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The second main theorem for holomorphic curves into semi-Abelian varieties

by

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1. Introduction and main result

Let $f: \mathbf{C} \to A$ be an entire holomorphic curve from the complex plane $\mathbf{C}$ into a semi-Abelian variety A. It was proved by [No2] that the Zariski closure of $f(\mathbf{C})$ is a translate of a semi-Abelian subvariety of A (logarithmic Bloch–Ochiai's theorem). Let D be an effective algebraic divisor on A which is compactified to $\overline{D}$ on a natural compactification $\bar{A}$ of A (see §3). If f omits D, i.e., $f(\mathbf{C}) \cap D = \varnothing$, then $f(\mathbf{C})$ is contained in a translate of a closed subgroup of A that has no intersection with D (see [No5], [SY1]). Note that the same holds for complex semi-tori defined in §3 (see [NW]). In particular, if A is Abelian and D is ample, then f is constant. This was called Lang's conjecture. A similar statement, however, is found in Bloch [Bl, p. 55, Théorème K] without much proof, and it is not clear what his Théorème K really means (cf. [Bl]).

The purpose of the present paper is to establish the quantitative version of the above result for f whose image may intersect D, i.e., the second main theorem and the defect relation (cf. §§ 2 and 3 for the notation):

Main Theorem. *Let $f: \mathbf{C} \to M$ be a holomorphic curve into a complex semi-torus M such that the image $f(\mathbf{C})$ is Zariski dense in M. Let D be an effective divisor on M. If M is not compact, we assume that $\overline{D}$ is a divisor on $\overline{M}$ satisfying general position condition* 4.11 *with respect to ∂M. Then we have the following.*

(i) *Suppose that f is of finite order ϱ_f. Then there is a positive integer $k_0 = k_0(\varrho_f, D)$ depending only on ϱ_f and D such that*

$$T_f(r; c_1(\overline{D})) = N_{k_0}(r; f^*D) + O(\log r).$$

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(ii) *Suppose that f is of infinite order. Then there is a positive integer $k_0 = k_0(f, D)$ depending on f and D such that*

$$T_f(r; c_1(\bar{D})) = N_{k_0}(r; f^*D) + O(\log T_f(r; c_1(\bar{D}))) + O(\log r)\|_E.$$

In particular, $\delta(f; \bar{D}) = \delta_{k_0}(f; \bar{D}) = 0$ in both cases.

See Examples 4.13, 5.22 and Proposition 5.17 that show the necessity of condition 4.11. The most essential part of the proof is the proof of an estimate of the proximity function (see Lemma 5.1):

$$m_f(r; \bar{D}) = O(\log r) \quad \text{or} \quad O(\log T_f(r; c_1(\bar{D}))) + O(\log r)\|_E. \tag{1.1}$$

In the course, the notion of logarithmic jet spaces due to [No3] plays a crucial role (cf. [DL] for an extension to the case of directed jets). We then use the jet projection method developed by [NO, Chapter 6, §3] (cf. [No1], [No2] and [No5]).

In §6 we will discuss some applications of the Main Theorem.

In [Kr2] R. Kobayashi claimed (1.1) for Abelian A, but there is a part of the arguments which is heuristic, and hard to follow rigorously. Siu–Yeung [SY2] claimed that for Abelian A

$$m_f(r; D) \leqslant \varepsilon T_f(r; c_1(D)) + O(\log r)\|_{E(\varepsilon)}, \tag{1.2}$$

where ε is an arbitrarily given positive number, but unfortunately there was a gap in the proof (see Remark 5.34). M. McQuillan [M] dealt with an estimate of type (1.2) for some proper monoidal transformation of $\bar{D} \subset \bar{A}$ with semi-Abelian A by a method different to those mentioned above and ours (see [M, Theorem 1]).

It might be appropriate at this point to recall the higher dimensional cases in which the second main theorem has been established. There are actually only a few such cases that have provided fundamental key steps. The first was by H. Cartan [Ca] for $f: \mathbf{C} \to \mathbf{P}^n(\mathbf{C})$ and hyperplanes in general position, where $\mathbf{P}^n(\mathbf{C})$ is the n-dimensional complex projective space. The Weyls–Ahlfors theory [Ah] dealt with the same case and the associated curves as well. W. Stoll [St] generalized the Weyls–Ahlfors theory to the case of $f: \mathbf{C}^m \to \mathbf{P}^n(\mathbf{C})$. Griffiths et al., [CG], [GK], established the second main theorem for $f: W \to V$ with a complex affine algebraic variety W and a general complex projective manifold V such that $\operatorname{rank} df = \dim V$, which was developed well by many others. For $f: \mathbf{C} \to V$ in general, only an inequality of the second main theorem type such as (5.33) was proved ([No1]–[No4], [AN]). Eremenko and Sodin [ES] proved a weak second main theorem for $f: \mathbf{C} \to \mathbf{P}^n(\mathbf{C})$ and hypersurfaces in general position, where the counting functions are not truncated. In this sense, the Main Theorem adds a new case in which an explicit second main theorem is established.

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2. Order functions

(a) For a general reference of items presented in this section, cf., e.g., [NO]. First we recall some standard notation. Let ϕ and ψ be functions in a variable $r>0$ such that $\psi>0$. Let E be a measurable subset of real positive numbers with finite measure. Then the expression

$$\phi(r) = O(\psi(r)) \quad (\text{resp. } \phi(r) = O(\psi(r))||_E)$$

stands for

$$\overline{\lim_{r\to\infty}} \frac{|\phi(r)|}{\psi(r)} < \infty \quad \Big(\text{resp. } \overline{\lim_{\substack{r\to\infty \\ r\notin E}}} \frac{|\phi(r)|}{\psi(r)} < \infty\Big).$$

In particular, $O(1)$ denotes a bounded term.

We use the superscript $^+$ to denote the positive part, e.g., $\log^+ r = \max\{0, \log r\}$. We write $\mathbf{R}^+$ for the set of all real positive numbers. We denote by $\operatorname{Re} z$ (resp. $\operatorname{Im} z$) the real (resp. imaginary) part of a complex number $z\in\mathbf{C}$.

(b) Let X be a compact Kähler manifold and let ω be a real $(1,1)$-form on X. For an entire holomorphic curve $f:\mathbf{C}\to X$ we first define the *order* function of f with respect to ω by

$$T_f(r;\omega) = \int_0^r \frac{dt}{t} \int_{\Delta(t)} f^*\omega,$$

where $\Delta(t) = \{z\in\mathbf{C} : |z|<t\}$ is the disk of radius t with center at the origin of the complex plane $\mathbf{C}$. Let $[\omega]\in H^2(X,\mathbf{R})$ be a second cohomology class represented by a closed real $(1,1)$-form ω on X. Then we set

$$T_f(r;[\omega]) = T_f(r;\omega).$$

Let $[\omega'] = [\omega]$ be another representation of the class. Since X is compact Kähler, there is a smooth function b on X such that $(i/2\pi)\partial\bar{\partial} b = \omega' - \omega$. There is a positive constant C with $|b|\leqslant C$. Then by Jensen's formula (cf. [NO, Lemma (3.39) and Remark (5.2.21)]) we have

$$|T_f(r;\omega') - T_f(r;\omega)| \leqslant C.$$

Therefore, the order function $T_f(r;[\omega])$ of f with respect to the cohomology class $[\omega]$ is well-defined up to a bounded term. Taking a positive definite form ω on X, we define the *order* of f by

$$\varrho_f = \overline{\lim_{r\to\infty}} \frac{\log T_f(r;\omega)}{\log r} \leqslant \infty,$$

which is independent of the choice of such ω. We say that f is of finite order if $\varrho_f < \infty$.

Let D be an effective divisor on X. We denote by $\operatorname{Supp} D$ the support of D, but sometimes write simply D for $\operatorname{Supp} D$ if there is no confusion. Assume that $f(\mathbf{C}) \not\subset D$. Let $L(D)$ be the line bundle determined by D and let $\sigma \in H^0(X, L(D))$ be a global holomorphic section of $L(D)$ whose divisor (σ) is D. Take a Hermitian fiber metric $\|\cdot\|$ in $L(D)$ with curvature form ω, normalized so that ω represents the first Chern class $c_1(L(D))$ of $L(D)$; $c_1(L(D))$ will be abbreviated to $c_1(D)$. Set

$$T_f(r;c_1(D)) = T_f(r;\omega),$$

$$m_f(r;D) = \frac{1}{2\pi}\int_0^{2\pi} \log \frac{1}{\|\sigma(f(re^{i\theta}))\|}\, d\theta.$$

It is known that if D is ample, then f is rational if and only if

$$\underline{\lim_{r\to\infty}} \frac{T_f(r;c_1(D))}{\log r} < \infty.$$

One sometimes writes $T_f(r;L(D))$ for $T_f(r;c_1(D))$, but it is noted that $T_f(r;c_1(D))$ is not depending on a specific choice of D in the homology class. We call $m_f(r;D)$ the *proximity* function of f for D. Denoting by $\operatorname{ord}_z f^*D$ the order of the pull-backed divisor f^*D at $z \in \mathbf{C}$, we set

$$n(t;f^*D) = \sum_{z\in\Delta(t)} \operatorname{ord}_z f^*D,$$

$$n_k(t;f^*D) = \sum_{z\in\Delta(t)} \min\{k, \operatorname{ord}_z f^*D\},$$

$$N(r;f^*D) = \int_1^r \frac{n(t;f^*D)}{t}\, dt,$$

$$N_k(r;f^*D) = \int_1^r \frac{n_k(t;f^*D)}{t}\, dt.$$

These are called the *counting* functions of f^*D. Then we have the *F.M.T.* (*First Main Theorem*, cf. [NO, Chapter V]):

$$T_f(r;c_1(D)) = N(r;f^*D) + m_f(r;D) + O(1). \tag{2.1}$$

The quantities

$$\delta(f;D)=1-\overline{\lim_{r\to\infty}}\frac{N(r;f^*D)}{T_f(r;L(D))}\in[0,1],$$

$$\delta_k(f;D)=1-\overline{\lim_{r\to\infty}}\frac{N_k(r;f^*D)}{T_f(r;L(D))}\in[0,1]$$

are called the *defects* of f for D.

(c) Let $F(z)$ be a meromorphic function, and let $(F)_\infty$ (resp. $(F)_0$) denote the polar (resp. zero-)divisor of F. Define the proximity function of $F(z)$ by

$$m(r,F)=\frac{1}{2\pi}\int_0^{2\pi}\log^+|F(re^{i\theta})|\,d\theta.$$

Nevanlinna's order function is defined by

$$T(r,F)=m(r,F)+N(r;(F)_\infty).$$

Cf., e.g., [NO, Chapter 6] for the basic properties of $T(r,F)$. For instance, let $T_F(r;\omega)$ be the order function of holomorphic $F:\mathbf{C}\to\mathbf{P}^1(\mathbf{C})$ with respect to the Fubini–Study metric form ω. Then Shimizu–Ahlfors' theorem says that

$$T_F(r;\omega)-T(r,F)=O(1).$$

If $F\not\equiv 0$, $T(r,1/F)=m(r,1/F)+N(r;(F)_0)$, and then by Nevanlinna's F.M.T. (cf. [H], [NO])

$$T(r,F)=T(r,1/F)+O(1). \tag{2.2}$$

For several meromorphic functions F_j, $1\leqslant j\leqslant l$, on $\mathbf{C}$ we have

$$\begin{aligned} T\left(r,\prod_{j=1}^{l}F_j\right)&\leqslant\sum_{j=1}^{l}T(r,F_j),\\ T\left(r,\sum_{j=1}^{l}F_j\right)&\leqslant\sum_{j=1}^{l}T(r,F_j)+\log l,\\ T(r,R(F_1,\dots,F_l))&\leqslant O\left(\sum_{j=1}^{l}T(r,F_j)\right)+O(1),\end{aligned} \tag{2.3}$$

where $R(F_1,\dots,F_l)$ is a rational function in $F_1,\dots,F_l$ and $R(F_1(z),\dots,F_l(z))\not\equiv\infty$.

