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A criterion of algebraicity for Lelong classes and analytic sets

by

AHMED ZERIAHI

Université Paul Sabatier
Toulouse, France

1. Introduction

Global extremal functions were first introduced by J. Siciak [Sic1], in the spirit of the classical work of F. Leja [Lej], in order to extend classical results of approximation and interpolation to holomorphic functions of several complex variables. Later V. P. Zahariuta [Za2] gave another definition of the global extremal function based on the following class of plurisubharmonic functions on $\mathbf{C}^N$:

$$\mathcal{L}(\mathbf{C}^N) := \{v \in \mathrm{PSH}(\mathbf{C}^N) : \exists c_v \in \mathbf{R},\, v(z) \leqslant c_v + \log^+|z|,\, \forall z \in \mathbf{C}^N\}. \tag{1.1}$$

This class is called the class of plurisubharmonic functions of logarithmic growth (or minimal growth) on $\mathbf{C}^N$.

Then given a compact set $K \subset \mathbf{C}^N$, we define its global extremal function on $\mathbf{C}^N$ by the formula

$$L_K(z) = L_K(z; \mathbf{C}^N) := \sup\{v(z) : v \in \mathcal{L}(\mathbf{C}^N),\, v|K \leqslant 0\}, \quad z \in \mathbf{C}^N. \tag{1.2}$$

It has been proved by Siciak that the function L_K is locally bounded on $\mathbf{C}^N$ if and only if K is nonpluripolar in $\mathbf{C}^N$. In this case, the upper semi-continuous regularization L_K^* of the function L_K in $\mathbf{C}^N$ belongs to the class $\mathcal{L}(\mathbf{C}^N)$ (see [Sic2], [Kl]). Moreover, if $U \Subset \mathbf{C}^N$ is a domain and $K \subset U$ is a nonpluripolar compact subset, then the following fundamental inequality, known as the *Bernstein–Walsh inequality*, holds: there exists a constant $R = R(K;U) > 1$ such that

$$\|f\|_U \leqslant \|f\|_K R^d, \quad \forall f \in \mathcal{P}_d(\mathbf{C}^N),\, \forall d \geqslant 1, \tag{1.3}$$

where $\mathcal{P}_d(\mathbf{C}^N)$ is the space of holomorphic polynomials on $\mathbf{C}^N$ of degree at most d. It is known ([Sic2]) that the best constant $R := R(K;U)$ in the inequality (1.3) is related to the L-extremal function by

$$R(K;U) = \exp\bigl(\sup_{z\in U} L_K(z)\bigr) =: \frac{1}{\mathrm{cap}_L(K;U)}. \tag{1.4}$$

The constant $\mathrm{cap}_L(K;U)$ is called the L-capacity of the compact set K with respect to U.

The class of plurisubharmonic functions with logarithmic growth, which was considered earlier by P. Lelong in another context ([Lel2]), plays a fundamental role in pluripotential theory (see [BT2]) as does the class of logarithmic potentials in logarithmic potential theory (see [Ran]). For instance, for any fixed domain $U \Subset \mathbf{C}^N$, the set function $\mathrm{cap}_L(\cdot\,;U)$ is a Choquet capacity on $\mathbf{C}^N$ ([BT1], [Sic3]) which is comparable to the Monge–Ampère condenser capacity ([AT]).

On the other hand, the class of plurisubharmonic functions of logarithmic growth can also be defined on an algebraic subvariety of $\mathbf{C}^N$ (see [Sa2], [Ze2]), and it turns out that in this case again, the associated extremal functions play a fundamental role (see [Ze1], [Ze2], [Ze3]).

Now suppose that X is an irreducible (proper) analytic subvariety of $\mathbf{C}^N$ and K is a nonpluripolar compact subset of X. Then, since K is now pluripolar in $\mathbf{C}^N$, it follows from the above-mentioned result of Siciak that the upper regularization L_K^* of the function L_K in $\mathbf{C}^N$ is identically equal to $+\infty$. Nevertheless, it is a natural question to ask whether the semi-local Bernstein–Walsh inequality (1.3) holds for a pair (K,U), where $U \Subset X$ is a domain in X and K is a nonpluripolar compact subset of U. The answer to this question was given by the beautiful criterion of algebraicity of A. Sadullaev [Sa2], which says that such an inequality holds on the analytic set X if and only if X is algebraic.

Our first motivation was to make this criterion more effective by understanding the algebraicity of a local analytic set in $\mathbf{C}^N$ in terms of the semi-local behaviour of its natural class of plurisubharmonic functions of restricted logarithmic growth.

It turns out that this investigation can be carried out in a more general context where precise results can be obtained. Namely, in the spirit of P. Lelong [Lel2], we introduce an abstract definition of a *Lelong class* of plurisubharmonic functions on a complex analytic space X, and investigate the main properties of their associated extremal functions.

In this general context we first obtain, in the spirit of the classical works of S. N. Bernstein [Ber] and J. L. Walsh [W], an abstract semi-local *Bernstein–Walsh inequality* for a natural graded sequence of complex conic spaces of holomorphic functions associated to a given Lelong class on X. Actually this new approach provides us with a general and natural framework for more general "Bernstein inequalities", which have been recently proved in special cases by C. Fefferman and R. Narasimhan for algebraic manifolds ([FeN1], [FeN2]). This point of view will be developed later in a subsequent paper.

Then we prove a fundamental theorem of algebraicity, which gives a sharp asymptotic upper bound of the Hilbert function associated to a given Lelong class $\mathcal{L}$ on the complex space X in terms of its so-called *minimal Lelong number*, which describes the semi-local behaviour of $\mathcal{L}$ on X. This result seems to be new even in the case of an (irreducible) algebraic subvariety $Z \subset \mathbf{C}^N$, where we obtain sharp estimates comparing the degree of algebraicity of Z with the minimal Lelong number of the Lelong class of plurisubharmonic functions of logarithmic growth on Z.

Finally, from our fundamental theorem of algebraicity, we deduce a new semi-local criterion of algebraicity which contains the local criterion of A. Sadullaev [Sa2] as well as the global criterion of W. Stoll [St2] thanks to a fundamental estimate of J. P. Demailly [D1].

2. Abstract Lelong classes and associated extremal functions

It turns out that the class $\mathcal{L}(\mathbf{C}^N)$ defined by the formula (1.1) has some interesting properties which make the theory of extremal functions with logarithmic growth useful. These properties will be taken as axioms and will permit us to develop a semi-local version of the theory of extremal functions with growth based on the fundamental concept of "Lelong classes".

2.1. Admissible classes and the Lelong property

All the complex analytic spaces considered here will be reduced and irreducible. Plurisubharmonic functions on a complex space have been studied by J. E. Fornæss and R. Narasimhan [FoN], and also by J. P. Demailly [D1]. Pluripotential theory in complex spaces has been investigated in [Bed], [D1], [Ze2].

Let X be a complex space of dimension n and X_{reg} the complex manifold of its regular points. Recall that a function $u: X \to [-\infty, +\infty[$ is said to be (weakly) *plurisubharmonic* on X if and only if u is locally bounded above on X and plurisubharmonic on X_{reg}.

For any function $u: X \to [-\infty, +\infty]$ defined on X, it is convenient to consider the (generalized) upper regularization of u on X defined by the formula

$$u^*(x) := \limsup_{\substack{y \to x \\ y \in X_{\mathrm{reg}}}} u(y), \quad \text{for } x \in X.$$

If u is locally bounded above on X then u^* is upper semi-continuous on X and is called the upper semi-continuous regularization of u on X.

If u is plurisubharmonic on X, then the function u^* is upper semi-continuous on X and coincides with u on X_{reg}.

Let us denote by $\mathrm{PSH}(X)$ the cone of plurisubharmonic functions on X which are not identically $-\infty$ on X, so that $\mathrm{PSH}(X)\subset L^1_{\mathrm{loc}}(X)$. The space $L^1_{\mathrm{loc}}(X)$ will be endowed with the L^1_{loc}-topology induced by local embeddings of X into complex Hermitian spaces. Then it is well known that the cone $\mathrm{PSH}(X)$ endowed with the L^1_{loc}-topology is closed in $L^1_{\mathrm{loc}}(X)$ (see [Ho2]).

In order to extend the theory of global extremal functions, we need to introduce the following important definition.

Definition 2.1.1. Let $\mathcal{L}\subset\mathrm{PSH}(X)$ be a class of plurisubharmonic functions on a complex space X.

(1) The class $\mathcal{L}$ is said to be an *admissible class* of plurisubharmonic functions on X if $\mathcal{L}$ contains all the (real) constants and is translation-invariant, i.e. if $u\in\mathcal{L}$ and $\alpha\in\mathbf{R}$ then $u+\alpha\in\mathcal{L}$.

(2) If $\Omega\subset X$ is an open subset of X, the class $\mathcal{L}$ is said to satisfy the *Lelong property* on Ω if the following condition holds:

Lelong property (LP): For any subfamily $\mathcal{M}\subset\mathcal{L}$, define the set of pointwise boundedness of $\mathcal{M}$ in Ω by

$$B_{\mathcal{M}}:=\{x\in\Omega:\sup_{v\in\mathcal{M}}v(x)<+\infty\}.$$

Then either the set $B_{\mathcal{M}}$ is pluripolar in Ω or the family $\mathcal{M}$ is locally bounded above on Ω.

(3) An admissible class $\mathcal{L}\subset\mathrm{PSH}(X)$ will be called an *abstract Lelong class* (or simply a Lelong class) on X if it is a closed subset of $\mathrm{PSH}(X)$ which satisfies the Lelong property (LP) on X.

Let $\mathcal{L}\subset\mathrm{PSH}(X)$ be an admissible class of plurisubharmonic functions. Then for a given subset $E\Subset X$, we define the $\mathcal{L}$-extremal function of E on X by

$$\Lambda_E(x)=\Lambda_E(x;\mathcal{L}):=\sup\{v(x):v\in\mathcal{L},\ v|E\leqslant 0\},\quad x\in X. \tag{2.1}$$

Moreover, as before, given a subdomain $U\Subset X$, we associate with the set E a "capacitary" constant defined by

$$\mathrm{cap}_{\mathcal{L}}(E;U):=\exp\bigl(-\sup_{x\in U}\Lambda_E(x)\bigr), \tag{2.2}$$

and call it the $\mathcal{L}$-capacity of the set E with respect to U. This number always satisfies the inequalities $0\leqslant\mathrm{cap}_{\mathcal{L}}(E;U)\leqslant 1$.

Let us quote here some known examples for later references (see [Ze2]).

Examples. Let X be a Stein space admitting a continuous plurisubharmonic exhaustion $p: X \to [-\infty, +\infty[$ which is maximal off some compact subset of X in the sense of A. Sadullaev (see [Sa3], [Kl]). Following the terminology of W. Stoll [St1], such a function will be called a (weak) *parabolic potential* on X, and (X, p) will be called a *parabolic space*.

Then it is possible to prove that the associated class of plurisubharmonic functions,

$$\mathcal{L}(X;p) := \{v \in \mathrm{PSH}(X) : \exists c_v \in \mathbf{R},\, v(x) \leqslant c_v + p^+(x),\, \forall x \in X\}, \tag{2.3}$$

is an abstract Lelong class on X (see [Ze2]).

