+t^2)\,dt<+\infty$.

5. The Main Theorem on completeness of translates in $L^p(\mathbf{R}_+,\omega)$

We shall now reformulate the Main Theorem from the introduction in terms of the weight classes $\mathfrak{W}_s$ of §4, and derive a corollary from it. First, however, we need the following lemma.

LEMMA 5.1 ($1\leqslant p<+\infty$). *Fix an $\alpha\in[0,+\infty[$, and a weight ω in the class $\mathfrak{W}$ introduced in §4. Denote by $L^p([\alpha,+\infty[,\omega)$ the subspace of $L^p(\mathbf{R}_+,\omega)$ consisting of all functions that vanish (almost everywhere) on the interval $[0,\alpha[$. Then the image $T_\alpha L^p(\mathbf{R}_+,\omega)$ of $L^p(\mathbf{R}_+,\omega)$ under the right translation operator T_α coincides with $L^p([\alpha,+\infty[,\omega)$.*

Proof. By Proposition 4.3, the image of $L^p(\mathbf{R}_+,\omega)$ under T_α is contained in $L^p(\mathbf{R}_+,\omega)$, and hence in $L^p([\alpha,+\infty[,\omega)$. On the other hand, since the weight ω increases, the left translation operator $T_{-\alpha}$ is norm contractive $L^p([\alpha,+\infty[,\omega)\rightarrow L^p(\mathbf{R}_+,\omega)$. That does it. □

The following statement is an equivalent formulation of the Main Theorem (see §1). The condition $\omega\in\mathfrak{W}_s$ requires not only that $\log\omega(t)-s\log(1+t)$ be concave, but also that it tend to $+\infty$ as $t\rightarrow+\infty$. What allows us to say that the two formulations are equivalent is that the prescribed intervals of s and ε are open.

THEOREM 5.2 ($1\leqslant p<+\infty$). *Put $\theta(p)=2+1/p'$ for $1<p<2$ and $2<p<+\infty$, $\theta(2)=\frac{1}{2}$, and $\theta(1)=3$. Suppose $\omega\in\mathfrak{W}_s$ for some s, $\theta(p)<s<+\infty$. Let $\mathfrak{S}$ be a collection of elements in $L^p(\mathbf{R}_+,\omega)$, and denote by $\mathfrak{T}_+(\mathfrak{S})$ the set of all (finite) linear combinations of right translates T_xf, $x\in\mathbf{R}_+$, of elements $f\in\mathfrak{S}$. Then $\mathfrak{T}_+(\mathfrak{S})$ is dense in $L^p(\mathbf{R}_+,\omega)$ if and only if*

(a) *the functions in $\mathfrak{F}(\mathfrak{S})$ have no common zeros in $\overline{\mathbf{C}}_-$, and*
(b) *there is no δ, $0<\delta$, such that $\mathfrak{S}$ is contained in $L^p([\delta,+\infty[,\omega)$.*

We obtain this theorem by proving a corresponding approximation statement in the context of the spaces $Q^p(\mathbf{C}_-,\sigma)$, for weights σ satisfying $1/\sigma^{p'}\in\mathfrak{P}_{sp'}$. In view of Theorems 2.1 and 3.1, proving an approximation theorem for the space $Q^p(\mathbf{C}_-,\sigma)$ (suitably modified for $p=1$) is equivalent via the Fourier transform to proving one for the space $L^p(\mathbf{R}_+,\omega_{\sigma,p})$, where $\omega_{\sigma,p}$ is connected with σ as in §§2 and 3. Moreover, from §4 we know that the requirements $1/\sigma^{p'}\in\mathfrak{P}_{sp'}$ and $\omega\in\mathfrak{W}_s$ correspond.

In an earlier paper [5], we studied certain spaces $Q(\mathbf{C}_-,w)$, which are similar to the spaces $Q^p(\mathbf{C}_-,\sigma)$ investigated here. They were defined as spaces of bounded holomorphic functions in the lower half-plane, having extensions to the whole complex plane that are bounded and asymptotically holomorphic in the sense that their $\bar\partial$ derivatives are controlled by a weight function w. It is possible to regard each such space $Q(\mathbf{C}_-,w)$ as consisting of Cauchy transforms of a space of densities φ satisfying a weighted uniform norm condition

$$|\varphi(z)| \leqslant C(\varphi)w(\operatorname{Im} z), \quad z \in \mathbf{C}_+.$$

In the setting of the present paper, the function w corresponds to $1/\sigma$. Our space $Q^p(\mathbf{C}_-, \sigma)$ is defined as the image under the Cauchy transform of the density space $\mathfrak{L}^p(\Sigma, \sigma)$, which contains unbounded functions. This difference causes us some difficulty, but it is not too serious. What we actually need is that the image $\mathfrak{C}_*\mathcal{H}^p(\Sigma, \sigma)$ under the global Cauchy transform of the space $\mathcal{H}^p(\Sigma, \sigma)$ of canonical densities (see §2) consists of sufficiently smooth functions.

It is easy to derive from Theorem 5.2 the following consequence, which at first glance seems much stronger. All one needs to do, however, is to apply a couple of translation operators, together with Lemma 5.1.

COROLLARY 5.3 ($1 \leqslant p < +\infty$). *Suppose that the weight ω is as in Theorem 5.2, and that we have a collection $\mathfrak{S}$ of elements in $L^p(\mathbf{R}_+, \omega)$ whose Fourier transforms lack common zeros in $\overline{\mathbf{C}}_-$. Then the closure of $\mathfrak{T}_+(\mathfrak{S})$ in $L^p(\mathbf{R}_+, \omega)$ coincides with the subspace $L^p([\alpha(\mathfrak{S}), +\infty[, \omega)$, where $\alpha(\mathfrak{S}) = \sup\{\alpha \in [0, +\infty[: \mathfrak{S} \subset L^p([\alpha, +\infty[, \omega)\}$.*

6. Translation invariance versus convolution invariance in the space $L^p(\mathbf{R}_+, \omega_{\sigma,p})$

Fix a p, $1 \leqslant p < +\infty$, and write $p' = p/(p-1)$, with the usual convention that $p' = \infty$ if $p = 1$. The function $\sigma\colon]0,1] \to]0, +\infty[$ is continuous, and satisfies (2.1) for $1 < p < +\infty$, and the requirements of §3 for $p = 1$. The related weight function $\omega_{\sigma,p}$ on $\mathbf{R}_+$, as defined in §§ 2 and 3, belongs to $\mathfrak{W}$, that is, it is increasing to $+\infty$, has a logarithm $\log \omega_{\sigma,p}$ that is concave, and satisfies

$$\log \omega_{\sigma,p}(t) = o(t), \quad \text{as } t \to +\infty.$$

Our aim is to show, for closed subspaces of $L^p(\mathbf{R}_+, \omega_{\sigma,p})$, the equivalence of right translation invariance and invariance with respect to convolution with the functions e_λ, for $\lambda \in \mathbf{C}_+$, which are defined in the proposition below. We shall see that we can do this without making any additional regularity assumption on σ.

PROPOSITION 6.1 ($1 \leqslant p < +\infty$). *For $\lambda \in \mathbf{C}_+$, the function*

$$e_\lambda(t) = -i \exp(i\lambda t), \quad t \in \mathbf{R}_+,$$

is an element of $L^p(\mathbf{R}_+, \omega_{\sigma,p})$, with norm $\hat{\sigma}(p, \operatorname{Im} \lambda)^{1/p}$, where $\hat{\sigma}$ is given by (4.5). Moreover, every function in $L^p(\mathbf{R}_+, \omega_{\sigma,p})$ can be approximated in norm by finite linear combinations of the functions e_λ, $\lambda \in \mathbf{C}_+$. Denote the Fourier transform of e_λ by E_λ. We then *have* $E_\lambda \in Q^p(\mathbf{C}_-, \sigma)$ *for* $\lambda \in \mathbf{C}_+$*, and the explicit formula*

$$E_\lambda(z) = (\lambda - z)^{-1}, \quad z \in \mathbf{C}_-.$$

Proof. We noted in §4 that, on its interval of definition, $\hat{\sigma}(p, \cdot\,)$ is strictly decreasing, and has image contained in $]0, +\infty[$. It is clear by (4.5) and the definition of the norm in $L^p(\mathbf{R}_+, \omega_{\sigma,p})$ that e_λ has norm $\hat{\sigma}(p, \operatorname{Im}\lambda)^{1/p}$, so we immediately obtain that $e_\lambda \in L^p(\mathbf{R}_+, \omega_{\sigma,p})$ for $\lambda \in \mathbf{C}_+$. The formula for E_λ is obtained by a straightforward calculation.

We now check the statement on approximation. Suppose that the functional $\phi \in L^{p'}(\mathbf{R}_+, \omega_{\sigma,p}^{-1})$ annihilates all e_λ, with $\lambda \in \mathbf{C}_+$. Because of the way we defined the resolvent transform, this means that $\mathfrak{R}\phi(\lambda) = 0$ for all $\lambda \in \mathbf{C}_+$. But then $\phi = 0$, by the uniqueness theorem for Laplace transforms. By duality, the functions e_λ, with $\lambda \in \mathbf{C}_+$, must span a dense subspace of $L^p(\mathbf{R}_+, \omega_{\sigma,p})$. □

We need an upper estimate for the norm of the translation operator T_x ($x \in \mathbf{R}_+$), on the space $L^p(\mathbf{R}_+, \omega_{\sigma,p})$. For $p=1$, the translation operator T_x also acts on $\mathcal{M}(\mathbf{R}_+, \omega_{\sigma,1})$, $T_x\xi(E) = \xi((E-x) \cap \mathbf{R}_+)$, where $E - x = \{t \in \mathbf{R} : t + x \in E\}$.

Proposition 6.2 ($1 \leqslant p < +\infty$). *For* $x \in \mathbf{R}_+$ *and* $f \in L^p(\mathbf{R}_+, \omega_{\sigma,p})$*, we have the inequality*

$$\int_0^{+\infty} |T_x f(t)|^p \omega_{\sigma,p}(t)^p \, dt \leqslant \left(\frac{\omega_{\sigma,p}(x)}{\omega_{\sigma,p}(0)}\right)^p \int_0^{+\infty} |f(t)|^p \omega_{\sigma,p}(t)^p \, dt.$$

For measures $\xi \in \mathcal{M}(\mathbf{R}_+, \omega_{\sigma,1})$,

$$\int_0^{+\infty} \omega_{\sigma,1}(t)\, d|T_x\xi|(t) \leqslant \frac{\omega_{\sigma,p}(x)}{\omega_{\sigma,p}(0)} \int_0^{+\infty} \omega_{\sigma,1}(t)\, d|\xi|(t).$$

Proof. The assertion is an easy consequence of Proposition 4.3, if we note that

$$\int_0^{+\infty} |T_x f(t)|^p \omega_{\sigma,p}(t)^p \, dt = \int_0^{+\infty} |f(t)|^p \omega_{\sigma,p}(t+x)^p \, dt.$$

This also works for measures, which completes the proof. □

Now we set $Q_0^p(\mathbf{C}_-, \sigma) = \mathfrak{F}L^p(\mathbf{R}_+, \omega_{\sigma,p})$. We note that, for $1 < p < +\infty$, $Q_0^p(\mathbf{C}_-, \sigma)$ coincides with $Q^p(\mathbf{C}_-, \sigma)$, but that $Q_0^1(\mathbf{C}_-, \sigma)$ is a proper closed subspace of $Q^1(\mathbf{C}_-, \sigma)$. The statement corresponding to Proposition 6.2 on the Fourier transform side is as follows.

COROLLARY 6.3 ($1\leqslant p<+\infty$). *For $x\in\mathbf{R}_+$ and $f\in Q^p(\mathbf{C}_-,\sigma)$, the function*

$$M_x f(z)=e^{-ixz}\cdot f(z),\quad z\in\mathbf{C}_-,$$

belongs to $Q^p(\mathbf{C}_-,\sigma)$; in fact, we have

$$\|M_x f\|_{Q^p(\mathbf{C}_-,\sigma)}\leqslant\frac{\omega_{\sigma,p}(x)}{\omega_{\sigma,p}(0)}\|f\|_{Q^p(\mathbf{C}_-,\sigma)},\quad x\in\mathbf{R}_+.$$

(This follows from the identity $M_x\mathfrak{F}g=\mathfrak{F}T_xg$.) Moreover, if $f\in Q^1_0(\mathbf{C}_-,\sigma)$, then M_xf also belongs to $Q^1_0(\mathbf{C}_-,\sigma)$.

The above result permits us to estimate the norm of the operator of multiplication by E_λ on $Q^p(\mathbf{C}_-,\sigma)$.

PROPOSITION 6.4 ($1\leqslant p<+\infty$). *For $\lambda\in\mathbf{C}_+$, multiplication by the function E_λ is a bounded operator on $Q^p(\mathbf{C}_-,\sigma)$, which for $p=1$ preserves $Q^1_0(\mathbf{C}_-,\sigma)$, and we have in fact*

$$\begin{aligned}\|E_\lambda f\|_{Q^p(\mathbf{C}_-,\sigma)}&\leqslant\omega_{\sigma,p}(0)^{-1}\left(\int_0^{+\infty}e^{-t\operatorname{Im}\lambda}\omega_{\sigma,p}(t)\,dt\right)\|f\|_{Q^p(\mathbf{C}_-,\sigma)}\\&\leqslant\omega_{\sigma,p}(0)^{-1}(\operatorname{Im}\lambda)^{-1/p'}\hat{\sigma}(p,\operatorname{Im}\lambda/p)^{1/p}\|f\|_{Q^p(\mathbf{C}_-,\sigma)},\quad f\in Q^p(\mathbf{C}_-,\sigma).\end{aligned}$$

Proof. The first inequality is immediate from the identity

$$E_\lambda f(z)=-i\int_0^\infty M_xf(z)\cdot e^{ix\lambda}\,dx,\quad z\in\mathbf{C}_-,\tag{6.1}$$

and the estimate of the norm of M_x obtained in Corollary 6.3; analogous formulas work for measures. The second estimate results from an application of Hölder's inequality. □

We now state the promised result on the equivalence of translation and convolution invariance for closed subspaces.

PROPOSITION 6.5 ($1\leqslant p<+\infty$). *Let $\mathfrak{S}$ be a collection of elements in $Q^p_0(\mathbf{C}_-,\sigma)$. Denote by $J_m(\mathfrak{S})$ the closure of the set of all (finite) linear combinations of functions of the type M_xf with $f\in\mathfrak{S}$ and $x\in\mathbf{R}_+$, and by $J_e(\mathfrak{S})$ the closure of the set of finite linear combinations of functions of the type $E_\lambda f$, with $\lambda\in\mathbf{C}_+$ and $f\in\mathfrak{S}$. Then $J_m(\mathfrak{S})=J_e(\mathfrak{S})$.*

Remark 6.6. If $Q^p_0(\mathbf{C}_-,\sigma)$ is a Banach algebra when supplied with pointwise multiplication of functions, it follows from Proposition 6.1 that $J_e(\mathfrak{S})$ equals the closure of the ideal in $Q^p_0(\mathbf{C}_-,\sigma)$ generated by $\mathfrak{S}$.

Proof of Proposition 6.5. Let us first show that $J_m(\mathfrak{S})$ is contained in $J_e(\mathfrak{S})$. Given $f\in\mathfrak{S}$ and $x\in\mathbf{R}_+$, it suffices to approximate M_xf by elements of $J_e(\mathfrak{S})$. By Corollary 6.3

and Proposition 6.4, the operator of multiplication by the function $M_x E_{i\alpha}$ is bounded for $0<\alpha$. Moreover, if $0<\alpha<\beta$, and Γ is the line $\Gamma=\{\lambda\in\mathbf{C}:\operatorname{Im}\lambda=\frac{1}{2}\alpha\}$, oriented from the right to the left, the identity

$$M_x E_{i\alpha}(z)-M_x E_{i\beta}(z)=\frac{1}{2\pi i}\int_\Gamma \frac{i(\beta-\alpha)e^{-ix\lambda}}{(i\alpha-\lambda)(i\beta-\lambda)}E_\lambda(z)\,d\lambda,$$

where the integral is norm convergent in the space of multipliers on $Q^p(\mathbf{C}_-,\sigma)$ (use Proposition 6.4), shows, by approximation with Riemann sums, that $M_x E_{i\alpha} f-M_x E_{i\beta} f$ belongs to $J_e(\mathfrak{S})$. But if we let $\beta\to+\infty$, $E_{i\beta}$ tends to 0 as a multiplier, by Proposition 6.4 and the fact that $\hat\sigma(p,t)\to 0$ as $t\to+\infty$, so we arrive at the conclusion that $M_x E_{i\alpha} f\in J_e(\mathfrak{S})$. We would now be done if we could show that

$$\|(i\alpha E_{i\alpha}-1)\,M_x f\|_{Q^p(\mathbf{C}_-,\sigma)}\to 0 \quad \text{as } 0<\alpha\to+\infty. \tag{6.2}$$

We shall in fact prove that (6.2) holds for general $f\in Q^p_0(\mathbf{C}_-,\sigma)$, not just members of $\mathfrak{S}$. Since we know M_x is a bounded operator on $Q^p(\mathbf{C}_-,\sigma)$ (Corollary 6.3), and a simple calulation based on (4.5) reveals that $\hat\sigma(p,x)=O(1/x)$ as $x\to+\infty$, so that by Proposition 6.4, the multiplier norm of $i\alpha E_{i\alpha}$ remains bounded as $\alpha\to+\infty$, it suffices to obtain (6.2) on a dense set of f in $Q^p_0(\mathbf{C}_-,\sigma)$. By Proposition 6.1, finite linear combinations of the functions E_λ, with $\lambda\in\mathbf{C}_+$, span a dense subspace of $Q^p_0(\mathbf{C}_-,\sigma)$, so we shall be content to check that (6.2) holds for f of the form $f=E_\lambda$, with $\lambda\in\mathbf{C}_+$. Since the norm of E_λ in $Q^p(\mathbf{C}_-,\sigma)$ is $\tilde\sigma(p,\operatorname{Im}\lambda)^{1/p}$ (Theorems 2.1 and 3.1, Proposition 6.1), and $\hat\sigma(p,t)\to 0$ as $t\to+\infty$, we have that

$$i\alpha E_{i\alpha}(z)E_\lambda(z)-E_\lambda(z)=\frac{i\alpha}{i\alpha-\lambda}E_{i\alpha}(z)-\frac{\lambda}{i\alpha-\lambda}E_\lambda(z)\to 0 \quad \text{as } \mathbf{R}_+\ni\alpha\to+\infty,$$

in the norm of $Q^p(\mathbf{C}_-,\sigma)$. By applying the operator M_x to both sides, we arrive at (6.2) for the function $f=E_\lambda$.

We now check that $J_e(\mathfrak{S})$ is contained in $J_m(\mathfrak{S})$. It suffices to show that for $\lambda\in\mathbf{C}_+$ and $f\in\mathfrak{S}$, the function $E_\lambda f$ may be approximated with finite linear combinations of $M_x f$. This, however, is possible by relation (6.1), if we use Riemann sums to approximate the integral. □

7. Multipliers on spaces of densities

A C^∞ function φ on $\mathbf{C}$ is said to be a *multiplier on* $\mathfrak{L}^p(\Sigma,\sigma)$, provided that $\varphi\tilde g\in\mathfrak{L}^p(\Sigma,\sigma)$ whenever $\tilde g\in\mathfrak{L}^p(\Sigma,\sigma)$, and a norm estimate holds:

$$\|\varphi\tilde g\|_{\mathfrak{L}^p(\Sigma,\sigma)}\leqslant C(\varphi)\|\tilde g\|_{\mathfrak{L}^p(\Sigma,\sigma)}. \tag{7.1}$$

In the special case $p=2$, the norm on $\mathfrak{L}^p(\Sigma,\sigma)$ is given by identity (2.5) in terms of the modulus of $\tilde{g}$, and thus (7.1) holds for all $\varphi\in L^\infty(\Sigma)$; in fact, $L^\infty(\Sigma)$ is the space of multipliers on $\mathfrak{L}^2(\Sigma,\sigma)$. For other p, no such simple characterization holds. What one would need is a description of the convolution multipliers on $L^p(\mathbf{R})$: the space $\mathfrak{M}(\Sigma)$ of multipliers on $\mathfrak{L}^p(\Sigma,\sigma)$ would then consist of functions $\varphi(x+iy)$ that for fixed y, $0<y<1$, are Fourier transforms of convolution multipliers on $L^p(\mathbf{R})$, with norm uniformly bounded in y. On the other hand, we do not really need a complete description of convolution multipliers. We make use of the following simple observation.

LEMMA 7.1. *For $\mu\in\mathcal{M}(\mathbf{R})$, the space of finite complex-valued Borel measures on $\mathbf{R}$, and $f\in L^p(\mathbf{R})$, the convolution*

$$\mu*f(x)=\int_{-\infty}^{+\infty} f(x-t)\,d\mu(t),\quad x\in\mathbf{R},$$

is in $L^p(\mathbf{R})$, and $\|\mu*f\|_{L^p(\mathbf{R})}\leqslant\|\mu\|_{\mathcal{M}(\mathbf{R})}\|f\|_{L^p(\mathbf{R})}$. Moreover, $\mathcal{M}(\mathbf{R})$ is a convolution algebra, that is, for $\mu,\nu\in\mathcal{M}(\mathbf{R})$, $\mu*\nu\in\mathcal{M}(\mathbf{R})$, and $\|\mu*\nu\|_{\mathcal{M}(\mathbf{R})}\leqslant\|\mu\|_{\mathcal{M}(\mathbf{R})}\|\nu\|_{\mathcal{M}(\mathbf{R})}$.

This result is well known.

Introduce the space $\mathfrak{M}(\Sigma)$ of all functions $\varphi\in L^\infty(\Sigma)$ with the following properties: for almost all y, $0<y<1$, the function $\varphi_y(x)=\varphi(x+iy)$ belongs to the Fourier image of $\mathcal{M}(\mathbf{R})$, and $y\mapsto\varphi_y$ is a Borel measurable uniformly bounded mapping $]0,1[\to\mathfrak{F}\mathcal{M}(\mathbf{R})$,

$$\|\varphi\|_{\mathfrak{M}(\Sigma)}=\operatorname{ess\,sup}\{\|\varphi_y\|_{\mathfrak{F}\mathcal{M}(\mathbf{R})}:0<y<1\}<+\infty.$$

Define the space $\mathfrak{M}(\check{\Sigma})$ as follows: $\varphi\in\mathfrak{M}(\check{\Sigma})$ means that φ is a Borel measurable bounded function on $\check{\Sigma}$, such that $y\mapsto\varphi_y$ is an everywhere defined Borel measurable uniformly bounded mapping $]0,1]\to\mathfrak{F}\mathcal{M}(\mathbf{R})$,

$$\|\varphi\|_{\mathfrak{M}(\check{\Sigma})}=\sup\{\|\varphi_y\|_{\mathfrak{F}\mathcal{M}(\mathbf{R})}:0<y\leqslant 1\}<+\infty.$$

By Lemma 7.1 and the well-known fact that the Fourier transform turns convolution into pointwise multiplication, and vice versa, it is clear that the functions in $\mathfrak{M}(\Sigma)$ are multipliers on $\mathfrak{L}^p(\Sigma,\sigma)$ (for $1<p<+\infty$), and that the functions in $\mathfrak{M}(\check{\Sigma})$ are multipliers on $\mathfrak{L}^1(\check{\Sigma},\sigma)$. However, for $1<p<+\infty$, there are plenty of multipliers on $\mathfrak{L}^p(\Sigma,\sigma)$ which do not belong to $\mathfrak{M}(\Sigma)$.

We note that this way of multiplying elements of $\mathfrak{M}(\Sigma)$ with densities coincides with the usual multiplication of functions, if the density is a sufficiently well-behaved function. This follows from Lemma 7.2 below. Define $L^1_\infty(\mathbf{R})$ to be the space of all (equivalence classes of) Borel measurable complex-valued functions f on $\mathbf{R}$ for which

$$\sup_{x\in\mathbf{R}}\int_0^1|f(x+t)|\,dt<+\infty.$$

LEMMA 7.2 ($1\leqslant p<+\infty$). *If $f\in L^p(\mathbf{R})$ and $g\in L^1(\mathbf{R})$, and the Fourier transform $\mathfrak{F}f$ is an element of $L^1_\infty(\mathbf{R})$, then $\mathfrak{F}(f*g)=\mathfrak{F}f\cdot\mathfrak{F}g\in L^1_\infty(\mathbf{R})$.*

The proof is an exercise in distribution theory.

