Acta Math., 169 (1992), 131–151

Volume comparison à la Aleksandrov

by

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Introduction

The purpose of this paper is to obtain optimal volume estimates for Riemannian manifolds curved from below, and to prove corresponding topological as well as metric stability theorems.

This program is in part inspired by a classical area problem for convex surfaces in $\mathbf{R}^3$ raised by A. D. Aleksandrov in [A]: Is the area of such a surface less than twice the area of a disc $D^2 \subset \mathbf{R}^2$ with the same intrinsic diameter? There are obviously convex bodies containing D^2 with surface area arbitrarily close to this.

It is likely that Aleksandrov's problem has an affirmative solution even in the more abstract context of Riemannian metrics on the 2-sphere S^2 with nonnegative curvature (cf. [CC], [S], [Sh] and Section 4). However, if in this framework one does not restrict the topological type of the surface, it is clearly wrong. For example, the real projective plane of constant curvature 1 has diameter $=\pi/2$ and area $=2\pi$, exceeding the corresponding "Aleksandrov estimate", $2\cdot\pi(\pi/4)^2=\pi^3/8$.

For arbitrary $k\in\mathbf{R}$ and integers $n\geqslant 2$, we consider closed, connected Riemannian n-manifolds, M whose sectional curvatures satisfy $\sec M\geqslant k$. The complete 1-connected n-dimensional space form of constant curvature k will be denoted by S^n_k, and $v^n_k(r)$ will be the volume of an r-ball in S^n_k. Standard volume comparison then yields, $\operatorname{vol} D(p,r)\leqslant v^n_k(r)$, where $D(p,r)$ is the closed r-ball in M centered at $p\in M$. With this in mind we consider the *radius* of M defined by

$$\operatorname{rad} M = \min_p \max_q \operatorname{dist}(p,q),$$

(¹) Supported in part by a grant from the National Science Foundation.
(²) Supported in part by a grant from the National Science Foundation and the Sloan Foundation.

i.e. $\operatorname{rad} M$ is the smallest $r>0$ so that $M=D(p,r)$ for some $p\in M$. This invariant was introduced in [SY] where it was denoted by $\underline{\operatorname{diam}}(M)$. Obviously, radius and diameter are related by

$$\operatorname{rad} M \leq \operatorname{diam} M \leq 2 \operatorname{rad} M,$$

where the last inequality is strict for Riemannian manifolds. From the discussion above

$$(*) \qquad \operatorname{vol} M \leq v_k^n(r),$$

for any Riemannian n-manifold, M with $\sec M \geq k$ and $\operatorname{rad} M \leq r$. It is not difficult to show that equality in (*) occurs only in the two cases $k>0$ and $r\geq \pi/\sqrt{k}$, $r=\pi/2\sqrt{k}$ corresponding to the sphere S_k^n and real projective space $\mathbf{R}P_k^n$ of constant curvature k.

We prove that (*) is optimal, except when $k>0$ and $r>\pi/2\sqrt{k}$, and determine correspondingly the possible topological types of manifolds with large volume.

THEOREM A. *Fix a real number k, a positive r ($\leq \pi/2\sqrt{k}$ if $k>0$), and an integer $n\geq 2$. Then:*

(i) *There is an $\varepsilon=\varepsilon(k,r,n)>0$ such that any Riemannian n-manifold M with* $\sec M\geq k$, $\operatorname{rad} M\leq r$ *and* $\operatorname{vol} M \geq v_k^n(r)-\varepsilon$ *is topologically either S^n or $\mathbf{R}P^n$. Moreover:*

(ii) *For every $\varepsilon>0$ there are Riemannian metrics on $M=S^n$, $\mathbf{R}P^n$ with* $\sec M\geq k$, $\operatorname{rad} M\leq r$ *and* $\operatorname{vol} M\geq v_k^n(r)-\varepsilon$.

In this theorem topological equivalence refers to homeomorphism type, except possibly in the case of $M=S^3$, where our proof gives homotopy type only (cf. 2.8, however).

For manifolds M not covered by Theorem A, i.e., $\sec M\geq k>0$ and $\operatorname{rad} M>\pi/2\sqrt{k}$, we prove in 3.1 that (*) can be improved to

$$(**) \qquad \operatorname{vol} M \leq \frac{r}{\pi/\sqrt{k}}\, v_k^n(\pi/\sqrt{k}) =: w_k^n(r),$$

whenever $\pi/2\sqrt{k} < \operatorname{rad} M \leq r \leq \pi/\sqrt{k}$. This on the other hand is optimal.

THEOREM B. *Fix an integer $n\geq 2$, a positive k and $\pi/2\sqrt{k}<r\leq\pi/\sqrt{k}$. Then:*

(i) *Any Riemannian n-manifold M with* $\sec M\geq k$ *and* $\operatorname{rad} M>\pi/2\sqrt{k}$ *is topologically S^n. Moreover:*

(ii) *For every $\varepsilon>0$ there is a Riemannian metric on $M=S^n$, with* $\sec M\geq k$, $\operatorname{rad} M\leq r$ *and* $\operatorname{vol} M\geq w_k^n(r)-\varepsilon$.

Here the first claim is of course an immediate consequence of the diameter sphere

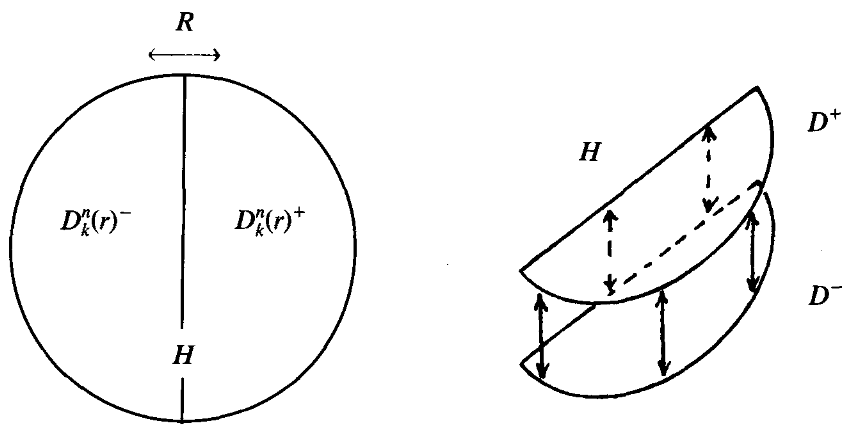


Fig. 1.

theorem [GS], since $\operatorname{diam} M \geq \operatorname{rad} M > \pi/2\sqrt{k}$. As in Theorem A our main concern here are manifolds M with large volume. This is addressed in Theorem C below.

Although the volume estimates (*) and (**) are virtually always strict for Riemannian manifolds, we exhibit now three important singular manifolds with the same basic geometric properties where equality holds.

Example I (curvture k crosscaps). For real k, and positive r ($\leq \pi/2\sqrt{k}$ if $k>0$) let $D_k^n(r)$ be the closed r-ball in S_k^n. Let $A: D_k^n(r) \to D_k^n(r)$ be the reflection in the center. The real projective space $C_{k,r}^n = D_k^n(r)/u \sim A(u)$, $u \in \partial D_k^n(r)$, has Toponogov curvature $\geq k$ (cf. [GP3] and 1.11), radius $=r$, and volume $=v_k^n(r)$. Note that $C_{k,\pi/2\sqrt{k}}^n = \mathbf{R}P_k^n$ when $k>0$.

Example II (curvature k purses). Let k and r be as in Example I. Let $R: D_k^n(r) \to D_k^n(r)$ be a reflection in a totally geodesic hyperplane H through the center. The sphere, $P_{k,r}^n = D_k^n(r)/v \sim R(v)$, $v \in \partial D_k^n(r)$, has Toponogov curvature $\geq k$, radius $=r$, and volume $=v_k^n(r)$ (see Figure 1).

Example III (curvature k lemons). Fix $k>0$ and a totally geodesic $S_k^{n-2} \subset S_k^n$. Let $W_{k,\theta}^n \subset S_k^n$ be the region between two totally geodesic $(n-1)$-discs in S_k^n with common boundary S_k^{n-2} making an angle $\theta \in (0, 2\pi]$ (see Figure 2).

