

The second main theorem for small functions and related problems

by

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1. Introduction

1.1. Results

One of the most interesting results in value distribution theory is the defect relation obtained by R. Nevanlinna: If f is a non-constant meromorphic function on the complex plane \mathbf{C} , then for an arbitrary collection of distinct $a_1, \dots, a_q \in \mathbf{P}^1$, the following defect relation holds:

$$\sum_{i=1}^q (\delta(a_i, f) + \theta(a_i, f)) \leq 2. \quad (1.1.1)$$

Here, as usual in Nevanlinna theory, the terms $\delta(a_i, f)$ and $\theta(a_i, f)$ are defined by

$$\delta(a_i, f) = \liminf_{r \rightarrow \infty} \left(1 - \frac{N(r, a_i, f)}{T(r, f)} \right),$$
$$\theta(a_i, f) = \liminf_{r \rightarrow \infty} \frac{N(r, a_i, f) - \bar{N}(r, a_i, f)}{T(r, f)},$$

and hence satisfy $0 \leq \delta(a_i, f) \leq 1$ and $0 \leq \theta(a_i, f) \leq 1$. For the definitions of the terms $T(r, f)$, $N(r, a_i, f)$ and $\bar{N}(r, a_i, f)$, we refer the reader to [Ne2], [Hay] and the following subsections.

A problem, suggested by Nevanlinna, is whether the defect relation is still true when we replace the constants a_i by an arbitrary collection of distinct *small functions* a_i with respect to f (cf. [Ne1]). Here we say that a meromorphic function a on \mathbf{C} is a small function with respect to f if a satisfies the condition $T(r, a) = o(T(r, f))$ as $r \rightarrow \infty$. Nevanlinna pointed out that the case $q=3$ for this question is valid, because we may reduce the problem to the case that a_1, a_2 and a_3 are all constants by using a Möbius transform. But for the case $q>3$, this method does not work.

Later, Steinmetz [St] and Osgood [O] proved that

$$\sum_{i=1}^q \delta(a_i, f) \leq 2$$

for distinct small functions a_i . Their methods, which may be regarded as generalizations of Nevanlinna's original proof of (1.1.1), are based on the consideration of differential polynomials in f and a_i , $1 \leq i \leq q$. Though Nevanlinna used only the first-order derivative of f , Steinmetz and Osgood used higher-order derivatives of f . Hence the truncation level of the counting function is greater than one in general. See also Chuang [C] and Frank–Weissenborn [FW].

However, it is hoped that the generalization of (1.1.1) for small functions is true with the form including the term $\theta(a_i, f)$ (cf. [D]). In this paper, we give a solution for this problem by the following theorem.

THEOREM 1. *Let Y and B be Riemann surfaces with proper, surjective holomorphic maps $\pi_Y: Y \rightarrow \mathbf{C}$ and $\pi_B: B \rightarrow \mathbf{C}$. Assume that π_Y factors through π_B , i.e., there exists a proper, surjective holomorphic map $\pi: Y \rightarrow B$ such that $\pi_Y = \pi_B \circ \pi$. Let f be a non-constant meromorphic function on Y . Let a_1, \dots, a_q be distinct meromorphic functions on B . Assume that $f \neq a_i \circ \pi$ for $i=1, \dots, q$. Then for all $\varepsilon > 0$, there exists a positive constant $C(\varepsilon) > 0$ such that the following inequality holds:*

$$(q-2-\varepsilon)T(r, f) \leq \sum_{i=1}^q \bar{N}(r, a_i \circ \pi, f) + N_{\text{ram } \pi_Y}(r) + C(\varepsilon) \left(\sum_{i=1}^q T(r, a_i) + N_{\text{ram } \pi_B}(r) \right) \|. \quad (1.1.2)$$

Here the symbol $\|$ means that the stated estimate holds when $r \notin E$ for some exceptional set $E \subset \mathbf{R}_{>0}$ with $\int_E d \log \log r < \infty$.