In particular, if $X = Z$ is an irreducible algebraic subvariety of $\mathbf{C}^N$, then it is known that there exists on Z a special parabolic potential p_0 such that $p_0(x) - \log|x|$ is bounded off some compact subset of Z, where $|\cdot|$ is any complex norm on $\mathbf{C}^N$ (see [Sa2], [Ze2]). Then the associated class $\mathcal{L}(Z;p_0)$ coincides with the class $\mathcal{L}(Z) := \mathcal{L}(Z;l)$ defined by the formula (2.3) with p replaced by the function $l(x) := \log|x|$ for $x \in Z$.

Therefore the class $\mathcal{L}(Z)$ is a Lelong class on Z, which will be called the class of plurisubharmonic functions of logarithmic growth on Z. Observe that in the particular case where $X = \mathbf{C}^n$, the logarithmic potential itself is a parabolic potential on $\mathbf{C}^n$.

The following result, which gives several characterizations of the Lelong property, will be useful later.

THEOREM 2.1.2. *Let $\mathcal{L} \subset \mathrm{PSH}(X)$ be an admissible class of plurisubharmonic functions on a complex space X. Then the following properties are equivalent*:

(i) *The class $\mathcal{L}$ satisfies the Lelong property (LP) on X.*

(ii) *For each $a \in X$ there exists an open neighbourhood ω of a such that the class $\mathcal{L}$ satisfies the Lelong property (LP) on ω.*

(iii) *For any nonpluripolar subset $E \Subset X$, Λ_E is locally bounded above on X.*

(iii′) *For any nonpluripolar subset $E \Subset X$ and any domain $U \Subset X$, $\mathrm{cap}_{\mathcal{L}}(E; U) > 0$.*

(iv) *There exists a subset $E \Subset X$ such that Λ_E is locally bounded above on X.*

(iv′) *There exists a subset $E \Subset X$ such that for any subdomain $U \Subset X$, $\mathrm{cap}_{\mathcal{L}}(E; U) > 0$.*

(v) *There exists a subset $E \subset X$ and a real-valued function q, locally bounded on X, such that the following inequality holds*:

$$w(x) \leqslant \sup_E w + q(x), \quad \forall x \in X,\, \forall w \in \mathcal{L}. \tag{2.4}$$

Proof. Let us first prove that all the conditions but (ii) are equivalent. Observe that all the implications (iii) $\Rightarrow$ (iv) $\Rightarrow$ (v) and the equivalences (iii) $\Leftrightarrow$ (iii′) and (iv) $\Leftrightarrow$ (iv′) are obvious.

The implication (i) $\Rightarrow$ (iii) is easy to prove. Indeed, let $E \Subset X$ be a nonpluripolar subset. Then consider the subclass of plurisubharmonic functions

$$\mathcal{M} := \{w \in \mathcal{L} : w|E \leqslant 0\}.$$

Since $E \subset B_{\mathcal{M}}$, $B_{\mathcal{M}}$ is also nonpluripolar in X, and then the condition (i) and the definition of the Lelong property (LP) imply that $\mathcal{M}$ is locally bounded above on X, which proves (iii).

Now let us prove the implication (v) $\Rightarrow$ (i). Let $\mathcal{M} \subset \mathcal{L}$ be a subclass such that $B := \{x \in X : u(x) < +\infty\}$ is nonpluripolar, where $u(x) := \sup\{w(x) : w \in \mathcal{M}\}$. We want to prove that u is locally bounded above on X. By condition (iv), it is enough to prove that $\sup_{x \in E} u(x) < +\infty$. Assume that the opposite holds. Then there exists a sequence $(w_j)_{j \geqslant 1}$ in the class $\mathcal{M}$ such that $m_j := \sup_E w_j \geqslant 2^j$ for $j \geqslant 1$. From the condition (iv), we deduce the inequality

$$w_j(x) \leqslant m_j + q(x), \quad \forall x \in X, \forall j \geqslant 1. \tag{2.5}$$

From (2.5) it follows that the sequence of plurisubharmonic functions $(w_j - m_j)$ is locally bounded above on X.

Since $\limsup_{j \to +\infty} \sup_E (w_j - m_j) = 0$, applying Hartogs's lemma, we conclude that there exists $x_0 \in X$ such that $\limsup_{j \to +\infty} (w_j(x_0) - m_j) > -1$. Taking a subsequence if necessary, we can assume that the following inequalities hold:

$$w_j(x_0) - m_j \geqslant -1, \quad \forall j \geqslant 1. \tag{2.6}$$

Now the function $w := \sum_{j=1}^{+\infty} 2^{-j}(w_j - m_j)$ is plurisubharmonic on X. Moreover, by the definition of the set B, for any $x \in B$ we have $u(x) < +\infty$, and then

$$w(x) \leqslant u(x) - \sum_j 2^{-j} m_j = -\infty,$$

since $m_j \geqslant 2^j$ for any $j \geqslant 1$. Moreover, from (2.6) it follows that $w(x_0) \geqslant -1$. Thus the set B is pluripolar in X, which proves our claim and then (i).

Now since the implication (i) $\Rightarrow$ (ii) is obvious, it is enough to prove that (ii) $\Rightarrow$ (iii). Let $E \subset X$ be a nonpluripolar compact set. Take an open set V such that $E \subset V \Subset X$. By the condition (ii), for any point $a \in \overline{V}$, there exists a neighbourhood ω_a of a such that the class $\mathcal{L}$ satisfies the Lelong property (LP) on ω_a. Since we know that (i) and (iv) are equivalent, it follows that there exists a compact set $K_a \subset \omega_a$ such that the $\mathcal{L}$-extremal function of K_a is locally bounded on ω_a. Then, using a standard compactness argument, we deduce that there exists a neighbourhood Ω of the compact set $\overline{V}$ and a compact set $K\subset\Omega$ such that the $\mathcal{L}$-extremal function of K is locally bounded on Ω. From the equivalence of the conditions (iii) and (iv), it follows that the $\mathcal{L}$-extremal function Λ_E is locally bounded on Ω, and then it is bounded on V. Since $V\Subset X$ is an arbitrary open subset such that $E\subset V$, we conclude that the $\mathcal{L}$-extremal function Λ_E is locally bounded on X, which proves the condition (iii). $\square$

The following result gives a quantitative version of the Lelong property.

COROLLARY 2.1.3. *Let $\mathcal{L}$ be an admissible class of plurisubharmonic functions on a complex space X which satisfies the Lelong property (LP) on X, and let $G\Subset X$ be a fixed nonpluripolar subset of X. Then for any subset $E\Subset X$, the following inequality holds*:

$$\Lambda_E(x)\leqslant\varrho(E;G)+\Lambda_G(x),\quad \forall x\in X, \tag{2.7}$$

where $\varrho(E;G)=\varrho_{\mathcal{L}}(E;G):=\max_G\Lambda_E$ is a positive number, which is finite if E is nonpluripolar in X.

More precisely, the $\mathcal{L}$-extremal function Λ_E of the set E is locally bounded above on X if and only if $\varrho(E;G)<+\infty$.

Proof. From the definition of the $\mathcal{L}$-extremal function of E, we deduce immediately the inequality

$$w(x)\leqslant\max_E w+\Lambda_E(x),\quad \forall x\in X,\ \forall w\in\mathcal{L}. \tag{2.8}$$

If we define the number $\varrho(E;G):=\max_G\Lambda_E$, then the inequality (2.8) implies

$$\max_G w\leqslant\max_E w+\varrho(E;G),\quad \forall w\in\mathcal{L}. \tag{2.9}$$

Then from (2.9) and the definition of the $\mathcal{L}$-extremal function Λ_G, we obtain (2.7). By Theorem 2.1.2, if E is nonpluripolar then Λ_E is locally bounded, and so $\varrho(E;G)<+\infty$. Again by Theorem 2.1.2, we know that Λ_G is locally bounded on X since $G\Subset X$ is nonpluripolar. Therefore, if $\varrho(E;G)<+\infty$ then (2.7) implies that the function Λ_E is also locally bounded on X. $\square$

It is interesting to observe that, under the assumptions of Corollary 2.1.3, for a given subset $E\Subset X$ the finiteness condition $\varrho_{\mathcal{L}}(E;G)=\varrho(E;G)<+\infty$ is independent on the particular choice of the nonpluripolar subset $G\Subset X$, which can be taken to be an open domain in X. Moreover, this condition is equivalent to the condition $\operatorname{cap}_{\mathcal{L}}(E;G):=\exp(-\varrho_{\mathcal{L}}(E;G))>0$. A subset $E\subset X$ satisfying this condition will be called a *non-$\mathcal{L}$-polar* subset of X. From Corollary 2.1.3, it follows that any nonpluripolar subset of X is non-$\mathcal{L}$-polar, but the converse is not true in general.

The following result is of particular interest in our theory, since it shows the semi-local character of the notion of Lelong class and how this notion is inherited by subspaces of X. Moreover, this will be the first step in the proof of our generalization of Sadullaev's criterion of algebraicity in §4.

THEOREM 2.1.4. *Let $\mathcal{L}\subset \mathrm{PSH}(X)$ be an admissible class of plurisubharmonic functions on a complex space X, and Y an analytic subspace of X. Let us define the class $\mathcal{L}(Y)$ to be the closure in $L^1_{\mathrm{loc}}(Y)$ of the the induced class*

$$\mathcal{L}_Y := \{u\in \mathrm{PSH}(Y) : \exists w\in\mathcal{L},\ w|Y=u\}.$$

Then the following properties are equivalent:

(1) *The class $\mathcal{L}_Y$ satisfies the Lelong property (LP) on Y.*

(2) *The class $\mathcal{L}(Y)$ is a Lelong class on Y.*

(3) *For some compact (and then for any nonpluripolar) subset $K\subset Y$, the restriction to Y of the $\mathcal{L}$-extremal function of K on X is locally bounded on Y.*

Moreover, for any nonpluripolar compact subset $K\subset Y$ the restriction to Y of the $\mathcal{L}$-extremal function associated to K as a subset of X coincides on Y with the $\mathcal{L}(Y)$-extremal function associated to K as a subset of Y, up to upper regularizations (in the generalized sense) on Y.

Proof. It is clear that $\mathcal{L}_Y\subset \mathrm{PSH}(Y)$ is translation-invariant and contains the constants as does $\mathcal{L}$. Thus $\mathcal{L}(Y)$ is a closed admissible class of plurisubharmonic functions on Y. Let us denote by $\Lambda_{K,X}$ the $\mathcal{L}$-extremal function of K on X, and by $\Lambda_{K,Y}$ the $\mathcal{L}(Y)$-extremal function of K on Y.

First we claim that that the following identity holds:

$$\Lambda^*_{K,Y}(y)=\Lambda^*_{K,X}(y),\quad \forall y\in Y, \tag{2.10}$$

where the upper regularizations are understood in the generalized sense on Y, which means that these numbers are equal even when they are infinite.