Let $R: S_k^n \to S_k^n$ be the reflection in the totally geodesic $S_k^{n-1} \subset S_k^n$ mapping $\partial W_{k,\theta}^n$ into itself. The sphere, $L_{k,\theta}^n = W_{k,\theta}^n / w \sim R(w)$, $w \in \partial W_{k,\theta}^n$, has Toponogov curvature $\geq k$, volume $=(\theta/2\pi) \operatorname{vol} S_k^n = w_k^n(\theta/2\sqrt{k})$, and radius $= \max\{\pi/2\sqrt{k}, \theta/2\sqrt{k}\}$. Note that $L_{k,2\pi}^n = S_k^n$ and $L_{k,\pi}^n = P_{k,\pi/2\sqrt{k}}^n$.

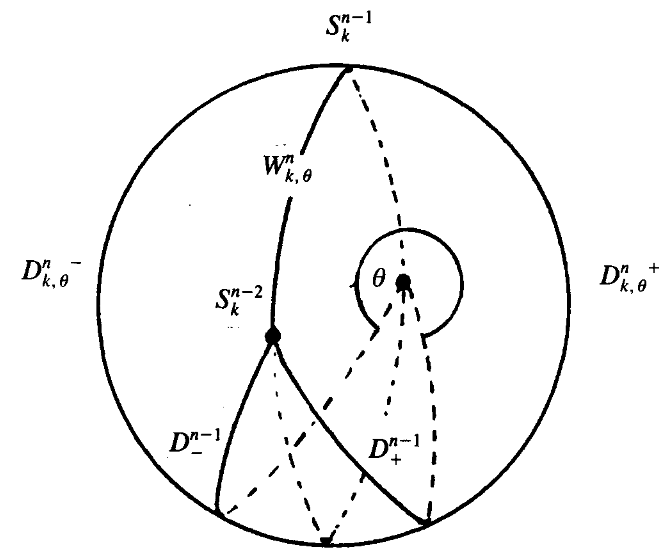


Fig. 2

Each of these examples can be viewed also as the result of a doubling construction: In the first example extend the action of $A: \partial D_k^n(r) \to \partial D_k^n(r)$ to the obvious antipodal map of the double $D_k^n(r) \cup D_k^n(r)$, and identify antipodal points. The two last examples are obtained by doubling one side, e.g., D^+ (see Figures 1 and 2), of the hyperplane defining the reflection R.

Based on this description it is now easy to exhibit smooth perturbations of Examples I, II, and III, thus proving part (ii) of Theorems A and B: Simply embed isometrically $D_k^n(r)$, $D_k^n(r)^+$, and $D_{k,\theta}^n{}^+$ into a totally geodesic $S_k^n \subset S_k^{n+1}$ and consider boundaries of smooth, symmetric (locally) convex neighborhoods. This suffices for the last two examples. In the first, one must in addition identify antipodal points.

The proof of Theorem A, part (i) is based on the following *metric stability theorem* for manifolds with nearly optimal volume.

THEOREM C. *Fix $n \geq 2$, $k \in \mathbf{R}$, $r > 0$ and let $\{M_i\}$ be a sequence of closed Riemannian n-manifolds with* $\sec M_i \geq k$ *and* $\operatorname{rad} M_i \leq r$.

(a) *Suppose* $\{\operatorname{vol} M_i\}$ *converges to $v_k^n(r)$, where $r \leq \pi/2\sqrt{k}$ if $k > 0$. Then a subsequence of $\{M_i\}$ converges to either the curvature k crosscap, $C_{k,r}^n$ or the purse, $P_{k,r}^n$ in the Gromov–Hausdorff topology.*

(b) *For $k > 0$ and $\pi/2\sqrt{k} < r \leq \pi/\sqrt{k}$, suppose* $\{\operatorname{vol} M_i\}$ *converges to $w_k^n(r)$. Then $\{M_i\}$ converges to the curvature k lemon $L_{k,2r}^n$ in the Gromov–Hausdorff topology.*

As an immediate consequence of the above theorems we also get the following solution to a generalized analogue of Aleksandrov's problem.

COROLLARY D. *Let S be a closed locally convex codimension* 1 *submanifold of* S_k^{n+1} *with* $\operatorname{rad} S=r$. *Then*

(i) $\operatorname{vol} S \leq v_k^n(r)$, *where* $r \leq \pi/2\sqrt{k}$ *if* $k>0$ *and*

(ii) $\operatorname{vol} S \leq w_k^n(r)$, *if* $k>0$ *and* $\pi/2\sqrt{k} \leq r \leq \pi/\sqrt{k}$.

Moreover, these inequalities are optimal and strict except for the case $k>0$ *and* $r=\pi/k$ *corresponding to* $S=S_k^n \subset S_k^{n+1}$.

This result provides in particular a solution to Problem 61 as worded in [Y].

We conclude by pointing out that, dual to the problem considered here, a lower bound for curvature and radius does not in general give a lower bound on volume (cf. e.g. the lemons, $L_{k,\theta}^n$, θ small). For an interesting exceptional case see 4.3.

The proofs of our main results A, B and C above utilizes many of the tools developed in [G1], [GP1,2,3], and [GPW]. In Section 1 we briefly summarize what we need from these papers and fix *non standard conventions* important to the exposition. The general case represented by Theorem A and C(a) is treated in Section 2. Although the topology of manifolds represented in Theorem B and C(b) is well understood and simple, the metric properties are more delicate. This is discussed in Section 3.

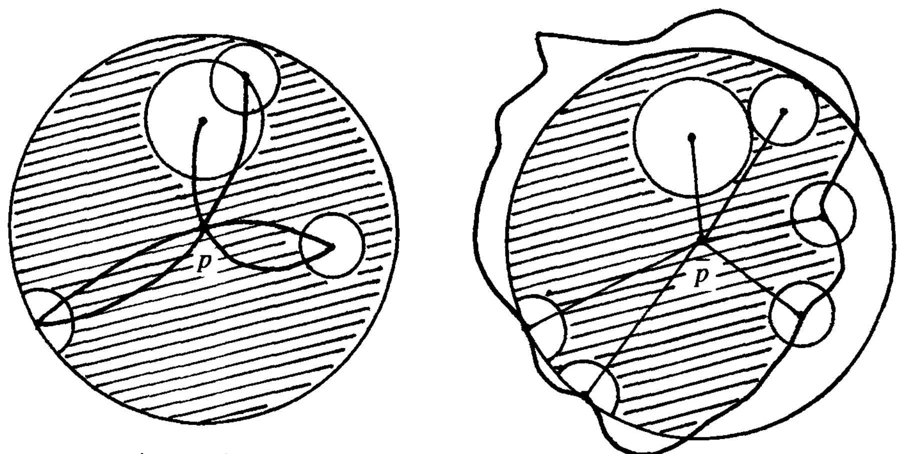
We thank A. Treibergs for bringing Aleksandrov's problem to our attention in connection with [GP3].

1. Basic tools and conventions

Throughout the paper we let M denote a closed, connected Riemannian n-manifold, $n \geq 2$, with sectional curvature, $\sec M \geq k$ for some real k. Following Rinow [R], S_k^n will be the complete, simply connected n-dimensional space form of constant curvature k. The distance function on M, S_k^n, or any other metric space will be denoted by d. To distinguish points in S_k^n from points in other metric spaces we use the notation $\bar{p}, \bar{q}, \bar{u}, \ldots$, etc., rather than $p, q, u, \ldots$, etc.

For each $p \in M$ and $r>0$, let $B(p,r)$, respectively, $D(p,r)$ denote the open, respectively, closed r-ball in M centered at p. Let $\exp_p \colon T_pM \to M$ be the exponential map at p, and $\operatorname{Seg}(p) \subset T_pM$ the star shaped region bounded by the tangent cut locus. We will refer to $\operatorname{Seg}(p)$ as the *segment domain* at p.

Now for each $p \in M$ replace the *euclidean metric* on $B(O_p, r) \subset T_pM$ by a *constant curvature k metric* via a radial conformal change ($r \leq \pi/\sqrt{k}$ if $k>0$). Viewed this way,



in $D(p,R)\subset M$

in $D(\bar{p},R)\cap\mathrm{Seg}(p)\subset S^n_k$

Fig. 1.2

$\mathrm{Seg}(p)$ is a proper closed subset of S^n_k for every $p\in M$, except when $k>0$ and M is isometric to S^n_k. In the latter case we interpret $\mathrm{Seg}(p)$ as S^n_k.

With the interpretaton given above we will throughout view the domain of $\exp_p$, $p\in M$ to be the closed subset $\mathrm{Seg}(p)\subset S^n_k$. When giving $\mathrm{Seg}(p)$ the metric induced from S^n_k the exponential map, $\exp_p\colon \mathrm{Seg}(p)\to M$ is distance nonincreasing by standard distance comparison.

