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Characteristic numbers of bounded domains

by

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1. Introduction

A fundamental problem in several complex variables is to find computable invariants of complex manifolds with strictly pseudoconvex boundaries. The foci of this subject have been: the construction of canonical metrics on the interior and the study of the finitely determined geometry of the boundary. The metrics studied are the Bergman metric, Einstein–Kähler metric, Kobayashi metric, etc. The geometry on the boundary is couched in the language of bundles with connections and normal forms. It was realized early on that there is a connection between the finitely determined part of the Einstein–Kähler metric at the boundary and the intrinsically defined structure bundle. Some of these connections were worked out in [F2], [BDS] and [W1].

In the work which follows, we will continue our study of global invariants started in [BE]. In [BE] we associated Chern–Simons type secondary characteristic forms to non-degenerate codimension one CR manifolds. In real dimension three, under suitable topological conditions, we could integrate this form, and we studied the resulting biholomorphic invariant. (Cheng and Lee have independently found this invariant, and found some interesting further properties of it, cf. [CL].)

Here we propose to study characteristic numbers of a strictly pseudoconvex domain N coming from the integrals of characteristic forms in the Einstein–Kähler metric on N. Of course, most such integrals will diverge, the most obvious example being $c_1{}^n$, which is a multiple of the volume form: it behaves like ψ^{-n-1} at the boundary, if ψ is a defining function for ∂N. However, the curvature matrix Ω_{EK} of the Einstein–Kähler metric can be written as a sum of two terms,

$$\Omega_{\mathrm{EK}} = -K + W$$

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where $-K$ is the constant negative holomorphic sectional curvature tensor, and W is the trace free part, or Bochner tensor, of Ω_{EK}. The term W is continuous up to the boundary ∂N, even though it has an apparent logarithmic singularity if $n=2$, while $-K$ has a second order pole at ∂N. We will sometimes call W the finite part of Ω_{EK}. It is clear that any Ad-invariant polynomial P, homogeneous of degree k on $\mathfrak{gl}(n,\mathbf{C})$, will give rise to a biholomorphically invariant (k,k)-form on N, $P(W)$, which will be integrable if $k=n$. These integrals are the characteristic numbers alluded to in the title above.

We check in § 2 below that these forms have the continuity properties asserted above, and prove that they are ordinary characteristic forms, i.e., polynomials in the Chern classes of the Einstein–Kähler metric. For P as above, we call the characteristic form $P(W)$ the renormalization of the characteristic form $P(\Omega_{\mathrm{EK}})$. We denote by $\tilde{c}_k$ the renormalized kth Chern form. As examples, $\tilde{c}_1=0$, while

$$\tilde{c}_2 = c_2(\Omega_{\mathrm{EK}}) - \frac{n}{2(n+1)} c_1(\Omega_{\mathrm{EK}})^2.$$

(Note that the renormalization depends on the dimension of N.) In much of what follows, it is much easier to work with the renormalized trace powers,

$$\tau_j \overset{\mathrm{def}}{=} \left(\frac{i}{2\pi}\right)^j \operatorname{tr}(W^j),$$

Theorem 2.1 expresses the τ_j in terms of ordinary characteristic forms.

The integrals of renormalized characteristic forms are not readily accessible, since they depend a priori on the global solution of the Einstein–Kähler equation. Theorem 2.2 gives an integration by parts formula which shows that these numbers can be evaluated on the boundary, using only Fefferman's approximate solution of the Monge–Ampère equation (2.1).

At the end of § 2 we consider briefly how much the characteristic numbers depend on the Einstein–Kähler metric, in particular we point out what happens for the characteristic forms of the Bergman metric.

Sections 3 to 5 below are directed towards relating the boundary integrals in Theorem 2.2 to intrinsic CR invariants on ∂N. The problem here is that in most reasonable cases, one will not be able to find a section of the CR structure bundle to produce a form of top degree on ∂N from the secondary characteristic forms defined on the structure bundle. Section 3 compares several different structure bundles related to the boundary ∂N. It turns out that the most natural place to prove general transgression, or integration by parts, formulas is on the structure bundle of Fefferman's Lorentz metric on the $(n+1)$st root of the canonical bundle on N. The finiteness at the boundary of the forms in question is also transparent from this point of view. Section 4 shows that while there will rarely be a section of the CR structure bundle over ∂N, for N in $\mathbf{C}^n$, there exists a homological section, i.e., a $(2n-1)$-cycle over which we can integrate secondary characteristic forms. Section 5 proves that the numbers so obtained are independent of the cycle over which we integrated, and goes on to complete the identification of the boundary integrals in § 2 with an expression in topological invariants of N and CR invariants of ∂N. As an example, if $n=2$, we consider the invariant of [BE], $\mu(\partial N)$, arising from a secondary characteristic form for c_2 on ∂N. In this case, the final integration by parts formula of Theorem 5.2 reads

$$\int_N c_2(\Omega_{\mathrm{EK}}) - \frac{1}{3} c_1{}^2(\Omega_{\mathrm{EK}}) = \mu(\partial N) + \chi(N),$$

where $\chi(N)$ is the Euler characteristic of N. An application of this to the problem of embedding abstract CR manifolds into $\mathbf{C}^2$ is mentioned in § 5.

The final § 6 contains a different method of proof of the basic integration by parts formula for $n=2$, which applies to more general compact complex manifolds with strictly pseudoconvex boundaries than domains in $\mathbf{C}^2$. The proof method here is more classical, along the lines of Chern's proof of the Gauss–Bonnet theorem. From this point of view, the secondary characteristic numbers on the boundary ∂N are analogous to the second fundamental form contributions on the boundary to the Gauss–Bonnet formula for a manifold with boundary. This method requires a very tedious pole-by-pole analysis of the singularity of the characteristic forms at the boundary. One does not yet have a formalism as simple as that of § 2 in the more general geometric case. We conclude with the reconsideration of some example manifolds whose boundary invariants we calculated in [BE]. An interesting question is left open here about the relationship of these invariants to the Kähler geometry of the interior manifold, and the behavior of developing maps for CR manifolds which are locally CR equivalent to the standard sphere.

It would be very interesting to obtain further analytic interpretations of the renormalized characteristic classes. In dimension two, the class $\tilde{c}_2$ leads to the consideration of certain spectral problems on N. We hope to be able to discuss this at a later date.

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We follow the summation convention: an index which appears as an upper and lower index should be summed, e.g.:

$$\varphi^i\varphi_i=\sum_{i=1}^{n}\varphi^i\varphi_i.$$

We shall use the notation:

$$\varphi_i=\frac{\partial\varphi}{\partial z^i},\quad \varphi_{\bar{i}}=\frac{\partial\varphi}{\partial\bar{z}^i}$$

$$\varphi_{i\bar{j}}=\frac{\partial^2\varphi}{\partial z^i\partial\bar{z}^j},\quad \text{etc.}$$

§ 2. Renormalized Chern classes

Let $N\subset\mathbf{C}^n$ be a bounded, smooth strictly pseudoconvex (s.ψ.c.) domain. In [CY] it is shown that on such a domain there is a unique, complete Einstein–Kähler metric. Obtaining this metric is equivalent to solving the complex Monge–Ampère equation:

$$\begin{cases} J(\varphi)=\det\begin{bmatrix}\varphi & \varphi_{\bar{j}}\\ \varphi_i & \varphi_{i\bar{j}}\end{bmatrix}=-1\\ \varphi=0 \text{ on } \partial N,\quad \varphi<0 \text{ on } N\\ \log(-1/\varphi) \text{ strictly plurisubharmonic on } N.\end{cases}\tag{2.1}$$

If φ satisfies (2.1) then

$$g=\frac{\partial^2\log(-1/\varphi)}{\partial z^i\partial\bar{z}^j}dz^i\cdot d\bar{z}^j,\tag{2.2}$$

or

$$g_{i\bar{j}}=\frac{\varphi_{i\bar{j}}}{-\varphi}+\frac{\varphi_i\varphi_{\bar{j}}}{\varphi^2},\tag{2.2'}$$

is the complete Einstein–Kähler metric. The solution to (2.1) is in general not in $C^\infty(\bar{N})$.

Fefferman, however, proved that one can find an approximate solution φ_0 in $C^\infty(\bar{N})$ which satisfies:

$$J(\varphi_0) = -1+O((\varphi_0)^{n+1}) \quad \text{near } \partial N. \tag{2.3}$$

The approximate solution is produced via a finite algorithm, see [F]. Lee and Melrose have shown the exact solution to (2.1) has an asymptotic expansion at ∂N of the form:

$$\varphi \sim \varphi_0+\varphi_0\left(\sum_{j=1}^{\infty} a_j({\varphi_0}^{n+1}\log \varphi_0)^j\right) \tag{2.4}$$

where $a_j \in C^\infty(\bar{N})$. In this section φ will denote the solution to (2.1) and φ_0 a Fefferman approximate solution.

We include here some formulas from [LM], pp. 163–164. Given the defining function φ, we define, in a neighborhood of ∂N, a $(1,0)$ vector field ξ by

$$\langle \xi, \partial\varphi\rangle = 1 \quad \text{and} \quad \partial\bar{\partial}\varphi(\xi,\cdot) = \lambda\langle\cdot,\partial\varphi\rangle. \tag{2.5}$$

Define r by $r=\varphi_{i\bar{j}}\xi^i\xi^{\bar{j}}$. One then solves for λ above:

$$\varphi_{i\bar{j}}\xi^i = r\varphi_{\bar{j}}. \tag{2.6}$$

The matrix $\psi_{i\bar{j}}$ defined by

$$\psi_{i\bar{j}} = \varphi_{i\bar{j}}+(1-r)\,\varphi_i\varphi_{\bar{j}} \tag{2.7}$$

is positive definite near ∂N, and defining $g^{i\bar{j}}$ as usual, so that $g^{i\bar{j}}g_{k\bar{j}}=\delta_k{}^i$, one can check that

$$g^{i\bar{j}} = (-\varphi)\,[\psi^{i\bar{j}}-(1-r\varphi+\varphi)/(1-r\varphi)\,\xi^i\xi^{\bar{j}}]. \tag{2.8}$$

From (2.5–2.7) we see that $\psi_{i\bar{j}}\xi^{\bar{j}}=\varphi_i$, and it therefore follows that

$$\psi^{i\bar{j}}\varphi_{\bar{j}} = \xi^i. \tag{2.9}$$

From this and (2.8) we get (2.9) of [LM]:

$$g^{i\bar{j}}\varphi_{\bar{j}} = -\xi^i\varphi^2/(r\varphi-1). \tag{2.10}$$

For later convenience, set

$$h^{i\bar{j}} = [\psi^{i\bar{j}}-(1-r\varphi+\varphi)/(1-r\varphi)\,\xi^i\xi^{\bar{j}}] = g^{i\bar{j}}/(-\varphi). \tag{2.11}$$

Define $A^{i\bar{j}}$ so that

$$A^{i\bar{j}}\varphi_{k\bar{j}} = \det(\varphi_{s\bar{t}})\,\delta_k{}^i, \tag{2.12}$$

and set

$$A = A^{i\bar{j}}\varphi_i\varphi_{\bar{j}} = -\det\begin{pmatrix} 0 & \varphi_{\bar{j}} \\ \varphi_i & \varphi_{i\bar{j}} \end{pmatrix} = -J(\varphi)+\varphi\det(\varphi_{s\bar{t}}), \tag{2.13}$$

the second equality by Cramer's rule. From (2.6) and (2.12) it follows that

$$r\varphi_{\bar{j}}A^{k\bar{j}} = \varphi_{i\bar{j}}\xi^i A^{k\bar{j}} = \det(\varphi_{s\bar{t}})\,\xi^k.$$

Contracting this with φ_k we get

$$rA = r\varphi_{\bar{j}}A^{k\bar{j}}\varphi_k = \det(\varphi_{s\bar{t}}). \tag{2.14}$$

Using (2.7) and (2.13–2.14), one can check that

$$\varphi_{\bar{j}}A^{k\bar{j}} = A\xi^k. \tag{2.15}$$

(2.14) also implies, with (2.13), that

$$r\varphi - 1 = J(\varphi)/A. \tag{2.16}$$

The Einstein–Kähler metric defines a torsion free (1,0) connection, which can be calculated directly, using (2.2′) and (2.8) above:

$$\begin{aligned}\omega_{\mathrm{EK}} = \omega_i{}^j &= g^{j\bar{r}}\partial g_{i\bar{r}} \\ &= (\delta_i{}^j\varphi_k+\delta_k{}^j\varphi_i)\,dz^k/(-\varphi)+g^{j\bar{r}}[-\varphi\varphi_{i\bar{r}k}+\varphi_{\bar{r}}\varphi_{ik}]\,dz^k/\varphi^2.\end{aligned}$$

Define

$$Y_i{}^j{}_k = (\delta_i{}^j\varphi_k+\delta_k{}^j\varphi_i)/(-\varphi), \tag{2.17}$$

and

$$\theta_i{}^j{}_k = g^{j\bar{r}}[-\varphi\varphi_{i\bar{r}k}+\varphi_{\bar{r}}\varphi_{ik}]/\varphi^2, \tag{2.18}$$

so that $\omega_i{}^j = Y_i{}^j{}_k\,dz^k+\theta_i{}^j{}_k\,dz^k$. Note that, by (2.10–2.11) above, we can calculate directly that:

$$\theta_i{}^j{}_k = h^{j\bar{r}}\varphi_{i\bar{r}k}+\xi^j\varphi_{ik}/(1-r\varphi). \tag{2.19}$$