LEMMA 2.4 (cf. [NO, Theorem (5.2.29)]). *Let X be a compact Kähler manifold, let L be a Hermitian line bundle on X, and let $\sigma_1, \sigma_2 \in H^0(X, L)$ with $\sigma_1 \not\equiv 0$. Let $f: \mathbf{C} \to X$ be a holomorphic curve such that $f(\mathbf{C}) \not\subset \mathrm{Supp}(\sigma_1)$. Then we have*

$$T\left(r, \frac{\sigma_2}{\sigma_1} \circ f\right) \leqslant T_f(r; c_1(L)) + O(1).$$

Proof. It follows from the definition that

$$N\left(r; \left(\frac{\sigma_2}{\sigma_1} \circ f\right)_\infty\right) \leqslant N(r; f^*(\sigma_1)).$$

Moreover, we have

$$m\left(r, \frac{\sigma_2}{\sigma_1} \circ f\right) = \frac{1}{2\pi} \int_{\{|z|=r\}} \log^+ \frac{\|\sigma_2 \circ f\|}{\|\sigma_1 \circ f\|} \, d\theta \leqslant \frac{1}{2\pi} \int_{\{|z|=r\}} \log^+ \frac{1}{\|\sigma_1 \circ f\|} \, d\theta + O(1).$$

Thus the required estimate follows from these and (2.1). □

(d) We begin with introducing a notation for a *small term*.

Definition. We write $S_f(r; c_1(D))$, sometimes $S_f(r; L(D))$, to express a small term such that

$$S_f(r; c_1(D)) = O(\log r),$$

if $T_f(r; c_1(D))$ is of finite order, and

$$S_f(r; c_1(D)) = O(\log T_f(r; c_1(D))) + O(\log r) \|_E.$$

We use the notation $S_f(r; \omega)$ in the same sense as above with respect to $T_f(r; \omega)$. For a meromorphic function F on $\mathbf{C}$, the notation $S(r, F)$ is used to express a small term with respect to $T(r, F)$ as well.

LEMMA 2.5. (i) *Let F be a meromorphic function and let $F^{(k)}(z)$ be the k-th derivative of F for $k = 1, 2, \ldots$. Then*

$$m\left(r, \frac{F^{(k)}}{F}\right) = S(r, F).$$

Moreover, if F is entire,

$$T(r, F^{(k)}) = T(r, F) + S(r, F), \quad k \geqslant 1.$$

(ii) *Let the notation be as in Lemma 2.4, and set $\varphi(z) = (\sigma_2/\sigma_1) \circ f(z)$. Suppose that $\varphi \not\equiv 0$. Then*

$$m\left(r, \frac{\varphi^{(k)}}{\varphi}\right) = S_f(r, c_1(D)), \quad k \geqslant 1.$$

Proof. The item (i) is called Nevanlinna's lemma on logarithmic derivatives (cf. [NO, Corollary (6.1.19)]). Then (ii) follows from (i) and Lemma 2.4. □

The following is called Borel's lemma (cf. [H, p. 38, Lemma 2.4]).

LEMMA 2.6. *Let $\phi(r)$ be a continuous, increasing function on $\mathbf{R}^+$ such that $\phi(r_0)>0$ for some $r_0\in\mathbf{R}^+$. Then we have*

$$\phi\left(r+\frac{1}{\phi(r)}\right)<2\phi(r)\|_E.$$

For a later use we show

LEMMA 2.7. *Let F be an entire function, and let $0<r<R$.*

(i) $T(r,F)=m(r,F)\leqslant \max_{|z|=r}\log|F(z)|\leqslant \dfrac{R+r}{R-r}m(R,F)$.

(ii) $m(r,F)=S(r,e^{2\pi iF})$.

Proof. (i) See [NO, Theorem (5.3.13)] or [H, p. 18, Theorem 1.6].

(ii) Using the complex Poisson kernel, we have

$$F(z)=\frac{i}{2\pi}\int_{\{|\zeta|=R\}}\frac{\zeta+z}{\zeta-z}\operatorname{Im}F(\zeta)\,d\theta+\operatorname{Re}F(0).$$

Therefore, using (i) and the F.M.T. (2.2) with $0<r<R<R'$ we obtain

$$\begin{aligned}\max_{|z|=r}|F(z)|&\leqslant\frac{R+r}{R-r}\max_{|\zeta|=R}|\operatorname{Im}F(\zeta)|+|\operatorname{Re}F(0)|\\&\leqslant\frac{R+r}{R-r}\Big(\max_{|\zeta|=R}\operatorname{Im}F(\zeta)+\max_{|\zeta|=R}\operatorname{Im}(-F(\zeta))\Big)+|\operatorname{Re}F(0)|\\&\leqslant\frac{R+r}{R-r}\cdot\frac{R'+R}{R'-R}\cdot\frac{1}{2\pi}(m(R',e^{-2\pi iF})+m(R',e^{2\pi iF}))+|\operatorname{Re}F(0)|\\&\leqslant\frac{R+r}{R-r}\cdot\frac{R'+R}{R'-R}\cdot\frac{1}{\pi}(T(R',e^{2\pi iF})+O(1))+O(1).\end{aligned}\tag{2.8}$$

If $T(r,e^{2\pi iF})$ has finite order, then setting $R=2r$ and $R'=3r$, we see by (2.8) that

$$m(r,F)\leqslant\log\max_{|z|=r}|F(z)|=O(\log r).$$

In the case where $T(r,e^{2\pi iF})$ has infinite order, we write $T(r)=T(r,e^{2\pi iF})$ for the sake of simplicity. Setting $R=r+1/2T(r)$ and $R'=r+1/T(r)$, we have by (2.8) and Lemma 2.6

$$\begin{aligned}m(r,F)&\leqslant\log\max_{|z|=r}|F(z)|\\&\leqslant\log\left((4rT(r)+1)(4rT(r)+3)\left(T\left(r+\frac{1}{T(r)}\right)+O(1)\right)+O(1)\right)\\&\leqslant\log((4rT(r)+1)(4rT(r)+3)(2T(r)+O(1))+O(1))\|_E=S(r,e^{2\pi iF}).\end{aligned}$$

□

3. Complex semi-torus

Let M be a complex Lie group admitting the exact sequence

$$0 \to (\mathbf{C}^*)^p \to M \xrightarrow{\eta} M_0 \to 0, \tag{3.1}$$

where $\mathbf{C}^*$ is the multiplicative group of non-zero complex numbers, and M_0 is a (compact) complex torus. Such an M is called a *complex semi-torus* or a *quasi-torus.* If M_0 is algebraic, i.e., an Abelian variety, M is called a *semi-Abelian variety* or a *quasi-Abelian variety.* In this section and in the next, we assume that M is a complex semi-torus.

Taking the universal coverings of (3.1), one gets

$$0 \to \mathbf{C}^p \to \mathbf{C}^n \to \mathbf{C}^m \to 0,$$

and an additive discrete subgroup Λ of $\mathbf{C}^n$ such that

$$\begin{aligned} \pi &: \mathbf{C}^n \to M = \mathbf{C}^n/\Lambda, \\ \pi_0 &: \mathbf{C}^m = (\mathbf{C}^n/\mathbf{C}^p) \to M_0 = (\mathbf{C}^n/\mathbf{C}^p)/(\Lambda/\mathbf{C}^p), \\ &(\mathbf{C}^*)^p = \mathbf{C}^p/(\Lambda \cap \mathbf{C}^p). \end{aligned}$$

We fix a linear complex coordinate system $x=(x', x'')=(x'_1, \dots, x'_p, x''_1, \dots, x''_m)$ on $\mathbf{C}^n$ such that $\mathbf{C}^p \cong \{x''_1 = \dots = x''_m = 0\}$ and

$$\Lambda \cap \mathbf{C}^p = \mathbf{Z}\begin{pmatrix} 1 \\ \vdots \\ 0 \end{pmatrix} + \dots + \mathbf{Z}\begin{pmatrix} 0 \\ \vdots \\ 1 \end{pmatrix}.$$

The covering mapping $\mathbf{C}^p \to (\mathbf{C}^*)^p$ is given by

$$\begin{pmatrix} x_1 \\ \vdots \\ x_p \end{pmatrix} \in \mathbf{C}^p \to \begin{pmatrix} e^{2\pi i x_1} \\ \vdots \\ e^{2\pi i x_p} \end{pmatrix} = \begin{pmatrix} u_1 \\ \vdots \\ u_p \end{pmatrix} \in (\mathbf{C}^*)^p.$$

We may regard $\eta: M \to M_0$ to be a flat $(\mathbf{C}^*)^p$-principal fiber bundle. By a suitable change of coordinates $(x'_1, \dots, x'_p, x''_1, \dots, x''_m)$ the discrete group Λ is generated over $\mathbf{Z}$ by the column vectors of the matrix of the type

$$\begin{pmatrix} 1 & \dots & 0 & \\ \vdots & \ddots & \vdots & A \\ 0 & \dots & 1 & \\ & O & & B \end{pmatrix}, \tag{3.2}$$

where A is a $(p, 2m)$-matrix and B is an $(m, 2m)$-matrix. Therefore the transition matrix-valued functions of the flat $(\mathbf{C}^*)^p$-principal fiber bundle $\eta: M \to M_0$ are expressed by a diagonal matrix such that

$$\begin{pmatrix} a_1 & 0 & \dots & 0 \\ 0 & a_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & a_p \end{pmatrix}, \quad |a_1| = \dots = |a_p| = 1. \tag{3.3}$$

Taking the natural compactification $\mathbf{C}^* = \mathbf{P}^1(\mathbf{C}) \setminus \{0, \infty\} \hookrightarrow \mathbf{P}^1(\mathbf{C})$, we have a compactification of M,

$$\bar{\eta}: \overline{M} \to M_0,$$

which is a flat $(\mathbf{P}^1(\mathbf{C}))^p$-fiber bundle over M_0. Set

$$\partial M = \overline{M} \setminus M,$$

which is a divisor on $\overline{M}$ with only simple normal crossings.

Let Ω_1 be the product of the Fubini–Study metric forms on $(\mathbf{P}^1(\mathbf{C}))^p$,

$$\Omega_1 = \frac{i}{2\pi} \sum_{j=1}^{p} \frac{du_j \wedge d\bar{u}_j}{(1+|u_j|^2)^2}.$$

Because of (3.3) Ω_1 is well-defined on $\overline{M}$. Let $\Omega_2 = (i/2\pi)\partial\bar{\partial} \sum_j |x_j''|^2$ be the flat Hermitian metric form on $\mathbf{C}^m$, and as well on the complex torus M_0. Then we set

$$\Omega = \Omega_1 + \bar{\eta}^* \Omega_2, \tag{3.4}$$

which is a Kähler form on $\overline{M}$.

Remark. The same complex Lie group M may admit several such exact sequences as (3.1) which are quite different. For instance, let τ be an arbitrary complex number with $\operatorname{Im} \tau > 0$. Let Λ be the discrete subgroup of $\mathbf{C}^2$ generated by

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad \begin{pmatrix} i \\ \tau \end{pmatrix}.$$

Then $M = \mathbf{C}^2/\Lambda$ is a complex semi-torus, and the natural projection of $\mathbf{C}^2 = \mathbf{C} \times \mathbf{C}$ onto the first and the second factors induce respectively exact sequences of the forms

$$0 \to \mathbf{C}^* \to M \to \mathbf{C}/\langle 1, i \rangle_{\mathbf{Z}} \to 0,$$
$$0 \to \mathbf{C}^* \to M \to \mathbf{C}/\langle 1, \tau \rangle_{\mathbf{Z}} \to 0.$$

In the sequel we always consider a complex semi-torus M with a *fixed* exact sequence as in (3.1) and with the discrete subgroup Λ satisfying (3.2).