8. Smoothness properties of canonical densities and their Cauchy transforms: $1<p<+\infty$

Throughout this section, $1<p<+\infty$, and $\sigma\colon]0,1]\to]0,+\infty[$ is continuous, and satisfies (2.1). The weight $\omega_{\sigma,p}$ is related to σ by formula (2.2). We note that $\omega_{\sigma,p}\in\mathfrak{W}$, by §4. In this section we study how smoothness properties of the functions in the spaces $Q^p(\mathbf{C}_-,\sigma)$ and $\mathcal{L}^p(\Sigma,\sigma)$ vary with the weight σ. For instance, we want to know when the functions in $Q^p(\mathbf{C}_-,\sigma)$ extend continuously up to the real line, and when the product of two functions in $Q^p(\mathbf{C}_-,\sigma)$ remains in $Q^p(\mathbf{C}_-,\sigma)$.

PROPOSITION 8.1 ($1<p<+\infty$). *If the condition*

$$\int_0^1\frac{dt}{t\sigma(t)^{p'}}<+\infty$$

holds, then the functions in $Q^p(\mathbf{C}_-,\sigma)$, initially defined in the open lower half-plane $\mathbf{C}_-$, extend continuously to the compactified lower half-plane $\overline{\mathbf{C}}_-\cup\{\infty\}$, and assume the value 0 at ∞.

Note. (a) The above condition may be phrased in terms of $\omega_{\sigma,p}$:

$$\int_0^{+\infty}\frac{dt}{\omega_{\sigma,p}(t)^{p'}}=\frac{2}{p'}\int_0^1\frac{dt}{t\sigma(t)^{p'}}.$$

(b) The condition of the lemma is fulfilled if $1/\sigma^{p'}$ belongs to $\mathfrak{P}_{sp'}$, $1/p'<s<+\infty$ (notation as in Appendix B).

Proof. If $\tilde g\in\mathfrak{L}^p(\Sigma)$ has compact support in Σ, its Cauchy transform extends analytically to $\overline{\mathbf{C}}_-\cup\{\infty\}$ with value 0 at ∞. For a general $\tilde g\in\mathfrak{L}^p(\Sigma,\sigma)$ we have, by inequality (2.10),

$$|\mathfrak{C}\tilde g(z)|\leqslant\|\tilde g\|_{\mathfrak{L}^p(\Sigma,\sigma)}\left(\int_0^1\frac{dt}{t\sigma(t)^{p'}}\right)^{1/p'}<\infty,\quad z\in\mathbf{C}_-.$$

Approximating $\tilde g$ in $\mathfrak{L}^p(\Sigma,\sigma)$ by densities with compact support, we get the desired result. □

We study when the space of canonical extensions (see §2) to $\mathbf{C}$ of the functions in $Q^p(\mathbf{C}_-,\sigma)$, denoted $\mathfrak{C}_*\mathcal{H}^p(\Sigma,\sigma)=\mathfrak{F}^p_*L^p(\mathbf{R}_+,\omega_{\sigma,p})$, is contained in $L^\infty(\mathbf{C})\cap C(\mathbf{C})$. Here,

$C(\mathbf{C})$ is the Fréchet space of continuous functions on $\mathbf{C}$, with the topology of uniform convergence on compact subsets. The following lemma explains what properties of $\mathcal{H}^p(\Sigma,\sigma)$ we should look for. The functions in the space $L^q(\Sigma)$ are tacitly assumed extended to $\mathbf{C}$ by declaring them equal to 0 on $\mathbf{C}\setminus\Sigma$.

LEMMA 8.2. *Let q satisfy $2<q<+\infty$. If $h\in L^q(\Sigma)$, then $\mathfrak{C}_*h\in L^\infty(\mathbf{C})\cap C(\mathbf{C})$, and there is a constant $C(q)$, $0<C(q)<+\infty$, such that*

$$\|\mathfrak{C}_*h\|_{L^\infty(\mathbf{C})}\leqslant C(q)\|h\|_{L^q(\Sigma)},\quad h\in L^q(\Sigma).$$

Proof. The integral defining $\mathfrak{C}_*h$ is

$$\mathfrak{C}_*h(z)=\int_\Sigma \frac{h(\zeta)}{z-\zeta}\,dS(\zeta),\quad z\in\mathbf{C},$$

and it is absolutely convergent, because of the estimates (q' is the dual exponent to q: $1/q+1/q'=1$)

$$\int_\Sigma\left|\frac{h(\zeta)}{z-\zeta}\right|dS(\zeta)\leqslant\|h\|_{L^q(\Sigma)}\left(\int_\Sigma\frac{dS(\zeta)}{|z-\zeta|^{q'}}\right)^{1/q'},$$

and

$$\int_\Sigma\frac{dS(\zeta)}{|z-\zeta|^{q'}}\leqslant\int_\Sigma\frac{dS(\zeta)}{\left|\zeta-\frac{1}{2}i\right|^{q'}}<+\infty.$$

The last estimate holds because of symmetry, and because $2<q<+\infty$ is equivalent to $1<q'<2$. This shows that $\mathfrak{C}_*h$ is bounded, and that the uniform norm estimate holds. To see that $\mathfrak{C}_*h$ is continuous, consider a translate $T_\eta\mathfrak{C}_*h(z)=\mathfrak{C}_*h(z-\eta)$ of $\mathfrak{C}_*h$, where $\eta\in\mathbf{C}$. Computations analogous to those we have performed already show that $T_\eta\mathfrak{C}_*h$ is close to $\mathfrak{C}_*h$ in $L^\infty(\mathbf{C})$ if η is close to the origin. This demonstrates that $\mathfrak{C}_*h$ is in fact uniformly continuous on $\mathbf{C}$. □

Recall from Appendix B the definition of the sets $\mathfrak{P}_s$, with $0<s<+\infty$, of fractional integrals of positive Borel measures in $\mathfrak{P}$. The easiest estimate of the size of canonical densities is the following; it shows that they are bounded provided $\sigma^{-p'}\in\mathfrak{P}_{1+p'}$.

PROPOSITION 8.3 ($1<p<+\infty$). *If $\sigma^{-p'}$ belongs to $\mathfrak{P}_{sp'}$ for some s, $1/p'<s<+\infty$, we have for $f\in L^p(\mathbf{R}_+,\omega_{\sigma,p})$*

$$|\mathfrak{D}^pf(z)|\leqslant C\sigma(\operatorname{Im} z)^{-r},\quad z\in\Sigma,$$

where $r=(s-1-1/p')/(s-1/p')$, and $C=C(s,\sigma,p,f)$ is a positive constant.

Proof. The assertion is immediate from Proposition 4.4 and (2.15), with $n=0$. □

Our next result provides L^q norm estimates of canonical densities. For $1\leqslant q\leqslant+\infty$ and r real, the space $L^q(\Sigma,\sigma^r)$ consists of all (equivalence classes of) Lebesgue area measurable functions f on Σ with the function $f(z)\sigma(\operatorname{Im} z)^r$ in $L^q(\Sigma)$. The norm of f in this space is the $L^q(\Sigma)$ norm of $f(z)\sigma(\operatorname{Im} z)^r$ (in §2, we multiplied by $2^{1/p}$ to make the isometry work).

PROPOSITION 8.4. *Suppose σ is such that $\sigma^{-p'}$ belongs to $\mathfrak{P}_{sp'}$ for some s, $1/p'<s<+\infty$, and that q and r are two real numbers, subject to the conditions* $\max\{p,p'\}<q<+\infty$ *and*

$$0\leqslant r\leqslant 1-\frac{1-2/q}{s-1/p'}. \tag{8.1}$$

We then have the estimate

$$\|\mathfrak{D}^p f\|_{L^q(\Sigma,\sigma^r)}\leqslant C\|f\|_{L^p(\mathbf{R}_+,\omega_{\sigma,p})},\quad f\in L^p(\mathbf{R}_+,\omega_{\sigma,p}),$$

where $C=C(s,\sigma,p,q,r)$ is a positive real constant which does not depend on f.

Proof. We begin by remarking that with the notation as in §2,

$$\mathfrak{D}^p f(x+iy)=\tilde{g}[f,p](x+iy),\quad x+iy\in\Sigma,$$

where

$$g[f,p](t+iy)=-ie^{-typ'/p}f(t)\omega_{\sigma,p}(t)^{p'}/\sigma(y)^{p'},\quad t+iy\in\Sigma;$$

recall that the tilde stands for taking the Fourier transform in the variable t. An application of the Hausdorff–Young theorem [28, p. 178] shows that $(2<q)$

$$\|\mathfrak{D}^p f(\,\cdot+iy)\|_{L^q(\mathbf{R})}=\|\mathfrak{F}g[f,p](\,\cdot+iy)\|_{L^q(\mathbf{R})}\leqslant(2\pi)^{1/q}\|g[f,p](\,\cdot+iy)\|_{L^{q'}(\mathbf{R}_+)},$$

where q' is the dual exponent to q: $1/q+1/q'=1$. By applying Hölder's inequality (with three factors), we get

$$\begin{aligned}\int_{-\infty}^{\infty}\sigma(y)^{rq}|\mathfrak{D}^p f(x+iy)|^q\,dx&\leqslant\frac{2\pi}{\sigma(y)^{qp'-rq}}\left(\int_0^\infty e^{-tyq'p'/p}|f(t)|^{q'}\omega_{\sigma,p}(t)^{p'q'}\,dt\right)^{q/q'}\\&\leqslant\frac{2\pi}{\sigma(y)^{qp'-rq}}\left(\int_0^\infty|f(t)|^p\omega_{\sigma,p}(t)^p\,dt\right)^{q/p-1}\left(\int_0^\infty e^{-typ'R}\omega_{\sigma,p}(t)^{p'R}\,dt\right)^{q/p'-1}\\&\qquad\times\int_0^\infty e^{-typ'}|f(t)|^p\omega_{\sigma,p}(t)^{p+p'}\,dt\\&=2\pi\|f\|^{q-p}_{L^p(\mathbf{R}_+,\omega_{\sigma,p})}\,\check{\sigma}(p,R,y)^{q/p'-1}\sigma(y)^{p'+rq-qp'}\\&\qquad\times\int_0^\infty e^{-typ'}|f(t)|^p\omega_{\sigma,p}(t)^{p+p'}\sigma(y)^{-p'}\,dt,\end{aligned}\tag{8.2}$$

where $R=(1/p-1/q)/(1/p'-1/q)$, and $\check{\sigma}$ is as in (4.5). Proposition 4.5 supplies us with an estimate of $\check{\sigma}$, so that (8.2) condenses to

$$\begin{aligned}\int_{-\infty}^{\infty} \sigma(y)^{rq}|\mathfrak{D}^p f(x+iy)|^q\,dx \\ \leqslant C\|f\|^{q-p}_{L^p(\mathbf{R}_+,\omega_{\sigma,p})}\,\sigma(y)^A\int_0^\infty e^{-typ'}|f(t)|^p\omega_{\sigma,p}(t)^{p+p'}\sigma(y)^{-p'}\,dt,\end{aligned}\tag{8.3}$$

where $C=C(s,\sigma,p,q)$ is a positive constant, and

$$A=q\left(r-1+\frac{1-2/q}{s-1/p'}\right)\leqslant 0,$$

by assumption (8.1). Since $\sigma^{-p'}\in\mathfrak{P}_{sp'}\subset\mathfrak{P}_1$, σ is decreasing, and the exponent A is $\leqslant 0$, $\sigma(y)^A\leqslant\sigma(1)^A$, so that if we introduce another constant,

$$C=\tfrac{1}{2}C(s,\sigma,p,q)\sigma(1)^A,$$

we may therefore obtain from (8.3) the simpler estimate

$$\begin{aligned}\int_{-\infty}^{\infty} \sigma(y)^{rq}|\mathfrak{D}^p f(x+iy)|^q\,dx \\ \leqslant C\|f\|^{q-p}_{L^p(\mathbf{R}_+,\omega_{\sigma,p})}\cdot 2\int_0^\infty e^{-typ'}|f(t)|^p\omega_{\sigma,p}(t)^{p+p'}\sigma(y)^{-p'}\,dt,\end{aligned}\tag{8.4}$$

On integrating (8.4) with respect to the y variable, we get

$$\begin{aligned}&\int_\Sigma \sigma(\operatorname{Im} z)^{rq}|\mathfrak{D}^p f(z)|^q\,dS(z)\\ &\quad\leqslant C\|f\|^{q-p}_{L^p(\mathbf{R}_+,\omega_{\sigma,p})}\cdot 2\int_0^1\int_0^\infty e^{-typ'}|f(t)|^p\omega_{\sigma,p}(t)^{p+p'}\sigma(y)^{-p'}\,dt\,dy\\ &\quad= C\|f\|^q_{L^p(\mathbf{R}_+,\omega_{\sigma,p})}.\end{aligned}$$

The proof is complete. □

In view of Lemma 8.2, Proposition 8.4 has the following corollary.

COROLLARY 8.5 ($1<p<+\infty$). *Suppose that* $\sigma^{-p'}\in\mathfrak{P}_{sp'}$ *with* $\max\{1/p,2-3/p\}<s$. *We then have* $\mathfrak{C}_*\mathcal{H}^p(\Sigma,\sigma)\subset L^\infty(\mathbf{C})\cap C(\mathbf{C})$, *and*

$$\|\mathfrak{F}^p_* f\|_{L^\infty(\mathbf{C})}\leqslant C\|f\|_{L^p(\mathbf{R}_+,\omega_{\sigma,p})},\quad f\in L^p(\mathbf{R}_+,\omega_{\sigma,p}),$$

holds for some constant $C=C(s,\sigma,p)$, $0<C<+\infty$.

Proof. Pick $q>\max\{p,p'\}$ very close to $\max\{p,p'\}$. Proposition 8.4 (with $r=0$) then asserts that $\mathfrak{D}^pL^p(\mathbf{R}_+,\omega_{\sigma,p})\subset L^q(\Sigma)$. In terms of norms, we have the corresponding inequality

$$\|\mathfrak{D}^p f\|_{L^q(\Sigma)}\leqslant C\|f\|_{L^p(\mathbf{R}_+,\omega_{\sigma,p})},\quad f\in L^p(\mathbf{R}_+,\omega_{\sigma,p}),$$

for some constant $C=C(s,\sigma,p,q)$, $0<C<+\infty$. The assertion of the corollary now follows from Lemma 8.2. □

Recall the definition of the space $\mathfrak{M}(\Sigma)$ in §7. The next result follows from Corollary 8.5 and Proposition 2.9.

COROLLARY 8.6 ($1<p<+\infty$). *Suppose that* $\sigma^{-p'}\in\mathfrak{P}_{sp'}$ *with* $\max\{1/p,2-3/p\}<s$. *Then the restriction of* $\mathfrak{F}_*^p f$ *to* Σ *is in* $\mathfrak{M}(\Sigma)$ *for* $f\in L^p(\mathbf{R}_+,\omega_{\sigma,p})$, *and*

$$\|\mathfrak{F}_*^p f\|_{\mathfrak{M}(\Sigma)}\leqslant C\|f\|_{L^p(\mathbf{R}_+,\omega_{\sigma,p})},\quad f\in L^p(\mathbf{R}_+,\omega_{\sigma,p}),$$

for some constant $C=C(s,\sigma,p)$, $0<C<+\infty$.

To handle multiplicative properties of the spaces $Q^p(\mathbf{C}_-,\sigma)$, we need to know when Leibniz' rule applies in the sense of distribution theory.

LEMMA 8.7. *Let* Ω *be a domain in the complex plane, and let* $f,g\in C(\Omega)$ *be functions whose* $\bar\partial$ *derivatives,* $\bar\partial f$ *and* $\bar\partial g$, *taken in the sense of distribution theory, belong to* $L^1_{\mathrm{loc}}(\Omega)$. *Then* $f\bar\partial g$ *and* $g\bar\partial f$ *are also in* $L^1_{\mathrm{loc}}(\Omega)$, *and Leibniz' formula holds:* $\bar\partial(fg)=f\bar\partial g+g\bar\partial f$. *Both sides of this identity are to be interpreted in the sense of distribution theory.*

Proof. Smooth up f and g by convolving them with a C^∞ approximate convolution identity. Then apply Leibniz' rule to the smoothed up functions, and make an appropriate passage to the limit. Interpreted in the sense of distribution theory, this yields the result. □

We now deal with the multiplicative properties of the spaces $Q^p(\mathbf{C}_-,\sigma)$.

PROPOSITION 8.8 ($1<p<+\infty$). *Suppose that* $\sigma^{-p'}\in\mathfrak{P}_{sp'}$ *with* $\max\{1/p,2-3/p\}<s$. *Then if* $f,g\in Q^p(\mathbf{C}_-,\sigma)$, *we have* $fg\in Q^p(\mathbf{C}_-,\sigma)$, *and moreover, there exists a constant* $C=C(s,\sigma,p)$, $0<C<+\infty$, *such that*

$$\|fg\|_{Q^p(\mathbf{C}_-,\sigma)}\leqslant C\|f\|_{Q^p(\mathbf{C}_-,\sigma)}\|g\|_{Q^p(\mathbf{C}_-,\sigma)}.$$

It follows that $Q^p(\mathbf{C}_-,\sigma)$ *is a commutative Banach algebra under pointwise multiplication of functions.*

Note. We consider the more general definition of a Banach algebra with the norm of a product bounded by a constant times the norms of the factors.

Proof. By Theorem 2.1, we can find functions $\varphi,\psi\in L^p(\mathbf{R}_+,\omega_{\sigma,p})$ with $\mathfrak{F}\varphi=f$ and $\mathfrak{F}\psi=g$ which have the same norms as f and g in $Q^p(\mathbf{C}_-,\sigma)$, respectively. The functions

$f_*=\mathfrak{F}_*^p\varphi$ and $g_*=\mathfrak{F}_*^p\psi$ then provide extensions to the whole complex plane of the functions f and g (see §2), which are bounded and continuous (Corollary 8.5), and satisfy $\bar\partial f_*=\mathfrak{D}^p\varphi\in\mathfrak{L}^p(\Sigma,\sigma)$ and $\bar\partial g_*=\mathfrak{D}^p\psi\in\mathfrak{L}^p(\Sigma,\sigma)$. By Proposition 8.4, $\bar\partial f_*$ and $\bar\partial g_*$ belong to $L^q(\Sigma)$ for some $q\in]2,+\infty[$. By Lemma 8.7 (with $\Omega=\mathbf{C}$), $\bar\partial(f_*g_*)=f_*\bar\partial g_*+g_*\bar\partial f_*$, and consequently, $\bar\partial(f_*g_*)\in\mathfrak{L}^p(\Sigma,\sigma)$, in view of Corollary 8.6. An application of Proposition 2.7 shows that fg is in $Q^p(\mathbf{C}_-,\sigma)$, and it is easy to verify that the claimed norm inequality holds. □

If we recall the well-known fact that the Fourier transform turns convolution into ordinary pointwise multiplication of functions, Proposition 8.8 has the following consequence, in view of Theorem 2.1.

COROLLARY 8.9 ($1<p<+\infty$). *Suppose* $\sigma^{-p'}\in\mathfrak{P}_{sp'}$ *with* $\max\{1/p,2-3/p\}<s$. *Then if* $f,g\in L^p(\mathbf{R}_+,\omega_{\sigma,p})$, *we have* $f*g\in L^p(\mathbf{R}_+,\omega_{\sigma,p})$, *and moreover, there exists a constant* $C=C(s,\sigma,p)$, $0<C<+\infty$, *such that*

$$\|f*g\|_{L^p(\mathbf{R}_+,\omega_{\sigma,p})}\leqslant C\|f\|_{L^p(\mathbf{R}_+,\omega_{\sigma,p})}\|g\|_{L^p(\mathbf{R}_+,\omega_{\sigma,p})}.$$

Remark 8.10. It is possible to arrive at the conclusion of Corollary 8.8 in a more elementary way, as we shall see. By Proposition 4.2, the assumption $\sigma^{-p'}\in\mathfrak{P}_{sp'}$, for some s, $0<s<+\infty$, entails that there exists a weight $\bar\omega$ in $\mathfrak{W}_s$ with $\bar\omega\asymp\omega_{\sigma,p}$.

Claim. For $1/p'<s<+\infty$, there is a constant C, $0<C<+\infty$, such that

$$(\bar\omega^{-p'}*\bar\omega^{-p'})(x)\leqslant C\bar\omega(x)^{-p'},\quad x\in\mathbf{R}_+.$$

To this end, note that $\bar\omega(t)=(1+t)^s\tilde\omega_s(t)$, where $\tilde\omega_s\in\mathfrak{W}$, so that what needs to be checked is

$$\int_0^x\frac{(1+x)^{sp'}\tilde\omega_s(x)^{p'}\,dt}{(1+x-t)^{sp'}(1+t)^{sp'}\tilde\omega_s(x-t)^{p'}\tilde\omega_s(t)^{p'}}\leqslant C,\quad x\in\mathbf{R}_+.$$

By Proposition 4.3, $\tilde\omega_s(0)\tilde\omega_s(x)\leqslant\tilde\omega_s(x-t)\tilde\omega_s(t)$, and consequently, it is sufficient to verify that

$$\int_0^x\frac{(1+x)^{sp'}\,dt}{(1+x-t)^{sp'}(1+t)^{sp'}}\leqslant C,\quad x\in\mathbf{R}_+,$$

if we change the value of the constant C by multiplying it with $\tilde\omega_s(0)^{p'}$. This is very easy to check, if we use that $1/p'<s<+\infty$; notice the symmetry of the integrand, and split the integral at the middle point $\frac{1}{2}x$. □

Since $\omega_{\sigma,p}\asymp\bar\omega$, the assertion of Corollary 8.9 follows from the above claim and the following lemma.

LEMMA 8.11 ($1<p<+\infty$). *If a weight function ω on $\mathbf{R}_+$ satisfies $(\omega^{-p'}*\omega^{-p'})^{1/p'}\leqslant C\omega^{-1}$ on $\mathbf{R}_+$, then*

$$\|f*g\|_{L^p(\mathbf{R}_+,\omega)}\leqslant C\|f\|_{L^p(\mathbf{R}_+,\omega)}\|g\|_{L^p(\mathbf{R}_+,\omega)},\quad f,g\in L^p(\mathbf{R}_+,\omega).$$

Proof. By Hölder's inequality,

$$\begin{aligned}\int_0^{+\infty}|(f*g)(x)|^p\omega(x)^p\,dx&=\int_0^{+\infty}\left|\int_0^x f(y)g(x-y)\,dy\right|^p\omega(x)^p\,dx\\&\leqslant\int_0^{+\infty}\omega(x)^p\left(\int_0^x\frac{dy}{\omega(y)^{p'}\omega(x-y)^{p'}}\right)^{p/p'}\\&\quad\times\left(\int_0^x|f(y)|^p|g(x-y)|^p\omega(y)^p\omega(x-y)^p\,dy\right)dx\\&\leqslant C^p\int_0^{+\infty}\int_0^x|f(y)|^p\omega(y)^p|g(x-y)|^p\omega(x-y)^p\,dy\,dx\\&=C^p\|f\|^p_{L^p(\mathbf{R}_+,\omega)}\|g\|^p_{L^p(\mathbf{R}_+,\omega)},\end{aligned}$$

as claimed. □

Remark 8.12. By Proposition 4.2 and Lemma 8.11, the assertion of Proposition 8.8 actually holds for all s, $1/p'<s<+\infty$. The point with the approach chosen here is that with the excessive regularity condition, we know that the reason why $Q^p(\mathbf{C}_-,\sigma)$ is a Banach algebra is that the canonical extensions to $\mathbf{C}$ of the functions in $Q^p(\mathbf{C}_-,\sigma)$ are sufficiently smooth.

We need to identify the maximal ideal space of the algebra $Q^p(\mathbf{C}_-,\sigma)$; however, since this algebra lacks a unit, it is preferable to consider instead the unitization $Q^p_e(\mathbf{C}_-,\sigma)$ of $Q^p(\mathbf{C}_-,\sigma)$.

