Acta Math., 169 (1992), 105–130

The structure of the automorphism group of an injective factor and the cocycle conjugacy of discrete abelian group actions

by

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§0. Introduction

The purpose of this paper is to give a proof of Connes' announcement on approximately inner automorphisms and centrally trivial automorphisms of an injective factor of type III for the first time, and to provide a classification, up to cocycle conjugacy, of actions of a discrete abelian or finite group on the unique injective factor of type III_1, which completes the final step of classification of actions of such groups on injective factors.

The study of automorphism groups has been a powerful method for understanding the structure of von Neumann algebras. Connes magnificently developed this approach in [4, 6, 7, 8]. Jones [15] and Ocneanu [18] followed the line of Connes [4, 6] and completed the classification of discrete amenable group actions on the unique approximately finite dimensional (AFD) factor of type II_1. Their work also provides useful tools for the case of type III. Sutherland–Takesaki [20] gave a classification of discrete amenable group actions on AFD factors of type III_λ, $0\leq\lambda<1$. Through their and Ocneanu's work, importance of two special classes of automorphisms became clear. The classes are the approximately inner automorphisms $\overline{\mathrm{Int}}(\mathcal{M})$ and the centrally trivial automorphisms $\mathrm{Cnt}(\mathcal{M})$ of a factor $\mathcal{M}$. Connes [5] announced a characterization of these classes for AFD factors of type III, but the proof has been unavailable for more than ten years since then, though this result was used in Lemma 2(a) of Connes [8], which together with Haagerup [13] established the uniqueness of AFD factors of type III_1, and also in the above-mentioned paper [20]. The characterization, announced in Connes [5, section 3.8] without proof, is as follows. (See [11] and [4] for notations.)

THEOREM 1. *For* AFD *factors* $\mathcal{M}$ *of type* III, *we have:*

(i) $\mathrm{Ker(mod)}=\overline{\mathrm{Int}}(\mathcal{M})$;

(ii) *An automorphism* α *of* $\mathcal{M}$ *is centrally trivial if and only if* α *is of the form* $\alpha=\mathrm{Ad}(u)\cdot\bar{\sigma}_c^{\varphi}$, *where* $\bar{\sigma}_c^{\varphi}$, *is an extended modular automorphism for a dominant weight* φ *on* $\mathcal{M}$, c *is a* θ*-cocycle on* $\mathcal{U}(\mathcal{C}_{\varphi})$, *and* $u\in\mathcal{U}(\mathcal{M})$.

We give a complete proof of this characterization in § 3. The centrally trivial automorphisms are also related to pointwise inner automorphism of Haagerup–Størmer [14].

In the classification of discrete amenable group actions on AFD factors of type III in [20], the case of type III_1 was left open. Here we now classify actions of discrete *abelian* groups and finite groups on the AFD factor of type III_1. Thus the classification of actions of discrete abelian or finite groups is complete, and this will be enough to accomplish classification of compact abelian group actions on AFD factors in Kawahigashi–Takesaki [17] along the lines of Jones–Takesaki [16] and Sutherland–Takesaki [20].

For the proof of Connes' announcement, we make use of the discrete decomposition and stability of the automorphism θ in it for the III_0 and III_λ $(0<\lambda<1)$ cases. For the type III_1 case, we will show that the algebra of strongly central sequences at a free ultrafilter is a factor, and will use Araki's property L'_λ [1]. For the cases of type III_0 and III_1, we need several preparatory lemmas, so we spend the first two sections § 1 and § 2 for these, respectively. The main idea for the type III_λ $(0\leqslant\lambda<1)$ case is reducing the problem to the type II_∞ case by using a discrete decomposition after an appropriate inner perturbation of a given automorphism. For the type III_1 case, we split out an automorphism of an AFD factor of type III_λ $(0<\lambda<1)$ after inner perturbation. In § 3, we complete the proof of Theorem 1. The proof is divided into three cases: type III_0, III_λ $(0<\lambda<1)$, and III_1. In § 4, we give a classification result for discrete abelian groups. The invariants in the case of type III_1 are exactly the same as in Sutherland–Takesaki [20], and, are complete. After applying Theorem 1, we can reduce the problem to a theorem of Ocneanu, [18].

The basic references are Connes [5], Connes–Takesaki [11], and Sutherland–Takesaki [20]. We use notations and results from these freely.

This work was started when the first and third named authors stayed at the Mittag-Leffler Institute, continued while they stayed at Institut des Hautes Études Scientifiques and all the three stayed at the Mittag-Leffler Institute, and completed when the third named author visited Japan. We are grateful to these institutes for their hospitality. The first named author was supported by the Alfred P. Sloan doctoral dissertation Fellowship and the Mittag-Leffler Institute, the second by the Mittag-Leffler Institute and an A. R. C. Grant, and the third by the Mittag-Leffler Institute, I.H.E.S., N.S.F. Grant-DMS-8908281 and JSPS.

Added in proof. We are thankful to Professor Y. Katayama for pointing out an error in the proof of Lemma 18 in the original version of this paper.

§1. Preliminaries on automorphisms of AFD factors of type III_0

Here we prepare technical lemmas for AFD factors of type III_0. In the Lemmas 2, 3, 4, 5, and 6, we will show some inner perturbation of a given automorphism of an AFD factor of type III_0 has a special property, which makes our later task easier.

LEMMA 2. *If an automorphism α of an* AFD *factor $\mathcal{M}$ of type* III_0 *belongs to* Ker(mod) *then there exists a faithful lacunary weight ψ with infinite multiplicity on $\mathcal{M}$ and a unitary $u\in\mathcal{U}(\mathcal{M})$ with the following properties:*

(1) *In the discrete decomposition $\mathcal{M}=\mathcal{M}_\psi\rtimes_\theta\mathbf{Z}$, we have* $\mathrm{Ad}(u)\cdot\alpha|_{\mathscr{C}_\psi}=\mathrm{id}$*;*

(2) $\psi|_{\mathcal{M}_\psi}\cdot\mathrm{Ad}(u)\cdot\alpha=\psi|_{\mathcal{M}_\psi}$*;*

(3) $\mathrm{Ad}(u)\cdot\alpha(U)=U$*, where U is the unitary implementing θ in the decomposition* (1).

Proof. By [11, p. 555], we can achieve (1) and (2). We replace α by $\mathrm{Ad}(u)\cdot\alpha$. Now for any $x\in\mathscr{C}_\psi$, we get

$$\alpha(U)U^*x=\alpha(U)\theta^{-1}(x)U^*=\alpha(U\theta^{-1}(x))U^*=\alpha(xU)U^*=x\alpha(U)U^*,$$

thus by the relative commutant theorem [11, Corollary I.2.10], $\alpha(U)U^*\in\mathcal{M}_\psi$. By stability of θ, [11, p. 544], there exists a unitary $v\in\mathcal{M}_\psi$ with $\alpha(U)U^*=v^*\theta(v)=v^*UvU^*$. Now $\mathrm{Ad}(v)\cdot\alpha$ satisfies the desired properties. Q.E.D.

LEMMA 3. *Let $\mathcal{M}$, θ be as in Lemma* 2, *and set $\mathcal{N}=\mathcal{M}_\psi$ and choose a free ultrafilter ω on* $\mathbf{N}$. *Then for any $n\in\mathbf{N}$, and any countable subset $(x_j)_{j\in\mathbf{N}}$ of $\mathcal{N}_\omega$, there exists a partition of unity $(F_k)_{k=1,\dots,n}$ in $\mathcal{N}_\omega$ such that each F_k commutes with all x_j and such that $\theta_\omega(F_k)=F_{k+1}$, $k=1,\dots,n$, where $F_{n+1}=F_1$.*

Proof. Because θ on $\mathscr{C}_\psi$ is ergodic, we can apply the proof of Lemma 2.1.4 in Connes [4] by using the usual Rohlin Lemma instead of Theorem 1.2.5 in [4]. Because $(\mathscr{C}_\psi)^\omega\subset\mathscr{Z}(\mathcal{N}_\omega)$, we are done. Q.E.D.

LEMMA 4. *Let $\mathcal{M}, \mathcal{N}, \theta, \omega$ be as in Lemma 3. Then for any unitary $u \in \mathcal{N}_\omega$, there is a unitary $v \in \mathcal{N}_\omega$ such that $\theta_\omega(v) = uv$.*

Proof. The proof of Theorem 2.1.3 in [4] works with our Lemma 3 instead of Lemma 2.1.4 in [4]. Q.E.D.

For the proof of Theorem 1(ii) for the AFD factors of type III_0, we would like to perturb a given automorphism α by a unitary so that the centralizer is globally fixed by α.

LEMMA 5. *If ψ is a lacunary weight with infinite multiplicity on $\mathcal{M}$ and α is an automorphism of $\mathcal{M}$, then there exists a projection $e \in \mathcal{C}_\psi$ and a partial isometry u such that*

(1) *$u^*u = \alpha(e)$ and $uu^* = e$;*
(2) *$u\alpha(\mathcal{M}_{\psi,e})u^* = \mathcal{M}_{\psi,e}$.*

Proof. Let $\delta > 0$ be such that

$$[-\delta, \delta] \cap \mathrm{Sp}(\sigma^\psi) = \{0\}.$$

Let $\bar{\psi} = \psi \cdot \alpha^{-1} \oplus \psi$ on $\mathcal{M} \otimes M_2(\mathbf{C})$. Choose a non-zero element

$$\bar{x} = x \otimes e_{12} \in \mathcal{M}_{\bar{\psi}}[c - \delta/3, c + \delta/3],$$

the Arveson spectral subspace for $\mathcal{M}_{\bar{\psi}}$ for some $c \in \mathbf{R}$. For every $a \in \mathcal{M}_\psi$, we have

$$xax^* \otimes e_{11} = \bar{x}(a \otimes e_{22})\bar{x}^* \in \mathcal{M}_{\psi\cdot\alpha^{-1}}[-2\delta/3, 2\delta/3] \otimes e_{11}.$$

By the choise of δ, we have

$$\mathcal{M}_{\psi\cdot\alpha^{-1}}[-2\delta/3, 2\delta/3] = \mathcal{M}_{\psi\cdot\alpha^{-1}}$$

so that

$$x\mathcal{M}_\psi x^* \subset \mathcal{M}_{\psi\cdot\alpha^{-1}};$$

$$x^*\mathcal{M}_{\psi\cdot\alpha^{-1}}x \subset \mathcal{M}_\psi.$$

Let $x = uh$ be the polar decomposition. Then we have $h \in \mathcal{M}_\psi$, $u^*u = f \in \mathcal{M}_\psi$ and $uu^* = g \in \mathcal{M}_{\psi\cdot\alpha^{-1}}$, and that

$$uau^* \in \mathcal{M}_{\psi\cdot\alpha^{-1}}, \quad a \in \mathcal{M}_\psi.$$

Since we may replace x by $x\otimes 1$ and ψ by $\psi\otimes\mathrm{Tr}$ on $\mathcal{M}\bar{\otimes}\mathcal{L}(l^2)$, we may assume that f (resp. g) is properly infinite in $\mathcal{M}_\psi$ (resp. $\mathcal{M}_{\psi\cdot\alpha^{-1}}$). Let $\bar{f}$ (resp. $\bar{g}$) be the central support of f in $\mathcal{M}_\psi$ (resp. g in $\mathcal{M}_{\psi\cdot\alpha^{-1}}$). Then $f\sim\bar{f}$ in $\mathcal{M}_\psi$ and $g\sim\bar{g}$ in $\mathcal{M}_{\psi\cdot\alpha^{-1}}$. Therefore, there exists a partial isometry $v\in\mathcal{M}_\psi$ (resp. $w\in\mathcal{M}_{\psi\cdot\alpha^{-1}}$) such that

$$vv^* = f \quad\text{and}\quad v^*v = \bar{f};$$

$$w^*w = g \quad\text{and}\quad ww^* = \bar{g}.$$

Let $U=wuv$. Then we have $U^*U=\bar{f}$, $UU^*=\bar{g}$ and $U(\mathcal{M}_{\psi,\bar{f}})U^*=\mathcal{M}_{\psi\cdot\alpha^{-1},\bar{g}}$. Since $\mathcal{M}_{\psi\cdot\alpha^{-1}}=\alpha(\mathcal{M}_\psi)$, $\bar{g}$ is of the form $\bar{g}=\alpha(e)$, $e\in\mathrm{Proj}(\mathscr{C}_\psi)$. We now want to compare e and $\bar{f}$ in $\mathscr{C}_\psi$ under the Hopf equivalence given by the ergodic automorphism θ on $\mathscr{C}_\psi$, where θ is the automorphism of $\mathcal{M}_\psi$ such that

$$\mathcal{M} = \mathcal{M}_\psi \rtimes_\theta \mathbf{Z}.$$

Let $p\simeq q$ denote the Hopf equivalence of $p, q\in\mathrm{Proj}(\mathscr{C}_\psi)$. Decompose $e=e_1+e_2+\dots+e_n+e_{n+1}$ in such a way that

$$e_i\simeq\bar{f} \quad\text{and}\quad e_{n+1}\simeq\bar{f}'\leqslant\bar{f},\ \bar{f}'\in\mathrm{Proj}(\mathscr{C}_\psi).$$