Let $f: \mathbf{C} \to M$ be a holomorphic curve. We regard f as a holomorphic curve into $\overline{M}$ equipped with the Kähler form Ω, and define the order function by

$$T_f(r;\Omega) = \int_0^r \frac{dt}{t} \int_{\Delta(t)} f^*\Omega.$$

Let $\tilde{f}: \mathbf{C} \to \mathbf{C}^n$ be the lift of f, and set

$$\tilde{f}(z) = (F_1(z), \dots, F_p(z), G_1(z), \dots, G_m(z)),$$

where $F_i(z)$ and $G_j(z)$ are entire functions. Extending the base M_0 of the fiber bundle $M \to M_0$ to the universal covering $\pi_0: \mathbf{C}^m \to M_0$, we have

$$M \times_{M_0} \mathbf{C}^m \cong (\mathbf{C}^*)^p \times \mathbf{C}^m, \quad \overline{M} \times_{M_0} \mathbf{C}^m \cong (\mathbf{P}^1(\mathbf{C}))^p \times \mathbf{C}^m.$$

Set

$$\widehat{M} = (\mathbf{P}^1(\mathbf{C}))^p \times \mathbf{C}^m.$$

Then $\widehat{M}$ is the universal covering of $\overline{M}$, and then $\tilde{f}$ induces a lifting $\hat{f}$ of $f: \mathbf{C} \to M \hookrightarrow \overline{M}$,

$$\hat{f}: z \in \mathbf{C} \to (e^{2\pi i F_1(z)}, \dots, e^{2\pi i F_p(z)}, G_1(z), \dots, G_m(z)) \in (\mathbf{C}^*)^p \times \mathbf{C}^m = \widehat{M}.$$

Set

$$\hat{f}_{(1)}: z \in \mathbf{C} \to (e^{2\pi i F_1(z)}, \dots, e^{2\pi i F_p(z)}) \in (\mathbf{C}^*)^p,$$
$$\hat{f}_{(2)}: z \in \mathbf{C} \to (G_1(z), \dots, G_m(z)) \in \mathbf{C}^m.$$

By definition we have

$$T_f(r;\Omega) = T_{\hat{f}_{(1)}}(r;\Omega_1) + T_{\hat{f}_{(2)}}(r;\Omega_2). \tag{3.5}$$

By Shimizu–Ahlfors' theorem we have

$$T_{\hat{f}_{(1)}}(r;\Omega_1) = \sum_{j=1}^{p} T(r, e^{2\pi i F_j}) + O(1). \tag{3.6}$$

By Jensen's formula (cf. [NO, Lemma (3.3.17)]) we have

$$\begin{aligned} T_{\hat{f}_{(2)}}(r;\Omega_2) &= \int_0^r \frac{dt}{t} \int_{\Delta(t)} \frac{i}{2\pi} \partial\bar{\partial} \sum_{j=1}^{m} |G_j(z)|^2 \\ &= \frac{1}{4\pi} \int_0^{2\pi} \left(\sum_{j=1}^{m} |G_j(re^{i\theta})|^2 \right) d\theta - \frac{1}{2} \sum_{j=1}^{m} |G_j(0)|^2. \end{aligned} \tag{3.7}$$

LEMMA 3.8. *Let the notation be as above. Then for $k \geqslant 0$ we have*

$$T(r, F_j^{(k)}) = T(r, F_j) + kS(r, F_j) \leqslant S_{\hat{f}_{(1)}}(r; \Omega_1) \leqslant S_f(r; \Omega),$$
$$T(r, G_j^{(k)}) = T(r, G_j) + kS(r, G_j) \leqslant S_{\hat{f}_{(2)}}(r; \Omega_2) \leqslant S_f(r; \Omega).$$

Proof. By Lemma 2.5 it suffices to show the case of $k=0$. By Lemma 2.7 and (3.6),

$$T(r, F_j) = m(r, F_j) = S(r, e^{2\pi i F_j}) \leqslant S_{\hat{f}_{(1)}}(r; \Omega_1) \leqslant S_f(r; \Omega).$$

For G_j we have by making use of (3.7) and the concavity of the logarithmic function

$$\begin{aligned} T(r; G_j) = m(r, G_j) &= \frac{1}{2\pi} \int_{\{|z|=r\}} \log^+ |G_j(z)| \, d\theta \\ &= \frac{1}{4\pi} \int_{\{|z|=r\}} \log^+ |G_j(z)|^2 \, d\theta \leqslant \frac{1}{4\pi} \int_{\{|z|=r\}} \log(1 + |G_j(z)|^2) \, d\theta \\ &\leqslant \frac{1}{2} \log\left(1 + \frac{1}{2\pi} \int_{\{|z|=r\}} |G_j(z)|^2 \, d\theta\right) = S_{\hat{f}_{(2)}}(r; \Omega_2) \leqslant S_f(r; \Omega). \qquad \square \end{aligned}$$

LEMMA 3.9. *Let the notation be as above. Assume that $f: \mathbf{C} \to M$ has a finite order ϱ_f. Then $F_j(z)$, $1 \leqslant j \leqslant p$, are polynomials of degree at most ϱ_f, and G_k, $1 \leqslant k \leqslant m$, are polynomials of degree at most $\frac{1}{2}\varrho_f$; moreover, at least one of F_j has degree ϱ_f, or at least one of G_k has degree $\frac{1}{2}\varrho_f$.*

Proof. Let $\varepsilon > 0$ be an arbitrary positive number. Then there is an $r_0 > 0$ such that

$$T_f(r; \Omega) \leqslant r^{\varrho_f + \varepsilon}, \quad r \geqslant r_0.$$

It follows from (3.5)–(3.7) that for $r \geqslant r_0$

$$\begin{gathered} T_{\hat{f}_{(1)}}(r; \Omega_1) \leqslant r^{\varrho_f + \varepsilon}, \\ \frac{1}{2\pi} \int_0^{2\pi} \left(\sum_{j=1}^m |G_j(re^{i\theta})|^2 \right) d\theta \leqslant r^{\varrho_f + \varepsilon}. \end{gathered} \tag{3.10}$$

It follows from (3.10), (3.6) and (2.8) applied with $R=2r$ and $R'=3r$ that there is a positive constant C such that

$$\max_{|z|=r} |F(z)| \leqslant C r^{\varrho_f + \varepsilon}.$$

Therefore, $F_j(z)$ is a polynomial of degree at most ϱ_f.

Expand $G_j(z)=\sum_{\nu}^{\infty} c_{j\nu} z^{\nu}$. Then one gets

$$\frac{1}{2\pi}\int_0^{2\pi}\left(\sum_{j=1}^{m}|G_j(re^{i\theta})|^2\right)d\theta=\sum_{j=1}^{m}\sum_{\nu=0}^{\infty}|c_{j\nu}|^2 r^{2\nu}.$$

It follows that

$$\sum_{j=1}^{m}\sum_{\nu=1}^{\infty}|c_{j\nu}|^2 r^{2\nu}\leqslant r^{\varrho_f+\varepsilon},\quad r\geqslant r_0.$$

Hence, $c_{j\nu}=0$ for all $\nu>\frac{1}{2}\varrho_f$ and $1\leqslant j\leqslant m$. We see that $G_j(z)$, $1\leqslant j\leqslant m$, are polynomials of degree at most $\frac{1}{2}\varrho_f$. The remaining part is clear. □

In the language of Lie group theory we obtain the following characterization of holomorphic curves of finite order:

PROPOSITION 3.11. *Let M be an n-dimensional complex semi-torus with the above compactification $\overline{M}$, let* Lie(M) *be its Lie algebra, and let* $\exp\colon \mathrm{Lie}(M)\to M$ *be the exponential map. Let $f\colon \mathbf{C}\to M$ be a holomorphic curve. Then f is of finite order considered as a holomorphic curve into $\overline{M}$ if and only if there is a polynomial map $P\colon \mathbf{C}\to \mathrm{Lie}(M)\cong\mathbf{C}^n$ such that $f=\exp\circ P$, and hence the property of f being of finite order is independent of the choice of the compactification $\overline{M}\supset M$.*

4. Divisors on semi-tori

(a) Let M be a complex semi-torus as before:

$$0\to(\mathbf{C}^*)^p\to M\xrightarrow{\eta} M_0\to 0,$$
$$\bar\eta\colon \overline{M}\to M_0.$$

Let D be an effective divisor on M such that D is compactified to $\overline{D}$ in $\overline{M}$; i.e., roughly speaking, D is algebraic along the fibers of $M\to M_0$. If M is a semi-Abelian variety, then this condition is equivalent to the algebraicity of D. We equip $L(\overline{D})\to\overline{M}$ with a Hermitian fiber metric. Let $f\colon\mathbf{C}\to M$ be a holomorphic curve such that $f(\mathbf{C})\not\subset D$. Let Ω be as in (3.4). Then there is a positive constant C independent of f such that

$$T_f(r;L(\overline{D}))=N(r;f^*D)+m_f(r;\overline{D})+O(1)\leqslant CT_f(r;\Omega)+O(1). \tag{4.1}$$

LEMMA 4.2. *Let $M,\overline{M},M_0$ be as above. Let $L\to\overline{M}$ be a line bundle on $\overline{M}$. Then there exist a divisor E with* $\mathrm{Supp}\,E\subset\partial M$ *and a line bundle $L_0\to M_0$ such that $L\cong L(E)\otimes\bar\eta^*L_0$ (in the sense of bundle isomorphism or linear equivalence); moreover, such an $L_0\to M_0$ is uniquely determined (up to isomorphism).*

Proof. Note that $\bar{\eta}: \overline{M} \to M_0$ is a topologically trivial $\mathbf{P}_1(\mathbf{C})^p$-bundle over M_0. Hence by the Künneth formula we have

$$H^2(\overline{M}, \mathbf{Z}) = H^2(\mathbf{P}_1(\mathbf{C}), \mathbf{Z})^p \oplus H^2(M_0, \mathbf{Z}). \tag{4.3}$$

Since the higher direct image sheaves $\mathcal{R}^q \eta_* \mathcal{O}$, $q \geqslant 1$, vanish, it follows that $H^*(\overline{M}, \mathcal{O}) \cong H^*(M_0, \mathcal{O})$. We deduce that the Picard group $\mathrm{Pic}(\overline{M})$ is generated by $\bar{\eta}^* \mathrm{Pic}(M_0)$ and the subgroup of $\mathrm{Pic}(\overline{M})$ generated by the irreducible components of $\partial M = \overline{M} \setminus M$. Thus for $\bar{L} \to \overline{M}$ there exists a divisor E with $\mathrm{Supp}\, E \subset \partial M$ such that $L \otimes L(-E) \in \bar{\eta}^* \mathrm{Pic}(M_0)$; the assertion follows. □

We denote by

$$\mathrm{St}(D) = \{x \in M : x + D = D\}^0 \tag{4.4}$$

the identity component of those $x \in M$ which leaves D invariant by translation. The complex semi-subtorus $\mathrm{St}(D)$ (cf. [NW]) is called the *stabilizer* of D.

LEMMA 4.5. (i) *Let Z be a divisor on $\overline{M}$ such that $Z \cap M$ is effective. Let $L_0 \in$ $\mathrm{Pic}(M_0)$ such that $L(Z) \otimes \bar{\eta}^* L_0^{-1} \cong L(E)$ with* $\mathrm{Supp}\, E \subset \partial M$. *Then $c_1(L_0) \geqslant 0$.*

(ii) *Let D be an effective divisor on M with compactification $\bar{D}$ as above. Assume that* $\mathrm{St}(D) = \{0\}$. *Then $\bar{D}$ is ample on $\overline{M}$.*

Proof. (i) Assume the contrary. Recall that M_0 is a compact complex torus with universal covering $\pi_0: \mathbf{C}^m \to M_0$. We may regard the Chern class $c_1(L_0)$ as a bilinear form on the vector space $\mathbf{C}^m$. Suppose that $c_1(L_0)$ is not semi-positive definite. Let $v \in \mathbf{C}^m$ with $c_1(L_0)(v, v) < 0$ and let W denote the orthogonal complement of v (i.e., $W = \{w \in \mathbf{C}^m : c_1(L_0)(v, w) = 0\}$). Let μ be a semi-positive skew-Hermitian form on $\mathbf{C}^m$ such that $\mu(v, \cdot) \equiv 0$ and $\mu|_{W \times W} > 0$. Now consider the $(n-1, n-1)$-form ω on $\overline{M}$ given by

$$\omega = \Omega^p \wedge \bar{\eta}^* \mu^{m-1}. \tag{4.6}$$

By construction we have $\omega \wedge \bar{\eta}^* c_1(L_0) < 0$. Let $Z = Z' + Z''$ so that Z' is effective and has no component of ∂M, and $\mathrm{Supp}\, Z'' \subset \partial M$.