PROPOSITION 8.13 ($1<p<+\infty$). *Suppose $\sigma^{-p'}\in\mathfrak{P}_{sp'}$ for some s, $1/p'<s<+\infty$, so that $Q^p(\mathbf{C}_-,\sigma)$ is a Banach algebra. Then the maximal ideal space of the unitization $Q^p_e(\mathbf{C}_-,\sigma)$ of $Q^p(\mathbf{C}_-,\sigma)$ coincides with $\overline{\mathbf{C}}_-\cup\{\infty\}$, in the sense that every nontrivial complex homomorphism $\tau\colon Q^p_e(\mathbf{C}_-,\sigma)\to\mathbf{C}$ has the form*

$$\tau(f)=f(z_0),\quad f\in Q^p_e(\mathbf{C}_-,\sigma),$$

for some $z_0\in\overline{\mathbf{C}}_-\cup\{\infty\}$.

Proof sketch. Check first that a nontrivial complex homomorphism is a point evaluation on the functions E_λ, $\lambda\in\mathbf{C}_+$. Then use Proposition 6.1 to show that it is a point evaluation on all functions. □

Remark 8.14. Some condition on σ is needed to ensure that $Q^p(\mathbf{C}_-,\sigma)$ is a Banach algebra. To illustrate this point, take $\sigma(t)\equiv 1$, so that $Q^2(\mathbf{C}_-,\sigma)$ becomes the *Dirichlet space*, which contains unbounded functions, and hence is no Banach algebra, yet σ^{-2} is in $\mathfrak{P}_{2s}$ for all s, $0<s<\frac{1}{2}$.

9. More on smoothness properties

The parameter p has $1\leqslant p<+\infty$ (p' is the dual exponent), and $\omega_{\sigma,p}$ and σ are as in §§ 2 and 3.

As in §8, we study canonical densities, that is, elements of

$$\mathcal{H}^p(\Sigma,\sigma)=\mathfrak{D}^p L^p(\mathbf{R}_+,\omega_{\sigma,p}),$$

and their global Cauchy transforms. This time we want to get more smoothness than that which is needed to have a Banach algebra structure.

PROPOSITION 9.1 ($1<p<+\infty$). *Suppose $\sigma^{-p'}$ is in $\mathfrak{P}_{sp'}$ for some s, $1/p'<s<+\infty$. Fix θ_0 and θ_1, $\theta_0=1-(s-1/p')^{-1}$, $\theta_1=1-2(s-1/p')^{-1}$. Then the estimates*

$$\|\mathfrak{D}^p f\|_{L^\infty(\Sigma,\sigma^{\theta_0})}\leqslant C\|f\|_{L^p(\mathbf{R}_+,\omega_{\sigma,p})},$$
$$\|\partial_x\mathfrak{D}^p f\|_{L^\infty(\Sigma,\sigma^{\theta_1})}\leqslant C\|f\|_{L^p(\mathbf{R}_+,\omega_{\sigma,p})},$$

hold, for some positive real constant $C=C(s,\sigma,p)$.

Proof. The assertion of the proposition follows from (2.15) and Proposition 4.4. □

For $f\in L^1(\mathbf{R}_+,\omega_{\sigma,1})$, we identify the function with the corresponding absolutely continuous measure, and thus write $\mathfrak{D}^1 f$ for the density associated with the measure $f(t)\,dt$,

$$\mathfrak{D}^1 f(x+iy)=\frac{1}{2i}\exp(yt_\sigma(y))\exp(-it_\sigma(y)x)f\circ t_\sigma(y)|t'_\sigma(y)|,\quad x+iy\in\check{\Sigma}.$$

By (3.3), this may also be written as

$$\mathfrak{D}^1 f(x+iy)=\frac{\omega_{\sigma,1}(t_\sigma(y))}{2i\sigma(y)}\exp(-it_\sigma(y)x)\,f\circ t_\sigma(y)|t'_\sigma(y)|,\quad x+iy\in\check{\Sigma}.$$

For $p=1$, the following result is helpful.

The analog of Proposition 9.1 for $p=1$ is obtained through an argument slightly different from what was used for $1<p<+\infty$. The space $L^\infty(\mathbf{R}_+,\omega_{\sigma,1})$ is needed, which consists of all Lebesgue measurable functions f on $\mathbf{R}_+$, subject to the norm condition

$$\|f\|_{L^\infty(\mathbf{R}_+,\omega_{\sigma,1})}=\operatorname{ess\,sup}\{\omega_{\sigma,1}(t)|f(t)|:t\in\mathbf{R}_+\}<+\infty.$$

PROPOSITION 9.2 ($p=1$). *Suppose* $\omega_{\sigma,1}\in\mathfrak{W}_s$ *for some* s *with* $0<s<+\infty$. *Then, with* $\theta_0=1-2/s$ *and* $\theta_1=1-3/s$, *the following estimates hold, for* $f\in L^1(\mathbf{R}_+,\omega_{\sigma,1})\cap L^\infty(\mathbf{R}_+,\omega_{\sigma,1})$,

$$\|\mathfrak{D}^1 f\|_{L^\infty(\Sigma,\sigma^{\theta_0})}\leqslant C\|f\|_{L^\infty(\mathbf{R}_+,\omega_{\sigma,1})},$$
$$\|\partial_x\mathfrak{D}^1 f\|_{L^\infty(\Sigma,\sigma^{\theta_1})}\leqslant C\|f\|_{L^\infty(\mathbf{R}_+,\omega_{\sigma,1})},$$

where $C=C(s,\sigma)$ *is a positive real constant.*

Proof. Note first that by the definition of the density $\mathfrak{D}^1 f$, we have for integers $n=0,1,2,\ldots$,

$$\begin{aligned}\partial_x^n\mathfrak{D}^1 f(x+iy)&=(-it_\sigma(y))^n\mathfrak{D}^1 f(x+iy)\\&=\frac{(-i)^n}{2i\sigma(y)}f\circ t_\sigma(y)\omega_{\sigma,1}\circ t_\sigma(y)\exp(-ixt_\sigma(y))t_\sigma(y)^n|t'_\sigma(y)|,\end{aligned}$$

so that

$$|\partial_x^n\mathfrak{D}^1 f(x+iy)|\leqslant\frac{t_\sigma(y)^n|t'_\sigma(y)|}{2\sigma(y)}\|f\|_{L^\infty(\mathbf{R}_+,\omega_{\sigma,1})},\quad x+iy\in\Sigma,$$

at least almost everywhere. The assertion now follows from Lemma 4.6. □

LEMMA 9.3 ($p=1$). *For each* $f\in L^1(\mathbf{R}_+,\omega_{\sigma,1})$, *the restriction to* $\check{\Sigma}$ *of* $\mathfrak{F}^1_* f$ *is in* $\mathfrak{M}(\check{\Sigma})$; *in fact,* $\|\mathfrak{F}^p_* f\|_{\mathfrak{M}(\check{\Sigma})}\leqslant\|f\|_{L^1(\mathbf{R}_+,\omega_{\sigma,1})}$.

Proof. This follows from Proposition 3.5. □

As in §6, the space $Q^1_0(\mathbf{C}_-,\sigma)$ is the Fourier image of $L^1(\mathbf{R}_+,\omega_{\sigma,1})$, which is a proper closed subspace of $Q^1(\mathbf{C}_-,\sigma)$.

PROPOSITION 9.4 ($p=1$). *When equipped with pointwise multiplication of functions, the space* $Q^1(\mathbf{C}_-,\sigma)$ *is a commutative Banach algebra with unit, and* $Q^1_0(\mathbf{C}_-,\sigma)$ *is a closed ideal in it.*

Proof. The space $\mathcal{M}(\mathbf{R}_+,\omega_{\sigma,1})$ is a convolution algebra, because $\omega_{\sigma,1}$ is submultiplicative, by Proposition 4.3. Moreover, $L^1(\mathbf{R}_+,\omega_{\sigma,1})$ is the closed ideal of all absolutely continuous measures in $\mathcal{M}(\mathbf{R}_+,\omega_{\sigma,1})$. The result follows by taking Fourier transforms. □

Remark 9.5. It is possible to obtain the assertion that $Q^1_0(\mathbf{C}_-,\sigma)$ is a Banach algebra by working with canonical extensions, as in §8.

We identify the unitization $Q^1_e(\mathbf{C}_-,\sigma)$ of $Q^1_0(\mathbf{C}_-,\sigma)$ with the subalgebra of $Q^1(\mathbf{C}_-,\sigma)$ generated by $Q^1_0(\mathbf{C}_-,\sigma)$ and the constant functions.

PROPOSITION 9.6 ($p=1$). *In the sense of point evaluations, the closed lower half-plane $\overline{\mathbf{C}}_-$ is an open subset of the maximal ideal space of $Q^1(\mathbf{C}_-,\sigma)$. Moreover, in the same sense, the maximal ideal space of $Q^1_e(\mathbf{C}_-,\sigma)$ is $\overline{\mathbf{C}}_-\cup\{\infty\}$.*

This result is well known.

10. Holomorphic continuation of the resolvent transform

Let p be in the interval $1\leqslant p<+\infty$, p' be the dual exponent to p, and $\omega_{\sigma,p}$ be related to σ as in §§ 2 and 3 for appropriate weights σ.

The *resolvent transform* of an element ϕ of the dual space to $L^p(\mathbf{R}_+,\omega_{\sigma,p})$, identified with the weighted space $L^{p'}(\mathbf{R}_+,\omega^{-1}_{\sigma,p})$ on $\mathbf{R}_+$, was introduced back in §§ 2 and 3 (relations (2.21) and (3.11)):

$$\mathfrak{R}\phi(\lambda)=-i\int_0^{+\infty}\exp(it\lambda)\phi(t)\,dt,\quad \lambda\in\mathbf{C}_+. \tag{10.1}$$

The function $\mathfrak{R}\phi$ defined by relation (10.1) is analytic on the upper half-plane $\mathbf{C}_+$, and by the uniqueness theorem for the Fourier transform, it vanishes identically if and only if $\phi=0$. The idea to use such a transform to gather information about the structure of closed ideals in a Banach algebra goes back to Arne Beurling, Torsten Carleman, and Izrail Gel'fand [13]; as far as we know, Gel'fand has the earliest paper using this method, and he also has, in a simple special case, the elegant way of getting the analytic continuation which was later rediscovered by Yngve Domar [8]. Beurling's and Carleman's contributions have been more influential, and what we refer to here as the resolvent transform is frequently called the Carleman transform in other research papers.

In terms of the bilinear form

$$\langle f,g\rangle=\int_0^{+\infty}f(t)g(t)\,dt,$$

we may rewrite (10.1) as

$$\mathfrak{R}\phi(\lambda)=\langle e_\lambda,\phi\rangle,\quad \lambda\in\mathbf{C}_+, \tag{10.2}$$

with e_λ as in Proposition 6.1:

$$e_\lambda(t)=-i\exp(i\lambda t),\quad t\in\mathbf{R}_+.$$

For $1<p<+\infty$, the Fourier transform maps $L^p(\mathbf{R}_+,\omega_{\sigma,p})$ isometrically onto $Q^p(\mathbf{C}_-,\sigma)$, by Theorem 2.1. As a consequence, the functional ϕ on $L^p(\mathbf{R}_+,\omega_{\sigma,p})$ corresponds to a bounded linear functional $\hat{\phi}$ on $Q^p(\mathbf{C}_-,\sigma)$, and in terms of bilinear forms, we write $\langle \mathfrak{F} f, \hat{\phi}\rangle = \langle f, \phi\rangle$. We may thus write (10.2) as

$$\mathfrak{R}\phi(\lambda) = \langle E_\lambda, \hat{\phi}\rangle, \quad \lambda \in \mathbf{C}_+, \tag{10.3}$$

where $E_\lambda(z) = \mathfrak{F} e_\lambda(z) = (\lambda - z)^{-1}$, as in Proposition 6.1. For $p=1$, the Fourier transform maps $\mathcal{M}(\mathbf{R}_+, \omega_{\sigma,1})$ isometrically onto $Q^1(\mathbf{C}_-,\sigma)$, by Theorem 3.1. Extend the functional ϕ to all of $\mathcal{M}(\mathbf{R}_+, \omega_{\sigma,1})$ by declaring $\langle \xi, \phi\rangle = 0$ for measures ξ that are singular to the linear Lebesgue measure. This extended ϕ then corresponds to a bounded linear functional $\hat{\phi}$ on $Q^1(\mathbf{C}_-,\sigma)$ via $\langle \mathfrak{F}\xi, \hat{\phi}\rangle = \langle \xi, \phi\rangle$, and (10.3) holds for $p=1$ as well. We intend to study resolvent transforms $\mathfrak{R}\phi$ of mean-periodic functions ϕ, that is, functions in the annihilator of a non-zero, closed, right translation invariant subspace of $L^p(\mathbf{R}_+, \omega_{\sigma,p})$. In particular, we investigate the possibility of holomorphically extending $\mathfrak{R}\phi$ across parts of the real axis. The purpose is to show that under certain conditions, $\mathfrak{R}\phi$ is an entire function; we shall later show that $\mathfrak{R}\phi(\lambda) \equiv 0$, which by the uniqueness theorem for the Laplace transform implies that $\phi = 0$.

Consider a closed, right translation invariant, nonzero subspace I of $L^p(\mathbf{R}_+, \omega_{\sigma,p})$, other than the zero subspace $\{0\}$, and assume that $\phi \in L^{p'}(\mathbf{R}_+, \omega_{\sigma,p}^{-1})$ annihilates I. In other words, $J = \mathfrak{F}(I)$ is a closed (nontrivial) subspace of $Q_0^p(\mathbf{C}_-,\sigma)$, invariant under the operators M_x, $x \in \mathbf{R}_+$, which were introduced in Corollary 6.3. It is a consequence of Proposition 6.5 that J is then also invariant under multiplication by the functions E_λ, for $\lambda \in \mathbf{C}_+$. Introduce, for every $\lambda \in \mathbf{C}$, the function

$$A_\lambda(z) = 1 + (\lambda - i) E_i(z) = \frac{\lambda - z}{i - z}, \quad z \in \mathbf{C}_-,$$

and note that it is clear from this formula and Proposition 6.4 that A_λ is a multiplier on $Q^p(\mathbf{C}_-,\sigma)$, which for $p=1$ preserves $Q_0^1(\mathbf{C}_-,\sigma)$, for every $\lambda \in \mathbf{C}$. Again by Proposition 6.4, A_λ is invertible as a multiplier provided that $\lambda \in \mathbf{C}_+$, if we use the identity

$$A_\lambda(z)^{-1} = \frac{i - z}{\lambda - z} = 1 - (\lambda - i) E_\lambda(z), \quad z \in \mathbf{C}_-.$$

Now, $E_\lambda = A_\lambda^{-1} E_i$ for $\lambda \in \mathbf{C}_+$, so that relation (10.3) may be written as

$$\mathfrak{R}\phi(\lambda) = \langle A_\lambda^{-1} E_i, \hat{\phi}\rangle, \quad \lambda \in \mathbf{C}_+. \tag{10.4}$$

Since we know J is invariant under multiplication by the function E_i, it makes sense to consider the operator $A_\lambda[J]\colon Q^p(\mathbf{C}_-,\sigma)/J \to Q^p(\mathbf{C}_-,\sigma)/J$, as given by

$$A_\lambda[J](f + J) = A_\lambda f + J, \quad f \in Q^p(\mathbf{C}_-,\sigma).$$

The condition that ϕ annihilates I is equivalent to the requirement that $\hat{\phi}$ annihilates J, which permits us to think of $\hat{\phi}$ as acting on the quotient space $Q^p(\mathbf{C}_-,\sigma)/J$. We are now in a position to extend the definition of the resolvent transform via the formula

$$\mathfrak{R}\phi(\lambda)=\langle A_\lambda[J]^{-1}(E_i+J),\hat{\phi}\rangle,\quad \lambda\in\Omega(J), \tag{10.5}$$

where $\Omega(J)$ denotes the *invertibility domain* for $A_\lambda[J]$, that is, the set of $\lambda\in\mathbf{C}$ for which the operator $A_\lambda[J]$ is invertible.

PROPOSITION 10.1 ($1\leqslant p<+\infty$). *The set $\Omega(J)$ is open, and it contains $\mathbf{C}_+$.*

Proof. Since A_λ itself is invertible as multiplier on $Q^p(\mathbf{C}_-,\sigma)$ for $\lambda\in\mathbf{C}_+$, it is evident that $\Omega(J)$ contains $\mathbf{C}_+$. We proceed to show that $\Omega(J)$ is open. To this end, suppose $\lambda_0\in\Omega(J)$, so that $A_{\lambda_0}[J]^{-1}$ exists. We shall show that there exists an ε, $0<\varepsilon$, such that if $\lambda\in\mathbf{C}$ has $|\lambda-\lambda_0|<\varepsilon$, then $\lambda\in\Omega(J)$. Let $E_i[J]$ denote the operator $Q^p(\mathbf{C}_-,\sigma)/J\rightarrow Q^p(\mathbf{C}_-,\sigma)/J$ that multiplies the cosets by E_i. Then, if

$$\varepsilon\cdot\|A_{\lambda_0}[J]^{-1}\|\cdot\|E_i[J]\|<1,$$

we have the identity

$$A_\lambda[J]^{-1}=A_{\lambda_0}[J]^{-1}\sum_{n=0}^{\infty}(\lambda_0-\lambda)^n(A_{\lambda_0}[J]^{-1}E_i[J])^n, \tag{10.6}$$

for $|\lambda-\lambda_0|<\varepsilon$, which does it. □

Remark 10.2. It is clear from (10.6) that the function $\mathfrak{R}\phi$, given by (10.5), is holomorphic on $\Omega(J)$.

The following lemma explains what we need to check to know whether a point is in $\Omega(J)$.

LEMMA 10.3 ($1\leqslant p<+\infty$). *A point $\lambda\in\mathbf{C}$ belongs to $\Omega(J)$ if and only if*
(a) *$A_\lambda Q^p(\mathbf{C}_-,\sigma)+J=Q^p(\mathbf{C}_-,\sigma)$, and*
(b) *if $f\in Q^p(\mathbf{C}_-,\sigma)$, and $A_\lambda f\in J$, then $f\in J$.*

Proof. By the definition of the set $\Omega(J)$, the point λ belongs to it if and only if $A_\lambda[J]$ is invertible, which means that it is one-to-one and onto. The assumption (a) of the lemma is exactly the condition that $A_\lambda[J]$ be onto, and (b) means that $A_\lambda[J]$ is one-to-one. □

For $p=1$, consider the set $\Omega_0(J)$ of all $\lambda\in\mathbf{C}$ for which $A_{\lambda,0}[J]$, the restriction of $A_\lambda[J]$ to $Q^1_0(\mathbf{C}_-,\sigma)/J$, is invertible $Q^1_0(\mathbf{C}_-,\sigma)/J\rightarrow Q^1_0(\mathbf{C}_-,\sigma)/J$.

LEMMA 10.4 ($p=1$). *The space $Q^1(\mathbf{C}_-,\sigma)$ is a commutative Banach algebra with unit under pointwise multiplication of functions, and $Q^1_0(\mathbf{C}_-,\sigma)$ is a closed ideal in it. Thus, the operators $A_\lambda[J]$ and $A_{\lambda,0}[J]$ are invertible simultaneously, that is, $\Omega(J)=\Omega_0(J)$.*

Proof. That $Q^1(\mathbf{C}_-,\sigma)$ is a commutative Banach algebra with unit follows immediately from the fact that $\mathcal{M}(\mathbf{R}_+,\omega_{\sigma,1})$ is a commutative convolution Banach algebra with unit, which is verified using the essential submultiplicativity of $\omega_{\sigma,1}$ stated in Proposition 4.3. Hence $Q^1(\mathbf{C}_-,\sigma)/J$ is a quotient Banach algebra.

Suppose $A_\lambda[J]$ is invertible; then multiplication by $A_\lambda+J$ maps $Q^1(\mathbf{C}_-,\sigma)/J$ onto $Q^1(\mathbf{C}_-,\sigma)/J$, which is only possible if $A_\lambda+J$ is invertible in $Q^1(\mathbf{C}_-,\sigma)/J$. Since $Q^1_0(\mathbf{C}_-,\sigma)$ is a closed ideal in $Q^1(\mathbf{C}_-,\sigma)$, and J is contained in it, multiplication by $(A_\lambda+J)^{-1}$ on $Q^1_0(\mathbf{C}_-,\sigma)/J$ is the inverse of $A_{\lambda,0}[J]$.

On the other hand, suppose $A_{\lambda,0}[J]$ is invertible. Then the image of $Q^1_0(\mathbf{C}_-,\sigma)/J$ under the operator $A_\lambda[J]$ is all of $Q^1_0(\mathbf{C}_-,\sigma)/J$, and since $A_\lambda[J](1+J)=A_\lambda+J$ belongs to the unitization $Q^1_{0,e}(\mathbf{C}_-,\sigma)/J$ of $Q^1_0(\mathbf{C}_-,\sigma)/J$, but not to $Q^1_0(\mathbf{C}_-,\sigma)/J$ itself, the image of $Q^1_{0,e}(\mathbf{C}_-,\sigma)/J$ under the multiplication by $A_\lambda+J$ is all of $Q^1_{0,e}(\mathbf{C}_-,\sigma)/J$. Then $A_\lambda+J$ is invertible in $Q^1_{0,e}(\mathbf{C}_-,\sigma)/J$, and thus in the larger algebra $Q^1(\mathbf{C}_-,\sigma)/J$. This supplies the inverse to $A_\lambda[J]$. □

In view of Lemma 10.3, we define $\Omega'(J)$, the *weak invertibility domain* for $A_\lambda[J]$, as consisting of all points $\lambda\in\mathbf{C}$ for which

$$A_\lambda Q^p(\mathbf{C}_-,\sigma)+J=Q^p(\mathbf{C}_-,\sigma). \tag{10.7}$$

Then clearly $\Omega(J)$ is a subset of $\Omega'(J)$. To better understand this definition, we need the following result, which is of independent interest.

PROPOSITION 10.5 ($1<p<+\infty$). *For $\lambda\in\mathbf{C}_+$, we have $A_\lambda Q^p(\mathbf{C}_-,\sigma)=Q^p(\mathbf{C}_-,\sigma)$, and for $\lambda\in\mathbf{C}_-$,*

$$A_\lambda Q^p(\mathbf{C}_-,\sigma)=\{f\in Q^p(\mathbf{C}_-,\sigma):f(\lambda)=0\}.$$

For $\lambda\in\mathbf{R}$, we have either that $A_\lambda Q^p(\mathbf{C}_-,\sigma)$ is dense in $Q^p(\mathbf{C}_-,\sigma)$, which happens when

$$\int_0^1\frac{dt}{t\sigma(t)^{p'}}=+\infty, \tag{10.8}$$

or that its closure coincides with

$$\{f\in Q^p(\mathbf{C}_-,\sigma):f(\lambda)=0\},$$

which occurs in case

$$\int_0^1\frac{dt}{t\sigma(t)^{p'}}<+\infty. \tag{10.9}$$

Proof. Since A_λ is an invertible multiplier on $Q^p(\mathbf{C}_-,\sigma)$ for $\lambda\in\mathbf{C}_+$, we immediately have $A_\lambda Q^p(\mathbf{C}_-,\sigma)=Q^p(\mathbf{C}_-,\sigma)$. So, let $\lambda\in\mathbf{C}_-$. Since A_λ vanishes at the point λ, we clearly have that $A_\lambda Q^p(\mathbf{C}_-,\sigma)$ is contained in the subspace of all functions in $Q^p(\mathbf{C}_-,\sigma)$ that vanish at λ. To obtain the reverse inclusion, we argue as follows. Note that $|A_\lambda(z)|$ is bounded away from 0 off a neighborhood of λ, and bounded away from $+\infty$ off a neighborhood of i. Suppose $f\in Q^p(\mathbf{C}_-,\sigma)$ has $f(\lambda)=0$. In view of the results in §2, there exists a canonical extension $f_*\in\mathfrak{C}_*\mathfrak{L}^p(\Sigma,\sigma)$ to the whole complex plane of the function f, and by Proposition 2.7, it satisfies $f_*\in L^1_\infty(\mathbf{C})$ and $\bar\partial f_*\in\mathfrak{L}^p(\Sigma,\sigma)$. Since $f(\lambda)=0$, the function $g=f_*/A_\lambda$ is holomorphic off the closure of Σ, and it also belongs $\mathfrak{C}_*\mathfrak{L}^p(\Sigma,\sigma)$, because $g\in L^1_\infty(\mathbf{C})$, and $\bar\partial g=\bar\partial f_*/A_\lambda\in\mathfrak{L}^p(\Sigma,\sigma)$ (use Proposition 2.7).

