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Homology of Euclidean groups of motions made discrete and Euclidean scissors congruences

by

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The present work continues our investigations on the Scissors Congruence Problems. These investigations originated with the Third Problem of Hilbert that dealt with the scissors congruence problem in Euclidean 3-space. As indicated in [1], [5], [16], the non-Euclidean versions are just as interesting. They have an intimate connection with the Eilenberg–MacLane homology of certain classical Lie groups (namely, the isometry groups of the appropriate classical geometries) and come into contact with algebraic K-theory, Cheeger–Chern–Simons characteristic classes, as well as other topics. In all three series of classical geometries, the spherical version enters because the basic Dehn invarients require an understanding of the spherical scissors congruence problem. In a number of recent works, we have concentrated our efforts on the non-Euclidean cases, see [6], [18] for results and summaries in these directions. In spite of our efforts, the most complete results remain to be the theorems of Dehn–Sydler–Jessen showing that volume and Dehn invariants form a complete system of invariants for the scissors congruence problem in Euclidean spaces of dimensions 3 and 4. The original work of Sydler [19] was an incredible tour de force geometric arument in Euclidean 3-space. It was rapidly simplified by Jessen in [9] and extended to Euclidean 4-space. The simplification by Jessen employed techniques from homological algebra. Nevertheless, two of the geometric arguments of Sydler were retained in Jessen's work. The present work continues in the direction of the general theme that the scissors congruence problems should be formulated and solved in terms of the Eilenberg–MacLane homology of classical groups (with appropriate coefficients). The principal goal in the present

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work is to return to the Euclidean case. Specifically, after proving a number of theorems concerning the Eilenberg–MacLane homology of various Euclidean motion groups, we apply the results in conjunction with some results on the Hochschild homology theory of the quaternions to obtain a new proof of the above mentioned theorems of Dehn–Sydler–Jessen. We note that the particular Hochschild homology theory involved in our arguments is intimately connected with the cyclic homology theory which is a theory of noncommutative differential forms, cf. Connes [4], Karoubi [7]. The fact that there should be such a connection is not unexpected because the scissors congruence problem is really an algebraic investigation of the domain of integration in a geometrically restricted setting. The present investigation is a first step in the direction of trying to pin down the relation. The main accomplishment is that we have a proof of the theorem of Dehn–Sydler–Jessen that is essentially homological thus answering affirmatively a question in [5]. The difficult geometric lemmas of Sydler retained by Jessen are no longer necessary. The only geometric arguments needed are completely elementary. Namely, in addition to perpendicularity and the parallel axiom, we only need to know that Euclidean simplices have circumscribed and inscribed spheres. It should be pointed out that one of the difficulties in hyperbolic geometry is the existence of simplices that cannot be circumscribed on a sphere. A second difficulty in hyperbolic geometry is the fact that orthogonal trajectories to a hyperplane do not focus in the finite part of the space. In this respect, the Euclidean case is half way between the hyperbolic and the spherical case. Namely, Euclidean simplices can be circumscribed on a sphere but the orthogonal trajectories to a hyperplane do not focus in the finite part of the space. The first part of the present work shows that we can get around the problem on the non-focusing of the orthogonal trajectories by taking advantage of the possibility of circumscribing a simplex on a sphere.

We rapidly summarize the present work. In Section 1, we show that the Eilenberg–MacLane homology of the full Euclidean motion group coincides with that of the corresponding orthogonal group in degree not exceeding the dimension of the Euclidean space. We also examine the stability range for the special orthogonal groups. In Section 2, we prove a number of vanishing theorems for the Eilenberg–MacLane homology of various Euclidean groups of motions with appropriate coefficients. In Section 3, we recall the identification of the Euclidean scissors congruence groups in terms of the investigations on the Eilenberg–MacLane homology of suitable Euclidean groups of motions. In Sections 4 and 5, we present the new proof of the theorems of Dehn–Sydler–Jessen. We conclude the present work with some unsolved problems in Section 6.

1. Euclidean motion groups

Let $E(n)$ denote the group of all isometries of the Euclidean n-space $\mathbf{R}^n$. $E(n)$ is therefore the semidirect product $T(n)\rtimes O(n)$ of the normal subgroup $T(n)$ formed by the translations and the subgroup $O(n)$ of all orthogonal transformations fixing a suitable origin. When there is no chance of confusion, we will identify $T(n)$ with $\mathbf{R}^n$. The principal result in the present section is the following assertion:

THEOREM 1.1. *The inclusion map from $O(n)$ into $E(n)$ induces injective maps from $H_i(O(n))$ to $H_i(E(n))$ for all i. This map is surjective, hence bijective, when $i\leq n$.*

The injectivity assertion is a trivial consequence of the semidirect product decomposition. We need to show the surjectivity assertion for the range indicated. The argument is based on the use of a transposed spectral sequence. In the literature, this is usually called an equivariant spectral sequence. The difference is that we have transposed the indices to emphasize the idea of comparison of two different filtrations on the same double complex in the first quadrant. Instead of transposing the terms to make the higher differentials go the same way, we prefer to keep the terms and change the directions of the differentials. Since none of the spectral sequences is displayed in the present work, we leave the burden of keeping track of the transposition of the indices to the readers.

Let $C_*(n)=C_*(\mathbf{R}^n)$ denote the normalized acyclic Eilenberg–MacLane chain complex based on the set $\mathbf{R}^n$. Thus $C_i(n)$ is the free Abelian group based on the set of i-cells where each i-cell is an ordered $(i+1)$-tuple $(v_0, \ldots, v_i)$ of points in $\mathbf{R}^n$ with the understanding that each such i-cell is set equal to 0 whenever $v_s=v_{s-1}$ holds for at least one s with $1\leq s\leq i$. The acyclicity (with augmentation $\mathbf{Z}$) of this chain complex is well-known and depends only on the fact that $\mathbf{R}^n$ is a set. Let $C_*^{\text{gen}}(\mathbf{R}^n)$ denote the subcomplex of $C_*(n)$ spanned by the set of all generic cells. Here an i-cell $(v_0, \ldots, v_i)$ is called *generic* when each of its j-faces with $j\leq\min(n, i)$ spans an *affine* subspace of dimension j. By using the fact that $\mathbf{R}$ is an *infinite* field, it is easy to see that this subcomplex is also acyclic with augmentation $\mathbf{Z}$. We let $Q_*(n)$ denote the quotient chain complex so that we have the following exact sequence of $E(n)$-chain complexes:

$$0\to C_*^{\text{gen}}(n)\to C_*(n)\to Q_*(n)\to 0. \tag{1.2}$$

By using the associated long homology exact sequence, $Q_*(n)$ is acyclic with augmentation 0. Since $E(n)$ permutes the cells, (1.2) splits as an exact sequence of $E(n)$-modules.

This immediately yields the following exact sequences of Abelian groups for any group G acting on the chain complexes through a homomorphism of G into $E(n)$:

$$(1.3) \qquad 0 \to H_i(G, C_*^{\mathrm{gen}}(n)) \to H_i(G, C_*(n)) \to H_i(G, Q_*(n)) \to 0, \quad i \geq 0.$$

It is then clear that the terms of (1.3) are nothing more than the ith column of transposed spectral sequences $''E^1_{i,*}$ that converge respectively to $H_*(G, A)$ where A is respectively $\mathbf{Z}$, $\mathbf{Z}$, and 0. Since G also permutes the cells, the terms $''E^1_{i,j}$ can be described by using Shapiro's lemma. For example, when $G=E(n)$,

$$H_i(E(n), C_0(n)) \cong H_i(O(n)) = ''E^\infty_{i,0}.$$

This last part is a consequence of the fact that $O(n)$ has the normal complement $T(n)$ in $E(n)$. A similar assertion holds for any subgroup G of $E(n)$ that contains $T(n)$. For example, this holds for the subgroup $SE(n)$ consisting of all the orientation-preserving isometries of $\mathbf{R}^n$. In order to prove Theorem 1.1, we only have to show that

LEMMA 1.4. *In the transposed spectral sequence associated to $E(n)$ and $C_*(n)$, the i-th column $''E^1_{i,*}(n)$ is $(n-i)$-acyclic with augmentation $''E^1_{i,0}(n) \cong H_i(O(n))$.*

The proof of Lemma 1.4 begins by an examination of the transposed spectral sequence associated to $E(n)$ and $C_*^{\mathrm{gen}}(n)$.

LEMMA 1.5. *$C_*^{\mathrm{gen}}(n) \otimes_G \mathbf{Z}$ is $(n-1)$-acyclic for $G=SE(n)$ or $E(n)$.*

Proof. For $j<n$, any j-cycle of $''E^{1,\mathrm{gen}}_{0,*}(n)$ is made up of a finite number of generic j-cells. Each such j-cell has a unique circumscribed $(j-1)$-sphere with a positive radius r and circumscribed center p in the affine j-subspace spanned by its vertices in $\mathbf{R}^n$. By moving p in a direction perpendicular to this affine subspace, we can circumscribe our j-cell on a j-sphere in $\mathbf{R}^n$. This requires $j<n$. Since $SE(n)$ is transitive on $\mathbf{R}^n$, we may move the j-cells appearing in our j-cycle c until all of them are circumscribed on a common j-sphere of radius strictly larger than the finite number of radii associated to the j-cells appearing in c. If we let p denote the center of this circumscribed sphere, then we can form the $(j+1)$-chain $p*c$. Since p is not on the affine j-subspace spanned by the vertices of any of the j-cells appearing in c, $p*c$ is a generic $(j+1)$-chain. We can now compute the boundary $\partial(p*c)=c-p*\partial c$. Since c is a j-cycle, the $(j-1)$-cells appearing in ∂c must cancel in pairs by using elements of $SE(n)$. Indeed, since $j-1 \leq n-2$, the element needed for the cancellation can be assumed to lie in $SE(n)$ and to carry p onto itself. In other words, c is the boundary of $p*c$. □

Our next task is to improve Lemma 1.5 in the case of $G=E(n)$. In particular, we will prove the special case of Lemma 1.4 when $i=0$. This will be accomplished by an induction argument.