In the context of manifolds with positive curvature an interesting volume estimate for the complement of a ball was observed in [D]. A straightforward but rather powerful extension was presented in [GP3; 1.1]. Here we need only two special versions of it. In the *swiss cheese* version, one estimates the volume of a ball from which a family of balls have been removed (see Figure 1.2):

Let $Q\subset M$ and $r\colon Q\to\mathbf{R}_+$ a function. Define the swiss cheese, K relative to $D(p,R)$ and (Q,r) as

$$K((Q,r);(p,R))=D(p,R)-\bigcup_{q\in Q}B(q,r(q)).$$

When $D(p,R)=M$ we may choose to omit (p,R), i.e.,

$$K(Q,r)=M-\bigcup B(q,r(q)).$$

The desired volume estimate is

$$\text{(1.1)} \qquad \operatorname{vol} K((Q,r);(p,R)) \leqslant \operatorname{vol} K((\exp_p^{-1} Q, r\circ \exp_p);(\bar p, R)),$$

where $\bar p = \exp_p^{-1}(p) \in \operatorname{Seg}(p) \subset S_k^n$.

Clearly, the way to think of, and use (1.1) is that the volumes of swiss cheeses $K\subset M$ are smaller, the more geodesics come together at Q. Another complementary application of [GP3; 1.1] is to estimate the volume of the union $D(Q,r)=\bigcup_{q\in Q} D(q,r(q))$:

$$\text{(1.3)} \qquad \operatorname{vol} D(Q,r) \leqslant \operatorname{vol} D(I(Q), r\circ I^{-1}),$$

provided $I: Q\to I(Q)\subset S_k^n$ is an isometry. This is particularly useful when Q consists of 2 or 3 points since any such set can be isometrically embedded in S_k^n.

Finally, we need to be able to compare volumes of special swiss cheeses in S_k^n. To describe these suppose $\bar Q\subset S_k^n$ has the properties (i) $d(\bar p, \bar q)=c$ for all $\bar q\in \bar Q$, and (ii) any direction at $\bar p$ makes an angle $\leqslant \pi/2$ to some segment $\overline{pq}, \bar q\in \bar Q$. Then for any constant $r>0$ we have

$$\text{(1.4)} \qquad \operatorname{vol} K((\bar Q, r);(\bar p, R)) \leqslant \operatorname{vol} K((\{\bar q_1, \bar q_2\}, r);(\bar p, R)),$$

where the segments $\overline{pq_1}, \overline{pq_2}$ have length c and makes an angle π at $\bar p$. Moreover, equality holds in (1.4) only if $\bar Q=\{\bar q_1, \bar q_2\}$ up to an isometry fixing $\bar p$. This follows from [GP1; appendix] and plays a central role here when $k>0$ and $\operatorname{rad} M>\pi/2\sqrt{k}$ (cf. Section 3).

The remaining part of this section is devoted to a brief discussion of the Gromov–Hausdorff topology suited for our purposes (cf. [G1]).

Let X, Y, Z, X_i, $i=1,2,3,\ldots$ be compact metric spaces. If X, Y are isometrically embedded in Z, the classical Hausdorff distance d_H^Z satisfies

$$\text{(1.5)} \qquad d_H^Z(X,Y)<\varepsilon \quad \text{if and only if} \quad Y\subset B(X,\varepsilon),\ X\subset B(Y,\varepsilon),$$

where $B(X,\varepsilon)=\{z\in Z \mid d(z,X)<\varepsilon\}$. The Gromov–Hausdorff distance d_{GH} satisfies

$$\text{(1.6)} \qquad \begin{array}{c} d_{GH}(X,Y)<\varepsilon \ \text{ if and only if } d_H^Z(X,Y)<\varepsilon \ \text{ for some} \\ \text{metric on } Z=X\amalg Y \text{ extending the ones on } X, Y. \end{array}$$

Similarly, Gromov–Hausdorff convergence is characterized as

$$\text{(1.7)} \qquad \begin{array}{c} X=\lim X_i \text{ if and only if the metrics on } X, X_i \\ \text{extend to a metric on } Z=X\amalg_i X_i \text{ and } d_H^Z(X,X_i)\to 0. \end{array}$$

Moreover:

(1.8) A class $\mathcal{M}$ of compact metric spaces is precompact if and only if there is a function $N(\varepsilon)$ so that for every $\varepsilon > 0$, any $X \in \mathcal{M}$ can be covered by less than $N(\varepsilon)$ (closed) balls of radius ε.

In the context of Riemannian manifolds this yields:

(1.9) For fixed $k \in \mathbf{R}$ and $D > 0$ the class of closed Riemannian n-manifolds M, $n \geqslant 2$, with Ricci curvature $\operatorname{Ric} M \geqslant (n-1)k$ and $\operatorname{diam} M \leqslant D$ is precompact.

In this paper we are interested in the subclass $\mathcal{M}_k{}^D{}_v(n)$ of (1.9) where in addition $\sec M \geqslant k$ and $\operatorname{vol} M \geqslant v$. If $X = \lim M_i$, $M_i \in \mathcal{M}_k{}^D{}_v(n)$, we will always equip $Z = X \amalg_i M_i$ with a metric as in (1.6). Then for any $p \in X$, $p = \lim p_i$, $p_i \in M_i$ and $\exp_{p_i}\colon \operatorname{Seg}(p_i) \to M_i$ is distance nonincreasing on $\operatorname{Seg}(p_i) \subset S_k^n$. By [GP3] we can assume:

(1.10) For any $p \in X$, $\bar{p} \in S_k^n$ there is a compact subset $\bar{p} \in \operatorname{Seg}(p) \subset S_k^n$ and a distance nonincreasing map $\exp_p\colon \operatorname{Seg}(p) \to X$. Moreover, $\exp_p$ maps segments from $\bar{p} \in \operatorname{Seg}(p) \subset S_k^n$ to segments in X from p, and any segment from p is the image of a segment from $\bar{p}$.

Here by possibly passing to a subsequence $\operatorname{Seg}(p) = \lim \operatorname{Seg}(p_i)$ and $\exp_p = \lim \exp_{p_i}$, i.e., $\exp_p(\bar{u}) = \lim \exp_{p_i}(\bar{u}_i)$, when $\bar{u} = \lim \bar{u}_i$.

(1.11) X has Toponogov curvature, $\sec X \geqslant k$, i.e., standard distance comparison holds for geodesic triangles in X (cf. [GP3]).

The topological properties of $X = \lim M_i$ needed in this paper can be summarized:

(1.12) For every $\varepsilon > 0$ there is an i_0 so that X and M_i are ε-homotopy equivalent for $i \geqslant i_0$ (cf. [GP1], [P] and [GPW]).

For further metric and topological properties of limit spaces X, the reader may want to consult [BGP], [GP3,4] and [GPW].

2. Metric polarity: the crosscap and purse case

In this section we prove the general volume pinching theorem formulated as A, and C(a) in the introduction.

Thus we fix $k\in\mathbf{R}$, $r>0$ ($\leq\pi/2\sqrt{k}$ if $k>0$) and an integer $n\geq2$, and consider the Gromov–Hausdorff precompact class of closed Riemannian n-manifolds M having $\sec M\geq k$ and $\operatorname{rad} M\leq r$ (cf. (1.9)).

Fix a Gromov–Hausdorff convergent sequence $\{M_i\}$ of closed Riemannian n-manifolds M_i, where

(2.1) $$\sec M_i\geq k,\quad \operatorname{rad} M_i\leq r \quad\text{and}\quad \operatorname{vol} M_i\to v_k^n(r).$$

To prove part (a) of Theorem C we must show that

(2.2) $$X=\lim M_i \text{ is isometric to either } C_{k,r}^n \text{ or } P_{k,r}^n,$$

which we now proceed to do.

In each M_i choose a point p_i realizing the radius of M_i, i.e., $D(p_i,\operatorname{rad} M_i)=M_i$. Then $\operatorname{vol} M_i\leq v_k^n(\operatorname{rad} M_i)\leq v_k^n(r)$ by standard volume comparison and hence $r=\lim\operatorname{rad} M_i=\operatorname{rad} X$ by (2.1) and (1.7). In view of (1.7) and (1.10) we can assume that

(2.3) $$\{p_i\} \text{ converges to } p\in X \text{ and } D(p,r)=X.$$

(2.4) $\{\operatorname{Seg}(p_i)\}$ converges to $\operatorname{Seg}(p)\subset S_k^n$ and $\{\exp_{p_i}\}$ converges to $\exp_p\colon \operatorname{Seg}(p)\to X$.