We finally turn to the case $\lambda\in\mathbf{R}$. Simple algebra shows that the function

$$N_\lambda f(z)=f(z)-f(\lambda)E_i(z)/E_i(\lambda),\quad z\in\mathbf{C}_-,$$

belongs to $A_\lambda Q^p(\mathbf{C}_-,\sigma)$ for all f in $Q^p(\mathbf{C}_-,\sigma)$ that are finite linear combinations of the functions E_λ, with $\lambda\in\mathbf{C}_+$. If condition (10.9) is fulfilled, then by Proposition 8.1, point evaluation at λ is a bounded linear functional, so that an approximation argument based on Proposition 6.1 proves that $N_\lambda f$ is in the closure of $A_\lambda Q^p(\mathbf{C}_-,\sigma)$, for every $f\in Q^p(\mathbf{C}_-,\sigma)$. But it is easy to see that $N_\lambda Q^p(\mathbf{C}_-,\sigma)$ consists of all functions in $Q^p(\mathbf{C}_-,\sigma)$ that vanish at λ.

If $\lambda\in\mathbf{R}$, but (10.8) holds, we adopt a different *modus operandi*, based on duality. Let $\psi\in L^{p'}(\mathbf{R}_+,\omega^{-1}_{\sigma,p})$ be such that the associated bounded linear functional $\hat\psi$ on $Q^p(\mathbf{C}_-,\sigma)$ annihilates $A_\lambda Q^p(\mathbf{C}_-,\sigma)$. If we can show that this ψ must equal 0, then the claim that $A_\lambda Q^p(\mathbf{C}_-,\sigma)$ is dense in $Q^p(\mathbf{C}_-,\sigma)$ follows immediately. The resolvent transform of ψ is given by the formula $\mathfrak{R}\psi(\zeta)=\langle E_\zeta,\hat\psi\rangle$ for $\zeta\in\mathbf{C}_+$. Since ψ is assumed to annihilate $A_\lambda Q^p(\mathbf{C}_-,\sigma)$, we have $\langle A_\lambda E_\zeta,\hat\psi\rangle=0$ for $\zeta\in\mathbf{C}_+$, and in view of the identity

$$E_\zeta(z)-A_\lambda(z)E_\zeta(z)/A_\lambda(\zeta)=\frac{\lambda-i}{\lambda-\zeta}E_i(z),\quad z\in\mathbf{C}_-,$$

we can now assert that

$$\mathfrak{R}\psi(\zeta)=\langle E_\zeta-A_\lambda E_\zeta/A_\lambda(\zeta),\hat\psi\rangle=\frac{\lambda-i}{\lambda-\zeta}\langle E_i,\hat\psi\rangle=\frac{\lambda-i}{\lambda-\zeta}\mathfrak{R}\psi(i),\quad \zeta\in\mathbf{C}_+.$$

By the uniqueness theorem for the Fourier (or Laplace) transform, this entails that as an element of $L^{p'}(\mathbf{R}_+,\omega^{-1}_{\sigma,p})$,

$$\psi(t)=(i-\lambda)\mathfrak{R}\psi(i)\exp(-i\lambda t),\quad t\in\mathbf{R}_+;$$

its norm is

$$\|\psi\|_{L^{p'}(\mathbf{R}_+,\omega^{-1}_{\sigma,p})}=|\lambda-i|\cdot|\mathfrak{R}\psi(i)|\left(\int_0^{+\infty}\frac{dt}{\omega_{\sigma,p}(t)^{p'}}\right)^{1/p'},$$

which is finite only if $\mathfrak{R}\psi(i)=0$, by the note right after Proposition 8.1. Thus $\mathfrak{R}\psi(\zeta)\equiv 0$, so that $\psi=0$. The proof is complete. □

The details of the proofs of the following two propositions are left to the interested reader. The results are of value when one tries to work with weights σ for which $Q^p(\mathbf{C}_-,\sigma)$ is not a Banach algebra.

PROPOSITION 10.6 ($1<p<+\infty$). *The weak invertibility domain* $\Omega'(J)$ *is open. Moreover, we have the relation* $\mathbf{C}_-\cap\Omega'(J)=\mathbf{C}_-\setminus Z(J,\mathbf{C}_-)$, *where*

$$Z(J,\mathbf{C}_-)=\{z\in\mathbf{C}_-: f(z)=0 \textit{ for all } f\in J\}.$$

Proof sketch. To see that $\Omega'(J)$ is open, one expresses (10.7) in terms of the adjoint operator to $A_\lambda[J]$. For the proof of the identity $\mathbf{C}_-\cap\Omega'(J)=\mathbf{C}_-\setminus Z(J,\mathbf{C}_-)$, appeal to Proposition 10.5. □

PROPOSITION 10.7 ($1<p<+\infty$). *If the set* $\mathbf{R}\cap\Omega'(J)$ *is nonempty, we have* $\Omega(J)=\Omega'(J)$.

Proof sketch. As in the proof of Proposition 10.6, work with the adjoint operator to $A_\lambda[J]$. Standard operator theory arguments then show that each boundary point of $\Omega(J)$ on the real line $\mathbf{R}$ must also be a boundary point of $\Omega'(J)$, whence the assertion follows. □

At this point, it is helpful to assume that $Q^p(\mathbf{C}_-,\sigma)$ has a Banach algebra structure, in order to identify $\Omega(J)$ and $\Omega'(J)$.

PROPOSITION 10.8 ($1\leqslant p<+\infty$). *Suppose either* $p=1$, *or* $\sigma^{-p'}\in\mathfrak{P}_{sp'}$ *for some* s, $1/p'<s$, *so that* $Q^p(\mathbf{C}_-,\sigma)$ *is a commutative Banach algebra under pointwise multiplication of functions. The weak invertibility domain* $\Omega'(J)$ *and the invertibility domain* $\Omega(J)$ *for* $A_\lambda[J]$ *coincide, and equal* $\Omega'(J)=\Omega(J)=\mathbf{C}\setminus Z(J,\overline{\mathbf{C}}_-)$, *where*

$$Z(J,\overline{\mathbf{C}}_-)=\{z\in\overline{\mathbf{C}}_-: f(z)=0 \textit{ for all } f\in J\}.$$

Proof. We first consider $1<p<+\infty$, assuming only that (10.9) holds, which ensures the boundedness of point evaluations on the real line (Proposition 8.1), so that the definition of $Z(J,\overline{\mathbf{C}}_-)$ makes sense. By Proposition 10.6, all we need to check is that $\mathbf{R}\cap\Omega'(J)=\mathbf{R}\cap\Omega(J)=\mathbf{R}\setminus Z(J,\overline{\mathbf{C}}_-)$. Given the elementary property of bounded holomorphic functions on $\mathbf{C}_-$, continuous in the closed lower half-plane $\overline{\mathbf{C}}_-$, that they either vanish identically, or vanish on the real line at most on a closed set having one-dimensional Lebesgue measure 0, Proposition 10.7 shows that we in fact only need to check that $\mathbf{R}\cap\Omega'(J)=\mathbf{R}\setminus Z(J,\overline{\mathbf{C}}_-)$.

If $\lambda\in\mathbf{R}\cap Z(J,\bar{\mathbf{C}}_-)$, then (10.7) cannot hold, because the functions on the left hand side all vanish at λ, whereas this is not so for the functions on the right hand side. This shows the inclusion $\mathbf{R}\cap\Omega'(J)\subset\mathbf{R}\setminus Z(J,\bar{\mathbf{C}}_-)$.

To obtain the reverse inclusion, suppose $\lambda\in\mathbf{R}\setminus Z(J,\mathbf{C}_-)$. We note that by Proposition 10.5, $A_\lambda Q^p(\mathbf{C}_-,\sigma)$ is dense in the subspace with codimension 1 in $Q^p(\mathbf{C}_-,\sigma)$ consisting of all functions vanishing at λ, so that since J contains a function that assumes a non-zero value at λ, we must have that $A_\lambda Q^p(\mathbf{C}_-,\sigma)+J$ is dense in $Q^p(\mathbf{C}_-,\sigma)$.

We now make use of the powerful Banach algebra assumption, for general p. The maximal ideal space of $Q^p_e(\mathbf{C}_-,\sigma)$ was identified in Propositions 8.13 and 9.6. The maximal ideal space of $Q^p_e(\mathbf{C}_-,\sigma)/J$ is then (canonically) identifiable with $Z(J,\bar{\mathbf{C}}_-)\cup\{\infty\}$, and the operator $A_\lambda[J]$ is identified with multiplication with the element $A_\lambda+J$ in the space $Q^p_e(\mathbf{C}_-,\sigma)/J$. One checks that $A_\lambda+J$ is invertible in $Q^p_e(\mathbf{C}_-,\sigma)/J$ if and only if $\lambda\in\mathbf{C}\setminus Z(J,\bar{\mathbf{C}}_-)$, by verifying that this is precisely when the Gelfand transform of $A_\lambda+J$ lacks zeros. However, $A_\lambda[J]$ is invertible if and only if $A_\lambda+J$ is invertible in $Q^p_e(\mathbf{C}_-,\sigma)/J$. □

11. The quick estimate of the resolvent transform

We continue our discussion from §10, with a particular interest in obtaining size estimates of the resolvent transform $\mathfrak{R}\phi$. The function $\phi\in L^{p'}(\mathbf{R}_+,\omega_{\sigma,p}^{-1})$ of course remains in the annihilator of I, and thus the associated functional $\hat{\phi}$ remains in the annihilator of J. By Proposition 6.1, we have the estimate

$$|\mathfrak{R}\phi(\lambda)|\leqslant C(\phi)\hat{\sigma}(p,\operatorname{Im}\lambda)^{1/p},\quad \lambda\in\mathbf{C}_+, \tag{11.1}$$

where $C(\phi)=\|\phi\|_{L^{p'}(\mathbf{R}_+,\omega_{\sigma,p}^{-1})}$. This only estimates $\mathfrak{R}\phi$ in the upper half-plane, but as we know, this function has a holomorphic continuation to the invertibility domain $\Omega(J)$ (Remark 10.2), since $\hat{\phi}$ annihilates J. We assume in the sequel that the weak invertibility domain $\Omega'(J)$ has the property that $\Omega'(J)\cap\mathbf{R}$ is nonempty, which, by Proposition 10.7, ensures that $\Omega(J)=\Omega'(J)$. Proposition 10.6 (and Proposition 10.8 for $p=1$) informs us that in our situation we have $\Omega(J)\cap\mathbf{C}_-=\mathbf{C}_-\setminus Z(J,\mathbf{C}_-)$. We would like to have an estimate of the size of $\mathfrak{R}\phi$ in $\mathbf{C}_-\setminus Z(J,\mathbf{C}_-)$. To this end, we introduce, given a function $f\in Q^p(\mathbf{C}_-,\sigma)$, its *backward shift with respect to the point* λ, $\lambda\in\mathbf{C}_-$,

$$S_\lambda f(z)=\frac{f(\lambda)-f(z)}{\lambda-z},\quad z\in\mathbf{C}_-\setminus\{\lambda\},$$

and we note that λ is a removable singularity of this holomorphic function. One interesting observation about $S_\lambda f$ is that it automatically belongs to $Q^p(\mathbf{C}_-,\sigma)$, for the following reason. If $f_*\in\mathfrak{C}_*\mathcal{H}^p(\Sigma,\sigma)$ ($\mathcal{H}^p(\Sigma,\sigma)$ is the subspace of $\mathfrak{L}^p(\Sigma,\sigma)$ of canonical densities) is the canonical extension of the function f to the complex plane, then the function

$$S_\lambda f_*(z)=\frac{f_*(\lambda)-f_*(z)}{\lambda-z},\quad z\in\mathbf{C}_-\backslash\{\lambda\},$$

is an extension of $S_\lambda f$, which belongs to $L^1_\infty(\mathbf{C})$ (f_* does, see Propositions 2.7 and 3.3), and its $\bar\partial$ derivative is

$$\bar\partial S_\lambda f_*(z)=-\frac{\bar\partial f_*(z)}{\lambda-z},\quad z\in\mathbf{C}_-\backslash\{\lambda\},$$

which clearly is in $\mathfrak{L}^p(\Sigma,\sigma)$, because $\bar\partial f_*$ is, and the function $(\lambda-z)^{-1}$ is an element of the multiplier space $\mathfrak{M}(\Sigma)$. Moreover, we have the norm control

$$\|\bar\partial S_\lambda f_*\|_{\mathfrak{L}^p(\Sigma,\sigma)}\leqslant\frac{\|\bar\partial f_*\|_{\mathfrak{L}^p(\Sigma,\sigma)}}{|\operatorname{Im}\lambda|}=\frac{\|f\|_{Q^p(\mathbf{C}_-,\sigma)}}{|\operatorname{Im}\lambda|}.$$

Propositions 2.7 and 3.3, together with the definition of the norm in $Q^p(\mathbf{C}_-,\sigma)$, now assert that

$$\|S_\lambda f\|_{Q^p(\mathbf{C}_-,\sigma)}\leqslant\|\bar\partial S_\lambda f_*\|_{\mathfrak{L}^p(\Sigma,\sigma)}\leqslant\frac{\|f\|_{Q^p(\mathbf{C}_-,\sigma)}}{|\operatorname{Im}\lambda|},\tag{11.2}$$

from which it is immediate that $S_\lambda f$ is in $Q^p(\mathbf{C}_-,\sigma)$. Let $Z(f,\mathbf{C}_-)$ denote the zero set of the function f in $\mathbf{C}_-$. It is now claimed that for $f\in J$ and $\lambda\in\mathbf{C}_-\backslash Z(f,\mathbf{C}_-)$,

$$A_\lambda[J]^{-1}(E_i+J)=\frac{S_\lambda f}{f(\lambda)}+J\tag{11.3}$$

holds. We know that $A_\lambda[J]$ is invertible, because $\lambda\in\Omega(J)$, so it would be sufficient just to check that $A_\lambda S_\lambda f/f(\lambda)-E_i\in J$. But this is obviously true, by Proposition 6.5, because

$$\frac{A_\lambda(z)S_\lambda f(z)}{f(\lambda)}=E_i(z)-\frac{E_i(z)f(z)}{f(\lambda)},\quad z\in\mathbf{C}_-.$$

If we combine identity (11.3) with norm estimate (11.2) and the definition (10.5) of $\mathfrak{R}\phi$, we are left with the estimate

$$|\mathfrak{R}\phi(\lambda)|\leqslant\frac{C(\phi,f)}{|f(\lambda)|\cdot|\operatorname{Im}\lambda|},\quad\lambda\in\mathbf{C}_-\backslash Z(f,\mathbf{C}_-),\tag{11.4}$$

where $C(\phi,f)=\|\phi\|_{L^{p'}(\mathbf{R}_+,\omega_{\sigma,p}^{-1})}\|f\|_{Q^p(\mathbf{C}_-,\sigma)}$, and $f\in J$ is arbitrary.

12. The holomorphization process: $p=2$

In this section we provide (for $p=2$) the tool which enables us to prove the Main Theorem without resorting to the log-log theorem, which permits us to work with quasianalytic weights.

We proceed as in [5] to obtain more information about the resolvent transform $\mathfrak{R}\phi$ of $\phi \in L^2(\mathbf{R}_+, \omega_{\sigma,2}^{-1})$, which is an element perpendicular to the closed right translation invariant subspace I, $I \neq \{0\}$. We recall that J stands for the Fourier image $\mathfrak{F}(I)$ of I. At this point, it is convenient to make the assumption that $\sigma^{-2} \in \mathfrak{P}_{2s}$ for some s, $\frac{1}{2} < s < +\infty$, so that the Fourier image of $L^2(\mathbf{R}_+, \omega_{\sigma,2})$, $Q^2(\mathbf{C}_-, \sigma)$, will be a Banach algebra (Corollary 8.8), and (Proposition 10.8 and Remark 10.2) $\mathfrak{R}\phi$ is holomorphic on $\Omega(J) = \mathbf{C} \setminus Z(J, \overline{\mathbf{C}}_-)$.

Let f be a function in J which does not vanish identically, and let f_* denote the canonical extension of f to $\mathbf{C}$, which is an element of the space $\mathfrak{C}_* \mathcal{H}^2(\Sigma, \sigma)$ (see §2). By Proposition 2.7, $\bar{\partial} f_* \in \mathcal{H}^2(\Sigma, \sigma)$, and $f_* = \mathfrak{C}_* \bar{\partial} f_*$, and according to Corollary 8.5, we also have $f_* \in L^\infty(\mathbf{C}) \cap C(\mathbf{C})$, because of the assumption made on σ.

Let q and r be real numbers such that $2<q$, $0<r$, and (8.1) holds (just choose q sufficiently close to 2 to ensure that the right hand side of (8.1) is (strictly) positive). Proposition 8.4 then tells us that $\bar{\partial} f_* \in L^q(\Sigma, \sigma^r)$ (because $\bar{\partial} f_*$ is a canonical density), and that

$$\|\bar{\partial} f_*\|_{L^q(\Sigma, \sigma^r)} \leqslant C(s, \sigma, q, r) \|f\|_{Q^2(\mathbf{C}_-, \sigma)}. \tag{12.1}$$

Now, let $\tau: \mathbf{C} \to [0, +\infty[$ be defined by $\tau(z) = \sigma(\operatorname{Im} z)^{-1}$ for $z \in \Sigma$, and by $\tau(z) = 0$ for $z \in \mathbf{C} \setminus \Sigma$. For $\lambda \in \mathbf{C}$ and $\gamma \in]0, +\infty[$, $D(\lambda, \gamma)$ denotes the open disk centered at $\lambda \in \mathbf{C}$ with radius γ; we agree that $D(\lambda, 0) = \varnothing$ and $D(\lambda, +\infty) = \mathbf{C}$. Given a real parameter ε, $0<\varepsilon$, and a point $\lambda \in \mathbf{C}$, let $\gamma(\lambda, \varepsilon; f_*)$ be the largest number γ, $0 \leqslant \gamma \leqslant +\infty$, with the property that

$$|f_*(z)| \geqslant \max\{\tau(z)^r, \varepsilon\}, \quad z \in D(\lambda, \gamma), \tag{12.2}$$

holds. In the sequel, we shall assume that we have picked a point λ for which $0 < \gamma(\lambda, \varepsilon; f_*)$. Denote by χ_λ^ε the characteristic function of $D(\lambda, \gamma(\lambda, \varepsilon; f_*))$, and consider the function

$$F_\lambda^\varepsilon(z) = -\chi_\lambda^\varepsilon(z) \frac{\bar{\partial} f_*(z)}{f_*(z)}, \quad z \in \mathbf{C},$$

where we treat the right hand side as identically 0 off the disk $D(\lambda, \gamma(\lambda, \varepsilon; f_*))$. Since $0 < \varepsilon \leqslant |f_*|$ on $D(\lambda, \gamma(\lambda, \varepsilon; f_*))$ by (12.2), and $\bar{\partial} f_* \in \mathfrak{L}^2(\Sigma, \sigma)$, we have that F_λ^ε vanishes on $\mathbf{C} \setminus \Sigma$, and also that $F_\lambda^\varepsilon \in \mathfrak{L}^2(\Sigma, \sigma)$. If we use (12.2) and the properties of τ, we get

$$\int_\Sigma |F_\lambda^\varepsilon(z)|^q \, dS(z) \leqslant \int_\Sigma |\bar{\partial} f_*(z)|^q \sigma(\operatorname{Im} z)^{rq} \, dS(z),$$

so that if we invoke (12.1), we arrive at

$$\|F^{\varepsilon}_{\lambda}\|_{L^q(\Sigma)} \leqslant C(s,\sigma,q,r)\,\|f\|_{Q^2(\mathbf{C}_-,\sigma)}. \tag{12.3}$$

The global Cauchy transform $\mathfrak{C}_*F^{\varepsilon}_{\lambda}$ belongs to the space $\mathfrak{C}_*\mathfrak{L}^2(\Sigma,\sigma)$, and consequently, its restriction to $\mathbf{C}_-$, $\mathfrak{C}F^{\varepsilon}_{\lambda}$, is in $Q^2(\mathbf{C}_-,\sigma)$. By (12.3) and Lemma 8.2, $\mathfrak{C}_*F^{\varepsilon}_{\lambda}$ is in $L^{\infty}(\mathbf{C})\cap C(\mathbf{C})$, and its uniform norm is controlled by

$$\|\mathfrak{C}_*F^{\varepsilon}_{\lambda}\|_{L^{\infty}(\mathbf{C})} \leqslant C(s,\sigma,q,r)\,\|f\|_{Q^2(\mathbf{C}_-,\sigma)}, \tag{12.4}$$

where the constant appearing in (12.4) may be different from the one in (12.3). Let the function $G^{\varepsilon}_{\lambda}$ be defined by the relation

$$G^{\varepsilon}_{\lambda}(z) = \frac{1}{f_*(\lambda)}\exp(\mathfrak{C}_*F^{\varepsilon}_{\lambda}(z) - \mathfrak{C}_*F^{\varepsilon}_{\lambda}(\lambda)), \quad z\in\mathbf{C}.$$

This is a bounded and continuous function, and we would like to identify its $\bar\partial$ derivative. For this, we need the following lemma.