Since $\mathcal{M}_\psi$ is properly infinite, there exists a partition:

$$\bar{f}=f_1+f_2+\dots+f_n+f_{n+1}$$

such that $\bar{f}\sim f_i$, $1\leqslant i\leqslant n$, and $f_{n+1}\sim\bar{f}'$ in $\mathcal{M}_\psi$. The Hopf equivalence $e_i\simeq\bar{f}$ implies the existence of partial isometries v_i such that $v_i^*v_i=e_i$ and $v_iv_i^*=\bar{f}$ and $v_i\mathcal{M}_\psi v_i^*\subset\mathcal{M}_\psi$. Putting these things together, we get a partial isometry V such that

$$V^*V = e,\quad VV^* = \bar{f},\quad V\mathcal{M}_\psi V^* = \mathcal{M}_{\psi,\bar{f}}.$$

Thus, we come to the situation that

$$UV\mathcal{M}_{\psi,e}V^*U^* = \alpha(\mathcal{M}_{\psi,e}) = \mathcal{M}_{\psi\cdot\alpha^{-1},\bar{g}}. \qquad \text{Q.E.D.}$$

LEMMA 6. *In the context of Lemma 5, there exists a lacunary weight $\bar{\psi}$ with infinite multiplicity and a unitary $U\in\mathcal{M}$ such that* $\mathrm{Ad}(U)\cdot\alpha(\mathcal{M}_{\bar{\psi}})=\mathcal{M}_{\bar{\psi}}$.

Proof. By Lemma 5, there exists a partial isometry u such that

$$uu^* = e\in\mathrm{Proj}(\mathscr{C}_\psi),\quad u^*u = \alpha(e),$$

$$u\alpha(\mathcal{M}_{\psi,e})u^* = \mathcal{M}_{\psi,e}.$$

Let v be an isometry of $\mathcal{M}$ with $e=vv^*$. Set

$$\bar{\psi}(x)=\psi(vxv^*),\quad x\in\mathcal{M}$$

$$U=v^*u\alpha(v)\in\mathcal{U}(\mathcal{M}).$$

Then we have $\mathrm{Ad}(U)\cdot\alpha(\mathcal{M}_{\bar{\psi}})=\mathcal{M}_{\bar{\psi}}$. Q.E.D.

We must deal with strongly central sequences for the study of centrally trivial automorphisms. The next lemma reduces the study for factors of type III_λ, $\lambda\neq 1$, to semifinite algebras.

LEMMA 7. *Let $\mathcal{M}=\mathcal{N}\rtimes_\theta\mathbf{Z}$ be the discrete decomposition of a factor $\mathcal{M}$ of type* III_λ, $0\leq\lambda<1$. *Every strongly central sequence $\{x_n\}$ in $\mathcal{M}$ is equivalent to a strongly central sequence $\{y_n\}$ in $\mathcal{N}$ with $\theta(y_n)-y_n\to 0$ *-strongly.*

Proof. Let ψ be a faithful normal state on $\mathcal{M}$ and $\{x_n\}$ a strongly central sequence of $\mathcal{M}$. Define a one-parameter automorphism group β_t by $\beta_t(x)=x$ for $x\in\mathcal{N}$ and $\beta_t(U)=e^{2\pi it}U$ for the implementing unitary U in the discrete decomposition. Then each β_t is centrally trivial, and the conditional expectation $\mathcal{E}_0$ of $\mathcal{M}$ onto $\mathcal{N}$ is given by

$$\mathcal{E}_0(x)=\int_0^1\beta_t(x)\,dt,\quad x\in\mathcal{M}.$$

The Lebesgue dominated convergence theorem ensures that $\|\mathcal{E}_0(x_n)-x_n\|_\psi^\#\to 0$, so that $\{x_n\}\sim\{\mathcal{E}_0(x_n)\}$. Since $[x_n, U]\to 0$, *-strongly, we get $\{\theta(\mathcal{E}_0(x_n))\}\sim\{\mathcal{E}_0(x_n)\}$.

Conversely, suppose that $\{y_n\}$ is a strongly central sequence in $\mathcal{N}$ such that $\|\theta(y_n)-y_n\|_\psi^\#\to 0$. We want to prove that $\|[y_n,\varphi]\|\to 0$ for every $\varphi\in\mathcal{M}_*$. Since the maps: $\varphi\mapsto[y_n,\varphi]$ are uniformly bounded, it suffices to prove that $\|[y_n,\varphi]\|\to 0$ for a dense subset of φ in $\mathcal{M}_*$. Let $\mathcal{E}_k(x)=\mathcal{E}_0(xU^{-k})U^k$, $k\in\mathbf{Z}$. Then $\{\varphi\cdot\mathcal{E}_k|\ k\in\mathbf{Z},\ \varphi\in\mathcal{M}_*\}$ is total in $\mathcal{M}_*$. Thus, we will show that $\|[y_n,\varphi\cdot\mathcal{E}_k]\|\to 0$ as $n\to\infty$. Fix $x\in\mathcal{M}$ and $k\in\mathbf{Z}$. Set $z_k=\mathcal{E}_0(xU^{-k})\in\mathcal{N}$. We then have

$$\begin{aligned}|\langle x,[y_n,\varphi\cdot\mathcal{E}_k]\rangle| &= |\langle xy_n-y_nx,\varphi\cdot\mathcal{E}_k\rangle|\\ &= |\langle z_kU^ky_n-y_nz_kU^k,\varphi\rangle|\\ &\leq |\langle(z_k\theta^k(y_n)-z_ky_n)U^k,\varphi\rangle|+|\langle z_ky_n-y_nz_k,U^k\varphi\rangle|\\ &\leq \|(\theta^k(y_n)-y_n)U^k\varphi\|_{\mathcal{N}}\cdot\|z_k\|+\|[y_n,U^k\varphi]\|_{\mathcal{N}}\cdot\|z_k\|,\end{aligned}$$

which converges to 0 uniformly in x with $\|x\|\leq 1$. Therefore, $\{y_n\}$ is strongly central in $\mathcal{M}$. Q.E.D.

We would like to further reduce the problem to semifinite factors by representing the automorphism by a field of automorphisms of fibres in the central decomposition of the centralizer. To this end, we need $\alpha|_{\mathscr{C}_\psi}=\mathrm{id}$.

LEMMA 8. *If an automorphism α of an* AFD *factor $\mathcal{M}$ of type* III_0 *is in* $\mathrm{Cnt}(\mathcal{M})$, *then there exists a lacunary weight ψ with infinite multiplicity on $\mathcal{M}$ and a unitary $u\in\mathcal{M}$ such that* $\mathrm{Ad}(u)\cdot\alpha$ *is trivial on* $\mathscr{C}_\psi$, $\mathrm{Ad}(u)\cdot\alpha(\mathcal{M}_\psi)=\mathcal{M}_\psi$, *and* $\mathrm{Ad}(u)\cdot\alpha(U)=U$, *where U is the implementing unitary in the discrete decomposition of $\mathcal{M}$.*

Proof. By Lemma 6, there is ψ such that $\alpha(\mathcal{M}_\psi)=\mathcal{M}_\psi$. If $\{x_n\}$ is a bounded sequence in $\mathscr{C}_\psi$ such that $\theta(x_n)-x_n\to 0$, $*$-strongly as $n\to\omega$ for some fixed free ultrafilter ω on $\mathbf{N}$, then this $\{x_n\}$ is strongly ω-central by Lemma 7, so we get $\alpha(x_n)-x_n\to 0$, $*$-strongly, as $n\to\omega$. This means $(\alpha|_{\mathscr{C}_\psi})_\omega=\mathrm{id}$ on $(\mathscr{C}_\psi)_\omega$. Because α is an automorphism of $\mathcal{M}$, we know $\alpha|_{\mathscr{C}_\psi}\in N[\theta|_{\mathscr{C}_\psi}]$, the normalizer of $\theta|_{\mathscr{C}_\psi}$. These imply $\alpha|_{\mathscr{C}_\psi}\in[\theta|_{\mathscr{C}_\psi}]$ by Lemma 2.4 in [9]. (See section 2 of [9] for notations.) Thus there exists a unitary u in $\mathcal{M}$ such that $\mathrm{Ad}(u)\cdot\alpha|_{\mathscr{C}_\psi}=\mathrm{id}$ and we still have $\mathrm{Ad}(u)\cdot\alpha(\mathcal{M}_\psi)=\mathcal{M}_\psi$. Now we can fix U by the same method as in the proof of Lemma 2. Q.E.D.

The next lemma shows that the field of automorphisms in the central decomposition of the centralizer may be chosen to be constant.

LEMMA 9. *In the context of Lemma 8, we can take α of the form* $\alpha_0\otimes\mathrm{id}$ *on* $\mathcal{M}_\psi\cong\mathscr{R}_{0,1}\bar{\otimes}L^\infty(X)$, *where $\mathscr{R}_{0,1}$ is the* AFD *factor of type* II_∞, *after inner perturbation.*

Proof. By Lemma 8, we may assume α and θ define a $\mathbf{Z}^2$-action on $\mathcal{M}_\psi$. Then by Theorems 1.2 and 3.1 in [19], we can take α of the desired form. Q.E.D.

The following is a slight modification of the standard Rohlin lemma. This will be used for construction of some central sequence.

LEMMA 10. *Let T be a non-singular ergodic transformation on a probability space (X,μ). For any $n\in\mathbf{N}$, there exists a subset E of X such that*

(1) *$E, T^{-1}E, \dots, T^{-n}E$ are mutually disjoint;*

(2) $\mu(\bigcup_{j=0}^{n-1}T^{-j}E)\geq 1-1/n$;

(3) $\mu(E)\leq 1/n, \mu(T^{-n}E)\leq 2/n$.

Proof. Choose a measurable subset A_0 of X such that $\mu(T^{-j}A_0)<1/(2n^2)$ for all $j=0,1,\dots,2(n-1)$. Then set

$$A_m=\{x\in X \mid T^m(x)\in A_0, T^jx\notin A_0, 0\leqslant j\leqslant m-1\}.$$

We have $X=\bigcup_{m\geqslant 0}A_m$, and set $F=\bigcup_{k=1}^{\infty}A_{kn}$. As in the usual proof of the Rohlin lemma, we can see $F, T^{-1}F,\dots, T^{-(n-1)}F$ are mutually orthogonal and

$$\bigcup_{j=0}^{n-1}T^{-j}F\supset X-\bigcup_{j=0}^{n-1}T^{-j}A_0.$$

Then there exists j_0, $0\leqslant j_0\leqslant n-1$, such that $\mu(T^{-j_0}F)\leqslant 1/n$. We set $E=T^{-j_0}F$. By the same proof as usual, we can see $E, T^{-1}E,\dots, T^{-(n-1)}E$ are mutually orthogonal. Note that

(a) $T^{-n}F\subset F\cup A_0\cup\dots\cup A_{n-1}$;

(b) $T^nF\subset F\cup A_0$.