In particular, $\theta_i{}^j{}_k$ is continuous up to ∂N in all dimensions. Note also that

$$\theta_i{}^j{}_k = \theta_k{}^j{}_i. \tag{2.20}$$

Next, we calculate the curvature Ω_{EK} of the Einstein–Kähler metric:

$$\begin{aligned}\Omega_{\mathrm{EK}} &= d\omega_{\mathrm{EK}} - \omega_{\mathrm{EK}} \wedge \omega_{\mathrm{EK}} \\ &= d(Y+\theta) - (Y+\theta) \wedge (Y+\theta) \\ &= dY - Y \wedge Y + d\theta - \theta \wedge \theta - \theta \wedge Y - Y \wedge \theta.\end{aligned}$$

Expanding $dY - Y \wedge Y$, and using (2.10) and (2.18) above, we obtain:

$$\begin{aligned}\Omega_{\mathrm{EK}} = &-(\delta_i{}^j g_{k\bar t} + \delta_k{}^j g_{i\bar t})\, dz^k \wedge d\bar z^t \\ &+ d\theta_i{}^j - \theta_i{}^s \wedge \theta_s{}^j + [\varphi_{it}\, dz^t \wedge dz^j - \varphi_t \theta_i{}^t \wedge dz^j]/(-\varphi).\end{aligned}$$

Set

$$K_i{}^j = (\delta_i{}^j g_{k\bar t} + \delta_k{}^j g_{i\bar t})\, dz^k \wedge d\bar z^t, \tag{2.21}$$

and

$$\mathrm{W}_i{}^j = (\Omega_{\mathrm{EK}})_i{}^j + K_i{}^j. \tag{2.22}$$

We simplify W by the following manipulation:

$$\begin{aligned}[\varphi_{it}\, dz^t - \varphi_s \theta_i^s]/(-\varphi) &= [\varphi_{it}\, dz^t - \varphi_s (h^{s\bar r}\varphi_{i\bar r t} + \xi^s \varphi_{it}/(1-r\varphi))\, dz^t]/(-\varphi) \\ &= [\varphi \xi^{\bar r}\varphi_{i\bar r t}/(1-r\varphi) - r\varphi\varphi_{it}/(1-r\varphi)]\, dz^t/(-\varphi) \\ &= [-\xi^{\bar r}\varphi_{i\bar r t} + r\varphi_{it}]\, dz^t/(1-r\varphi) \\ &= -[-\xi^{\bar r}\varphi_{i\bar r t} + r\varphi_{it}]\, dz^t A/J(\varphi).\end{aligned}$$

For later convenience, set

$$u_i = (n+1)[-\xi^{\bar r}\varphi_{i\bar r t} + r\varphi_{it}]\, dz^t A/J(\varphi). \tag{2.23}$$

Note that $u_i = u_{ij} dz^j$, with $u_{ij} = u_{ji}$, so that

$$dz^i \wedge u_i \equiv 0. \tag{2.24}$$

Returning to our calculation of $W_i{}^j$, we get that

$$W_i{}^j = d\theta_i{}^j - \theta_i{}^s \wedge \theta_s{}^j - u_i \wedge \frac{1}{n+1}\, dz^j. \tag{2.25}$$

If $n \geqslant 3$ it follows from (2.4), (2.19) and (2.25) that $W_i{}^j$ is continuous in the closure of N and depends on the 4-jet of φ_0 at ∂N. In two dimensions, since $W_i{}^j$ depends on the derivatives of φ of order less than or equal to four, the singularity that arises in $W_i{}^j$ at ∂N is at worst logarithmic, and therefore polynomials in W are integrable on N in this dimension, too. W is continuous on $\bar{N}$ for $n=2$ as well: cf. the proof of Proposition 2.1 below.

For convenience in some of the formulas which follow, we call $\omega = g_{i\bar{j}} dz^i \wedge d\bar{z}^j$ the Kähler form. Note that it differs by a factor $\sqrt{-1}/2$ from the usual definition. The Einstein–Kähler equation is:

$$\Omega_i{}^i = -(n+1)\,\omega. \tag{2.26}$$

An easy calculation shows that $K_i{}^i = (n+1)\omega$. From this we see that $W_i{}^j$ is the trace free part of the curvature. If we let $W_i{}^j = W_i{}^j{}_{k\bar{l}} dz^k \wedge d\bar{z}^l$, then $W_{i\bar{j}k\bar{l}} = g_{m\bar{j}} W_i{}^m{}_{k\bar{l}}$ has the following symmetries:

$$W_{j\bar{i}k\bar{l}} = W_{k\bar{i}j\bar{l}} = W_{j\bar{l}k\bar{i}}. \tag{2.27}$$

These follow from the fact that both $\Omega_{i\bar{j}}$ and $K_{i\bar{j}}$ have these symmetries. The following algebraic lemma is what allows one to renormalize the characteristic classes explicitly:

LEMMA 2.1. *With $W_i{}^j$ and $K_i{}^j$ as above, the following identities hold:*

$$\begin{aligned} &\text{(a) } W_i{}^k K_k{}^j = \omega W_i{}^j = K_i{}^k W_k{}^j. \\ &\text{(b) } K_i{}^k K_k{}^j = \omega K_i{}^j. \end{aligned} \tag{2.28}$$

Proof. The proof of (a):

$$\begin{aligned} W_i{}^k K_k{}^j &= W_i{}^k{}_{l\bar{m}} dz^l \wedge d\bar{z}^m \wedge [\delta_k{}^j g_{p\bar{q}} + \delta_p{}^j g_{k\bar{q}}]\, dz^p \wedge d\bar{z}^q \\ &= W_i{}^j{}_{l\bar{m}} dz^l \wedge d\bar{z}^m \wedge \omega + W_{i\bar{q}l\bar{m}} dz^l \wedge d\bar{z}^m \wedge dz^j \wedge d\bar{z}^q. \end{aligned}$$

Since $W_{i\bar{q}l\bar{m}} = W_{i\bar{m}l\bar{q}}$ the second term is zero and this proves the first equation in (a). Note that as $W_i{}^j$ is a matrix of two forms and ω is a two form, they commute. To prove the second statement in (a) we observe

$$\begin{aligned} K_i{}^k W_k{}^j &= (\delta_i{}^k g_{p\bar{q}} + \delta_p{}^k g_{i\bar{q}})\, dz^p \wedge d\bar{z}^q \wedge W_k{}^j{}_{l\bar{m}} dz^l \wedge d\bar{z}^m \\ &= \omega W_i{}^j + g_{i\bar{q}} dz^p \wedge d\bar{z}^q \wedge W_p{}^j{}_{l\bar{m}} dz^l \wedge d\bar{z}^m. \end{aligned}$$

Since $W_p{}^j{}_{l\bar{m}} = W_l{}^j{}_{p\bar{m}}$ the second term is zero and this proves the second part of (a). The proof of (b) is quite similar and is left to the reader.

Symbolically, we have shown that

$$WK = KW = \omega W$$

and

$$K^2 = \omega K.$$

Evidently we can iterate the second formula to obtain:

$$K^j = \omega^{j-1} K.$$

Since $\Omega = W - K$ it follows that

$$K\Omega = \Omega K = \omega \Omega. \tag{2.29}$$

From the Chern–Weil theory it follows that the characteristic classes of Ω can be generated by the trace powers:

$$T_k(\Omega) \overset{\text{def}}{=} \left(\frac{i}{2\pi}\right)^k \operatorname{tr} \Omega^k. \tag{2.30}$$

Note that:

$$c_1 = T_1(\Omega) = -(n+1)\left(\frac{i}{2\pi}\right)\omega. \tag{2.31}$$

We can now construct the renormalized characteristic classes of the Einstein–Kähler metric; we define

$$\tau_j = \left(\frac{i}{2\pi}\right)^j \operatorname{tr} W^j. \tag{2.32}$$

THEOREM 2.1. *Let N be a strictly pseudoconvex domain in $\mathbf{C}^n$ with complete Einstein–Kähler metric g, and let Ω denote its curvature form. Then if W is defined by $W = \Omega + K$ we have:*

$$\tau_j = \sum_{k=0}^{j-2} (-1)^k \binom{j}{k} \frac{T_{j-k}(\Omega)\, T_1^k(\Omega)}{(n+1)^k} + \frac{(-1)^{j-1}(j-1) T_1^j(\Omega)}{(n+1)^{j-1}}, \quad j = 2, \ldots, n. \tag{2.33}$$

Proof. From (2.28) it follows that K and Ω generate a commutative ring and thus that

$$W^j = (\Omega+K)^j = \sum_{k=0}^{j} \binom{j}{k} \Omega^{j-k} K^k;$$

we use $K^k = \omega^{k-1} K$ to rewrite this as:

$$W^j = \Omega^j + j\Omega^{j-1}K + \sum_{k=2}^{j} \binom{j}{k} \Omega^{j-k} K \omega^{k-1}.$$

Now using (2.28) we obtain:

$$W^j = \sum_{k=0}^{j-1} \binom{j}{k} \Omega^{j-k}\omega^k + K\omega_{j-1}. \tag{2.34}$$

Multiplying by $(i/2\pi)^j$ and taking the trace in (2.19) leads to:

$$\tau_j = \sum_{k=0}^{j-2} \left(\frac{i}{2\pi}\right)^k \binom{j}{k} T_{j-k}(\Omega)\,\omega^k + (n+1)\left(\frac{i}{2\pi}\right)^j (1-j)\,\omega^j.$$

Using (2.31), we easily complete the proof of (2.33).

Remarks. (1) We would like to thank Troels Jorgensen for simplifying the proof of Theorem 2.1.

(2) If we express τ_2 in terms of Chern classes we obtain:

$$\tau_2 = -2\left[c_2 - \frac{nc_1^{\,2}}{2(n+1)}\right].$$

This is the characteristic class which arises in the work of Yau and others, cf. [Y]. When $n=2$ it reduces to $-2[c_2 - \frac{1}{3}c_1^{\,2}]$, which is known to be negative semi-definite, vanishing if and only if g has constant holomorphic sectional curvature. The classes constructed in the theorem give potential generalizations of this class in higher dimensions. Each vanishes if g has constant holomorphic sectional curvature; in fact τ_j vanishes to order j at such a metric.

(3) We can rewrite Theorem 2.1 in terms of the basic Chern classes, although we cannot make it quite as explicit as (2.33):

THEOREM 2.1′. *Let N be a strictly pseudoconvex domain in $\mathbf{C}^n$. Let c_k denote the k-th Chern form of the complete Einstein–Kähler metric on N. Then we can define* *inductively renormalized Chern forms $\tilde{c}_k$, $k=2, \ldots, n$, which are continuous on $\bar{N}$ by the formulas:*

$$c_k = \sum_{i=1}^{k} P_{k,i}(\tilde{c}_2, \ldots, \tilde{c}_{k-1})(c_1)^i + \tilde{c}_k. \tag{2.35}$$

Here the $P_{k,i}$ are uniquely determined polynomials, which depend on the dimension n.

Theorem 2.1′ is a restatement of Theorem 2.1. The c_k can be expressed in terms of the $T_j(\Omega)$, $j=1, \ldots, k$. Each $T_j(\Omega)$, in turn, can be solved for in terms of the $\tau_2, \ldots, \tau_j$ and powers of $T_1(\Omega)=c_1$, as follows inductively from (2.15). For $n=2$, the continuity is proved below (Proposition 2.1).

Here are the explicit formulas for the first three $\tilde{c}_k$:

$$\begin{aligned}
\tilde{c}_2 &= c_2 - \frac{1}{2(n+1)} c_1^2 \\
\tilde{c}_3 &= c_3 - \frac{n-1}{n+1} c_2 c_1 - \frac{n(n-1)}{6(n+1)^2} c_1^3 \\
\tilde{c}_4 &= c_4 - \frac{n-2}{n+1} c_3 c_1 + \frac{n^2+n+2}{(n+1)^2} c_2 c_1^2 - \frac{n(7n^2-9n+8)}{8(n+1)^3} c_1^4.
\end{aligned} \tag{2.36}$$

(4) If we choose indices $2 \leq i_1 \leq i_2 \leq \ldots \leq i_p$, so that $i_1 + \ldots + i_p = n$, then we can define renormalized characteristic numbers:

$$c_{i_1 \ldots i_p} = c_{i_1 \ldots i_p}(N) = \int_N \tilde{c}_{i_1} \cdot \ldots \cdot \tilde{c}_{i_p}. \tag{2.37}$$

Since the Einstein–Kähler metric is biholomorphically invariant and the $\tilde{c}_j$ are given by universal polynomials in its curvature Ω it follows that the characteristic numbers are real-valued biholomorphic invariants.

As a corollary of this and the theorem on the positive semi-definiteness of $c_2 - \frac{1}{3}c_1^2$, we have a very easy proof of:

COROLLARY 2.1. *If N is a strictly pseudoconvex domain in $\mathbf{C}^2$ not covered by the unit ball, then* Aut(N) *is a compact group.*

Proof. If N is not covered by the ball then $c_2 - \frac{1}{3}c_1^2$ is positive almost everywhere. Let $p \in N$ be a point where $c_2 - \frac{1}{3}c_1^2 > 0$. If Aut (N) is not compact then there is an $\varepsilon > 0$ and an infinite sequence of elements $\gamma_i \in \mathrm{Aut}(N)$ so that

$$\gamma_i(B_\varepsilon(p)) \cap \gamma_j(B_\varepsilon(p)) = \emptyset \quad \text{if} \quad i \neq j. \tag{2.38}$$

Since $c_2-\frac{1}{3}c_1{}^2>0$ is biholomorphically invariant:

$$0<\int_{B_\varepsilon(p)} c_2-\frac{1}{3}c_1{}^2=\int_{\gamma_i(B_\varepsilon(p))} c_2-\frac{1}{3}c_1{}^2. \tag{2.39}$$

Together (2.38) and (2.39) imply that

$$\int_N c_2-\frac{1}{3}c_1{}^2=+\infty,$$

a contradiction.