Indeed, observe first that the inequality $\Lambda_{K,X}\leqslant\Lambda_{K,Y}$ on Y is obvious. On the other hand, for each function $w\in\mathcal{L}(Y)$, there exists a sequence $(w_j)_{j\geqslant 1}$ from $\mathcal{L}_Y$ such that $w:=(\limsup_{j\to+\infty} w_j)^*$ on Y. Now fix $w\in\mathcal{L}(Y)$ with $w|K\leqslant 0$ and $\varepsilon>0$. Applying Hartogs's lemma, we see that there exists $j_0\geqslant 1$ such that $w_j\leqslant\varepsilon$ on K for $j\geqslant j_0$. Since each w_j is the restriction to Y of a function from $\mathcal{L}$, we conclude that $w_j\leqslant\varepsilon+\Lambda_{K,X}$ on Y for $j\geqslant j_0$, which implies that $w\leqslant\Lambda^*_{K,X}$ on Y. Therefore we obtain $\Lambda^*_{K,Y}\leqslant\Lambda^*_{K,X}$ in the generalized sense on Y, which proves the formula (2.10).

Now the equivalence of the properties stated in the theorem follows from (2.10) by applying Theorem 2.1.2. The last assertion of our theorem means exactly that the formula (2.10) holds. □

2.2. Abstract semi-local Bernstein–Walsh inequalities

As a first application of Theorem 2.1.2, we will give a semi-local abstract version of the (well-known) important polynomial inequalities called Bernstein–Walsh inequalities in the literature (see [Ber], [W], [Sic2]).

Let X be a complex space and $\mathcal{L}$ an admissible class of plurisubharmonic functions on X. Then we can introduce the following natural graded sequence of spaces of holomorphic functions associated to the class $\mathcal{L}$ on X. For each integer $d\geqslant 1$ define

$$\mathcal{P}_d(X;\mathcal{L}):=\{f\in\mathcal{O}(X):(1/d)\log|f|\in\mathcal{L}\}\cup\{0\} \tag{2.11}$$

and $\mathcal{P}_0(X;\mathcal{L}):=\mathbf{C}$.

In the case where X admits a parabolic potential p, the space (2.11) associated to the class $\mathcal{L}=\mathcal{L}(X;p)$ is the complex linear space of holomorphic functions of polynomial growth (with respect to p) of degree at most d.

In the general case, it is not clear whether the set $\mathcal{P}_d(X;\mathcal{L})$ is a complex linear space, but it is a *complex conic space*, which means that if $f\in\mathcal{P}_d(X;\mathcal{L})$ and $\lambda\in\mathbf{C}$ then $\lambda\cdot f\in\mathcal{P}_d(X;\mathcal{L})$.

As an immediate application of Corollary 2.1.3, we will deduce the following Bernstein–Walsh inequalities, which will give a precise comparaison between two uniform norms on the complex conic spaces $\mathcal{P}_d(X;\mathcal{L})$.

PROPOSITION 2.2.1. *Let $\mathcal{L}\subset X$ be an admissible class satisfying the Lelong property (LP) on X. Then for any non-$\mathcal{L}$-polar subsets $A\Subset X$ and $B\Subset X$ of X, the following inequalities hold*:

$$\|f\|_A\cdot e^{-d\cdot\varrho(B;A)}\leqslant\|f\|_B\leqslant\|f\|_A\cdot e^{d\cdot\varrho(A;B)},\quad \forall f\in\mathcal{P}_d(X;\mathcal{L}),\ \forall d\in\mathbf{N}^*. \tag{2.12}$$

In particular, for each $d\in\mathbf{N}^*$, the complex linear space spanned by the complex conic space $\mathcal{P}_d(X;\mathcal{L})$ is of finite dimension.

Proof. From Corollary 2.1.3 we know that the constants $\varrho(A;B):=\max_{x\in B}\Lambda_A$ is finite, and then the last inequality in (2.12) follows easily from the definition of the $\mathcal{L}$-extremal function Λ_A and the definition of the space $\mathcal{P}_d(X;\mathcal{L})$. The first inequality in (2.12) follows from the last one by permuting A and B.

Now let $A\Subset X$ be a fixed nonpluripolar compact subset. Then from the inequalities (2.12) and Montel's theorem, it follows that the set

$$\mathcal{U}_d:=\{f\in\mathcal{P}_d(X;\mathcal{L}):\|f\|_A\leqslant 1\}$$

is a relatively compact neighbourhood of the origin in the space $\mathcal{P}_d(X;\mathcal{L})$ for the topology of local uniform convergence on X. Therefore the complex linear subspace

$\mathcal{P}_d(X;\mathcal{L})+\mathcal{P}_d(X;\mathcal{L})$ spanned by the complex conic space $\mathcal{P}_d(X;\mathcal{L})$ is of finite dimension, thanks to Riesz's theorem. □

Let us now see how our approch leads to abstract semi-local Bernstein–Walsh inequalities, which happen to be useful in applications (see [FeN2]).

PROPOSITION 2.2.2. *Let X be a complex space and U an open subset of X. Suppose that $\mathcal{F}=(f_i)_{i\in I}$ is a nonempty family of holomorphic functions on U for which there exists a compact set $K\subset U$, a family of positive integers $D=(D_i)_{i\in I}$ and a constant $M\geqslant 1$ such that the following Bernstein–Walsh inequality holds*:

$$\|f_i\|_U \leqslant M^{D_i}\cdot \|f_i\|_K, \quad \forall i\in I. \tag{2.13}$$

Then for any nonpluripolar subset $E\Subset U$ and any nonpluripolar subset $G\subset U$, there exists a positive constant $R=R(E;G)\geqslant 1$, depending only on E,G and K,M, such that the following Bernstein–Walsh inequalities hold:

$$\|f_i\|_G \leqslant R^{D_i}\cdot \|f_i\|_E, \quad \forall i\in I. \tag{2.14}$$

Proof. For each integer $d\geqslant 1$, consider the set of all holomorphic functions f on U defined as

$$\mathcal{Q}_d := \{f\in\mathcal{O}(U) : \|f\|_U \leqslant M^d\cdot \|f\|_K\}. \tag{2.15}$$

Then each $\mathcal{Q}_d$ is a complex conic space containing the constant functions, and $f_i\in\mathcal{Q}_{D_i}$ for any $i\in I$. Now let us consider the class $\mathcal{H}=\mathcal{H}(K,U,M)$ of Hartogs plurisubharmonic functions on U defined as

$$\mathcal{H} := \{v\in \mathrm{PSH}(U) : \exists d\in \mathbf{N}^*,\ \exists f\in\mathcal{Q}_d,\ v=(1/d)\log|f|\}. \tag{2.16}$$

Then it is clear from the definitions (2.16) and (2.15) that $\mathcal{H}$ is an admissible class of plurisubharmonic functions on U satisfying the estimates

$$\max_U v \leqslant \max_K v + \log M, \quad \forall v\in\mathcal{H}. \tag{2.17}$$

From (2.17) and Theorem 2.1.2, it follows that the class $\mathcal{H}$ satisfies the Lelong property (LP) on U. Then, applying Corollary 2.1.3, we conclude that $\varrho_{\mathcal{H}}(E;G)<+\infty$ and $\varrho_{\mathcal{H}}(G;E)<+\infty$. Therefore, using the inequalities (2.12) for $\mathcal{H}$, we obtain the inequalities (2.14) with the constant $R:=\exp\varrho_{\mathcal{H}}(E;G)$, since $f_i\in\mathcal{Q}_{D_i}\subset\mathcal{P}_{D_i}(U;\mathcal{H})$ for each $i\in I$. □

Observe that the constant $R(E;G)$ in Proposition 2.2.2 is related to the $\mathcal{H}$-extremal function of the set E, and then it is possible to compute it explicitly in some specific cases or, at least, to have a good estimate using the inequality (2.7). In this way, it is possible to deduce the so-called "doubling inequality" in the case of algebraic sets (see [FeN2]). In fact, our methods lead to more general "Bernstein inequalities", which will be investigated in a subsequent paper.

3. Semi-local behaviour of Lelong classes

In this section we will investigate the semi-local behaviour of admissible classes in terms of their Lelong numbers.

3.1. Lelong numbers associated to a Lelong class

Let us assume that X is a complex space and denote as before by X_{reg} the complex manifold of its regular points. Let $a \in X_{\mathrm{reg}}$ and let U be a coordinate neighbourhood of a such that there exists a homeomorphism $\phi: \bar{U} \to \bar{\Delta}^n$ onto the closed unit polydisc $\bar{\Delta}^n$ in $\mathbf{C}^n$ which is holomorphic on U and satisfies $\phi(a)=0$. Such a coordinate system (U, ϕ) will be called a *regular coordinate system* at the point a. Let Δ_s^n be the open polydisc of radius $s>0$ centred at the origin in $\mathbf{C}^n$ and consider the sets $\bar{U}_s := \phi^{-1}(\bar{\Delta}_s^n)$, $0<s<1$.

Let us denote by $\|z\|$ the sup-norm on $\mathbf{C}^n$. Then it is well known that for any plurisubharmonic function w on a neighbourhood of $\bar{U}$, the real-valued function

$$M_w(a, s) := \sup\{w(x) : x \in \bar{U}_s\} = \sup\{w \circ \phi^{-1}(z) : \|z\| \leqslant s\}$$

is an increasing and convex function of $\log s$ on the real interval $]0, 1[$, so that the following limit exists and is finite:

$$\nu(w; a) := \lim_{s \to 0} \frac{\sup_{\bar{U}_s} w}{\log s}. \tag{3.1}$$

By a result of Kiselman [Ki1], [Ki2], the positive real number $\nu(w; a)$ defined by (3.1) is equal to the Lelong number of the plurisubharmonic function w at the point a ([Lel1], [LG], [Ho1]), and by a result of Siu [Siu] it is independent of the coordinate system we choose at the point $a \in X_{\mathrm{reg}}$ (see also [D2], [Ho1]).

Now we proceed to prove the following fundamental result which describes the semi-local behaviour of an abstract Lelong class.

THEOREM 3.1.1. *Let X be a complex space, $\mathcal{L}$ a Lelong class on X and (U, ϕ) a given regular coordinate system at a fixed regular point of $a \in X_{\mathrm{reg}}$. Then the following properties are satisfied:*

(1) *The limit*

$$\varkappa_{\mathcal{L}}(a) = \varkappa_{X,\mathcal{L}}(a) := \lim_{s \to 0} \frac{\log \operatorname{cap}_{\mathcal{L}}(\bar{U}_s; U)}{\log s} = \inf_{s>0} \frac{\log \operatorname{cap}_{\mathcal{L}}(\bar{U}_s; U)}{\log s} \tag{3.2}$$

exists and is finite.

(2) *We have*

$$\varkappa_{\mathcal{L}}(a) = \nu_{\mathcal{L}}(a), \tag{3.3}$$

where the right-hand side is defined by

$$\nu_{\mathcal{L}}(a) := \sup\{\nu(w;a) : w \in \mathcal{L}\}. \tag{3.4}$$

In particular, the number $\varkappa_{\mathcal{L}}(a)=\nu_{\mathcal{L}}(a)$ *does not depend on the regular coordinate system* (U,ϕ) *at the point* $a\in X_{\mathrm{reg}}$.

(3) *The positive real-valued function* $\varkappa_{\mathcal{L}}=\nu_{\mathcal{L}}$ *is upper semi-continuous on* X_{reg}.