LEMMA 12.1. *Let $g\in C(\mathbf{C})$ have the property that $\bar\partial g\in L^1_{\mathrm{loc}}(\mathbf{C})$. Then $\bar\partial(\exp(g)) = \exp(g)\bar\partial g$ in the sense of distribution theory.*

Proof. Analogous to that of Lemma 8.7. □

We can now safely assert that

$$\bar\partial G^{\varepsilon}_{\lambda}(z) = G^{\varepsilon}_{\lambda}(z)F^{\varepsilon}_{\lambda}(z) = \frac{1}{f_*(\lambda)}\exp(\mathfrak{C}_*F^{\varepsilon}_{\lambda}(z) - \mathfrak{C}_*F^{\varepsilon}_{\lambda}(\lambda))F^{\varepsilon}_{\lambda}(z), \quad z\in\mathbf{C},$$

so that $\bar\partial G^{\varepsilon}_{\lambda}\in\mathfrak{L}^2(\Sigma,\sigma)$. Since $G^{\varepsilon}_{\lambda}$ is bounded, Proposition 2.7 therefore tells us that $G^{\varepsilon}_{\lambda}-\beta(G^{\varepsilon}_{\lambda})$ is in $\mathfrak{C}_*\mathfrak{L}^2(\Sigma,\sigma)$, where $\beta(G^{\varepsilon}_{\lambda})=\exp(-\mathfrak{C}_*F^{\varepsilon}_{\lambda}(\lambda))/f_*(\lambda)$ denotes "the value of $G^{\varepsilon}_{\lambda}$ at infinity". In particular, the restriction $G^{\varepsilon}_{\lambda}|_{\mathbf{C}_-}$ belongs to $Q^2_e(\mathbf{C}_-,\sigma)$. We have the uniform norm estimate

$$\|G^{\varepsilon}_{\lambda}\|_{L^{\infty}(\mathbf{C})} \leqslant \frac{1}{|f_*(\lambda)|}\exp(2\|\mathfrak{C}_*F^{\varepsilon}_{\lambda}\|_{L^{\infty}(\mathbf{C})}). \tag{12.5}$$

The function $G^{\varepsilon}_{\lambda}f_*$ is in $L^{\infty}(\mathbf{C})\cap C(\mathbf{C})$, and by Leibniz' rule, applicable by Lemma 8.7,

$$\begin{aligned}\bar\partial(G^{\varepsilon}_{\lambda}(z)f_*(z)) &= G^{\varepsilon}_{\lambda}(z)\bar\partial f_*(z) + f_*(z)\bar\partial G^{\varepsilon}_{\lambda}(z)\\ &= (\bar\partial f_*(z) + F^{\varepsilon}_{\lambda}(z)f_*(z))G^{\varepsilon}_{\lambda}(z) = (1-\chi^{\varepsilon}_{\lambda}(z))G^{\varepsilon}_{\lambda}(z)\bar\partial f_*(z), \quad z\in\mathbf{C}.\end{aligned}$$

It follows that $G^{\varepsilon}_{\lambda}f_*$ is holomorphic on $D(\lambda,\gamma(\lambda,\varepsilon;f_*))$, and by inspection,

$$G^{\varepsilon}_{\lambda}(\lambda)f_*(\lambda) = 1.$$

We conclude that the function given by the formula

$$H_\lambda^\varepsilon(z)=\frac{1-f_*(z)G_\lambda^\varepsilon(z)}{\lambda-z},\quad z\in\mathbf{C}\setminus\{\lambda\},\tag{12.6}$$

has a removable singularity at $z=\lambda$, and that it too is holomorphic on $D(\lambda,\gamma(\lambda,\varepsilon;f_*))$. We have moreover $H_\lambda^\varepsilon\in L^\infty(\mathbf{C})\cap C(\mathbf{C})$. On applying the $\bar\partial$-operator to the function H_λ^ε, we get the expression

$$\bar\partial H_\lambda^\varepsilon(z)=-G_\lambda^\varepsilon(z)\bar\partial f_*(z)\frac{1-\chi_\lambda^\varepsilon(z)}{\lambda-z},\quad z\in\mathbf{C}\setminus\{\lambda\},\tag{12.7}$$

so that since $\bar\partial f_*\in\mathfrak{L}^2(\Sigma,\sigma)$, the above formula shows that $\bar\partial H_\lambda^\varepsilon\in\mathfrak{L}^2(\Sigma,\sigma)$. *Identity* (12.7) *is what makes the whole construction tick!* For, as we compute the norm of $\bar\partial H_\lambda^\varepsilon$ in $\mathfrak{L}^2(\Sigma,\sigma)$, we are no longer concerned with the norm of $G_\lambda^\varepsilon|_{\mathbf{C}_-}$ in $Q_e^2(\mathbf{C}_-,\sigma)$, all that matters is its norm in the space of multipliers on densities, which in this case is $L^\infty(\Sigma)$. It is clear by (12.6) that H_λ^ε vanishes at infinity, whence $H_\lambda^\varepsilon\in\mathfrak{C}_*\mathfrak{L}^2(\Sigma,\sigma)$, in view of Proposition 2.7. Formula (12.7) permits us to make the estimate

$$\int_\Sigma|\bar\partial H_\lambda^\varepsilon(z)|^2\sigma(\operatorname{Im}z)^2\,dS(z)\leqslant\frac{\|G_\lambda^\varepsilon\|_{L^\infty(\Sigma)}^2}{\gamma(\lambda,\varepsilon;f_*)^2}\int_\Sigma|\bar\partial f_*(z)|^2\sigma(\operatorname{Im}z)^2\,dS(z),$$

or, using (2.5),

$$\|\bar\partial H_\lambda^\varepsilon\|_{\mathfrak{L}^2(\Sigma,\sigma)}\leqslant\frac{1}{\gamma(\lambda,\varepsilon;f_*)}\|G_\lambda^\varepsilon\|_{L^\infty(\Sigma)}\|f\|_{Q^2(\mathbf{C}_-,\sigma)}.\tag{12.8}$$

Estimates (12.4), (12.5), and (12.8) combine to yield the norm estimate

$$\begin{aligned}\|H_\lambda^\varepsilon|_{\mathbf{C}_-}\|&\leqslant\|\bar\partial H_\lambda^\varepsilon\|_{\mathfrak{L}^2(\Sigma,\sigma)}\\&\leqslant\frac{\|f\|_{Q^2(\mathbf{C}_-,\sigma)}}{\gamma(\lambda,\varepsilon;f_*)|f_*(\lambda)|}\cdot\exp(C(s,\sigma,q,r)\|f\|_{Q^2(\mathbf{C}_-,\sigma)}.\end{aligned}\tag{12.9}$$

The reason why we are interested in this function H_λ^ε is that if $\lambda\in\Omega(J)=\mathbf{C}\setminus Z(J,\overline{\mathbf{C}}_-)$, then

$$\mathfrak{R}\phi(\lambda)=\langle H_\lambda^\varepsilon,\hat\phi\rangle.\tag{12.10}$$

To see this, note that since $\lambda\in\Omega(J)$, the operator $A[J]$ is invertible (notation as in §10), and

$$A_\lambda[J]^{-1}(E_i+J)=H_\lambda^\varepsilon|_{\mathbf{C}_-}+J$$

holds, because

$$E_i(z)-A_\lambda(z)H_\lambda^\varepsilon(z)=f(z)G_\lambda^\varepsilon(z)E_i(z),\quad z\in\mathbf{C}_-,$$

so, since $f\in J$, and J is a closed ideal in $Q^2(\mathbf{C}_-,\sigma)$, we have $fG^\varepsilon_\lambda E_i\in J$. Now (12.10) follows from (10.5). It follows from (12.9) and (12.10) that

$$|\mathfrak{R}\phi(\lambda)|\leqslant\frac{C(\phi,f)}{\gamma(\lambda,\varepsilon;f_*)|f_*(\lambda)|}\cdot\exp(C(s,\sigma,q,r)\,\|f\|_{Q^2(\mathbf{C}_-,\sigma)}),\tag{12.11}$$

where $C(\phi,f)=\|\mathfrak{F}\phi\|_{Q^2(\mathbf{C}_-,\sigma)}\,\|f\|_{Q^2(\mathbf{C}_-,\sigma)}$. We recall that in the above estimate, f is an arbitrary nonzero element of J, $0<\varepsilon$, and $\lambda\in\Omega(J)\subset\mathbf{C}$ is a point for which $0<\gamma(\lambda,\varepsilon;f_*)$. The variable ε has all but disappeared in (12.11). If we set $\gamma(\lambda;f_*)=\sup\{\gamma(\lambda,\varepsilon;f_*):0<\varepsilon\}$, we obtain from (12.11) a simplified estimate:

$$|\mathfrak{R}\phi(\lambda)|\leqslant\frac{C(\phi,f)}{\gamma(\lambda;f_*)|f_*(\lambda)|}\cdot\exp(C(s,\sigma,q,r)\|f\|_{Q^2(\mathbf{C}_-,\sigma)}).\tag{12.12}$$

13. Modifications to the holomorphization process for p other than 2

Let us recall the general setting: φ is an arbitrary nonzero element of the norm closure of finite linear combinations of right translates of functions in the collection $\mathfrak{S}$ in $L^p(\mathbf{R}_+,\omega_{\sigma,p})$, and we put $f=\mathfrak{F}\varphi$, and $f_*=\mathfrak{F}^p_*\varphi$. In case $p=1$, we also need to assume that $\varphi\in L^1(\mathbf{R}_+,\omega_{\sigma,1})\cap L^\infty(\mathbf{R}_+,\omega_{\sigma,1})$, which is achieved by replacing φ with $\varphi*\psi$, where ψ is a compactly supported C^∞ function on $]0,+\infty[$, subject to the conditions $0\leqslant\psi$ and $\int_0^{+\infty}\psi(t)\,dt=1$. This smoothing of φ enables us to apply Proposition 9.2. It is clear that the new φ belongs to the same class of functions, and that it does not vanish almost everywhere. There is one additional modification of technical nature, which we should insert: we replace f_* with $\tilde f_*$,

$$\tilde f_*(z)=A(z)f_*(z),\quad z\in\mathbf{C},\tag{13.1}$$

where $A(z)$ equals $\left(z-\tfrac12 i\right)^{-2}$ off the disk $D\left(\tfrac12 i,\tfrac18\right)$ with radius $\tfrac18$ centered at $\tfrac12 i$, and is of class C^∞ in the whole complex plane. Thus, $\tilde f_*$ is small near ∞.

The function f is in $Q^p(\mathbf{C}_-,\sigma)$, f_* is in $\mathfrak{C}_*\mathfrak{L}^p(\Sigma,\sigma)$, the restriction of f_* to $\mathbf{C}_-$ is f, and f does not vanish identically, because φ is not the 0 element. The weight σ is assumed to be such that $1/\sigma^{p'}$ is in $\mathfrak{P}_{sp'}$ for some s, $2+1/p'<s<+\infty$, for $1<p<+\infty$; for $p=1$, σ is to conform with the requirement that $\omega_{\sigma,1}\in\mathfrak{W}_s$, where $3<s<+\infty$. Note that by the assumptions made on the size of the parameter s, Propositions 9.1 and 9.2 tell us that $\bar\partial f_*\in L^\infty(\Sigma,\sigma^{\theta_0})$ and $\partial_x\bar\partial f_*\in L^\infty(\Sigma,\sigma^{\theta_1})$, where $\frac12<\theta_0=1-(s-1/p')^{-1}$ and $0<\theta_1=1-2(s-1/p')^{-1}$ for $1<p<+\infty$, and $\frac13<\theta_0=1-2/s$ and $0<\theta_1=1-3/s$ for $p=1$.

The notation $D(\lambda,\gamma)$ and the function τ are defined as in §12. Suppose $\lambda\in\mathbf{C}$, $0<\gamma\leqslant\frac12$, and $0<\varepsilon$ are given, such that

$$|f_*(z)|\geqslant\max\{\gamma^{-1}\tau(z)^{\theta_0},\tau(z)^{\theta_0/2},\tau(z)^{\theta_1},\varepsilon\},\quad z\in D(\lambda,\gamma).\tag{13.2}$$

The function $\chi_\lambda \in C^\infty(\mathbf{C})$ is to be real-valued, and to have $0 \leqslant \chi_\lambda(z) \leqslant 1$ throughout $\mathbf{C}$, $\chi_\lambda(z)=0$ on $\mathbf{C}\setminus D(\lambda,\gamma)$, and $\chi_\lambda(z)=1$ on $D(\lambda, \frac{1}{2}\gamma)$. Clearly, a χ_λ can be found such that

$$|\nabla \chi_\lambda(z)| \leqslant 5/\gamma, \quad z \in \mathbf{C}, \tag{13.3}$$

where $\nabla=(\partial_x, \partial_y)$ is the gradient operator (where $z=x+iy$). As in §12, consider the function

$$F_\lambda(z) = -\chi_\lambda(z)\bar{\partial} f_*(z)/f_*(z), \quad z \in \mathbf{C},$$

where we treat the right hand side as identically 0 off the set $\Sigma \cap D(\lambda,\gamma)$. Since $0<\varepsilon\leqslant|f_*|$ on $D(\lambda,\gamma)$ by (13.2) and in addition, χ_λ/f_* is smooth enough to be in the space $\mathfrak{M}(\check{\Sigma})$ (to see this, use the results in §9), the fact that $\bar{\partial} f_* \in \mathfrak{L}^p(\Sigma,\sigma)$ entails that $F_\lambda \in \mathfrak{L}^p(\Sigma,\sigma)$. If we recall that $\bar{\partial} f_* \in L^\infty(\Sigma, \sigma^{\theta_0})$, and use (13.2), we get

$$\|F_\lambda\|_{L^\infty(\Sigma)} \leqslant \gamma \|\bar{\partial} f_*\|_{L^\infty(\Sigma,\sigma^{\theta_0})}. \tag{13.4}$$

We intend to show that the restriction to Σ of the global Cauchy transform $\mathfrak{C}_* F_\lambda$ is in $\mathfrak{M}(\check{\Sigma})$, and that we can control its norm in that space, with a bound that does not depend on the particular values of the parameters λ and γ. In §12, it was sufficient to estimate the supremum norm of $\mathfrak{C}_* F_\lambda$ on Σ, but here, we need more, due to the phenomenon that pops up in §7. To this end, the following lemma is useful.

LEMMA 13.1. *If $g \in L^2(\mathbf{R})$ and $g' \in L^2(\mathbf{R})$, then $g \in \mathfrak{F}L^1(\mathbf{R})$; in fact,*

$$\|g\|_{\mathfrak{F}L^1(\mathbf{R})} \leqslant 2^{-1/2}(\|g\|^2_{L^2(\mathbf{R})} + \|g'\|^2_{L^2(\mathbf{R})})^{1/2}.$$

Proof. By Parseval's identity, we have $g=\mathfrak{F}h$, where $h \in L^2(\mathbf{R})$, and $\|g\|_{L^2(\mathbf{R})} = (2\pi)^{1/2}\|h\|_{L^2(\mathbf{R})}$. The same argument applied to the derivative g' yields $\|g'\|_{L^2(\mathbf{R})} = (2\pi)^{1/2}\|t \mapsto th(t)\|_{L^2(\mathbf{R})}$. By the Cauchy–Schwarz–Bunyakovskiĭ inequality,

$$\begin{aligned}
\|g\|_{\mathfrak{F}L^1(\mathbf{R})} = \|h\|_{L^1(\mathbf{R})} &= \int_{-\infty}^{+\infty} |h(t)|\,dt \\
&\leqslant \left(\int_{-\infty}^{+\infty} (1+t^2)|h(t)|^2\,dt\right)^{1/2} \left(\int_{-\infty}^{+\infty} (1+t^2)^{-1}\,dt\right)^{1/2} \\
&= \pi^{1/2}(\|h\|^2_{L^2(\mathbf{R})} + \|t \mapsto th(t)\|^2_{L^2(\mathbf{R})})^{1/2}.
\end{aligned}$$

The proof is complete. □

PROPOSITION 13.2. *The restriction to* Σ *of* $\mathfrak{C}_*F_\lambda$ *is in* $\mathfrak{M}(\check{\Sigma})$ *for all pairs* (λ,γ) *such that* (13.2) *holds. Moreover, the norm is bounded by a constant independent of the particular* (λ,γ).

Proof. It is well-known that for functions $h\in L^\infty(\mathbf{C})$, with support contained in the closure of the disk $D(\lambda,\gamma)$ $\big($recall that $0<\gamma\leqslant\frac{1}{2}\big)$,

$$\|\mathfrak{C}_*h\|_{L^\infty(\mathbf{C})}\leqslant\|h\|_{L^\infty(\mathbf{C})}, \tag{13.5}$$

and

$$|\mathfrak{C}_*h(z)|\leqslant\frac{1}{4}\cdot\frac{\|h\|_{L^\infty(\mathbf{C})}}{d(z,D(\lambda,\gamma))},\quad z\in\mathbf{C}\setminus D(\lambda,\gamma). \tag{13.6}$$

Here, d is the usual Euclidean distance function. Thus, by (13.5) and (13.6), we can assert that

$$\sup\{\|\mathfrak{C}_*h(\cdot+iy)\|_{L^2(\mathbf{R})}:y\in\mathbf{R}\}\leqslant 2\|h\|_{L^\infty(\mathbf{C})}. \tag{13.7}$$

By Lemma 13.1,

$$\|\mathfrak{C}_*F_\lambda\|_{\mathfrak{M}(\check{\Sigma})}\leqslant\sup\{\|\mathfrak{C}_*F_\lambda(\cdot+iy)\|_{L^2(\mathbf{R})}+\|\partial_x\mathfrak{C}_*F_\lambda(\cdot+iy)\|_{L^2(\mathbf{R})}:0<y<1\}. \tag{13.8}$$

We already have good control of the first term of the expression on the right hand side, by (13.4), the fact that F_λ is supported in the closure of $D(\lambda,\gamma)$, and (13.7). To control the second term, we study ∂_xF_λ. Differential calculus gives us

$$\partial_xF_\lambda(z)=-\partial_x\chi_\lambda(z)\frac{\bar\partial f_*(z)}{f_*(z)}-\chi_\lambda(z)\frac{\partial_x\bar\partial f_*(z)}{f_*(z)}+\chi_\lambda(z)\partial_xf_*(z)\frac{\bar\partial f_*(z)}{f_*(z)^2}, \tag{13.9}$$

where the function ∂_xf_* is bounded, for the following reason. First of all, $\bar\partial f_*$ and $\partial_x\bar\partial f_*$ are bounded, and by the modification (13.1) of f_*, $\partial_x\bar\partial f_*$ is so small at infinity that we can apply the global Cauchy transform to it, and get $\partial_xf_*=\mathfrak{C}_*\partial_x\bar\partial f_*$ bounded. By (13.2) and (13.3), the right hand side of (13.9) is uniformly bounded, and the bound is independent of (λ,γ). The operator $\mathfrak{C}_*$ is of convolution type, whence it commutes with the differential operator ∂_x, $\partial_x\mathfrak{C}_*F_\lambda=\mathfrak{C}_*\partial_xF_\lambda$. The support of the function ∂_xF_λ is contained in the closure of $D(\lambda,\gamma)\cap\Sigma$, so that by (13.7), we have the desired control of the second term in (13.8). The proof is complete. □

The rest of the holomorphization process runs as in §12, except that where you use $L^\infty(\Sigma)$ in §12, it is usually necessary to replace it with $\mathfrak{M}(\check{\Sigma})$. For instance, in (12.7), the norm of $(1-\chi_\lambda(z))/(\lambda-z)$ in $\mathfrak{M}(\check{\Sigma})$ may be estimated,

$$\left\|\frac{1-\chi_\lambda}{\lambda-z}\right\|_{\mathfrak{M}(\check{\Sigma})}\leqslant 8\gamma^{-3/2},$$

by using (13.3) and Lemma 13.1. The resulting estimate of the resolvent transform is

$$|\mathfrak{R}\phi(\lambda)| \leqslant \frac{C(\phi,f)}{\gamma^{3/2}|f_*(\lambda)|}, \tag{13.10}$$

on the set of (λ,γ) with $\lambda\in\mathbf{C}$, $0<\gamma\leqslant\frac{1}{2}$, and (13.2) valid for some ε, $0<\varepsilon$. Although it is not indicated, the constant $C(\phi,f)$ may also depend on other parameters which are held constant, such as s, but not on (λ,γ).

14. The conclusion of the proof of the Main Theorem (Theorem 5.2)

We shall prove the statement as formulated in Theorem 5.2. By the results of §4, we may restrict ourselves to the case $\omega=\omega_{\sigma,p}$, with σ assumed to be such that $1/\sigma^{p'}$ is in $(p=2)$ $\mathfrak{P}_{2s}$ for some s, $\frac{1}{2}<s<+\infty$, $(1<p<+\infty, p\neq 2)$ $\mathfrak{P}_{sp'}$ for some s, $2+1/p'<s<+\infty$, and for $p=1$, σ is to conform with the requirement that $\omega_{\sigma,1}\in\mathfrak{W}_s$, with $3<s<+\infty$. The necessity of conditions (a) and (b) in Theorem 5.2 is clear, so we shall concentrate on proving their sufficiency. To this end, we use duality arguments. Assume that $\phi\in L^{p'}(\mathbf{R}_+,\omega_{\sigma,p}^{-1})$ annihilates $\mathfrak{T}_+(\mathfrak{S})$; we intend to show that every such ϕ must equal the 0 functional, which would entail that $\mathfrak{T}_+(\mathfrak{S})$ is dense in $L^p(\mathbf{R}_+,\omega_{\sigma,p})$. We shall in fact prove that $\mathfrak{R}\phi\equiv 0$, and use the fact that ϕ is uniquely determined by its resolvent transform $\mathfrak{R}\phi$ (this follows from the uniqueness theorem for Laplace transforms). Let us agree to write $I(\mathfrak{S})$ to denote the closure of $\mathfrak{T}_+(\mathfrak{S})$ in $L^p(\mathbf{R}_+,\omega_{\sigma,p})$, and let $J(\mathfrak{S})=\mathfrak{F}(I(\mathfrak{S}))$. By Proposition 10.8, formula (10.5) extends $\mathfrak{R}\phi$ to an entire function, because condition (a) may be written as $Z(J(\mathfrak{S}),\overline{\mathbf{C}}_-)=\varnothing$. For reasons of brevity, we carry out the remainder of the proof in the special case $p=2$ only. It is not difficult to generalize it to general p; in fact, the only place where it is necessary to make any essential modifations is in Lemma 14.3, where one could instead follow the line of argument in the proof of Theorem 8.2 [5].

As in §12, we choose real parameters q and r, subject to the conditions $2<q$, $0<r<1$, and (8.1). We also need θ, $0<\theta<r$, and $\beta=(s+\frac{1}{2})/(s-\frac{1}{2})>0$. Let $f\in\mathfrak{F}(\mathfrak{S})\subset Q^2(\mathbf{C}_-,\sigma)$ be a function which does not vanish identically on $\mathbf{C}_-$, and let f_* be its canonical extension to $\mathbf{C}$ (see §2), which is in $\mathfrak{C}_*\mathfrak{D}L^2(\mathbf{R}_+,\omega_{\sigma,2})$. Introduce the set

$$\mathcal{U}(f_*)=\{\lambda\in\mathbf{C}:0<\gamma(\lambda;f_*)\},$$

where $\gamma(\lambda;f_*)$ is as in §12. Contemplating for a moment on the definition of the function $\gamma(\lambda;f_*)$, we see that the set $\mathcal{U}(f_*)$ is open. In fact, for $\lambda\in\mathcal{U}(f_*)$, $\gamma(\lambda;f_*)$ expresses the distance from λ to the complement of $\mathcal{U}(f_*)$.

By (11.1) combined with Proposition 3.4, (11.4), and (12.12), we have the following estimates of the entire function $\mathfrak{N}\phi$:

$$|\mathfrak{N}\phi(\lambda)| \leqslant C(\phi)\hat{\sigma}(2, \operatorname{Im}\lambda)^{1/2}, \quad \lambda \in \mathbf{C}_+, \tag{14.1}$$

$$|\mathfrak{N}\phi(\lambda)| \leqslant C(\phi)\sigma(\operatorname{Im}\lambda)^{\beta}, \quad \lambda \in \Sigma, \tag{14.1'}$$

$$|\mathfrak{N}\phi(\lambda)| \leqslant \frac{C(f,\phi)}{|\operatorname{Im}\lambda|\cdot|f(\lambda)|}, \quad \lambda \in \mathbf{C}_-, \tag{14.2}$$

$$|\mathfrak{N}\phi(\lambda)| \leqslant \frac{C(f,\phi)}{\gamma(\lambda; f_*)|f_*(\lambda)|}, \quad \lambda \in \mathcal{U}(f_*). \tag{14.3}$$

Here, we no longer indicate when constants depend on the parameters s, σ, q, r, because they are kept fixed. Estimate (14.1) implies that $\mathfrak{N}\phi$ is bounded in every half-plane $\mathbf{C}_+ + i\varepsilon$, with $0<\varepsilon$. Estimate (14.2) entails that $\mathfrak{N}\phi$ belongs to the Nevanlinna class of holomorphic quotients of bounded analytic functions in every half-plane $\mathbf{C}_- - i\varepsilon$, with $0<\varepsilon$. By [21, pp. 184–185], condition (b) of Theorem 5.2 states on the Fourier transform side that to every δ, $0<\delta$, there exists an $f \in \mathfrak{F}(\mathfrak{S})$ which has

$$\limsup_{y\to+\infty} y^{-1}\log|f(-iy)| \geqslant -\delta.$$

Using this fact, it is not hard to verify using (14.2) that $\mathfrak{N}\phi$ belongs to the Smirnov class (which consists of quotients F/G, where F, G are bounded and holomorphic, and G is outer) in every half-plane $\mathbf{C}_- - i\varepsilon$, $0<\varepsilon$. We have in particular for all δ, $0<\delta$,

$$|\mathfrak{N}\phi(-iy)| = O(\exp(\delta y)), \quad \text{as } y \to +\infty. \tag{14.4}$$

Moreover, by (14.2) and Proposition 14.1 below, we also have

$$\log|\mathfrak{N}\phi(z)| \leqslant C(\phi, f)\frac{1+|z|^2}{|\operatorname{Im} z|}, \quad z \in \mathbf{C}_-, \tag{14.5}$$

for some constant $C(\phi, f)$, $0<C(\phi, f)<+\infty$. For a proof of Proposition 14.1, we refer to [5, Lemma 8.3].