Remark 1.1.3. (1) The term $N_{\text{ram } \pi_Y}(r)$ counts the ramification points of π_Y . In the case $Y = \mathbf{C}$ and $\pi_Y = \text{id}_{\mathbf{C}}$, we have $N_{\text{ram } \pi_Y}(r) = 0$. Similarly for $N_{\text{ram } \pi_B}(r)$.

(2) We can define the terms $T(r, f)$, $T(r, a_i)$ and $\bar{N}(r, a_i \circ \pi, f)$ for the algebroid functions f, a_1, \dots, a_q by a similar way for meromorphic functions on \mathbf{C} . See the following subsections.

Applying the theorem to the case when $Y = B = \mathbf{C}$, $\pi_Y = \pi_B = \text{id}_{\mathbf{C}}$ and all a_i are small functions with respect to f , we immediately obtain the following corollary.

COROLLARY 1. *Let f be a non-constant meromorphic function on \mathbf{C} , and let a_1, \dots, a_q be distinct meromorphic functions on \mathbf{C} . Assume that a_i are small functions with respect to f for all $i = 1, \dots, q$. Then we have the second main theorem,*

$$(q - 2 - \varepsilon)T(r, f) \leq \sum_{i=1}^q \bar{N}(r, a_i, f) \quad \text{for all } \varepsilon > 0,$$

and the defect relation,

$$\sum_{i=1}^q (\delta(a_i, f) + \theta(a_i, f)) \leq 2.$$

A special case of this corollary, when f is a transcendental meromorphic function and a_i are rational functions, was proved in [Y2] (see also [Sa] for an earlier result). The present paper is a development of the previous one.

We shall prove two other results. The first one is a corollary of the theorem above. This is suggested by Erëmenko [E]. Let \mathfrak{K}_Y and \mathfrak{K}_B be the fields of all meromorphic functions on Y and B , respectively. For a function $\psi: \mathbf{R}_{>0} \rightarrow \mathbf{R}_{>0}$, we define a subset \mathfrak{K}_B^ψ of \mathfrak{K}_B by

$$\mathfrak{K}_B^\psi = \{a \in \mathfrak{K}_B : T(r, a) = O(\psi(r)) \text{ as } r \rightarrow \infty\}.$$

Then this \mathfrak{K}_B^ψ is a subfield of \mathfrak{K}_B . For instance, if ψ is a bounded function, then \mathfrak{K}_B^ψ is the field of all constant functions, i.e., $\mathfrak{K}_B^\psi = \mathbf{C}$. Let $F(x, y) \in \mathfrak{K}_B^\psi[x, y]$ be a polynomial in two variables with coefficients in \mathfrak{K}_B^ψ . For general $z \in B$, we denote by $F_z(x, y) \in \mathbf{C}[x, y]$ the polynomial obtained by taking the values at z of the meromorphic functions in the coefficients of $F(x, y)$. Here the terminology “general” is used to indicate that the exceptional set is discrete.

COROLLARY 2. *Let Y, B and π be the same as in Theorem 1, and let $\psi: \mathbf{R}_{>0} \rightarrow \mathbf{R}_{>0}$. Let $F(x, y) \in \mathfrak{K}_B^\psi[x, y]$ be an irreducible polynomial such that, for general $z \in B$, the polynomial $F_z(x, y)$ is irreducible and the equation $F_z(x, y) = 0$ defines an algebraic curve of (topological) genus greater than one. Assume that $f_1, f_2 \in \mathfrak{K}_Y$ satisfy the functional*

equation $F(f_1, f_2)=0$, where we consider \mathfrak{K}_B^ψ as a subfield of \mathfrak{K}_Y by the natural inclusion defined by π . Then we have

$$T(r, f_i) = O(\psi(r) + N_{\text{ram } \pi_Y}(r)) \quad ||$$

when $r \rightarrow \infty$, for $i=1, 2$.