LEMMA 1.6. *$C_*^{\text{gen}}(n) \otimes_{E(n)} \mathbf{Z}$ and $C_*(n) \otimes_{E(n)} \mathbf{Z}$ are both n-acyclic.*

Proof. We proceed by induction on n. When $n=1$, the two chain complexes coincide in degree 1. Each 1-cell is automatically a 1-cycle. The reflection about the midpoint p of a 1-cell lies in $E(1)$ and the 1-cell (v_0, v_1) is the boundary of the generic 2-cell (p, v_0, v_1). Our induction hypothesis is that $C_*(k) \otimes_{E(k)} \mathbf{Z}$ is k-acyclic for $k<n$. By using the long homology exact sequence associated to the short exact sequence of chain complexes (1.3) with $i=0$, we claim that $C_*(n) \otimes_{E(n)} \mathbf{Z}$ is $(n-1)$-acyclic with augmentation $\mathbf{Z}$ and $Q_*(n) \otimes_{E(n)} \mathbf{Z}$ is $(n+1)$-acyclic with augmentation 0. We note that $(n-1)$-acyclicity of $C_*(n) \otimes_{E(n)} \mathbf{Z}$ and of $Q_*(n) \otimes_{E(n)} \mathbf{Z}$ follows from the induction hypothesis in conjunction with Lemma 1.5. The induction hypothesis does apply because $C_*(n) \otimes_{E(n)} \mathbf{Z}$ coincides with $C_*(n-1) \otimes_{E(n-1)} \mathbf{Z}$ through degree $n-1$ and the latter can be identified with a subcomplex of $C_*(n) \otimes_{E(n)} \mathbf{Z}$. This identification defines a dimension filtration $\mathscr{F}$. Namely, a cell has filtration $\mathscr{F}^j$ if the affine subspace of $\mathbf{R}^n$ spanned by its vertices has dimension at most j.

In order to show that $Q_*(n) \otimes_{E(n)} \mathbf{Z}$ is $(n+1)$-acyclic, we recall that the transposed spectral sequence associated to $E(n)$ and $Q_*(n)$ converges to 0. To get at the desired result, it is enough to show that, for $1 \leq i \leq n$, the column $''E^{1,Q}_{i,*}(n)$ is $(n-i)$-acyclic with augmentation 0. Since $Q_*(n)$ begins in degree 2, the assertion about the augmentation is trivial and we can assume that $1 \leq i \leq n-2$.

We now assume $1 \leq i \leq n-2$ and consider the subcomplex $\mathscr{F}^{n-i,Q}_{i,*}$ of $''E^{1,Q}_{i,*}(n)$ spanned by cells with filtration $n-i$. If c denotes a j-cell of dimension filtration exactly r, then $H_i(O(n-r)) \otimes c$ appears as a direct summand of $''E^{1,Q}_{i,j}(n)$. By letting c range over distinct $E(n)$-equivalence classes of j-cells that are not generic, we obtain a direct sum decomposition of $''E^{1,Q}_{i,j}(n)$. For $j \leq n$, this simply means that $r<j$. If $j \leq n-i$, then the coefficient group is $H_i(O(t))$ with $t=n-r>n-(n-i)=i$. By using the stability theorem proved in Sah [17, Theorem 1.1], we see that

$$\mathscr{F}^{n-i,Q}_{i,*} \cong H_i(O(\infty)) \otimes \mathscr{F}^{n-i,Q}_{0,*}.$$

Evidently, $\mathscr{F}^{n-i,Q}_{i,*}$ coincides with $''E^{1,Q}_{i,*}(n)$ through degree $n-i$ for any $i \geq 0$ (in fact, through degree $n-i+1$ when $i \leq n$). By combining the universal coefficient theorem with the $(n-1)$-acyclicity of $''E^{1,Q}_{0,*}(n)$, we see that $\mathscr{F}^{n-i,Q}_{i,*}$ is $(n-i)$-acyclic with augmentation

0. As indicated before, this means that $''E^{1,Q}_{0,*}(n)$ is $(n+1)$-acyclic with augmentation 0. It follows that

$$H_j(C^{\mathrm{gen}}_*(n) \otimes_{E(n)} \mathbf{Z}) \to H_j(C_*(n) \otimes_{E(n)} \mathbf{Z}) \tag{1.7}$$

is surjective for $j = n+1$ and bijective for $j = n$.

It is therefore enough to show that the map in (1.7) is 0 when $j=n$. Let c denote an n-cycle in $C^{\mathrm{gen}}_*(n) \otimes_{E(n)} \mathbf{Z}$. It is enough to show that c becomes a boundary in the complex $C_*(n) \otimes_{E(n)} \mathbf{Z}$. To see this, we perform a "bootstrap" argument. Namely, we look at the chain complex $C_*(n+1) \otimes_{E(n+1)} \mathbf{Z} = \mathscr{F}^{n+1}$ and the corresponding exact sequence of chain complexes:

$$0 \to \mathscr{F}^n \to \mathscr{F}^{n+1} \to \mathscr{F}^{n+1}/\mathscr{F}^n \to 0. \tag{1.8}$$

By the induction hypothesis, both of the first two chain complexes are $(n-1)$-acyclic. The last chain complex begins in degree $n+1$ and is a free Abelian group with basis consisting of generic $(n+1)$-cells ranging over distinct $E(n+1)$-equivalence classes. By using either the circumscribed center or the inscribed center construction, each such $(n+1)$-cell b is the boundary of an $(n+2)$-chain modulo $\mathscr{F}^n$ (cf. Sah [17, pp. 320–1]). In essence, the existence of reflections in $E(n)$ kills off this "scissors congruence group" $H_{n+1}(\mathscr{F}^{n+1}/\mathscr{F}^n)$. As a consequence of the long exact homology sequence, we see that

$$H_n(\mathscr{F}^n) \cong H_n(\mathscr{F}^{n+1}).$$

It is therefore enough to show that our generic n-cycle c becomes a boundary in the chain complex $\mathscr{F}^{n+1}$. Since $\mathscr{F}^{n+1}$ contains the generic subcomplex $C^{\mathrm{gen}}_*(n+1) \otimes_{E(n+1)} \mathbf{Z}$, evidently the n-cycle c bounds in this subcomplex by Lemma 1.4. □

Proof of Theorem 1.1. From the preceding argument, it is enough to show that the transposed spectral sequence associated to $E(n)$ and $C_*(n)$ is such that $''E^1_{i,*}(n)$ is $(n-i)$-acyclic with augmentation $H_i(O(n))$, $1 \le i \le n-1$. We may now imitate the argument used for $Q_*(n)$. For each such i, we consider the subcomplex of $''E^1_{i,*}(n)$ of dimension filtration $n-i$. This subcomplex $\mathscr{F}^{n-i}_{i,*}$ coincides with $''E^1_{i,*}(n)$ through degree $n-i$ so that it is enough to show that this subcomplex is $(n-i)$-acyclic. By using Shapiro's lemma, it is easy to see that we have an exact sequence of chain complexes of the form:

$$0 \to B(i) \otimes (\mathscr{F}^{n-i}_{i,*}/\mathscr{F}^{n-i-1}_{i,*}) \to \mathscr{F}^{n-i}_{i,*} \to H_i(O(\infty)) \otimes \mathscr{F}^{n-i}_{0,*} \to 0.$$

Here $B(i)$ denotes the kernel of the surjective homomorphism from $H_i(O(i))$ to the stable group $H_i(O(\infty))$. The surjectivity of this map was proved in Sah [17, Theorem 1.1]. By using the universal coefficient theorem, the first of the above chain complex begins in degree $n-i$ and is $(n-i)$-acyclic by means of the circumscribed center construction (or the inscribed center construction). The last of the above chain complex is $(n-i)$-acyclic by using the $(n-i)$-acyclicity of $\mathscr{F}^{n-i}_{0,*}$. The desired $(n-i)$-acyclicity of $\mathscr{F}^{n-i}_{i,*}$ now follows from the long homology exact sequence. □

The situation concerning $SE(n)$ and $SO(n)$ is somewhat different. We will first present the stability theorems for $SO(n)$ (this could have been stated and proven already in Sah [17]).

THEOREM 1.9. *The map $H_i(SO(n))\to H_i(SO(n+1))$ is surjective for $n\geq 2i$ and bijective for $n\geq 2i+1$. For $i<n<2i$, and n odd, $H_i(SO(n))\to H_i(SO(\infty))$ is bijective. For $i\leq n$, $H_i(SO(n))\to H_i(SO(\infty))$ is surjective.*

Proof. Let $C_*(S^n)$ denote the acyclic chain complex whose j-cells are the ordered $(j+1)$-tuples $(v_0, \ldots, v_j)$ of unit vectors in $\mathbf{R}^{n+1}$ such that no two of them are linearly dependent. The acyclicity follows easily fromt he fact that $\mathbf{R}$ is infinite. We now examine the transposed spectral sequence associated to $SO(n+1)$ and $C_*(S^n)$. We assert that

$$(1.10)\qquad {}''d^1_{i,1}=0 \quad \text{so that} \quad {}''E^2_{i,0}={}''E^1_{i,0}=H_i(SO(n)).$$

By Shapiro's lemma, ${}''E^1_{i,0}=H_i(SO(n))$ and ${}''E^1_{i,1}$ is the direct sum of terms of the form

$$H_i(SO(n-1))\otimes(a,b), \quad a,\ b \text{ are independent unit vectors in } \mathbf{R}^{n+1}.$$

If c is in $H_i(SO(n-1))$, then $d^1_{i,1}(c\otimes(a,b))=c\otimes(b)-c\otimes(a)$ where c is now viewed as lying in $H_i(SO(n))$. Evidently, we can find $\sigma\in SO(2)$ so that $\sigma(b)=a$ and so that σ commutes with $SO(n-1)$. It follows that $d^1_{i,1}(c\otimes(a,b))=0$.

The following assertion was proved in Sah [17, (1.5)] by using the orthogonal join construction

$$(1.11)\qquad {}''E^1_{0,*} \text{ is } (n-1)\text{-acyclic}.$$

Evidently, the first assertion in Theorem 1.9 follows from the assertion

$$(1.12)\qquad {}''E^1_{i,*} \text{ is } (n-2i-1)\text{-acyclic for } 1\leq i\leq (n-1)/2.$$

We will prove Theorem 1.9 and (1.12) in tandem by an induction argument. The induction hypothesis is that (1.12) and Theorem 1.9 have been verified for all $m<n$. Let us now consider the column $''E^1_{i,*}$ for $1\leqslant i\leqslant (n-1)/2$. By Shapiro's lemma, $''E^1_{i,j}$ is the direct sum of terms of the form $H_i(SO(n+1-r))\otimes c$ where c is a j-cell of rank r. The rank r denotes the dimension of the $\mathbf{R}$-subspace of $\mathbf{R}^{n+1}$ spanned by the vertices of c. Evidently, $r\leqslant \min(n+1,j+1)$. For $j\leqslant n-2i-1$, we have $n+1-r\geqslant n-j\geqslant 2i+1$. Thus the coefficient groups are the stable groups $H_i(SO(\infty))\cong H_i(SO(2i+1))$. For $j=n-2i$ and $r=j+1$, the coefficient group $H_i(SO(2i))$ maps surjectively onto $H_i(SO(2i+1))$. By using (1.11) and the universal coefficient theorem, it is easy to see that (1.12) holds. To see the last assertion, we note that $O(n)=SO(n)\times\langle \pm I_n\rangle$ when n is odd. The isomorphism between $H_i(SO(n))$ and $H_i(SO(\infty))$ for $i<n$ and odd n therefore follows from the bijective stability of $H_i(O(n))$ for $i<n$. In a similarly manner, we can deduce from the Hochschild–Serre spectral sequence associated to the semidirect product $O(n)=SO(n)\rtimes O(1)$ that $H_i(SO(n))$ maps surjectively onto $H_i(SO(\infty))$ for $i\leqslant n$. □

THEOREM 1.13. *The map $H_i(SO(n))\to H_i(SE(n))$ is always injective; it is surjective when $n\geqslant 2i$.*

Proof. As before, the injectivity is a consequence of the semidirect product splitting of $SE(n)$ as $T(n)\rtimes SO(n)$. For the surjectivity assertion, we examine the transposed spectral sequence associated to $SE(n)$ and $C_*(n)$. It is enough to show that $''E^1_{i,*}$ is $(n-2i-1)$-acyclic for $0\leqslant i\leqslant (n-1)/2$. The argument is entirely similar to Theorem 1.1. We leave the details to the careful reader. □

2. Some vanishing theorems

We will now verify some vanishing theorems for the homology of orthogonal groups with coefficients in suitable exterior powers (over the field $\mathbf{Q}$ of rational numbers). For this purpose, we will examine the Hochschild–Serre spectral sequence associated to a split exact sequence of groups.