In principle, it is not possible to compute the characteristic numbers directly from (2.37), as this formula requires a solution of the Monge–Ampère equation. We will next show that the computation in (2.37) can be reduced to a computation on ∂N which requires only the Fefferman asymptotic solution and is therefore, in principle, computable.

In the paper of Chern and Simons [CS] a general formula is given for the trangression $TP(\psi,\Psi)$ of a characteristic form $P(\Psi)$. Here ψ is a connection taking values in a Lie algebra $\mathfrak{g}$, Ψ the curvature of the connection defined by $\Psi=d\psi-\psi\wedge\psi$ and P is an Ad-invariant polynomial defined on $\mathfrak{g}$. The trangression satisfies $dTP=P$. If $P_j(\Psi)=T_j(\Psi)$, then:

$$TP_j(\psi,\Psi)=\left(\frac{i}{2\pi}\right)^j j\int_0^1 \mathrm{tr}[\psi\wedge(t\Psi+(t-t^2)\,\psi\wedge\psi)^{j-1}]\,dt, \tag{2.40}$$

and $dTP_j(\psi,\Psi)=P_j(\Psi)$. Setting

$$X_i{}^j=u_i\wedge\frac{1}{n+1}dz^j,$$

as in (2.25) above, we have

$$d\theta-\theta\wedge\theta=W+X,$$

so that

$$TP_j(\theta,W)=\left(\frac{i}{2\pi}\right)^j j\int_0^1 \mathrm{tr}[\theta\wedge(t(W+X)+(t-t^2)\,\theta\wedge\theta)^{j-1}]\,dt, \tag{2.41}$$

verifies

$$dTP_j(\theta,W)=P_j(W+X).$$

The following simple lemma, taken together with the cyclicity of the trace, shows that, for $j \geq 2$, X may be dropped from the last two formulas:

LEMMA 2.3. *The following identities hold:*
(a) $X \cdot W = 0$,
(b) $X^2 = 0$, (2.42)
(c) $X \cdot \theta = 0$.

Proof. By the definition of X, it suffices to show

$$dz^i \wedge W_i{}^j = dz^i \wedge X_i{}^j = dz^i \wedge \theta_i{}^j = 0.$$

The first and third vanish by (2.27) and (2.20), respectively, while the second vanishes by (2.24).

This proves most of the following proposition:

PROPOSITION 2.1. *With W the trace free part of the Einstein–Kähler curvature form and θ defined by* (2.20) *and* (2.25) *we define:*

$$T\tau_j = \left(\frac{i}{2\pi}\right)^j j \int_0^1 \operatorname{tr}[\theta \wedge (tW + (t-t^2)\,\theta \wedge \theta)^{j-1}]\, dt, \tag{2.43}$$

for $j=2,\dots,n$. Then $dT\tau_j = \tau_j$ in N and the $T\tau_j$ are continuous in $\bar{N}$ and along ∂N depend only on the four-jet of the Fefferman approximation, φ_0.

Proof. If $n \geq 3$, then the continuity and dependence on the four-jet of φ_0 at ∂N of $T\tau_k$ in N follow from (2.4).

In case $n=2$ it suffices to prove that W is continuous on $\bar{N}$. Since only derivatives of φ of order ≥ 4 are singular along ∂N, we can write, using (2.25), (2.19) and (2.11):

$$W_i{}^j \equiv g^{j\bar{i}}\bar{\partial}\partial\varphi_{i\bar{i}}/(-\varphi) = h^{j\bar{i}}\bar{\partial}\partial\varphi_{i\bar{i}}$$

mod terms involving ≤ 3 derivatives of φ, and, hence, continuous on $\bar{N}$. We examine this fourth-order term, using (2.4) and (2.11):

$$h^{j\bar{i}}\bar{\partial}\partial\varphi_{i\bar{i}} - h^{j\bar{i}}\bar{\partial}\partial(\varphi_0)_{i\bar{i}} \equiv b\log(\varphi_0)\,(\varphi_0)_i(\varphi_0)_{\bar{i}}\,h^{j\bar{i}}\partial(\varphi_0) \wedge \bar{\partial}(\varphi_0),$$

now modulo terms continuous on $\bar{N}$ and vanishing along ∂N, and where b is continuous on $\bar{N}$. Substituting $(\varphi)_{\bar{i}}$ for $(\varphi_0)_{\bar{i}}$, we get

$$h^{j\bar{i}}\bar{\partial}\partial\varphi_{i\bar{i}} - h^{j\bar{i}}\bar{\partial}\partial(\varphi_0)_{i\bar{i}} \equiv b\log(\varphi_0)(\varphi_0)_i(\varphi)_{\bar{i}}h^{j\bar{i}}\partial(\varphi_0)\wedge\bar{\partial}(\varphi_0),$$

$$\equiv b\log(\varphi_0)(\varphi_0)_i\varphi\xi^j\partial(\varphi_0)\wedge\bar{\partial}(\varphi_0)/(r\varphi-1),$$

by (2.4) and (2.10–2.11), both modulo terms continuous on $\bar{N}$, and vanishing on ∂N. This last expression is continuous on $\bar{N}$, and vanishes along ∂N. This proves that the value of W, and hence of $T\tau_k$, can be computed replacing φ by φ_0 in all the formulas above, especially those for θ and W, (2.19) and (2.25). When $n=2$, φ_0 is only completely well-defined up through third order terms. However, if we replace φ_0 by $\varphi_0+a\varphi_0{}^4$, the argument just given can be used to show that the value of W is independent of a. (In fact, a more careful examination of which fourth derivatives of φ_0 actually enter into the calculation of W along ∂N shows that these are precisely the fourth derivatives of φ_0 which can be determined from (2.3).)

From the proposition we easily derive the following theorem, stated in terms of the renormalized trace powers:

THEOREM 2.2. *The characteristic numbers are given by:*

$$\int_N \tau_{i_1}\cdot\tau_{i_1}\cdot\ldots\cdot\tau_{i_p} = \int_{\partial N} T\tau_{i_1}\cdot\tau_{i_1}\cdot\ldots\cdot\tau_{i_p}, \quad i_1+\ldots+i_p=n. \tag{2.44}$$

Here W and θ are computed from formulas (2.19) *and* (2.25), *using Fefferman's asymptotic solution.*

As a corollary we have:

COROLLARY 2.2. *The characteristic numbers are biholomorphic invariants which are computable from local data on the boundary of N.*

We would like to make a few remarks here on the use of other biholomorphically invariant metrics, and particularly the Bergman metric. It follows from the work of Fefferman [F1] that the connection form and curvature form of the Bergman metric can be decomposed into singular and bounded terms analogous to the decompositions above (for $n=2$, the "finite part" W_B of the curvature once again has a logarithmic singularity). In carrying out the analogy with the case treated above, one must replace the solution φ of (2.1) by $(K_B)^{-1/n+1}$, where K_B is the Bergman kernel function. Therefore, one has Bergman renormalized characteristic numbers as well, with integrands polynomials in the trace powers $\operatorname{tr}(W_B{}^j)$. One cannot, however, use (2.31), and in lieu of (2.33), one can only prove (2.34). Thus, the Bergman renormalized characteristic forms are in the ring generated by the Chern forms of the Bergman metric and its Kähler form.

The proof of Theorem 2.2 uses only the general symmetries which are shared by the Bergman metric, and is thus valid for the Bergman invariants as well. since, for $n=2$, it is known that

$$(K_B)^{-1/n+1} = \varphi_0 + O(\varphi_0{}^4 \log(-1/\varphi_0)) \tag{2.45}$$

the proof of Proposition 2.1 for $n=2$ also proves the following corollary.

COROLLARY 2.3. *For $N \subset \mathbf{C}^2$, we have*

$$\int_N \operatorname{tr}(W_{EK}{}^2) = \int_N \operatorname{tr}(W_B{}^2).$$

In higher dimensions the Bergman and Einstein–Kähler renormalized characteristic numbers may well be different.

It is clear that the renormalized characteristic classes exist on more general manifolds than domains in $\mathbf{C}^n$. For one variant of this, see § 6 below. The most precise theorem would require a complete Einstein–Kähler metric of asymptotically constant holomorphic sectional curvature with an estimate on the rate at which the curvature approaches the constant value.

In the sections which follow we will reexpress the renormalized characteristic numbers in terms of the connection and curvature form defined intrinsically by the CR structure induced from the embedding $\partial N \subset \mathbf{C}^n$.

§ 3. CR geometry: a review

In this section, M will denote a strongly pseudoconvex CR manifold of real dimension $2n-1$. We review quickly the geometric structures that arise naturally in this situation. For the intrinsic theory, we will follow [CM] and [W2]; for the extrinsic theory, we follow [W1].

If $T(M)$ denotes the tangent bundle of M, we denote by $T^{1,0}(M) \subset T(M) \otimes \mathbf{C}$ the complex subbundle of vectors of type (1,0): if X, Y are sections of $T^{1,0}(M)$, then so is $[X,Y]$. Let θ denote a real one-form such that $\ker \theta = T^{1,0}(M) \oplus \overline{T^{1,0}(M)}$. A complex one-form η on M is of type (1,0) if η annihilates $\overline{T^{1,0}(M)}$. Suppose $\theta^1, \ldots, \theta^{n-1}$ are (1,0)-forms locally on M whose restrictions to $T^{1,0}(M)$ are independent. The structure on M is strictly pseudoconvex if and only if

$$d\theta = i\varrho_{\alpha\bar\beta}\theta^{\alpha} \wedge \theta^{\bar\beta} + \theta \wedge \phi \tag{3.1}$$

where ϕ is a real one-form and $\varrho_{\alpha\bar\beta}$ is Hermitian and positive definite; θ will sometimes be called a contact one-form. In the sequel, Greek indices will run from 1 to $n-1$, Latin indices from 0 or 1 to n. To obtain a solution for the equivalence problem for CR manifolds, Cartan (and later Chern) introduced a trivial ray bundle $E \subset T^*(M)$, given by

$$M \times \mathrm{R} \ni (x,t) \to e^{t}\theta_x \in T^*(M). \tag{3.2}$$

If $p_0 : E \to M$ denotes the projection, we can define the tautological one-form π^n by

$$\pi^n_{(x,t)}(X) = e^t\theta_x(p_{0*}(X)). \tag{3.3}$$

Set $\pi^{\alpha} = p_0{}^*(e^{t/2}\theta^{\alpha})$ and $\pi^0 = p_0{}^*\phi - dt$. On E we have

$$d\pi^n = i\varrho_{\alpha\bar\beta}\pi^{\alpha} \wedge \pi^{\bar\beta} + \pi^n \wedge \pi^0. \tag{3.4}$$

We define a principal coframe bundle Y^* as the coframes $\{\pi^0, \pi^{\alpha}, \pi^{\bar\alpha}, \pi^n\}$ on E which satisfy (3.4) with $\varrho_{\alpha\bar\beta} = \delta_{\alpha\bar\beta}$. The structure group is

$$H = \left\{ \begin{pmatrix} 1 & 0 & 0 & 0 \\ v_{\bar\alpha} & a_{\alpha\beta} & 0 & 0 \\ v_{\bar\alpha} & 0 & a_{\bar\beta\bar\alpha} & 0 \\ a & w_{\alpha} & w_{\bar\beta} & 1 \end{pmatrix} \middle| \begin{array}{c} a_{\alpha\beta}a_{\bar\alpha\bar\gamma} = \delta_{\beta\bar\gamma} \\ \text{and} \\ w_{\beta} = -i\, a_{\alpha\beta} v_{\bar\alpha} \end{array} \right\}. \tag{3.5}$$

This is a subgroup of $SU(n,1)$. The Chern bundle $P_1 : Y \to E$ is the bundle of all frames dual to coframes in Y^*. The main result of Cartan and Chern is the existence of a canonical $\mathfrak{su}(n,1)$-valued Cartan connection π on Y with curvature

$$\Pi = d\pi - \pi \wedge \pi.$$

The curvature has the form

$$\Pi = \begin{bmatrix} \Pi^0_0 & 0 & 0 \\ \Pi^0_{\alpha} & \Pi^{\beta}_{\alpha} & 0 \\ \Pi^0_{n+1} & \Pi^{\beta}_{n+1} & -\bar\Pi^0_0 \end{bmatrix}$$

(viewed in $\mathfrak{su}(n, 1)$) where

$$\Pi \equiv 0 \mod (\pi^\alpha, \pi^{\bar\alpha}, \pi^n). \tag{3.6}$$

The connection is normalized by trace conditions on the (1,1)-components of Π.

From a topological point of view, the bundle Y is rather cumbersome. Because of this, we consider its pseudohermitian reduction X introduced in [W1]. Let θ be a fixed contact one-form and consider all solutions of the structure equation on M:

$$d\theta = i\delta_{\alpha\bar\beta}\theta^\alpha \wedge \theta^{\bar\beta}, \tag{3.7}$$

where θ^α are forms of type (1,0). The structure group of this bundle is $U(n-1)$. Define $P_0: X \to M$ as the dual frame bundle. We can use this choice of contact form to trivialize E, as in (3.3) above, and hence a splitting $T(E) = T(\mathbf{R}) \oplus T(M)$. Let Q denote the projection onto the $T(M)$ summand and set

$$\begin{cases} \tilde\theta(\xi) = \theta(Q(\xi)), \\ \tilde\theta^\alpha(\xi) = \theta^\alpha(Q(\xi)), \end{cases} \tag{3.8}$$

for $\xi \in T(E)$. From a solution to (3.7) we get a solution to (3.4) by

$$\begin{cases} \pi^0 = -dt, \\ \pi^\alpha = e^{t/2}\tilde\theta^\alpha, \\ \pi^n = e^t\tilde\theta. \end{cases} \tag{3.9}$$

We use (3.9) to include $X^* \times \mathbf{R}$ into Y^*. A glance at the structure group H in (3.5) shows that Y^* is topologically a vector bundle over $X^* \times \mathbf{R}$. We define $s: X \times \mathbf{R} \to Y$ by taking dual frame fields, and set $s_0: X \to Y$ the inclusion at $t=0$. Clearly, $s_0^*(\pi^0) = 0$. On the level of frames we define s_0 by

$$s_0(e_a) = \left(-\frac{\partial}{\partial t}, e_a, e_n\right), \tag{3.10}$$

where e_n is the vector in $T(M)$ such that $\theta(e_n) = 1, \theta^\alpha(e_n) = 0$. Here (e_a, e_n) are viewed in $T(E)$ via the splitting in (3.8).