By (b), we get

$$\begin{aligned}&E\cup T^{-1}E\cup\dots\cup T^{-(n-1)}\cup T^{-n}A_0\cup\dots\cup T^{-(n+j_0-1)}A_0\\ &=E\cup T^{-1}E\cup\dots\cup T^{-(n-1-j_0)}E\cup T^{-n}(F\cup A_0)\cup\dots\cup T^{-(n+j_0-1)}(F\cup A_0)\\ &\supset T^{-j_0}F\cup\dots\cup T^{-(n-1)}F\cup F\cup\dots\cup T^{-j_0+1}F.\end{aligned}$$

Thus we get

$$1-\frac{1}{2n}\leqslant\mu\left(\bigcup_{j=0}^{n-1}T^{-j}F\right)\leqslant\mu\left(\bigcup_{j=0}^{n-1}T^{-j}E\right)+n\cdot\frac{1}{2n^2},$$

which implies property (2). By (a), we get

$$\begin{aligned}\mu(T^{-n}E)&=\mu(T^{-j_0}T^{-n}F)\\ &\leqslant\mu(T^{-j_0}F)+\mu(T^{-j_0}A_0)+\dots+\mu(T^{-j_0}A_{n-1})\\ &\leqslant\frac{1}{n}+n\cdot\frac{1}{2n^2}\leqslant\frac{2}{n}.\end{aligned}$$

Q.E.D.

The following is an easy corollary of Connes' splitting of a model action.

LEMMA 11. *Let $\mathcal{M}$ be a separable strongly stable factor. If α is an automorphism of $\mathcal{M}$ and $\alpha\notin\mathrm{Cnt}(\mathcal{M})$, then for any free ultrafilter ω, there exist $\gamma\in\mathbf{C}$, $|\gamma|=1$, $\gamma\neq 1$, and $u\in\mathcal{U}(\mathcal{M}_\omega)$ such that $\alpha_\omega(u)=\gamma u$.*

Proof. Because $p_0(\alpha)\neq 1$, α is cocycle conjugate to $\alpha\otimes s_p$ for some $p>1$, in $\mathcal{M}\cong\mathcal{M}\otimes\mathcal{R}$, where $\mathcal{R}$ denotes the AFD factor of type II_1, by [4, Theorem 1]. (Here s_p denotes the free action of $\mathbf{Z}/p\mathbf{Z}$ on $\mathcal{R}$. See Theorem 5.1 in [6].) Thus we may assume that α is of the form $\alpha\otimes s_p$. Then, we can construct a central sequence $\{u_n\}$ of unitaries in $\mathcal{R}$ such that $s_p(u_n)=\gamma u_n$, $\gamma=\exp(2\pi i/p)$. The sequence $\{1\otimes u_n\}$ in $\mathcal{M}\otimes\mathcal{R}\cong\mathcal{M}$ is strongly central, and we can set this sequence to be u. Q.E.D.

With Lemmas 11 and 12, we show that each automorphism of a fibre of the central decomposition of the centralizer is centrally trivial if the original one is also.

LEMMA 12. *In the context of Lemma* 9, $\alpha_0\in\mathrm{Cnt}(\mathcal{R}_{0,1})$.

Proof. Since θ on $\mathcal{C}_\psi$ is ergodic, we can apply Lemma 10 to get a projection $e_n\in\mathcal{C}_\psi$ such that

(1) $e_n, \theta(e_n), \dots, \theta^{n-1}(e_n)$ are mutually orthogonal;

(2) $\mu(\Sigma_{j=0}^{n-1}\theta^j(e_n))\geq 1-1/n$;

(3) $\mu(\theta^n(e_n))\leq 2/n, \mu(e_n)\leq 1/n$.

Suppose $\alpha_0\notin\mathrm{Cnt}(\mathcal{R}_{0,1})$, and choose a strongly central sequence $\{u_m\}$ of unitaries in $\mathcal{R}_{0,1}$ such that $\alpha_0(u_m)-\gamma u_m\to 0$, $*$-strongly as $m\to\infty$, for some $\gamma\in\mathbf{C}$, $|\gamma|=1$, $\gamma\neq 1$, by Lemma 11. Choose a normal state φ on $\mathcal{R}_{0,1}$ and a dense sequence $\{\varphi_n\}$ in $(\mathcal{R}_{0,1}\bar{\otimes}L^\infty(X))_*$. For each n, the sequence $\{u_m\otimes e_n\}_{m=1}^\infty$ is strongly central in $\mathcal{M}_\psi$. Thus there exists an integer $m=m(n)$ such that

(a) $\|(\alpha_0(u_m)-\gamma u_m)\otimes e_n\|^\#_{(\varphi\otimes\mu)\cdot\theta^j}\leq 1/n^2$, for all $j=0,\dots,n-1$;

(b) $\|[u_m\otimes e_n, \varphi_k\cdot\theta^{-j}]\|\leq 1/n^2$, for all $j=0,\dots,n-1$, $k=1,\dots,n$.

Set $x_n=\Sigma_{j=0}^{n-1}\theta^j(u_{m(n)}\otimes e_n)$. We show $\{x_n\}$ is strongly central in $\mathcal{M}$. First note $\|x_n\|=1$. Next we have, for φ_k and $n>k$,

$$\|[x_n,\varphi_k]\|\leq\sum_{j=0}^{n-1}\|[\theta^j(u_{m(n)}\otimes e_n),\varphi_k]\|\leq\frac{1}{n}\to 0,$$

as $n\to\infty$, by (b). We also have

$$\begin{aligned}\|\theta(x_n)-x_n\|^\#_{\varphi\otimes\mu} &= \|\theta^n(u_{m(n)}\otimes e_n)-u_{m(n)}\,e_n\|^\#_{\varphi\otimes\mu}\\ &\leq\mu(\theta^n(e_n))^{1/2}+\mu(e_n)^{1/2}\\ &\leq 2\left(\frac{2}{n}\right)^{1/2}\to 0.\end{aligned}$$

Thus by Lemma 7, $\{x_n\}$ is strongly central in $\mathcal{M}$. But we have

$$\|(\alpha_0\otimes\mathrm{id})(x_n)-x_n\|^{\#}_{\varphi\otimes\mu} = \left\|\sum_{j=0}^{n-1}\theta^j((\alpha_0(u_{m(n)})-u_{m(n)})\otimes e_n)\right\|^{\#}_{\varphi\otimes\mu}$$

$$\geqslant|1-\gamma|\cdot\left\|\sum_{j=0}^{n-1}\theta^j(u_{m(n)}\otimes e_n)\right\|^{\#}_{\varphi\otimes\mu}-\left\|\sum_{j=0}^{n-1}\theta^j((\alpha_0(u_{m(n)})-\gamma u_{m(n)})\otimes e_n)\right\|^{\#}_{\varphi\otimes\mu}$$

$$\geqslant|1-\gamma|\left(1-\frac{1}{n}\right)^{1/2}-\sum_{j=0}^{n-1}\|(\alpha_0(u_{m(n)})-\gamma u_{m(n)})\otimes e_n\|^{\#}_{(\varphi\otimes\mu)\cdot\theta^j}$$

$$\geqslant|1-\gamma|\left(1-\frac{1}{n}\right)^{1/2}-\frac{1}{n}\to|1-\gamma|>0,$$

as $n\to\infty$ by (a). This contradicts $\alpha\in\mathrm{Cnt}(\mathcal{M})$. Q.E.D.

§ 2. Preliminaries on automorphisms of the AFD factors of type III_1

For the AFD factor of type III_1, we do not have the discrete decomposition, and the continuous decomposition is rather difficult to handle. Thus we will make a different approach based on the infinite tensor product expression. First, we show that the ultraproduct algebra is a factor.

PROPOSITION 13. *In an* AFD *factor* $\mathcal{M}$ *of type* III_1, *all strongly hypercentral sequences are equivalent to trivial ones. Therefore, for any free ultrafilter* ω *on* $\mathbf{N}$, $\mathcal{M}_\omega$ *is a factor of type* II_1.

Proof. By the uniqueness of AFD factors of type III_1, [8] and [13], $\mathcal{M}$ can be identified with an infinite tensor product of matrix algebras. Thus, $\mathcal{M}$ admits an increasing sequence $\{M_n\}$ of finite factors of type I and a faithful normal state φ such that $\mathcal{M}=(\bigcup M_n)''$ and each M_n is globally invariant under $\{\sigma_t^\varphi\}$. Each M_n is generated by two unitaries $u(n)$ and $v(n)$ such that $u(n)^{N(n)}=v(n)^{N(n)}=1$ and $u(n)v(n)=e^{2\pi i/N(n)}v(n)u(n)$, where M_n is isomorphic to the $N(n)\times N(n)$-matrix algebra. So with $\alpha_{k,l}=\mathrm{Ad}(u(n)^k v(n)^l)$, we get an action of $\mathbf{Z}_{N(n)}\times\mathbf{Z}_{N(n)}$ on $\mathcal{M}$ such that

$$\mathscr{E}_n(x)=\frac{1}{N(n)^2}\sum_{k,l=0}^{N(n)-1}\alpha_{k,l}(x),\quad x\in\mathcal{M},$$

is a projection of norm one from $\mathcal{M}$ onto M_n^c such that

$$\|x-\mathscr{E}_n(x)\|^{\#}_\varphi\leqslant\sup_{k,l}\|x-\alpha_{k,l}(x)\|^{\#}_\varphi.$$

Therefore, if $\{x_k\}$ is strongly central, then there exists a subsequence $\{x_{k_n}\}$ such that $\|x_{k_n}-\mathscr{E}_n(x_{k_n})\|_\varphi^\#\to 0$.

Suppose that $\{x_k\}$ is strongly central and not equivalent to a trivial sequence. With $\{k_n\}$ as above, set $y_n=\mathscr{E}_n(x_{k_n})$, $n\in\mathbf{N}$. If we have chosen $\{x_k\}$ so that

$$\liminf\|x_k-\varphi(x_k)\|_\varphi^\# = \alpha>0,$$

which is possible by passing to a subsequence because $\{x_k\}$ is not equivalent to a trivial one, we get $\liminf\|y_n-\varphi(y_n)\|_\varphi^\#\geqslant\alpha>0$. Replacing y_n by $y_n-\varphi(y_n)$, we have a strongly sequence $\{y_n\}$ such that $\varphi(y_n)=0$, $\lim_{n\to\infty}y_n\neq 0$, and $y_n\in M_n^c$. Since M_n^c is an AFD factor of type III_1, there exists a unitary $u_n\in M_n^c$ such that

$$\|y_n-u_ny_nu_n^*\|_\varphi^\#\geqslant\frac{1}{2}\|y_n\|_\varphi^\#;$$

$$\|[u_n,\varphi|_{M_n^c}]\|<\frac{1}{4^n},$$

because 0 is in the σ-week convex closure of $\{uy_nu^*|\ u\in\mathscr{U}(M_n^c),\ \|[u,\varphi|_{M_n^c}]\|<1/4^n\}$ by Haagerup [13, 1.4(c)]. Since $\varphi=\varphi_{M_n}\otimes\varphi|_{M_n^c}$ by assumption, we have $\|[u_n,\varphi]\|<1/4^n$. If ξ_φ is the implementing vector in the natural cone of a standard form, we have $\|u_n\xi_\varphi-\xi_\varphi u_n\|\leqslant\|[u_n,\varphi]\|^{1/2}<1/2^n$. We claim that $\{u_n\}$ is strongly central. Since $\{u_n\}$ commutes with φ asymptotically, we have only to show that $\{u_n\}$ is central. Given $\varepsilon>0$, and $a\in\mathscr{M}$, choose $a_0\in M_k$ such that $\|a-a_0\|_\varphi^\#<\varepsilon$ and $\|a_0\|\leqslant\|a\|$. Then we have

$$\begin{aligned}\|[a,u_n]\xi_\varphi\| &\leqslant \|[a-a_0,u_n]\xi_\varphi\|+\|[a_0,u_n]\xi_\varphi\|\\ &\leqslant \|(a-a_0)\xi_\varphi u_n\|+\|(a-a_0)[u_n,\xi_\varphi]\|\\ &\qquad+\|u_n(a-a_0)\xi_\varphi\|+\|[a_0,u_n]\xi_\varphi\|\\ &\leqslant 2\|a-a_0\|_\varphi^\#+2\|a\|\,\|[u_n,\xi_\varphi]\|+\|[a_0,u_n]\xi_\varphi\|\\ &\leqslant 2\varepsilon+\|a\|/2^{n-1}\end{aligned}$$

for $n\geqslant k$. Hence $\lim\|[a,u_n]\xi_\varphi\|=0$. Similarly we have $\lim\|[a,u_n]^*\xi_\varphi\|=0$. Thus $\{u_n\}$ is central. On the other hand, $\{u_n\}$ does not commute with $\{y_n\}$ asymptotically. Hence $\{y_n\}$ is not hypercentral. Q.E.D.