LEMMA 2.1. *Let $G=A\rtimes\Gamma$ denote a semidirect product of groups with A denoting a left Γ-module. Assume that A and $H_i(\Gamma, H_j(A,\mathbf{Z}))$ are torsionfree for all $j\geqslant 2$. Then the Hochschild–Serre spectral sequence associated to the semidirect product splitting of G has the property that $H_i(\Gamma, H_j(A,\mathbf{Z}))\cong {}'E^2_{i,j}\cong {}'E^\infty_{i,j}$. Moreover, we have a canonical "spectral" decomposition*

$$H_n(G,\mathbf{Z})\cong \coprod_{i+j=n} H_i(\Gamma, H_j(A,\mathbf{Z})).$$

Remark. To be precise, the Γ-endomorphism of A defined by multiplication by the integer m induces multiplication by m^j on $H_i(\Gamma, H_j(A, \mathbf{Z}))$, and $H_j(A, \mathbf{Z}) \cong \Lambda^j_{\mathbf{Z}}(A)$.

Proof. The Γ-endomorphism arising from multiplication by m evidently induces multiplication by m^j on $H_j(A, \mathbf{Z}) \cong \Lambda^j_{\mathbf{Z}}(A)$. It induces an endomorphism φ_m of G that is compatible with the semidirect product splitting. Hence it induces a map, again denoted by φ_m, on the Hochschild–Serre spectral sequence so that φ_m commutes with all the differentials $'d^r_{i,j}\colon {'E^r_{i,j}} \to {'E^r_{i-r,j+r-1}}$. We only have to show that all these differentials are 0 for $r \geq 2$. This is certainly true for $j=0$ by virtue of the semidirect splitting because

$$H_i(\Gamma, H_j(A, \mathbf{Z})) \cong {'E^2_{i,j}} \quad \text{and} \quad {'E^2_{i,0}} = {'E^\infty_{i,0}}.$$

By using the commutation relation, $\varphi_m \circ {'d^r_{i,j}} = {'d^r_{i,j}} \circ \varphi_m$, it becomes clear that the image of $'d^r_{i,j}$ is annihilated by the integer $m^j(m^{r-1}-1)$. Since $r \geq 2$, our assumption on the torsionfreeness of $'E^2_{i,j}$ for $j \geq 2$ shows that all the higher differentials are zero. The remaining assertions are now clear. □

The preceding argument apparently was first used by David Lieberman in the setting of the cohomology spectral sequence. It is possible (not needed in the present work) to get some more information on the relevant type of torsions that need to be avoided.

THEOREM 2.2. *Let* $0 \leq i \leq n$ *and* $1 \leq j \leq n-i$. *Then* $H_i(O(n), \Lambda^j_{\mathbf{Z}}(\mathbf{R}^n)) = 0$.

Proof. We recall that $T(n) \cong \mathbf{R}^n$. By using Theorem 1.1 and Lemma 2.1 for the semidirect product $E(n) = T(n) \rtimes O(n)$, our assertion follows. □

When $i=0$, Theorem 2.2 can be extended by a simple geometric argument.

THEOREM 2.3. *Let* $1 \leq j \leq 2n-1$. *Then* $H_0(O(n), \Lambda^j_{\mathbf{Z}}(\mathbf{R}^n)) = 0$.

Proof. There is nothing to prove when $n=1$. We therefore assume $n>1$.

In general, $\Lambda^j_{\mathbf{Z}}(\mathbf{R}^n)$ is spanned over $\mathbf{Q}$ by elements of the form $v_1 \wedge \ldots \wedge v_j$ where the vectors v_s may be assumed to lie on n one-dimensional $\mathbf{R}$-subspaces of $\mathbf{R}^n$ that are mutually orthogonal. Since the exterior product is over $\mathbf{Z}$, hence over $\mathbf{Q}$, it is possible for v_s and v_{s+1} to be $\mathbf{R}$-dependent but not $\mathbf{Q}$-dependent. If an odd number of the vectors, say $v_1, \ldots, v_{2t+1}$, lie on a subspace orthogonal to the subspace spanned by the remaining vectors $v_{2t+2}, \ldots, v_j$, then we can find σ in $O(n)$ so the $\sigma(v_i) = \pm v_i$ according to $i \geq 2t+2$ or $i \leq 2t+1$. Thus $\sigma(v_1 \wedge \ldots \wedge v_j) = -(v_1 \wedge \ldots \wedge v_j)$ so that this element is 0 in the group of coinvariants. In particular, j may be assumed to be even. Since $1 \leqslant j \leqslant 2n-1$, we may assign to each basic exterior product $v_1 \wedge \ldots \wedge v_j$ a decreasing sequence of integers

$$2k(1) \geqslant \ldots \geqslant 2k(s) > 0, \quad 2k(1)+\ldots+2k(s) = j.$$

Here the vectors $v_1, \ldots, v_j$ are such that $2k(t)$ of them lie on the one-dimensional subspace L_t of $\mathbf{R}^n$ and the L_t's are pairwise orthogonal. Evidently $s<n$ so that we can find a 1-dimensional subspace L orthogonal to each L_t. We assume that $v_1, \ldots, v_{2k(1)}$ lie in L_1. In the 2-dimensional subspace $L+L_1$, we can find w in L so that $u_1=v_1+w$ is perpendicular to $u_2=v_2-u_1$. Thus $v_1 \wedge v_2 = v_1 \wedge (v_2-v_1) = (u_1-w) \wedge (u_2+w) = u_1 \wedge u_2 + v_2 \wedge w$. Since $u_1 \perp u_2$ and $v_2 \perp w$, the product $v_1 \wedge \ldots \wedge v_{2k(1)}$ is the sum of two terms of the form $w_1 \wedge w_2 \wedge v_3 \wedge \ldots \wedge v_{2k(1)}$ where $w_1 \perp w_2$ lies in $L+L_1$. By breaking up each v_l, $3 \leqslant l \leqslant 2k(1)$, into components along w_1 and w_2 and multiply out the product, we see that $v_1 \wedge \ldots \wedge v_j$ becomes a sum of terms with strictly bigger s or terms that are 0 in the group of coinvariants. Thus, if $k(1)=1$, $w_1 \wedge w_2 \wedge v_3 \wedge \ldots \wedge v_j$ is 0 in the group of coinvariants. This completes the proof by induction. □

THEOREM 2.4. *Let $i \geqslant 0$ and $j>0$ so that $n \geqslant 2(i+j)$. Then $H_i(SO(n), \Lambda^j_{\mathbf{Z}}(\mathbf{R}^n))=0$.*

Proof. Use Theorem 1.13 and Lemma 2.1. □

THEOREM 2.5. *$H_i(O(n), \Lambda^j_{\mathbf{Z}}(\mathbf{R}^n))=0$ when j is odd. $H_i(SO(n), \Lambda^j_{\mathbf{Z}}(\mathbf{R}^n))=0$ when either* (a) *n is even and j is odd, or* (b) *n is odd, $i \geqslant 0$, $j>0$ is even, and $i+j \leqslant n$.*

Proof. The first two cases are consequences of the lemma on "center kills". In the last case, $O(n)=SO(n) \times \langle \pm I_n \rangle$ when n is odd. By using the Hochschild–Serre spectral sequence, $H_i(SO(n), \Lambda^j_{\mathbf{Z}}(\mathbf{R}^n)) \cong H_i(O(n), \Lambda^j_{\mathbf{Z}}(\mathbf{R}^n))$. We can now apply Theorem 2.2. □

THEOREM 2.6. *$H_0(SO(n), \Lambda^j_{\mathbf{Z}}(\mathbf{R}^n))=0$ for $1 \leqslant j \leqslant n-1$ and $H_0(SO(n), \Lambda^n_{\mathbf{Z}}(\mathbf{R}^n)) \cong \mathbf{R}$.*

Proof. For the first assertion, we may imitate the proof of Theorem 2.3. For the second assertion, we have the following exact sequence, see Dupont [5, p. 613, Remark 2]

$$\coprod_U \Lambda^n_{\mathbf{Z}}(U) \to \Lambda^n_{\mathbf{Z}}(\mathbf{R}^n) \to \Lambda^n_{\mathbf{R}}(\mathbf{R}^n) \to 0.$$

Here U ranges over all the codimensional 1 $\mathbf{R}$-subspaces of $\mathbf{R}^n$. We note that $SO(n)$ is transitive on these hyperplanes and the stability subgroup in $SO(n)$ of $\mathbf{R}^{n-1}$ is isomorphic to $O(n-1)$. We may apply the right exact functor $H_0(SO(n), —)$ to the preceding exact sequence and use Shapiro's lemma to get the exact sequence

$$H_0(O(n-1), \Lambda^n_{\mathbf{Z}}(\mathbf{R}^{n-1})) \to H_0(SO(n), \Lambda^n_{\mathbf{Z}}(\mathbf{R}^n)) \to \mathbf{R} \to 0.$$

The second assertion is trivial when $n=1$. When $n>2$ we have $n \leq 2n-3$ so that the first term in the above exact sequence is 0 by Theorem 2.3. We are left with the case of $n=2$. Here $O(1)$ acts trivially on the coefficient group $\Lambda^2_{\mathbf{Z}}(\mathbf{R})$ so the first term in the above exact sequence is $\Lambda^2_{\mathbf{Z}}(\mathbf{R})$. On the other hand, we know from Theorem 2.3 that

$$H_0(O(2), \Lambda^2_{\mathbf{Z}}(\mathbf{R}^2)) = H_0(O(2)/SO(2), H_0(SO(2), \Lambda^2_{\mathbf{Z}}(\mathbf{R}^2))) = 0.$$

Since $O(2)/SO(2)$ can be identified with $O(1)$, we conclude that $H_0(SO(n), \Lambda^2_{\mathbf{Z}}(\mathbf{R}^2))$ is the negative eigenspace for the action of $O(1)$. Since $O(2)$ acts on $\mathbf{R}$ through the determinant, the desired conclusion follows by taking the negative eigenspaces for the action of $O(1)$ in the above exact sequence in the case of $n=2$. □

3. Euclidean scissors congruences

The principal goal in the present section is to describe the connection between the results in the preceding sections and the study of the scissors congruence problem in Euclidean spaces, see Dupont [5, Section 4] for foundational information. For the two non-Euclidean cases, see Dupont–Parry–Sah [6, Section 5].