Proof. Let us denote by $\mathcal{L}_0$ the subclass consisting of functions $w\in\mathcal{L}$ such that $M_w(a,1)=\sup_{\bar U} w=0$. From the definition of the $\mathcal{L}$-capacity, it follows that

$$\chi(s) := \frac{\log \operatorname{cap}_{\mathcal{L}}(\bar U_s;U)}{\log s} = \sup_{w\in\mathcal{L}_0} \frac{M_w(a,s)}{\log s}, \quad \forall s\in\,]0,1[\,. \tag{3.5}$$

By Theorem 2.2.2, we have $\operatorname{cap}_{\mathcal{L}}(\bar U_s;U)>0$ for any $s\in\,]0,1[$, and then the function χ defined by (3.5) is a positive real-valued function on $]0,1[$. Since for each $w\in\mathcal{L}_0$ the function $s\mapsto M_w(a,s)$ is an increasing and convex function of $\log s$ on the real interval $]0,1[$, it follows that the function $s\mapsto M_w(a,s)/\log s$ is an increasing function on the real interval $]0,1[$. Hence the function χ defined by (3.5) is an increasing function on $]0,1[$ with positive real values. Thus the limit in (3.2) exists and can be expressed by

$$\varkappa_{\mathcal{L}}(a) = \inf_{s>0} \sup_{w\in\mathcal{L}_0} \frac{M_w(a,s)}{\log s}. \tag{3.6}$$

On the other hand, the same argument shows that for any $w\in\mathcal{L}_0$, we have

$$\nu(w;a) = \inf_{s>0} \frac{M_w(a,s)}{\log s}.$$

Since for any $w\in\mathcal{L}$ the function $w_0:=w-\sup\{w(x):x\in\bar U\}\in\mathcal{L}_0$ and satisfies $\nu(w;a)=\nu(w_0;a)$, it follows from the definition (3.4) that

$$\nu_{\mathcal{L}}(a) = \sup_{w\in\mathcal{L}_0}\left\{\inf_{s>0}\frac{M_w(a,s)}{\log s}\right\} = \sup_{w\in\mathcal{L}_0}\nu(w;a). \tag{3.7}$$

From (3.6) and (3.7), it follows clearly that $\nu_{\mathcal{L}}(a)\leqslant\varkappa_{\mathcal{L}}(a)$.

In order to prove the reverse inequality, we need the following property.

CLAIM 1. *The subset* $\mathcal{L}_0\subset\mathrm{PSH}(X)$ *is compact.*

Indeed, let $(w_j)_{j\geqslant 0}$ be a sequence of functions from $\mathcal{L}_0$. Since $\sup_{\bar U} w_j=0$ for any $j\geqslant 0$, it follows from the definition of $\mathcal{L}_0$ that $w_j\leqslant\Lambda_{\bar U}(\cdot\,;\mathcal{L})$ on X for any $j\geqslant 0$. Thus by Theorem 2.1.2, the sequence $(w_j)_{j\geqslant 0}$ is locally bounded above on X. Then from Hartogs's lemma, it follows that $\limsup_{j\to+\infty} w_j\not\equiv-\infty$. By [Ho2, Theorem 3.2.12], it follows that some subsequence $(v_k)_{k\geqslant 0}$ converges in $L^1_{\mathrm{loc}}(X)$ to a plurisubharmonic function v on X, and $v=(\limsup_{j\to+\infty} v_j)^*$. Since $K:=\overline{U}$ is pluriregular and $N:=\{x\in X: \limsup_{j\to+\infty} v_j(x)<v(x)\}$ is pluripolar by [BT1], [Bed], it follows from a generalized version of Hartogs's lemma (see [Ze2, Theorem 2.5]) that $\sup_K v=0$. Since $\mathcal{L}$ is closed, v belongs to $\mathcal{L}_0$, and the claim is proved.

Now let us prove that $\varkappa_{\mathcal{L}}(a)\leqslant\nu_{\mathcal{L}}(a)$. Indeed, let $\varkappa<\varkappa_{\mathcal{L}}(a)$. Then by (3.6), for every $s\in]0,1[$ there exists $w_s\in\mathcal{L}_0$ such that

$$\frac{M_{w_s}(a;s)}{\log s}>\varkappa. \tag{3.8}$$

Taking a decreasing sequence (s_j) of numbers in $]0,1[$ converging to 0, we obtain a sequence (w_{s_j}) from $\mathcal{L}_0$ satisfying the estimate (3.8) for $s=s_j$ and $j\geqslant 1$. From Claim 1, it follows that some subsequence converges to a function $w\in\mathcal{L}_0$. Taking a subsequence if necessary, we may assume that the sequence (w_{s_j}) itself converges to w. Then from [Ho2, Theorem 3.2.13], it follows that $w:=(\limsup_{j\to+\infty} w_{s_j})^*$. By (3.8), for each $t\in]0,1[$ and any j large enough so that $0<s_j<t$, we have

$$\frac{M_{w_{s_j}}(a;t)}{\log t}\geqslant\frac{M_{w_{s_j}}(a;s_j)}{\log s_j}>\varkappa. \tag{3.9}$$

By Hartogs's lemma (see [Ze2]), it is easy to see that for any $t\in]0,1[$ we have $M_w(a,t)=\limsup_{j\to+\infty} M_{w_{s_j}}(a,t)$, which by (3.9) implies

$$\frac{M_w(a,t)}{\log t}\geqslant\varkappa,\quad \forall t\in]0,1[. \tag{3.10}$$

From the formula (3.7) and the inequality (3.10), it follows that $\nu_{\mathcal{L}}(a)\geqslant\varkappa$. This proves that $\nu_{\mathcal{L}}(a)\geqslant\varkappa_{\mathcal{L}}(a)$. Thus we have proved that $\varkappa_{\mathcal{L}}(a)=\nu_{\mathcal{L}}(a)$. Therefore we obtain the formula (3.3).

Let us now prove (3), that the function $\nu_{\mathcal{L}}$ is upper semi-continuous on X_{reg}. Then we need the following known result.

CLAIM 2. *The mapping*

$$\begin{aligned}\nu\colon \mathrm{PSH}(X)\times X_{\mathrm{reg}}&\to\mathbf{R}^+,\\ (w,x)&\mapsto\nu(w;x)\end{aligned} \tag{3.11}$$

is upper semi-continuous on $\mathrm{PSH}(X)\times X_{\mathrm{reg}}$.

Indeed, let $a\in X_{\mathrm{reg}}$ be a fixed regular point and (U,ϕ) a regular coordinate system at the point $a\in X_{\mathrm{reg}}$ as before. First observe that the problem is local and the number

$\nu(w;x)$ is independent of the coordinate system so that $\nu(w;x)=\nu(w\circ\phi^{-1};\phi(x))$ for $x\in U$. Then we can assume that U is a fixed polydisc in $\mathbf{C}^n$ containing the closed unit polydisc $\bar{\Delta}^n$, and consider the class $\mathrm{PSH}(U)$ of plurisubharmonic functions w on U. Then there exists $s_0\in]0,1[$ small enough such that for a fixed $s\in]0,s_0[$, the function $(w,x)\mapsto M_w(x,s):=\sup\{w(z):\|z-x\|\leqslant s\}$ is a continuous function on $\mathrm{PSH}(U)\times\Delta^n$. Therefore the mapping ν defined in (3.11) is given by

$$\nu(w;x)=\inf_{s>0}\frac{M_w(x,s)-M_w(x,1)}{\log s} \tag{3.12}$$

on $\mathrm{PSH}(U)\times\Delta^n$. Then the formula (3.12) implies that the mapping ν is upper semi-continuous on $\mathrm{PSH}(U)\times\Delta^n$, which proves our second claim.

Now, from the upper semi-continuity of the mapping ν (Claim 2), the compactness of the set $\mathcal{L}_0$ (Claim 1) and the formula (3.7), it is easy to deduce that the function $\varkappa_{\mathcal{L}}=\nu_{\mathcal{L}}$ is upper semi-continuous on X_{reg}. □

Let us derive the following easy consequence of our theorem which will be important in subsequent considerations.

COROLLARY 3.1.2. *Let X be a complex space and $\mathcal{L}$ a Lelong class on X. Then for any regular point $a\in X_{\mathrm{reg}}$, the Lelong number $\nu_{\mathcal{L}}(a)$ of the class $\mathcal{L}$ at a is finite and we have*

$$m_f(a)\leqslant\nu_{\mathcal{L}}(a)\cdot d,\quad \forall f\in\mathcal{P}_d(X;\mathcal{L})\setminus\{0\},\ \forall d\in\mathbf{N}^*, \tag{3.13}$$

where $m_f(a)$ is the order of vanishing of the holomorphic function $f\not\equiv0$ at the regular point $a\in X_{\mathrm{reg}}$ (with $m_f(a)=0$ if $f(a)\neq0$).

Proof. The finiteness of the Lelong numbers $\nu_{\mathcal{L}}(a)$ follows from Theorem 3.1.1. Moreover, it is well known that $m_f(a)=\nu(\log|f|;a)$ is the Lelong number of the plurisubharmonic function $\log|f|$ at the regular point $a\in X_{\mathrm{reg}}$ (see [LG], [Hol]). Therefore the estimates (3.13) follow from (3.4) and the fact that $(1/d)\log|f|\in\mathcal{L}$ for any $f\in\mathcal{P}_d(X;\mathcal{L})\setminus\{0\}$.

3.2. The minimal Lelong number of an admissible class

Let $\mathcal{L}$ be an admissible class of plurisubharmonic functions on a complex space X. We can still consider the two functions $\varkappa_{\mathcal{L}}$ and $\nu_{\mathcal{L}}$ defined on X_{reg} by the formulas (3.6) and (3.7) respectively. Then we get

$$\nu_{\mathcal{L}}(x)\leqslant\varkappa_{\mathcal{L}}(x),\quad \forall x\in X_{\mathrm{reg}}, \tag{3.14}$$

and these two numbers might be infinite. We do not know if they are equal in general.

The following definition will be important for the statement of our theorems of algebraicity.

Definition 3.2.1. Let $\mathcal{L}$ be an admissible class of plurisubharmonic functions on a complex space X. Then the positive (possibly infinite) number

$$\nu(\mathcal{L})=\nu(X,\mathcal{L}):=\inf\{\nu_{\mathcal{L}}(a):a\in X_{\mathrm{reg}}\} \tag{3.17}$$

will be called the *minimal Lelong number* of the class $\mathcal{L}$.

From Theorem 3.1.1, it follows that for a Lelong class $\mathcal{L}$ the Lelong function is finite everywhere on X_r, and then the minimal Lelong number of $\mathcal{L}$ is finite.

Let us consider the important particular case of an irreducible algebraic subvariety Z of $\mathbf{C}^N$.

Definition 3.2.2. Let Z be an irreducible algebraic subvariety of $\mathbf{C}^N$. The minimal Lelong number of the class $\mathcal{L}(Z)$ will be denoted by

$$\nu(Z)=\nu(\mathcal{L}(Z)):=\inf\{\nu_{\mathcal{L}(Z)}(x):x\in Z_{\mathrm{reg}}\} \tag{3.18}$$

and will be called the *minimal graded Lelong number* of Z. This terminology will be motivated later in §4.

Let us first give some simple examples of computation of minimal Lelong numbers, which will be used later.