By the Poincaré duality,

$$\int_{\overline{M}} c_1(L(Z)) \wedge \omega = \int_Z \omega.$$

Since $\omega \wedge c_1(L(E)) = 0$, we have

$$\int_{\overline{M}} c_1(L(Z)) \wedge \omega = \int_{\overline{M}} \bar{\eta}^* c_1(L_0) \wedge \omega < 0.$$

On the other hand,

$$\int_Z \omega = \int_{Z'} \omega + \int_{Z''} \omega.$$

Note that $\int_{Z'} \omega \geqslant 0$, because Z' is effective and $\omega \geqslant 0$, and that $\int_{Z''} \omega = 0$, because $\mathrm{Supp}\, Z'' \subset \partial M$, and Ω^p vanishes on ∂M by construction. Thus we deduced a contradiction.

(ii) When $p=0$, the assertion is well known ([W]). Assume $p>0$. Let $\mathbf{C}^*$ act on $\overline{M}$ as the kth factor of $(\mathbf{C}^*)^p \subset M$. Since $\mathrm{St}(D)=\{0\}$, one infers that there is an orbit whose closure intersects $\overline{D}$ transversally. Hence,

$$c_1(L) = (n_1, \ldots, n_p; c_1(L_0)) \tag{4.7}$$

in the form described in (4.3) with $n_1, \ldots, n_p > 0$.

Now let us consider L_0 as in (i) above. By (i) we know that $c_1(L_0) \geqslant 0$. Assume that there is a vector $v \in \mathbf{C}^m \setminus \{O\}$ with $c_1(L_0)(v, v) = 0$. Then we choose μ and ω as in (4.6). Because of the definition we have

$$\int_{\overline{D}} \omega = 0. \tag{4.8}$$

By the flat connection of the bundle $\eta: M \to M_0$, the vector v is identified as a vector field on M. Observe that $\bar{\eta}(\overline{D}) = M_0$. The construction of ω and (4.8) imply that $v \in T_x(D)$ for all $x \in D$. It follows that the one-parameter subgroup corresponding to v must stabilize D; this is a contradiction. Thus $c_1(L_0) > 0$ if $\mathrm{St}(D) = \{0\}$.

Since all $n_i > 0$ in (4.7) and $c_1(L_0) > 0$, it follows that $c_1(L(\overline{D}))$ is positive. Thus $\overline{D}$ is ample on $\overline{M}$. □

COROLLARY 4.9. *Let $f: \mathbf{C} \to M$ and D be as above, and let Ω be as in* (3.4). *Assume that* $\mathrm{St}(D) = \{0\}$. *Then we have the following.*

(i) *There is a positive constant C such that*

$$C^{-1} T_f(r; \Omega) + O(1) \leqslant T_f(r; c_1(\overline{D})) \leqslant C T_f(r; \Omega) + O(1).$$

(ii) $S_f(r; \Omega) = S_f(r; c_1(\overline{D}))$.

The proof is clear.

Remark. $\overline{D}$ may be ample even if $\mathrm{St}(D) \neq \{0\}$. For instance, this happens for the diagonal divisor D in $M = \mathbf{C}^* \times \mathbf{C}^* \hookrightarrow \overline{M} = \mathbf{P}_1 \times \mathbf{P}_1$.

(b) *Boundary condition for D.* We keep the previous notations. Let

$$\partial M = \bigcup_{j=1}^{p} B_j \tag{4.10}$$

be the Whitney stratification of the boundary divisor of M in $\bar{M}$; i.e., B_j consists of all points $x \in \partial M$ such that the number of irreducible components of ∂M passing x is exactly j. Set $B_0 = M$. A connected component of B_j, $0 \leqslant j \leqslant p$, is called a *stratum* of the stratification $\bar{M} = \bigcup_{j=0}^{p} B_j$. Observe that $\dim B_j = n - j$.

Note that the holomorphic action of M on M by translations is equivariantly extended to an action on $\bar{M}$, which preserves every stratum of B_j, $0 \leqslant j \leqslant p$.

Let D be an effective divisor of M which can be extended to a divisor $\bar{D}$ on $\bar{M}$ by taking its topological closure of the support. We consider the following boundary condition for D:

Condition 4.11. $\bar{D}$ does not contain any stratum of B_p.

Note that the strata of B_p are minimal.

LEMMA 4.12. *If condition* 4.11 *is fulfilled, then*

$$\dim \bar{D} \cap B_j < \dim B_j = n - j, \quad 0 \leqslant \forall j \leqslant p.$$

Proof. Assume the contrary. Then there exists a stratum $S \subset B_j$ such that $S \subset \bar{D}$. Clearly the closure $\bar{S}$ of S is likewise contained in $\bar{D}$. But the closure of any stratum contains a minimal stratum, i.e., contains a stratum of B_p. However, this is in contradiction to condition 4.11. □

Example 4.13. Take a classical case where M is the complement of $n+1$ hyperplanes H_j of $\mathbf{P}^n(\mathbf{C})$ in general position. Then $M \cong (\mathbf{C}^*)^n$. Let $D = H_{n+2}$ be an $(n+2)$nd hyperplane of $\mathbf{P}^n(\mathbf{C})$. Then condition 4.11 is equivalent to that all H_j, $1 \leqslant j \leqslant n+2$, are in general position.

Next we interpret boundary condition 4.11 in terms of local defining equations of $\bar{D}$. Take $\sigma \in H^0(\bar{M}, L(\bar{D}))$ such that $(\sigma) = \bar{D}$. Suppose that $p > 0$. Let $x_0 \in \partial M \cap \bar{D}$ be an arbitrary point. Let E and L_0 be as in Lemma 4.2 for $L = L(\bar{D})$. We take an open neighborhood U of $\bar{\eta}(x_0)$ such that the restrictions $\bar{M}|U$ and $L_0|U$ to U are trivialized. Write

$$x_0 = (u_0, x_0'') \in (\mathbf{P}^1(\mathbf{C}))^p \times U \cong \bar{M}|U.$$

We take an open neighborhood V of u_0 such that $V \cong \mathbf{C}^p \subset (\mathbf{P}^1(\mathbf{C}))^p$ with coordinates $(u_1, ..., u_p)$. Then $L(\bar{D})|(V \times U)$ is trivial, and hence $\sigma|(V \times U)$ is given by a polynomial function

$$\sigma(u, x'') = \sum_{\text{finite}} a_{l_1 \ldots l_p}(x'') u_1^{l_1} \ldots u_p^{l_p}, \quad (u, x'') \in V \times U, \tag{4.14}$$

with coefficients $a_{l_1 \ldots l_p}(x'')$ holomorphic in U. Since $\bar{D}$ has no component of ∂M, $\sigma(u, x'')$ is not divisible by any u_j. Set $u_0 = (u_{01}, ..., u_{0p})$. Then, after a change of indices of u_i

one may assume that $u_{01}=\dots=u_{0q}=0$, $u_{0i}\neq 0$, $1\leqslant q<i\leqslant p$. Expand $\sigma(u,x'')$ and set σ_1 and σ_2 as follows:

$$\begin{aligned}\sigma(u,x'') &= \sum_{l_1+\dots+l_q\geqslant 1} a_{l_1\dots l_p}(x'')u_1^{l_1}\dots u_p^{l_p}+\sum_{l_1=\dots=l_q=0} a_{0\dots 0 l_{q+1}\dots l_p}(x'')u_{q+1}^{l_{q+1}}\dots u_p^{l_p},\\ \sigma_1 &= \sum_{l_1+\dots+l_q\geqslant 1} a_{l_1\dots l_p}(x'')u_1^{l_1}\dots u_p^{l_p},\\ \sigma_2 &= \sum_{l_1=\dots=l_q=0} a_{0\dots 0 l_{q+1}\dots l_p}(x'')u_{q+1}^{l_{q+1}}\dots u_p^{l_p}.\end{aligned} \tag{4.15}$$

We have

LEMMA 4.16. *Let the notation be as above. Then condition* 4.11 *is equivalent to that for every* $x_0\in\partial M$, $\sigma_2\not\equiv 0$.

(c) *Regularity of stabilizers.* Let M be a complex semi-torus with fixed presentation as in (3.1):

$$0\to G=(\mathbf{C}^*)^p\to M\to M_0\to 0. \tag{4.17}$$

Definition. A closed complex Lie subgroup H of M is called *regular* if there is a subset $I\subset\{1,\dots,p\}$ such that

$$G\cap H=\{(z_1,\dots,z_p)\in G: z_i=1 \text{ for all } i\in I\}.$$

Regular subgroups are those compatible with the compactification induced by (4.17). The presentation (4.17) induces in a canonical way such presentations for H and M/H.

LEMMA 4.18. *Let H be a regular Lie subgroup of M. Then the quotient mapping $M\to M/H$ is extended holomorphically in a natural way to the compactification*

$$\overline{M}\xrightarrow{\bar{H}}\overline{(M/H)},$$

which is a holomorphic fiber bundle of compact complex manifolds with fiber $\overline{H}$.

We will prove the following proposition.

PROPOSITION 4.19. *Let M be a semi-torus with presentation* (4.17) *and let D be an effective divisor fulfilling condition* 4.11. *Then there exists a finite unramified covering $\mu': M_0'\to M_0$ such that* $\mathrm{St}(\mu^*D)$ *is regular in M', where $\mu: M'\to M$ is the finite covering of M induced by μ'; i.e., $M'=M\times_{M_0}M_0'$.*

Remark. Note that μ extends holomorphically to the unramified covering of the compactification $\overline{M}$, $\bar{\mu}: \overline{M}'\to\overline{M}$.

Proof. First, if D is invariant under one of the p direct factors of $G=(\mathbf{C}^*)^p$ in (4.17), we take the corresponding quotient. Thus we may assume that $\mathrm{St}(D)\cap G$ does not contain anyone of the p coordinate factors of G.

Assume that $\dim \mathrm{St}(D)\cap G>0$. Let I be a subgroup of $\mathrm{St}(D)\cap G$ isomorphic to $\mathbf{C}^*$. Then there are integers $n_1, \ldots, n_p$ such that

$$I=\{(t^{n_1}, \ldots, t^{n_p}) : t\in \mathbf{C}^*\}.$$

By rearranging indices and coordinate changes of type $z_i \mapsto 1/z_i$, we may assume that there is a natural number q such that $n_i>0$ for $i\leqslant q$ and $n_i=0$ for $i>q$. Let $G=G_1\times G_2$ with

$$G_1=\{(u_1, \ldots, u_q, \underbrace{1, \ldots, 1}_{p-q}) : u_i\in \mathbf{C}^*\}\subset G,$$

$$G_2=\{(\underbrace{1, \ldots, 1}_{q}, u_{q+1}, \ldots, z_p) : u_i\in \mathbf{C}^*\}\subset G.$$

Then $I\subset G_1$. Consider $\lambda: M\to M/G_1$. If $\lambda(D)\neq M/G_1$, then D would be G_1-invariant and in particular would be invariant under the coordinate factor groups contained in G_1. Since this was ruled out, we have $\lambda(D)=M/G_1$. Now observe that for every $u=(u_1, \ldots, u_p)\in \mathbf{C}^p\subset(\mathbf{P}^1(\mathbf{C}))^p$ we have

$$\lim_{t\to 0}(t^{n_1}, \ldots, t^{n_p})\cdot u=(0, \ldots, 0, u_{q+1}, u_{q+2}, \ldots, u_p).$$

Hence it follows from $I\subset \mathrm{St}(D)$ and $\lambda(D)=M/G_1$ that

$$\{0\}^q\times(\mathbf{P}^1(\mathbf{C}))^{p-q}\subset \bar{D}.$$

This violates condition 4.11 because of Lemma 4.12. Thus $G\cap \mathrm{St}(D)$ is zero-dimensional, and hence finite. As a consequence, $\mathrm{St}(D)$ is compact. After a finite covering, $\mathrm{St}(D)$ maps injectively in M_0 and therefore is regular. □

5. Proof of the Main Theorem

We first prove the following key lemma:

LEMMA 5.1. *Assume the same conditions as in the Main Theorem. Then,*

$$m_f(r;\bar{D})=S_f(r;c_1(\bar{D})).$$

Besides the conditions stated above, we may also assume by Proposition 4.19 and Lemma 4.5 (ii) that $\mathrm{St}(D)=\{0\}$, $\bar{D}$ is ample on $\bar{M}$, and hence M is a semi-Abelian variety A:

$$0 \to (\mathbf{C}^*)^p \to A \to A_0 \to 0.$$

We keep these throughout in this section.