If we apply this corollary to the case when $Y=B=\mathbf{C}$, $\pi_Y=\pi_B=\text{id}_{\mathbf{C}}$ and ψ is a bounded function, then we conclude that $T(r, f_i)=O(1) \quad ||$ for $i=1, 2$. Hence, both f_1 and f_2 are constant functions. This is equivalent to a result of E. Picard: If the equation $F(x, y)=0$, where $F(x, y)$ is an irreducible polynomial over \mathbf{C} , defines an algebraic curve of (topological) genus greater than one, then there is no pair of non-constant meromorphic functions $f_1(z)$ and $f_2(z)$ on \mathbf{C} such that $F(f_1(z), f_2(z))=0$ identically. (See also [Z].)

The next result is an algebraic analogue of the theorem above.

THEOREM 2. *Let q be a positive integer. For all $\varepsilon > 0$, there exists a positive constant $C(q, \varepsilon) > 0$ with the following property: Let Y and B be compact Riemann surfaces with a proper, surjective holomorphic map $\pi: Y \rightarrow B$. Let f be a rational function on Y . Let a_1, \dots, a_q be distinct rational functions on B . Assume that $f \neq a_i \circ \pi$ for all $i=1, \dots, q$. Then we have*

$$(q-2-\varepsilon) \deg f \leq \sum_{1 \leq i \leq q} \bar{n}(a_i \circ \pi, f, Y) + 2g(Y) + C(q, \varepsilon)(\deg \pi) \left(\max_{1 \leq i \leq q} (\deg a_i) + g(B) + 1 \right). \quad (1.1.4)$$

Here we put $\bar{n}(a_i \circ \pi, f, Y) = \text{card}\{z \in Y : f(z) = a_i \circ \pi(z)\}$ and denote by $g(Y)$ (resp. $g(B)$) the genus of the compact Riemann surface Y (resp. B). Using this theorem, we can prove the height inequality for curves over function fields, which is a geometric analogue of a conjectural Diophantine inequality in number theory proposed by P. Vojta ([V1], [V3]). Since the formulation of this height inequality requires some notation, we postpone stating it until §9 (cf. Theorem 5). A proof of Theorem 2 is similar to that of Theorem 1. But we do not need Nevanlinna theory in this case. The following scheme for the proof of Theorem 1 also works for that of Theorem 2, if we replace “ $B(R)$ ” by “ B ”. We also note that the inequality (1.1.4) is an analogue of the unintegrated version of (1.1.2).

Remark 1.1.5. The reader who is not familiar with Nevanlinna theory may skip §1.4, §§ 2 and 7 to read the proofs of Theorem 2 (§8) and Theorem 5 (§9).

1.2. A rough outline of the proof of Theorem 1

We use Ahlfors’s theory of covering surfaces (cf. [A], [Ne2], [Hay]) and the geometry of the moduli space of q -pointed stable curves of genus 0 (cf. [Kn]), especially properties around the degenerate locus whose point corresponds to a degenerate, nodal curve.

We first divide \mathbf{P}^1 by a finite union of curves γ such that $\mathbf{P}^1 \setminus \gamma$ is a finite disjoint union of sufficiently small Jordan domains D_k , $1 \leq k \leq K$, i.e., $\mathbf{P}^1 \setminus \gamma = \bigcup_{1 \leq k \leq K} D_k$. This division of \mathbf{P}^1 gives the division of $(\mathbf{P}^1)^q$ in the form of open subsets

$$D_{k_1} \times \dots \times D_{k_q}, \quad 1 \leq k_i \leq K \text{ for } 1 \leq i \leq q.$$

Then this division and the holomorphic map

$$\underline{a} = (a_1, \dots, a_q): B \longrightarrow (\mathbf{P}^1)^q$$

give the division of the open set

$$B(R) = \pi_B^{-1}(\{z \in \mathbf{C} : |z| < R\})$$

by the open subsets

$$F(\underline{k}) = F(k_1, \dots, k_q) = B(R) \cap \underline{a}^{-1}(D_{k_1} \times \dots \times D_{k_q}).$$

Note that on each $F(\underline{k})$, the move of a_i is bounded in \mathbf{P}^1 . Hence the situation becomes closer to the case that a_i are all constants. We apply Ahlfors’s theory of covering surfaces to the subcovering $f: \pi^{-1}(F(\underline{k})) \rightarrow \mathbf{P}^1$ and q -Jordan domains D_{k_i} , $1 \leq i \leq q$, on \mathbf{P}^1 . Then we obtain the unintegrated version of (1.1.2) for each domain $F(\underline{k})$. By adding over all \underline{k} , we get the unintegrated version of (1.1.2) for $B(R)$. Using the Schwarz inequality, we conclude the inequality (1.1.2). This is the very rough plan of our proof (we use the moduli space of q -pointed stable curves of genus 0 instead of the space $(\mathbf{P}^1)^q$ above).