PROPOSITION 14.1. *Let F and G be holomorphic in $\mathbf{C}_+$, with $F \not\equiv 0$, and F bounded on $\mathbf{C}_+$, and suppose*

$$|F(z)G(z)| \leqslant 1/\operatorname{Im} z, \quad z \in \mathbf{C}_+.$$

Then, for some constant $C(F)$, $0<C(F)<+\infty$, the following estimate holds:

$$\log|G(z)| \leqslant C(F)\frac{(1+|z|^2)}{\operatorname{Im} z} \leqslant C(F)\cdot(1+|z|^4+(\operatorname{Im} z)^{-2}), \quad z \in \mathbf{C}_+.$$

In order to be able to effectively apply estimate (14.3), we need to show that $\mathcal{U}(f_*)$ contains certain massive subsets; as it turns out, these will be rectangles. In the sequel, q, r, s, θ, σ, ϕ, and f are fixed ($f\in\mathfrak{F}(\mathfrak{S})$ is not identically 0), and dependence of constants on any of them (with the exception of f and ϕ) will not be indicated explicitly. For big positive integers k, say $k\geqslant k_0\geqslant 1$, it is always possible to find real numbers α_k, $0<\alpha_k\leqslant\frac{1}{2}$, such that

$$\sigma(\alpha_k)=\exp(e^{2k}/\theta),$$

because the condition $1/\sigma\in\mathfrak{P}_{2s}$, for some s, $\frac{1}{2}<s<+\infty$, implies that the continuous function σ is strictly decreasing, and that $\sigma(t)\to+\infty$ as $t\to 0$. The numbers α_k are actually uniquely determined by the above equation, and they form a strictly decreasing sequence, with limit $\alpha_k\to 0$ as $k\to+\infty$. Given that $k\geqslant k_0$, we introduce, for $j=0,\ldots,3^k-1$, the thin rectangles

$$\mathfrak{N}(j,k)=\{z\in\mathbf{C}:k+j3^{-k}<\operatorname{Re}z<k+(j+1)3^{-k},\ -\tfrac{1}{2}<\operatorname{Im}z<\alpha_k-2^{-k}\}.$$

We quote here Theorem 8.1 [5], specialized to the case $a=b=1$, $h=1/(2N+1)$; it should be mentioned that Semen Khavinson studied the same basic problem in [20].

PROPOSITION 14.2. *Let N be a positive integer, and let ε,δ be real parameters with $0<\varepsilon,\delta<1$. Suppose we have a finite sequence $\zeta_0,\ldots,\zeta_N$ of points in the upper half-plane $\mathbf{C}_+$, with the property that $\varepsilon\leqslant\operatorname{Im}\zeta_j\leqslant 1$, and*

$$\frac{2j}{2N+1}\leqslant\operatorname{Re}\zeta_j\leqslant\frac{2j+1}{2N+1},\quad j=0,\ldots,N.$$

Introduce the rectangles

$$R(\varepsilon)=\{z\in\mathbf{C}_+:0\leqslant\operatorname{Re}z\leqslant 1,\ \varepsilon\leqslant\operatorname{Im}z\leqslant 1\}.$$

There exists an absolute constant C such that the following holds: if F is a holomorphic function on $\mathbf{C}_+$ with $|F(z)|\leqslant 1$ on $\mathbf{C}_+$, and $|F(\zeta_j)|\leqslant\delta$ for all $j=0,\ldots,N$, then if $M_0=C\exp(78(N+2)\log(2N+1))$, we have the estimate

$$|F(z)|\leqslant(1+\delta M_0)\exp\left(-\tfrac{2}{5}(N+1)\varepsilon\cdot\operatorname{Im}z\right)+\delta M_0,\quad z\in R(\varepsilon).$$

We use Proposition 14.2 to show that $\mathcal{U}(f_*)$ contains many rectangles of the type $\mathfrak{N}(j,k)$.

LEMMA 14.3. *For all big integers k, say $k\geqslant k_1$ ($\geqslant 1$), having $\left(\frac{5}{6}\right)^k\leqslant\alpha_k$, there exists a j_k, $0\leqslant j_k\leqslant 3^k-1$, such that*

(a) $\exp(-e^{2k})=\sigma(\alpha_k)^{-\theta}\leqslant|f_*(z)|$, $z\in\mathfrak{N}(j_k,k)$.

It follows that $\mathfrak{N}(j_k,k)\subset\mathcal{U}(f_*)$, *whence*

(b) $d(z,\mathbf{C}_+\setminus\mathfrak{N}(j_k,k))\leqslant\gamma(z;f_*)$, $z\in\mathfrak{N}(j_k,k)$, *where* $d(z,E)$ *denotes the Euclidean distance from* $z\in\mathbf{C}$ *to a set* $E\subset\mathbf{C}$.

Proof. We first give some motivation. By the definition of τ in §12, the fact that σ decreases, and our assumption $0<\theta<r$, the inequality (12.2) is fulfilled for all $z\in\mathfrak{N}(j_k,k)$ with the choice $0<\varepsilon=\exp(-e^{2k})$, whence $\mathfrak{N}(j_k,k)\subset\mathcal{U}(f_*)$. Inequality (b) is then a consequence of our interpretation of $\gamma(z;f_*)$ as the distance from z to $\mathbf{C}\setminus\mathcal{U}(f_*)$.

We now turn to the main assertion, (a). Let $\mathfrak{K}$ be the set of all k for which the inequalities $\left(\frac{5}{6}\right)^k\leqslant\alpha_k$ and

$$\inf\{|f_*(z)|:z\in\mathfrak{N}(j,k)\}<\sigma(\alpha_k)^{-\theta}=\exp(-e^{2k}),\quad j=0,\dots 3^k-1,\tag{14.6}$$

both hold. It is maintained that $\mathfrak{K}$ is finite. The intuitive reason why this is true is that (14.6) forces f_* to be very small on all line segments $\{x\in\mathbf{R}:k\leqslant x\leqslant k+1\}$, with $k\in\mathfrak{K}$, which is not possible for the boundary values of a bounded holomorphic function (other than the 0 function), unless $\mathfrak{K}$ is finite.

Let $\Sigma_{(0,\alpha_k)}$ denote the infinite strip

$$\Sigma_{(0,\alpha_k)}=\{z\in\mathbf{C}:0<\operatorname{Im}z<\alpha_k\},$$

let H_k be the product of $\bar\partial f_*$ and the characteristic function of $\Sigma_{(0,\alpha_k)}$, and consider its global Cauchy transform

$$G_k(z)=\mathfrak{C}_*H_k(z)=\int_{\Sigma_{(0,\alpha_k)}}\frac{\bar\partial f_*(\zeta)}{z-\zeta}\,dS(\zeta),\quad z\in\mathbf{C}.$$

By Proposition 8.4, $\bar\partial f_*\in L^q(\Sigma,\sigma^r)$, with norm control

$$\|\bar\partial f_*\|_{L^q(\Sigma,\sigma^r)}\leqslant C\|f\|_{Q^2(\mathbf{C}_-,\sigma)},$$

whence

$$\|\bar\partial f_*\|_{L^q(\Sigma_{(0,\alpha_k)})}=\left(\int_{\Sigma_{(0,\alpha_k)}}|\bar\partial f_*(z)|^q\,dS(z)\right)^{1/q}\leqslant C\sigma(\alpha_k)^{-r}\|f\|_{Q^2(\mathbf{C}_-,\sigma)}.$$

We conclude from Lemma 8.2 that $G_k\in L^\infty(\mathbf{C})\cap C(\mathbf{C})$, and

$$\|G_k\|_{L^\infty(\mathbf{C})}\leqslant C\sigma(\alpha_k)^{-r}\|f\|_{Q^2(\mathbf{C}_-,\sigma)},$$

with a different constant than before. Because $0<\theta<r$, we can therefore safely assert that for sufficiently large k, say $k\geqslant k_2$, the estimate

$$|G_k(z)|\leqslant\sigma(\alpha_k)^{-\theta}=\exp(-e^{2k}),\quad z\in\mathbf{C},\tag{14.7}$$

holds. By (14.6), we can, for $k\in\mathfrak{K}$, and $j=0,...,3^k-1$, find points $\xi(j,k)$ in $\mathfrak{N}(j,k)$, such that

$$|f_*(\xi(j,k))|<\sigma(\alpha_k)^{-\theta}=\exp(-e^{2k}),$$

so that if we introduce the function $\tilde{f}_k=f_*-G_k$, and require $k\in\mathfrak{K}$ to meet $k\geqslant k_2$, then estimate (14.7) informs us that $\|\tilde{f}_k\|_{L^\infty(\mathbf{C})}\leqslant\|f_*\|_{L^\infty(\mathbf{C})}+1$, and

$$|\tilde{f}_k(\xi(j,k))|<2\sigma(\alpha_k)^{-\theta}=2\exp(-e^{2k}),\quad j=0,...,3^k-1.$$

The function $\tilde{f}_k$ has, by the construction of G_k, $\bar{\partial}\tilde{f}_k=0$ on the half-plane $\mathbf{C}_-+i\alpha_k$, and is therefore holomorphic there. For this reason, the function

$$F(z)=\frac{\tilde{f}_k(k+1-z+i\alpha_k)}{C(f)},\quad z\in\mathbf{C}_+,$$

where $C(f)=\|f_*\|_{L^\infty(\mathbf{C})}+1$, is holomorphic on $\mathbf{C}_+$, and its uniform norm there is bounded by 1. At this point, we find it convenient to make the additional normalizing assumption that $\|f_*\|_{L^\infty(\mathbf{C})}=1$, so that $C(f)=2$. If we plug in the parameter choices $N=\frac{1}{2}(3^k-1)$, $\varepsilon=2^{-k}$, and $\delta=\exp(-e^{2k})$ into Proposition 14.2, noting in passing that $M_0\leqslant C\exp(43k(3^k+3))$, where C is the absolute constant mentioned in Proposition 14.2, so that we must have $\delta M_0\leqslant\exp\left(-\frac{1}{2}e^{2k}\right)$, for large k, say $k\geqslant k_3$ (we use here the fact that $3<e^2$), we obtain, for real x, $k\leqslant x\leqslant k+1$, the estimate

$$|\tilde{f}_k(x)|=2|F(k+1-x+i\alpha_k)|\leqslant 4\exp\left(-\tfrac{1}{5}\left(\tfrac{3}{2}\right)^k\alpha_k\right)+2\exp\left(-\tfrac{1}{2}e^{2k}\right),$$

valid under the assumption on k that $k\in\mathfrak{K}$, and $k\geqslant k_4=\max\{k_2,k_3\}$. If we also use the property of $\mathfrak{K}$ that if $k\in\mathfrak{K}$, then $\left(\frac{5}{6}\right)^k\leqslant\alpha_k$, we get

$$|\tilde{f}_k(x)|\leqslant 6\exp\left(-\tfrac{1}{5}\left(\tfrac{5}{4}\right)^k\right),\quad k\leqslant x\leqslant k+1,$$

whence

$$|f_*(x)|\leqslant 7\exp\left(-\tfrac{1}{5}\left(\tfrac{5}{4}\right)^k\right),\quad k\leqslant x\leqslant k+1,$$

in view of (14.7), again for $k\in\mathfrak{K}$ with $k\geqslant k_4$. But if the set $\mathfrak{K}$ were infinite, this would then force

$$\int_{-\infty}^{+\infty}\frac{\log|f_*(t)|}{1+t^2}\,dt=-\infty,$$

which indeed is not possible for the boundary values of a bounded holomorphic function in $\mathbf{C}_-$, unless it vanishes identically. The set $\mathfrak{K}$ must therefore be finite. $\square$

We proceed with the proof of Theorem 5.2. Consider the auxiliary entire function

$$\Phi(z)=\exp(-z^6)\cdot\mathfrak{R}\phi(z),\quad z\in\mathbf{C},$$

and the infinite strip

$$\Sigma_{(-1/4,1/4)} = \{z \in \mathbf{C} : -\tfrac{1}{4} < \operatorname{Im} z < \tfrac{1}{4}\}.$$

Elementary computations show that we have the estimate

$$\tfrac{15}{16}(\operatorname{Re} z)^6 - \tfrac{1}{16} \leqslant \operatorname{Re}(z^6) \leqslant \tfrac{17}{16}(\operatorname{Re} z)^6 + \tfrac{1}{16}, \quad z \in \Sigma_{(-1/4,1/4)}, \tag{14.8}$$

so that by (14.1), and by (14.2) together with Proposition 14.1, the function Φ is bounded on the boundary $\partial\Sigma_{(-1/4,1/4)}$ of $\Sigma_{(-1/4,1/4)}$, and enjoys the estimate

$$|\Phi(z)| \leqslant \exp(C(1+|\operatorname{Im} z|^{-2})), \quad z \in \Sigma_{(-1/4,1/4)} \cap \mathbf{C}_-, \tag{14.9}$$

for some constant C, $0<C<+\infty$, which may depend on all the fixed parameters.

LEMMA 14.4. *If Φ is bounded on $\Sigma_{(-1/4,1/4)}$, then $\mathfrak{R}\phi(z) \equiv 0$.*

Proof. If Φ is bounded on $\Sigma_{(-1/4,1/4)}$, then the entire function $\mathfrak{R}\phi$ enjoys the following properties. It is bounded in the shifted upper half-space $\mathbf{C}_+ + \frac{1}{4}i$, and by (14.8), it satisfies the estimate

$$|\mathfrak{R}\phi(z)| \leqslant C\exp(2(\operatorname{Re} z)^6), \quad z \in \Sigma_{(-1/4,1/4)},$$

for some constant C, $0<C<+\infty$. Furthermore, from (14.5) we know that $\mathfrak{R}\phi$ is of slow growth in $\mathbf{C}_-$, which allows us to appeal to the Phragmén–Lindelöf principle for angles, and if we do this successively for various angles, we see that there exists a constant C, $0<C<+\infty$, such that

$$|\mathfrak{R}\phi(z)| = O(\exp(C|z|)), \quad \text{as } |z| \to \infty.$$

By (14.4), we may appeal once more to the Phragmén–Lindelöf for angles, to get that for each fixed δ with $0<\delta$,

$$|\mathfrak{R}\phi(z)| = O(\exp(\delta|z|)), \quad \text{as } |z| \to \infty,$$

holds. This growth is, however, insufficient for $\mathfrak{R}\phi$ not to be bounded in the whole complex plane, by the Phragmén–Lindelöf for half-planes, because $\mathfrak{R}\phi$ is known to be bounded in the shifted upper half-plane $\mathbf{C}_+ + \frac{1}{4}i$. Liouville's theorem then asserts that $\mathfrak{R}\phi$ must be constant. This constant must equal 0, for the following reason: by (14.1), and the fact that $\hat{\sigma}(2,t) \to 0$ as $t \to +\infty$, we have $\mathfrak{R}\phi(iy) \to 0$ as $y \to +\infty$. $\square$

Lemma 14.4 shows that to finish the proof of Theorem 5.2, we just need to check that Φ is bounded on the strip $\Sigma_{(-1/4,1/4)}$. To this end, let us introduce two sets of positive integers

$$\mathcal{X}=\left\{k\in\mathbf{Z}_+ : k_1\leqslant k,\, \left(\tfrac{5}{6}\right)^k\leqslant\alpha_k\right\},$$

and

$$\mathcal{Y}=\{k\in\mathbf{Z}_+ : k_1\leqslant k,\, \alpha_{2k}\leqslant\alpha_k-2^{1-k}\},$$

where k_1 is the positive integer that appears in Lemma 14.3. For convenience, we assume $k_1\geqslant 4$. For $k\in\mathcal{X}$, consider the line segments

$$\mathcal{I}_k=\left\{z\in\mathbf{C} : \operatorname{Re} z=k+\left(j_k+\tfrac{1}{2}\right)3^{-k},\ -\tfrac{1}{4}\leqslant\operatorname{Im} z\leqslant\alpha_k-2^{1-k}\right\},$$
$$\mathcal{J}_k=\left\{z\in\mathbf{C} : \operatorname{Re} z=k+\left(j_k+\tfrac{1}{2}\right)3^{-k},\ \alpha_k-2^{1-k}\leqslant\operatorname{Im} z\leqslant\tfrac{1}{4}\right\},$$

where $0\leqslant j_k\leqslant 3^k-1$ is as in the formulation of Lemma 14.3. Then, by Lemma 14.3, $\frac{1}{2}3^{-k}\leqslant\gamma(z;f_*)$ for all $z\in\mathcal{I}_k$ and $k\in\mathcal{X}$, and we conclude from (14.3) that

$$|\mathfrak{R}\phi(z)|\leqslant C\exp(2e^{2k}),\quad z\in\mathcal{I}_k, \tag{14.10}$$

for some constant C, $0<C<+\infty$, provided that $k\in\mathcal{X}$. If we apply (14.1′), using that σ is decreasing, we obtain for $k\in\mathcal{X}\cap\mathcal{Y}$ the estimate

$$|\mathfrak{R}\phi(z)|\leqslant C\sigma(\alpha_{2k})^{\beta}=C\exp(\beta e^{4k}/\theta),\quad z\in\mathcal{J}_k, \tag{14.11}$$

where $C=C(\phi)$ is as in (14.1′), and we as usual drop dependence on fixed parameters. If we combine our estimates (14.10) and (14.11), and use (14.8), we get for some constant C, $0<C<+\infty$,

$$\sup\{|\Phi(z)| : z\in\mathcal{K}_k\}\leqslant C\cdot\exp(\beta e^{4k}/\theta),\quad k\in\mathcal{X}\cap\mathcal{Y}, \tag{14.12}$$

where $\mathcal{K}_k$ denotes the line segment that is the union of $\mathcal{I}_k$ and $\mathcal{J}_k$:

$$\mathcal{K}_k=\mathcal{I}_k\cup\mathcal{J}_k=\left\{z\in\mathbf{C} : \operatorname{Re} z=k+\left(j_k+\tfrac{1}{2}\right)3^{-k},\ -\tfrac{1}{4}\leqslant\operatorname{Im} z\leqslant\tfrac{1}{4}\right\}.$$

It was observed earlier that the entire function Φ is bounded on the boundary of $\Sigma_{(-1/4,1/4)}$. If the number of integers in $\mathcal{X}\cap\mathcal{Y}$ is infinite, then the growth that estimate (14.12) permits in $\Sigma_{(-1/4,1/4)}$ is too small for Φ to be unbounded in the part of

$\Sigma_{(-1/4,1/4)}$ that lies in the right half-plane (because $4<2\pi$); the critical growth rate required by the Phragmén–Lindelöf principle in the strip $\Sigma_{(-1/4,1/4)}$ is in fact of the order of magnitude $\exp(\exp(2\pi z))$. We conclude that Φ is bounded on the portion of $\Sigma_{(-1/4,1/4)}$ that lies in the right half-plane, provided $\mathcal{X}\cap\mathcal{Y}$ is not finite. We proceed to prove the following.

Claim. If the set $\mathcal{X}\cap\mathcal{Y}$ finite, then $\mathcal{X}$ is finite, too.

We argue as follows. Since $\mathcal{X}\cap\mathcal{Y}$ is finite, there exists an integer $l\in\mathcal{X}\cap\mathcal{Y}$ such that $k\leqslant l$ for all (other) $k\in\mathcal{X}\cap\mathcal{Y}$. If $\mathcal{X}$ contains an integer k which is bigger than l, then k cannot belong to $\mathcal{Y}$, and hence

$$\left(\tfrac{5}{6}\right)^{2k} < \left(\tfrac{5}{6}\right)^{k} - 2^{1-k} \leqslant \alpha_k - 2^{1-k} < \alpha_{2k},$$

because $4\leqslant k$. We see that $2k$ also belongs to $\mathcal{X}$, and $2k$ being bigger than l, we must have $2k\in\mathcal{X}\setminus\mathcal{Y}$. If we continue inductively, it follows that for $n=0,1,2,\ldots$, we have $2^n k\in\mathcal{X}\setminus\mathcal{Y}$, and we also get the inequality

$$\alpha_k - 2/(2^k-1) < \alpha_k - 2\cdot(2^{-k}+2^{-2k}+\ldots+2^{-2^n k}) < \alpha_{2^n k}.$$

Since $k\in\mathcal{X}$ and $4\leqslant k$, we have

$$0 < \left(\tfrac{5}{6}\right)^{k} - 2/(2^k-1) \leqslant \alpha_k - 2/(2^k-1),$$

which is absurd together with the previous inequality, because we know that $\alpha_{2^n k}\to 0$ as $n\to\infty$. This shows that our assumption concerning the existence of a $k\in\mathcal{X}$ bigger than l must be wrong. Therefore, the set $\mathcal{X}$ is also finite. □

We finally treat the case when $\mathcal{X}$ is finite. Then $\alpha_k < \left(\tfrac{5}{6}\right)^k$, and its immediate consequence,

$$\sigma\left(\left(\tfrac{5}{6}\right)^k\right) < \sigma(\alpha_k) = \exp(e^{2k}/\theta),$$

hold for all but finitely many positive integers k. This last inequality implies that

$$\sigma(t) \leqslant \exp(-1/t^{11}), \quad 0<t<\varepsilon_0,$$

holds for some small but positive real number ε_0. The weight σ is thus of non-quasi-analytic type, so by the estimates (14.1′) and (14.9), the log-log theorem applies to the function Φ, and proves that it is bounded in the whole strip $\Sigma_{(-1/4,1/4)}$.

We conclude that the function Φ is bounded at least on the portion of the strip $\Sigma_{(-1/4,1/4)}$ that lies in the right half-plane, no matter whether $\mathcal{X}\cap\mathcal{Y}$ is finite or not. However, the right half-plane plays no special role here. Lemma 14.3 has a version with rectangles approaching infinity in the left half-plane, or if you prefer, it is possible to apply it as it stands to the function $\bar{f}_*(-\bar{z})$, and the conclusion is that $|f_*|$ is reasonably big on a union of rectangles moving toward infinity in the left half-plane. All the computations we have made for the right half-plane carry through analogously, and show that Φ is bounded on the portion of $\Sigma_{(-1/4,1/4)}$ that lies in the left half-plane as well. The function Φ is therefore bounded on the whole infinite strip $\Sigma_{(-1/4,1/4)}$.

In view of Lemma 14.4, the proof of Theorem 5.2 is now complete. □

Remark 14.5. For $p=2$, there is an alternative way of getting the holomorphic continuation of the resolvent transform and estimates similar to (14.1)–(14.3), without appealing to the Gelfand theory of commutative Banach algebras. The argument is difficult to generalize to other p, but works very smoothly for this particular value of p. It runs as follows. Suppose, to begin with, that $\phi\in L^2(\mathbf{R}_+,\omega_{\sigma,2}^{-1})$ is arbitrary. By the isometry in Lemma 2.2, its resolvent transform satisfies

$$\int_\Sigma |\mathfrak{R}\phi(z)|^2\sigma(\operatorname{Im} z)^{-2}\,dS(z)<+\infty,$$

as is seen by noting that $\mathfrak{R}\phi=\mathfrak{A}^2 f$, with $f(t)=\phi(t)/\omega_{\sigma,2}(t)$. Thus, if $\tilde{g}$ is a density in $\mathfrak{L}^2(\Sigma,\sigma)$, the product $\mathfrak{R}\phi\cdot\tilde{g}$ belongs to $L^1(\Sigma)$, and, moreover,

$$\int_0^{+\infty}\mathcal{E}\tilde{g}(t)\phi(t)\,dt=-\frac{1}{\pi}\int_\Sigma \mathfrak{R}\phi(z)\tilde{g}(z)\,dS(z). \tag{14.13}$$

Let $f\in Q^2(\mathbf{C}_-,\sigma)$, and write f_* for its canonical extension to the whole complex plane, so that $f_*(z)=0$ for $1<\operatorname{Im} z$, and $\bar{\partial}f_*$ is the associated density in $\mathfrak{L}^2(\Sigma,\sigma)$. Let $\hat{\phi}$ be the bounded linear functional on $Q^2(\mathbf{C}_-,\sigma)$ induced by ϕ via the Fourier transform, as in §10. Then, by the identity (14.13),

$$\langle f,\hat{\phi}\rangle=-\frac{1}{\pi}\int_\Sigma \mathfrak{R}\phi(z)\bar{\partial}f_*(z)\,dS(z).$$

Moreover, with $E_\lambda(z)=(\lambda-z)^{-1}$, the function $E_\lambda(z)f_*(z)$ has no singularity at $z=\lambda$ provided that $1<\operatorname{Im}\lambda$ (recall that $f_*(z)=0$ for $1<\operatorname{Im} z$), so that it is the global Cauchy transform of its density $\bar{\partial}(E_\lambda f_*)=E_\lambda\bar{\partial}f_*$. If we again appeal to (14.13), we see that

$$\langle E_\lambda f,\hat{\phi}\rangle=-\frac{1}{\pi}\int_\Sigma \mathfrak{R}\phi(z)E_\lambda(z)\bar{\partial}f_*(z)\,dS(z),\quad 1<\operatorname{Im}\lambda. \tag{14.14}$$

If, as in §12, $\hat{\phi}$ annihilates an invariant subspace J in $Q^2(\mathbf{C}_-,\sigma)$, and $f\in J$, then (14.14) shows that the global Cauchy transform $\mathfrak{C}_*(\mathfrak{R}\phi\,\bar{\partial}f_*)$ vanishes on $\{z\in\mathbf{C}:1<\operatorname{Im} z\}$. The function $\mathfrak{C}_*(\mathfrak{R}\phi\,\bar{\partial}f_*)$ belongs to $\mathfrak{C}_*L^1(\Sigma)$, and hence in particular to $L^r_\infty(\mathbf{C})$, for every

$r<2$. The functions $\mathfrak{R}\phi\cdot f_*$ and $\mathfrak{C}_*(\mathfrak{R}\phi\,\bar\partial f_*)$ have the same $\bar\partial$ on $\mathbf{C}_+$, and both vanish on $\{z\in\mathbf{C}:1<\operatorname{Im} z\}$, so they must coincide area-almost everywhere in $\mathbf{C}_+$. To get the holomorphic continuation, we assume f_* is continuous on $\mathbf{C}$, and that $\bar\partial f_*\in L^q(\Sigma)$ for some q, $2<q<+\infty$. Declare $\mathfrak{R}\phi(z)=\mathfrak{C}_*(\mathfrak{R}\phi\,\bar\partial f_*)(z)/f_*(z)$ off the zero set $Z(f_*)$ of f_*, which then is a function in L^q locally on $\mathbf{C}\setminus Z(f_*)$. Since $\mathfrak{C}_*(\mathfrak{R}\phi\,\bar\partial f_*)$ is in $L^{q'}_{\infty}(\mathbf{C})$, we can apply elementary distribution theory arguments analogous to those of Lemma 8.7, to obtain $\bar\partial\mathfrak{R}\phi(z)=0$ througout $\mathbf{C}\setminus Z(f_*)$. This gives us the analytic continuation; size estimates of $\mathfrak{R}\phi(\lambda)$ are obtained by using the mean value property of holomorphic functions on disks, to get an estimate of $\mathfrak{R}\phi(\lambda)$ in terms of the L^1 norm of $\mathfrak{C}_*(\mathfrak{R}\phi\,\bar\partial f_*)$ on the disk in question.