Here we need the notion of logarithmic jet spaces due to [No3]. Since ∂A has only normal crossings, we have the logarithmic kth jet bundle $J_k(\bar{A}; \log \partial A)$ over $\bar{A}$ along ∂A, and a morphism

$$\psi_k: J_k(\bar{A}; \log \partial A) \to J_k(\bar{A})$$

such that the sheaf of germs of holomorphic sections of $J_k(\bar{A}; \log \partial A)$ is isomorphic to that of logarithmic k-jet fields (see [No3, Proposition (1.15)]; there, a "subbundle" $J_k(\bar{A}; \log \partial A)$ of $J_k(\bar{A})$ should be understood in this way). Because of the flat structure of the logarithmic tangent bundle $\mathbf{T}(\bar{A}; \log \partial A)$,

$$J_k(\bar{A}; \log \partial A) \cong \bar{A} \times \mathbf{C}^{nk}.$$

Let

$$\begin{aligned} \pi_1 &: J_k(\bar{A}; \log \partial A) \cong \bar{A} \times \mathbf{C}^{nk} \to \bar{A}, \\ \pi_2 &: J_k(\bar{A}; \log \partial A) \cong \bar{A} \times \mathbf{C}^{nk} \to \mathbf{C}^{nk} \end{aligned} \tag{5.2}$$

be the first and the second projections. For a k-jet $y \in J_k(\bar{A}; \log \partial A)$ we call $\pi_2(y)$ the *jet part* of y.

Let $x \in \bar{D}$ and let $\sigma=0$ be a local defining equation of $\bar{D}$ about x. For a germ $g: (\mathbf{C}, 0) \to (A, x)$ of a holomorphic mapping we denote its k-jet by $j_k(g)$ and write

$$d^j\sigma(g) = \left.\frac{d^j}{d\zeta^j}\right|_{\zeta=0} \sigma(g(\zeta)).$$

We set

$$\begin{aligned} J_k(\bar{D})_x &= \{j_k(g) \in J_k(\bar{A})_x : d^j\sigma(g) = 0,\ 1 \leqslant j \leqslant k\}, \\ J_k(\bar{D}) &= \bigcup_{x \in \bar{D}} J_k(\bar{D})_x, \\ J_k(\bar{D}; \log \partial A) &= \psi_k^{-1} J_k(\bar{D}). \end{aligned}$$

Then $J_k(\bar{D}; \log \partial A)$ is a subspace of $J_k(\bar{A}; \log \partial A)$, which is depending in general on the embedding $\bar{D} \hookrightarrow \bar{A}$ (cf. [No3]). Note that $\pi_2(J_k(\bar{D}; \log \partial A))$ is an algebraic subset of $\mathbf{C}^{nk}$.

Let $J_k(f): \mathbf{C} \to J_k(\bar{A}; \log \partial A) = \bar{A} \times \mathbf{C}^{nk}$ be the kth jet lifting of f. Then by [No5] the Zariski closure of $J_k(f)(\mathbf{C})$ in $J_k(\bar{A}; \log \partial A)$ is of the form, $\bar{A} \times W_k$, with an affine irreducible subvariety $W_k \subset \mathbf{C}^{nk}$. Let $\pi: \mathbf{C}^n \to A$ be the universal covering and let

$$\tilde{f}: z \in \mathbf{C} \to (\tilde{f}_1(z), \ldots, \tilde{f}_n(z)) \in \mathbf{C}^n$$

be the lifting of f. Assume that f is of finite order. Then $\tilde{f}(z)$ is a vector-valued polynomial by Lemma 3.9. Note that every non-constant polynomial map from $\mathbf{C}$ to $\mathbf{C}^n$ is proper, and hence the image is an algebraic subset. It follows that

$$W_k = \overline{\{(\tilde{f}'(z), ..., \tilde{f}^{(k)}(z)), z \in \mathbf{C}\}} = \{(\tilde{f}'(z), ..., \tilde{f}^{(k)}(z)), z \in \mathbf{C}\},$$

and hence $\dim W_k \leqslant 1$. Thus we deduced the following lemma.

LEMMA 5.3. *Let the notation be as above. If $f: \mathbf{C} \to A$ is of finite order, then* $\dim W_k \leqslant 1$ *and for every point $w_k \in W_k$ there is a point $a \in \mathbf{C}$ with $\pi_2 \circ J_k(f)(a) = w_k$.*

We recall the logarithmic Bloch–Ochiai theorem as a lemma ([No1], [No2]; cf. also [NW]):

LEMMA 5.4. *Let $g: \mathbf{C} \to A$ be an arbitrary holomorphic curve into an Abelian subvariety A. Then the Zariski closure of $g(\mathbf{C})$ is a translate of a semi-Abelian subvariety of A.*

We also need

LEMMA 5.5. (i) *Let Y be an irreducible subvariety of A such that* $\mathrm{St}(Y) = \{0\}$. *Let Z be the set of those points $x \in Y$ such that there exists a translate T of a non-trivial semi-Abelian subvariety of A with $x \in T \subset Y$. Then, Z is a proper algebraic subset of Y, and decomposes to finitely many irreducible components Z_i such that* $\mathrm{St}(Z_i) \neq \{0\}$.

(ii) *Furthermore, there are finitely many non-trivial semi-Abelian subvarieties A'_j of A such that every T as above is contained in a translate of some A'_j.*

Proof. The statement (i) is Lemma (4.1) of [No2] (cf. [Ka] for the Abelian case).

For (ii) we first note that $T \subset Z_i$ for some Z_i. Then we consider the quotients $T/\mathrm{St}(Z_i) \subset Z_i/\mathrm{St}(Z_i)$. We have that $\dim Z_i/\mathrm{St}(Z_i) < \dim Z_i$ and $\mathrm{St}(Z_i/\mathrm{St}(Z_i)) = \{0\}$. If $T/\mathrm{St}(Z_i)$ is a point, then T is contained in a translate of $\mathrm{St}(Z_i)$. Otherwise, we repeat this quotient process by making use of (i). Because the dimension of the quotient space strictly decreases at every step, we find finitely many non-trivial semi-Abelian subvarieties A'_j of A such that every T as in (i) is contained in a translate of some A'_j. □

LEMMA 5.6. *Let the notation be as above. Then there is a number $k_0 = k_0(f, D)$ such that*

$$\pi_2(J_k(\bar{D}; \log \partial A)) \cap W_k \neq W_k, \quad k \geqslant k_0.$$

Moreover, if f is of finite order ϱ_f, then k_0 depends only on ϱ_f and D.

Proof. (a) We first assume that f is of finite order ϱ_f. We see by Lemma 3.9 that $\varrho_f \in \mathbf{Z}$, and $\tilde{f}(z)$ is a vector-valued polynomial of order $\leqslant \varrho_f$. Thus, W_k, $k \geqslant \varrho_f$, is of form

$$W_k = (W_{\varrho_f}, \underbrace{O, ..., O}_{k-\varrho_f}).$$

Take an arbitrary point $\xi_{\varrho_f} \in W_{\varrho_f}$, and set

$$\xi_k = (\xi_{\varrho_f}, \underbrace{O, \dots, O}_{k-\varrho_f}) \in W_k \subset \mathbf{C}^{nk}, \quad k \geqslant \varrho_f.$$

Assume that

$$\xi_k \in \pi_2(J_k(\bar{D}; \log \partial A)) \quad \text{for all } k \geqslant \varrho_f. \tag{5.7}$$

We identify ξ_k with a logarithmic k-jet field on $\bar{A}$ along ∂A (see [No3]). Set

$$S_k = \pi_1(J_k(\bar{D}; \log \partial A) \cap \pi_2^{-1}(\xi_k)).$$

Then,

$$\bar{D} \supset S_{\varrho_f} \supset S_{\varrho_f+1} \supset \dots,$$

which stabilize to $S_0 = \bigcap_{k=\varrho_f}^{\infty} S_k \neq \varnothing$. Let $x_0 \in S_0$. If $x_0 \in D$, it follows from Lemma 5.3 that there are points $a \in \mathbf{C}$ and $y_0 \in A$ such that

$$f(a) + y_0 = x_0 \in D,$$
$$\left.\frac{d^k}{dz^k}\right|_{z=a} \sigma(f(z) + y_0) = 0 \quad \text{for all } k \geqslant 1,$$

where σ is a local defining function of D about x_0. Therefore

$$f(\mathbf{C}) + y_0 \subset D,$$

and hence this contradicts the Zariski denseness of $f(\mathbf{C})$ in A. Moreover in this case, it follows from Lemma 5.4 that $f(\mathbf{C}) + y_0$ is contained in a translate of a semi-Abelian subvariety of A contained in D.

Suppose now that $x_0 \in \bar{D} \setminus A$. Let $\partial A = \bigcup B_j$ be the Whitney stratification as in (4.10), and let $x_0 \in B_q$. Let B be the stratum of B_q containing x_0. Then B itself is a semi-Abelian variety such that

$$0 \to (\mathbf{C}^*)^{p-q} \to B \to A_0 \to 0.$$

Let $\sigma(u, x'') = \sigma_1(u, x'') + \sigma_2(u, x'')$ be as in (4.15) and define $\bar{D}$ in a neighborhood W of x_0 such that W is of type $V \times U$ as in (4.14). It follows from Lemma 4.16 that $\sigma_2 \not\equiv 0$. Note that $\bar{D} \cap W \cap B$ is defined by $\sigma_2 = 0$ in B. There is a point $a \in \mathbf{C}$ such that $\pi_2 \circ J_{\varrho_f}(f)(a) = \xi_{\varrho_f}$. Dividing the coordinates into three blocks, we set

$$x_0 = (\underbrace{0, \dots, 0}_{q}, x_0', x_0'').$$

We may regard $w_0=(x'_0,x''_0)\in B$. Taking a shift $f(z)+y_0$ with $y_0\in A$ so that $f(a)+y_0\in W$, we set in a neighborhood of $a\in \mathbf{C}$

$$\begin{aligned} f(z)+y_0&=(u_1(z),\dots,u_q(z),u_{q+1}(z),\dots,u_p(z),x''(z))\in W,\\ g(z)&=(u_{q+1}(z),\dots,u_p(z),x''(z))\in W\cap B.\end{aligned}\tag{5.8}$$

Here we may choose y_0 so that $g(a)=w_0$.