Example 3.2.3. Let $c>0$ be a real number and consider on the space $\mathbf{C}^n$ the Lelong class

$$\mathcal{L}_c(\mathbf{C}^n):=\{v\in \mathrm{PSH}(\mathbf{C}^n):v(z)\leqslant c\log^+|z|+O(1)\}.$$

If $w\in\mathcal{L}_c:=\mathcal{L}_c(\mathbf{C}^N)$, then for any $a\in\mathbf{C}^n$ and any $t\in]0,1[$, the one-variable function defined by $s\mapsto(M_w(a,s)-M_w(a,t))/(\log s-\log t)$ is an increasing function on $]t,+\infty[$, and its limit as $s\to+\infty$ is at most equal to c, so that $\nu(w;a)\leqslant c$. Thus by Theorem 3.1.1, we deduce that $\varkappa_{X,\mathcal{L}_c}(a)\leqslant c$. On the other hand, the function $z\mapsto c\log|z-a|$ belongs to $\mathcal{L}_c$, and its Lelong number at a is equal to c. Therefore we have $\varkappa_{\mathcal{L}_c}(a)=c$ for any $a\in\mathbf{C}^n$. Thus $\nu(\mathcal{L}_c)=c$.

Example 3.2.4. Consider the smooth algebraic curve $Z:=\{(x,y)\in\mathbf{C}^2:y=P(x)\}$, where $P(x)$ is a polynomial of one complex variable of degree $m\geqslant1$. Given a (pluri)-subharmonic function $u\in\mathcal{L}(Z)$, we can define a (pluri)-subharmonic function on $\mathbf{C}$ by the formula $\tilde{u}(x):=u(x,P(x))$, $x\in\mathbf{C}$. Since $y=P(x)\sim c_m x^m$ at infinity on Z, it is clear that $\tilde{u}\in\mathcal{L}_m(\mathbf{C})$ and the map $u\mapsto\tilde{u}$ is a bijection of $\mathcal{L}(Z)$ onto $\mathcal{L}_m(\mathbf{C})$. Moreover, if

$z_0:=(x_0,y_0)\in X$, we have $\nu(u;z_0)=\nu(\tilde{u};x_0)$. Therefore, from Example 3.2.3, we deduce the identity $\varkappa_{\mathcal{L}(Z)}(z_0)=\varkappa_{\mathcal{L}_m(\mathbf{C})}(x_0)=m$ for any $z_0\in Z$. Thus $\nu(Z)=m$.

Recall that for an arbitrary (irreducible) algebraic subset $Z\subset\mathbf{C}^N$ of dimension n, the degree of algebraicity of Z, which will be denoted here by $\delta(Z)$, is the number of points of intersection of Z with a generic $(N-n)$-plane of $\mathbf{C}^N$ (see [Ha], [C]).

The last example suggests that the minimal graded Lelong number $\nu(Z)$ of an algebraic curve in $\mathbf{C}^2$ coincides with its degree of algebraicity $\delta(Z)$. This will be proved in the next section (see Corollary 4.1.3).

Now let us prove the following important estimate for the general case.

PROPOSITION 3.2.5. *Let Z be an irreducible algebraic subvariety of dimension n in $\mathbf{C}^N$. Then we have $\nu(Z)\leqslant\delta(Z)$.*

Proof. By [C, Corollary 11.3.1], there exists an $(N-n)$-plane Γ in $\mathbf{C}^N$ such that the projection $\pi: Z\to\Gamma^\perp$ is a δ-sheeted analytic cover, where $\delta:=\delta(Z)$, and moreover, after a unitary change of variables in $\mathbf{C}^N$, we can assume that for some constant $c>0$, the following inclusion holds:

$$Z\subset\{\zeta=(\zeta',\zeta'')\in\mathbf{C}^n\times\mathbf{C}^{N-n}:|\zeta''|\leqslant c(1+|\zeta'|)\},\tag{3.19}$$

where $\zeta':=\pi(\zeta)=(\zeta_1,...,\zeta_n)$ and $\zeta'':=(\zeta_{n+1},...,\zeta_N)$ (see also [Ru], [Sa1]).

Let S be the critical set of the projection π. Then we claim that for any $a\in Z\setminus S$ and any $w\in\mathcal{L}(Z)$, we have $\nu(w,a)\leqslant\delta$. Indeed, let $w\in\mathcal{L}(Z)$ and consider the function

$$\pi_*w(z):=\sum_{\pi(\zeta)=z}w(\zeta),\quad z\in\mathbf{C}^n.\tag{3.20}$$

Since $\pi: Z\to\mathbf{C}^n$ is a δ-sheeted analytic cover which satisfies (3.19), it follows that $\pi_*w\in\mathcal{L}_\delta(\mathbf{C}^n)$ (see Example 3.2.3). Let $a\in Z\setminus S$ and $b:=\pi(a)$. Then there exists an open neighbourhood V of b in $\mathbf{C}^n$ and an open neighbourhood $U\Subset Z$ of a such that the restriction $\pi_U: U\to V$ is biholomorphic. To estimate $\nu(w,a)$, we can assume that $w\leqslant 0$ on U. Then it follows from (3.20) that $\pi_*w\leqslant w\circ\pi_U^{-1}$ on V, which implies the estimate

$$\nu(\pi_*w,b)\geqslant\nu(w,a).\tag{3.21}$$

Since $\pi_*w\in\mathcal{L}_\delta(\mathbf{C}^n)$, it follows from (3.21) and Example 3.2.3 that $\nu(w,a)\leqslant\delta$, which implies that $\varkappa_{\mathcal{L}(Z)}(a)\leqslant\delta$. This proves our claim, from which the proposition follows. □

The following formula is important for computing the minimal graded Lelong number and will be useful later.

PROPOSITION 3.2.6. *For $i=1,2$, let Z_i be an irreducible algebraic subvariety of dimension n_i in $\mathbf{C}^{N_i}$. Then the minimal graded Lelong number of the algebraic subvariety $Z:=Z_1\times Z_2$ is given by*

$$\nu(Z)=\max\{\nu(Z_1),\nu(Z_2)\}. \tag{3.22}$$

Proof. Fix a regular point $a=(a_1,a_2)$ of Z, and for each $i=1,2$ consider a regular coordinate system (U^i,h_i) at the point a_i. Put $U:=U^1\times U^2$ and $\phi:=\phi_1\times\phi_2$. Then (U,ϕ) is a regular coordinate system at the point a, and for every $s\in]0,1[$, we have $U_s:=\phi^{-1}(\Delta_s^{n_1+n_2})=U_s^1\times U_s^2$. From [Ze2, Theorem 4.5] we get the product property

$$\Lambda_{\bar{U}_s}(\zeta;Z):=\sup\{\Lambda_{\bar{U}_s^1}(\zeta_1;Z_1),\Lambda_{\bar{U}_s^2}(\zeta_2;Z_2)\},\quad \forall\zeta=(\zeta_1,\zeta_2)\in Z. \tag{3.23}$$

Now from Definition 3.2.1 and the formulas (3.1) and (3.23), we deduce

$$\varkappa_{\mathcal{L}(Z)}(a)=\sup\{\varkappa_{\mathcal{L}(Z_1)}(a_1),\varkappa_{\mathcal{L}(Z_2)}(a_2)\}. \tag{3.24}$$

Then (3.22) follows from the definition (3.18) and the formulas (3.24) and (3.3). □

Using (3.22) and Example 3.2.4, we can now produce a simple example which shows that for an algebraic subvariety of high dimension, the minimal graded Lelong number and the degree of algebraicity are not equal in general.

Example 3.2.7. Let us take two algebraic curves, C_1 of degree $m_1\geqslant 2$ and C_2 of degree $m_2\geqslant 2$ as in Example 3.2.4, and put $Z:=C_1\times C_2$. Then it is well known that the degree of algebraicity of Z is given by the formula $\delta(Z)=m_1\cdot m_2$, while the graded minimal Lelong number of Z is given by $\nu(Z)=\max\{m_1,m_2\}$, thanks to (3.22). Thus $\nu(Z)<\delta(Z)$.

4. Algebraicity of Lelong classes and analytic sets

In this section we will prove a theorem of algebraicity for an admissible class $\mathcal{L}$ which gives a sharp asymptotic estimate on the Hilbert function associated to the class $\mathcal{L}$ in terms of its minimal Lelong number. This result will be the main step in the proof of our new semi-local criterion of algebraicity in §4.2.

4.1. A theorem of algebraicity for Lelong classes

In this subsection, we will prove our first main result which is an algebraicity theorem for admissible classes with a finite minimal Lelong number.

Let X be a complex space of dimension n and $\mathcal{L}\subset \mathrm{PSH}(X)$ an admissible class on X. Recall the definition of the natural graded sequence of spaces of $\mathcal{L}$-polynomial holomorphic functions on X:

$$\mathcal{P}(X;\mathcal{L}) := \bigcup_{d\geqslant 1} \mathcal{P}_d(X;\mathcal{L}), \tag{4.1}$$

where

$$\mathcal{P}_d(X;\mathcal{L}) := \{f\in\mathcal{O}(X) : (1/d)\log|f|\in\mathcal{L}\}\cup\{0\}, \quad d\in\mathbf{N}^*, \tag{4.2}$$

and $\mathcal{P}_0:=\mathbf{C}$. We want to estimate the asymptotic behaviour of the "dimension" of the complex conic space $\mathcal{P}_d(X;\mathcal{L})$ with respect to the $\mathcal{L}$-degree d for an admissible class $\mathcal{L}$ of finite minimal Lelong number. By definition, the (complex) dimension of the conic space $\mathcal{P}_d(X;\mathcal{L})$ will be defined by

$$h_{X,\mathcal{L}}(d) := \dim_{\mathbf{C}} \mathcal{P}_d(X;\mathcal{L}) := \sup\{\dim\mathcal{E} : \mathcal{E}\in\widetilde{\mathcal{P}}_d\}, \tag{4.3}$$

where $\widetilde{\mathcal{P}}_d$ is the family of all complex linear spaces of finite dimension contained in $\mathcal{P}_d(X;\mathcal{L})$. If the class $\mathcal{L}$ is a Lelong class, then we know that the complex linear subspace spanned by the conic space $\mathcal{P}_d(X;\mathcal{L})$ is of finite dimension, which implies that the function defined by (4.3) takes finite integer values.

In any case, by analogy to the case of affine algebraic varieties (see [Ha]), the function defined by (4.3) will be called the *Hilbert function* of $(X,\mathcal{L})$.

Our main goal here is to give a sharp asymptotic estimate of the Hilbert function (4.3) in terms of the minimal Lelong number of the class $\mathcal{L}$. In fact, we will need a slightly more general version of such an estimate.

Let $(\mathcal{C}_d)_{d\in\mathbf{N}}$ be a graded sequence of complex conic subspaces of $\mathcal{O}(X)$, which means that each space $\mathcal{C}_d$ is a complex conic space such that $\mathcal{C}_0=\mathbf{C}\subset\mathcal{C}_d$ for any $d\in\mathbf{N}^*$. Then we will say that $\mathcal{C}:=\bigcup_{d\in\mathbf{N}}\mathcal{C}_d$ is a graded complex conic space.