Let $C_*(n)=C_*(\mathbf{R}^n)$ denote the normalized acyclic Eilenberg–MacLane chain complex based on the set $\mathbf{R}^n$ as in Section 1. We will filter $C_*(n)\otimes_{SE(n)}\mathbf{Z}$ by the subcomplexes $\mathcal{F}^i_+$, $0 \leq i \leq n$, through the $\mathbf{R}$-dimension filtration in analogy with Section 1. Evidently, we have a natural surjective map from $\mathcal{F}^i_+$ to $\mathcal{F}^i = H_0(E(n)/SE(n), \mathcal{F}^i_+)$. Since any congruence between simplices with $\mathbf{R}$-dimension less than n can be realized by elements of $SE(n)$, we see that:

(3.1) For $0 \leq i < n$, $\mathcal{F}^i_+ \to \mathcal{F}^i$ is an isomorphism of chain complexes.

We note that the chain complex $\mathcal{F}^i$ does not depend on n while $\mathcal{F}^i_+$ does depend on n. By fixing n, we can use the i-acyclicity of $\mathcal{F}^i$ and the long exact homology sequence associated to the exact sequence of chain complexes:

$$0 \to \mathcal{F}^{n-1}_+ \to \mathcal{F}^n_+ \to \mathcal{F}^n_+/\mathcal{F}^{n-1}_+ \to 0$$

to obtain the exact sequence:

$$(3.2) \qquad \to H_n(\mathcal{F}_+^{n-1}) \to H_n(\mathcal{F}_+^n) \to H_n(\mathcal{F}_+^n/\mathcal{F}_+^{n-1}) \to 0.$$

From Dupont [5, Theorem 2.3], we know that $H_n(\mathcal{F}_+^n/\mathcal{F}_+^{n-1}) \cong \mathcal{P}E^n =:$ the scissors congruence group in euclidean n-space. In particular, the isomorphism carries the n-simplex A onto $\varepsilon(A)\cdot[A]$ where $\varepsilon(A)$ denotes the orientation of A and $[A]$ denotes the scissors congruence class of the convex closure of A. We note that $\varepsilon(A)=0$ if and only if A lies in $\mathcal{F}_+^{n-1}$. When $n>0$, $\mathcal{P}E^n$ is known to be a $\mathbf{Q}$-vector space by a theorem of Hadwiger. It is in fact an $\mathbf{R}$-vector space by a theorem of Jessen–Thorup [10] (also see Sah [16, Chapters 3 and 4]). We assert that

THEOREM 3.3. (a) *$\mathcal{F}_+^n$ is $(n-1)$-acyclic and*
(b) *$H_n(\mathcal{F}_+^n) \cong H_n(\mathcal{F}_+^n/\mathcal{F}_+^{n-1}) \cong \mathcal{P}E^n$ holds for $n \geq 0$.*

Proof. (a) is just a summary of the discussions preceding the theorem and follows from Lemma 1.6 and (3.1).

(b) When $n=0$, all three groups are naturally isomorphic to $\mathbf{Z}$ so that the assertion is trivial. We consider the exact sequence of chain complexes:

$$(3.4) \qquad 0 \to R_* \to \mathcal{F}_+^n \to \mathcal{F}^n \to 0$$

R_* is $\mathbf{Z}$-free and begins in degree n. In particular, R_n has a free $\mathbf{Z}$-basis consisting of chains of the form $A-A'$ where A is a generic n-simplex and A' denotes its mirror image with A ranging over a complete set of positively oriented $SE(n)$ inequivalent generic n-simplices. We note that our n-simplices are ordered so that many such n-simplices are mapped to the same geometric n-simplex in Euclidean n-space. Since $\mathcal{F}^n$ is n-acyclic, $H_n(R_*)=R_n/\partial R_{n+1}$ maps surjectively onto $H_n(\mathcal{F}_+^n)$. It follows that $A-A'$ is mapped onto $2[A] \in \mathcal{P}E^n$. We now go in the reverse direction and associate to each abstract simplex A in $\mathcal{F}_+^n$ the chain $A-A'$ where A' denotes the image of A under a reflection of Euclidean n-space. If A lies in $\mathcal{F}_+^{n-1}$, then $A-A'=0$. This then defines a chain map so that we have a homomorphism from $H_n(\mathcal{F}_+^n/\mathcal{F}_+^{n-1})$ to $H_n(R_*)$. Following this map by the two surjective maps described above, we clearly have the map that is multiplication by 2 on $\mathcal{P}E^n$. Since the present map is evidently surjective, the absence of 2 torsion in $\mathcal{P}E^n$ gives us the desired assertion. □

Remark. The preceding argument is fairly formal. As indicated, it is the analogue of the case of spherical n-space where the scissors congruence group has to be reduced modulo suspensions, cf. Dupont–Parry–Sah [6; Proposition 5.6]. If we turn to hyperbolic n-space, then the principal problem is the n-acyclicity of $\mathcal{F}^n$. This is known to hold for $n \leq 3$, see Dupont–Parry–Sah [6, Remark 4.8]. If we assume this result for $n \leq m$, then $H_n(R_*)$ maps onto $H_n(\mathcal{F}^n_+)$ for $n \leq m$. The image of $H_m(\mathcal{F}^{m-1}_+)$ in $H_m(\mathcal{F}^m_+)$ is then annihilated by 2.

4. The theorems of Sydler and Jessen: beginning

In the present section, we review the background materials needed to give a direct homological proof of the theorem of Sydler that the scissors congruence classes of polytopes in $\mathbf{R}^3$ are determined by their volume and their Dehn invariant. We first recall from Dupont [5, Corollary 1.2] the following exact sequence:

$$(4.1) \qquad 0 \to H_2(SO(3), \mathbf{R}^3) \to \mathcal{P}(\mathbf{R}^3)/\mathcal{Z}_2(\mathbf{R}^3) \xrightarrow{D} \mathbf{R} \otimes (\mathbf{R}/\mathbf{Z}) \xrightarrow{J} H_1(SO(3), \mathbf{R}^3) \to 0,$$

where $SO(3)$ acts in the natural way on $\mathbf{R}^3$, D is the Dehn invariant, J is defined below, and $\mathcal{Z}_2(\mathbf{R}^3)$ is the subgroup generated by all the prisms. Sydler's theorem [9], [19] is therefore equivalent to showing $H_2(SO(3), \mathbf{R}^3)=0$ because the scissors congruence class of a prism is determined by its volume. Our direct homological proof of this fact is based on Theorem 2.2. Thus we shall show:

THEOREM. 4.2. (a) $H_1(SO(3), \mathbf{R}^3) \cong \Omega^1_{\mathbf{R}}$, *the set of all absolute Kähler differentials of* $\mathbf{R}$. *Furthermore, by means of this isomorphism, the map* J *in* (4.1) *is given by*

$$J(l \otimes (\vartheta/2\pi)) = \frac{1}{2} l \frac{d \cos\vartheta}{\sin\vartheta}$$

(b) $H_2(SO(3), \mathbf{R}^3) = 0$,

Remark. Theorem 4.2 (a) together with (4.1) give the theorem of Jessen [9, Theorem 6] that the image of D is the kernel of J (see also Cathelineau [2], [3]).

For the proof of Theorem 4.2, we first recall a few facts about the Hochschild homology of an algebra A with unit over the ground field $\mathbf{Q}$. In the following all tensor products will be over $\mathbf{Q}$ unless explicitly stated otherwise. Following the notation in Loday–Quillen [13], the Hochschild homology group $H_n(A, A)$ is the homology of the complex $(A^{\otimes(n+1)}, b)$ where the boundary map b is given by

$$(4.3) \qquad b(a_0 \otimes \ldots \otimes a_n) = \sum_{0 \leq i \leq n-1} (-1)^i a_0 \otimes \ldots \otimes a_i a_{i+1} \otimes \ldots \otimes a_n + (-1) a_n a_0 \otimes a_1 \otimes \ldots \otimes a_{n-1}.$$

Following Karoubi [12] we will express $H_n(A, A)$ also as the homology of the complex $\Omega_*(A)$ of *non-commutative differential forms* defined as follows:

Let $\varepsilon_i : A^{\otimes(n+1)} \to A^{\otimes n}, 0 \leq i \leq n-1$, be given by

$$\varepsilon_i(a_0 \otimes \ldots \otimes a_n) = a_0 \otimes \ldots \otimes a_i a_{i+1} \otimes \ldots \otimes a_n, \tag{4.4}$$

and put

$$\Omega_n(A) = \cap_{0 \leq i \leq n-1} \operatorname{Ker} \varepsilon_i, \quad \Omega_0(A) = A.$$

Then $b|\Omega_n(A)$ agrees with $(-1)^n \varepsilon_n$ given by

$$\varepsilon_n(a_0 \otimes \ldots \otimes a_n) = a_n a_0 \otimes \ldots \otimes a_{n-1} \tag{4.5}$$

and it follows (see e.g. May [15, Theorem 22.1]) that

$$H_n(A, A) \cong H_n(\Omega_*(A), b).$$

Notice that $A^{\otimes(*+1)}$ is a graded algebra with product given by

$$(a_0 \otimes \ldots \otimes a_n)(a_0' \otimes \ldots \otimes a_m') = a_0 \otimes \ldots \otimes a_{n-1} \otimes a_n a_0' \otimes \ldots \otimes a_m'$$

and $\Omega_*(A)$ is a subalgebra. Now if we introduce the differentials

$$da = 1 \otimes a - a \otimes 1 \in \Omega_1(A), \quad a \in A,$$

then it is not difficult to show that the elements of $\Omega_n(A)$ are sums of terms of the form

$$\omega = a_0 \cdot da_1 \ldots da_n, \quad a_i \in A, \ 0 \leq i \leq n.$$

More precisely, the natural projection $A^{\otimes(n+1)} \to A \otimes (A/\mathbf{Q})^{\otimes n}$ restricts to an isomorphism,

$$\Omega_n(A) \cong A \otimes (A/\mathbf{Q})^{\otimes n}, \tag{4.6}$$

that maps $a_0 \cdot da_1 \ldots da_n$ to $a_0 \otimes \ldots \otimes a_n$. The differential $b: \Omega_n(A) \to \Omega_{n-1}(A)$ is given by

$$b(\omega \cdot da) = (-1)^n (a\omega - \omega a) \quad \omega \in \Omega_{n-1}(A), \ a \in A. \tag{4.7}$$

Let us put $I_n(A) = \operatorname{Ker}\{b: \Omega_n(A) \to \Omega_{n-1}(A)\}$ and $B_n(A) = \operatorname{Im}\{b: \Omega_{n+1}(A) \to \Omega_n(A)\}$ so that $B_n(A) \subset I_n(A) \subset \Omega_n(A)$ and $H_n(A, A) \cong I_n(A)/B_n(A)$. Note that if $d: \Omega_n(A) \to \Omega_{n+1}(A)$ is defined by $d(a_0 \cdot da_1 \ldots da_n) = da_0 \cdot da_1 \ldots da_n$, then $(\Omega_*(A), d)$ is the *universal differential graded noncommutative algebra* of A.