We set $\xi_k=\pi_2\circ J_k(f)(a)$ for all $k\geqslant 1$. Using the same coordinate blocks as (5.8), we set

$$\xi_k=(\xi'_{k(1)},\xi'_{k(2)},\xi''_k),\quad k\geqslant \varrho_f,$$
$$\xi_{k(2)}=\text{the jet part of } J_k(g)(a)=(\xi'_{k(2)},\xi''_k).$$

Since the logarithmic term (e.g., $z_j\,\partial/\partial z_j$, $1\leqslant j\leqslant q$, in the case of 1-jets) of a logarithmic jet field vanishes on the corresponding divisor locus (e.g., $\bigcup_{j=1}^q\{z_j=0\}$) (see [No3, §1 and (1.14)] for more details), we have $\xi_k(\sigma_1)(x_0)=0$ by (4.15), and hence $\xi_{k(2)}(\sigma_2)(x_0)=0$ for all $k\geqslant 1$; i.e.,

$$\left.\frac{d^k}{dz^k}\right|_{z=a}\sigma_2(g(z))=0\quad \text{for all } k\geqslant 0.\tag{5.9}$$

Let $(\mathbf{C}^*)^q$ be the first q-factor of the subgroup $(\mathbf{C}^*)^p\subset A$, and let $\lambda_B\colon A\to A/(\mathbf{C}^*)^q\cong B$ be the quotient map. Set

$$f_B=\lambda_B\circ f\colon \mathbf{C}\to B.$$

It follows from (5.8) and (5.9) that the composed map, $\lambda_B\circ(f(z)+y_0)=f_B(z)+\lambda_B(y_0)$, has an image contained in $\bar{D}\cap B$; furthermore in this case, by Lemma 5.4, $f_B(\mathbf{C})+\lambda_B(y_0)$ is contained in a translate of a semi-Abelian subvariety of B contained in $B\cap\bar{D}$. Thus, it has no Zariski dense image in $A/(\mathbf{C}^*)^q$, and hence so is f; this is a contradiction.

Summarizing what was proved, we have

SUBLEMMA 5.10. *Assume* (5.7) *for* $\xi_k=(\xi_{\varrho_f},O,\dots,O)$. *Then a translate of* $f(\mathbf{C})$ *is contained in a translate of a non-trivial semi-Abelian subvariety of* A *contained in* D, *or the same holds for* f_B *and* $B\cap\bar{D}$, *where* B *is a boundary stratum of* A *as above.*

(b) Here we show that k_0 depends only on the order $\varrho\,(<\infty)$ of f and D. As the claimed property is invariant by translates of f, one may assume that $f(0)=0$ and $\tilde f(0)=0$. As $\tilde f$ runs over some vector-valued polynomials of order at most ϱ with $\tilde f(0)=0$, one may parameterize them by their coefficients which are points of $\mathbf{C}^{n\varrho}$. We denote by $f_P\colon\mathbf{C}\to A$ and $\tilde f_P\colon \mathbf{C}\to\mathbf{C}^n$ the mappings defined by a point $P\in\mathbf{C}^{r\varrho}\setminus\{0\}$ such that $f_P(0)=0$ and $\tilde f_P(0)=0$.

If a translate of f_P is contained in D, it follows from Lemma 5.4 that there is a non-trivial semi-Abelian subvariety A' of A such that a translate of A' is contained in D

and $f_P(\mathbf{C}) \subset A'$. By Lemma 5.5 (ii) applied to $Y=D$, there are finitely many non-trivial semi-Abelian subvarieties A'_j such that $f_P(\mathbf{C}) \subset A'_j$ for some A'_j. The condition that $f_P(\mathbf{C}) \subset A'_j$ is a non-trivial linear condition for $P \in \mathbf{C}^{n\varrho} \setminus \{0\}$. One may apply the same for f_B and $B \cap \bar{D}$, and notes that the number of those B is finite. It is deduced that the set Ξ of points $P \in \mathbf{C}^{n\varrho} \setminus \{0\}$ such that the conclusion of Sublemma 5.10 applied to f_P and $\xi_k = J_k(f_P)(0)$ does not hold is a non-empty Zariski open subset of $\mathbf{C}^{n\varrho}$.

Set $W_k(f_P) = \pi_2(J_k(f_P)(\mathbf{C}))$. It follows from Sublemma 5.10 that for every $P \in \Xi$ there is a number k satisfying

$$\pi_2(J_k(\bar{D}; \log \partial A)) \cap W_k(f_P) \neq W_k(f_P).$$

This is a Zariski open condition for $P \in \Xi$, which defines a Zariski open neighborhood $\Xi_k \subset \Xi$ of the point P. By the Noetherian property, Ξ is covered by finitely many Ξ_{k_ν}'s. Set $k_0 = \max\{k_\nu\}$.

Since our f has a Zariski dense image in A, we have $P \in \Xi$ with writing $f = f_P$; this completes the proof for f of finite order.

(c) Let f be of infinite order. Assume contrarily that $\pi_2(J_k(\bar{D}; \log \partial A)) \cap W_k = W_k$ for all $k \geqslant 1$. Since $\pi_2 \circ J_k(f)(0) \in W_k$ for all $k \geqslant 1$, we apply the same argument as in (a) with setting $\xi_k = \pi_2 \circ J_k(f)(0)$. Then we deduce a contradiction that f has no Zariski dense image. □

Remark 5.11. By the proof of Lemma 5.6 (a) we see that if $\varrho_f < \infty$, there is a number $k_1(f, D)$ satisfying

$$\pi_2(J_k(\bar{D}; \log \partial A)) \cap W_k = \varnothing, \quad k \geqslant k_1(f, D).$$

This implies that f has no intersection point of order $\geqslant k_1(f, D)$ with D.

Proof of Lemma 5.1. For a multiple $l\bar{D}$ of $\bar{D}$ we have

$$m_f(r; l\bar{D}) = l m_f(r; \bar{D}).$$

Thus we may assume that $\bar{D}$ is very ample on $\bar{A}$. Let $\{\tau_j\}_{j=1}^N$ be a base of $H^0(\bar{A}, L(\bar{D}))$ such that $\mathrm{Supp}(\tau_j) \not\supset f(\mathbf{C})$ for all $1 \leqslant j \leqslant N$. Since $\bar{D}$ is very ample, the sections τ_j, $1 \leqslant j \leqslant N$, have no common zero. Set

$$U_j = \{\tau_j \neq 0\}, \quad 1 \leqslant j \leqslant N.$$

Then $\{U_j\}$ is an affine open covering of $\bar{A}$. Let $\sigma \in H^0(\bar{A}, L(\bar{D}))$ be a section such that $(\sigma) = \bar{D}$. We define a regular function σ_j on every U_j by

$$\sigma_j(x) = \frac{\sigma(x)}{\tau_j(x)}.$$

Note that σ_j is a defining function of $\bar{D}\cap U_j$. Let us now fix a Hermitian metric $\|\cdot\|$ on $L(\bar{D})$. Then there are positive smooth functions h_j on U_j such that

$$\frac{1}{\|\sigma(x)\|}=\frac{h_j(x)}{|\sigma_j(x)|}, \quad x\in U_j.$$

It follows from Lemma 5.6 that there exists a polynomial function $R(w)$ in $w\in W_{k_0}$ such that

$$\pi_2(J_{k_0}(\bar{D};\log\partial A))\cap W_{k_0}\subset\{w\in W_{k_0}:R(w)=0\}\neq W_{k_0}.$$

We regard R as a regular function on every $U_j\times W_{k_0}$. Then we have the following equation on every $U_j\times W_{k_0}$:

$$b_{j0}\sigma_j+b_{j1}d\sigma_j+...+b_{jk_0}d^{k_0}\sigma_j=R(w), \tag{5.12}$$

where $b_{ji}=\sum_{\text{finite}}b_{jil\beta_l}(x)w_l^{\beta_l}$ are regular functions on $U_j\times W_{k_0}$. Thus we infer that in every $U_j\times W_{k_0}$

$$\frac{1}{\|\sigma\|}=\frac{1}{|R|}\frac{h_j}{|\sigma_j|}=\frac{1}{|R|}\left|h_jb_{j0}+h_jb_{j1}\frac{d\sigma_j}{\sigma_j}+...+h_jb_{jk_0}\frac{d^{k_0}\sigma_j}{\sigma_j}\right|. \tag{5.13}$$

Take relatively compact open subsets $U'_j\Subset U_j$ so that $\bigcup U'_j=\bar{A}$. For every j there is a positive constant C_j such that for $x\in U'_j$

$$h_j|b_{ji}|\leqslant\sum_{\text{finite}}h_j|b_{jil\beta_l}(x)|\cdot|w_l|^{\beta_l}\leqslant C_j\sum_{\text{finite}}|w_l|^{\beta_l}.$$

Thus, after making C_j larger if necessary, there is a number $d_j>0$ such that for $f(z)\in U'_j$

$$h_j(f(z))|b_{ji}(J_{k_0}(f)(z))|\leqslant C_j\Bigg(1+\sum_{\substack{1\leqslant l\leqslant n\\1\leqslant k\leqslant k_0}}|\tilde{f}_l^{(k)}(z)|\Bigg)^{d_j}.$$

We deduce that

$$\begin{aligned}\frac{1}{\|\sigma(f(z))\|}\leqslant{}&\frac{1}{|R(\tilde{f}'(z),...,\tilde{f}^{(k_0)}(z))|}\sum_{j=1}^{N}C_j\Bigg(1+\sum_{\substack{1\leqslant l\leqslant n\\1\leqslant k\leqslant k_0}}|\tilde{f}_l^{(k)}(z)|\Bigg)^{d_j}\\&\times\left(1+\left|\frac{d\sigma_j}{\sigma_j}(J_1(f)(z))\right|+...+\left|\frac{d^{k_0}\sigma_j}{\sigma_j}(J_{k_0}(f)(z))\right|\right).\end{aligned} \tag{5.14}$$

Hence one gets

$$\begin{aligned}
m_f(r;\bar{D}) &= \frac{1}{2\pi}\int_0^{2\pi}\log^+\frac{1}{\|\sigma(f(re^{i\theta}))\|}\,d\theta+O(1)\\
&\leqslant m\Bigl(r,\frac{1}{R(\tilde{f}',\dots,\tilde{f}^{(k_0)})}\Bigr)+O\Bigl(\sum_{\substack{1\leqslant l\leqslant n\\ 1\leqslant k\leqslant k_0}}\frac{1}{2\pi}\int_0^{2\pi}\log^+|\tilde{f}_l^{(k)}(re^{i\theta})|\,d\theta\\
&\quad+\sum_{\substack{1\leqslant j\leqslant N\\ 1\leqslant k\leqslant k_0}}\frac{1}{2\pi}\int_0^{2\pi}\log^+\Bigl|\frac{d^k\sigma_j}{\sigma_j}(J_k(f)(re^{i\theta}))\Bigr|\,d\theta\Bigr)+O(1) \qquad (5.15)\\
&\leqslant T(r,R(\tilde{f}',\dots,\tilde{f}^{(k_0)}))\\
&\quad+O\Bigl(\sum_{\substack{1\leqslant l\leqslant n\\ 1\leqslant k\leqslant k_0}}m(r,\tilde{f}_l^{(k)})+\sum_{\substack{1\leqslant j\leqslant N\\ 1\leqslant k\leqslant k_0}}m\Bigl(r,\frac{d^k\sigma_j}{\sigma_j}(J_k(f))\Bigr)\Bigr)+O(1).
\end{aligned}$$

Recall that the rational functions σ_j are equal to quotients of two holomorphic sections σ and τ_j of $L(\bar{D})$. By Lemma 2.5 (ii) we see that

$$m\Bigl(r,\frac{(\sigma_j\circ f)^{(k)}}{\sigma_j\circ f}\Bigr)=S_f(r;c_1(\bar{D})).$$

This combined with (5.15) and Lemma 3.8 implies that $m_f(r;\bar{D})=S_f(r;c_1(\bar{D}))$. □

Proof of the Main Theorem. We keep the notation used above. Thanks to Lemma 5.1 the only thing we still have to show is the statement on the truncation, i.e., the bounds on $N(r;f^*D)-N_{k_0}(r;f^*D)$. Observe that $\operatorname{ord}_z f^*D>k$ if and only if $J_k(f)(z)\in J_k(\bar{D};\log\partial A)$. We infer from (5.14) that

$$\operatorname{ord}_z f^*D-\min\{\operatorname{ord}_z f^*D,k_0\}\leqslant\operatorname{ord}_z(R(\tilde{f}',\dots,\tilde{f}^{k_0}))_0.$$

Thus we have after integration that

$$N(r;f^*D)-N_{k_0}(r;f^*D)\leqslant N(r;(R(\tilde{f}',\dots,\tilde{f}^{(k_0)}))_0).$$

It follows from (2.2), (2.3) and Lemma 3.8 that

$$\begin{aligned}
N(r;(R(\tilde{f}',\dots,\tilde{f}^{(k_0)}))_0)&\leqslant T(r,R(\tilde{f}',\dots,\tilde{f}^{(k_0)}))+O(1)\\
&\leqslant O\Bigl(\sum_{\substack{1\leqslant l\leqslant n\\ 1\leqslant k\leqslant k_0}}T(r,\tilde{f}_l^{(k)})\Bigr)=S_f(r;\Omega).
\end{aligned}$$

Furthermore, $S_f(r;\Omega)=S_f(r,c_1(\bar{D}))$ by Corollary 4.9 (ii), because $\bar{D}$ is ample. Hence,

$$N(r;f^*D)\leqslant N_{k_0}(r;f^*D)+S_f(r;c_1(\bar{D})).$$

The proof is completed. □

We have the following immediate consequence of the Main Theorem.