We will study how an automorphism of the AFD factor of type III_1 acts on its tensor product factor of type III_λ $(0<\lambda<1)$. To this end, we need the following Lemma 14 and Corollary 15.

LEMMA 14. *Fix* $0<\lambda<1$. *If* $\{e_{i,j}(k)|\ 1\leq i,j\leq 2\}$ *and* $\{f_{i,j}(k)|\ 1\leq i,j\leq 2\}$ *are respectively mutually commuting sequences of* 2×2*-matrix units in a separable factor* $\mathcal{M}$ *such that*

$$\lim_{k\to\infty}\|\psi e_{i,j}(k)-\lambda^{i-j}e_{i,j}(k)\psi\|=0;$$

$$\lim_{k\to\infty}\|\psi f_{i,j}(k)-\lambda^{i-j}f_{i,j}(k)\psi\|=0,$$

then there exists $\sigma\in\overline{\mathrm{Int}}(\mathcal{M})$ *and an increasing sequence* $\{k_n|\ n\in\mathbf{N}\}$ *in* $\mathbf{N}$ *such that*

$$\sigma(e_{i,j}(k_n))=f_{i,j}(k_n),\quad n\in\mathbf{N},\ i,j=1,2.$$

Proof. Let $\{\psi_j\}$ be a dense sequence in the space of normal states on $\mathcal{M}$. Passing to subsequences, we assume that

$$\sum_{k=1}^{\infty}\|\psi_\nu e_{i,j}(k)-\lambda^{i-j}e_{i,j}(k)\psi_\nu\|<+\infty;$$

$$\sum_{k=1}^{\infty}\|\psi_\nu f_{i,j}(k)-\lambda^{i-j}f_{i,j}(k)\psi_\nu\|<+\infty$$

for $i,j=1,2$ and $\nu\in\mathbf{N}$, so that the subfactor $\mathcal{P}$ (resp. $\mathcal{Q}$) generated by $\{e_{i,j}(k)|\ k\in\mathbf{N}, i,j=1,2\}$ (resp. $\{f_{i,j}(k)\}$) decomposes $\mathcal{M}$ into a tensor product: $\mathcal{M}=\mathcal{P}\otimes\mathcal{P}^c$ (resp. $\mathcal{M}=\mathcal{Q}\otimes\mathcal{Q}^c$) by [1, Theorem 1.3]. Let ω be a fixed free ultrafilter on $\mathbf{N}$. Since every strongly central sequence of $\mathcal{P}$ (resp. $\mathcal{Q}$) is strongly central in $\mathcal{M}$, $\mathcal{P}_\omega$ and $\mathcal{Q}_\omega$ are both von Neumann subalgebras of $\mathcal{M}_\omega$. Since $\mathcal{P}_\omega$ (resp. $\mathcal{Q}_\omega$) is a factor, all tracial states on $\mathcal{M}_\omega$ take the same values on $\mathcal{P}_\omega$ (resp. $\mathcal{Q}_\omega$). This means that to prove the equivalence of the projection E and F represented respectively by $\{e_{11}(k)\}$ and $\{f_{11}(k)\}$ we need only to show that E and F take the same trace value. Let φ be a faithful normal states on $\mathcal{M}$. Then we have

$$\begin{aligned}\tau_\omega(E)&=\lim_{k\to\omega}\varphi(e_{11}(k))=\lim_{k\to\omega}\varphi(e_{12}(k)\,e_{21}(k))\\&=\lim_{k\to\omega}\lambda^{-1}\varphi(e_{21}(k)\,e_{12}(k))\\&=\frac{1}{\lambda}\tau_\omega(1-E),\end{aligned}$$

so that $\tau_\omega(E)=1/(1+\lambda)$. Similarly, $\tau_\omega(F)=1/(1+\lambda)$. Hence E and F are equivalent in $\mathcal{M}_\omega$.

By induction, we construct sequences $\{k_n\}\subset\mathbf{N}$ and $\{u_n\}\subset\mathcal{U}(\mathcal{M})$ such that

(a) $[u_n, f_{i,j}(k_\nu)]=0$, $\nu=1,2,\dots,n-1$;

(b) with $v_n=u_nu_{n-1}\dots u_i$, $v_\nu e_{i,j}(k_\nu)v_\nu^*=f_{i,j}(k_\nu)$, $\nu=1,2,\dots,n$;

(c) $\|\psi_\nu\cdot\mathrm{Ad}(v_n)-\psi_\nu\cdot\mathrm{Ad}(v_{n-1})\|<2^{-n}$, $\|\psi_\nu\cdot\mathrm{Ad}(v_n^*)-\psi_\nu\cdot\mathrm{Ad}(v_{n-1}^*)\|<2^{-n}$, $\nu=1,2,\dots,n$.

Suppose that $\{k_\nu\}$ and $\{u_\nu\}$ have been constructed for $\nu=1,2,\dots,n-1$. Let

$$\mathcal{N}=\{f_{i,j}(k_\nu)|\ \nu=1,2,\dots,n-1,\ i,j=1,2\}^c.$$

Since $v_{n-1}e_{i,j}(k_\nu)v_{n-1}^*=f_{i,j}(k_\nu)$, $1\leqslant\nu\leqslant n-1$, we have $v_{n-1}e_{i,j}(k)v_{n-1}^*\in\mathcal{N}$ for $k>k_{n-1}$. Let E and F be the projections of $\mathcal{N}_\omega$ considered above, corresponding to

$$\{v_{n-1}e_{11}(k)v_{n-1}^*|\,k>k_{n-1}\}\quad\text{and}\quad\{f_{11}(k)|\,k>k_{n-1}\}.$$

Then $E\sim F$ in $\mathcal{N}_\omega$. Hence there exists a strongly central sequence $\{w_k\}$, passing to a subsequence if necessary, such that

$$w_k^*w_k=v_{n-1}e_{11}(k)v_{n-1}^*,\quad k>k_n;$$

$$w_kw_k^*=f_{11}(k).$$

Put

$$x_k=\sum_{j=1}^{2}f_{j,1}(k)\,w_k\,v_{n-1}e_{i,j}(k)\,v_{n-1}^*.$$

Then $\{x_k\}\subset\mathcal{U}(\mathcal{N})$ is strongly central. If k is sufficiently large, then $u_n=x_k$ satisfies the above (a), (b) and (c).

By (c), $\{\mathrm{Ad}(v_n)\}$ is a Cauchy sequence in $\mathrm{Aut}(\mathcal{M})$. With $\sigma=\lim_{n\to\infty}\mathrm{Ad}(v_n)\in\overline{\mathrm{Int}}(\mathcal{M})$, we have

$$\sigma(e_{i,j}(k_n))=f_{i,j}(k_n).$$

Q.E.D.

COROLLARY 15. *Let $\mathcal{M}$ be a separable factor. Let $\mathcal{P}$ and $\mathcal{Q}$ and $\mathcal{Q}$ be* AFD *subfactors of type* III_λ, $0<\lambda<1$. *If $\mathcal{M}=\mathcal{P}\vee\mathcal{P}^c$ and $\mathcal{M}=\mathcal{Q}\vee\mathcal{Q}^c$ are both tensor product factorizations such that $\mathcal{P}^c\cong\mathcal{Q}^c\cong\mathcal{M}$, then there exists $\sigma\in\overline{\mathrm{Int}}(\mathcal{M})$ such that $\sigma(\mathcal{P})=\mathcal{Q}$.*

Proof. The proof is similar to the first part of the proof of [4, Proposition 2.2.3]. Q.E.D.

Next, we consider a centrally trivial automorphism of the AFD factor of type III_1. We show that the automorphism splits on a tensor product factorization.

LEMMA 16. *Fix* $0<\lambda<1$. *Let* $\mathcal{M}$ *be an* AFD *factor of type* III_1. *For each* $\alpha\in\mathrm{Cnt}(\mathcal{M})$, *there exists a unitary* $a\in\mathcal{M}$ *and a tensor product factorization* $\mathcal{M}=\mathcal{P}_1\bar{\otimes}\mathcal{P}_2$ *such that*

(a) $\mathrm{Ad}(a)\cdot\alpha=\alpha_1\otimes\alpha_2$ *relative to* $\mathcal{P}_1\bar{\otimes}\mathcal{P}_2$;

(b) $\mathcal{P}_1$ *is an* AFD *factor of type* III_λ; α_1 *is of the form* $\mathrm{Ad}(u)\cdot\sigma_{T_1}^{\varphi_1}$ *with* $u\in\mathcal{U}(\mathcal{P}_1)$, $T_1\in\mathbf{R}$ *and* φ_1 *a faithful normal state on* $\mathcal{P}_1$;

(c) $\mathcal{P}_2\cong\mathcal{M}$.

Proof. By assumption, there exists a mutually commuting sequences $\{e_{i,j}(k)\}$ of 2×2-matrix units such that

$$\lim_{k\to\infty}\|\psi e_{i,j}(k)-\lambda^{i-j}e_{i,j}(k)\psi\|=0,$$

for a normal state ψ on $\mathcal{M}$. Passing to a subsequence, we may assume that $\{e_{i,j}(k)\}$ generates an AFD subfactor $\mathcal{P}_1$ of type III_λ such that $\mathcal{M}=\mathcal{P}_1\bar{\otimes}\mathcal{P}_1^c$ and $\mathcal{P}_1^c\cong\mathcal{M}$. Since $\alpha\in\mathrm{Cnt}(\mathcal{M})$, and $\{e_{11}(k)\}$ and $\{e_{22}(k)\}$ are both strongly central, we have

$$\lim_{k\to\infty}\|\alpha(e_{ii}(k))-e_{ii}(k)\|_\varphi^\#=0,\quad i=1,2.$$

We want to show that there exists $\gamma\in\mathbf{C}$, $|\gamma|=1$, such that $\lim_{k\to\infty}\|\alpha(e_{12}(k))-\gamma e_{12}(k)\|_\varphi^\#=0$. First, observe that $\{\alpha(e_{12}(k))e_{21}(k)\}$ is strongly central. Fix a free ultrafilter ω on $\mathbf{N}$, and set

$$E=\pi_\omega(\{e_{11}(k)\}),\quad U=\pi_\omega(\{\alpha(e_{12}(k))e_{21}(k)\}),$$

where π_ω is the canonical map from the C^*-algebra of strongly ω-central sequences onto $\mathcal{M}_\omega$. By Proposition 13, we know that $\mathcal{M}_\omega$ is a factor. To prove that $U=\gamma E$ for some $\gamma\in\mathbf{C}$, $|\gamma|=1$, we show that U is in the center of $\mathcal{M}_{\omega,E}$. Let X be an element of $\mathcal{M}_{\omega,E}$, and represent X by a sequence $\{x(k)\}$ such that $x(k)=e_{11}(k)x(k)e_{11}(k)$. Set $y(k)=e_{21}x(k)e_{12}(k)$. Then $\{y(k)\}$ is strongly ω-central. Now, we see that u and X commute as follows:

$$\begin{aligned}XU&=\pi_\omega(\{x(k)\alpha(e_{12}(k))e_{21}(k)\})\\&=\pi_\omega(\{\alpha(x(k)e_{12}(k))e_{21}(k)\}),\quad\text{since }\alpha\in\mathrm{Cnt}(\mathcal{M}),\\&=\pi_\omega(\{\alpha(e_{12}(k)y(k))e_{21}(k)\})\\&=\pi_\omega(\{\alpha(e_{12}(k))y(k))e_{21}(k)\})\\&=\pi_\omega(\{\alpha(e_{12}(k))e_{21}(k)x(k)\})=UX.\end{aligned}$$

Therefore $U=\gamma E$ for some $\gamma\in\mathbf{C}$, $|\gamma|=1$. This means that we have a subsequence $\{e_{i,j}(k_n)\}$ such that

$$\lim_{n\to\infty}\|\alpha(e_{12}(k_n))-\gamma e_{12}(k_n)\|_\varphi^\# = 0.$$

Passing to a subsequence, we obtain a sequence $\{e_{i,j}(k)\}$ of mutually commuting 2×2-matrix units such that

$$\lim_{k\to\infty}\|\psi e_{i,j}(k)-\lambda^{i-j}e_{i,j}(k)\psi\| = 0;$$

$$\lim_{k\to\infty}\|\alpha(e_{12}(k))-\gamma e_{12}(k)\|_\varphi^\# = 0;$$

$$\lim_{k\to\infty}\|\alpha(e_{21}(k))-\bar{\gamma} e_{21}(k)\|_\varphi^\# = 0;$$

$$\lim_{k\to\infty}\|\alpha(e_{i,i}(k))-e_{i,i}(k)\|_\varphi^\# = 0, \quad i=1,2.$$

We now adopt the arguments of Lemma 14. In the construction of the sequences $\{k_n\}$ and $\{u_n\}$, we require

(a) $[u_n, e_{i,j}(k_\nu)]=0$, $1\leqslant\nu\leqslant n-1$;

(b) with $v_n=u_n u_{n-1}\dots u_1$, $v_\nu\alpha(e_{i,j}(k_\nu))v_\nu^*=\gamma^{j-i}e_{i,j}(k_\nu)$, $1\leqslant\nu\leqslant n$;

(c) $\|u_n-1\|_\varphi^\#\leqslant 1/2^n$.