For a *commutative algebra* A, this should *not* be confused with Ω^*_A, the *graded commutative algebra* of absolute Kähler differentials of A. In fact, in this case, the map $b:\Omega_1(A)\to A$ is zero and clearly $H_1(A,A)\cong\Omega_1(A)/B_1(A)\cong\Omega^1_A$. Moreover, $\Omega^n_A=\Lambda^n_A(\Omega^1_A)$ and the shuffle product defines a natural map $\gamma:\Omega^n_A\to H_n(A,A)$ that maps injectively onto a direct summand. In addition, the composite map

$$\Omega^n_A \xrightarrow{\gamma} H_n(A,A) \to \Omega_n(A)/B_n(A) \to \Omega^n_A$$

is multiplication by $n!$. The next result is a direct consequence of the work of Hochschild–Kostant–Rosenberg [8, Theorems 2.2 and 3.1]:

PROPOSITION 4.8. *If A is a field of characteristic zero, then γ: $\Omega^n_A \to H_n(A,A)$ is an isomorphism.*

An essential step in our proof of Theorem 4.2 is the calculation of the Hochschild homology of the real quaternion division algebra $\mathbf{H}$ considered as an algebra over $\mathbf{Q}$. As ususal, $\mathbf{H}=\mathbf{R}\cdot 1+\mathbf{R}\cdot i+\mathbf{R}\cdot j+\mathbf{R}\cdot k$ and we show

PROPOSITION 4.9. *The inclusion of $\mathbf{Q}$-algebras $\mathbf{R}\subset\mathbf{H}$ induces an isomorphism*

$$H_n(\mathbf{R},\mathbf{R})\cong H_n(\mathbf{H},\mathbf{H}), \quad n\geq 0,$$

and thus

$$H_n(\mathbf{H},\mathbf{H})\cong\Omega^n_{\mathbf{R}}, \quad n\geq 0.$$

Proof. The second statement follows from the first and Proposition 4.8. For the proof of the first statement, we let $\mathbf{H}_0=\mathbf{Q}\cdot 1+\mathbf{Q}\cdot i+\mathbf{Q}\cdot j+\mathbf{Q}\cdot k$ be the quaternion algebra over $\mathbf{Q}$ so that $\mathbf{H}\cong\mathbf{R}\otimes\mathbf{H}_0$. By the Künneth theorem (see e.g. MacLane [14, Theorem X.7.4])

$$H_*(\mathbf{H},\mathbf{H})\cong H_*(\mathbf{H}_0,\mathbf{H}_0).$$

Hence it suffices to show that $H_0(\mathbf{H}_0,\mathbf{H}_0)=\mathbf{Q}$ and $H_n(\mathbf{H}_0,\mathbf{H}_0)=0$ for $n>0$. However, $\mathbf{H}_0\otimes\mathbf{H}_0\cong M_4(\mathbf{Q})$, the full 4×4 matrix algebra over $\mathbf{Q}$. Since Hochschild homology is a Morita invariant, we can again use the Künneth theorem to obtain

$$H_*(\mathbf{H}_0,\mathbf{H}_0)\otimes H_*(\mathbf{H}_0,\mathbf{H}_0)\cong H_*(M_4(\mathbf{Q}),M_4(\mathbf{Q}))\cong H_*(\mathbf{Q},\mathbf{Q}). \tag{4.10}$$

Since $H_0(\mathbf{Q},\mathbf{Q})=\mathbf{Q}$ and $H_n(\mathbf{Q},\mathbf{Q})=0$ for $n>0$, Proposition 4.9 follows. □

We next reformulate Theorem 4.2 in terms of the quaternions: Spin(3), the universal (double) covering group of $SO(3)$, is identified with the group $Sp(1)\subset\mathbf{H}$ of unit quaternions. Namely, let q^* denote the usual quaternion conjugate of $q\in\mathbf{H}$. Identify $\mathbf{R}^3$ with the space $\mathbf{H}^-=\mathbf{R}\cdot i+\mathbf{R}\cdot j+\mathbf{R}\cdot k$ of pure quaternions. Then the covering map $\varrho:Sp(1)\to SO(3)$ is given by the representation $\varrho(q)(v)=qvq^*$ for $v\in\mathbf{H}^-$. The inner product on $\mathbf{H}=\mathbf{R}^4$ is defined by $\langle q_1,q_2\rangle=(q_1q_2^*+q_2,q_1^*)/2$. Similarly, Spin(4), the universal (double) covering group of $SO(4)$, is naturally identified with $S_1\times S_2$ where $S_i=Sp(1), i=1,2$, and the usual action via the covering σ: Spin(4)$\to SO(4)$ is given by the rule:

$$\sigma(q_1,q_2)(v)=q_1vq_2^*,\quad q_i\in S_1,\ i=1,2,\ v\in\mathbf{H}=\mathbf{R}^4.$$

This covering map actally extends to σ: Pin(4)$\to O(4)$ where Pin(4)$=(S_1\times S_2)\rtimes\mathbf{Z}_2$ is the semidirect product with $\mathbf{Z}_2$ acting on $S_1\times S_2$ by interchanging the factors. The corresponding orientation reversing involution in $O(4)$ is the quaternion conjugation map.

For the induced action of Pin(4) on $\Lambda^2_{\mathbf{Z}}(\mathbf{H})$, we deduce from Theorem 2.2 and the Hochschild–Serre spectral sequence for the extension

$$1\to\{\pm1\}\to\mathrm{Pin}(4)\to O(4)\to 1$$

that

(4.11) $$H_i(\mathrm{Pin}(4),\Lambda^2_{\mathbf{Z}}(\mathbf{H}))=0\quad\text{for}\quad i=0,1,2.$$

It is convenient to identify $\Lambda^2_{\mathbf{Z}}(\mathbf{H})$ with the (-1)-eigenspace in $\mathbf{H}\otimes\mathbf{H}$ for the involution τ given by

(4.12) $$\tau(a_0\otimes a_1)=a_1^*\otimes a_0^*.$$

The isomorphism

(4.13) $$(\mathbf{H}\otimes\mathbf{H})^-\cong\Lambda^2_{\mathbf{Z}}(\mathbf{H})$$

is given by $a_0\otimes a_1\to a_0\wedge a_1^*$. This isomorphism is Pin(4)-equivariant if we define the action $\tilde{\sigma}$ on $\mathbf{H}\otimes\mathbf{H}$ by

(4.14) $$\tilde{\sigma}(q_1,q_2)(a_0\otimes a_1)=q_1a_0q_2^*\otimes q_2a_1q_1^*,\quad q_i\in S_i,\ i=1,2,\ a_j\in\mathbf{H},\ j=0,1,$$

and the $\mathbf{Z}_2$-factor acts by $a_0\otimes a_1\to a_0^*\otimes a_1^*$.

We now consider the map

$$\varepsilon = \varepsilon_0 \amalg (-\varepsilon_1) : \mathbf{H} \otimes \mathbf{H} \to \mathbf{H} \amalg \mathbf{H} \tag{4.15}$$

where ε_0 and ε_1 are given by (4.4) and (4.5) above. We note that ε is Pin(4)-equivariant if the action on the right-hand side is given by

$$\bar{\sigma}(q_1, q_2)(v_1, v_2) = (\varrho(q_1)(v_1), \varrho(q_2)(v_2)), \quad q_i \in S_i, \ v_i \in \mathbf{H}, \ i = 1, 2,$$

and if the $\mathbf{Z}_2$-factor acts by $(v_1, v_2) \to (-v_2^*, -v_1^*)$. Also ε commutes with the involution τ if τ is defined on the right-hand side by the quaternion conjugation: $\tau(v_1, v_2) = (v_1^*, v_2^*)$. Note also that on both sides of (4.15) τ commutes with the Pin(4)-action.

By Proposition 4.9, $H_0(\mathbf{H}, \mathbf{H}) = \mathbf{R}$ is the $(+1)$-eigenspace for τ on $\mathbf{H}$. We therefore obtain from (4.15) the following exact sequences of Pin(4)-modules:

$$0 \to I_1(\mathbf{H})^- \to (\mathbf{H} \otimes \mathbf{H})^- \to \mathbf{H}^- \amalg \mathbf{H}^- \to 0 \tag{4.16}$$

$$0 \to I_1(\mathbf{H})^+ = \Omega_1(\mathbf{H})^+ \to (\mathbf{H} \otimes \mathbf{H})^+ \to \mathbf{H}^+ = \mathbf{R} \to 0 \tag{4.17}$$

where the superscripts $\pm$ indicate the (± 1)-eigenspaces for τ. Note that the last map in (4.17) is given by $\varepsilon_0^+ = \varepsilon_1^+$.

From the Hochschild–Serre spectral sequence and Künneth's theorem (cf. Dupont [5, p. 619]) we obtain

$$H_i(\mathrm{Pin}(4), \mathbf{H}^- \amalg \mathbf{H}^-) \cong H_i(\mathrm{Spin}(3), \mathbf{H}^-) \cong H_i(SO(3), \mathbf{R}^3), \quad i = 0, 1, 2. \tag{4.18}$$

By using the long homology sequence associated to (4.16), we conclude from (4.11) and (4.13) that

$$H_i(SO(3), \mathbf{R}^3) \cong H_{i-1}(\mathrm{Pin}(4), I_1(\mathbf{H})^-), \quad i = 1, 2. \tag{4.19}$$

Thus, the proof of Theorem 4.2 is reduced to calculating $H_i(\mathrm{Pin}(4), I_1(\mathbf{H})^-)$, for $i=0,1$. This will be done in the next section.