COROLLARY 5.16. *Let M be a complex torus and let $f: \mathbf{C} \to M$ be an arbitrary holomorphic curve. Let D be an effective divisor on M such that $D \not\supset f(\mathbf{C})$. Then we have the following.*

(i) *Suppose that f is of finite order ϱ_f. Then there is a positive integer $k_0 = k_0(\varrho_f, D)$ such that*

$$T_f(r; c_1(D)) = N_{k_0}(r; f^*D) + O(\log r).$$

(ii) *Suppose that f is of infinite order. Then there is a positive integer $k_0 = k_0(f, D)$ such that*

$$T_f(r; c_1(D)) = N_{k_0}(r; f^*D) + S_f(r; c_1(D)).$$

In particular, $\delta(f; D) = \delta_{k_0}(f; D) = 0$ in both cases.

Proof. Since the Zariski closure of $f(\mathbf{C})$ is a translation of a complex subtorus of M (cf., e.g., [NO, Chapter VI], [Ks, Chapter 3, §9], [NW]), we may assume that $f(\mathbf{C})$ is Zariski dense. Hence this statement is a special case of the Main Theorem. □

PROPOSITION 5.17. *Let M be a complex semi-torus M and let D be an effective divisor on M such that its topological closure $\bar{D}$ is a divisor in $\bar{M}$. Assume that D violates condition 4.11. Then there exists an entire holomorphic curve $f: \mathbf{C} \to M$ of an arbitrarily given integral order $\varrho \geqslant 2$ in general, and $\varrho \geqslant 1$ in the case of $M_0 = \{0\}$ such that $f(\mathbf{C})$ is Zariski dense in M and $\delta(f; \bar{D}) > 0$.*

Proof. Let $\widehat{M} = (\mathbf{P}^1(\mathbf{C}))^p \times \mathbf{C}^m \to \bar{M}$ (resp. $\mathbf{C}^m \to M_0$) be the universal covering of $\bar{M}$ (resp. M_0), and $\widehat{D} \subset \widehat{M}$ the preimage of $\bar{D}$. We may assume that

$$\{(\infty)\}^p \times \mathbf{C}^m \subset \widehat{D}.$$

Let $c_1, ..., c_p$ be $\mathbf{Q}$-linear independent real numbers with

$$0 < c_1 < c_2 < ... < c_p. \tag{5.18}$$

Let $\varrho \geqslant 2$ or $\varrho \geqslant 1$ be an arbitrary integer as assumed in the proposition, and set

$$\hat{f}: z \mapsto ([1 : e^{c_1 z^\varrho}], [1 : e^{c_2 z^\varrho}], ..., [1 : e^{c_p z^\varrho}]; L(z)), \tag{5.19}$$

where $L: \mathbf{C} \to \mathbf{C}^m$ is a linear map such that the image of $L(\mathbf{C})$ in M_0 is Zariski dense. Moreover, by a generic choice of c_j and L we have that $f(\mathbf{C})$ is Zariski dense in M. Let

$$U_i \Subset V_i \Subset M_0$$

be a finite collection of relatively compact holomorphically convex open subsets of M_0 such that there are sections $\mu_i: V_i \xrightarrow{\sim} \widehat{V}_i \subset \mathbf{C}^m$ and such that the U_i cover M_0. Set $\widehat{U}_i = \mu_i(U_i)$.

For every i the restricted divisor $\widehat{D}|((\mathbf{P}^1(\mathbf{C}))^p\times\widehat{V}_i)$ is defined by a homogeneous polynomial P_{i0} of multidegree $(d_1,\ldots,d_p)$, where the coefficients are holomorphic functions on V_i. Let P_i denote the associated inhomogeneous polynomial. Then P_i is a polynomial of multidegree $(d_1,\ldots,d_p)$. Due to $\{\infty\}^p\times\mathbf{C}^m\subset\widehat{D}$, P_i does not carry the highest degree monomial, $u_1^{d_1}\ldots u_p^{d_p}$.

Recall that $\overline{M}=\widehat{M}/\Lambda_0$, where Λ_0 is a lattice in $\mathbf{C}^m$ and acts on $\widehat{M}$ via

$$\lambda\colon(u_1,\ldots,u_p;x'')\mapsto\lambda\cdot(u;x'')=(\beta_1(\lambda)u_1,\ldots,\beta_p(\lambda)u_p;x''+\lambda),$$

where $\beta\colon\Lambda_0\to(S^1)^p$ is a group homomorphism into the product of $S^1=\{|z|=1:z\in\mathbf{C}^*\}$.

Together with (5.19) and (5.18) this implies that there is a constant $C>0$ such that

$$|P_i(\lambda\cdot\hat{f}(z))|\leqslant C|e^{(\sum_j d_jc_j)z^\varrho-c_1z^\varrho}| \tag{5.20}$$

for all $\lambda\in\Lambda_0$ and $z\in\mathbf{C}$ with $\operatorname{Re}z^\varrho>0$ and $\lambda\cdot\hat{f}(z)\in(\mathbf{P}^1(\mathbf{C}))^p\times\widehat{U}_i$. Note that for every $z\in\mathbf{C}$ there exists an element $\lambda\in\Lambda_0$ and an index i such that $\lambda\cdot\hat{f}(z)\in(\mathbf{P}^1(\mathbf{C}))^p\times\widehat{U}_i$. Then there is a constant $C'>0$ such that

$$\|\sigma(x)\|^2\leqslant C'\frac{|P_i(\lambda\cdot x)|^2}{\prod_j(1+|u_j|^2)^{d_j}} \tag{5.21}$$

for all $x\in\widehat{M}$, $\lambda\in\Lambda_0$ with $\lambda\cdot x\in U_i$. From (5.20) and (5.21) it follows that for $\operatorname{Re}z^\varrho>0$

$$\begin{aligned}\|\sigma(f(z))\|^2&\leqslant C'C^2\frac{|e^{(\sum_j d_jc_j)z^\varrho-c_1z^\varrho}|^2}{\prod_j(1+|e^{2c_jz^\varrho}|)^{d_j}}\leqslant C'C^2\frac{|e^{(\sum_j d_jc_j)z^\varrho-c_1z^\varrho}|^2}{\prod_j|e^{2c_jd_jz^\varrho}|}\\&=C'C^2|e^{-c_1z^\varrho}|^2=C'C^2e^{-2c_1\operatorname{Re}z^\varrho}.\end{aligned}$$

Hence,

$$\log^+\frac{1}{\|\sigma(f(z))\|}\geqslant c_1\operatorname{Re}z^\varrho+O(1)$$

for all $z\in\mathbf{C}$ with $\operatorname{Re}z^\varrho>0$. Therefore,

$$\begin{aligned}m_f(r;\overline{D})&=\frac{1}{2\pi}\int_{\{|z|=r\}}\log\frac{1}{\|\sigma(f(z))\|}\,d\theta=\frac{1}{2\pi}\int_{\{|z|=r\}}\log^+\frac{1}{\|\sigma(f(z))\|}\,d\theta+O(1)\\&\geqslant\frac{1}{2\pi}\int_{\{|z|=r:\operatorname{Re}z^\varrho>0\}}\log^+\frac{1}{\|\sigma(f(z))\|}\,d\theta+O(1)\\&=\frac{1}{2\pi}\int_{\{|z|=r\}}c_1\cdot(\operatorname{Re}z^\varrho)^+\,d\theta+O(1)\\&=\frac{1}{2\pi}\int_0^{2\pi}c_1r^\varrho\cos^+\varrho\theta\,d\theta+O(1)=\frac{c_1}{\pi}r^\varrho+O(1).\end{aligned}$$

On the other hand one deduces easily from (5.19) that $T_f(r;D)=O(r^\varrho)$. Hence,

$$\delta(f;\overline{D})=\varliminf_{r\to\infty}\frac{m_f(r;\overline{D})}{T_f(r;\overline{D})}>0.$$ □

We will now give an explicit example with $\operatorname{St}(D)=\{0\}$.

Example 5.22. Let A be the semi-Abelian variety $A=\mathbf{C}^*\times\mathbf{C}^*$, compactified by $\mathbf{P}^1(\mathbf{C})\times\mathbf{P}^1(\mathbf{C})$ with a pair of homogeneous coordinates, $([x_0:x_1],[y_0:y_1])$. For a pair of natural numbers (m,n) with $m<n$, let $\bar{D}$ be the divisor given by

$$\bar{D}=\{([x_0:x_1],[y_0:y_1]):y_0^n x_1+y_0^{n-m}y_1^m x_0+y_1^n x_0=0\}.$$

Set $D=\bar{D}\cap A$. Note that $\mathrm{St}(D)=\{0\}$. Moreover, D violates condition 4.11, since $\bar{D}\ni([1:0],[1:0])$. Let c be a positive irrational real number such that

$$0<cm<1<cn. \tag{5.23}$$

Let $f\colon\mathbf{C}\to A$ be the holomorphic curve given by

$$f\colon z\mapsto([1:e^z],[1:e^{cz}]).$$

Let Ω_i, $i=1,2$, be the Fubini–Study metric forms of the two factors of $(\mathbf{P}^1(\mathbf{C}))^2$. Then $c_1(\bar{D})=\Omega_1+n\Omega_2$. By an easy computation one obtains

$$T_f(r;c_1(\bar{D}))=\frac{1+nc}{\pi}r+O(1). \tag{5.24}$$

Thus, $\varrho_f=1$, and the image $f(\mathbf{C})$ is Zariski dense in A, because c is irrational.