In [F2], [W2], [BDS] a bundle is defined for $M = \partial N$, N a strictly pseudoconvex domain in $\mathbf{C}^n$. We will follow Webster's conventions. Adjoin an extra variable z^0, and consider $N \times \mathbf{C}^*$, $z^0 \neq 0$. For a defining function φ for N, define

$$\Phi(z^0, z^1, \ldots, z^n) = |z^0|^{2/n+1}\varphi(z^1, \ldots, z^n).$$

One obtains a Kähler metric of signature $(n, 1)$ on $N\times\mathbf{C}^*$ by

$$G=\Phi_{i\bar{j}}\,dz^i\cdot d\bar{z}^j. \tag{3.11}$$

Normalize Hermitian frames $(e_0, e_1, \ldots, e_n)$ so that

$$\begin{aligned} &G(e_\alpha, e_\beta)=\delta_{\alpha\bar{\beta}},\\ &G(e_\alpha, e_0)=G(e_\alpha, e_n)=0,\\ &G(e_0, e_0)=G(e_n, e_n)=0,\\ &G(e_0, e_n)=-i. \end{aligned} \tag{3.12}$$

This defines a $U(n, 1)$ bundle over $N\times\mathbf{C}^*$.

Throughout §§ 3–5, ω will denote the connection matrix of the metric connection of type $(1,0)$ associated to G, both on its $U(n,1)$-frame bundle, and its extension to the full $Gl(n+1,\mathbf{C})$-frame bundle. Likewise, Ω will stand for the curvature of this connection.

Relative to the local coframe near ∂N:

$$\omega^0=\frac{1}{n+1}\frac{dz^0}{z^0}-\eta_\alpha\,dz^\alpha+iQ\omega^n$$

$$\omega^\alpha=dz^\alpha, \quad \omega^n=-iu\partial\varphi,$$

where $u=|z^0|^{2/n+1}$, Webster shows the curvature components $\Omega_0{}^j=0$. As a result, we easily obtain:

LEMMA 3.1. *The curvature* $\Omega=(\Omega_i{}^j)$ *of* G *is independent of* $dz^0, d\bar{z}^0$.

Webster defines a bundle over $\partial N\times\mathbf{C}^*$ by adapting G-frames to the boundary as follows:

(a) $e_0=(n+1)\,z^0\,\partial/\partial z^0$,

(b) e_0, e_α span $T^{1,0}(\partial N\times\mathbf{C}^*)$, (3.13)

(c) $\mathrm{Re}(e_n)$ is tangent to $\partial N\times\mathbf{C}^*$ and $\mathrm{Im}(e_n)$ is transverse.

Call the bundle of such adapted frames $P_2\colon Z\to\partial N\times\mathbf{C}^*$. Map $\partial N\times\mathbf{C}^*$ to E by

$$p_1(p, z^0)=u\theta_p\in E$$

where $\theta=-i\partial\varphi$, and $u=|z^0|^{2/n+1}$. Cover this map by $\hat{p}_1\colon Z\to Y$ defined as:

$$\hat{p}_1(e_0, e_\alpha, e_n)=\left(-\frac{1}{2}p_{1*}(e_0), p_{1*}(e_\alpha), p_{1*}(e_n)\right).$$

Z is an S^1-bundle over Y. Webster compares the connection and curvature forms on Y and Z as follows: if φ is a 3rd order approximate solution of Fefferman's Monge–Ampère equation, as in (2.1) and (2.3), then

$$\begin{aligned} &\text{(a) } \hat{p}_1{}^*(\Pi_i{}^j) = \Omega_i{}^j \\ &\text{(b) } \hat{p}_1{}^*(\pi_i{}^j) = \omega_i{}^j - \mu\delta_i{}^j \end{aligned} \tag{3.14}$$

where $\mu+\bar{\mu}=0$, $d\mu=0$. The form μ accounts for the fiber of $\hat{p}_1$. The Monge–Ampère condition on φ at ∂N implies $\Omega_i{}^i=0$ on $\partial N\times\mathbf{C}^*$.

In what follows, it will be useful to have a section s_1 of $\hat{p}_1\colon Z\to Y$. For our purposes it suffices to construct the section over $X\times\mathbf{R}\subset Y$; it can be extended to all of Y using the structure groups. If $\{f_a\}$ is a frame in X, with $f_a=a_\alpha{}^\gamma\,\partial/\partial z^\gamma$, then (3.4) implies

$$(f_0, f_\alpha, f_n) = \left(-\frac{\partial}{\partial t}, e^{-t/2}a_\alpha{}^\gamma\frac{\partial}{\partial z^\gamma}, 2e^{-t}\,\mathrm{Re}\left[ir\varphi_j\frac{\partial}{\partial z^j}+b^\alpha f_\alpha\right]\right) \tag{3.15}$$

defines a frame in Y over $(p, e^t\theta_p)$, if the b^α verify

$$\theta^\beta\left(ir\varphi_j\frac{\partial}{\partial z^j}+b^\alpha f_\alpha\right)=0, \quad \beta=1,\dots,n-1. \tag{3.16}$$

(Here r is as in (2.6).) From (3.15), (3.16) we construct a frame in Z over the point $(p, e^{(n+1)t/2})$ by $s_1(f_0, f_\alpha, f_n)=(e_0, e_\alpha, e_n)$ with

$$\begin{aligned} &\text{(a) } e_0 = (n+1)\,z^0\,\partial/\partial z^0 \\ &\text{(b) } e_\alpha = e^{-t/2}a_\alpha{}^\gamma\,\partial/\partial z^\gamma \\ &\text{(c) } e_n = e^{-t}ir\varphi_j\,\partial/\partial z^j + b^\alpha e_\alpha + e^{-t}be_0. \end{aligned} \tag{3.17}$$

The constant b is determined by the conditions

$$\begin{aligned} &\text{(a) } \mathrm{Re}(b)=0 \\ &\text{(b) } G(e_n, e_n)=0. \end{aligned} \tag{3.18}$$

An easy calculation shows $\hat{p}_1\circ s_1=\mathrm{id}$ on Y, and Webster has shown that

$$s_1{}^*(\Omega)=\Pi. \tag{3.19}$$

The bundle Z is constructed as a sub-bundle of $\partial N\times\mathbf{C}^*\times Gl(n+1,\mathbf{C})$. The bundle X can be included into $\partial N\times Gl(n,\mathbf{C})$ by

$$i_0(\{f_\alpha\}) = \left(f_\alpha, ir\varphi_j \frac{\partial}{\partial z^j} + b^\alpha f_\alpha\right) \tag{3.20}$$

with b^α as in (3.15–3.16).

Putting all of the above together, we have the following diagram which summarizes the comparisons made above:

$$\begin{tikzcd}
\partial N \times Gl(n, \mathbf{C}) \arrow[rrrr, "j"] & & & & \partial N \times \mathbf{C}^* \times Gl(n+1, \mathbf{C}) \\
& X \arrow[ul, "i_0"'] \arrow[r, shift left, "s_0"] \arrow[d, "P_0"'] & Y \arrow[l, shift left, "\hat{p}_0"] \arrow[r, shift left, "s_1"] \arrow[d, "P_1"'] & Z \arrow[l, shift left, "\hat{p}_1"] \arrow[ur, "i_1"] \arrow[d, "P_2"'] & \\
& \partial N & E \arrow[l, "p_0"] & \partial N \times \mathbf{C}^* \arrow[l, "p_1"] &
\end{tikzcd} \tag{3.21}$$

Here j is the inclusion:

$$j(p, A) = \left(p, 1, \left[\begin{array}{c|c} A & 0 \\ \hline 0 & 1 \end{array}\right]\right). \tag{3.22}$$

We define a last map σ from $N \times \mathbf{C}^*$ to $N \times \mathbf{C}^* \times Gl(n+1, \mathbf{C})$ simply by $\sigma(p, z^0) = (p, z^0, I)$. The two inclusions of X into $\partial N \times \mathbf{C}^* \times Gl(n+1, \mathbf{C})$, $i_1 \circ s_1 \circ s_0$ and $j \circ i_0$, are homotopy equivalent.

We can complete this circle of comparisons by calculating the connection ω and curvature Ω on $\partial N \times \mathbf{C}^* \times Gl(n+1, \mathbf{C})$ in the standard frame of $\mathbf{C}^{n+1}$, i.e., their pull-backs via σ.

LEMMA 3.2 (a)

$$\sigma^*(\omega) = \begin{bmatrix} 0 & \frac{1}{n+1} dz^j \\ u_i & \theta_i{}^j \end{bmatrix}$$

(b)

$$\sigma^*(\Omega) = \begin{bmatrix} 0 & 0 \\ V_i & W_i{}^j \end{bmatrix}.$$

Here u_i is as in (2.24), *$\theta_i{}^j$ is as in* (2.19), *$W_i{}^j$ is as in* (2.25), *and*

$$V_i = du_i - \theta_i{}^k u_k. \tag{3.23}$$

Proof. We use the standard formulas for Kähler metrics relative to holomorphic frames. Thus,

$$\sigma^*(\omega) = \partial\sigma^*(G)\,\sigma^*(G)^{-1}. \tag{3.24}$$

Pulling back G amounts to setting $z^0=1$:

$$\sigma^*(G) = \begin{bmatrix} \varphi/(n+1)^2 & \varphi_{\bar j}/(n+1) \\ \varphi_i/(n+1) & \varphi_{i\bar j} \end{bmatrix} \tag{3.25}$$

and one can check that

$$\sigma^*(G)^{-1} = \begin{bmatrix} \dfrac{(n+1)^2}{J(\varphi)}\det(\varphi_{s\bar t}) & \dfrac{-\bar\xi^i B(n+1)}{J(\varphi)} \\ \dfrac{-\xi^j B(n+1)}{J(\varphi)} & h^{i\bar j} \end{bmatrix}. \tag{3.26}$$

Using (3.24–3.26), one can calculate

$$\sigma^*(\omega_i{}^k) = \begin{bmatrix} \dfrac{\det(\varphi_{s\bar t})\,\partial\varphi - A\bar\xi^j\partial\varphi_{\bar j}}{J(\varphi)} & \dfrac{h^{k\bar t}\partial\varphi_{\bar t}-(A/J(\varphi))\xi^k\partial\varphi}{n+1} \\ \dfrac{(n+1)[\det(\varphi_{s\bar t})\,\partial\varphi_i - A\bar\xi^j\partial\varphi_{i\bar j}]}{J(\varphi)} & h^{k\bar t}\partial\varphi_{i\bar t} - \dfrac{A}{J(\varphi)}\xi^k\partial\varphi_i \end{bmatrix}. \tag{3.27}$$

One uses the following easily verified facts:

$$\begin{aligned} &\text{(a)}\ \ \bar\xi^j\partial\varphi_{\bar j} = r\partial\varphi \\ &\text{(b)}\ \ \psi^{k\bar j}\partial\varphi_{\bar j} = dz^k-(1-r)\,\partial\varphi\,\xi^k \end{aligned} \tag{3.28}$$

as well as (2.11), (2,14), (2.19) and (2.24) to reduce (3.27) to the form given in the statement of the lemma.

The proof of Lemma 3.2(b) follows from part (a) of the lemma, the definition of Ω, the formula (2.25) for W and the fact that

$$dz^i\wedge u_i = dz^i\wedge\theta_i{}^j = 0.$$

Finally, we calculate the forms necessary for explicit transgression calculations. First let us define a non-commutative monomial $M(x,y)$ in two (non-commuting) variables x, y as a product of k factors, where each factor is either x or y. We will call k the degree of $M(x,y)$.

PROPOSITION 3.1. *For any non-commutative monomial $M(x, y)$, we have*

(a) $\sigma^*(\operatorname{tr} M(\omega, \Omega)) = \operatorname{tr}(M(\theta, W))$

(b) $(i/2\pi)^k \sigma^*(T \operatorname{tr}(\Omega^k)) = T\tau_k$.

Proof. We prove part (a) by induction on the degree k of M. For $k=1$, Lemma 3.2 implies (a). For $k \geq 2$, we claim

$$\sigma^*(M(\omega, \Omega)) = \begin{bmatrix} 0 & 0 \\ A_i & B_i \wedge dz^j + M(\theta, W) \end{bmatrix}, \tag{3.29}$$

where $dz^i \wedge A_i = dz^i \wedge B_i = 0$. If $k=2$, it is trivial to check (3.29), using as in §2 above:

$$dz^i \wedge u_i = dz^i \wedge \theta_i{}^j = dz^i \wedge W_i{}^j = 0. \tag{3.30}$$

Suppose that $M'(x, y) = M(x, y)x$ or $M(x, y)y$ is a monomial of degree $k+1$, where $M(x, y)$ is of degree k. By induction, we may assume (3.29) holds for $M(x, y)$. If $M'(x, y) = M(x, y)x$, say, then

$$\begin{aligned} \sigma^*(M'(\omega, \Omega)) &= \begin{bmatrix} 0 & 0 \\ A_i & B_i \wedge dz^j + M(\theta, W) \end{bmatrix} \cdot \begin{bmatrix} 0 & \dfrac{dz^l}{n+1} \\ u_j & V_j{}^l \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 \\ B_i \wedge dz^j \wedge u_j + M \wedge u & \dfrac{A_i \wedge dz^l}{n+1} + B_i \wedge dz^j \wedge \theta_j{}^l + M'(\theta, W) \end{bmatrix}. \end{aligned}$$

(3.30) now shows that the induction is complete. The case $M'(s, y) = M(s, y) \cdot y$ is treated in a completely similar manner, proving (3.29). Taking traces in (3.29) proves part (a) of the lemma. Part (b) follows directly from part (a) and the definition of TP_k in (2.40).