5. The theorem of Sydler and Jessen: conclusion

The goal in the present section is to prove

THEOREM 5.1. (a) *The natural map* $I_1(\mathbf{H}) \to H_1(\mathbf{H}, \mathbf{H}) = H_1(\mathbf{R}, \mathbf{R}) = \Omega^1_{\mathbf{R}}$ *induces an isomorphism* $H_0(\mathrm{Spin}(4), I_1(\mathbf{H})^-) \cong \Omega^1_{\mathbf{R}}$ *and* Pin(4)/Spin(4)$\cong \mathbf{Z}_2$ *acts trivially on both.*

(b) $H_1(\mathrm{Spin}(4), I_1(\mathbf{H})^-)=0$.

For the proof we first give the complex $(\Omega_*(\mathbf{H}), b)$ a Spin(4)-module structure compatible on $\Omega_1(\mathbf{H})\subset\mathbf{H}\otimes\mathbf{H}$ with the action $\bar\sigma$ (4.14). Since $\Omega_n(\mathbf{H})\subset\mathbf{H}^{\otimes(n+1)}$, we define the action on $\mathbf{H}^{\otimes(n+1)}$ by setting

$$\text{(5.2)}\quad \begin{aligned} &\text{for } n=0, \quad \sigma(q_1,q_2)(a_0)=q_2 a_0 q_2^*, \quad a_0\in\mathbf{H},\\ &\text{for } n>0, \quad \sigma(q_1,q_2)(a_0\otimes a_1\otimes\ldots\otimes a_n)\\ &\qquad\qquad = q_1 a_0 q_2^*\otimes q_2 a_1 q_1^*\otimes q_1 a_2 q_1^*\otimes\ldots\otimes q_1 a_n q_1^*, \quad a_i\in\mathbf{H}.\end{aligned}$$

Here $q_i\in S_i$, $i=1,2$. Notice that $\Omega_n\subset\mathbf{H}^{\otimes(n+1)}$ is stable under this action and that $b\colon\Omega_n(\mathbf{H})\to\Omega_{n-1}(\mathbf{H})$ is a Spin(4)-module map. By (4.6), we have a natural isomorphism of Spin(4)-modules:

$$\text{(5.3)}\qquad \Omega_n(\mathbf{H})\cong\Omega_1(\mathbf{H})\otimes(\mathbf{H}/\mathbf{Q})^{\otimes(n-1)}, \quad n>0,$$

where $(q_1,q_2)\in S_1\times S_2$ acts on $\Omega_1(\mathbf{H})$ as in (4.14) and acts on $\mathbf{H}/\mathbf{Q}$ by $\varrho(q_1)$, i.e.

$$\text{(5.4)}\qquad \bar\sigma(q_1,q_2)(\omega\cdot da_2\ldots da_n)=\bar\sigma(q_1,q_2)(\omega)\cdot d(\varrho(q_1)(a_2))\ldots d(\varrho(q_1)(a_n)),$$

where $\omega\in\Omega_1(\mathbf{H})$, $a_2,\ldots,a_n\in\mathbf{H}$.

We next extend the involution τ to all of $\Omega_n(\mathbf{H})$:

We already defined τ on $\Omega_0(\mathbf{H})=\mathbf{H}$ by $\tau(a_0)=a_0^*\in\mathbf{H}$ for $a_0\in\mathbf{H}$ and on $\Omega_1(\mathbf{H})\subset\mathbf{H}\otimes\mathbf{H}$ by (4.12), i.e., $\tau(a_0da_1)=\tau(a_0\otimes a_1-a_0a_1\otimes 1)=a_1^*\otimes a_0^*-1\otimes a_1^*a_0^*=-(da_1^*)a_0^*$. Hence, if we define the conjugation on $\Omega_n(\mathbf{H})$ by $(a_0da_1\ldots da_n)^*=(da_n^*\ldots da_1^*)a_0^*$, $a_i\in\mathbf{H}$, then $\tau(\omega)=-\omega^*$ for $\omega\in\Omega_1(\mathbf{H})$. We now define τ on $\Omega_n(\mathbf{H})$ for $n>1$ by using the isomorphism (5.3) so that

$$\text{(5.5)}\qquad \tau(\omega\cdot da_2\ldots da_n)=(-1)^{(n-2)(n-3)/2}\cdot\omega^*\cdot da_n^*\ldots da_2^*, \quad \omega\in\Omega_1(\mathbf{H}),\ a_i\in\mathbf{H}.$$

With this definition, τ commutes with $b\colon\Omega_n(\mathbf{H})\to\Omega_{n-1}(\mathbf{H})$, and in view of (5.4), τ also commutes with the Spin(4)-action.

It follows that the complex $(\Omega_*(\mathbf{H}), b)$ splits as a complex of Spin(4)-modules into (±1)-eigenspaces for τ,

$$\Omega_*(\mathbf{H})=\Omega_*(\mathbf{H})^+\amalg\Omega_*(\mathbf{H})^-$$

and similarly for $I_*(\mathbf{H})$, $B_*(\mathbf{H})$, and $H_*(\mathbf{H},\mathbf{H})$.

We now prove two useful lemmas:

LEMMA 5.6. (a) *The* Spin(4)*-action on* $H_*(\mathbf{H},\mathbf{H})$ *is trivial.*
(b) *On* $H_n(\mathbf{H},\mathbf{H})$ *the involution* τ *is given by* $(-1)^n\cdot\mathrm{id}$.

Proof. (a) By Proposition 4.9, every element in $H_n(\mathbf{H},\mathbf{H})=I_n(\mathbf{H})/B_n(\mathbf{H})$ is represented by elements in $\Omega_n(\mathbf{R})$. Let us show first that $q\in S_1$ acts trivially on $H_n(\mathbf{H},\mathbf{H})$. Clearly $\{q\}$ and $\mathbf{R}$ are contained in a *commutative* subfield $C\subset\mathbf{H}$ so it is enough to show that q acts trivially on $H_n(C,C)$. Now by Proposition 4.7 the composition of the maps

$$H_n(C,C)\to\Omega_n(C)/B_n(C)\to\Omega_C^n$$

is an isomorphism, and clearly, by (4.14) and (5.4), $\bar\sigma(q,1)$ is given on $\Omega_n(C)$ by

$$\bar\sigma(q,1)(\omega\cdot da_2\dots da_n)=q\omega q^*\cdot da_2\dots da_n\quad\text{for}\quad\omega\in\Omega_1(C),\ a_i\in C.$$

Since this element goes to $\omega\wedge da_2\wedge\dots\wedge da_n$ in Ω_C^n, we have shown that $q\in S_1$ acts trivially on $H_n(C,C)$. For $q\in S_2$, the proof is the same since for C commutative and $q\in\mathrm{Sp}(1)\cap C$, $\bar\sigma(1,q)=\bar\sigma(q^*,1)$. This proves (a).

(b) Again by Propositions 4.7 and 4.8 we shall just calculate τ when it is restricted to $H_n(\mathbf{R},\mathbf{R})\cong\Omega_\mathbf{R}^n$ where by (5.5)

$$\begin{aligned}\tau(\omega\wedge da_2\wedge\dots\wedge da_n)&=(-1)^{(n-2)(n-3)/2}\cdot\omega\wedge da_n\wedge\dots da_2\\&=(-1)^{n-2}\cdot\omega\wedge da_2\wedge\dots\wedge da_n,\quad\omega\in\Omega_\mathbf{R}^1,\ a_i\in\mathbf{R}.\end{aligned}$$

This proves Lemma 5.6. □

LEMMA 5.7. *For the action of* $S_2\subset\mathrm{Spin}(4)$ *on* $\Omega_n(\mathbf{H})$), *we have* $H_0(S_2,\Omega_0(\mathbf{H}))=\mathbf{R}$ *and* $H_0(S_2,\Omega_n(\mathbf{H}))=0$ *for* $n>0$.

Proof. For $n=0$, this is clear from (5.2). For $n>0$, it follows from (5.3) and (5.4) that it suffices to prove

$$H_0(S_2,\Omega_1(\mathbf{H}))=0.$$

For this, consider the exact sequence of S_2-modules

$$0\to\Omega_1(\mathbf{H})\to\mathbf{H}\otimes\mathbf{H}\xrightarrow{\varepsilon_0}\mathbf{H}\to 0\tag{5.8}$$

where the action in the middle is given by (4.14) and where S_2 acts trivially on the last module $\mathbf{H}$. Since S_2 is a perfect group the long exact sequence for (5.8) yields

$$0\to H_0(S_2,\Omega_1(\mathbf{H}))\to H_0(S_2,\mathbf{H}\otimes\mathbf{H})\xrightarrow{\varepsilon_{0*}}\mathbf{H}\to 0$$

and we must show that ε_{0*} is injective. Now ε_{0*} is given by $\varepsilon_{0*}(a_0 \otimes a_1) = a_0 a_1$ for $a_0, a_1 \in \mathbf{H}$, and is split by $\eta: \mathbf{H} \to H_0(S_2, \mathbf{H} \otimes \mathbf{H})$ defined by

$$\eta(a) = a \otimes 1, \quad a \in \mathbf{H}. \tag{5.9}$$

We show that η is surjective, i.e., that every element is of the form (5.9). This in turn follows once we show that in $H_0(S_2, \mathbf{H} \otimes \mathbf{H})$

$$a_0 x \otimes a_1 \equiv a_0 \otimes x a_1 \quad \text{for all } a_0, a_1, x \in \mathbf{H}. \tag{5.10}$$

For this, we observe that (5.10) is clearly true for $x = q \in Sp(1)$ by (4.13), and hence also for $x = q^*$. By adding these two equations, it follows that (5.10) is true for $x = q + q^*$. This latter ranges over all real numbers in the closed interval from -2 to 2. It follows that (5.10) holds for any real number. Hence, writing any $x \in \mathbf{H}$ in the form rq with $r \in \mathbf{R}$ and $q \in Sp(1)$, we conclude that (5.10) hods in general so that Lemma 5.7 holds

Proof of Theorem 5.1. (a) Consider the exact sequences of Spin(4)-modules:

$$0 \to B_1(\mathbf{H})^- \to I_1(\mathbf{H})^- \to H_1(\mathbf{H}, \mathbf{H})^- \to 0, \tag{5.11}$$

$$0 \to I_2(\mathbf{H})^- \to \Omega_2(\mathbf{H})^- \xrightarrow{b} B_1(\mathbf{H})^- \to 0. \tag{5.12}$$

By Lemma 5.6, we have

$$H_0(\mathrm{Spin}(4), H_1(\mathbf{H}, \mathbf{H})^-) = H_1(\mathbf{H}, \mathbf{H})^- \cong \Omega^1_{\mathbf{R}}$$

so that (5.11) yields the exact sequence.

$$H_0(\mathrm{Spin}(4), B_1(\mathbf{H})^-) \to H_0(\mathrm{Spin}(4), I_1(\mathbf{H})^-) \to \Omega^1_{\mathbf{R}} \to 0.$$

By (5.12), $H_0(\mathrm{Spin}(4), B_1(\mathbf{H})^-)$ is a quotient of $H_0(\mathrm{Spin}(4), \Omega_2(\mathbf{H})^-)$ which is 0 by Lemma 5.7. This proves the first statement of Theorem 5.1 (a). Since Pin(4)/Spin(4) acts on $I_1(\mathbf{H})$ by conjugation and since every element in $H_1(\mathbf{H}, \mathbf{H})$ is real, the second statement of Theorem 5.1 (a) is obvious.