We compute $N(r;f^*D)$ as follows. Note the following identity for divisors on $\mathbf{C}$:

$$f^*D=(e^z+e^{mcz}+e^{ncz})_0. \tag{5.25}$$

We consider a holomorphic curve g in $\mathbf{P}^2(\mathbf{C})$ with the homogeneous coordinate system $[w_0:w_1:w_2]$ defined by

$$g\colon z\in\mathbf{C}\to[e^z:e^{mcz}:e^{ncz}]\in\mathbf{P}^2(\mathbf{C}).$$

By computing the Wronskian of e^z, e^{mcz} and e^{ncz} one sees that they are linearly independent over $\mathbf{C}$; i.e., g is linearly non-degenerate. Let $T_g(r)$ be the order function of g with respect to the Fubini–Study metric form on $\mathbf{P}^2(\mathbf{C})$. It follows that

$$\begin{aligned}T_g(r)&=\frac{1}{4\pi}\int_{\{|z|=r\}}\log(|e^z|^2+|e^{mcz}|^2+|e^{ncz}|^2)\,d\theta+O(1)\\&=\frac{1}{4\pi}\int_{\{|z|=r\}}\log(1+|e^{(mc-1)z}|^2+|e^{(nc-1)z}|^2)\,d\theta+O(1).\end{aligned} \tag{5.26}$$

If $\operatorname{Re} z\geqslant 0$ (resp. $\leqslant 0$), then $|e^{(mc-1)z}|\leqslant 1$ (resp. $\geqslant 1$) and $|e^{(nc-1)z}|\geqslant 1$ (resp. $\leqslant 1$). Therefore, if $z=re^{i\theta}$ and $\operatorname{Re} z\geqslant 0$,

$$\log(1+|e^{(mc-1)z}|^2+|e^{(nc-1)z}|^2)=2\log^+|e^{(nc-1)z}|+O(1)=2(nc-1)r\cos\theta+O(1).$$

If $z=re^{i\theta}$ and $\operatorname{Re} z\leqslant 0$,

$$\log(1+|e^{(mc-1)z}|^2+|e^{(nc-1)z}|^2)=2\log^+|e^{(mc-1)z}|+O(1)=2(mc-1)r\cos\theta+O(1).$$

Combining these with (5.26), we have

$$T_g(r)=\frac{(n-m)c}{\pi}r+O(1). \tag{5.27}$$

We consider the following four lines H_j, $1\leqslant j\leqslant 4$, of $\mathbf{P}^2(\mathbf{C})$ in general position:

$$\begin{aligned}&H_j=\{w_{j-1}=0\}, \quad 1\leqslant j\leqslant 3,\\&H_4=\{w_0+w_1+w_2=0\}.\end{aligned}$$

Noting that g is linearly non-degenerate and has a finite order (in fact, $\varrho_g=1$), we infer from Cartan's S.M.T. [Ca] that

$$T_g(r)\leqslant\sum_{j=1}^{4}N_2(r;g^*H_j)+O(\log r). \tag{5.28}$$

Since $N_2(r;g^*H_j)=0$, $1\leqslant j\leqslant 3$, we deduce from (5.28), (5.27) and (2.1) that

$$N(r;g^*H_4)=\frac{(n-m)c}{\pi}r+O(\log r).$$

By (5.25), $N(r;g^*H_4)=N(r;f^*D)$, and so

$$N(r;f^*D)=\frac{(n-m)c}{\pi}r+O(\log r). \tag{5.29}$$

It follows from (5.24) and (5.29) that

$$\delta(f;\bar{D})=\frac{1+mc}{1+nc}. \tag{5.30}$$

By elementary calculations one shows that $\operatorname{ord}_z f^*D\geqslant 2$ implies

$$(mc-1)(e^{cz})^m+(nc-1)(e^{cz})^n=0.$$

Furthermore, $f(z)\in D$ if and only if $e^z+e^{mcz}+e^{ncz}=0$. Combined, these two relations imply that there is a finite subset $S\subset\mathbf{C}^2$ such that $\operatorname{ord}_z f^*D\geqslant 2$ implies $(e^z,e^{cz})\in S$. Since $z\mapsto(e^z,e^{cz})$ is injective, it follows that $\{z:\operatorname{ord}_z f^*D\geqslant 2\}$ is a finite set. Therefore,

$$\begin{aligned}N_1(r,f^*D)&=N(r,f^*D)+O(\log r),\\\delta_1(f;\bar{D})&=\delta(f;\bar{D})=\frac{1+mc}{1+nc}.\end{aligned} \tag{5.31}$$

Let $c'>1$ be an irrational number, and set

$$c=1/c', \quad m=[c'], \quad n=[c']+1,$$

where $[c']$ denotes the integral part of c'. Then m, n and c satisfy (5.23), and by (5.30)

$$\delta(f;\bar{D})=\frac{1+[c']/c'}{1+([c']+1)/c'}\to 1 \quad (c'\to\infty).$$

Thus $\delta(f;\bar{D})$ $(=\delta_1(f;\bar{D})$ by (5.31)) takes values arbitrarily close to 1.

Remark 5.32. In [No5], the first author proved that for D without condition 4.11 a holomorphic curve $f\colon \mathbf{C}\to A$, omitting D, has no Zariski dense image, and is contained in a translate of a semi-Abelian subvariety which has no intersection with D. What was proved in [No5] applied to $f\colon \mathbf{C}\to A$ with Zariski dense image yields that there is a positive constant $\varkappa$ such that

$$\varkappa T_f(r;c_1(\bar{D}))\leqslant N_1(r;f^*D)+S_f(r;c_1(\bar{D})), \tag{5.33}$$

provided that $\mathrm{St}(D)=\{0\}$. The above $\varkappa$ may be, in general, very small because of the method of the proof. One needs more detailed properties of $J_k(D)$ to get the best bound such as in the Main Theorem than to get (5.33); this is the reason why we need boundary condition 4.11 for D.

Remark 5.34. In [SY2] Siu and Yeung claimed (1.2) for Abelian A of dimension n. The most essential part of their proof was Lemma 2 of [SY2], but the claimed assertion does not hold. The cause of the trouble is due to the application of the semi-continuity theorem to a non-flat family of coherent ideal sheaves. But, it is a bit delicate, and so we give a counter-example to their lemma.

Let $f\colon \mathbf{C}\to A$ be a one-parameter subgroup with Zariski dense image. Let D be an ample divisor on A containing $0\in A$ such that $f(\mathbf{C})$ is tangent highly enough to D at 0 so that $J_kf(0)\in J_k(D)$, but $f(\mathbf{C})\not\subset D$. Let $\mathfrak{m}_0$ be the maximal ideal sheaf of the structure sheaf $\mathcal{O}_A$ at 0. Since W_k consists of only one point, we can identify $A\times W_k$ with A. Then, $0\in J_k(D)\cap(A\times W_k)$. Let $\mathcal{I}_k=\mathcal{I}(J_k(D)\cap(A\times W_k))$ be the ideal sheaf of $J_k(D)\cap(A\times W_k)$. Then $\mathcal{I}_k\subset\mathfrak{m}_0$. If the claimed Lemma 2 of [SY2] were correct, it should follow that

$$H^0(A,\mathcal{O}((L(D))^\delta)\otimes\mathfrak{m}_0^q)\supset H^0(A,\mathcal{O}((L(D))^\delta)\otimes\mathcal{I}_k^q)\neq\{0\} \quad \text{for all } q\geqslant 1.$$

Thus we would obtain that $\dim H^0(A,\mathcal{O}((L(D))^\delta))=\infty$; this is a contradiction.

Remark 5.35. It is an interesting problem to see if the truncation level k_0 of the counting function $N_{k_0}(r;f^*D)$ in the Main Theorem can be taken as a function only in $\dim A$. By the above proof, it would be sufficient to find a natural number k such that $\pi_2(J_k(\bar{D};\log \bar{A}\cap\partial A))\cap W_k\neq W_k$. Note that

$$\dim \pi_2(J_k(\bar{D};\log \bar{A}))\leqslant \dim J_k(\bar{D};\log \bar{A})=(n-1)(k+1).$$

Thus, if $\dim W_k>(n-1)(k+1)$ we may set $k_0=k$. For example, if $J_n(f)(\mathbf{C})$ is Zariski dense in $J_n(A)$, then $\dim W_n=n^2$. Since $\dim \pi_2(J_n(\bar{D};\log \partial A))=n^2-1$, we may set $k_0=n$. In general, it is impossible to choose k_0 depending only on n because of the following example.

Example 5.36. Let $E=\mathbf{C}/(\mathbf{Z}+i\mathbf{Z})$ be an elliptic curve, and let D be an irreducible divisor on E^2 with cusp of order N at $0\in E^2$. Let $f: z\in\mathbf{C}\to(z,z^2)\in E^2$. Then $f(\mathbf{C})$ is Zariski dense in E^2, and

$$T_f(r,L(D))\sim r^4(1+o(1)).$$

Note that $f^{-1}(0)=\mathbf{Z}+i\mathbf{Z}$ and $f^*D\geqslant N(\mathbf{Z}+i\mathbf{Z})$. For an arbitrary fixed k_0, we take $N>k_0$, and then have

$$N(r,f^*D)-N_{k_0}(r,f^*D)\geqslant (N-k_0)r^2(1+o(1)).$$

The above left-hand side cannot be bounded by $S_f(r,c_1(D))=O(\log r)$. This gives also a counter-example to [Kr1, Lemma 4].

6. Applications

Let the notation be as in the previous section. Here we assume that A is an Abelian variety and D is reduced and hyperbolic; in this special case, D is hyperbolic if and only if D contains no translate of a one-parameter subgroup of A. Cf. [NO], [L] and [Ks] for the theory of hyperbolic complex spaces.

THEOREM 6.1. *Let $D\subset A$ be hyperbolic and let d_0 be the highest order of tangency of D with translates of one-parameter subgroups. Let $\pi: X\to A$ be a finite covering space such that its ramification locus contains D and the ramification order over D is greater than d_0. Then X is hyperbolic.*

Proof. By Brody's theorem (cf. [Br], [NO, Theorem (1.7.3)]) it suffices to show that there is no non-constant holomorphic curve $g:\mathbf{C}\to X$ such that the length $\|g'(z)\|$ of the derivative $g'(z)$ of $g(z)$ with respect to an arbitrarily fixed Finsler metric on X is bounded. Set $f(z)=\pi(g(z))$. Then the length $\|f'(z)\|$ with respect to the flat metric is bounded, too, and hence $f'(z)$ is constant. Thus, $f(z)$ is a translate of a one-parameter subgroup. By definition we may take $k_0=d_0+1$ in (5.12). Take $d\,(>d_0)$ so that X ramifies over D

with order at least d. Then we have that $N_1(r;f^*D)\leqslant(1/(d+1))N(r;f^*D)$. Hence it follows from the Main Theorem that

$$\begin{aligned}T_f(r;L(D)) &= N_{d_0+1}(r,f^*D)+O(\log r)\leqslant(d_0+1)N_1(r;f^*D)+O(\log r)\\ &\leqslant\frac{d_0+1}{d+1}N(r;f^*D)+O(\log r)\leqslant\frac{d_0+1}{d+1}T_f(r;L(D))+O(\log r).\end{aligned}$$

Since $T_f(r;L(D))\geqslant c_0r^2$ with a constant $c_0>0$, $d\leqslant d_0$; this is a contradiction. □

Remark. In the special case of $\dim X=\dim A=2$, C. G. Grant [Gr] proved that if X is of general type and $X\to A$ is a finite (ramified) covering space, then X is hyperbolic. When $\dim X=\dim A=2$, D is an algebraic curve, and hence the situation is much simpler than the higher dimensional case.

THEOREM 6.2. *Let $f:\mathbf{C}\to A$ be a 1-parameter analytic subgroup in A with $a=f'(0)$. Let D be an effective divisor on A with the Riemann form $H(\cdot,\cdot)$ such that $D\not\supset f(\mathbf{C})$. Then we have*

$$N(r;f^*D)=H(a,a)\pi r^2+O(\log r).$$

Proof. Taking the Zariski closure of $f(\mathbf{C})$, we may assume that $f(\mathbf{C})$ is Zariski dense in A. Note that the first Chern class $c_1(L(D))$ is represented by $i\partial\bar\partial H(w,w)$. It follows from (2.1) and Lemma 5.1 that

$$\begin{aligned}N(r;f^*D) &= T_f(r;L(D))+O(\log r)\\ &=\int_0^r\frac{dt}{t}\int_{\Delta(t)}iH(a,a)\,dz\wedge d\bar z+O(\log r)=H(a,a)\pi r^2+O(\log r).\end{aligned}$$ □

Remark 6.3. In the case where $f(\mathbf{C})$ is Zariski dense in A, Ax ([Ax]) proved the estimate

$$0<\varliminf_{r\to\infty}\frac{n(r,f^*D)}{r^2}\leqslant\varlimsup_{r\to\infty}\frac{n(r,f^*D)}{r^2}<\infty,$$

which is equivalent to

$$0<\varliminf_{r\to\infty}\frac{N(r,f^*D)}{r^2}\leqslant\varlimsup_{r\to\infty}\frac{N(r,f^*D)}{r^2}<\infty.$$

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