The claim (2.2) is now a straightforward consequence of the next three lemmas.

LEMMA 2.5. *The exponential map in* (2.4) *satisfies:*

(i) $\operatorname{Seg}(p)=D(\bar p,r)\subset S_k^n$.

(ii) $\exp_p\colon B(\bar p,r)\to X$ *is injective.*

(iii) $\exp_p\colon D(\bar u,\varepsilon)\to X$ *is an isometry whenever* $D(\bar u,2\varepsilon)\subset D(\bar p,r)$.

(iv) $\exp_p\colon \partial D(\bar p,r)\to X$ *is two to one, i.e.,* $\exp_p^{-1}(q)$ *is one or two points for all* $q\in X$ *with* $d(p,q)=r$.

Proof. For every i

$$\operatorname{vol} M_i\leq \operatorname{vol}\operatorname{Seg}(p_i)\leq \operatorname{vol} D(\bar p,r)=v_k^n(r)$$

by standard volume comparison. In particular

$$\operatorname{vol}\operatorname{Seg}(p_i)\to\operatorname{vol} D(\bar p,r)$$

by (2.1). Since $\operatorname{Seg}(p_i)\subset D(\bar p,r)\subset S_k^n$ converges to $\operatorname{Seg}(p)\subset D(\bar p,r)$, (cf. (2.4)), we conclude that $D(\bar p,r)-\operatorname{Seg}(p)$ has no interior points and hence is empty. This proves (i).

To prove (ii) assume $\bar u\neq\bar v\in B(\bar p,r)$ and $\exp_p(\bar u)=\exp_p(\bar v)$. Choose $\varepsilon>0$ so that

$D(\bar{u},\varepsilon), D(\bar{v},\varepsilon)\subset B(\bar{p},r)$ are disjoint. If $\bar{u}=\lim \bar{u}_i$, $\bar{v}=\lim \bar{v}_i$ with $\bar{u}_i, \bar{v}_i \in \mathrm{Seg}(p_i)$ then $\lim \exp_{p_i}(\bar{u}_i)=\lim \exp_{p_i}(\bar{v}_i)$ by (2.4). Moreover, using

$$M_i = K(\{\exp_{p_i}\bar{u}_i, \exp_{p_i}\bar{v}_i\},\varepsilon)\cup D(\{\exp_{p_i}\bar{u}_i, \exp_{p_i}\bar{v}_i\},\varepsilon)$$

together with (1.1) and (1.3) gives

$$\lim \mathrm{vol}\, M_i \leq (v_k^n(r) \dot{-} 2v_k^n(\varepsilon)) + v_k^n(\varepsilon)$$

contradicting (2.1). Hence $\exp_p(\bar{u})\neq \exp_p(\bar{v})$.

Now let $\bar{u},\bar{v}\in B(\bar{p},r)$, $d(\bar{u},\bar{v})=2c>0$ and suppose $B(\bar{u},c)$, $B(\bar{v},c)\subset B(\bar{p},r)$. We then claim that $d(\exp_p(\bar{u}),\exp_p(\bar{v}))=d(\bar{u},\bar{v})$ from which (iii) follows. If indeed

$$d(\exp_p(\bar{u}),\exp_p(\bar{v})) = d(\bar{u},\bar{v})-\delta$$

for some $\delta>0$ (cf. (1.10)), we argue as follows. Pick $\bar{u}_i,\bar{v}_i\in \mathrm{Seg}(p_i)$ with $\bar{u}=\lim \bar{u}_i$, $\bar{v}=\lim \bar{v}_i$ and $\exp_p \bar{u}=\lim \exp_{p_i}\bar{u}_i$, $\exp_p \bar{v}=\lim \exp_{p_i}\bar{v}_i$. Write M_i as above with ε replaced by c. Again using (1.1) and (1.3) we get

$$\lim \mathrm{vol}\, M_i \leq (v_k^n(r)-2v_k^n(c))+\mathrm{vol}\, D(\{\bar{q}_1, q_2\},c),$$

for some $\bar{q}_1,\bar{q}_2\in S_k^n$ with $d(\bar{q}_1,\bar{q}_2)=2c-\delta$. This clearly contradicts (2.1) and proves our claim. Also (iv) is proved by contradiction. Thus assume $q\in X-B(p,r)$ and $\bar{u},\bar{v},\bar{w}\in \exp_p^{-1}(q)\subset\partial D(\bar{p},r)$ are distinct points. Let $D(\bar{u},\varepsilon)$, $D(\bar{v},\varepsilon)$, $D(\bar{w},\varepsilon)$ be disjoint balls and $\{\bar{u}_i\}$, $\{\bar{v}_i\}$, $\{\bar{w}_i\}$ be sequences as above. From $M_i=K(q,\varepsilon)\cup D(q,\varepsilon)$ and (1.1) we deduce

$$\lim \mathrm{vol}\, M_i \leq (v_k^n(r)-3\cdot \bar{v}_k^n(\varepsilon))+v_k^n(\varepsilon),$$

where $\bar{v}_k^n(\varepsilon)=\mathrm{vol}\, D(\bar{p},r)\cap D(\bar{q},\varepsilon)$, $\bar{q}\in\partial D(\bar{p},r)$. Since k, r and n are fixed this inequality contradicts (2.1) when ε is chosen sufficiently small. □

Using (iv) of 2.5 we define a relation R in $\partial D(\bar{p},r)$ by $\bar{u}R\bar{v}$ if and only if $\exp_p(\bar{u})=\exp_p(\bar{v})$.

LEMMA 2.6. *Give the $(n-1)$-sphere $\partial D(\bar{p},r)$ the constant curvature Riemannian metric induced from $D(\bar{p},r)\subset S_k^n$.*

(i) *The relation R defines an isometric involution on $\partial D(\bar{p},r)$.*

(ii) *$\exp_p\colon D(\bar{p},r)\to X$ induces an isometry between the inner metric spaces $D(\bar{p},r)/R$ and X.*

Proof. First observe that any path in $D(\bar{p}, r)$ can be uniformly approximated by paths in $B(\bar{p}, r)$. In particular $\exp_p: D(\bar{p}, r) \to X$ preserves lengths of paths. In view of 2.5(ii) it therefore remains to prove (i) only.

For this, define $R(\bar{u}) = \bar{u}$, if $\exp_p^{-1}(\exp_p(\bar{u})) = \{\bar{u}\}$, and $R(\bar{u}) = \bar{v} \neq \bar{u}$, if $\exp_p(\bar{u}) = \exp_p(\bar{v})$. It follows from (1.11) that $R: \partial D(\bar{p}, r) \to \partial D(\bar{p}, r)$ is continuous, since a point of discontinuity would lead to the existence of bifurcating geodesics (cf. [GP3; § 2]). Then clearly R is an involution that preserves lengths of paths. In particular R is distance non-increasing and hence an isometry since $R^2 = \mathrm{id}$. □

In view of 2.6 we consider now the unit disc $D^n = D_0^n(1) \subset S_0^n = \mathbf{R}^n$ in euclidean n-space. For each $m = 0, 1, \ldots, n$ we write $\mathbf{R}^n = \mathbf{R}^m \oplus \mathbf{R}^{n-m}$ and let $R_m: \mathbf{R}^n \to \mathbf{R}^n$ be the linear involution determined by $R_m | \mathbf{R}^m = \mathrm{id}$ and $\mathbf{R}_m | \mathbf{R}^{n-m} = -\mathrm{id}$.

LEMMA 2.7. *The identification space $X_m^n = D^n / u \sim R_m u$, $u \in S^{n-1} = \partial D^n$ is homeomorphic to the m-th suspension $\Sigma^m \mathbf{R}P^{n-m}$. In particular, X_m^n has the homology of a manifold if and only if $m = 0$ or $n-1$, corresponding to $\mathbf{R}P^n$ and S^n.*

Proof. The first claim is obvious if one exhibits D^n as $\Sigma^m D^{n-m}$ with $\partial D^n = \Sigma^m \partial D^{n-m}$ and correspondingly R_m as $\Sigma^m(-\mathrm{id} | \partial D^{n-m})$.

Since $X_0^n \cong \mathbf{R}P^n$, $X_{n-1}^n \cong S^n$ and $X_n^n \cong D^n$ it remains only to show that X_m^n does not have the homology of a manifold when $1 \leq m \leq n-2$. In these cases

$$H_*(X_m^n) \cong H_{*-m}(\mathbf{R}P^{n-m}).$$

In particular, X_m^n does not satisfy Poincaré duality with $\mathbf{Z}_2$-coefficients. □

According to 2.6, 2.7 and (1.12), $X = \lim M_i$ is isometric to either $C_{k,r}^n$ or $P_{k,r}^n$. By (1.9) this completes the proof of part (a), Theorem C.