For each regular point $x\in X_{\mathrm{reg}}$, we can define the positive number (possibly infinite)

$$\mu_{\mathcal{C}}(x) := \sup\{m_f(x)/d : f\in\mathcal{C}_d,\ f\neq 0,\ d\geqslant 1\}, \tag{4.4}$$

which will be called the *graded multiplicity* of the graded space $\mathcal{C}$ at the regular point x. Recall that $m_f(x)$ denotes the order of vanishing of the holomorphic function $f\neq 0$ at the regular point x, and $m_f(x)=\nu(\log|f|;x)$ is the Lelong number of the plurisubharmonic function $\log|f|$ at the point x.

Then the positive number

$$\mu(\mathcal{C}) := \inf\{\mu_{\mathcal{C}}(x) : x\in X_{\mathrm{reg}}\}, \tag{4.5}$$

which might be infinite, will be called the *minimal graded multiplicity* of the graded space (or sequence) $\mathcal{C}=\bigcup_{d\in\mathbf{N}}\mathcal{C}_d$. We also define the *Hilbert function* of the graded sequence $\mathcal{C}$ of complex conic spaces by

$$h_{\mathcal{C}}(d):=\sup\{\dim_{\mathbf{C}}(\mathcal{E}):\mathcal{E}\in\tilde{\mathcal{C}}_d\},\quad d\in\mathbf{N},\tag{4.6}$$

where $\tilde{\mathcal{C}}_d$ is the family of all complex linear subspaces of $\mathcal{C}_d$ of finite dimension.

Now we are ready to prove our fundamental theorem of algebraicity.

THEOREM 4.1.1. *Let X be a complex space of dimension n, and $\mathcal{C}=\bigcup_{d\in\mathbf{N}}\mathcal{C}_d$ a graded sequence of complex conic subspaces of $\mathcal{O}(X)$ with finite minimal graded multiplicity, i.e. $\mu(\mathcal{C})<+\infty$. Then the Hilbert function of the graded sequence $\mathcal{C}$ defined by (4.6) satisfies the asymptotic upper estimate*

$$\limsup_{d\to+\infty}\frac{h_{\mathcal{C}}(d)}{d^n}\leqslant\frac{\mu^n}{n!},\tag{4.7}$$

where $\mu:=\mu(\mathcal{C})$ is the minimal graded multiplicity of the graded sequence $\mathcal{C}$.

Proof. Fix a regular point $a\in X_{\mathrm{reg}}$, an open neighbourhood $U\Subset X$ of a, and a biholomorphic mapping ϕ from U onto the open unit polydisc Δ^n in $\mathbf{C}^n$, which extends continuously to $\bar{U}$. For each $f\in\mathcal{C}_d$, define $\tilde{f}:=f\circ\phi^{-1}$, which is holomorphic on Δ^n and continuous on $\bar{\Delta}^n$. Let us denote $\mathcal{Q}_d:=\{\tilde{f}:f\in\mathcal{C}_d\}$ for $d\in\mathbf{N}$.

The main idea of the proof is to compare the dimension of any complex linear subspace of the complex conic space $\mathcal{Q}_d$ with the dimension of the complex linear space $\mathcal{P}_m(\mathbf{C}^n)$ of polynomials in n complex variables of degree not greater than m for an optimal value of the integer $m:=m_d$.

To this end we first consider these spaces as subspaces of the Banach space $\mathcal{B}_s$ of complex-valued continuous functions on the compact polydisc $\bar{\Delta}_s^n$ endowed with the norm of uniform convergence on $\bar{\Delta}_s^n$ defined by $\|g\|_s:=\max\{|g(z)|:z\in\bar{\Delta}_s^n\}$ for $g\in\mathcal{B}_s$. Then we will estimate the distance of any element of $\mathcal{Q}_d$ to the finite-dimensional space $\mathcal{P}_m(\mathbf{C}^n)$.

So expand each function $F\in\mathcal{Q}_d$, which is holomorphic on the polydisc Δ^n, in Taylor series on the polydisc Δ^n as

$$F(z)=\sum_{\alpha\in\mathbf{N}^n}c_\alpha z^\alpha,\quad z\in\Delta^n,$$

with uniform convergence on compact subsets of Δ^n.

Let us consider the Taylor polynomials of the function F given by the formula $T_m(z):=\sum_{|\alpha|\leqslant m}c_\alpha z^\alpha$, for $m\in\mathbf{N}$.

Fix a real number θ such that $0<\theta<\frac{1}{2}$, take a real number $0<s<\theta$ and put $t:=s/\theta<1$. Then using Cauchy inequalities, we get the estimates

$$\|F-T_m\|_s \leqslant \|F\|_t \sum_{|\alpha|=m+1}^{+\infty} \theta^{|\alpha|}, \quad \forall F\in \mathcal{Q}_d,\ \forall m\in \mathbf{N}. \tag{4.8}$$

An easy computation shows that

$$\sum_{|\alpha|=m+1}^{+\infty} \theta^{|\alpha|} = \sum_{k=m}^{+\infty} \binom{n+k}{k+1} \theta^{k+1} = \frac{1}{(n-1)!} D_\theta^{(n-1)} \left(\frac{\theta^{n+m}}{1-\theta}\right), \tag{4.9}$$

where $D_\theta^{(n-1)}$ stands for the derivative of order $n-1$ with respect to θ. Then, since $0<\theta<\frac{1}{2}$, from the equation (4.9) and the estimate (4.8), it becomes clear that there exists a constant c_n depending only on the dimension n such that we have the estimates

$$\|F-T_m\|_s \leqslant c_n (n+m)^{n-1} \theta^{m+1} \|F\|_t, \quad \forall F\in \mathcal{Q}_d,\ \forall m\in \mathbf{N},$$

which imply immediately the estimates

$$\operatorname{dist}_{\mathcal{B}_s}(F;\mathcal{P}_m(\mathbf{C}^n)) \leqslant c_n (n+m)^{n-1}\theta^{m+1}\|F\|_t, \tag{4.10}$$

for any $F\in\mathcal{Q}_d$, $m\in\mathbf{N}$ and $d\in\mathbf{N}$, the distance being calculated in the Banach space $\mathcal{B}_s$.

On the other hand, fix an integer $d\in\mathbf{N}^*$ and an arbitrary subspace $\mathcal{E}_d$ of finite dimension contained in $\mathcal{C}_d$. Then we can associate the Chebyshev constant

$$\tau_d(U_s;U) := \inf\{\|f\|_{U_s}^{1/d} : f\in\mathcal{E}_d,\ \|f\|_U = 1\}, \tag{4.11}$$

for $s\in\,]0,1[$. Since $\mathcal{E}_d$ is of finite dimension, a compactness argument shows that the Chebyshev constant defined by (4.11) is nonzero, and the class

$$\mathcal{H}_d := \{u\in \mathrm{PSH}(U) : \exists f\in\mathcal{E}_d,\ u=(1/d)\log|f|\}$$

is closed in $\mathrm{PSH}(U)$. Then by Theorem 2.1.2, it follows that $\mathcal{H}_d$ is a Lelong class on U. Therefore, applying Theorem 3.1.1, we conclude that the limit

$$\varkappa_d(a) := \lim_{s\to 0^+} \frac{\log \tau_d(U_s;U)}{\log s} = \inf_{s>0} \frac{\log\tau_d(U_s;U)}{\log s} \tag{4.12}$$

exists and is finite. Moreover,

$$\varkappa_d(a) = \sup\{(1/d)m_f(a) : f\in\mathcal{E}_d\setminus\{0\}\}. \tag{4.13}$$

The identity (4.13) and the definition (4.4) yield the inequality

$$\varkappa_d(a) \leqslant \mu_C(a). \tag{4.14}$$

Let us define the space $\mathcal{F}_d := \{f\circ\phi^{-1} : f \in \mathcal{E}_d\}$, which is isomorphic to $\mathcal{E}_d$, and consider the numbers

$$\alpha_d(s) := \sup_{F\in\mathcal{F}_d}\left\{\frac{\log\|F\|_s - \log\|F\|_1}{\log s}\right\} = \frac{d\cdot\log\tau_d(U_s;U)}{\log s}, \tag{4.15}$$

where the last identity follows immediately from the definition (4.11). Since for each $F\in\mathcal{F}_d$ the function $r\mapsto\log\|F\|_r$ is a convex function of the variable $\log r$ for $r\in\,]0,1]$, it is easy to derive from (4.15) the inequality

$$\|F\|_t \leqslant \|F\|_s\,\theta^{-\alpha_d(s)}, \quad \forall F\in\mathcal{F}_d. \tag{4.16}$$

Then by combining the inequalities (4.10) and (4.16), we deduce the fundamental estimates

$$\operatorname{dist}_{\mathcal{B}_s}(F;\mathcal{P}_m(\mathbf{C}^n)) \leqslant c_n(n+m)^{n-1}\|F\|_s\,\theta^{m+1-\alpha_d(s)}, \tag{4.17}$$

for any $F\in\mathcal{F}_d$ and any $m\in\mathbf{N}^*$.

Now take a real number $\mu>\mu(C)$ and, according to the definition (4.5), choose the regular point $a\in X_{\mathrm{reg}}$ so that $\mu_C(a)<\mu$. Then fix $\varepsilon>0$ and take a large integer d_0 such that

$$\eta_d := c_n(n+\mu d+\varepsilon d)^{n-1}\theta^{\varepsilon d} < 1, \quad \forall d\geqslant d_0.$$

Now fix $d\geqslant d_0$ and let m_d be the unique integer satisfying the inequalities $m_d\leqslant(\mu+\varepsilon)\cdot d< m_d+1$. Moreover, observe that $\lim_{s\to 0^+}\alpha_d(s)=d\cdot\varkappa_d(a)\leqslant d\cdot\mu_C(a)<d\cdot\mu$, thanks to the definition (4.12) and the inequality (4.14). Then it is possible to choose s so small that $0<s<\theta$ and $\alpha_d(s)<d\cdot\mu$, which implies that $m_d+1-\alpha_d(s)\geqslant\varepsilon d$.

Therefore from (4.17) and the fact that $\eta_d<1$ for the fixed integer $d\geqslant d_0$, we deduce the estimates

$$\operatorname{dist}_{\mathcal{B}_s}(F;\mathcal{P}_{m_d}(\mathbf{C}^n)) \leqslant \eta_d\cdot\|F\|_s < \|F\|_s, \quad \forall F\in\mathcal{F}_d\setminus\{0\}. \tag{4.18}$$

Using the estimate (4.18), we want to conclude that $\dim\mathcal{F}_d\leqslant\dim\mathcal{P}_{m_d}(\mathbf{C}^n)$. Assume that the converse is true, i.e. $\dim\mathcal{F}_d>\dim\mathcal{P}_{m_d}(\mathbf{C}^n)$. Since $\mathcal{P}_{m_d}(\mathbf{C}^n)$ is a subspace of finite dimension of the Banach space $\mathcal{B}_s$, we can apply the "projection theorem" in Banach spaces, known as the Krein–Krasnoselski–Milman theorem (see [Sin]), to obtain a function $F_0\in\mathcal{F}_d\setminus\{0\}$ which is "orthogonal" to the subspace $\mathcal{P}_{m_d}(\mathbf{C}^n)$ in the Banach space $\mathcal{B}_s$ in the sense that

$$\|F_0\|_s = \operatorname{dist}_{\mathcal{B}_s}(F_0;\mathcal{P}_{m_d}(\mathbf{C}^n)).$$

This contradicts the estimate (4.18) and proves the inequality

$$\dim \mathcal{E}_d = \dim \mathcal{F}_d \leqslant \dim \mathcal{P}_{m_d}(\mathbf{C}^n) = \binom{m_d+n}{n}, \quad \forall d \geqslant d_0. \tag{4.19}$$

Since $m_d \sim (\mu+\varepsilon)^n d^n$ as $d \to +\infty$, and $\mu > \mu(\mathcal{C})$ and $\varepsilon > 0$ are arbitrary, (4.19) implies clearly (4.7), which proves the theorem. □

As an easy consequence of the theorem let us deduce the following result.