We conclude this section with several remarks on the comparisons and calculations above. In invariant terms, $N \times \mathbf{C}^*$ is a trivialization of the $(n+1)$st root of the canonical bundle, $K^{1/n+1}$, of $\mathbf{C}^n$ (with the zero section removed). All of the frame bundles above are subbundles of the full holomorphic frame bundle of $K^{1/n+1}$. There is really just one transgression formula, defined on this last bundle:

$$T(\operatorname{tr}(\Omega^j)) = j \int_0^1 \operatorname{tr}[\omega \wedge (t\Omega + (t-t^2)\,\omega \wedge \omega)^{j-1}]\, dt. \tag{3.31}$$

Formula (3.31) is valid for any choice of framing of type $(1, 0)$. All the other transgression formulas are consequences of (3.31). For domains $N \subset \mathbf{C}^n$, the holomorphic frame bundle of $T^{1,0}(K^{1/n+1})$ is just $K^{1/n+1} \times Gl(n+1, \mathbf{C})$. The comparisons above can be made in more general contexts, but the non-trivial topology of $K^{1/n+1}$ will lead to additional terms in the integration by parts formulas analogous to those of § 2 above.

Since the formula in Theorem 2.2 makes explicit use of the frame $(\partial/\partial z^1, \ldots, \partial/\partial z^n)$, and this frame doesn't belong to any of X, Y or Z, this formula does not have any intrinsic meaning on ∂N. In general, $Y \to \partial N$ will *not* have a section, and so there will not be a direct way to compare the holomorphic transgression defined on ∂N with transgression forms on Y. In § 4 we circumvent this difficulty by finding a homological section of Y over ∂N, i.e., a $(2n-1)$-cycle C in Y such that $P_{1*}[C]=[M] \in H_{2n-1}(E, \mathbf{Z}) \cong H_{2n-1}(M, \mathbf{Z})$. (In general, such cycles don't exist either, but for $M=\partial N$ in $\mathbf{C}^n$, we construct one below.)

A second issue arises: in general, the Chern–Simons theory [CS] says that for $[\tilde{C}] \in H_{2n-1}(Y, \mathbf{Z})$ such that $P_{1*}[\tilde{C}]=0$. Then

$$\int_{\tilde{C}} TP \in \mathbf{Z}$$

if $dTP=0$, and P is an integral class. In general this integer is non-zero, and if this is the case, the class $[TP] \in H^{2n-1}(Y; \mathbf{R})$ is not the pull-back of a class in $H^{2n-1}(M, \mathbf{R})$. Again, for $M=\partial N$ in $\mathbf{C}^n$ we show below that these integers are all 0. We thus will have created canonical classes $[\tilde{T}P] \in H^{2n-1}(M; \mathbf{R}) \cong H^{2n-1}(E; \mathbf{R})$ such that $P_1{}^*[\tilde{T}P]=[TP]$. Since Y doesn't have a section over M, this class may differ from that given by the holomorphic framing in § 2 above.

Finally, we remark that our calculations in the CR case are quite similar to a procedure outlined in [FG] for constructing scalar invariants of a conformal structure. The complete Einstein–Kähler structure on N can essentially be realized as a structure induced on a hypersurface $i(N)$ in $K^{1/n+1}$ by the Ricci-flat Lorentz metric. As one approaches infinity in $N, i(N)$ approaches infinity in the fiber of $K^{1/n+1}$. If P is an invariant polynomial of degree n then $P(\Omega)$ defines an invariant of weight zero in the terminology of [FG]. This explains why $P(\Omega)$ is finite: it is constant along the fibers of $K^{1/n+1}$ and obviously bounded along the section σ defined above. A propos the comments at the beginning of section III of [FG], we remark that the only invariants of the complete "Poincaré metric" with finite boundary values are a subset of the Weyl invariants of weight zero for the Ricci-flat "ambient metric".

§ 4. Boundary classes and homological sections

In the bundles Y and Z (notation as in § 3) we can define secondary characteristic forms using the curvature forms Π and Ω respectively. If Z is defined using a third order approximate solution of the Monge–Ampère equation, then we have from (3.14), (3.19) that

$$\hat{p}_1{}^*P(\Pi) = P(\Omega) \tag{4.1}$$

$$s_1{}^*P(\Omega) = P(\Pi) \tag{4.2}$$

for P any Ad-invariant polynomial on $\mathfrak{gl}(n+1, \mathbf{C})$.

Set $\tilde{\pi} = s_1{}^*(\omega)$. Then

$$d\tilde{\pi} - \tilde{\pi} \wedge \tilde{\pi} = s_1{}^*(\Omega) = \Pi.$$

By the Uniqueness theorem 5.1 of [CM], we conclude $\pi = s_1{}^*(\omega)$. Since $\hat{p}_1 \circ s_1 = \mathrm{id}_Y$, (3.19) implies

$$\pi = s_1{}^* \circ \hat{p}_1{}^*(\pi) = s_1{}^*(\omega - \mu\delta_i{}^j) = \pi - s_1{}^*(\mu)\,\delta_i{}^j.$$

Thus, $s_1{}^*(\mu) = 0$, and therefore,

$$s_1{}^*(TP(\omega, \Omega)) = TP(\pi, \Pi), \tag{4.3}$$

where TP is the canonical transgression of (3.5) in [CS].

Lemma 3.1 implies that $P(\Omega)|_{\partial\Omega \times \mathbf{C}^*} \equiv 0$, if P is an invariant polynomial of degree $\geqslant n$. Similarly, $\Pi \equiv 0 \bmod (\pi^n, \pi^\alpha, \pi^{\bar{\alpha}})$ implies $P(\Pi) \equiv 0$ on Y for P of degree $\geqslant n$. Thus, we conclude from (4.3)

$$s_1{}^* \circ i_1{}^*(TP(\omega, \Omega)) = TP(\pi, \Pi), \tag{4.4}$$

and we have classes $[TP(\omega, \Omega)] \in H^{2n-1}(\partial N \times \mathbf{C}^* \times Gl(n+1, \mathbf{C}); \mathbf{R})$, and $[TP(\pi, \Pi)] \in H^{2n-1}(Y; \mathbf{R})$. Since the Chern bundle Y is functorial for biholomorphic maps, $[TP(\pi, \Pi)]$ is a biholomorphic invariant.

We would like to pull the class $[TP(\pi, \Pi)]$ down to ∂N so that we may define CR-characteristic numbers for ∂N which we can compare with the boundary integrals of § 2 and relate to the renormalized Chern numbers of N. As already noted at the end of § 3, we will here show that there exist homologial sections of Y over ∂N (if $\partial N \subset \mathbf{C}^n$), enabling us to define characteristic numbers. We postpone until § 5 the proof that these numbers are independent of the homological section chosen, and the comparison of these numbers with the renormalized Chern numbers.

In this section, M will be a closed s.ψ.c. hypersurface in $\mathbf{C}^n$. If φ is any defining function for M, let θ be the contact one-form $-i\partial\varphi$ on M. In § 3 above we recalled the construction from θ of the $U(n-1)$ principle bundle X over M. Our purpose here is to prove the following theorem on the existence of a homological section of X over M:

THEOREM 4.1. *Let M be a closed s.ψ.c. hypersurface in $\mathbf{C}^n$. Then there exists a class $\hat{x}_{2n-1} \in H_{2n-1}(X;\mathbf{Z})$ such that $p_{0*}(\hat{x}_{2n-1})=[M] \in H_{2n-1}(M;\mathbf{Z})$.*

Remarks. (1) Since X is homotopy equivalent to Y, one can consider $\hat{x}_{2n-1}$ in $H_{2n-1}(Y;\mathbf{Z})$.

(2) This provides a topological obstruction to the codimension 1 embedding (or even immersion) in $\mathbf{C}^n$ of an abstract compact s.ψ.c. CR-manifold M. In the case of M of real dimension 3, this reduces to the condition that $T^{1,0}(M)$ be trivial, as in [BE]. It is known that this obstruction to embedding is non-vacuous in this case.

The bundle X could be described equivalently as a bundle of unitary frames in $T^{1,0}(M)$ for the Levi-form of φ as a metric on $T^{1,0}(M)$. The homotopy type of X is independent of Hermitian metric chosen on $T^{1,0}(M)$. Thus, for the problem at hand, we can consider $\hat{X}$=the bundle of unitary frames in $T^{1,0}(M)$ for the Euclidean metric in $\mathbf{C}^n$.

$\hat{X}$ has a simple "universal" description in terms of the Gauss map. Let S^{2n-1} be the unit sphere in $\mathbf{C}^n$. The group $U(n)$ acts on S^{2n-1}, and defines a $U(n-1)$-principle bundle over S^{2n-1} via

$$U(n) \ni g \overset{q_0}{\mapsto} g\cdot(1,0,\ldots,0) \in S^{2n-1}. \tag{4.5}$$

This is simply X over $M=S^{2n-1}$ for the contact form $\theta=-i\bar{z}^j dz^j$. For a general M, with defining function φ, and $p\in M$, define the Gauss map by:

$$g(p) = (\varphi_{\bar{1}}(p), \ldots, \varphi_{\bar{n}}(p))/|d\varphi|^2 \tag{4.6}$$

where $|d\varphi|^2$ is here measured with respect to the Euclidean metric. $\hat{X}$ is the pull-back by g of q_0: $Y(n)\to S^{2n-1}$:

$$\begin{tikzcd} \hat{X} \arrow[r, "g_\#"] \arrow[d, "q_1"'] & U(n) \arrow[d, "q_0"] \\ M \arrow[r, "g"'] & S^{2n-1}. \end{tikzcd}$$

To define a cycle in $H_{2n-1}(\hat{X};\mathbf{Z})$ we will use Poincaré duality, and we start with a description of the homology and cohomology of $U(n)$. The basic facts are these:

$$\begin{aligned} H_*(U(n)) &= \wedge(x_1, \dots, x_{2n-1}) \\ H^*(U(n)) &= \wedge(y^1, \dots, y^{2n-1}) \end{aligned} \tag{4.7}$$

where each x_i is a primitive cycle in dimension i, and the y^i are the dual primitive cocycles. The cycles $x_1, \dots, x_{2n-3}$ lie in the fiber of q_0, but:

$$q_0^* V_{S^{2n-1}} = y^{2n-1}, \tag{4.8}$$

where $\langle [S^{2n-1}], V_{S^{2n-1}} \rangle = 1$. From this it follows that on $U(n)$, the Poincaré dual of x_{2n-1} is $y^1 \dots y^{2n-3}$.

For a general M, define $\hat{x}_{2n-1}$ to be the Poincaré dual of $g_\#^*(y^1 \dots y^{2n-3})$. One of two possibilities must occur:

$$\begin{aligned} &\text{(a) } \deg g = 0, \text{ and } \hat{X} \text{ has a section, or} \\ &\text{(b) } \deg g \neq 0, \text{ and } q_{1*}(\hat{x}_{2n-1}) = [M]. \end{aligned} \tag{4.9}$$

Since (a) is clear, consider case (b):

$$\begin{aligned} \langle q_{1*}(\hat{x}_{2n-1}), V_M \rangle &= \langle \hat{x}_{2n-1}, q_1^* V_M \rangle \\ &= \frac{1}{\deg g} \langle \hat{x}_{2n-1}, q_1^* \circ g^* V_{S^{2n-1}} \rangle \\ &= \frac{1}{\deg g} \langle \hat{x}_{2n-1}, g_\#^* \circ q_0^* V_{S^{2n-1}} \rangle \\ &= \frac{1}{\deg g} \langle [\hat{X}], g_\#^*(y^1 \dots y^{2n-3}) \cdot g_\#^* \circ q_0^* V_{S^{2n-1}} \rangle \\ &= \frac{\deg g_\#}{\deg g} \langle [U(n)], y^1 \dots y^{2n-3} \cdot q_0^* V_{S^{2n-1}} \rangle \\ &= \langle [U(n)], y^1 \dots y^{2n-1} \rangle = 1. \end{aligned}$$

We have used $\deg g = \deg g_\#$ (because $g_\#$ is a bundle map), and (4.8).

In the sequel we will need a little more precision than the statement above.

PROPOSITION 4.1. *Let $M = \partial N$, $N \subset \mathbf{C}^n$, and let $i_0 : X \to \partial N \times Gl(n, \mathbf{C})$ be the map given in* (3.20) *above. Then if the Euler characteristic $\chi(N) \neq 0$,*

$$i_{0*}(\hat{x}_{2n-1}) = [\partial N] - \chi(N) x_{2n-1} \tag{4.10}$$

in $H_{2n-1}(\partial N \times Gl(n;\mathbf{C}), Q)$, *where* x_{2n-1} *is the universal class in* $H_{2n-1}(U(n))$ *described above.*

Proof. We start by denoting by q_2, resp. q_3, the projection of $\partial N \times U(n)$ to ∂N, resp. $U(n)$. Calculating as in the proof of Theorem 4.1, one sees directly that $g_{\#*}(\hat{x}_{2n-1}) = (\deg g)\, x_{2n-1} = -\chi(N)\, x_{2n-1}$, the last equality by the Gauss–Bonnet theorem Since $q_1 = q_2 \circ i_0$, and $q_{1*}(\hat{x}_{2n-1}) = [\partial N]$, one has

$$i_{0*}(\hat{x}_{2n-1}) = [\partial N] - \chi(N)\, x_{2n-1} + c \tag{4.11}$$

where c is annihilated by both q_{2*} and q_{3*}.