The passage from Theorem C(a) to Theorem A(i) is provided by (1.12). From this we know that any closed Riemannian n-manifold M with $\sec M \geq k$, $\mathrm{rad}\, M \leq r$ ($\leq \pi/2\sqrt{k}$ if $k > 0$) and $\mathrm{vol}\, M$ sufficiently close to $v_k^n(r)$ is controlled homotopy equivalent to either $C_{k,r}^n \simeq \mathbf{R}P^n$ or $P_{k,r}^n \simeq S^n$. The homeomorphism claim follows from this in all dimensions $n \geq 4$ by appealing to the controlled h-cobordism theorem [Q1,2] and in [GPW]; see also [CF] and [F].

In the case were $M_i \to C_{k,r}^n \simeq \mathbf{R}P^n$, we can also apply more direct geometric arguments as in [GP2] based on understanding critical, or nearly critical points for the distance functions $d(p_i, \cdot)$. Using this one exhibits M_i, i large, as the union of a disc and a homotopy Möbius band. In the 3-dimensional case this proves that M_i is homeomorphic to $\mathbf{R}P^3$ (cf. also [L]).

Remark 2.8. In the case where $M_i \to P^n_{k,r} \simeq S^n$, the geometric arguments alluded to above become considerably more complicated. It seems plausible, however, that one can develop such arguments and as a consequence exhibit M_i, i large, as a (simply connected) union of two trivial bundles $S^{n-2} \times D^2$ and $D^{n-1} \times S^1$. This would yield homeomorphism also in the 3-dimensional case. We will not elaborate further on this here. Another approach is suggested by the developments announced in [BGP].

We conclude this section by pointing out that, knowing the possible limit spaces $C^n_{k,r}$ and $P^n_{k,r}$ explicitly opens up the possibility of understanding the Gromov–Hausdorff convergence better. For example in the unique nonsingular example $C^n_{k,\pi/\sqrt{k}} \equiv \mathbf{R}P^n_k$, $k>0$, we know from [OSY] that the convergence is Lipschitz and in particular any M^n with sec $M \geq k$ close to $\mathbf{R}P^n_k$ is diffeomorphic to $\mathbf{R}P^n$, (cf. also [Y1,2]).

3. Metric uniqueness: the lemon case

The object of this section is to prove part (b) of the metric stability Theorem C. We therefore consider Riemannian n-manifolds M, $n \geq 2$, with $\sec M \geq k > 0$ and $\operatorname{rad} M > \pi/2\sqrt{k}$, or equivalently (after scaling) $\sec M \geq 1$ and $\operatorname{rad} M > \pi/2$.

For any $p \in M$ standard distance comparison implies that $K(p, \pi/2) = M - B(p, \pi/2)$ is π-convex, i.e., any geodesic of length $<\pi$ joining points in $K(p, \pi/2)$ remains in $K(p, \pi/2)$, and there is a unique point $q = A(p) \in K(p, \pi/2)$ at maximal distance from p (cf. [GS]). The decomposition $M = D(p, \pi/2) \cup K(p, \pi/2)$ will play a key role throughout this section. First we shall use it to prove the volume estimate (**) in the introduction (in a slightly different context compare also with [W] and [GP3].

PROPOSITION 3.1. *Let M be a closed Riemannian n-manifold, n≥2, with* $\sec M \geq 1$ *and* $\operatorname{rad} M > \pi/2$. *Then for any* $p \in M$,

(i) $\operatorname{vol} D(p, \pi/2) \leq (1/2) \operatorname{vol} S^n_1$, *and*

(ii) $\operatorname{vol} K(p, \pi/2) \leq (r/\pi - 1/2) \operatorname{vol} S^n_1$, $r = d(p, A(p))$.

In particular, $\operatorname{vol} M \leq (\operatorname{rad} M/\pi) \operatorname{vol} S^n_1 = w^n_1(\operatorname{rad} M)$.

Proof. (i) is obvious by standard volume comparison. To prove (ii) note that $q = A(p)$ is a critical point for p, i.e., any direction at q makes an angle $\leq \pi/2$ with a segment $\bar{q}\bar{p}$ for some $\bar{p} \in \exp_q^{-1}(p)$. It is now immediate from (1.1) and (1.4) that

$$\operatorname{vol} K(p, \pi/2) \leq \operatorname{vol} K(\{\bar{p}_1, \bar{p}_2\}, \pi/2),$$

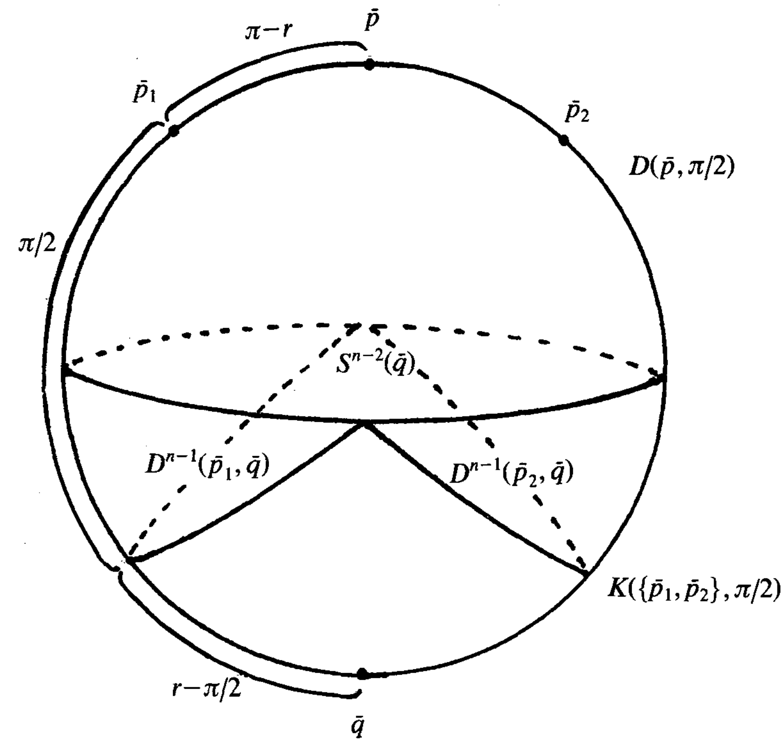


Fig. 3.2

for some $\bar{p}_1, \bar{p}_2 \in S_1^n$ ($\supset$Seg(q)) with $d(\bar{p}_1,\bar{p}_2)=2\pi-2r$. Since

$$K(\{\bar{p}_1,\bar{p}_2\},\pi/2) = S_1^n - B(\bar{p}_1,\pi/2) \cup B(\bar{p}_2,\pi/2)$$

is a wedge of angle $2r-\pi$ in S_1^n (see Figure 3.2), the proof of (ii) and hence 3.1 is complete. □

As a subset of S_1^n the boundary ∂K of the convex set $K(\{\bar{p}_1,\bar{p}_2\},\pi/2)$ decomposes into $D^{n-1}(\bar{p}_1,\bar{q})=\partial K \cap \partial D(\bar{p}_1,\pi/2)$ and $D^{n-1}(\bar{p}_2,\bar{q})=\partial K \cap \partial D(\bar{p}_2,\pi/2)$ hinged at $S^{n-2}(\bar{q})= D^{n-1}(\bar{p}_1,\bar{q}) \cap D^{n-1}(\bar{p}_2,\bar{q})=\partial D(\bar{p}_1,\pi/2) \cap \partial D(\bar{p}_2,\pi/2)$ (see Figure 3.2).

Intrinsically, however, ∂K is isometric to S_1^{n-1}. Moreover, the constant curvature 1 lemon $L_{1,2r}^n$ of Example III can be viewed as $D(\bar{p},\pi/2) \cup K(\{\bar{p}_1,\bar{p}_2\},\pi/2)$ with the obvious identification of boundaries.

This observation plays an important role in our proof of Theorem C, part (b), which we now proceed to give.

Fix $r>\pi/2$ and a Gromov–Hausdorff convergent sequence $\{M_i\}$ of closed Riemannian n-manifolds M_i, where

$$\sec M_i \geq 1, \quad \operatorname{rad} M_i \leq r < \pi \quad \text{and} \quad \operatorname{vol} M_i \to w_1^n(r). \tag{3.3}$$

In order to prove part (b) of Theorem C we must show that

$$X = \lim M_i \text{ is isometric to } L^n_{1,2r}. \tag{3.4}$$

The case $r=\pi$ follows directly from our treatment of the general case $r<\pi$, and is covered also by [OSY].