Condition (c) guarantees the convergence $v=\lim_{n\to\infty}v_n\in\mathcal{U}(\mathcal{M})$ and we have $\mathrm{Ad}(v)\cdot\alpha(e_{i,j}(k_n))=\gamma^{j-i}e_{i,j}(k_n)$.

Now, $\mathrm{Ad}(v)\cdot\alpha$ leaves the von Neumann algebra $\mathcal{P}_1$ generated by $\{e_{i,j}(k_n)|\,n\in\mathbf{N}\}$ globally invariant. If we choose further a subsequence from $\{e_{i,j}(k_n)\}$ denoted by $\{e_{i,j}(k_n)\}$ again, then $\mathcal{P}_1$ factorizes $\mathcal{M}$ and $\mathcal{P}_1^c\cong\mathcal{M}$, and also $\mathcal{P}_1$ is an AFD factor of type III_λ. We know that $\mathrm{Ad}(v)\cdot\alpha$ is of the form $\alpha_1\otimes\alpha_2$ relative to the factorization $\mathcal{M}=\mathcal{P}_1\bar{\otimes}\mathcal{P}_1^c$. Furthermore, if φ_1 is the periodic state on $\mathcal{P}_1$, then α_1 is given by $\sigma_{T_1}^{\varphi_1}$ where $\gamma=\lambda^{-iT_1}$. Q.E.D.

We will make the above splitting twice. The next lemma shows a relation between the two centrally trivial automorphisms obtained by tensor product factorizations.

LEMMA 17. *Suppose $\mathcal{P}$ and $\mathcal{Q}$ are* AFD *factors of type* III_λ *and* III_μ *respectively,* $0<\lambda,\mu<1$, *and that* $\log\lambda/\log\mu\notin\mathbf{Q}$. *Let φ and ψ be respectively faithful normal states on $\mathcal{P}$ and $\mathcal{Q}$. Then* (a) $\sigma_T^\varphi\otimes\mathrm{id}\notin\mathrm{Cnt}(\mathcal{P}\bar{\otimes}\mathcal{Q})$ *unless* $\log\mu=(2\pi n\log\lambda)/(T\log\lambda+2\pi m)$ *for some*

$m, n \in \mathbf{Z}$; (b) *if* $\sigma_T^\varphi \otimes \mathrm{id} \in \mathrm{Cnt}(\mathscr{P} \bar{\otimes} \mathscr{Q})$, *then* $\sigma_T^\varphi \otimes \mathrm{id} \sim \sigma_{T'}^\varphi \otimes \sigma_{T'}^\psi$, *where* $T' = T + 2\pi m/\log\lambda$ (*m as in* (a)).

Proof. We assume $T\log\lambda \notin 2\pi\mathbf{Z}$, otherwise $\sigma_T^\varphi \in \mathrm{Int}(\mathscr{P})$. If

$$\log\mu = (2\pi n \log\lambda)/(T\log\lambda + 2\pi m) \quad \text{for some } m, n \in \mathbf{Z},$$

then we set

$$T' = \frac{2\pi n}{\log\mu} = \frac{T\log\lambda + 2\pi m}{\log\lambda}.$$

It then follows that $\sigma_{T'}^\varphi \equiv \sigma_T^\varphi \bmod(\mathrm{Int}(\mathscr{P}))$ and $\sigma_{T'}^\psi \in \mathrm{Int}(\mathscr{Q})$. We now assume that $\log\mu \neq (2\pi n \log\lambda)/(T\log\lambda + 2\pi m)$ for any $m, n \in \mathbf{Z}$. We will derive a contradiction.

Setting $\mathbf{T} = \mathbf{R}/(2\pi\mathbf{Z})$, we define a subgroup:

$$A = \left\{ \left(2\pi k \frac{\log\lambda}{\log\mu}, kT\log\lambda \right) \middle| \, k \in \mathbf{Z} \right\}$$

of the Cartesian product $\mathbf{T}^2$. Let $B = \bar{A}$. First we show that there exists $x \in \mathbf{T}$, $x \neq 0$, such that $(0, x) \in B$. Suppose that $(0, x) \in B$ implies $x = 0$. Since the projection of A to the first coordinate is a dense subgroup of $\mathbf{T}$ by the irrationality of $\log\lambda/\log\mu$, the projection of B to the first coordinate covers the entire $\mathbf{T}$. Hence the assumption that $(0, x) \notin B$ for any $x \neq 0$ means that B is the graph of a continuous homomorphism of $\mathbf{T}$ into $\mathbf{T}$, so that there exists $n \in \mathbf{Z}$ such that $B = \{(\alpha, n\alpha) | \, \alpha \in \mathbf{T}\}$. In particular, we have

$$T\log\lambda = 2\pi n \frac{\log\lambda}{\log\mu} - 2\pi m$$

for some $m \in \mathbf{Z}$, which means precisely that $\log\mu = (2\pi n \log\lambda)/(T\log\lambda + 2\pi m)$, the case we have excluded. Thus, there exists a non-zero $x \in \mathbf{T}$ such that $(0, x) \in B$. We choose and fix such an $x \in \mathbf{T}$. Since $B = \bar{A}$, there exist two sequences $k(n), l(n)$ of integers such that

$$2\pi k(n) \frac{\log\lambda}{\log\mu} - 2\pi l(n) \to 0 \quad \text{in } \mathbf{R};$$

$$k(n) T\log\lambda \to x \quad \text{in } \mathbf{T}.$$

We may take both $k(n)$ and $l(n)$ in $\mathbf{N}$. Note that the above two convergences mean that $\lambda^{k(n)}\mu^{-l(n)} \to 1$ and $\lambda^{iTk(n)} \to e^{ix} \neq 1$. Let $\mathscr{P}_0$ and $\mathscr{Q}_0$ be AFD factors of type III_λ and III_μ respectively. Choose faithful normal states φ_0 on $\mathscr{P}_0$ and ψ_0 on $\mathscr{Q}_0$ such that their modular automorphism groups σ^{φ_0} and σ^{ψ_0} have respectively the period $-2\pi/\log\lambda$ and

$-2\pi/\log\mu$. It then follows that the centralizers $\mathscr{P}_{0,\varphi_0}$ and $\mathscr{Q}_{0,\psi_0}$ have both trivial relative commutants. Suppose $k, l \in \mathbf{N}$ are given. Then there exist isometries $u_1 \in \mathscr{P}_0$ and $v_1 \in \mathscr{Q}_0$ such that

$$\varphi_0 u_1 = \lambda^k u_1 \varphi_0, \quad u_1^* u_1 = 1, \quad u_1 u_1^* = e_1 \in \mathscr{P}_{0,\varphi_0};$$

$$\psi_0 v_1 = \mu^l v_1 \psi_0, \quad v_1^* v_1 = 1, \quad v_1 v_1^* = f_1 \in \mathscr{P}_{0,\psi_0};$$

In the above procedure, the projections $e_1 \in \mathscr{P}_{0,\varphi_0}$ and $f_1 \in \mathscr{Q}_{0,\psi_0}$ can be arbitrary subject to the condition: $\varphi_0(e_1)=\lambda^k$ and $\psi_0(f_1)=\mu^l$. Considering the reduced algebras, $\mathscr{P}_{0,1-e_1}$ and $\mathscr{Q}_{0,1-e_1}$ and repeating the same process inductively, we obtain sequences of partial isometries $\{u_n\}\subset\mathscr{P}_0$ and $\{v_n\}\subset\mathscr{Q}_0$ such that with $e_n=u_nu_n^*$ and $f_n=v_nv_n^*$,

(1) $\{e_n\}$ and $\{f_n\}$ are both orthogonal sequences in $\mathscr{P}_{0,\varphi_0}$ and $\mathscr{Q}_{0,\psi_0}$ respectively;

(2) $\varphi_0 u_n=\lambda^k u_n \varphi_0$, $u_n^*u_n=1-\Sigma_{j=1}^{n-1} e_j$, $\psi_0 v_n=\mu^l v_n \psi_0$, $v_n^* v_n=1-\Sigma_{j=1}^{n-1} f_j$.

Set $w=\Sigma_{n=1}^{\infty} u_n\otimes v_n^* \in \mathscr{P}_0\bar{\otimes}\mathscr{Q}_0$. Then we have

$$(\varphi_0\otimes\psi_0)\, w = \lambda^k\mu^{-l} w(\varphi_0\otimes\psi_0),$$

$$(\sigma_T^{\varphi_0}\otimes\mathrm{id})(w) = \lambda^{ikT} w.$$

Since $\varphi_0(e_n)=\lambda^k(1-\lambda^k)^{n-1}$ and $\psi_0(f_n)=\mu^l(1-\mu^l)^{n-1}$, we have

$$\begin{aligned}(\|w\|^{\#}_{\varphi_0\otimes\psi_0})^2 &= \frac{1}{2}(\varphi_0\otimes\psi_0)\left(\sum_{n=1}^{\infty}(u_n^*u_n\otimes v_nv_n^* + u_nu_n^*\otimes v_n^*v_n)\right)\\ &= \frac{1}{2}\sum_{n=1}^{\infty}((1-\lambda^k)^{n-1}\mu^l(1-\mu^l)^{n-1}+\lambda^k(1-\lambda^k)^{n-1}(1-\mu^l)^{n-1})\\ &= \frac{1}{2}\frac{\lambda^k+\mu^l}{2\lambda^k+\mu^l-\lambda^k\mu^l} \geqslant \frac{1}{2}.\end{aligned}$$

With $\mathscr{P}_n=\mathscr{P}_0$, $\mathscr{Q}_n=\mathscr{Q}_0$, $\varphi_n=\varphi_0$, and $\psi_n=\psi_0$, we regard

$$\{\mathscr{P},\varphi\} = \prod_{n=1}^{\infty}{}^{\otimes}\{\mathscr{P}_n,\varphi_n\}, \quad \{\mathscr{Q},\psi\} = \prod_{n=1}^{\infty}{}^{\otimes}\{\mathscr{Q}_n,\psi_n\}.$$

For the sequences $k(n)$ and $l(n)$ obtained in the first part of the proof, we apply the above construction to get $w(n)$ in the nth factor $\mathscr{P}_n\bar{\otimes}\mathscr{Q}_n\subset\mathscr{P}\bar{\otimes}\mathscr{Q}$. Because

$$(\varphi\otimes\psi)\, w(n) = \lambda^{k(n)}\mu^{-l(n)} w(n)(\varphi\otimes\psi),$$

$\lambda^k(n)\mu^{-l(n)}\to 1$, $w(n)\in\mathscr{P}_n\bar{\otimes}\mathscr{Q}_n$, and $\|w(n)\|^{\#}_{\varphi\otimes\psi}\geq 1/\sqrt{2}$, we get a non-zero strongly central sequence $\{w(n)\}$ in $\mathscr{P}\bar{\otimes}\mathscr{Q}$. But $(\sigma^\varphi_T\otimes\mathrm{id})(w(n))=\lambda^{ik(n)T}w(n)$ and $\lambda^{ik(n)T}\to e^{ix}\neq 1$, contradicting the assumption $\sigma^\varphi_T\otimes\mathrm{id}\in\mathrm{Cnt}(\mathscr{P}\bar{\otimes}\mathscr{Q})$. Q.E.D.