(b) We first notice that by Lemma 5.6 (b)

$$I_2(\mathbf{H})^- = B_2(\mathbf{H})^-,$$

thus, similar to (5.11) and (5.12), we also have the exact sequence of Spin(4)-modules

$$0 \to I_3(\mathbf{H})^- \to \Omega_3(\mathbf{H})^- \xrightarrow{b} I_2(\mathbf{H})^- \to 0.$$

By Lemma 5.7, $H_0(\mathrm{Spin}(4), \Omega_3(\mathbf{H})^-)=0$, it follows that $H_0(\mathrm{Spin}(4), I_2(\mathbf{H})^-)=0$. We deduce from the long homology sequence for (5.12) that

$$b_* : H_1(\mathrm{Spin}(4), \Omega_2(\mathbf{H})^-) \to H_1(\mathrm{Spin}(4), I_1(\mathbf{H})^-) \quad \text{is } \textit{surjective}.$$

The long homology sequence for (5.11) therefore gives the exact sequence

$$H_1(\mathrm{Spin}(4), \Omega_2(\mathbf{H})^-) \xrightarrow{b_*} H_1(\mathrm{Spin}(4), I_1(\mathbf{H})^-) \to H_1(\mathrm{Spin}(4),\ H_1(\mathbf{H}, \mathbf{H}))$$

where the last group vanishes because of Lemma 5.6 (a) and the fact that Spin(4) is a perfect group. Hence it only remains to prove that the induced map

$$(5.13) \qquad b_*: H_1(\mathrm{Spin}(4), \Omega_2(\mathbf{H})^-) \to H_1(\mathrm{Spin}(4), I_1(\mathbf{H})^-)$$

is zero. For this we first observe that the Hochschild–Serre spectral sequence for the extension

$$1 \to S_2 \to \mathrm{Spin}(4) \to S_1 \to 1$$

and the module $\Omega_2(\mathbf{H})^-$ yields an exact sequence

$$H_2(S_1, H_0(S_2, \Omega_2(\mathbf{H})^-)) \to H_0(S_1, H_1(S_2, \Omega_2(\mathbf{H})^-)) \to H_1(\mathrm{Spin}(4), \Omega_2(\mathbf{H})^-)$$
$$\to H_1(S_1, H_0(S_2, \Omega_2(\mathbf{H})^-)) \to 0.$$

By Lemma 5.7, we obtain an isomorphism

$$H_0(S_1, H_1(S_2, \Omega_2(\mathbf{H})^-)) \xrightarrow{\cong} H_1(\mathrm{Spin}(4), \Omega_2(\mathbf{H})^-).$$

The vanishing of b_* in (5.13) therefore follows from the following

LEMMA 5.14. *The composition of the maps below is zero*:

$$H_1(S_2, \Omega_2(\mathbf{H})) \xrightarrow{b_*} H_1(S_2, I_1(\mathbf{H})) \to H_1(\mathrm{Spin}(4), I_1(\mathbf{H})).$$

For this we first notice that (5.3) for $n=2$ gives and isomorphism of Spin(4)-modules: $\Omega_2(\mathbf{H}) \cong \Omega_1(\mathbf{H}) \otimes (\mathbf{H}/\mathbf{Q})$, where S_2 acts trivially on the second factor. We then prove the weaker statement

LEMMA 5.15. *The composition of the maps below is zero*

$$H_1(S_2, I_1(\mathbf{H})\otimes\mathbf{H}^-) \to H_1(S_2, \Omega_2(\mathbf{H})) \xrightarrow{b_*} H_1(S_2, I_1(\mathbf{H})) \to H_1(\mathrm{Spin}(4), I_1(\mathbf{H})).$$

Proof. Let $\xi \in H_1(S_2, I_1(\mathbf{H}))$. We will show that for $t \in \mathbf{R}$ the class $\xi \cdot d(it) \in H_1(S_2, \Omega_2(\mathbf{H}))$ maps to zero in $H_1(\mathrm{Spin}(4), I_1(\mathbf{H}))$. Replacing i by j and k will then prove the lemma. We begin by showing that $b_*(\xi \cdot d(it))$ is zero in $H_0(S_1, H_1(S_2, I_1(\mathbf{H})))$. Let us write $\equiv$ (mod S_1) for the equivalence relation in this group. Notice that for $q \in Sp(1)$ and $\omega \in \Omega_1(\mathbf{H})$, $\bar{\sigma}(q, 1)(\omega) = q\omega q^*$ so clearly

$$q\xi q^* \equiv \xi \pmod{S_1}. \tag{5.16}$$

Now let $q \in Sp(1) \cap \mathbf{C} = U(1)$ and write $q = r + is$ so that (5.16) becomes

$$(r+is)\xi(r-is) \equiv \xi \pmod{S_1}$$

that is,

$$r\xi r - is\xi is + (is\xi r - r\xi is) \equiv \xi \pmod{S_1}. \tag{5.17}$$

Replacing q by q^*, that is, replacing s by $-s$ in (5.17) and subtracting from (5.17), we obtain

$$2(is\xi r - r\xi is) \equiv 0 \pmod{S_1}. \tag{5.18}$$

Now $I_1(\mathbf{H})$ is both a left and a right $\mathbf{R}$-module (but *not* an $\mathbf{H}$-module!) so that there is an induced left and right $\mathbf{R}$-multiplication on $H_0(S_1, H_1(S_2, I_1(\mathbf{H})))$. Hence we can mutiply (5.18) on the left and on the right by $(2r)^{-1}$ respectively r^{-1} and obtain

$$isr^{-1}\xi - \xi isr^{-1} \equiv 0 \pmod{S_1}$$

i.e., by (4.7),

$$b_*(\xi \cdot d(it)) \equiv 0 \pmod{S_1}, \quad t = s/r.$$

Since $t = \tan\vartheta$ for $r + is = \exp(i\vartheta)$, it can be any real number and we have proved Lemma 5.15. □

Proof of Lemma 5.14. An easy calculation gives the identity

$$b(\omega \cdot dt) = b(t\omega idi) - b(\omega id(it)), \quad \omega \in \Omega_1(\mathbf{H}), \ t \in \mathbf{R}.$$

Hence it suffices to evalutate b_* on elements of the form

$$\xi \cdot da \in H_1(S_2, \Omega_2(\mathbf{H})), \quad \xi \in H_1(S_2, \Omega_1(\mathbf{H})) \quad \text{and} \quad a \in \mathbf{H}^-.$$

Lemma 5.15 therefore follows from Lemma 5.14 once we show that the inclusion map $I_1(\mathbf{H}) \to \Omega_1(\mathbf{H})$ induces a surjective map

$$H_1(S_2, I_1(\mathbf{H})) \to H_1(S_2, \Omega_1(\mathbf{H})).$$

Since $\mathrm{Re}(a_0 a_1) = \mathrm{Re}(a_1 a_0)$ for $a_0, a_1 \in \mathbf{H}$, $I_1(\mathbf{H})^+ = \Omega_1(\mathbf{H})^+$ so it suffices to show surjectivity of the map

$$H_1(S_2, I_1(\mathbf{H})^-) \to H_1(S_2, \Omega_1(\mathbf{H})^-). \tag{5.19}$$

Now consider the commutative diagram with exact rows

$$\begin{array}{ccccccccc}
0 & \longrightarrow & I_1(\mathbf{H})^- & \longrightarrow & \Omega_1(\mathbf{H})^- & \xrightarrow{\;b\;} & \mathbf{H}^- & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow{\scriptstyle i_2} & & \\
0 & \longrightarrow & I_1(\mathbf{H})^- & \longrightarrow & (\mathbf{H} \otimes \mathbf{H})^- & \xrightarrow{\;\varepsilon\;} & \mathbf{H}^- \amalg \mathbf{H}^- & \longrightarrow & 0
\end{array}$$

where ε is given by (4.15) and $i_2(v) = (0, v)$, $v \in \mathbf{H}^-$. Here the top row is a sequence of Spin(4)-modules whereas the bottom row is a sequence of Pin(4)-modules. Together with the remarks following (4.16) we obtain a commutative diagram with exact rows

$$\begin{array}{ccccccc}
H_1(S_2, I_1(\mathbf{H})^-) & \longrightarrow & H_1(S_2, \Omega_1(\mathbf{H})^-) & \xrightarrow{\;b_*\;} & H_1(S_2, \mathbf{H}^-) & \xrightarrow{\;\partial\;} & H_0(S_2, I_1(\mathbf{H})^-) \\
 & & & & \downarrow{\scriptstyle i_2} & & \downarrow \\
 & & 0 & \longrightarrow & H_1(\mathrm{Pin}(4), \mathbf{H}^- \amalg \mathbf{H}^-) & \longrightarrow & H_0(\mathrm{Pin}(4), I_1(\mathbf{H})^-)
\end{array} \tag{5.20}$$

Since i_2 is an isomorphism by (4.18), it follows from (5.20) that the boundary map ∂ in the top row is injective. Hence $b_* = 0$ and we have proved the surjectivity of the map in (5.19). This ends the proof of Lemma 5.15 and hence also the proof of Theorem 5.1. □

Proof of Theorem 4.2. In view of (4.19) and Theorem 5.1 it only remains to determine the map J. For this we recall from Dupont [5, Example 4.11] that the map

$$\mathbf{R} \otimes (\mathbf{R}/\mathbf{Z}) \to H_1(SO(3), \mathbf{R}^3)$$

is the induced map

(5.21) $$H_1(SO(2),\mathbf{R})\to H_1(SO(3),\mathbf{R}^3)$$

where $SO(2)\subset SO(3)$ is the natural inclusion and $\mathbf{R}\subset\mathbf{R}^3$ is the inclusion of the line normal to the plane in which $SO(2)$ acts by rotations. Thus we have made the usual identifications

$$\mathbf{R}/\mathbf{Z}\cong SO(2)\cong H_1(SO(2),\mathbf{Z}).$$

Now we identify $O(2)$ with the semidirect product $O(2)=U(1)\rtimes\mathbf{Z}_2$ where $SO(2)\cong U(1)$ is acting by complex multiplication on $\mathbf{C}=\mathbf{R}^2$ and the reflection in the real line is just complex conjugation. We also put $\mathrm{Pin}(2)=U(1)\rtimes\mathbf{Z}_2$ where $\varrho:\mathrm{Pin}(2)\to O(2)$ is just given by the squaring map on $U(1)$. We next include Pin(2) into Pin(4) by sending $z\in U(1)$ to $(z,z^*)\in S_1\times S_2$ and we have a map of exact sequences