Let $\beta = q_2^*(z) \otimes q_3^*(y^{i_1} \dots y^{i_k}) \in H^{2n-1}(M \times U(n))$, where $z \in H^i(M)$, and $i + i_1 + \dots + i_k = 2n-1$, $i \neq 0, 2n-1$. To show $c=0$, it suffices to show $\langle c, \beta \rangle = 0$. If $\chi(N) \neq 0$, we use $\hat{x}_{2n-1}$, as defined above, and compute:

$$\begin{aligned}
\langle c, \beta \rangle &= \langle i_{0*}(\hat{x}_{2n-1}), \beta \rangle \\
&= \langle [\hat{X}],\ g_{\#}^*(y^1 \dots y^{2n-3}) \cdot i_0^*(\beta) \rangle \\
&= \langle i_{0*}[\hat{X}],\ q_3^*(y^1 \dots y^{2n-3}) \cdot \beta \rangle \\
&= 0,
\end{aligned}$$

since $q_3^*(y^1 \dots y^{2n-3}) \cdot \beta = 0$.

Remark. Note that (4.11) holds even if $\chi(N) = 0$.

§ 5. Homotopy of connections and independence of homological section

In this section our first goal is to prove the following theorem.

THEOREM 5.1. *Let* $c \in H_{2n-1}(Y;\mathbf{Q})$ *project to* 0 *in* $H_{2n-1}(\partial N;\mathbf{Q})$. *Then*

$$\int_c TP(\pi, \Pi) = 0 \tag{5.1}$$

for any ad-*invariant polynomial* P.

The proof will be based on the relation

$$TP(\pi, \Pi) = s_1^* \circ i_1^* TP(\omega, \Omega), \tag{5.2}$$

of (4.4) above. We will deform the connection ω on $\partial N\times\mathbf{C}^*\times Gl(n+1,\mathbf{C})$ to a family of flat connections on $\partial N\times\mathbf{C}^*\times Gl(n+1,\mathbf{C})$, which will reduce the evaluation of (5.1) to some variants of standard facts in the classical development of characteristic classes.

We will first deform ω by deforming the underlying Fefferman metric $h_0=G$, of § 3 above. For h a Hermitian metric on $N\times\mathbf{C}^*$, of signature $(n,1)$, let $\mathcal{U}(h)$ denote the corresponding bundle of $(1,0)$-frames normalized as in (3.12). Topologically, $\mathcal{U}(h_0)$ is diffeomorphic to $N\times\mathbf{C}^*\times U(n,1)$. We set $U(h)=\mathcal{U}(h)|_{\partial N\times\mathbf{C}^*}$. Let φ_0 be a defining function for N which is a Fefferman approximate solution for the Monge–Ampère equation along ∂N. We begin our deformation by homotoping φ_0 to a defining function φ_1 which is strictly plurisubharmonic in a neighborhood of $\bar{N}$, e.g., by

$$\psi_t=(1-t)\,\varphi_0+t\varphi_1,\quad 0\leqslant t\leqslant 1.$$

Next, let R be a constant large enough that $|z|<R$ on N. Set

$$\psi_t=(2-t)\,\varphi_1+(t-1)(|z|^2-R^2),\quad 1\leqslant t\leqslant 2, \tag{5.3}$$

in a neighborhood of N, and set $\Psi_t=|z^0|^{2/n+1}\psi_t(z)$ on $N\times\mathbf{C}^*$. Defining $h_t=(\Psi_t)_{i\bar{j}}dz^i d\bar{z}^j$, $0\leqslant i,j\leqslant n$. By computing $J(\psi_t)$ on ∂N one easily sees that we have a 1-parameter family of non-degenerate Kähler–Lorentz metrics of signature $(n,1)$ in a neighborhood of $\partial N\times\mathbf{C}^*$, for $t\in[0,2]$. Let $\omega_t=\omega(h(t))$ be the connection form for h_t, and $\Omega_t=d\omega_t-\omega_t\wedge\omega_t$ its curvature form.

We will need to calculate the matrix ω_2 with respect to the standard frame of $N\times\mathbf{C}^*$. Call this Γ_2. One computes readily that

$$\Gamma_2=\begin{pmatrix}\dfrac{-n}{n+1}\dfrac{dz^0}{z^0} & \dfrac{1}{n+1}\dfrac{dz^j}{z^0}\\ 0 & \dfrac{1}{n+1}\dfrac{dz^0}{z^0}\delta_{ij}\end{pmatrix},\quad 1\leqslant i,j\leqslant n, \tag{5.4}$$

and, in particular, $\Omega_2=0$.

LEMMA 5.1. *$P(\Omega_t)|_{U(h)}\equiv 0$, where P is any* Ad*-invariant polynomial of degree $\geqslant n$, and $t\in[0,2]$.*

Proof. Indeed, the calculation of Webster's referred to in Lemma 3.1 above shows the Ω_t do not involve dz^0, $d\bar{z}^0$, whence the lemma.

Thus, if we extend the connections ω_t to all of $\partial N\times\mathbf{C}^*\times Gl(n+1,\mathbf{C})$, $TP(\omega_t,\Omega_t)$ defines a cohomology class in $H^{2n-1}(\partial N\times\mathbf{C}^*\times Gl(n+1,\mathbf{C}))$.

We are next going to simplify ω_2 further. We deform ω_2 through two families of connections on $\mathbf{C}^n\times\mathbf{C}^*\times Gl(n+1,\mathbf{C})$, preserving the condition of Lemma 5.1 along $\partial N\times\mathbf{C}^*$. We will write these deformations out in terms of the Christoffel matrices with respect to the standard frame, as in (5.4) above.

Define first

$$\Gamma_t=\begin{pmatrix} \dfrac{-n}{n+1}\dfrac{dz^0}{z^0} & \dfrac{(3-t)}{n+1}\dfrac{dz^j}{z^0} \\ 0 & \dfrac{1}{n+1}\dfrac{dz^0}{z^0}\delta_{ij} \end{pmatrix}, \quad 2\leqslant t\leqslant 3,$$

and then

$$\Gamma_t=\begin{pmatrix} \dfrac{-(4-t)\,n}{n+1}\dfrac{dz^0}{z^0} & 0 \\ 0 & \dfrac{(4-t)}{n+1}\dfrac{dz^0}{z^0}\delta_{ij} \end{pmatrix}, \quad 3\leqslant t\leqslant 4.$$

Note that $\Omega_t\equiv 0$ over $\mathbf{C}^n\times\mathbf{C}^*$, $2\leqslant t\leqslant 4$, and that $\Gamma_4\equiv 0$ on $\mathbf{C}^n\times\mathbf{C}^*$.

LEMMA 5.2. *For*

$$c\in H_{2n-1}(\mathbf{C}^n\times\mathbf{C}^*\times Gl(n+1,\mathbf{C}))$$

which projects to 0 *in* $H_{2n-1}(\partial N\times\mathbf{C}^*)$, *and* P *an* Ad-*invariant polynomial of degree* n,

$$\int_c TP(\omega_t,\Omega_t)$$

is independent of $t\in[0,4]$.

Proof. Without loss of generality, we may suppose that c is an integral cycle, and P an integral invariant polynomial. Then Theorem 3.16 of [CS] says that $\int_c TP(\omega_t,\Omega_t)\in\mathbf{Z}$, $t\in[0,4]$. Since the formula for $TP(\omega_t,\Omega_t)$ is continuous in t, the integrals are constant.

We can now use (5.2) and Lemma 5.2 to conclude

$$\int_c TP(\pi,\Pi)=\int_{(i_1\circ s_1)_*(c)} TP(\omega_4,\Omega_4). \tag{5.5}$$

Since $\Gamma_4\equiv 0$ on $\partial N\times\mathbf{C}^*\times Gl(n+1,\mathbf{C})$, ω_4 is simply the Maurer–Cartan form of $Gl(n+1,\mathbf{C})$ pulled up to $\partial N\times\mathbf{C}^*\times Gl(n+1,\mathbf{C})$. Since $\Omega_4\equiv 0$,

$$TP(\omega_4, \Omega_4) = k(P)\,\mathrm{tr}(\omega_4{}^{2n-1}), \tag{5.6}$$

where the constant $k(P)$ depends on P. In any event,

$$\int_c TP(\omega_t, \Omega_t) = 0$$

for any class $c \in H_{2n-1}(\mathbf{C}^n \times \mathbf{C}^* \times Gl(n+1, \mathbf{C}))$ which is not a pure fiber class, i.e., unless

$$c = 1 \otimes a, \quad a \in H_{2n-1}(Gl(n+1, \mathbf{C})). \tag{5.7}$$

To evaluate (5.1) via (5.5), since Y is homotopy equivalent to X (as in § 3), we can assume that a in (5.7) in fact comes from $H_{2n-1}(U(n-1))$. Using the notation of § 4, then, the following lemma is the key evaluation we need.

LEMMA 5.3. *For* $x_1, \ldots, x_{2n-1} \in H_*(U(n))$,

(a)

$$\int_{x_{i_1} \cdots x_{i_l}} \mathrm{tr}(\omega_{\mathrm{MC}}^{2n-1}) = 0 \quad \textit{if} \quad i_1 + \ldots + i_l = 2n-1,$$

and two i_j are non-zero, while

(b)

$$\int_{x_{2n-1}} \mathrm{tr}(\omega_{\mathrm{MC}}^{2n-1}) = n\binom{2n-1}{n}(2\pi i)^n.$$

Here ω_{MC} is the Maurer–Cartan form on $U(n)$.

Proof. Introduce the Grassmannian $\mathrm{Gr}(n, N)$ of n planes in $\mathbf{C}^N$, $n \ll N$, and let E_n be the canonical n-plane bundle on $\mathrm{Gr}(n, N)$, $F(E_n)$ its bundle of unitary frames. We have a diagram:

$$\begin{tikzcd}
U(n-1) \arrow[rr] \arrow[dr] & & U(n) \arrow[dr] & \\
& F(E_{n-1}) \arrow[rr, "\tilde{j}"] \arrow[d] & & F(E_n) \arrow[d, "\pi"] \\
& \mathrm{Gr}(n-1, N-1) \arrow[rr, "j"] & & \mathrm{Gr}(n, N)
\end{tikzcd} \tag{5.8}$$

where j includes $\mathrm{Gr}(n-1, N-1)$ as all n-planes containing a fixed vector in $\mathbf{C}^N$. Let ω_n be the standard connection on E_n, Ω_n its curvature. Then $dTc_n(\omega_n, \Omega_n)=\pi^* c_n(\Omega_n)$. Note that

$$Tc_n(\omega_n, \Omega_n) \equiv (-1)^{n+1} \frac{1}{n} \left(\frac{i}{2\pi}\right)^n \frac{1}{\binom{2n-1}{n}} \operatorname{tr}(\omega_n^{2n-1}) \tag{5.9}$$

modulo terms which are exact when retricted to the fiber $U(n)$. Note also that ω_n restricts to ω_{MC} on $U(n)$. Since $\tilde{j}^*(TC_n(\omega_n, \Omega_n))$ is universally transgressive on $F(E_{n-1})$, in the sense of [B, § 19], its restriction to the fiber $U(n-1)$ is primitive ([B], Proposition 20.1). Since $H^*(U(n-1))$ has no primitive class of degree $2n-1$, $\tilde{j}^*(Tc_n(\omega_n, \Omega_n))$ is exact on $U(n-1)$. Restricting $Tc_n(\omega_n, \Omega_n)$ first to $U(n)$ then $U(n-1)$ in (5.8) and using (5.9) proves (a) of the lemma.

To prove (b), let $S(E_n)$ be the bundle of unit vectors in E_n. We have a diagram:

$$\begin{tikzcd}
U(n) \arrow[r] \arrow[d] & S^{2n-1} \arrow[d] \\
F(E_n) \arrow[r, "p"] \arrow[dr, "\pi"'] & S(E_n) \arrow[d, "q"] \\
& \mathrm{Gr}(n, N)
\end{tikzcd} \tag{5.10}$$

where the top horizontal map is that of § 4 above. As in [BC], there is a canonical $2n-1$ form Φ on $S(E_n)$ such that $d\Phi=q^*c_n(\Omega_n)$. Then $dTc_n(\omega_n, \Omega_n)=\pi^*c_n(\Omega_n)=dp^*(\Phi)$. Since $H^{2n-1}(F(E_n))=0$, $Tc_n(\omega_n, \Omega_n)-p^*(\Phi)$ is exact, and

$$\int_{x_{2n-1}} Tc_n(\omega_n, \Omega_n) = \int_{S_o(E_n)} \Phi \tag{5.11}$$

where $S_o(E_n)$ is the fiber of $S(E_n)$ over any point $o \in \mathrm{Gr}(n, N)$.