§ 3. Proof of Theorem 1

We know $\overline{\mathrm{Int}}(\mathscr{M})\subset\mathrm{Ker}(\mathrm{mod})$ by [11] and $\mathrm{Ad}(u)\cdot\bar{\sigma}^\varphi_c\in\mathrm{Cnt}(\mathscr{M})$ by [3]. Thus we need only prove the other implications.

We handle three cases of type III_0, III_λ $(0<\lambda<1)$, and III_1, separately.

Proof of Theorem 1 *for* AFD *factors of type* III_0. (i) If α is in Ker(mod), then we may assume the three properties in Lemma 2. Because ψ on $\mathscr{N}$ is a semifinite α-invariant trace and $\mathscr{N}$ is isomorphic to $L^\infty(X)\bar{\otimes}\mathscr{R}_{0,1}$, where $\mathscr{R}_{0,1}$ is the AFD factor of type II_∞, we know $\alpha|_\mathscr{N}\in\overline{\mathrm{Int}}(\mathscr{N})$ by Corollary 6 in [4]. Thus there exists a sequence $\{u_n\}$ of unitaries in $\mathscr{N}$ such that $\alpha|_\mathscr{N}=\lim_{n\to\infty}\mathrm{Ad}(u_n)$. Since $\theta\alpha=\alpha\theta$, $\{u_n\theta(u_n)^*\}$ is in $\mathscr{N}_\omega$ so there exists a sequence $\{v_n\}\subset\mathscr{N}_\omega$ such that $\pi_\omega(\{u_n\theta(u_n)^*\})=\pi_\omega(\{v_n\theta(v_n)^*\})$ in $\mathscr{N}_\omega$ by Lemma 4, where π_ω is as in the proof of Lemma 16. Replacing $\{u_n\}$ by $\{v_n^*u_n\}$ and choosing a subsequence, we may assume $\alpha|_\mathscr{N}=\lim_{n\to\infty}\mathrm{Ad}(u_n)$, $u_n-\theta(u_n)$ approaches zero $*$-strongly, and $u_n\in\mathscr{U}(\mathscr{N})$. We prove $\alpha=\lim_{n\to\infty}\mathrm{Ad}(u_n)$ in $\mathrm{Aut}(\mathscr{M})$. It suffices to prove that $\|\varphi\cdot\alpha-u_n^*\varphi u_n\|\to 0$ and $\|\varphi\cdot\alpha^{-1}-u_n\varphi u_n^*\|\to 0$ for a dense subset of φ in $\mathscr{M}_*$. Let $\mathscr{E}_0$ be the normal conditional expectation of $\mathscr{M}$ onto $\mathscr{N}$. We also define $\mathscr{E}_k(x)=\mathscr{E}_0(xU^{-k})U^k$, $k\in\mathbf{Z}$. Then $\{\varphi\cdot\mathscr{E}_k|\ \varphi\in\mathscr{M}_*,\ k\in\mathbf{Z}\}$ is total in $\mathscr{M}_*$. Fix $x\in\mathscr{M}$ and $k\in\mathbf{Z}$. Setting $z_k=\mathscr{E}_0(xU^{-k})\in\mathscr{N}$, we have

$$\begin{aligned}|\langle x,\varphi\cdot\mathscr{E}_k\cdot\alpha-u_n^*(\varphi\cdot\mathscr{E}_k)u_n\rangle| &= |\langle\alpha(x)-u_nxu_n^*,\varphi\cdot\mathscr{E}_k\rangle|\\ &= |\langle\alpha(z_k)U^k-u_nz_kU^ku_n^*,\varphi\rangle|\\ &= |\langle(\alpha(z_k)-u_nz_ku_n^*u_n\theta^k(u_n^*))U^k,\varphi\rangle|\\ &\leq |\langle\alpha(z_k)-u_nz_ku_n^*,U^k\varphi\rangle|+|\langle u_nz_ku_n^*,(u_n\theta^k(u_n^*)-1)U^k\varphi\rangle|\\ &\leq \|U^k\varphi|_\mathscr{N}\cdot(\alpha-\mathrm{Ad}(u_n))\|\cdot\|z_k\|+\|(u_n\theta^k(u_n^*)-1)U^k\varphi\|\cdot\|z_k\|,\end{aligned}$$

which converges to zero uniformly in x with $\|x\|\leq 1$. The other convergence follows similarly. This completes the proof.

(ii) By Lemma 5 in [4] and Lemma 12, we know that $\alpha|_{\mathscr{M}_\psi}$ is inner. So by inner perturbation, we get $\alpha|_{\mathscr{M}_\psi}=\mathrm{id}$. Thus α must be an extended modular automorphism, up to inner perturbation, by Theorem 3.1 and Theorem 5.5 in Haagerup–Størmer [14]. Q.E.D.

Next we consider AFD factors of type III_λ $(0<\lambda<1)$. We need a lemma.

LEMMA 18. *Let* $\mathcal{R}_{0,1}$ *be the* AFD *factor of type* II_∞. *If* β *is an action of a discrete countable abelian group* G *such that* $\beta^{-1}(\mathrm{Cnt}(\mathcal{M}))=H$, *then for any free ultrafilter* ω *on* $\mathbf{N}$ *and any character* $p\in(G/H)^\wedge=H^\perp$, *there exists* $x\in(\mathcal{R}_{0,1})_\omega$, $x\neq 0$ *such that* $\beta_g^\omega(x)=\langle g,p\rangle x$, $g\in G$.

Proof. By Theorem 2.9 in [18], an appropriate product type action of G/H on the AFD factor of type II_1 splits from β as a tensor product factor. Q.E.D.

Proof of Theorem 1 *for* AFD *factors of type* III_λ $(0<\lambda<1)$. (i) By [11, p. 554], we know that $\overline{\mathrm{Int}}(\mathcal{M})\subset\mathrm{Ker}(\mathrm{mod})$. Suppose $\mathrm{mod}(\alpha)=1$, $\alpha\in\mathrm{Aut}(\mathcal{M})$. Then for a lacunary weight φ, we have $\varphi\cdot\alpha\cdot\mathrm{Ad}(u)=\varphi$ for some $u\in\mathcal{U}(\mathcal{M})$. Replacing α by $\alpha\cdot\mathrm{Ad}(u)$, we assume $\varphi\cdot\alpha=\varphi$, which implies that α and $\{\sigma_t^\varphi\}$ commute. Hence $\alpha(\mathcal{N})=\mathcal{N}$ in the discrete decomposition $\mathcal{M}=\mathcal{N}\rtimes_\theta\mathbf{Z}$, $\mathcal{N}\cong\mathcal{R}_{0,1}$, and $\alpha(U)U^*=v\in\mathcal{N}$ for the implementing unitary U. By the stability of θ again, there exists $w\in\mathcal{U}(\mathcal{N})$ with $v=w^*\theta(w)$, which means that $\mathrm{Ad}(w)\cdot\alpha$ leaves U fixed. Replace α again by $\mathrm{Ad}(w)\cdot\alpha$, so that $\varphi\cdot\alpha=\varphi$ and $\alpha(U)=U$. Since $\varphi|_\mathcal{N}=\tau$, $\mathrm{mod}(\alpha|_\mathcal{N})=1$ so that $\alpha_0=\alpha|_\mathcal{N}\in\overline{\mathrm{Int}}(\mathcal{N})$ by Corollary 6 in [4]. Let $\{u_n\}$ be a sequence in $\mathcal{U}(\mathcal{N})$ such that $\alpha_0=\lim_{n\to\infty}\mathrm{Ad}(u_n)$ in $\mathrm{Aut}(\mathcal{N})$. Since θ and α_0 commute, we have also $\alpha_0=\lim_{n\to\infty}\mathrm{Ad}(\theta(u_n))$. Hence $\{u_n^*\theta(u_n)\}$ is strongly central in $\mathcal{N}$. By Theorem 2.1.3 in [4], there exists a strongly central sequence $\{v_n\}$ such that $\{u_n^*\theta(u_n)\}\sim\{v_n^*\theta(v_n)\}$. Hence we have the $*$-strong convergence of $\{u_nv_n^*-\theta(u_nv_n^*)\}$ to zero, and

$$\lim_{n\to\infty}\mathrm{Ad}(u_nv_n^*)=\lim_{n\to\infty}\mathrm{Ad}(u_n)\,\mathrm{Ad}(v_n^*)=\alpha_0.$$

By an argument similar to the type III_0 case, we get $\lim_{n\to\infty}\mathrm{Ad}(u_nv_n^*)=\alpha$ in $\mathrm{Aut}(\mathcal{M})$.

(ii) We know the inclusion: $\sigma^\varphi(\mathbf{R})\cdot\mathrm{Int}(\mathcal{M})\subset\mathrm{Cnt}(\mathcal{M})$ by [3, Proposition 2.3]. Suppose $\alpha\in\mathrm{Cnt}(\mathcal{M})$. Let $\varphi=\hat{\tau}$ for a trace τ on $\mathcal{N}$. We first prove that $\mathrm{mod}(\alpha)=1$. Suppose that $\mathrm{mod}(\alpha)\neq1$. Then we have $\varphi\cdot\alpha\sim\mu\varphi$ for some $\lambda<\mu<1$, i.e., $F_{-\log\mu}=\mathrm{mod}(\alpha)$. Thus $\varphi\cdot\alpha\cdot\mathrm{Ad}(u)=\mu\varphi$ for some $u\in\mathcal{U}(\mathcal{M})$. Replacing α by $\alpha\cdot\mathrm{Ad}(u)$, we may assume $\varphi\cdot\alpha=\mu\varphi$, $\lambda<\mu<1$. It then follows that α and $\{\sigma_t^\varphi\}$ commute, so that $v=\alpha(U)U^*\in\mathcal{N}$. As seen in the proof of (1), $v=w^*\theta(w)$ for some $w\in\mathcal{U}(\mathcal{N})$ and $\mathrm{Ad}(w)\cdot\alpha$ leaves U fixed. Replacing α by $\mathrm{Ad}(w)\cdot\alpha$, we can assume that α is the canonical extension of $\alpha_0=\alpha|_\mathcal{N}$, i.e., $\alpha(U)=U$. Since $\varphi\cdot\alpha=\mu\varphi$, $\tau\cdot\alpha_0=\mu\tau$. Furthermore, α_0 and $\theta=\theta_{-\log\lambda}$ commute. Since $\mathrm{mod}(\alpha_0)\notin(\log\lambda)\mathbf{Z}$, $\alpha_{0,\omega}$ is not trivial on the fixed point subalgebra $(\mathcal{N}_\omega)^\theta$ of $\mathcal{N}_\omega$ for a free ultrafilter ω on $\mathbf{N}$ by Lemma 18. But this means by Lemma 7 that α does not belong to $\mathrm{Cnt}(\mathcal{M})$. Therefore we have proved $\mathrm{mod}(\alpha)=1$.

After all, we come to the situation that $\varphi\cdot\alpha=\varphi$ and $\alpha(U)=U$. We claim that α_0 is inner. Suppose that $\alpha_0\notin\mathrm{Int}(\mathcal{N})$. Let $\beta_{m,n}=\alpha_0^n\theta^m$, $(n,m)\in\mathbf{Z}^2$. By the assumption, $\mathbf{Z}\times\{0\}\not\subset\beta^{-1}(\mathrm{Int}(\mathcal{N}))=H$. By Lemma 18, $\alpha_{0,\omega}$ cannot be trivial on $(\mathcal{N}_\omega)^\theta$, which means $\alpha\notin\mathrm{Cnt}(\mathcal{M})$. Thus $\alpha_0=\mathrm{Ad}(u)$ for some $u\in\mathcal{U}(\mathcal{N})$. Since θ and α_0 commute, $\theta(u)=\lambda^{is}u$ for some $s\in\mathbf{R}$, so that $\mathrm{Ad}(u)U=\lambda^{-is}U$. Hence $\mathrm{Ad}(u^*)\cdot\alpha$ is trivial on $\mathcal{N}$ and $\mathrm{Ad}(u^*)\cdot\alpha(U)=\lambda^{is}U$. Therefore we conclude that $\mathrm{Ad}(u^*)\cdot\alpha=\sigma_s^\varphi$. Hence $\alpha=\mathrm{Ad}(u)\cdot\sigma_s^\varphi$. Q.E.D.