Remark 14.6. As the referee kindly pointed out to us, it is possible to replace the operator-theoretic considerations in §10 with purely analytical arguments based on the holomorphization process. In fact, the holomorphization process supplies us with an explicit extension of the resolvent transform $\mathfrak{R}\phi$, so it is not surprising that it can be used to show that the extension is holomorphic. We explain the details in the setting of §12. Fix a point $\lambda_0\in\mathbf{C}$ such that $f_*(\lambda_0)\neq 0$, and if necessary, multiply f by a suitable constant multiple so that (12.2) holds for some radius γ, $0<\gamma$. To simplify the notation, let us write $f_{\lambda_0}(z)=f_*(z)G^{\varepsilon}_{\lambda_0}(z)$, and note that the restriction of f_{λ_0} to $\mathbf{C}_-$ belongs to J, because f does. The function f_{λ_0} is holomorphic and zero-free on the small disk $D(\lambda_0,\gamma)$ centered at λ_0. If we write

$$\widetilde{H}_\lambda(z)=\frac{1-f_{\lambda_0}(z)/f_{\lambda_0}(\lambda)}{\lambda-z},\quad z\in\mathbf{C},$$

then $\widetilde{H}_\lambda$ is holomorphic in λ on the disk $D(\lambda_0,\gamma)$, and just as in §12, $\mathfrak{R}\phi(\lambda)=\langle\widetilde{H}_\lambda,\hat\phi\rangle$ holds. The assertion about analytic continuation follows. Since the regularity assumptions on the weight σ for the holomorphization process (§§ 12 and 13) are generally more restrictive than those of §10, no gain is made in that respect with this method.

Appendix A. Moment problems, part I

The problem to characterize the class of functions $F_{d\mu}$ on $\mathbf{R}_+$ that arise from a finite positive Borel measure μ on $\mathbf{R}_+$ with finite moments

$$\int_0^{+\infty} t^n\,d\mu(t)<+\infty,\quad n=0,1,2,\ldots,$$

via the moment-type formula

$$F_{d\mu}(x)=\int_0^{+\infty} e^{-tx}\,d\mu(t),\quad x\in\mathbf{R}_+,$$

was solved by Sergeĭ Bernshteĭn [1]. The description is in terms of total monotonicity; a function f on the real line is said to be totally monotone provided that $(-1)^n f^{(n)}(x) \geqslant 0$ for all real x in the relevant interval and all $n=0,1,2,\ldots$.

PROPOSITION A.1. *Let μ be a positive finite Borel measure on the real line, supported on a compact interval. We denote by $\operatorname{supp}\mu$ the support of the measure μ, and by $\alpha(\mu)$ and $\beta(\mu)$ the infimum and supremum of the set $\operatorname{supp}\mu$, respectively. The function*

$$F_{d\mu}(x) = \int_{-\infty}^{+\infty} e^{-tx}\, d\mu(t), \quad x \in \mathbf{R},$$

is then the restriction to the real line of an entire function of finite exponential type, it is positive, and the function $G_\mu = -\log F_{d\mu}$ is concave. We also have, with $G'_\mu(\infty)$ and $G'_\mu(-\infty)$ denoting the asymptotical slopes of the concave function G_μ at plus and minus infinity, respectively,

$$G'_\mu(\infty) = \lim_{x\to+\infty} G'_\mu(x) = \lim_{x\to+\infty} x^{-1} G_\mu(x) = \alpha(\mu),$$

and

$$G'_\mu(-\infty) = \lim_{x\to-\infty} G'_\mu(x) = \lim_{x\to-\infty} x^{-1} G_\mu(x) = \beta(\mu).$$

We finally note that the function $x \mapsto \log F_{d\mu}(x) + \alpha(\mu)x$ decreases on the whole real line $\mathbf{R}$.

Proof. The measure μ is supported on the interval $[\alpha(\mu), \beta(\mu)]$, and by definition, this is the smallest closed interval with that property. The formula

$$F_{d\mu}(z) = \int_{0}^{+\infty} e^{-tz}\, d\mu(t), \quad z \in \mathbf{C},$$

defines an entire function of finite exponential type, and the type is the maximum of the two numbers $|\alpha(\mu)|$ and $|\beta(\mu)|$. In fact, if $\gamma(\mu)$ denotes this maximum, then

$$|F_{d\mu}(z)| \leqslant C\cdot \exp(\gamma(\mu)|\operatorname{Re} z|), \quad z \in \mathbf{C},$$

holds for some constant C, $0<C<+\infty$. The function $F_{d\mu}$ has the property that

$$|F_{d\mu}(z)| \leqslant F_{d\mu}(\operatorname{Re} z), \quad z \in \mathbf{C},$$

and it is bounded in every region

$$\{z \in \mathbf{C} : a < \operatorname{Re} z < b\}$$

with a, b, $a<b$, arbitrary, so an application of the corollary following Theorem 12.8 [27, p. 275] yields that $G_\mu = -\log F_{d\mu}$ is concave on all of $\mathbf{R}$. The asymptotics of G_μ near infinity follows from [21, pp. 184–187]. That the function $x \mapsto \log F_{d\mu}(x) + \alpha(\mu)x$ decreases on the whole real line $\mathbf{R}$ is a consequence of the formula

$$\exp(\alpha(\mu)x)F_{d\mu}(x) = \int_0^{+\infty} e^{-tx}\, d\mu(t+\alpha(\mu)),$$

valid because the measure $d\mu(t+\alpha(\mu))$ is supported on the interval $[0, \beta(\mu)-\alpha(\mu)]$. □

THEOREM A.2. *Let ν be a positive concave function on $\mathbf{R}_+$ with $\nu'(0)<+\infty$. We can then find a finite compactly supported positive Borel measure μ on $\mathbf{R}_+$, such that*

$$\tfrac{1}{3} < e^{\nu(x)} \int_0^{+\infty} e^{-xt}\, d\mu(t) < 3, \quad x \in \mathbf{R}_+.$$

If $\nu'(\infty)$ denotes the slope of the function ν at infinity,

$$\nu'(\infty) = \lim_{t\to+\infty} \nu'(t) = \lim_{t\to+\infty} \nu(t)/t \geqslant 0,$$

the measure μ may be chosen to have support contained in the interval $[\nu'(\infty), \nu'(0)]$.

Proof. Without loss of generality, we may assume ν to be of class C^1 on $\mathbf{R}_+$. Consider the tangent line function

$$L(x,t) = (x-t)\nu'(t) + \nu(t), \quad (x,t) \in \mathbf{R}_+ \times \mathbf{R}_+,$$

and note that by the concavity of the function ν, we have

$$\nu(x) \leqslant L(x,t), \quad (x,t) \in \mathbf{R}_+ \times \mathbf{R}_+.$$

Let λ, $0<\lambda$, be a real parameter, to be determined later. We now introduce two sequences of points t_j, s_j in $\mathbf{R}_+$, with $t_0 < s_1 < t_1 < s_2 < t_2 < s_3 < \ldots$, via an iterative process. These sequences may be finite or infinite, depending on the outcome of the process. The starting point is $t_0 = 0$, and the other t_j's and s_j's are defined recursively by $(j=0,1,2,\ldots)$

(i) $L(s_{j+1}, t_j) - \nu(s_{j+1}) = \lambda$, and

(ii) $L(s_{j+1}, t_{j+1}) - \nu(s_{j+1}) = \lambda$;

(i) and the assumption $t_j < s_{j+1}$ determine s_{j+1} from t_j, and (ii) with $s_{j+1} < t_{j+1}$ gives t_{j+1} from s_{j+1}. Geometrically, condition (i) determines s_{j+1} uniquely as the point to the right of t_j where the tangent line $y = L(x, t_j)$ deviates vertically (upward) by λ units from the curve $y = \nu(x)$. Moreover, condition (ii) determines t_{j+1} uniquely as the point to the right of s_{j+1} whose associated tangent line exceeds the curve $y = \nu(x)$ by λ units at the point $x=s_{j+1}$. There are two possible obstacles in this construction: (a) it is possible that the solution s_{j+1} to (i) or the solution t_{j+1} to (ii) is not unique, and (b) it is also possible that the iterative process stops, which happens if for some index j, there is no solution s_{j+1} or t_{j+1} to (i) or (ii), respectively. If (a) occurs, this would mean that the curve ν is affine on some segment, in which case the set of solutions forms a closed interval. To make the numbers s_{j+1} and t_{j+1} unique, we choose them as the leftmost point on the respective solution interval. As to whether (b) occurs, we separate between the following cases.

Case I. The iterative process continues indefinitely.

Case II. The iterative process comes to a stop.

We first deal with Case I, and later indicate what modifications have to be made in order to handle Case II. Note that the numbers t_j and s_j cannot tend to a finite limit as $j\to+\infty$, because in this case (i) and (ii) lead to a contradiction. Taking into account the positivity and concavity of the function ν, and the construction of the sequences $\{t_j\}_j$ and $\{s_j\}_j$, we see that

$$0\leqslant \nu'(t_{j+1})<\nu'(s_{j+1})<\nu'(t_j), \quad j=0,1,2,\ldots . \tag{A.1}$$

By the definition of the tangent line function $L(x,t)$ and the concavity of ν, the function

$$y=L(x,t)-\nu(x), \quad x\in\mathbf{R}_+,$$

is convex, attains the value 0 at $x=t$, decreases on the interval $0\leqslant x<t$, and increases on the interval $t<x<+\infty$. One consequence of this is that for $j=1,2,3,\ldots$,

$$\nu(x)\leqslant L(x,t_j)\leqslant \nu(x)+\lambda, \quad s_j\leqslant x\leqslant s_{j+1}; \tag{A.2}$$

actually, (A.2) holds for $j=0$ as well, if we accept the definition $s_0=0$. Another consequence is that for $j=0,1,2,\ldots$,

$$\lambda+\nu(x)\leqslant L(x,t_j), \quad s_{j+1}<x<+\infty,$$

and since $L(x,x)=\nu(x)$, we have in particular

$$\lambda+L(t_{j+1},t_{j+1})\leqslant L(t_{j+1},t_j),$$

from which the estimate

$$\lambda+\nu(x)\leqslant \lambda+L(x,t_{j+1})\leqslant L(x,t_j), \quad t_{j+1}\leqslant x<+\infty, \tag{A.3}$$

is immediate, because the slope of the line $x \mapsto L(x,t_j)$ is bigger than that of $x \mapsto L(x,t_{j+1})$ (this is simply the statement that $\nu'(t_{j+1}) < \nu'(t_j)$), by (A.1). If we iterate (A.3) m times, we arrive at

$$m\lambda+\nu(x) \leqslant m\lambda+L(x,t_{j+m}) \leqslant L(x,t_j), \quad t_{j+m} \leqslant x < +\infty. \tag{A.4}$$

An argument analogous to the one used to obtain (A.3) leads to

$$\lambda+\nu(x) \leqslant \lambda+L(x,t_j) \leqslant L(x,t_{j+1}), \quad 0 \leqslant x \leqslant t_j,$$

and iterated n times, this estimate becomes

$$n\lambda+\nu(x) \leqslant n\lambda+L(x,t_j) \leqslant L(x,t_{j+n}), \quad 0 \leqslant x \leqslant t_j. \tag{A.5}$$

Consider the functions

$$E_j(x) = \exp(-L(x,t_j)), \quad x \in \mathbf{R},$$

which we shall find it advantageous to write in the form

$$E_j(x) = a_j \exp(-b_j x), \quad x \in \mathbf{R},$$

where $b_j = \nu'(t_j) \geqslant 0$ and

$$a_j = \exp(-L(0,t_j)) = \exp(t_j \nu'(t_j) - \nu(t_j)) > 0.$$

By (A.1), the sequence $\{b_j\}_j$ is strictly decreasing. We may also add, since we are in Case I, that $b_j > 0$, because if we had $b_j = \nu'(t_j) = 0$ for some index j, then the iterative process would come to an end in the next iteration. The function

$$E(x) = \sum_{j=0}^{+\infty} E_j(x), \quad x \in \mathbf{R},$$

will prove essential to us; in fact, our next step is to obtain the estimate

$$e^{-\lambda} \leqslant \exp(\nu(x)) E(x) < 2(1-e^{-\lambda})^{-1}, \quad x \in \mathbf{R}_+. \tag{A.6}$$

By (A.2),

$$\exp(-\lambda-\nu(x)) \leqslant E_j(x), \quad s_j \leqslant x \leqslant s_{j+1},$$

and consequently,

$$\exp(-\lambda-\nu(x)) \leqslant \sum_{j=0}^{\infty} E_j(x) = E(x), \quad x \in \mathbf{R}_+,$$

which is the left hand side inequality of (A.6). If we rewrite (A.4) and (A.5) in terms of the functions $E_j(x)$, we have, for $m=0,1,\dots,k$,

$$E_{k-m}(x) \leqslant e^{-m\lambda}\exp(-\nu(x)), \quad t_k \leqslant x < +\infty, \tag{A.7}$$

and for $n=0,1,2,\dots$,

$$E_{k+n+1}(x) \leqslant e^{-n\lambda}\exp(-\nu(x)), \quad 0 \leqslant x < t_{k+1}. \tag{A.8}$$

To get the remaining inequality of (A.6), we invoke (A.7) and (A.8), and obtain, for x in the interval $[t_k, t_{k+1}]$,

$$\begin{aligned} E(x) &= \sum_{j=0}^{+\infty} E_j(x) = \sum_{m=0}^{k} E_{k-m}(x) + \sum_{n=0}^{+\infty} E_{k+n+1}(x) \\ &\leqslant \exp(-\nu(x)) \sum_{m=0}^{k} e^{-m\lambda} + \exp(-\nu(x)) \sum_{n=0}^{+\infty} e^{-n\lambda} < 2(1-e^{-\lambda})^{-1}\exp(-\nu(x)). \end{aligned} \tag{A.9}$$

This completes the verification of (A.6). We now plug in the numerical value $\lambda=\log 2$, and see that (A.6) simplifies to

$$\tfrac{1}{2} \leqslant \exp(\nu(x))E(x) < 4, \quad x \in \mathbf{R}_+,$$

that is,

$$2^{-3/2} \leqslant 2^{1/2}\exp(\nu(x))E(x) < 2^{3/2}, \quad x \in \mathbf{R}_+.$$

The assertion of the lemma,

$$\tfrac{1}{3} < \exp(\nu(x)) \int_0^{+\infty} e^{-xt}\, d\mu(t) < 3, \quad x \in \mathbf{R}_+,$$

is now evident if we put

$$d\mu(x) = 2^{1/2} \sum_{k=0}^{+\infty} a_k\, d\delta(x-b_k),$$

where $d\delta$ stands for the Dirac unit point mass at the origin. The statement on the support of μ is an immediate consequence of this definition, if we note that $b_0=\nu'(0)$, and that the sequence $b_j=\nu'(t_j)$ decreases down to the slope at infinity $\nu'(\infty)$. This completes the proof in Case I.

We now turn to Case II. This time the iterative process defined by (i) and (ii) stops for some index N, which means that we have been able to find the numbers $t_1, t_2, \dots, t_N$, but unable to get t_{N+1}. This can happen in two ways, one is that (i) fails to deliver an s_{N+1}, in which case ν is affine on the interval $[t_N,+\infty[$, or, expressed differently,

$$\nu(x)=L(x,t_N),\quad t_N\leqslant x<+\infty,$$

and we then write $s_{N+1}=+\infty$. The computations carried out in Case I cover the present situation as well, with the obvious necessary modifications, such as the definition

$$E(x)=\sum_{j=0}^{N}E_j(x),\quad x\in\mathbf{R},$$

and they show that with $\lambda=\log 2$,

$$\tfrac12\leqslant\exp(\nu(x))E(x)<4,\quad x\in\mathbf{R}_+,$$

from which it is immediate that

$$\tfrac13<\exp(\nu(x))\int_0^{+\infty}e^{-xt}\,d\mu(t)<3,\quad x\in\mathbf{R}_+,$$

holds with

$$d\mu(x)=2^{1/2}\sum_{k=0}^{N}a_k\,d\delta(x-b_k).$$

The other way for the iterative process to stop is that (ii) fails to deliver a t_{N+1} out of s_{N+1}. We shall now see that this can happen only if ν is almost affine near infinity, that is, with a higher level of precision,

$$0\leqslant L_{N+1}(x)-\nu(x)\leqslant\lambda,\quad s_{N+1}\leqslant x<+\infty,\tag{A.10}$$

where

$$L_{N+1}(x)=\nu(s_{N+1})+(x-s_{N+1})\nu'(\infty)+\lambda,\quad x\in\mathbf{R},$$

is the "tangent line at infinity". Consider the function

$$\eta(x)=L_{N+1}(x-s_{N+1})-\nu(x-s_{N+1}),\quad x\in[0,+\infty[,$$

and note that what is claimed is that $0\leqslant\eta(x)\leqslant\lambda$ on $[0,+\infty[$. It is convex, has slope $\eta'(\infty)=0$, and attains the value $\eta(0)=\lambda$. The nonexistence of a finite t_{N+1} translates to the statement that none of the lines $x\mapsto\beta x$, with $\beta<0$, can be tangent to η. But then none of these lines can even intersect η, because if one did, then by continuously lowering β, we would eventually have a tangent to η, by convexity, and $0<\eta(0)$. This then implies that $0 \leqslant \eta(x)$ on $[0,+\infty[$. Since the convex function η has slope 0 at infinity, it has to be decreasing, so we also get $\eta(x) \leqslant \eta(0)=\lambda$ on $[0,+\infty[$. Hence (A.10) holds. Put

$$E(x)=\sum_{j=0}^{N} E_j(x)+E_{N+1}(x), \quad x \in \mathbf{R},$$

where

$$E_{N+1}(x)=\exp(-L_{N+1}(x)), \quad x \in \mathbf{R},$$

and note that by (A.2) and (A.10), we have, just as in Case I,

$$\exp(-\lambda-\nu(x)) \leqslant E(x), \quad x \in \mathbf{R}_+. \tag{A.11}$$

From the treatment of Case I we pick up the inequalities

$$E_{k-m}(x) \leqslant e^{-m\lambda} \exp(-\nu(x)), \quad t_k \leqslant x < +\infty, \tag{A.12}$$

for $k=0,1,\ldots,N$ and $m=0,1,\ldots,k$, and

$$E_{k+n+1}(x) \leqslant e^{-n\lambda} \exp(-\nu(x)), \quad 0 \leqslant x < t_{k+1}. \tag{A.13}$$

for $k=0,1,\ldots,N-1$ and $n=0,1,\ldots,N-k$. We need to verify that (A.13) holds for $k=N$ as well, that is, since we should think of t_{N+1} as $+\infty$,

$$E_{N+1}(x) \leqslant \exp(-\nu(x)), \quad 0 \leqslant x < +\infty.$$

This, however, is an obvious consequence of the fact that

$$\nu(x) \leqslant L_{N+1}(x), \quad x \in \mathbf{R}_+,$$

which follows straightforwardly from (A.10), by the convexity of the function $L_{N+1}-\nu$. Using these inequalities (A.12)–(A.13), we obtain the analog of (A.9), that is, for x in the interval $[t_k, t_{k+1}[$, with $k=0,1,\ldots,N$, and the convention $t_{N+1}=+\infty$, we have

$$\begin{aligned} E(x) &= \sum_{j=0}^{N+1} E_j(x) = \sum_{m=0}^{k} E_{k-m}(x)+\sum_{n=0}^{N-k} E_{k+n+1}(x) \\ &\leqslant \exp(-\nu(x)) \sum_{m=0}^{k} e^{-m\lambda}+\exp(-\nu(x)) \sum_{n=0}^{\infty} e^{-n\lambda} < 2(1-e^{-\lambda})^{-1} \exp(-\nu(x)). \end{aligned}$$

If we plug in $\lambda=\log 2$, this inequality combined with (A.11) yields

$$\tfrac{1}{2} \leqslant \exp(\nu(x)) E(x) < 4, \quad x \in \mathbf{R}_+.$$

If we write

$$E_{N+1}(x) = a_{N+1} \exp(-b_{N+1}x), \quad x \in \mathbf{R},$$

with $b_{N+1} = \nu'(\infty)$ and $a_{N+1} = \exp(-\nu(s_{N+1}) - s_{N+1}\nu'(\infty) + \lambda)$, the measure

$$d\mu(x) = 2^{1/2} \sum_{k=0}^{N+1} a_k \, d\delta(x - b_k),$$

now fulfils the property of the lemma:

$$\tfrac{1}{3} < \exp(\nu(x)) \int_0^{+\infty} e^{-xt} \, d\mu(t) < 3, \quad x \in \mathbf{R}_+.$$

This concludes the proof of Theorem A.2. □

Appendix B. Moment problems, part II

Consider the collection $\mathfrak{P}$ of finite positive Borel measures μ on $[0,1]$, placing no mass at the point 0, but having $\mu(]0,\varepsilon]) > 0$ for every ε, $0 < \varepsilon$. We shall also be concerned with the collection $\mathfrak{V}$ of all continuous functions $F: \mathbf{R}_+ \to]0, +\infty[$, subject to the conditions that F is decreasing, $F(t) \to 0$ as $t \to +\infty$, $\log F$ is convex, and

$$\log F(t) = o(t) \quad \text{as } t \to +\infty.$$

Recall from Appendix A the convention to assign a function

$$F_{d\mu}(t) = \int_0^1 e^{-tx} \, d\mu(x), \quad t \in \mathbf{R}_+, \tag{B.1}$$

to a given measure μ in $\mathfrak{P}$. Let us introduce the notation $f \asymp g$, and say in words that f and g are *comparable* on $\mathbf{R}_+$, if f and g are two functions on $\mathbf{R}_+$ with values in $[0, +\infty[$, which satisfy

$$C_1 f(t) \leqslant g(t) \leqslant C_2 f(t), \quad t \in \mathbf{R}_+,$$

for some constants C_1, C_2, $0 < C_1 \leqslant C_2 < +\infty$. The following statement contains most of the information from Appendix A that we shall need.

PROPOSITION B.1. *If μ is in $\mathfrak{P}$, then $F_{d\mu}$ belongs to $\mathfrak{V}$. If, on the other hand, F is in $\mathfrak{V}$, then a $\mu \in \mathfrak{P}$ can be found such that $F_{d\mu} \asymp F$; there exists in fact a μ of the form $d\mu(x) = \varphi(x)\,dx$, where φ is continuous on $]0,1]$, and enjoys the additional condition $0 < \varphi(x)$, for all $x \in]0,1]$.*

Proof. Let us start with having a μ, and try to prove $F_{d\mu} \in \mathfrak{V}$. It is clear from the definition that $F_{d\mu}$ is decreasing, and that $F_{d\mu}(t) \to 0$ as $t \to +\infty$. Proposition A.1 tells us that $\log F_{d\mu}$ is convex, and since we know $\alpha(\mu)=0$ (in the terminology of Proposition A.1), we also have

$$\log F_{d\mu}(t) = o(t), \quad t \to +\infty.$$

This completes the verification that $F_{d\mu}$ is in $\mathfrak{V}_s$.