COROLLARY 4.1.2. *Let X be a complex space of dimension n, and let $\mathcal{L}$ be an admissible class of plurisubharmonic functions on X with finite minimal Lelong number, i.e. $\nu := \nu(\mathcal{L}) < +\infty$. Then the minimal graded multiplicity of the associated graded sequence $\mathcal{P} = \mathcal{P}(X; \mathcal{L})$ is finite, and its Hilbert function $h_{\mathcal{P}} = h_{X,\mathcal{L}}$ defined by (4.3) satisfies the asymptotic upper estimate*

$$\limsup_{d \to +\infty} \frac{h_{X,\mathcal{L}}(d)}{d^n} \leqslant \frac{\mu^n}{n!}, \tag{4.20}$$

where $\mu = \mu(\mathcal{P})$ is the minimal graded multiplicity of the graded sequence $\mathcal{P} = \mathcal{P}(X; \mathcal{L})$.

Proof. Since $\mathcal{L}$ is an admissible class with minimal Lelong number $\nu(\mathcal{L}) < +\infty$, then by the estimate (3.13) of Corollary 3.1.2, the minimal graded multiplicity of the graded sequence $\mathcal{P} := \mathcal{P}(X; \mathcal{L})$ satisfies the inequality $\mu(\mathcal{P}) \leqslant \nu(\mathcal{L})$, which proves that the graded sequence $\mathcal{P}(X; \mathcal{L})$ has a finite graded multiplicity, and then the estimate (4.7) of Theorem 3.1.1 implies (4.20). □

The general idea that Bernstein–Walsh inequalities for a sequence of linear spaces of holomorphic functions should imply an upper bound on their dimensions was pointed out earlier by W. Plesniak in a different context (see [P]). Later the author used this idea to prove a weaker version of Corollary 4.1.2 in the case of parabolic spaces (see [Ze2]).

It is interesting to apply Theorem 4.1.1 to the particular case of an (irreducible) algebraic subvariety Z of $\mathbf{C}^N$.

Let $\mathcal{A}(Z) := \bigcup_{d \in \mathbf{N}} \mathcal{A}_d(Z)$ be the graded algebra of regular functions on Z, and denote by $h_Z(d) := \dim_{\mathbf{C}} \mathcal{A}_d(Z)$ the Hilbert function of the algebraic subvariety Z. The minimal graded multiplicity of the graded sequence $\mathcal{A}(Z)$ will be denoted by $\mu(Z)$ and will be called the *minimal graded multiplicity* of Z. Clearly we have $\mu(Z) \leqslant \nu(Z)$, and equality holds for algebraic curves as we will show later (see Corollary 4.1.4). We do not know if equality holds in general. It seems, however, reasonable to conjecture that $\mu(Z) = \nu(Z)$. With this in mind, it is quite natural to call $\nu(Z)$ the *minimal graded Lelong number* of the algebraic subvariety Z.

Anyway, we obtain the following interesting result.

COROLLARY 4.1.3. *Let Z be an algebraic subvariety of dimension n in $\mathbf{C}^N$. Then we have the asymptotic estimate*

$$\limsup_{d\to+\infty} \frac{h_Z(d)}{d^n} \leqslant \frac{\mu^n}{n!}, \tag{4.21}$$

where $\mu:=\mu(Z)$ is the minimal graded multiplicity of Z.

Furthermore, the degree of algebraicity $\delta(Z)$ of Z satisfies the estimates

$$\mu(Z) \leqslant \nu(Z) \leqslant \delta(Z) \leqslant \mu(Z)^n. \tag{4.22}$$

In particular, if C is an irreducible algebraic curve of $\mathbf{C}^N$ then $\mu(C)=\nu(C)=\delta(C)$.

Proof. Since Z is an algebraic subvariety, we know that the class $\mathcal{L}(Z)$ of plurisubharmonic functions with logarithmic growth on Z is a Lelong class on Z. Moreover, for each $d\in\mathbf{N}$, $\mathcal{A}_d(Z)$ is a complex linear subspace of $\mathcal{P}_d(Z;\mathcal{L}(Z))$. Thus the required estimate (4.21) follows from Corollary 4.1.2. Now combining this estimate with a well-known fact from algebraic geometry, we will obtain the last estimate in (4.22). Indeed, it is well known that the Hilbert function of the algebraic subvariety Z, defined by $h_Z(d):=\dim \mathcal{A}_d(Z)$, is a polynomial in d of degree $n=\dim Z$ for d large enough. Moreover, the leading coefficient of the Hilbert polynomial of Z is known to be $\delta(Z)/n!$, where $\delta(Z)$ is the degree of algebraicity of Z (see [Ha]). So our claim follows immediately, and then, taking into account the estimate of Proposition 3.2.5, we obtain the estimates (4.22).

Remark. The last result (Corollary 4.1.3) shows that the identities $\delta(C)=\nu(C)=\mu(C)$ are true for any algebraic curve C in $\mathbf{C}^N$. In higher dimension, Example 3.3.7 shows that the situation is different. The inequalities (4.22) are, however, optimal, since if C is an algebraic curve, we have $\mu(C)=\nu(C)=\delta(C)$, and for the algebraic subvariety $Z:=C^n$ of dimension n, we have $\nu(Z)=\nu(C)$ by Proposition 3.3.5 and $\delta(Z)=\nu(C)^n$ by the multiplicative property of the degree (see Example 3.3.7).

Let us now consider a more general situation where Theorem 4.1.1 can be applied. This was suggested by the fundamental work of Demailly [D1]. Let us first recall some facts from [D1] with slightly different notations. Let X be a Stein space of dimension n and $\varphi: X\to[-\infty+\infty[$ a continuous plurisubharmonic exhaustion, i.e. $B_r:=\{x\in X:\varphi(x)<\log r\}\Subset X$, for any $r>0$. Then Demailly introduced in [D1] a continuous family (σ_r) of Monge–Ampère measures on X associated to the exhaustion φ. More precisely, if $r_0:=\min_X\varphi$ then for each real number $r>r_0$, the measure σ_r is a positive Borel measure on X supported on the pseudosphere $S_r:=\partial B_r$ with total mass $\|\sigma_r\|=\int_{B_r}(dd^c\varphi)^n$.

Moreover, any $w \in \mathrm{PSH}(X)$ is σ_r-integrable for any $r > r_0$, and a generalized Lelong–Jensen formula is satisfied (see [D1, théorème 3.4]).

Now we need the following growth condition on (X, φ):

$$\lim_{r \to +\infty} \frac{\int_{B_r} (dd^c \varphi)^n}{\log r} = 0. \tag{4.23}$$

Observe that this condition is clearly satisfied if φ is a parabolic potential on X, since in this case the integral $\int_{B_r} (dd^c \varphi)^n$ is constant for r large enough.

Under the condition (4.23), Demailly introduced an interesting graded algebra of holomorphic functions on X.

A holomorphic function f on X is said to be of finite degree (with respect to φ) if the condition

$$\deg_\varphi(f) := \limsup_{r \to +\infty} \frac{\sigma_r(\log^+ |f|; \varphi)}{\log r} < +\infty$$

is satisfied. For each integer $d \geqslant 1$ let $\mathcal{A}_d(X; \varphi)$ be the space of all holomorphic functions f on X with finite degree $\deg_\varphi(f) \leqslant d$, and put $\mathcal{A}_0(X; \varphi) := \mathbf{C}$. Then using the condition (4.23), it is easy to see that each $\mathcal{A}_d(X; \varphi)$ is a complex conic space, and the set $\mathcal{A}(X; \varphi) := \bigcup_{d \geqslant 0} \mathcal{A}_d(X; \varphi)$ is a graded algebra of holomorphic functions on X.

On the other hand, using the condition (4.23) and his generalized Lelong–Jensen formula, Demailly proved the fundamental inequalities

$$m_f(a) \leqslant C(a) \cdot \deg_\varphi(f), \quad \forall f \in \mathcal{A}(X; \varphi), \tag{4.24}$$

for any regular point $a \in X_{\mathrm{reg}}$, where $C(a)$ is a positive constant which depends only on $a \in X_{\mathrm{reg}}$ (see [D1, corollaire 8.4]).

The inequalities (4.24) implies immediately that the graded sequence $\mathcal{A}(X; \varphi) = \bigcup_{d \geqslant 0} \mathcal{A}_d(X; \varphi)$ has a finite minimal graded multiplicity, i.e.

$$\mu_\varphi(X) := \inf_{a \in X_{\mathrm{reg}}} (\sup\{(1/d) m_f(a) : f \in \mathcal{A}_d(X; \varphi),\, d \in \mathbf{N}^*\}) < +\infty. \tag{4.25}$$

Then from the condition (4.25) and Theorem 4.1.1, we deduce the following "algebraicity theorem" for the space (X, φ), which may have some interest in connection with the work of Demailly [D1].

PROPOSITION 4.1.4. *Assume that (X, φ) satisfies the growth condition (4.23). Then the the graded sequence of complex conic spaces $\mathcal{A}(X; \varphi) = \bigcup_d \mathcal{A}_d(X; \varphi)$ has a finite minimal graded multiplicity, i.e. $\mu = \mu_\varphi(X) < +\infty$, and its Hilbert function satisfies the asymptotic upper estimate*

$$\limsup_{d \to +\infty} \frac{\dim_{\mathbf{C}} \mathcal{A}_d(X; \varphi)}{d^n} \leqslant \frac{\mu^n}{n!}.$$

It is interesting to observe that the inequalities (4.24) are analogous to our estimate (3.13) for the complex conic spaces associated to a Lelong class. In fact, given a space (X,φ) satisfying the growth condition (4.23), it is possible to define an admissible class of plurisubharmonic functions on X for which the associated complex conic spaces are precisely the spaces $\mathcal{A}_d(X;\varphi)$. Moreover, using the same method as in [D1, corollaire 8.5], we can prove that this class is an admissible class with finite Lelong numbers on X_{reg}, which implies the condition (4.25). Therefore the inequalities (4.24) and (3.13) are both consequences of the same result (compare with Corollary 4.1.2). Unfortunately we do not know if this class satisfies the Lelong property, so we will omit these details here.

4.2. A semi-local criterion of algebraicity for analytic sets

In this section we are going to deduce from Theorem 4.1.1 a new semi-local criterion of algebraicity which contains the criterion of algebraicity of A. Sadullaev [Sa2] as well as the global criterion of W. Stoll [St2].

A piece of an algebraic set in $\mathbf{C}^N$ will be, by definition, a local irreducible analytic subset of some algebraic subvariety of the same dimension.