We finally turn to the III_1 case. We use splitting factors of type III_λ, $0<\lambda<1$.

Proof of Theorem 1 *for the* AFD *factors of type* III_1. (i) Let $\alpha\in\mathrm{Aut}(\mathcal{M})$ and $\mathcal{M}$ be an AFD factor of type III_1. Fix λ, $0<\lambda<1$. Since $\mathcal{M}$ is strongly λ-stable, it contains a sequence $\{e_{i,j}(k)|\ 1\leq i,j\leq 2,\ k\in\mathbf{N}\}$ of mutually commuting 2×2-matrix units such that

$$\lim_{k\to\infty}\|\psi e_{i,j}(k)-\lambda^{i-j}e_{i,j}(k)\psi\|=0,$$

for every normal state ψ on $\mathcal{M}$ by [1, Theorem 1.3]. Apply Lemma 14 to $\{e_{i,j}(k)\}$ and $\{f_{i,j}(k)\}$ with $f_{i,j}(k)=\alpha(e_{i,j}(k))$ to find $\sigma\in\overline{\mathrm{Int}}(\mathcal{M})$ and a sequence $\{k_n\}\subset\mathbf{N}$ such that $\sigma(e_{ij}(k_n))=\alpha(e_{i,j}(k_n))$. To prove $\alpha\in\overline{\mathrm{Int}}(\mathcal{M})$, we may replace α by $\sigma^{-1}\alpha$ since $\overline{\mathrm{Int}}(\mathcal{M})$ is a subgroup. Then we come to the situation $\alpha(e_{i,j}(k_n))=e_{i,j}(k_n)$. Considering further a subsequence, we may assume that $\{e_{i,j}(k_n)\}$ generates a subfactor $\mathcal{P}$ of type III_λ which factorizes $\mathcal{M}$ tensorially in such a way that $\mathcal{M}\cong\mathcal{P}^c$. With the decomposition $\mathcal{M}=\mathcal{P}\bar{\otimes}\mathcal{P}^c$, α is of the form: $\alpha=\mathrm{id}_{\mathcal{P}}\otimes\alpha|_{\mathcal{P}^c}$. Repeating the same arguments for $\{\mathcal{P}^c,\alpha|_{\mathcal{P}^c}\}$ with $0<\mu<1$ such that $\log\lambda/\log\mu\notin\mathbf{Q}$, and obtain a decomposition of an $\overline{\mathrm{Int}}(\mathcal{M})$-perturbation of α:

$$\mathcal{M}=\mathcal{P}\bar{\otimes}\mathcal{Q}\bar{\otimes}\mathcal{N},\quad \mathcal{N}\cong\mathcal{M};$$

$$\alpha=\mathrm{id}_{\mathcal{P}\bar{\otimes}\mathcal{Q}}\otimes\alpha|_{\mathcal{M}}.$$

We know, however, that $\mathcal{P}\bar{\otimes}\mathcal{Q}\cong\mathcal{M}$. Therefore, α is, modulo $\overline{\mathrm{Int}}(\mathcal{M})$, of the form: $\mathcal{M}\cong\mathcal{M}\bar{\otimes}\mathcal{M}$, $\alpha\sim\alpha_1\otimes\mathrm{id}$. Let $\mathcal{M}_n$ be the replica of $\mathcal{M}$ and write

$$\mathcal{M}=\prod_{n=1}^{\infty}{}^{\otimes}\{\mathcal{M}_n,\varphi_n\},$$

where $\varphi_n=\varphi$ is a fixed faithful normal state on $\mathcal{M}_n$ for ech n, and choose an approximately inner automorphism σ_n, which exchanges $\mathcal{M}_1$ and $\mathcal{M}_n$ and leaves the other components fixed (see [12, Lemma 2.1]). We assume that α is of the form $\alpha_1\otimes\mathrm{id}$, where $\alpha_1\in\mathrm{Aut}(\mathcal{M}_1)$ and id acts on $\Pi^{\otimes\infty}_{n=2}\mathcal{M}_n$. Since $\overline{\mathrm{Int}}(\mathcal{M})$ is a normal subgroup of $\mathrm{Aut}(\mathcal{M})$, $\alpha\sigma_n\alpha^{-1}\sigma_n^{-1}$ belongs to $\overline{\mathrm{Int}}(\mathcal{M})$. But $\alpha\sigma_n\alpha^{-1}\sigma_n^{-1}$ is of the form:

$$\alpha\sigma_n\alpha^{-1}\sigma_n^{-1}=\alpha_1\otimes\mathrm{id}\otimes\alpha_1^{-1}\otimes\mathrm{id},$$

where α_1^{-1} appears on the nth component. Therefore, it remains only to prove that for any $\alpha_1\in\mathrm{Aut}(\mathcal{M}_1)$ and a decomposition

$$\mathcal{M}=\prod_{n=1}^{\infty}{}^{\otimes}\{\mathcal{M}_n,\varphi_n\}$$

such that $\mathcal{M}_n=\mathcal{M}_1\cong\mathcal{M}$ and $\varphi_n=\varphi$, there exists a sequence $\{u_n\}\subset\mathcal{U}(\mathcal{M}_1)$ such that $\lim_{n\to\infty}\sigma_n\cdot((\mathrm{Ad}(u_n)\cdot\alpha_1)\otimes\mathrm{id})\cdot\sigma_n^{-1}=\mathrm{id}$ in $\mathrm{Aut}(\mathcal{M})$, because this will show

$$\alpha_1\otimes\mathrm{id}=\lim_{n\to\infty}(\alpha_1\otimes\mathrm{id})\cdot\sigma_n\cdot(\alpha_1^{-1}\otimes\mathrm{id})\cdot\sigma_n^{-1}\cdot\sigma_n\cdot(\mathrm{Ad}(u_n^*)\otimes\mathrm{id})\cdot\sigma_n^{-1}\in\overline{\mathrm{Int}}(\mathcal{M}).$$

By the density of the orbit of φ under $\mathrm{Int}(\mathcal{M}_1)$ by [10, Theorem 4], there exists a sequence $\{u_n\}\subset\mathcal{U}(\mathcal{M}_1)$ such that $\|\varphi\cdot\mathrm{Ad}(u_n)\cdot\alpha_1-\varphi\|<1/2^n$. In the space of normal states on $\mathcal{M}$, the set of states of the form: $\psi\otimes\Pi^{\otimes\infty}_{n=N+1}\varphi_n$ with ψ a normal state on $\Pi^{\otimes n}_{k=1}\mathcal{M}_k$ is dense. Then we have for $n>N$,

$$\left\|\psi\otimes\prod_{k=N+1}^{\infty}{}^{\otimes}\varphi_k-\left(\psi\otimes\prod_{k=N+1}^{\infty}{}^{\otimes}\varphi_k\right)\cdot\sigma_n((\mathrm{Ad}(u_n)\cdot\alpha)\otimes\mathrm{id})\cdot\sigma_n^{-1}\right\|$$
$$=\|\varphi-\varphi\cdot\mathrm{Ad}(u_n)\cdot\alpha\|<\frac{1}{2^n}\to 0.$$

This shows that $\lim_{n\to\infty}\sigma_n\cdot((\mathrm{Ad}(u_n)\cdot\alpha_1)\otimes\mathrm{id})\cdot\sigma_n^{-1}=\mathrm{id}$.

(ii) Let $\mathcal{M}$ be an AFD factor of type III_1 and $\alpha\in\mathrm{Cnt}(\mathcal{M})$. By Lemma 16, there exists an AFD subfactor $\mathcal{P}_1$ of type III_λ such that $\mathcal{M}=\mathcal{P}_1\bar{\otimes}\mathcal{P}_1^c$, $\mathcal{P}_1^c\cong\mathcal{M}$ and $\alpha\sim\alpha_1\otimes\alpha'$ where "$\sim$" means congruence modulo $\mathrm{Int}(\mathcal{M})$. Since $\alpha_1\otimes\alpha'$ is in $\mathrm{Cnt}(\mathcal{M})$, we have $\alpha_1\in\mathrm{Cnt}(\mathcal{P}_1)$ and $\alpha'\in\mathrm{Cnt}(\mathcal{P}_1^c)$. Applying the same arguments to α', we obtain a factorization of $\{\mathcal{P}_1^c,\alpha'\}$:

$$\mathcal{P}_1^c=\mathcal{P}_2\bar{\otimes}\mathcal{Q},\quad \mathcal{Q}\cong\mathcal{M},\quad \alpha'\sim\alpha_2\otimes\beta,$$

where $\mathscr{P}_2$ is an AFD subfactor of type III_μ with $\log\lambda/\log\mu \notin \mathbf{Q}$. Thus we obtain a factorization:

$$\mathcal{M} = \mathscr{P}_1 \bar{\otimes} \mathscr{P}_2 \bar{\otimes} \mathscr{Q}, \quad \alpha \sim \alpha_1 \otimes \alpha_2 \otimes \beta.$$

Since $\alpha_1 \in \mathrm{Cnt}(\mathscr{P}_1)$ and $\alpha_2 \in \mathrm{Cnt}(\mathscr{P}_2)$, we have

$$\alpha_1 \sim \sigma_{T_1}^{\varphi_1} \quad \text{and} \quad \alpha_2 \sim \sigma_{T_2}^{\varphi_2},$$

where φ_1 and φ_2 are respectively normal states on $\mathscr{P}_1$ and $\mathscr{P}_2$. Since $\alpha_1 \otimes \alpha_2 \in \mathrm{Cnt}(\mathscr{P}_1 \bar{\otimes} \mathscr{P}_2)$, $\sigma_{T_1}^{\varphi_1} \otimes \sigma_{T_2}^{\varphi_2}$ must belong to $\mathrm{Cnt}(\mathscr{P}_1 \bar{\otimes} \mathscr{P}_2)$. Since $\sigma_{-T_2}^{\varphi_1} \otimes \sigma_{-T_2}^{\varphi_2} = \sigma_{-T_2}^{\varphi_1 \otimes \varphi_2}$ is an element of $\mathrm{Cnt}(\mathscr{P}_1 \bar{\otimes} \mathscr{P}_2)$, $\sigma_{T_1 - T_2}^{\varphi_1} \otimes \mathrm{id}$ must belong to $\mathrm{Cnt}(\mathscr{P}_1 \bar{\otimes} \mathscr{P}_2)$, which means that we may assume $T_1 = T_2$ by Lemma 17. Hence we get the decomposition: $\alpha \sim \sigma_T^{\varphi_1} \otimes \sigma_T^{\varphi_2} \otimes \beta$. Since $\mathcal{M} \cong \mathscr{P}_1 \bar{\otimes} \mathscr{P}_2$, with $\varphi = \varphi_1 \otimes \varphi_2$ we come to the situation that $\mathcal{M} = \mathscr{P} \bar{\otimes} \mathscr{Q}, \alpha \sim \sigma_T^{\varphi} \otimes \beta$, and $\mathcal{M} \cong \mathscr{P} \cong \mathscr{Q}$. Now, let σ be the flip of $\mathscr{P} \bar{\otimes} \mathscr{Q}$ after identifying $\mathscr{P}$ and $\mathscr{Q}$ i.e., $\sigma(x \otimes y) = y \otimes x$. Since $\sigma \in \overline{\mathrm{Int}}(\mathcal{M})$, and $\mathrm{Cnt}(\mathcal{M})$ and $\overline{\mathrm{Int}}(\mathcal{M})$ commute modulo $\mathrm{Int}(\mathcal{M})$, we have

$$\sigma_T^{\varphi} \cdot \beta^{-1} \otimes \beta \cdot \sigma_{-T}^{\varphi} = (\sigma_T^{\varphi} \otimes \beta) \cdot \sigma \cdot (\sigma_T^{\varphi} \otimes \beta)^{-1} \cdot \sigma \in \mathrm{Int}(\mathcal{M}),$$

which means that $\sigma_T^{\varphi} \sim \beta$. Thus, we finally conclude

$$\alpha \sim \sigma_T^{\varphi} \otimes \beta \sim \sigma_T^{\varphi} \otimes \sigma_T^{\varphi} = \sigma_T^{\varphi \otimes \varphi}.$$

This completes the proof. Q.E.D.