In each M_i pick a point p_i, realizing the radius of M_i, and let $q_i=A(p_i)\in K(p_i,\pi/2)$ be the point at maximal distance from p_i. Since $\operatorname{vol} M_i \leq w_1^n(\operatorname{rad} M_i) \leq w_1^n(r)$ by 3.1, we conclude $\operatorname{rad} X = \lim \operatorname{rad} M_i = r$ using (3.3). Moreover, using (1.7), (1.10) and (1.11) we may assume that

$$\{p_i\}, \{q_i\} \text{ converges to } p, q \in X \text{ and } d(p,q)=r. \tag{3.5}$$

$$K(p,\pi/2) \text{ is convex and } \{K(p_i,\pi/2)\} \text{ converges to it.} \tag{3.6}$$

(3.7)
$$\{\operatorname{Seg}(p_i)\}, \{\operatorname{Seg}(q_i)\} \text{ converges to } \operatorname{Seg}(p), \operatorname{Seg}(q) \subset S_1^n \text{ and}$$
$$\{\exp_{p_i}\}, \{\exp_{q_i}\} \text{ converges to } \exp_p\colon \operatorname{Seg}(p) \to X, \exp_q\colon \operatorname{Seg}(q) \to X.$$
$$\text{The points } \bar{p}, \bar{q} \in \operatorname{Seg}(p), \operatorname{Seg}(q) \subset S_1^n \text{ corresponding to } p, q$$
$$\text{are chosen so that } \bar{q}=A(\bar{p})=-\bar{p} \text{ as in Figure 3.2.}$$

$$\{\exp_{q_i}^{-1}(p_i)\} \text{ converges and } \lim \exp_{q_i}^{-1}(p_i) \subset \exp_q^{-1}(p). \tag{3.8}$$

(3.9) $\{\exp_{q_i}^{-1}(K(p_i,\pi/2))\}$ converges and $\lim \exp_{q_i}^{-1}(K(p_i,\pi/2)) \subset \exp_q^{-1}(K(p,\pi/2))$.

The proof of (3.4) is divided into four lemmas:

LEMMA 3.10. *With $M_i=D(p_i,\pi/2)\cup K(p_i,\pi/2)$ as above we have*
(i) $\operatorname{vol} D(p_i,\pi/2) \to \operatorname{vol} D(\bar{p},\pi/2)$,
(ii) $\operatorname{vol} K(p_i,\pi/2) \to \operatorname{vol} K(\{\bar{p}_1,\bar{p}_2\},\pi/2)$, *where* $\bar{p}_1,\bar{p}_2 \in S_1^n$ *satisfy* $d(\bar{p}_1,\bar{p}_2)=2\pi-2r$ (*and* $d(\bar{p}_1,\bar{q})=d(\bar{p}_2,\bar{q})=r$ *as in Figure* 3.2).

Proof. This is clear from (3.3) since $\operatorname{vol} M_i = \operatorname{vol} D(p_i,\pi/2) + \operatorname{vol} K(p_i,\pi/2)$ and $\operatorname{vol} D(p_i,\pi/2) \leq \operatorname{vol} D(\bar{p},\pi/2)$, $\operatorname{vol} K(p_i,\pi/2) \leq \operatorname{vol} K(\{\bar{p}_1,\bar{p}_2\},\pi/2)$ according to 3.1(i), (ii). □

LEMMA 3.11. *The exponential maps of* (3.7) *satisfy:*

(i) $D(\bar{p},\pi/2)\subset \mathrm{Seg}(p)\subset S_1^n$, $\exp_p(D(\bar{p},\pi/2))=D(p,\pi/2)$ *and* $\exp_p(B(\bar{p},\pi/2))=B(p,\pi/2)$.

(ii) $\exp_q^{-1}(p)=\{\bar{p}_1,\bar{p}_2\}\subset \mathrm{Seg}(q)\subset S_1^n$, *where* $d(\bar{p}_1,\bar{p}_2)=2\pi-2r$ *and* $d(\bar{q},\bar{p}_1)=d(\bar{q},\bar{p}_2)=r$.

(iii) $K(\{\bar{p}_1,\bar{p}_2\},\pi/2)\subset \mathrm{Seg}(\bar{q})$, $\exp_q(K(\{\bar{p}_1,\bar{p}_2\},\pi/2))=K(p,\pi/2)$ *and*

$$\exp_q(\operatorname{int} K(\{\bar{p}_1,\bar{p}_2\},\pi/2)) = X-D(p,\pi/2).$$

Proof. (i) Once $D(p,\pi/2)\subset \mathrm{Seg}(p)$ has been established the rest follows from (1.10). Now for every i, $\operatorname{vol} D(p_i,\pi/2)\leqslant \operatorname{vol} D(\bar{p}_i,\pi/2)\cap \mathrm{Seg}(p_i)\leqslant \operatorname{vol} D(\bar{p},\pi/2)$ by standard volume comparison. From this, 3.10(i) and $\lim D(\bar{p}_i,\pi/2)\cap \mathrm{Seg}(p_i)=D(\bar{p},\pi/2)\cap \mathrm{Seg}(p)$ we conclude that $D(\bar{p},\pi/2)-\mathrm{Seg}(\bar{p})$ has no interior points, and therefore is empty.

(ii) Since for each i, $q_i=A(p_i)$ is a critical point for p_i we get from (1.1) and (1.4) that

$$\operatorname{vol} K(p_i,\pi/2)\leqslant \operatorname{vol} K(\exp_{q_i}^{-1}(p_i),\pi/2)\cap \mathrm{Seg}(q_i)\leqslant \operatorname{vol} K(\{\bar{p}_1,\bar{p}_2\},\pi/2),$$

where $\bar{p}_1,\bar{p}_2\in S_1^n$ is a pair of points with $d(\bar{p}_1,\bar{p}_2)=2\pi-2r\leqslant 2\pi-2\operatorname{rad} M_i$. Clearly every direction at $\bar{q}$ makes an angle $\leqslant\pi/2$ with some segment $\bar{q}\bar{p}$, $\bar{p}\in\lim \exp_{q_i}^{-1}(p_i)$. From this, the above volume estimate and 3.10(ii) we conclude that

$$\lim \exp_{q_i}^{-1}(p_i)=\{\bar{p}_1,\bar{p}_2\}\subset S_1^n \quad \text{with} \quad d(\bar{p}_1,\bar{p}_2)=2\pi-2r \text{ and } d(\bar{q},\bar{p}_1)=d(\bar{q},\bar{p}_2)=r.$$

In particular, $\{\bar{p}_1,\bar{p}_2\}\subset \exp_q^{-1}(p)$ by (3.8).

Now suppose $\bar{u}\in \exp_q^{-1}(p)-\{\bar{p}_1,\bar{p}_2\}$, and choose $\bar{u}_i\in \mathrm{Seg}(q_i)\subset S_1^n$ with $\bar{u}=\lim u_i$. Then $d(\bar{u}_i,\exp_{q_i}^{-1}(p_i))\to d(\bar{u},\{\bar{p}_1,\bar{p}_2\})$ and $d(\exp_{q_i}(\bar{u}_i),p_i)\to 0$. By (1.1), (1.3) and (1.4)

$$\lim \operatorname{vol} M_i\leqslant \operatorname{vol} D(\bar{p},\pi/2)+\operatorname{vol} K(\{\bar{p}_1,\bar{p}_2,\bar{u}\},\pi/2).$$

This obviously (cf. (1.4)) contradicts (3.3), and therefore $\exp_q^{-1}(p)=\{\bar{p}_1,\bar{p}_2\}$ as claimed.