Let Y be a local and irreducible analytic subset of dimension n in $\mathbf{C}^N$. Since we are interested in algebraic properties of Y, it is natural to consider the following class of plurisubharmonic functions of "restricted logarithmic growth" on Y:

$$\mathcal{L}_Y := \mathrm{PSH}(Y)\cap\{v|Y : v\in\mathcal{L}(\mathbf{C}^N)\}.$$

The closure $\mathcal{L}(Y)$ of the induced class $\mathcal{L}_Y$ in $L^1_{\mathrm{loc}}(Y)$ will be called the class of plurisubharmonic functions of *restricted logarithmic growth* on Y. It is clear that $\mathcal{L}(Y)$ is a closed admissible class of plurisubharmonic functions on Y.

On the other hand, it is also natural to consider the graded algebra

$$\mathcal{A}(Y) = \bigcup_{d\geqslant 1} \mathcal{A}_d(Y)$$

of holomorphic functions on Y, where

$$\mathcal{A}_d(Y) := \{f|Y : f\in\mathcal{P}_d(\mathbf{C}^N)\},\quad d\geqslant 1. \tag{4.26}$$

It is clear that for each $d\in\mathbf{N}^*$, $\mathcal{A}_d(Y)$ is a complex linear subspace of finite dimension of the conic space $\mathcal{P}_d(Y;\mathcal{L}(Y))$.

Therefore we can consider, as in the case on an algebraic subvariety, two positive numbers (possibly infinite) attached to Y.

The *minimal graded Lelong number* of Y is defined by

$$\nu(Y) := \inf\{\nu_{\mathcal{L}(Y)}(x) : x \in Y_{\mathrm{reg}}\}, \tag{4.27}$$

and the *minimal graded multiplicity* of Y is defined by

$$\mu(Y) := \inf\{\mu_{\mathcal{A}(Y)}(x) : x \in Y_{\mathrm{reg}}\}. \tag{4.28}$$

These two positive numbers might be infinite, since the class $\mathcal{L}(Y)$ need not to satisfy the Lelong property (LP) as the next theorem will show. It is clear that $\mu(Y) \leqslant \nu(Y)$, but we do not know if there is equality here.

We can, however, prove the following criterion of algebraicity, which was the main goal of this paper.

THEOREM 4.2.1. *Let Y be a local and irreducible analytic set of dimension n in $\mathbf{C}^N$. Then the following conditions are equivalent*:

(i) *Y is a piece of an algebraic set in $\mathbf{C}^N$.*

(ii) *The class $\mathcal{L}_Y$ satisfies the Lelong property (LP) on Y.*

(iii) *The class $\mathcal{L}(Y)$ is a Lelong class on Y.*

(iv) *There exists a compact subset $E \subset Y$ such that L_E is locally bounded on Y.*

(v) *Y is of $\mathcal{L}$-positive capacity, i.e. there exists a subdomain $U \Subset Y$ and a compact subset $K \subset \Omega$ such that $\mathrm{cap}_{\mathcal{L}(Y)}(K; U) > 0$.*

(vi) *The minimal graded Lelong number of Y is finite, i.e. $\nu(Y) < +\infty$.*

(vii) *The minimal graded multiplicity of Y is finite, i.e. $\mu(Y) < +\infty$.*

Furthermore, if one of these equivalent properties is satisfied then Y is a piece of an irreducible algebraic subvariety Z of dimension n, whose degree of algebraicity satisfies the estimates

$$\mu(Y) \leqslant \delta(Z) \leqslant \mu(Y)^n.$$

Proof. First observe that (i) $\Rightarrow$ (ii) follows from the examples given after Definition 2.1.2, (ii) $\Rightarrow$ (iii) follows from Theorem 2.1.4, and (iii) $\Rightarrow$ (iv) $\Rightarrow$ (v) follows from Theorem 2.1.2.

If the condition (v) is satisfied, it follows from Theorem 2.1.2 and Theorem 2.1.4 that $\mathcal{L}(U)$ is a Lelong class on U, and then, by Theorem 3.2.1, the minimal Lelong number of the class $\mathcal{L}(U)$ is finite. Therefore the condition (vi) is satisfied since $\nu(Y) = \nu(\mathcal{L}(Y)) \leqslant \nu(\mathcal{L}(U))$. The implication (vi) $\Rightarrow$ (vii) is obvious since we know that $\mu(Y) \leqslant \nu(Y)$.

Now assume that the condition (vii) is satisfied. Let us consider the graded sequence of linear spaces of finite dimension $\mathcal{A}(Y) = \bigcup_{d \geqslant 1} \mathcal{A}_d(Y)$. Then by the condition (vii) we know that $\mu(\mathcal{A}(Y))=\mu(Y)<+\infty$. Thus from Theorem 4.1.1, it follows that the upper asymptotic estimate (4.7) is satified for the graded sequence $\mathcal{C}=\mathcal{A}(Y)$. Namely, we have

$$\limsup_{d\to+\infty} \frac{\dim \mathcal{A}_d(Y)}{d^n} \leqslant \frac{\mu^n}{n!}, \tag{4.29}$$

where $\mu:=\mu(Y)$. It is well known that an asymptotic estimate like (4.29) implies that Y is a piece of an algebraic set. Indeed, let J be the ideal of polynomials belonging to $\mathbf{C}[z_1, z_2, ..., z_N]$ which vanishes identically on Y, and let $Z=\operatorname{loc}(J)$, the set locus of the ideal J. Then Z is an irreducible algebraic subvariety of $\mathbf{C}^N$.

By the Nullstellensatz, the vanishing ideal of Z is given by $I(Z)=\operatorname{Rad} J=J$. Therefore we obtain

$$\mathcal{A}(Y)=\mathbf{C}[z_1, z_2, ..., z_N]/J=\mathbf{C}[z_1, z_2, ..., z_N]/I(Z).$$

Hence $\dim \mathcal{A}_d(Y)=\dim \mathcal{A}_d(Z)=:h_Z(d)$ coincides, for d large enough, with the Hilbert polynomial of the algebraic subvariety Z, whose degree is precisely $m:=\dim Z$ (see [Ha]). Then from the formula (4.29), it follows that for a fixed $\nu>\mu(Y)$ and d large enough, we have $h_Z(d)\leqslant \nu^n d^n/n!$, which implies that $m\leqslant n$. Since $Y\subset Z$ is an irreducible analytic set of dimension n, we conclude that $m=n$, and then Y is a piece of the irreducible algebraic subvariety Z of $\mathbf{C}^N$ of the same dimension n, which proves (i). Moreover, since we know that $h_Y(d)=h_Z(d)\sim\delta(Z)\cdot d^n/n!$ for d large enough, it follows from the estimate (4.29) that $\delta(Z)\leqslant\mu(Y)^n$, wich proves the second estimate stated in the theorem. To prove the inequality $\mu(Y)\leqslant\delta(Z)$, observe first that from the proof of Proposition 3.2.5 we deduce the inequalities $\nu(w;a)\leqslant\delta(Z)$ for any $w\in\mathcal{L}(Z)$ and almost any point $a\in Z_{\text{reg}}$. Since any $f\in\mathcal{A}_d(Y)$ is the restriction to Y of a function in $F\in\mathcal{A}_d(Z)$, we can apply the last inequalities to $w=(1/d)\log|f|$ with $f\in\mathcal{A}_d(Y)$ and almost any point $a\in Y_{\text{reg}}$. Then we immediately get the inequality $\mu_{\mathcal{A}}(a)\leqslant\delta(Z)$ for almost any point $a\in Y_{\text{reg}}$. This implies in particular that $\mu(Y)\leqslant\delta(Z)$, which completes the proof of the theorem. □

The above results show that on a transcendental analytic subvariety $Y\subset\mathbf{C}^N$ the class $\mathcal{L}(Y)$ is not a Lelong class, and then no local Bernstein–Walsh inequalities of type (2.12) are satisfied for the natural graded algebra $\mathcal{A}(Y)$ of holomorphic polynomials on Y.

From Theorem 4.2.1 and Proposition 4.1.4, we can also derive the following result which contains the classical criterion of W. Stoll [St2].

COROLLARY 4.2.2. *Let Y be an irreducible analytic subvariety of dimension n in $\mathbf{C}^N$. Then the following conditions are equivalent*:

(1) *The set Y is algebraic.*

(2) *The projective volume of Y is finite, i.e. $\int_Y (dd^c\log(1+|z|^2))^n<+\infty$.*

(3) *The minimal graded multiplicity of Y is finite, i.e. $\mu(Y)<+\infty$.*

Moreover, if one of these conditions is satisfied then Y is an algebraic subvariety whose degree of algebraicity satisfies the inequalities $\mu(Y)\leqslant\delta(Z)\leqslant\mu(Y)^n$.

Proof. The implication $(1)\Rightarrow(2)$ is well known (see [C]). Indeed, identifying $\mathbf{C}^N$ with the open set $\{\zeta\in\mathbf{P}^N:\zeta_0\neq 0\}$ in the complex projective space $\mathbf{P}^N$, it follows that $\overline{Y}$ is an algebraic subvariety of $\mathbf{P}^N$ of dimension n, so that its volume with respect to the metric induced by the Fubini–Study metric on $\mathbf{P}^N$ is finite, which means that

$$\int_Y (dd^c \log(1+|z|^2))^n = \int_{\overline{Y}} (dd^c \log|\zeta|)^n < +\infty.$$

To prove the implication $(2)\Rightarrow(3)$, first observe that the plurisubharmonic function defined by $\varphi(z):=\frac{1}{2}\log(1+|z|^2)$ is a continuous plurisubharmonic exhaustion on Y. Moreover, the property (2) of the theorem implies that the condition (4.23) is satisfied by (Y,φ) and $\mathcal{A}_d(Y)\subset\mathcal{A}_d(Y;\varphi)$ for any $d\in\mathbf{N}$. Therefore, applying Proposition 4.1.4 to (Y,φ), we conclude that $\mu(Y)\leqslant\mu(Y;\varphi)<+\infty$, which proves the condition (3) of the theorem. The implication $(3)\Rightarrow(1)$ and the estimates on $\delta(Y)$ follow from Theorem 4.2.1. $\square$

The equivalence of the conditions (i) and (vi) of Theorem 4.2.1 is known as the local criterion of algebraicity of Sadullaev. The proof of Sadullaev uses methods from Padé approximation (see [Sa2]). A particular case of this theorem is also contained in our earlier paper with incomplete proof (see [Ze4]).

(2) Aytuna gave another proof of Sadullaev's criterion, which proceeds differently in spirit but leads to the same conclusion that $\dim\mathcal{A}_d(Y)=O(d^n)$ (see [Ay]). The idea of proving such an estimate on the dimension in order to deduce algebraicity was pointed out earlier by D. N. Ragozin [Rag] and has also been used more recently by L. Bos, N. Levenberg, P. Milman and B. A. Taylor in their characterization of real algebraicity in terms of "tangential Markov inequalities" (see [BLMT]).

(3) The equivalence of (1) and (2) in Corollary 4.2.2 is known as Stoll's criterion. The implication $(2)\Rightarrow(1)$ was also obtained by Demailly by quite a different method which is also based on the estimate (4.23) (see [D1]).

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AHMED ZERIAHI
Laboratoire de Mathématiques Emile Picard, UMR 5580
Université Paul Sabatier – Toulouse 3
118 route de Narbonne
FR-31062 Toulouse Cedex 04
France
zeriahi@picard.ups-tlse.fr

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