Remark 19. Haagerup–Størmer proved in Theorem 5.5 of [14] that an automorphism α of a general separable factor $\mathcal{M}$ of type III_λ, $0 \leq \lambda < 1$, is pointwise inner if and only if there is a unitary $u \in \mathcal{M}$ and an extended modular automorphism $\bar{\sigma}_c^{\psi}$ such that $\alpha = \mathrm{Ad}(u) \cdot \bar{\sigma}_c^{\psi}$. Together with Theorem 1(ii) here, thus we know that an automorphism an AFD factor of type III_λ, $0 \leq \lambda < 1$, is centrally trivial if and only if it is pointwise inner.

§ 4. Actions of discrete abelian or finite groups

As an application of Theorem 1, we will classify actions of discrete (countable) abelian or finite groups on the AFD factor of type III_1, up to cocycle conjugacy. This completes the final step of the classification program of actions of such groups on injective factors initiated by Ocneanu [18] and Sutherland–Takesaki [20], though the classification of discrete amenable (non-abelian) group actions on the AFD factor of type III_1 is still open. This result will be used for the conjugacy classification of compact abelian group actions on AFD factors in Kawahigashi–Takesaki [17].

Let G be a discrete countable group, and α be an action of G on the AFD factor $\mathcal{M}$ of type III_1. Let $N=N(\alpha)=\alpha^{-1}(\mathrm{Cnt}(\mathcal{M}))$, then N is a normal subgroup of G and we can define $\chi_\alpha \in \Lambda(G, N, \mathbf{T})$ and a homomorphism $\nu_\alpha: N \to \mathbf{R}$ as on page 437 in [20]. (Here ν is actually a homomorphism into $\mathbf{R}$ because the flow of weights is now trivial.) Then we get the following theorem, corresponding to Theorem 5.9 in [20]. (For terminology and notations, see [16] and [20].)

THEOREM 20. *Let $\mathcal{M}$ be the* AFD *factor of type* III_1, *and let α, β be actions of a discrete countable group G on $\mathcal{M}$. Then if G is either abelian or finite, α and β are cocycle conjugate if and only if*

(1) $N(\alpha)=N(\beta)$;

(2) $(\chi_\alpha, \nu_\alpha)=(\chi_\beta, \nu_\beta)$.

We need the following lemma first.

LEMMA 21. *Let α be an action of a group G on a factor $\mathcal{M}$ of type* III, *and φ, ψ be α-invariant dominant weights on $\mathcal{M}$. Then for a homomorphism $\nu: G \to \mathbf{R}$, $\alpha_g \cdot \sigma^\varphi_{\nu(g)}$ is cocycle conjugate to $\alpha_g \cdot \sigma^\psi_{\nu(g)}$.*

Proof. Note that $\alpha_g \cdot \sigma^\varphi_{\nu(g)}$ and $\alpha_g \cdot \sigma^\psi_{\nu(g)}$ are actually G-actions because the modular automorphism groups commute with α. Since φ and ψ are both dominant, there is a unitary $v \in \mathcal{M}$ such that $\psi=\varphi \cdot \mathrm{Ad}(v)$. We have

$$\begin{aligned}\varphi \cdot \mathrm{Ad}(v) = \psi = \psi \cdot \alpha_g^{-1} &= \varphi \cdot \mathrm{Ad}(v) \cdot \alpha_g^{-1} \\ &= \varphi \cdot \alpha_g^{-1} \cdot \mathrm{Ad}(\alpha_g(v)) = \varphi \cdot \mathrm{Ad}(\alpha_g(v)),\end{aligned}$$

thus $v\alpha_g(v^*) \in \mathcal{M}_\varphi$. Then

$$\begin{aligned}\alpha_g \cdot \sigma^\psi_{\nu(g)} &= \alpha_g \cdot \mathrm{Ad}(v^*)\, \sigma^\varphi_{\nu(g)} \cdot \mathrm{Ad}(v) \\ &= \mathrm{Ad}(\alpha_g(v^*)) \cdot \alpha_g \cdot \sigma^\varphi_{\nu(g)} \cdot \mathrm{Ad}(v) \\ &= \mathrm{Ad}(v^*) \cdot \mathrm{Ad}(v\alpha_g(v^*)) \cdot \alpha_g \cdot \sigma^\varphi_{\nu(g)} \cdot \mathrm{Ad}(v).\end{aligned}$$

Here $v\alpha_g(v^*)$ is an α-cocycle, but this is also an $\alpha \cdot \sigma^\varphi$-cocycle because $v\alpha_g(v^*) \in \mathcal{M}_\varphi$. This shows the desired cocycle conjugacy. Q.E.D.

Proof of Theorem 20. The necessity of the two conditions follow from Proposition 5.7 in [20]. Thus we prove the other implication. We write ν for $\nu_\alpha=\nu_\beta$ and extend this to a homomorphism of G into $\mathbf{R}$. This is possible when G is finite as ν is then trivial, and also when G is discrete abelian by divisibility of $\mathbf{R}$, and we denote the extension by ν again. Choose an α-invariant dominant weight φ and a β-invariant dominant weight ψ by Lemma 5.10 in [20]. Define two new actions by $\tilde{\alpha}_g=\alpha_g\cdot\sigma^\varphi_{-\nu(g)}$ and $\tilde{\beta}_g=\beta_g\cdot\sigma^\psi_{-\nu(g)}$. These are actually actions by the invariance of φ,ψ. Now we have

$$\tilde{\alpha}^{-1}(\mathrm{Int}(\mathcal{M}))=\tilde{\alpha}^{-1}(\mathrm{Cnt}(\mathcal{M}))=\tilde{\beta}^{-1}(\mathrm{Int}(\mathcal{M}))=\tilde{\beta}^{-1}(\mathrm{Cnt}(\mathcal{M}))=N(\alpha)=N(\beta),$$

and $\chi_{\tilde{\alpha}}=\chi_\alpha=\chi_\beta=\chi_{\tilde{\beta}}$, thus by Theorem 2.7 in [18], $\tilde{\alpha}$ and $\tilde{\beta}$ are cocycle conjugate. Then the second dual actions $\tilde{\alpha}\otimes\mathrm{Ad}(\varrho_g)$ and $\tilde{\beta}\otimes\mathrm{Ad}(\varrho_g)$ are conjugate on $\mathcal{M}\bar{\otimes}\mathcal{L}(l^2(G))$, where ϱ denotes the right regular representation of G. Thus there exists an automorphism π of $\mathcal{M}\bar{\otimes}\mathcal{L}(l^2(G))$ such that $\pi\cdot(\tilde{\alpha}_g\otimes\mathrm{Ad}\,\varrho_g)\cdot\pi^{-1}=\tilde{\beta}_g\otimes\mathrm{Ad}\,\varrho_g$. For the Tr on $\mathcal{L}(l^2(G))$, $\varphi\otimes\mathrm{Tr}$ is $(\tilde{\alpha}\otimes\mathrm{Ad}\,\varrho)$-invariant and $\psi\otimes\mathrm{Tr}$ is $(\tilde{\beta}\otimes\mathrm{Ad}\,\varrho)$-invariant, hence $(\psi\otimes\mathrm{Tr})\cdot\pi$ is $(\tilde{\alpha}\otimes\mathrm{Ad}\,\varrho)$-invariant. By Lemma 21, $\alpha_g\otimes\mathrm{Ad}\,\varrho_g=(\tilde{\alpha}_g\otimes\mathrm{Ad}\,\varrho_g)\cdot\sigma^{\varphi\otimes\mathrm{Tr}}_{\nu(g)}$ is cocycle conjugate to $(\tilde{\alpha}_g\otimes\mathrm{Ad}\,\varrho_g)\cdot\sigma^{(\psi\otimes\mathrm{Tr})\cdot\pi}_{\nu(g)}$. Now

$$\begin{aligned}(\tilde{\alpha}_g\otimes\mathrm{Ad}\,\varrho_g)\cdot\sigma^{(\psi\otimes\mathrm{Tr})\cdot\pi}_{\nu(g)}&=\pi^{-1}\cdot(\tilde{\beta}_g\otimes\mathrm{Ad}\,\varrho_g)\cdot\pi\cdot\pi^{-1}\cdot\sigma^{\psi\otimes\mathrm{Tr}}_{\nu(g)}\cdot\pi\\&=\pi^{-1}\cdot(\beta_g\otimes\mathrm{Ad}\,\varrho_g)\cdot\pi,\end{aligned}$$

which shows the cocycle conjugacy of the second dual actions $\alpha\otimes\mathrm{Ad}\,\varrho$ and $\beta\otimes\mathrm{Ad}\,\varrho$. Then α and β are stably conjugate, hence, cocycle conjugate because the factor $\mathcal{M}$ is now infinite. Q.E.D.

PROPOSITION 22. *For any countable discrete group G, any normal subgroup N of G, and any choice of invariants $(\chi,\nu)\in\Lambda(G,N,\mathbf{T})\times\mathrm{Hom}(N,\mathbf{R})$, there exists an action α of G on the* AFD *factor $\mathcal{M}$ of type* III_1 *with $N(\alpha)=N,(\chi_\alpha,\nu_\alpha)=(\chi,\nu)$.*

Proof. Choose $\lambda,\mu\in(0,1)$ with $\log\lambda/\log\mu\notin\mathbf{Q}$ and let $\mathcal{P},\mathcal{Q}$ be AFD factors of type III_λ, III_μ, respectively. Viewing $\mathbf{T}$ and $\mathbf{R}$ imbedded in the obvious way in $\mathcal{UF}(\mathcal{P})$, $H^1(\mathcal{F}(\mathcal{P}))$ (and similarly for $\mathcal{Q}$—see [20, p. 442 for notations]), we note that $\delta_1(\chi)=\delta_2(\nu)=0$, where δ_1,δ_2 are as in [20, p. 421]. Thus by [20, Theorem 5.14], there are actions β,γ of G on $\mathcal{P},\mathcal{Q}$ with invariants (N,χ,ν) and $(N,0,\nu)$ respectively. Thus $\alpha=\beta\otimes\gamma$ is an action of G on $\mathcal{P}\bar{\otimes}\mathcal{Q}\cong\mathcal{M}$, with invariants (N,χ,ν). Q.E.D.

Remark 23. If the group G is abelian in Proposition 23, then unlike the other cases, the existence of an action α of G on an AFD factor $\mathcal{M}$ of type III_1 with a prescribed invariant (N, χ, ν) is very simple, thanks to the simple structure of $\mathrm{Aut}(\mathcal{M})$, as follows. Let χ be an element of $\Lambda(G, N, \mathbf{T})$ and $\nu \in \mathrm{Hom}(N, \mathbf{R})$. Extend first ν to an element of $\mathrm{Hom}(G, \mathbf{R})$, denoted by ν again. Let m be an action of G on the AFD factor $\mathcal{R}$ of type II_1 with $\chi_m = \chi$. The action α defined by

$$\alpha_g = m_g \otimes \sigma^{\varphi}_{\nu(g)}, \quad g \in G,$$

on $\mathcal{R} \otimes \mathcal{M} \cong \mathcal{M}$ has precisely the invariant:

$$(N(\alpha), \chi_\alpha, \nu_\alpha) = (N, \chi, \nu).$$

Remark 24. If we directly compare $\alpha_g = \tilde{\alpha}_g \cdot \sigma^{\varphi}_{\nu(g)}$ and $\beta_g = \tilde{\beta}_g \cdot \sigma^{\psi}_{\nu(g)}$ using the Radon–Nikodym cocycle $(D(\psi \cdot \pi) : D_\varphi)$ in the above proof, we get α_g and β_g are conjugate in $\mathrm{Out}(\mathcal{M})$, which is enough for $G = \mathbf{Z}$. But for general groups, this method does not produce a cocycle, and we have to use the second duals as above.

As an application, we have the following:

COROLLARY 25. *For an action α of a discrete abelian or finite group on the* AFD *factor of type* III, *there exists a cocycle perturbation β of α such that there is a Cartan subalgebra which is globally invariant under β.*

Proof. By Theorem 5.1 in [19] and by Theorem 5.9 in [20], we consider only the case of type III_1. Because the modular automorphism of an ITPFI factor fixes a Cartan subalgebra, we get the conclusion by Remark 23. Q.E.D.

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Received November 20, 1989
Received in revised form January 28, 1991