(5.22) $$\begin{array}{ccccccccc} 0 & \longrightarrow & I_1(\mathbf{C})^- & \longrightarrow & (\mathbf{C}\otimes\mathbf{C})^- & \xrightarrow{\varepsilon_0} & i\mathbf{R} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow{\scriptstyle\Delta} & & \\ 0 & \longrightarrow & I_1(\mathbf{H})^- & \longrightarrow & (\mathbf{H}\otimes\mathbf{H})^- & \xrightarrow{\varepsilon} & \mathbf{H}^-\amalg\mathbf{H}^- & \longrightarrow & 0 \end{array}$$

where Δ is given by $\Delta(v)=(v,-v),\ v\in i\mathbf{R}$. Here the top row is a sequence of Pin(2)-modules induced via $\varrho\otimes\varrho^*$ on $\mathbf{C}\otimes\mathbf{C}$; on $i\mathbf{R}$ the action is given by ϱ followed by the determinant (over $\mathbf{R}$). The bottom row is a sequence of Pin(4)-modules as usual and the vertical maps are clearly equivariant. From (5.22) we obtain a commutative diagram

(5.23) $$\begin{array}{ccccc} H_1(\mathrm{Pin}(2),i\mathbf{R}) & \xrightarrow{\partial} & H_0(\mathrm{Pin}(2),I_1(\mathbf{C})^-) & \longrightarrow & H_0(\mathrm{Pin}(2),H_1(\mathbf{C},\mathbf{C})^-) \\ \downarrow{\scriptstyle\Delta_*} & & \downarrow & & \downarrow \\ H_1(\mathrm{Pin}(4),\mathbf{H}^-\amalg\mathbf{H}^-) & \xrightarrow{\partial} & H_0(\mathrm{Pin}(4),I_1(\mathbf{H})^-) & \xrightarrow{\cong} & H_0(\mathrm{Pin}(4),H_1(\mathbf{H},\mathbf{H})^-) \end{array}$$

Here

$$H_1(\mathbf{C},\mathbf{C})^-=(\Omega^1_{\mathbf{C}})^-=\{\omega\in\Omega^1_{\mathbf{C}}\mid \omega^*=\omega\}=\Omega^1_{\mathbf{R}}$$

and Pin(2) evidently acts trivially. Together with Theorem 5.1 (a) it follows that the right most vertical map in (5.23) is an isomorphism. On the other hand

(5.24) $$H_1(\mathrm{Pin}(2),i\mathbf{R})\cong H_1(O(2),\mathbf{R}')\cong\mathbf{R}\otimes(\mathbf{R}/\mathbf{Z})$$

where $\mathbf{R}'$ donotes $\mathbf{R}$ with the determinant action. By way of the isomorphism (4.18) Δ_* is 2 times the map in (5.21) so it remains to determine the top horizontal map in (5.23) via the identifications in (5.24). Since all maps in (5.23) respects multiplication by $r \in \mathbf{R}$, it is enough to determine $J(1\otimes(\vartheta/2\pi))$ for $\vartheta \in [0, 2\pi]$. Let $z=\exp(i\vartheta) \in U(1) \cong SO(2)$ and observe that the corresponding 1-chain in $C_1(\mathrm{Pin}(2), i\mathbf{R})$ is $z^{1/2}\otimes i$. This lifts in $C_1(\mathrm{Pin}(2), (\mathbf{C}\otimes\mathbf{C})^-)$ to

$$z^{1/2}\otimes\tfrac{1}{2}(i\otimes 1+1\otimes i).$$

Hence $\partial(z^{1/2}\otimes i)$ is represented in $C_0(\mathrm{Pin}(2), I_1(\mathbf{C})^-)$ by

$$\tfrac{1}{2}(i\otimes 1+1\otimes i-iz\otimes z^*-z\otimes iz^*)=\tfrac{1}{2}(dz\cdot iz^*-iz\cdot dz^*).$$

It follows that in $(\Omega^1_{\mathbf{C}})^-$ we have

$$2J(1\otimes(\vartheta/2\pi))=\tfrac{i}{2}(z^*\cdot dz-z\cdot dz^*).$$

By using

$$(\cos\vartheta)\cdot d(\cos\vartheta)+(\sin\vartheta)\cdot d(\sin\vartheta)=\tfrac{1}{2}d(\cos^2\vartheta+\sin^2\vartheta)=0,$$

we have

$$\begin{aligned} z^*\cdot dz &= (\cos\vartheta-i\cdot\sin\vartheta)\cdot d(\cos\vartheta+i\cdot\sin\vartheta)=i\{\cos\vartheta\cdot d(\sin\vartheta)-\sin\vartheta\cdot d(\cos\vartheta)\}\\ &= i\left\{-\frac{\cos^2\vartheta}{\sin\vartheta}d(\cos\vartheta)-\sin\vartheta\, d(\cos\vartheta)\right\}=-i\frac{d(\cos\vartheta)}{\sin\vartheta}. \end{aligned}$$

Hence

$$2J(1\otimes(\vartheta/2\pi))=d(\cos\vartheta)/\sin\vartheta$$

which proves the remaining part of Theorem 4.2.

6. Remarks: unsolved problems

A result of Jessen [10] shows that the scissors congruence group in Euclidean 4-space is isomorphic to the scissors congruence group in Euclidean 3-space. For a homological proof of this reduction cf. Dupont [5, Corollary 4.28]. Geometrically every 4-dimensional Euclidean polytope is scissors congruent to an orthogonal cylinder of height 1 with base equal to some polytope of dimension 3. Since both the volume and the Dehn invariant easily "desuspends" to the base, this showed that the theorem of Dehn–Sydler–Jessen extends to Euclidean 4-space. For dimensions 5 or higher, Dehn invariants and volume are still invariants of the scissors congruence classes. It is not known if they are complete invariants. The "desuspension" theorem of Jessen extends weakly to higher dimension in the sense that any even dimensional Euclidean polytope is scissors congruent to a finite interior disjoint union of orthogonal product of properly lower dimensional polytopes, i.e., they are "generalized cylinders" or "decomposable". However, it is not known if they are scissors congruent to an orthogonal cylinder of height 1. The first unsolved case occurs in dimension 6. This problem is first posed by Jessen and can be phrased in the form:

Problem of Jessen: Is it true that every orthogonal product of 3-simplices in $\mathbf{R}^6$ is scissors congruent to an orthogonal cylinder of height 1 with a base equal to some polytope in $\mathbf{R}^5$?

A Dehn invariant computation for simple examples shows that the the converse of the preceding question is definitely false.

It should be noted that there are various possibilities in the definition of Dehn invariants. Moreover, the graded structures of the scissors congruence groups and the related definitions of the higher Dehn invariants have a formal similarity to the known cyclic homology groups of the field $\mathbf{R}$, cf. Cartier [1] as well as Sah [11, Chapters 6 and 7]. It should also be noted that the problem of Jessen can be rephrased in homological terms. In fact, a necessary condition for an affirmative answer is the injectivity of the natural map

$$H_3(O(5),\Lambda^4_{\mathbf{R}}(\mathbf{R}^5)) \rightarrow H_3(O(6),\Lambda^4_{\mathbf{R}}(\mathbf{R}^6)).$$

The above map is known to be surjective. A sufficient condition will also involve a surjective statement for suitable H_4, cf. Dupont [5, Example 5.39].

References

[1] Cartier, P., Décomposition des polyèdres: Le point sur le Troisième Problème de Hilbert. *Seminaire N. Bourbaki*, no. 646 (1984–85).

[2] Cathelineau, J.-L., Remarques sur l'homologie de $SO(n, \mathbf{R})$ considéré comme groupe discret. *C. R. Acad. Sci. Paris Sér. I*, 295 (1982), 281–283.

[3] — Sur l'homologie de SL_2 a coefficients dans l'action adjointe. *Math. Scand.*, 63 (1988), 51–86.

[4] Connes, A., Non commutative differential geometry. *Inst. Hautes Études Sci. Publ. Math.*, 62 (1985), 41–144, 257–360.

[5] Dupont, J. L., Algebra of polytopes and homology of flag complexes. *Osaka J. Math.*, 19 (1982), 599–641.

[6] DUPONT, J. L., PARRY, W. & SAH, C. H., Homology of classical Lie groups made discrete, II. H_2, H_3, and relations with scissors congruence. *J. Algebra,* 113 (1988), 215–260.
[7] DUPONT, J. L. & SAH, C. H., Scissors congruences, II. *J. Pure Appl. Algebra,* 25 (1982), 159–195.
[8] HOCHSCHILD, G., KOSTANT, B., & ROSENBERG, A., Differential forms on regular affine algebras. *Trans. Amer. Math. Soc.,* 102 (1962), 383–408.
[9] JESSEN, B., The algebra of polyhedra and the Dehn–Sydler theorem. *Math. Scand.,* 22 (1968), 241–256.
[10] — Zur Algebra der Polytope. *Nachr. Akad. Wiss. Göttingen Math.-Phys. Kl. II* (1972), 47–53.
[11] JESSEN, B. & THORUP, A., The algebra of polytopes in affine spaces. *Math. Scand.,* 43 (1978), 211–240.
[12] KAROUBI, M., *K*-théorie multiplicative et homologie cyclique. *C. R. Acad. Sci. Paris Sér. I,* 303 (1986), 507–510.
[13] LODAY, J.-L. & QUILLEN, D., Cyclic homology and the Lie algebra homology of matrices. *Commment. Math. Helv.,* 59 (1984), 565–591.
[14] MACLANE, S., *Homology*. Grundl. Math. Wiss. 114, SpringerVerlag, Berlin–Göttingen–Heidelberg, 1963.
[15] MAY, J. P., *Simplical Objects in Algebraic Topology*. D. Van Nostrand Co., Toronto–London–Melbourne, 1967.
[16] SAH, C. H., *Hilbert's Third Problem: Scissors Congruences*. Res. Notes in Math. 33, Pitman, London, 1979.
[17] — Homology of classical Lie groups made discrete, I. Stability theorems and Schur multipliers. *Comment. Math. Helv.,* 61 (1986), 308–347.
[18] — Homology of classical Lie groups made discrete, III. *J. Pure Appl. Algebra,* 56 (1989), 269–312.
[19] SYDLER, J. P., Conditions nécessaires et suffisantes pour l'équivalence des polyèdres de l'espace euclidien à trois dimensions. *Comment. Math. Helv.,* 40 (1965), 43–80.

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