(iii) To prove $K(\{\bar{p}_1,\bar{p}_2\},\pi/2)\subset \mathrm{Seg}(q)$ is analogous to the proof of $D(\bar{p},\pi/2)\subset \mathrm{Seg}(p)$, and is therefore left out. For each i,

$$\exp_{q_i}(K(\exp_{q_i}^{-1}(p_i),\pi/2))\cap \mathrm{Seg}(q_i)\supset K(p_i,\pi/2)$$

by standard distance comparison, and hence the limit

$$\exp_q K(\{\bar{p}_1,\bar{p}_2\},\pi/2)\supset K(p,\pi/2),$$

(cf. (3.6)). On the other hand an argument like the one given for

$$D(\bar{p},\pi/2)\subset \mathrm{Seg}(p)\quad (\text{and } K(\{\bar{p}_1,\bar{p}_2\},\pi/2)\subset \mathrm{Seg}(q))$$

shows that

$$\lim \exp_{q_i}^{-1}(K(p_i,\pi/2)) = K(\{\bar{p}_1,\bar{p}_2\},\pi/2)$$

and thus $\exp_q K(\{\bar{p}_1,\bar{p}_2\},\pi/2)\subset K(p,\pi/2)$, by (3.9). From $\exp_q K(\{\bar{p}_1,\bar{p}_2\},\pi/2)=K(p,\pi/2)$ and (ii) it follows via (1.10) that $\exp_q(\partial K)\subset D(p,\pi/2)-B(p,\pi/2)$. To finish the proof we need to see that $\exp_q(\operatorname{int} K)\subset X-D(p,\pi/2)$. For this let $\bar{u}\in \operatorname{int} K$ and consider the unique $\bar{v}\in\partial K$ so that $\bar{u}$ is on the segment $\bar{q}\bar{v}$ in $K\subset \mathrm{Seg}(q)\subset S_1^n$. Since $\exp_q(\bar{v})$, respectively q has distance $\pi/2$, respectively $r>\pi/2$ from p we get from (1.11) that $\exp_q \bar{u}$ has distance $>\pi/2$ from p. □

The next lemma will provide the necessary metric properties for rigidity.

LEMMA 3.12. *The exponential maps of* 3.11 *have the following metric properties.*

(i) $\exp_p\colon D(\bar{p},\pi/2)\to D(p,\pi/2)$ *is injective and an isometry on balls* $D(\bar{u},\varepsilon)$ *where* $D(\bar{u},2\varepsilon)\subset D(\bar{p},\pi/2)$.

(ii) $\exp_q\colon \operatorname{int} K(\{\bar{p}_1,\bar{p}_2\},\pi/2)\to X-D(p,\pi/2)$ *is injective and an isometry on balls* $D(\bar{v},\varepsilon)$ *whenever* $D(\bar{v},2\varepsilon)\subset K(\{\bar{p}_1,\bar{p}_2\},\pi/2)$.

Proof. The injectivity and local isometry properties of $\exp_p|\operatorname{int} D$ and $\exp_q|\operatorname{int} K$ are proved along the lines of 2.5(ii) and (iii). We therefore confine our attention to $\exp_p\colon \partial D(\bar{p},\pi/2)\to X$.

Suppose $\bar{u}\neq\bar{v}\in\partial D(\bar{p},\pi/2)$ and $\exp_p(\bar{u})=\exp_p(\bar{v})=x$. Using 3.11(iii) we find a $\bar{w}\in\partial K(\{\bar{p}_1,\bar{p}_2\},\pi/2)$ so that $\exp_q(\bar{w})=x$. Now choose $\varepsilon>0$ so that $D(\bar{u},\varepsilon)$, $D(\bar{v},\varepsilon)$ are disjoint in S_1^n. If $\bar{u}=\lim \bar{u}_i$, $\bar{v}=\lim\bar{v}_i$, and $\bar{w}=\lim\bar{w}_i$ with $\bar{u}_i,\bar{v}_i\in D(\bar{p}_i,\pi/2)\cap \mathrm{Seg}(p_i)$, and $\bar{w}_i\in K(\exp_{q_i}^{-1}(p_i),\pi/2)\cap\mathrm{Seg}(q_i)$ then $x_i=\exp_{p_i}\bar{u}_i$, $y_i=\exp_{p_i}\bar{v}_i$, $z_i=\exp_{q_i}\bar{w}_i$ converges to x by (3.7). Consider the decomposition

$$M_i=(D(p_i,\pi/2)\cup K(p_i,\pi/2)-B(\{x_i,y_i,z_i\},\varepsilon))\cup D(\{x_i,y_i,z_i\},\varepsilon)$$

or

$$M_i=K((\{x_i,y_i,z_i\},\varepsilon);(p_i,\pi/2))\cup K((\{x_i,y_i,z_i\},\varepsilon),(p_i,\pi/2))\cup D(\{x_i,y_i,z_i\},\varepsilon).$$

From this clearly

$$\operatorname{vol} M_i\leq \operatorname{vol} K((\{x_i,y_i\},\varepsilon);(p_i,\pi/2))+\operatorname{vol} K((z_i,\varepsilon),(p_i,\pi/2))+\operatorname{vol} D(\{x_i,y_i,z_i\},\pi/2).$$

Using (1.1) and (1.3) this yields

$$\lim \operatorname{vol} M_i \leq \left(v_1^n(\pi/2) - 2\cdot\frac{1}{2}v_1^n(\varepsilon)\right) + \operatorname{vol}(K(\{\bar{p}_1, \bar{p}_2\}, \pi/2) - B(\bar{w}, \varepsilon)) + v_1^n(\varepsilon)$$

$$= w_1^n(r) - \operatorname{vol}(K(\{\bar{p}_1, \bar{p}_2\}, \pi/2) \cap B(\bar{w}, \varepsilon)) < w_1^n(r)$$

contradicting (3.3). Thus $\exp_p: D(\bar{p}, \pi/2) \to X$ is injective. □

From 3.11 and 3.12 we know that X is obtained from $D(\bar{p}, \pi/2)$ and $K(\{\bar{p}_1, \bar{p}_2\}, \pi/2)$ by suitable gluing of boundaries. The precise gluing is expressed in:

LEMMA 3.13. *Here we endow $\partial K(\{\bar{p}_1, \bar{p}_2\}, \pi/2)$ and $\partial D(\bar{p}, \pi/2)$ with inner metrics induced from $K(\{\bar{p}_1, \bar{p}_2\}, \pi/2)$ and $D(\bar{p}, \pi/2)$ respectively. Then*

(i) $\exp_p^{-1} \circ \exp_q: \partial K \to \partial D = S_1^n$ *is an isometry. In particular,*

(ii) $\exp_q: K(\{\bar{p}_1, \bar{p}_2\}, \pi/2) \to K(p, \pi/2)$ *is injective, and*

(iii) $\exp_p \amalg \exp_q: D(\bar{p}, \pi/2) \amalg K(\{\bar{p}_1, \bar{p}_2\}, \pi/2) \to X$ *induces an isometry between the inner metric spaces $D(\bar{p}, \pi/2) \amalg K(\{\bar{p}_1, \bar{p}_2\}, \pi/2)/\sim$ and X. Here $\bar{u} \sim \bar{v}$, $\bar{u} \in \partial D(\bar{p}, \pi/2)$, $\bar{v} \in \partial K(\{\bar{p}_1, \bar{p}_2\}, \pi/2)$ if and only if $\exp_p \bar{u} = \exp_q \bar{v}$.*

Proof. (i) As in 2.6 we see from 3.12 that $\exp_p: D(\bar{p}, \pi/2) \to D(p, \pi/2) \subset X$ and $\exp_q: K(\{\bar{p}_1, \bar{p}_2\}, \pi/2) \to K(p, \pi/2)$ both preserve lengths of paths. In particular, (cf. 3.11(i), (iii) and 3.12(i)) the map $\exp_p^{-1} \circ \exp_q: \partial K \to \partial D$ is a surjective map which preserves lengths of paths. However, a surjective distance nonincreasing map of S_1^{n-1} is an isometry. To see this first observe that pairs of antipodal points must be mapped to pairs of antipodal points, and hence geodesics to geodesics.

(ii) Obvious from (i).

(iii) By definition of $\sim$ it is now clear that $\exp_p \amalg \exp_q$ induces a bijective map between $D(\bar{p}, \pi/2) \amalg K(\{\bar{p}_1, \bar{p}_2\}, \pi/2)/\sim$ and X. Since it also preserves lengths of paths, it is an isometry. □

Since the isometry class of $D \amalg K/\sim$ is independent of the isometry $\partial K \to \partial D$, the proof of (3.4) and hence Theorem C is complete.

We conclude the section with a more intrinsic description of $X = \lim M_i$. Let

$$E(p, q; \pi/2) = \{x \in X \mid d(p, x) = d(x, q) = \pi/2\}.$$

From 3.12 and (1.10) $\exp_q(S^{n-2}(\bar{q})) = E(p, q; \pi/2) = K(p, \pi/2) \cap K(q, \pi/2)$, which is convex by (1.11). Moreover, using 3.12(ii) and 3.13(ii) we conclude from this that $\exp_q: S^{n-2}(\bar{q}) \to E(p, q; \pi/2) \subset X$ is an isometry. Thus X contains a convex isometrically embedded $S_1^{n-2} \simeq E(p, q; \pi/2)$.