One can evaluate $\int_{S_o(E_n)} \Phi$ as in the proof of the generalized Gauss–Bonnet theorem in [BC]. One considers the standard n-plane bundle Q on $\mathbf{P}^n$ (which admits a holomorphic section with an isolated simple zero). Let $f: \mathbf{P}^n \to \mathrm{Gr}(n, N)$, $N \gg 0$, be a classifying map. We have a diagram:

$$\begin{tikzcd} S(Q) \arrow[r, "\tilde{f}"] \arrow[d, "f_q"'] & S(E_n) \arrow[d, "q"] \\ \mathbf{P}^n \arrow[r, "f"] & \mathrm{Gr}(n, N) \end{tikzcd} \tag{5.12}$$

and on $S(Q)$,

$$d\tilde{f}^*(\Phi) = f_q^* c_n(Q). \tag{5.13}$$

(We pull-back the metric and connection on E_n to Q.) Let s be a section of Q with one simple zero at $0 \in \mathbf{P}^n$, and let $\sigma = s/|s|$ be the corresponding section from $\mathbf{P}^n - \{0\}$ to $S(Q)|_{\mathbf{P}^n - \{0\}}$. Let $B(\varepsilon)$ be an ε-ball in $\mathbf{P}^n$ centered at 0. Then (5.13) and Stokes's theorem imply

$$\begin{aligned} 1 = \int_{\mathbf{P}^n} c_n(Q) &= -\lim_{\varepsilon \to 0} \int_{\partial B(\varepsilon)} \sigma^*(\tilde{f}^*\Phi) \\ &= -\int_{S_o(Q)} \tilde{f}^*\Phi \\ &= -\int_{S_o(E)} \Phi. \end{aligned} \tag{5.14}$$

Putting (5.14), (5.11) and (5.9) together, we see that part (b) of the lemma is proved.

One has only to remark that part (a) of the lemma suffices to complete the proof of Theorem 5.1.

As noted at the beginning of § 4, Theorem 5.1 and the results of § 4 show that to every Ad-invariant polynomial P of degree n, we can associate a CR-characteristic number $\int_c TP(\pi, \Pi)$, where c is any class in $H_{2n-1}(Y, \mathbf{Z})$ such that $P_{1*}(c) = [\partial N] \in H_{2n-1}(\partial N; \mathbf{Z})$. In particular, we can take $c = \dot{x}_{2n-1}$, as described in § 4. Equivalently, we can associate to $TP(\pi, \Pi)$ a CR-invariant cohomology class $[\tilde{T}P(\pi, \Pi)] \in H^{2n-1}(\partial N; \mathbf{R})$ such that $P_1^*[\tilde{T}P(\pi, \Pi)] = [TP(\pi, \Pi)]$.

We will next put this result together with the formulas of § 2 to derive our main theorems, which give a generalized Gauss–Bonnet theorem relating the renormalized characteristic numbers of N and the boundary characteristic numbers we have just defined. We will again express them explicitly in terms of the renormalized trace powers.

Let

$$P_I(A)=\left(\frac{i}{2\pi}\right)^n \operatorname{tr}(A^{i_1})\dots\operatorname{tr}(A^{i_p}),$$

where $I=\{i_1,\dots,i_p\}$, $2\leq i_1\leq\dots\leq i_p$, and $i_1+\dots+i_p=n$.

THEOREM 5.2. (a) *If* $p>1$

$$\int_N \tau_{i_1}\dots\tau_{i_p}=\int_{\hat{x}_{2n-1}} TP_I(\pi,\Pi);$$

(b) *if* $p=1$,

$$\int_N \tau_n=\int_{\hat{x}_{2n-1}} TP_{\{n\}}(\pi,\Pi)+\chi(N).$$

Proof. We know from (5.2) that

$$\int_{\hat{x}_{2n-1}} TP_I(\pi,\Pi)=\int_{(i_1\circ s_1)_*(\hat{x}_{2n-1})} TP_I(\omega,\Omega). \tag{5.15}$$

From (4.10) we have: (a) if $\chi(N)\neq 0$,

$$(i_1\circ s_1)_*(\hat{x}_{2n-1})=[\partial N]\otimes 1-\chi(N)\,1\otimes[x_{2n-1}],$$

or (b) if $\chi(n)=0$,

$$(i_1\circ s_1)_*(\hat{x}_{2n-1})=[\partial N]\otimes 1+c,$$

where $P_{1*}(c)=0$. Thus, (5.15) implies

$$\int_{\hat{x}_{2n-1}} TP(\pi,\Pi)=\int_{\partial N} TP(\omega,\Omega)-\chi(N)\int_{\hat{x}_{2n-1}} TP(\omega,\Omega). \tag{5.16}$$

Since Ω is identically zero when restricted to a fiber, the second term on the right is 0 for $P=P_I$, unless $I=\{n\}$. In case $P=P_{\{n\}}$, the integral on the right is -1, as in the proof of Lemma 5.3 (b). From Theorem 2.2 we conclude

$$\int_{\partial N} TP(\omega,\Omega)=\int_{\partial N} T(\tau_{i_1}\cdot\dots\cdot\tau_{i_p})=\int_N \tau_{i_1}\cdot\dots\cdot\tau_{i_p}$$

proving the theorem.

As a possible application of Theorem 5.2, consider the question of which abstract s.ψ.c. CR structures on the sphere S^3 may be embedded in $\mathbf{C}^2$ as a s.ψ.c. hypersurface. In dimension two, the formula of Theorem 5.2(b) becomes explicitly:

$$\int_N c_2-\tfrac{1}{3}c_1{}^2=\chi(N)+\mu(\partial N), \tag{5.17}$$

where $\mu(\partial N)$ is as in the introduction or [BE]. As shown in [BE], $\mu(\partial N)$ can be calculated from knowledge of the abstract CR structure given on $M=\partial N$. The region N bounded by such a hypersurface in $\mathbf{C}^2$ would have to be homeomorphic to the standard ball, so $\chi(N)$ is necessarily 1. On the other hand, the integrand on the left is well-known (cf., e.g., [Y]) to be $\geqslant 0$ everywhere, if calculated in the Einstein–Kähler metric of N, and $\equiv 0$ if and only if N is biholomorphic to the standard ball. Putting these facts together, we get the following corollary.

COROLLARY 5.1. *Let M be a s.ψ.c. CR manifold homeohorphic to S^3. A necessary condition for M to admit a CR embedding into $\mathbf{C}^2$ is the inequality:*

$$\mu(M)\geqslant -1. \tag{5.18}$$

If $\mu(M)=-1$, and M embeds in $\mathbf{C}^2$, then M is CR equivalent to the standard boundary of the ball $\mathbf{B}^2$.

We call this a potential application of Theorem 5.2 because we do not know of an example of a s.ψ.c. CR structure on S^3 which has $\mu<-1$. Indeed, Cheng and Lee have recently proven ([CL]) that at the standard structure on S^3 the functional μ has a non-negative second variation. Whether $\mu<-1$ for structures distant from the standard one is still an open question.

§ 6. Another method of proof for $n=2$

When $n=2$ we can prove a result like Theorem 5.2 in a slightly more general geometric setting. The proof follows the lines of Chern's classic argument, as in [BC], § 6, for example. As is often the case with secondary characteristic classes, it seems difficult to state optimal hypotheses for a theorem like Theorem 6.1 below. We offer this version as a sample, and will consequently be somewhat terse about the necessary computations. (They are elementary, if somewhat tedious.) We conclude this § by comparing some of the examples calculated in [BE] with the present work.

In this section we let N be a compact strictly pseudoconvex complex manifold with smooth boundary ∂N, which for convenience we will assume is contained in a slightly larger complex manifold N'. We will assume that N admits a volume form v whose Ricci form is identically zero in a neighborhood of the boundary.([1]) In this case, it is well known that we can find local holomorphic coordinates $z^1, \ldots, z^n$ in a neighborhood of any point where $\mathrm{Ric}(v) \equiv 0$ such that v is given locally as

$$v = (i/2)^n dz^1 \wedge d\bar{z}^1 \wedge \ldots \wedge dz^n \wedge d\bar{z}^n. \tag{6.1}$$

We will call such a coordinate system unimodular. Let φ be a defining function for ∂N, strictly plurisubharmonic in a neighborhood of ∂N. We look for Kähler metrics $g = g_{i\bar{j}}$ on N which verify the Einstein–Kähler equation

$$\mathrm{Ric}_{i\bar{j}}(g) = -(n+1)\, g_{i\bar{j}} \tag{6.2}$$

in a neighborhood of ∂N. If we look for g of the form

$$g_{i\bar{j}} = (\log(-1/\varphi))_{i\bar{j}} \tag{6.3}$$

in a neighborhood of ∂N, we find that the sufficient differential equation for φ is given locally, in a unimodular coordinate system, by the Monge–Ampère equation:

$$J(\varphi) = \det \begin{pmatrix} \varphi & \varphi_{\bar{j}} \\ \varphi_i & \varphi_{i\bar{j}} \end{pmatrix} = -1. \tag{6.4}$$

Using the algorithm described in [F1], one can construct approximate solutions to (6.4) up to order $n+1$. From now on, we will assume that φ is a third order approximation, i.e., satisfies

$$J(\varphi) = -1 + O(\varphi^3), \tag{6.5}$$

and that $g_{i\bar{j}}$ is a Hermitian metric on N which is as in (6.3) near the boundary. If N' admits a smooth volume form whose Ricci form is negative, then N admits a complete Einstein–Kähler metric [CY] which is asymptotic to (6.3) with φ as in (6.4).

We want to consider the renormalized Chern form $\tilde{c}_2 = c_2 - \frac{1}{3} c_1{}^2$, for N of dimension 2. The form $\tilde{c}_2$ is integrable on N and we want to integrate by parts to derive a Gauss–Bonnet theorem as in Theorem 5.2 expressing $\int_N \tilde{c}_2$ in terms of the invariants of

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([1]) We don't know how much more than $c_1(N) = 0$ when restricted to $H^2(\partial N; \mathbf{R})$ is necessary to guarantee this. If $c_1(N) = 0$ in $H^2(N; \mathbf{R})$, we can prove that there exists such a volume form on N; this seems, however, a very restrictive hypothesis on N.

∂N and the topology of N. In order to do this, we will treat the terms c_2 and c_1^2 separately for awhile.

Let us treat c_1^2 first. In a neighborhood of ∂N, we can write

$$c_1 = -\frac{3i}{2\pi}\partial\bar{\partial}\log(-1/\varphi)+\partial\bar{\partial}F, \tag{6.6}$$

where $F=O(\varphi^3)$ near the boundary, and the estimate can be differentiated. Define $N_\varepsilon=\{\varphi<-\varepsilon\}$, for ε small and positive. By (6.6) and Stokes's theorem, there is a compactly supported closed two-form $\bar{c}_1$ on N_ε, for ε sufficiently small, such that

$$\int_{N_\varepsilon} c_1^2 = \int_{N_\varepsilon}\bar{c}_1^2+\int_{\partial N_\varepsilon}\left(\frac{3i}{2\pi}\right)^2\partial\log(-1/\varphi)\wedge\partial\bar{\partial}\log(-1/\varphi)+O(\varepsilon). \tag{6.7}$$

(Note that this equation also holds true if c_1 is the first Chern form of the complete Einstein–Kähler metric on N, if N admits such a metric, by the results of [CY] and [LM].) The first term on the right is a topological invariant of the situation: $\bar{c}_1$ is a lifting of c_1 to $H^2(N,\partial N;\mathbf{R})$, and the integral is independent of this lifting and the ε. We are now in a position to state the main theorem of this section.

THEOREM 6.1. *With notation as above, we have*

$$\int_N c_2-\frac{1}{3}c_1^2=\chi(N)-\frac{1}{3}\int_N\bar{c}_1^2+\mu(\partial N), \tag{6.8}$$

where $\mu(\partial N)=$ *the secondary class as in* (5.17) *above.*

To prove Theorem 6.1, we will express the left-hand integral in (6.8) as the

$$\lim_{\varepsilon\to 0}\int_{N_\varepsilon} c_2-\frac{1}{3}c_1^2.$$

Equation (6.7) shows us how to begin treating the c_1^2 portion of the integral. Next let us consider c_2.

Let ψ be a Morse function on N which agrees with our function φ in a neighborhood of ∂N, and let X be the vector field on N of type $(1,0)$ given by the $(1,0)$-part of the gradient of φ with respect to the metric g on N. X has a finite number of isolated, non-degenerate zeroes, none of them on the boundary ∂N. Let $N_{\varepsilon,\delta}$ denote the manifold N_ε above with a smooth ball of radius δ removed about each zero of X. On $N_{\varepsilon,\delta}$ we can split the holomorphic tangent bundle smoothly into the subbundle τ

spanned by X, and the subbundle ν normal to τ. Over $N_{\varepsilon,\delta}$, therefore, we have two connections on the bundle of holomorphic tangents, $T^{1,0}(N)$, viz., the original connection ∇^0 associated to the Hermitian metric g, and the connection ∇^1 given as the direct sum of the connections induced by ∇^0 on τ and ν. We can calculate a relative transgression $Tc_2(\nabla^0, \nabla^1)$ on $N_{\varepsilon,\delta}$ such that

$$c_2(\nabla^0)-c_2(\nabla^1) = dTc_2(\nabla^0, \nabla^1). \tag{6.9}$$

Furthermore, since ∇^1 splits along τ and ν, we have

$$c_2(\nabla^1) = c_1(\tau) \wedge c_1(\nu). \tag{6.10}$$

Finally, since we have an explicit section X of τ, we can construct an explicit transgression one-form $Tc_1(\tau)$ such that

$$c_1(\tau) = dTc_1(\tau) \tag{6.11}$$

on $N_{\delta,\varepsilon}$. Putting (6.9–6.11) together, we have

$$\begin{aligned} \int_{N_\varepsilon} c_2 = \int_{N_\varepsilon} c_2(\nabla^0) &= \lim_{\delta\to 0}\int_{N_{\varepsilon,\delta}} c_2(\nabla^0) \\ &= \chi(N_\varepsilon)+\int_{\partial N_\varepsilon} Tc_2(\nabla^0, \nabla^1)+Tc_1(\tau)\wedge c_1(\nu). \end{aligned} \tag{6.12}$$

Putting (6.7) together with (6.12), we see that the proof of Theorem 6.1 reduces to showing

$$\mu(\partial N) = \lim_{\varepsilon\to 0}\int_{\partial N_\varepsilon} B, \tag{6.13}$$

where the integrand B is given by

$$B = Tc_2(\nabla^0, \nabla^1)+Tc_1(\tau)\wedge c_1(\nu)-\frac{1}{3}\left(\frac{3i}{2\pi}\right)^2 \partial\log(-1/\varphi)\wedge\partial\bar\partial\log(-1/\varphi), \tag{6.14}$$

restricted to ∂N_ε. To prove (6.13), we prove that the apparent pole singularities in B cancel, and that B converges smoothly to an integrand for the invariant $\mu(\partial N)$ calculated in [BE].