We proceed to the case when we have an F, and seek a μ. Let $A \in]0, +\infty[$, and put $\nu(t) = \log \omega(t)$ on the interval $[A, +\infty[$, noting that by assumption, ν is concave and increasing. We extend ν to be affine on the interval $[0, A]$, in such a way that ν becomes differentiable at A. The extended ν is concave and increasing on $[0, +\infty[$, with $\nu(t) \to +\infty$ and $\nu'(t) \to 0$ as $t \to +\infty$. We now stipulate that A be chosen so large that ν is positive throughout $\mathbf{R}_+$, and $\nu'(A) < \frac{1}{2}$. By Theorem A.2, a finite positive Borel measure μ, supported on $\left[0, \frac{1}{2}\right]$, may be found such that $F_{d\mu} \asymp \exp(-\nu) \asymp F$. Moreover, since $F(t) \to 0$ as $t \to +\infty$, μ cannot place a positive mass at 0. It is clear from Proposition A.1 that the assumption $\nu(t) = o(t)$ as $t \to +\infty$ forces μ to place positive mass on every interval $]0, \varepsilon]$, with $0 < \varepsilon$. We conclude that $\mu \in \mathfrak{P}$. If we analyze the proof of Theorem A.2, we see that μ is obtained as a sum of discrete point masses. We may then mollify each point mass a little bit, without changing the main relation $F_{d\mu} \asymp F$, to get μ of the form $d\mu(x) = \varphi(x)\, dx$, with φ continuous on $]0, 1]$. Similar reasoning permits us to add a little background noise, to ascertain that $0 < \varphi(x)$, for all $x \in]0, 1]$. The proof is complete. □

We now intend to use Proposition B.1 to study functions of the type

$$F_\varrho(t) = \int_0^1 e^{-tx} \varrho(x)\, dx, \quad t \in \mathbf{R}, \tag{B.2}$$

associated with functions $\varrho: [0, 1] \to]0, +\infty[$, belonging to the families $\mathfrak{P}_s$, $0 < s < +\infty$, which we are about to define. For a $\mu \in \mathfrak{P}$, and all real numbers s, $0 < s < +\infty$, we introduce the fractional integrals

$$I_s[\mu](x) = \frac{1}{\Gamma(s)} \int_0^x (x-t)^{s-1}\, d\mu(t), \quad 0 \leqslant x \leqslant 1,$$

where Γ denotes the gamma function:

$$\Gamma(s) = \int_0^{+\infty} e^{-t} t^{s-1}\, dt.$$

The above fractional integral, as all other integrals in this paper, should be thought of as ranging over the *closed* interval between the indicated endpoints, unless specifically indicated otherwise. We shall specify that an endpoint is not to be included with a superscript, a plus if it is the left endpoint, and minus if it is the right one. The set of all $I_s[\mu]$, with s fixed and μ ranging over all of $\mathfrak{P}$, is denoted by $\mathfrak{P}_s$. It is clear from the above definition that

$$I_{r+s}[\mu](x)=\frac{1}{\Gamma(r)}\int_0^x (x-t)^{r-1} I_s[\mu](t)\,dt, \quad 0\leqslant x\leqslant 1,\ 0<r,s<+\infty,$$

by the well-known fact that the fractional integral kernel functions $K_s(x)=x^{s-1}/\Gamma(s)$ ($x\in\mathbf{R}_+$) form a convolution semigroup: $K_{r+s}=K_r*K_s$, for $0<r,s<+\infty$. Note that it follows from the above observation that the classes $\mathfrak{P}_s$ are nested: $\mathfrak{P}_r\subset\mathfrak{P}_s\subset\mathfrak{P}$, $0<s<r<+\infty$.

PROPOSITION B.2. *Suppose $\varrho\in\mathfrak{P}_s$, for some s, $0<s<+\infty$, so that $\varrho=I_s[\mu]$ for some $\mu\in\mathfrak{P}$, and let the functions $F_{d\mu}$ and F_ϱ be as in Appendix* A *and formula* (B.2), *respectively. Then the functions F_ϱ and $F_{d\mu}$ both belong to $\mathfrak{V}$. Moreover, we have the following relationship between the functions F_ϱ and $F_{d\mu}$:*

$$0\leqslant F_{d\mu}(t)-t^s F_\varrho(t)\leqslant e^{1-t}(1+t)^s\mu([0,1]), \quad 1\leqslant t<+\infty.$$

Proof. By Proposition B.1, applied to the measures $d\mu(x)$ and $\varrho(x)\,dx$, the functions F_ϱ and $F_{d\mu}$ belong to the class $\mathfrak{V}$.

Since $\varrho=I_s[\mu]$, we have

$$\begin{aligned} F_\varrho(t)&=\int_0^1 e^{-tx} I_s[\mu](x)\,dx=\frac{1}{\Gamma(s)}\int_0^1 e^{-tx}\int_0^x (x-u)^{s-1}\,d\mu(u)\,dx \\ &=\frac{t^{-s}}{\Gamma(s)}\int_0^1 e^{-tu}\int_0^{t(1-u)} e^{-y}y^{s-1}\,dy\,d\mu(u), \quad t\in\mathbf{R}_+, \end{aligned}\tag{B.3}$$

by switching the order of integration, and applying the change of variables $y=t(x-u)$, $u=u$. If we use (B.3), the integral definition of the gamma function, and make the change of variables $v=y-t(1-u)$, $u=u$, we obtain

$$\begin{aligned} F_{d\mu}(t)-t^sF_\varrho(t)&=\frac{1}{\Gamma(s)}\int_0^1\left(\Gamma(s)-\int_0^{t(1-u)} e^{-y}y^{s-1}\,dy\right)e^{-tu}\,d\mu(u) \\ &=\frac{1}{\Gamma(s)}\int_0^1\int_{t(1-u)}^{+\infty} e^{-y}y^{s-1}\,dy\,e^{-tu}\,d\mu(u) \\ &=\frac{e^{-t}}{\Gamma(s)}\int_0^1\int_0^{+\infty} e^{-v}(v+t(1-u))^{s-1}\,dv\,d\mu(u) \\ &=e^{-t}\int_0^1\frac{\Gamma(s,t(1-u))}{\Gamma(s)}\,d\mu(u), \end{aligned}\tag{B.4}$$

where $\Gamma(s,\alpha)$ denotes the generalized gamma function:

$$\Gamma(s,\alpha)=\int_0^{+\infty} e^{-u}(u+\alpha)^{s-1}\,du.$$

The case $\alpha=0$, gives us the gamma function itself: $\Gamma(s)=\Gamma(s,0)$. To carry on the proof, we need a lemma concerning the growth of the generalized gamma function.

LEMMA B.3. *For $s,\alpha\in]0,+\infty[$, the following holds*:

$$\Gamma(s,\alpha)\leqslant e(1+\alpha)^s\Gamma(s).$$

Proof sketch. Break up the two integrals defining $\Gamma(s)$ and $\Gamma(s,\alpha)$ into two parts each, from 0 to 1, and then from 1 to $+\infty$. One then obtains

$$\int_0^1 e^{-u}(u+\alpha)^{s-1}\,du\leqslant e(1+\alpha)^s\int_0^1 e^{-u}u^{s-1}\,du,$$

and

$$\int_1^{+\infty} e^{-u}(u+\alpha)^{s-1}\,du\leqslant (1+\alpha)^s\int_1^{+\infty} e^{-u}u^{s-1}\,du,$$

from which the assertion is immediate. □

We now proceed with the proof of Proposition B.2. By (B.4), Lemma B.3, the estimate $t(1-u)\leqslant t$ (in the setting of (B.4)), and the fact that the expression $(1+\alpha)^s$ appearing in Lemma B.3 increases with α, we arrive at

$$0\leqslant F_{d\mu}(t)-t^sF_\varrho(t)\leqslant e^{1-t}(1+t)^s\mu([0,1]),$$

as asserted. □

For a fixed value of the real parameter s, $0\leqslant s<+\infty$, $\mathfrak{V}_s$ will denote the subset of $\mathfrak{V}$ consisting of all $F\in\mathfrak{V}$ for which $t\mapsto(1+t)^sF(t)$ belongs to $\mathfrak{V}$. From Proposition B.2 we may derive the following useful result.

COROLLARY B.4. *Suppose $\varrho=I_s[\mu]$ for some s, $0<s<+\infty$, and some $\mu\in\mathfrak{P}$. Denote by $\widetilde{F}_\varrho$ the function $\widetilde{F}_\varrho(t)=(1+t)^{-s}F_{d\mu}(t)$, $t\in\mathbf{R}_+$. Then $\widetilde{F}_\varrho$ belongs to $\mathfrak{V}_s$, and $\widetilde{F}_\varrho\asymp F_\varrho$, that is, the functions $\widetilde{F}_\varrho$ and F_ϱ are comparable on $\mathbf{R}_+$.*

Proof. It is clear from Proposition B.1 and the definition of the classes $\mathfrak{V}_s$ that $\widetilde{F}_\varrho$ is in $\mathfrak{V}_s$. Moreover, that $\widetilde{F}_\varrho\asymp F_\varrho$ follows from Proposition B.2, and the fact that both F_ϱ and $\widetilde{F}_\varrho$ are bounded away from 0 and $+\infty$ on a finite interval. □

PROPOSITION B.5. *Fix a real parameter s, $0<s<+\infty$, and let F be an element of $\mathfrak{V}_s$. Then there exists a continuous function $\varrho\colon]0,1]\to]0,+\infty[$ in $\mathfrak{P}_s$ such that $F_\varrho\asymp F$.*

Proof. Since F is in $\mathfrak{V}_s$, the associated function $F^s(t)=(1+t)^sF(t)$, $t\in\mathbf{R}_+$, belongs to $\mathfrak{V}$. By Proposition B.1, we may then find a $\mu\in\mathfrak{P}$ of the form $d\mu(t)=\varphi(t)\,dt$, where $\varphi\colon]0,1]\to]0,+\infty[$ continuous, with the property that $F_{d\mu}\asymp F^s$. Define $\varrho\in\mathfrak{P}_s$ by $\varrho=I_s[\mu]$, which is then continuous on $]0,1]$, and takes values in $]0,+\infty[$. By Corollary B.4, the function $\widetilde{F}_\varrho(t)=(1+t)^{-s}F_{d\mu}(t)$, $t\in\mathbf{R}_+$, has $\widetilde{F}_\varrho\asymp F_\varrho$. On the other hand, $\widetilde{F}_\varrho\asymp F$, which does it. □

LEMMA B.6. *Suppose $\varrho = I_s[\mu]$ for some s, $1 < s < +\infty$, and some $\mu \in \mathfrak{P}$. If two real parameters x and y are given, satisfying $0 < y < x < 1$, then the following estimate holds*:

$$\varrho(x) - 2^{s-1}\varrho(y) \leqslant \frac{2^{s-1}}{\Gamma(s)}\,\mu(]0,x])(x-y)^{s-1}.$$

Proof. An elementary estimate of the integral expression defining $I_s[\mu]$ yields

$$\varrho(x) = I_s[\mu](x) \leqslant \frac{1}{\Gamma(s)}\,\mu(]0,x])x^{s-1}, \quad 0 < x \leqslant 1. \tag{B.5}$$

We split the verification of the lemma into two cases.

Case 1. $y \leqslant \frac{1}{2}x$. Then by (B.5),

$$\varrho(x) \leqslant \frac{1}{\Gamma(s)}\,\mu(]0,x])x^{s-1} = \frac{2^{s-1}}{\Gamma(s)}\,\mu(]0,x])\left(\tfrac{1}{2}x\right)^{s-1} \leqslant \frac{2^{s-1}}{\Gamma(s)}\,\mu(]0,x])(x-y)^{s-1},$$

which is even better than requested.

Case 2. $\frac{1}{2}x < y$. We then have the partition $0 < 2y - x < y < x < 1$ of the interval $[0,1]$, and split the integral defining $\varrho(x) - \varrho(y)$ accordingly:

$$\begin{aligned}\Gamma(s)(\varrho(x)-\varrho(y)) &= \int_0^{2y-x} ((x-t)^{s-1} - (y-t)^{s-1})\,d\mu(t) \\ &\quad + \int_{2y-x^+}^{y} ((x-t)^{s-1} - (y-t)^{s-1})\,d\mu(t) + \int_{y^+}^{x} (x-t)^{s-1}\,d\mu(t).\end{aligned}$$

The last two integrals allow themselves to be estimated as follows:

$$\int_{2y-x^+}^{y} ((x-t)^{s-1} - (y-t)^{s-1})\,d\mu(t) + \int_{y^+}^{x} (x-t)^{s-1}\,d\mu(t) \leqslant 2^{s-1}(x-y)^{s-1}\mu(]2y-x,x]).$$

Moreover, the first integral is estimated by

$$\begin{aligned}\int_0^{2y-x} ((x-t)^{s-1} - (y-t)^{s-1})\,d\mu(t) &\leqslant (2^{s-1}-1)\int_0^{2y-x} (y-t)^{s-1}\,d\mu(t) \\ &\leqslant (2^{s-1}-1)\Gamma(s)\varrho(y).\end{aligned}$$

Adding these estimates together, we arrive at

$$\varrho(x) - \varrho(y) \leqslant \frac{2^{s-1}}{\Gamma(s)}(x-y)^{s-1}\mu(]2y-x,x]) + (2^{1-s}-1)\varrho(y),$$

from which it is immediate that

$$\varrho(x) - 2^{s-1}\varrho(y) \leqslant \frac{2^{s-1}}{\Gamma(s)}(x-y)^{s-1}\mu(]2y-x,x]),$$

and the assertion follows.

The proof of the lemma is complete. □

LEMMA B.7. *Suppose $\varrho\in\mathfrak{P}_s$ for some s, $1<s<+\infty$, and let θ satisfy $0\leqslant\theta<+\infty$. We then have the estimate*

$$\sup\{(x-y)^{1+\theta}\varrho(y):y\in]0,x[\}\geqslant C\varrho(x)^{(s+\theta)/(s-1)},\quad x\in]0,1],$$

where $C=C(s,\varrho,\theta)$, $0<C<+\infty$, is a constant.

Proof. The function $\varrho=I_s[\mu]\in\mathfrak{P}_s$ is continuous, because it is the integral of $I_{s-1}[\mu]$, which is in $L^1([0,1])$. We choose y, $0<y<x$, as a solution to the equation $\varrho(y)=2^{-s}\varrho(x)$, which is possible, by the intermediate value theorem. When we plug this y into Lemma B.6, we obtain

$$(x-y)^{s-1}\geqslant\frac{2^{-s}\Gamma(s)}{\mu(]0,1])}\varrho(x),$$

and if we raise both sides to the power $(1+\theta)/(s-1)$, and multiply by $\varrho(y)$, we get

$$(x-y)^{1+\theta}\varrho(y)\geqslant 2^{-s}\left(\frac{2^{-s}\Gamma(s)}{\mu(]0,1])}\right)^{(1+\theta)/(s-1)}\varrho(x)^{(s+\theta)/(s-1)},$$

as asserted. □

PROPOSITION B.8. *Given s, α, β, γ, and ϱ, subject to $1<s<+\infty$, $0\leqslant\alpha<+\infty$, $0<\beta,\gamma<+\infty$, and $\varrho\in\mathfrak{P}_s$, the following estimate holds,*

$$\int_0^{+\infty}t^{\alpha}(e^{-\gamma tx}F_\varrho(\gamma t)^{-1})^{\beta}\,dt\leqslant C\varrho(x)^{-(\beta s+\alpha+1)/(s-1)},\quad 0<x\leqslant1.$$

where $C=C(s,\varrho,\alpha,\beta,\gamma)$, $0<C<+\infty$, is a constant.

Proof. The change of variable $u=\gamma t$ shows that without loss of generality, we may take $\gamma=1$.

Let y be a real parameter satisfying $0<y\leqslant\frac{1}{2}$. Then, in view of the fact that ϱ is an increasing function, and the more or less trivial inequality

$$\begin{aligned}\int_y^1 e^{-tu}\,du&=\frac{e^{-ty}-e^{-t}}{t}=e^{-ty}\,\frac{1-e^{-(1-y)t}}{t}\\&\geqslant e^{-ty}\,\frac{1-e^{-t/2}}{t}\geqslant\frac{e^{-ty}}{2(t+1)},\quad 0\leqslant t<+\infty,\end{aligned}$$

we have

$$\begin{aligned}F_\varrho(t)&=\int_0^1 e^{-tu}\varrho(u)\,du>\int_y^1 e^{-tu}\varrho(u)\,du\\&\geqslant\varrho(y)\int_y^1 e^{-tu}\,du\geqslant\varrho(y)\frac{e^{-ty}}{2(t+1)},\quad 0\leqslant t<+\infty.\end{aligned}$$

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For a given x, $0<x\leqslant 1$, suppose that y in addition has $0<y<x$, so that by the above estimate,

$$\int_0^{+\infty} e^{-tx} F_\varrho(t)^{-1}\,dt \leqslant \left(\frac{2}{\varrho(y)}\right)^{\beta} \int_0^{+\infty} t^{\alpha}(t+1)^{\beta} \exp(-t\beta(x-y))\,dt \\ \leqslant \frac{C(\alpha,\beta)}{\varrho(y)^{\beta}(x-y)^{1+\alpha+\beta}},$$

for some positive constant $C(\alpha,\beta)$. The assertion of the proposition now follows from Lemma B.7. □

Acknowledgements. We thank Sten Kaijser for a remark he made concerning the second author's paper [18], which helped us find an isometry between weighted l^2 spaces on the nonnegative integers, and certain Hilbert spaces of asymptotically holomorphic functions. That isometry then suggested the possibility for a continuous analog, which is what we develop in §2 for $p=2$. We are grateful to Evseĭ Dyn'kin for pointing out to us the results contained in his thesis. We thank Yngve Domar and Svante Janson for valuable remarks as this paper has progressed, and Lennart Carleson for the interest he has taken in this work. We thank Elizabeth Strouse for offering constructive criticism. We also thank the referee for making several valuable comments.

References

[1] Bernstein, S. N., Sur les fonctions absolument monotones. *Acta Math.*, 52 (1929), 1–66.

[2] Beurling, A., Sur les intégrales de Fourier absolument convergentes et leur application à une transformation fonctionelle, in *Neuvième congrès des mathématiciens scandinaves, Helsingfors, 1938*, pp. 345–366; Reprinted in *The Collected Works of Arne Beurling*, Vol. 2, pp. 39–60. Birkhäuser, Boston, 1989.

[3] — On two problems concerning linear transformations in Hilbert space. *Acta Math.*, 81 (1949), 239–255; Reprinted in *The Collected Works of Arne Beurling*, Vol. 1, pp. 147–163. Birkhäuser, Boston, 1989.

[4] — Analytic continuation across a linear boundary. *Acta Math.*, 128 (1972), 153–182; Reprinted in *The Collected Works of Arne Beurling*, Vol. 1, pp. 309–338. Birkhäuser, Boston, 1989.

[5] Borichev, A. A. & Hedenmalm, H., Approximation in a class of Banach algebras of quasianalytically smooth analytic functions. *J. Funct. Anal.*, 115 (1993), 359–390.

[6] Dales, H. G. & Hayman, W. K., Esterle's proof of the Tauberian theorem for Beurling algebras. *Ann. Inst. Fourier (Grenoble)*, 31 (1981), 141–150.

[7] Ditkin, V. A., Investigation of the structure of ideals in some normed rings. *Uchenye Zapiski Moskov. Gos. Univ. Matematika*, 30 (1939), 83–130 (Russian; English summary).

[8] DOMAR, Y., On the analytic transform of bounded linear functionals on certain Banach algebras. *Studia Math.*, 53 (1975), 203–224.

[9] — Bilaterally translation-invariant subspaces of weighted $L^p(\mathbf{R})$, in *Radical Banach Algebras and Automatic Continuity*. Lecture Notes in Math., 975, pp. 210–213. Springer-Verlag, Berlin–New York, 1983.

[10] — Uniform boundedness in families related to subharmonic functions. *J. London Math. Soc.* (*2*), 38 (1988), 485–491.

[11] DYN'KIN, E. M., Operator calculus based on the Cauchy–Green formulae, and quasi-analyticity of the classes $\mathcal{D}(h)$. *Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov.* (*LOMI*), 19 (1970), 221–226; English translation in *Sem. Math. V. A. Steklov Math. Inst. Leningrad*, 19 (1972), 128–131.

[12] — Functions with a given estimate for $\partial f/\partial \bar{z}$ and N. Levinson's theorem. *Mat. Sb.*, 81 (1972), 182–190; English translation in *Math. USSR-Sb.*, 18 (1972), 181–189.

[13] GEL'FAND, I. M., Ideale und primäre Ideale in normierten Ringen. *Mat. Sb.*, 9 (1941), 41–47.

[14] GURARIĬ, V. P., Harmonic analysis in spaces with a weight. *Trudy Moskov. Mat. Obshch.*, 35 (1976), 21–76; English translation in *Trans. Moscow Math. Soc.*, 35 (1979), 21–75.

[15] — Completeness of translates of a given function in a weighted space, in *Linear and Complex Analysis Problem Book, 199 Research Problems*. Lecture Notes in Math., 1043, pp. 409–413. Springer-Verlag, Berlin–New York, 1984.

[16] GURARIĬ, V. P. & LEVIN, B. YA., On the completeness of a system of translates in the space $L(0,\infty)$ with a weight. *Zap. Mekh.-Mat. Fak. i Khar'kov. Mat. Obshch.*, 30 (1964), 178–185 (Russian).

[17] HEDENMALM, H., On the primary ideal structure at infinity for analytic Beurling algebras. *Ark. Mat.*, 23 (1985), 129–158.

[18] — Formal power series and nearly analytic functions. *Arch. Math.* (*Basel*), 57 (1991), 61–70.

[19] — Spectral properties of invariant subspaces in the Bergman space. *J. Funct. Anal.*, 116 (1993), 441–448.

[20] KHAVINSON, S. YA., The theory of extremal problems for bounded analytic functions satisfying additional conditions inside the domain. *Uspekhi Mat. Nauk*, 18 (110) (1963), 25–98 (Russian).

[21] KOOSIS, P., *Introduction to H_p Spaces*. London Math. Soc. Lecture Note Ser., 40. Cambridge Univ. Press, Cambridge–New York, 1980.

[22] — *The Logarithmic Integral*, Vol. 1. Cambridge Univ. Press, Cambridge–New York, 1988.

[23] LAX, P. D., Translation invariant spaces. *Acta Math.*, 101 (1959), 163–178.

[24] NEUMANN, J. VON, Zur Operatorenmethode in der klassischen Mechanik. *Ann. of Math.*, 33 (1932), 587–642; Reprinted in *John von Neumann: Collected Works*, Vol. II, pp. 307–362. Pergamon Press, New York–Oxford–London–Paris, 1961.

[25] NYMAN, B., *On the One-Dimensional Translation Group and Semi-Group in Certain Function Spaces*. Appelbergs Boktryckeri, Uppsala, 1950.

[26] PALEY, R. E. A. C. & WIENER, N., *Fourier Transforms in the Complex Domain*. Amer. Math. Soc. Colloq. Publ., 19. Amer. Math. Soc., New York, 1934.

[27] RUDIN, W., *Real and Complex Analysis*, 2nd edition. McGraw-Hill, New York–Düsseldorf–Johannesburg, 1974.

[28] STEIN, E. M. & WEISS, G., *Introduction to Fourier Analysis on Euclidean Spaces*. Princeton Univ. Press, Princeton, N.J., 1971.

[29] VOLBERG, A. L., The logarithm of an almost analytic function is summable. *Dokl. Akad. Nauk SSSR*, 265 (1982), 1297–1302; English translation in *Soviet Math. Dokl.*, 26 (1982), 238–243.

[30] WIENER, N., Tauberian theorems. *Ann. of Math.*, 33 (1932), 1–100; Reprinted in *Norbert Wiener: Collected Works with Commentaries*, Vol. II, pp. 519–618. MIT Press, Cambridge, Mass., 1979.

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Received May 10, 1993
Received in revised form April 11, 1994