In particular, $\operatorname{diam} X=\pi$ which by [GP4] implies that X is isometric to the sin-suspension (warped product) $X=\Sigma_{\sin} E$, where E is the convex equidistant set for two points at maximal distance π in X. This argument can be repeated $n-1$ times yielding $X=\Sigma_{\sin}^{n-1} S^1(r/\pi)$, where $S^1(r/\pi)$ is the circle of length $2r$.

Actually $S^1(r/\pi)\subset X$ is the convex set of points of distance $\pi/2$ from $E(p,q;\pi/2)$. It consists of the two segments from p to q, which together form a closed geodesic in X (cf. 3.11(ii) and 3.13(i)).

4. Concluding remarks

So far we have allowed all manifolds in our discussion. Suppose now instead, that M is a fixed closed, connected smooth n-manifold, $n\geqslant 2$. For given $k\in\mathbf{R}$ and $r>0$ we then let

$$v_M(k,r)=\sup\{\operatorname{vol} M \mid \sec M\geqslant k, \operatorname{rad} M\leqslant r\}.$$

Theorems A and B computes $v_{S^n}(k,r)$ $(=v_{\mathbf{R}P^n}(k,r)$ unless $k>0$ and $r>\pi/2\sqrt{k})$ and shows that $v_M(k,r)<v_{S^n}(k,r)$ for any M which is not topologically S^n or $\mathbf{R}P^n$. It is unclear, however, whether for instance exotic spheres or real projective spaces can have $v_M(k,r)=v_{S^n}(k,r)$. The results in [OSY] though, imply that at least $v_M(1,\pi)<v_{S^n}(1,\pi)$ unless M is diffeomorphic to S^n.

For general M we note that $v_M(k,r)\to\infty$ as $k\to-\infty$. To see this choose any Riemannian metric on M, scale it so that its radius is very small and then make connected sum with a sphere approximating the purse $P^n_{k,r}$. The metric constructed on M this way has $\operatorname{rad} M\sim r$, $\operatorname{vol} M\sim\operatorname{vol} P^n_{k,r}=v^n_k(r)$ and $\min\sec M\leqslant k$.

Of course this whole program can be considered with radius replaced by diameter. In this context a natural extension of Aleksandrov's problem may be formulated as follows:

Question 4.1. For any Riemannian metric on $M=S^n$ with $\sec M\geqslant k$ and $\operatorname{diam} M\leqslant D$, is $\operatorname{vol} M\leqslant 2V^n_k(D/2)$?

This would follow by standard volume comparison if any sequence $\{M_i=S^n\}$ with $\sec M_i\geqslant k$, $\operatorname{diam} M_i\leqslant D$ and $\operatorname{vol} M_i\to\sup\{\operatorname{vol} M \mid M=S^n, \sec M\geqslant k, \operatorname{diam} M\leqslant D\}$ has a subsequence, where $\operatorname{exc} M_i\to 0$. Here $\operatorname{exc} M$ denotes the excess of M defined by $\operatorname{exc} M=\min_{(p,q)}\max_x\{d(p,x)+d(x,q)-d(p,q)\}$ (cf. [GP4,5]).

For unrestricted M as in the present paper, the diameter–volume problem was investigated in [GP3]. It was shown there, that $\sup\{\operatorname{vol} M \mid \sec M\geqslant k, \operatorname{diam} M\leqslant D\}$ is strictly smaller than $v^n_k(D)$ except in the two cases where $k>0$ and $D=\pi/2\sqrt{k}$ or $\pi/\sqrt{k}$ corresponding to $\mathbf{R}P^n_k$ and S^n_k respectively.

A natural extension of the main problem considered here is of course to replace the sectional curvature bound $\sec M \geq k$ by the weaker Ricci curvature bound $\operatorname{Ric} M \geq (n-1)k$. In this generality, it is still true that $\operatorname{vol} M \leq v_k^n(\operatorname{rad} M)$. Again equality is only obtained when M is a sphere or real projection space of constant curvature. Furthermore, this inequality is optimal, unless $k>0$ and $\operatorname{rad} M > \pi/2\sqrt{k}$. In the case $\operatorname{Ric} M \geq (n-1)k>0$ and $\operatorname{rad} M>\pi/2\sqrt{k}$ the stronger inequality

$$\operatorname{vol} M \leq w_k^n(\operatorname{rad} M) = \frac{\operatorname{rad} M}{\pi/2\sqrt{k}} \cdot v_k^n(\pi/2\sqrt{k}) < v_k^n(\operatorname{rad} M)$$

derived for sectional curvature does not hold in general, as demonstrated, e.g., by $M=\mathbf{C}P^2$ with its standard Riemannian metric.

The stability question, regarding which manifolds have almost maximal volume, is clearly also very difficult in this situation. It is, for instance, not even known if a manifold M with $\operatorname{Ric} M \geq n-1$ and $\operatorname{vol} M \sim \operatorname{vol} S^n = v_1^n(\pi)$ must be a sphere.

We conclude with a brief discussion of possible lower volume bounds, in terms of lower bounds for curvature and radius. For general k and r ($\leq \pi/2\sqrt{k}$ if $k>0$), however,

$$\inf\{\operatorname{vol} M \mid \sec M \geq k, \operatorname{rad} M \geq r\} = 0. \tag{4.2}$$

For $k=1$ and $r=\pi/2$ this is illustrated by, e.g., the sequence of lens spaces, $S_1^3/\mathbf{Z}_k$, $k \geq 2$, where the action of $\mathbf{Z}_k \subset S^1$ is given by complex multiplication. For all other choices of k and r as above there are metrics on, e.g., $M=S^2$ with arbitrarily small volume. This is illustrated by the lemons $L_{1,\theta}^2$, θ small ($k>0$), capped off thin cylinders ($k=0$), and Q-tips ($k<0$).

The remaining class of manifolds M with $\sec M \geq k > 0$ and $\operatorname{rad} M > \pi/2\sqrt{k}$ exhibit an interesting exception.

THEOREM 4.3. *There exists a $v=v(n)>0$ such that any Riemannian n-manifold M with* $\sec M \geq 1$ *and* $\operatorname{rad} M > \pi/2$ *has* $\operatorname{vol} M \geq v$.

This is an immediate consequence of properties of the filling radius of M (see, [G2;4.5.C]) and the following estimate for the *criticality radius*, $\operatorname{crit} M$, of M. Here $\operatorname{crit} M$ is the smallest critical value for any of the distance functions $d(p,\cdot)$, $p \in M$. Since any ball of radius $r<\operatorname{crit} M$ is contractible, this is also sometimes being referred to as the contractibility radius.

LEMMA 4.4. *If* $\sec M \geq 1$ *and* $\operatorname{rad} M > \pi/2$, *then* $\operatorname{crit} M = \operatorname{rad} M$.

Proof. Obviously $\operatorname{crit} M \leq \operatorname{rad} M$. Now let $p, x \in M$ and suppose $d(p,x) < \operatorname{rad} M$.

To show that x is a regular point for p assume that $p=A(q)$ is the unique point at maximal distance from some $q\in M$. In particular, $d(p,q)>\max(d(x,p),d(x,q))$ and $d(p,q)\geqslant \operatorname{rad} M>\pi/2$. By standard distance comparison it then follows that the angle between any segments from x to p and x to q is larger than $\pi/2$. Thus x is a regular point for p as well as for q.

It remains to show that the map $A: M\to M$ is surjective. Since obviously $p\to \operatorname{rad}_p = \max_x d(p,x)$ is continuous, and there is only one point, $A(p)$ at maximal distance rad_p from p, it follows that A is continuous. If now $A: M\to M$, $M\simeq S^n$, were not surjective, it would have a fixed point according to the Brouwer fixed point theorem. □

Remark 4.5. When M is as in 4.4, an additional convexity argument due to Fred Wilhelm implies that for every $p\in M$ only $A(p)\in M$ is critical for $\operatorname{dist}(p,\cdot)$.

In view of 4.4 we propose the following

Conjecture 4.6. If M is a Riemannian n-manifold with $\sec M\geqslant 1$ and $\operatorname{rad} M\geqslant r>\pi/2$ then $\operatorname{vol} M\geqslant v^n_{(\pi/r)^2}(r)$.

From [SY] it follows that if $\sec M\geqslant 1$ and $\operatorname{rad} M\geqslant r\sim\pi$, then at least $\operatorname{vol} M\sim v^n_1(\pi)$.

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Received March 11, 1991