To evaluate $\mu(\partial N)$, we will use the pseudohermitian form of the invariant given in [BE]. We will use the pseudohermitian structure on ∂N given (notation as in § 3 above and [BE]) by the defining function φ. It will be particularly convenient to carry out our computations in a unimodular coordinate system z^1, z^2. We set

$$\theta = i\bar{\partial}\varphi, \tag{6.15}$$

and

$$\theta^1 = \xi^1 dz^2 - \xi^2 dz^1, \tag{6.16}$$

where the ξ^i are as in (2.5), while the connection form is given by

$$\theta_1{}^1 = -2ir\theta, \tag{6.17}$$

where r is as in (2.6). The reader may check that the form θ^1 is not necessarily well-defined independently of the unimodular coordinate system chosen, but changes only by multiplication by a unimodular complex constant factor under change of such coordinates. This is because our topological assumptions on ∂N only imply that some tensor multiple of $T^{1,0}(\partial N)$ is trivial. The connection form is well-defined, however, and as is remarked in [BE], Remark 1, p. 339, this is sufficient for defining the invariant $\mu(\partial N)$, and particularly the formula in Remark 1, p. 339 of [BE] remains valid in this context. Calculating the pseudohermitian invariants of this reduction, we find the CR curvature is given by

$$R = 2r, \tag{6.18}$$

while the torsion is given by

$$\tau^1 = i(\xi^1 \bar{\partial}\xi^2 - \xi^2 \bar{\partial}\xi^1). \tag{6.19}$$

We remark that these formulas are of peculiarly low order in the derivatives of φ, but this is because φ satisfies the Monge–Ampère equation at the boundary. Finally, we evaluate $\mu(\partial N)$ via the formula on p. 339 of [BE] cited above:

$$\mu(\partial N) = \frac{i}{8\pi^2} \int_{\partial N} \frac{-2i}{3} d\theta_1{}^1 \wedge \theta_1{}^1 + \frac{1}{6} R\theta \wedge d\theta_1{}^1 - 2\theta \wedge \tau^1 \wedge \tau^{\bar{1}}. \tag{6.20}$$

Call the integrand of the right hand side of (6.20) $\tilde{T}c_2(\pi)$, as in [BE]. The proof of Theorem 6.1 consists of showing that the integrand B of (6.14) above extends continuously to the boundary, where it agrees with $\tilde{T}c_2(\pi)$.

In order to simplify matters a little bit, we consider a unimodular coordinate system for N' at a point $p=z_0$ in ∂N which has been arranged so that

$$\varphi_{1\bar{2}}(z_0)=\varphi_{2\bar{1}}(z_0)=0, \quad \varphi_2(z_0)=\xi^2(z_0)=0. \tag{6.21}$$

Let us call such a coordinate system normalized at z_0. Substituting (6.15–6.19) into (6.20) proves the following:

LEMMA 6.1. *At the origin z_0 of normalized coordinates, one has*

$$\tilde{T}c_2(\pi)=-\frac{1}{8\pi^2}r^2[1-(1/A)^2|\varphi_{\bar{2}\bar{2}}\varphi_{1\bar{1}}-\varphi_{\bar{2}\bar{2}\bar{1}}\varphi_{\bar{1}}|^2](\partial\varphi-\bar{\partial}\varphi)\wedge\partial\bar{\partial}\varphi.$$

Next we must describe the computation of the integrand B. This naturally breaks up into three components, as in (6.14). The third term of (6.14) restricted to the surface $\{\varphi=-\varepsilon\}$ is given by

$$\frac{3}{8\pi^2}(1/\varphi^2)(\partial\varphi-\bar{\partial}\varphi)\wedge\partial\bar{\partial}\varphi. \tag{6.22}$$

It remains to evaluate the other two terms of B. Let us begin with $Tc_2(\nabla^0,\nabla^1)$. Define the difference form A, a section of $\mathrm{End}(T^{1,0}M)\otimes\Lambda^{1,0}$ in a neighborhood of ∂N, by $\nabla^0=\nabla^1+A$, and set $\nabla^t=\nabla^0+tA$. Let Ω_t be the curvature of ∇^t. The relative transgression is calculated using

$$\begin{aligned} c_2(\nabla^1)-c_2(\nabla^0)&=-\frac{1}{8\pi^2}\int_0^1\frac{d}{dt}[\mathrm{tr}(\Omega_t)^2-\mathrm{tr}(\Omega_t^2)]\,dt\\ &=d\left\{\frac{1}{8\pi^2}\mathrm{tr}(A\cdot\Omega_0)\right\},\end{aligned}$$

and we take

$$Tc_2(\nabla^0,\nabla^1)=-\frac{1}{8\pi^2}\mathrm{tr}(A\cdot\Omega_0). \tag{6.23}$$

This calculation relies on the Bianchi identity and the fact that $\mathrm{tr}(A)=\mathrm{tr}(A^3)\equiv 0$, since the operator A is purely "off-diagonal" when $T^{1,0}(N)$ is split into $\tau\otimes\nu$.

Let us indicate how to begin making the difference form explicit. In a unimodular coordinate system, define the vector field Z of type $(1,0)$ by the formula

$$Z=\varphi_1\frac{\partial}{\partial z^2}-\varphi_2\frac{\partial}{\partial z^1}. \tag{6.24}$$

Again Z is well-defined modulo multiplication by a constant unimodular factor. The fields X and Z are orthogonal to one another in the metric (6.3) and give a local framing for $T^{1,0}(N)$ in terms of which A can be characterized by

$$\begin{aligned} A(X) &= (1/(Z,Z))(\nabla^0 X, Z)Z, \\ A(Z) &= (1/(X,X))(\nabla^0 Z, X)X. \end{aligned} \tag{6.25}$$

Here the inner product is with respect to (6.3).

Finally, we can calculate the term $Tc_1(\tau)\wedge c_1(\nu)$ as follows. The field X is a section of τ, so we can write $c_1(\tau)=(i/2\pi)\,d\alpha$, where α is the connection form on τ in the frame given by X, namely

$$\alpha = (1/(X,X))(\nabla^0 X, X), \quad Tc_1(\tau) = \frac{i}{2\pi}\alpha. \tag{6.26}$$

The term $c_1(\nu)$ can be calculated similarly in terms of Z. It is, however, more useful to calculate it as

$$c_1(\nu) = c_1(\nabla^0) - c_1(\tau) = \frac{i}{2\pi}[-3\partial\bar\partial \log(-1/\varphi) + d\alpha] + \partial\bar\partial O(\varphi^3), \tag{6.27}$$

where the estimate can be differentiated. Note that it follows from (2.5) that α is $O(\varphi^{-1})$ near ∂N. Thus, along ∂N_ε, i.e., along the surface $\{\varphi=-\varepsilon\}$, one has

$$Tc_1(\tau)\wedge c_1(\nu) = -\frac{1}{4\pi^2}\alpha\wedge[-3\partial\bar\partial\log(-1/\varphi)+d\alpha]+O(\varepsilon). \tag{6.28}$$

For the rest of the proof of Theorem 6.1, one substitutes (6.22), (6.23) and (6.28) into B. One uses (2.17–2.18) to make ∇^0 explicit. One calculates B restricted to ∂N_ε at a point z_ε. It is useful to make this evaluation in a coordinate system normalized as in (6.21) above. One can thus calculate that the apparent "poles" cancel. In order to complete the comparison of the limit of B restricted to ∂N_ε with the expression for $\tilde{T}C_2(\pi)$ given in Lemma 6.1, one must recall that the defining function φ is a third order solution of the Monge–Ampère equation along ∂N, and use the consequent identities among the derivatives of φ at z_0. These calculations are tedious, but elementary, and are omitted. This completes the proof of Theorem 6.1.

We conclude with a reexamination of the examples of the invariant $\mu(\partial N)$ computed in § 4.3 of [BE] in light of Theorem 6.1. These examples were given as follows: Let Σ be a compact Riemann surface and ds^2 a conformal metric on Σ whose Gauss curvature K is nowhere vanishing on Σ, and let M be the unit circle bundle with respect to ds^2

inside the bundle $T^{1,0}(\Sigma)$. M is a s.ψ.c. CR-manifold, and its invariant is calculated (cf. equation (4.6) of [BE]) to be

$$\mu = -\frac{|\chi(\Sigma)|}{4} + \frac{1}{24\pi}\int_\Sigma [\Delta(\log|K|)]^2 \frac{d\,\text{Area}}{|K|}. \tag{6.29}$$

In comparing this formula with (6.8) above, care must be taken in choosing the appropriate N. If the genus of Σ is $\geqslant 2$, we take N to be the unit disk bundle in $T^{1,0}(\Sigma)$, with $\partial N = M$. If the genus of Σ is 0, we must take N to be the vectors in $T^{1,0}(\Sigma)$ of length $\geqslant 1$, compactified by adding the "section at infinity" of $T^{1,0}(\Sigma)$. (These choices are dictated so that N is to the s.ψ.c. side of M. The case of Σ of genus one does not occur.) Near ∂N there is a canonical non-vanishing holomorphic two-form, the symplectic form normally viewed on $\Lambda^{1,0}(\Sigma)$, so that N verifies all the hypotheses of the discussion around (6.8).

In order to have a wider class of examples, let us simultaneously consider the cyclic covers and quotients of the N just described, corresponding to taking roots and powers of the tangent bundle $T^{1,0}(\Sigma)$. In the resultant line bundle of degree d over Σ we denote by $N(d)$ the manifold of vectors of length $\leqslant 1$ or $\geqslant 1$ (compactified), according to the pseudoconvexity requirements described above. The canonical two-form on $\partial N \subset T^{1,0}(\Sigma)$ suffices to show that all $N(d)$ verify the assumptions of Theorem 6.1.

Let us first assume that the genus g of Σ is $\geqslant 2$. Then $N(1-g)$ is the unit disk bundle in a square root of the tangent bundle of Σ. Let us assume at first that Σ is equipped with a metric of constant sectional curvature. There is a representation of the fundamental group of Σ into $SU(1,1)$ sending γ in $\pi_1(\Sigma)$ to

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

such that $N(1-g)$ is the quotient of the ball $\mathbf{B}^2$ by the action of π_1 given by

$$\gamma\cdot(z,w) = (az+b/cz+d,\ w/cz+d).$$

In particular, $N(1-g)$ has a complete Kähler metric of constant negative holomorphic sectional curvature. By (6.29),

$$\mu(\partial N(1-g)) = \frac{\chi(\Sigma)}{2} + \frac{1}{12\pi}\int_\Sigma [\Delta \log|K|]^2 \frac{dA}{|K|}, \tag{6.30}$$

the last integral being zero when Σ is of constant curvature. The canonical divisor of $N(1-g)$ is readily calculated to be -3Σ, so that by Poincaré duality, we have

$$\int_{N(1-g)} \bar{c}_1^{\,2} = (3\Sigma)^2 = \frac{9}{2}\chi(\Sigma).$$

Here the self-intersection of Σ is computed in $N(1-g)$. Since $\chi(N(d))=\chi(\Sigma)$, independent of d, we get that the "topological terms" in (6.8) and (6.30) cancel, and for Σ with arbitrary metric with K nowhere zero one has

$$\int_{N(1-g)} c_2 - \frac{1}{3}c_1^{\,2} = \frac{1}{12\pi}\int_\Sigma [\Delta \log|K|]^2 \frac{dA}{|K|}. \tag{6.31}$$

On the other hand, if we compare (6.8) with (6.29) for $N(2-2g)$, using the fact that the canonical divisor is -2Σ and $(\Sigma)^2=2-2g$ on $N(2-2g)$, we get

$$\int_{N(2-2g)} c_2 - \frac{1}{3}c_1^{\,2} = -\frac{1}{12}\chi(\Sigma) + \frac{1}{24\pi}\int_\Sigma [\Delta \log|K|]^2 \frac{dA}{|K|}. \tag{6.32}$$

The topological term on the right is always >0, and represents an obstruction to extending the constant holomorphic sectional curvature metric inherited near $\partial N(2-2g)$ from $N(1-g)$ to all of $N(2-2g)$. It is interesting to note that, if the original metric on Σ were of contant curvature, then the Hermitian metric on $N(2-2g)$ in which we compute the left hand side of (6.32) could be taken to be the quotient metric (of constant holomorphic sectional curvature) from $N(1-g)$ outside an arbitrarily small neighborhood of $\Sigma \subset N(2-2g)$. In fact the value $-\frac{1}{12}\chi(\Sigma)$ can be computed, as a limit of regularizations, as the value of

$$\int_{N(2-2g)} c_2 - \frac{1}{3}c_1^{\,2}$$

for the singular quotient metric. It is interesting to speculate whether this number may be defined as a local invariant of the embedding of Σ in $N(2-2g)$; a priori, although the integral can be localized along Σ, the metrics used appear to depend on a global boundary condition (viz., that they agree with the constant holomorphic sectional curvature metric away from Σ). This would be interesting in connection with deciding which 2-dimensional compact s.ψ.c. manifolds with boundary could be covered by the ball $\mathbf{B}^2$. We do not pursue this here, but point out that a similar analysis can be made in the case of Σ of genus 0 as well.

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