There are several problems to work out the process above correctly. The major problem comes from the *degenerating points* $z \in B$, where the values of two distinct functions a_i and a_j degenerate into the same value $a_i(z) = a_j(z)$; the problem is how to *separate* the functions a_i and a_j at the degenerating points z in an appropriate way. To motivate the rest of this introduction, we only remark the following two points, which are closely related.

(1) If $z \in F(k_1, \dots, k_q)$ is a degenerating point such that $a_i(z) = a_j(z)$, then we have $D_{k_i} = D_{k_j}$. Hence we cannot apply the usual method of Ahlfors’s theory; we need to modify it. The idea of the modification is roughly as follows: We use Ahlfors’s theory in two steps (in several steps in general). First, we apply Ahlfors’s theory to the subcovering

$$f: \pi^{-1}(F(\underline{k})) \longrightarrow \mathbf{P}^1.$$

Secondly, we apply Ahlfors's theory to the covering

$$\frac{f-a_i}{a_j-a_i}: f^{-1}(D_{k_i}) \cap \pi^{-1}(F(\underline{k})) \longrightarrow \mathbf{P}^1.$$

Note that we choose the function $\lambda(w) = (w - a_i)/(a_j - a_i)$ so as to separate the functions a_i and a_j , i.e., $\lambda(a_i) \equiv 0$ and $\lambda(a_j) \equiv 1$. Combining these two steps, we get rid of the degenerating point z above. Hence, we can say that our idea is the systematic use of Ahlfors's theory in several steps for different functions which separate the degenerating functions in due order at a degenerating point. The *dual graph* (cf. §1.5) of the q -pointed stable curve associated to a degenerating point describes the combinatorial structure of the degeneration of the functions a_i at the point. In this paper, we use a system of *contraction maps* (cf. §1.5) instead of the functions of the form λ above.

(2) Let $\mathfrak{K} = \mathbf{C}(a_1, \dots, a_q)$ be a subfield of \mathfrak{K}_B generated over \mathbf{C} by the meromorphic functions a_1, \dots, a_q . In general, the transcendental degree of the field extension \mathfrak{K}/\mathbf{C} has high dimension, which requires us to use higher-dimensional algebraic geometry. The most natural way to control the degeneration such as $a_i(z) = a_j(z)$ in an appropriate way is to consider the moduli space of q -pointed stable curves of genus 0, denoted by $\overline{\mathcal{M}}_{0,q}$. Roughly speaking, this space is a quotient of $(\mathbf{P}^1)^q$ by the diagonal action of $\text{Aut}(\mathbf{P}^1)$. For generic $z \in B$, the points $a_1(z), \dots, a_q(z) \in \mathbf{P}^1$ are distinct. We consider these points as q marked points of \mathbf{P}^1 . Since the space $\overline{\mathcal{M}}_{0,q}$ is the classification space of q marked points of stable curves of genus 0, we have the classification map

$$\text{cl}_a: B \longrightarrow \overline{\mathcal{M}}_{0,q}.$$

This map is a modification of the map \underline{a} above. When we consider the degenerating point $z \in B$, then the image $\text{cl}_a(z)$ is contained in the degenerate locus $\mathcal{L}_q \subset \overline{\mathcal{M}}_{0,q}$. What is important is that we may consider the points $a_1(z), \dots, a_q(z)$ as distinct marked points of a degenerate, nodal curve instead of considering them as non-distinct points of \mathbf{P}^1 . Hence in this sense, we can say that the values $a_1(z), \dots, a_q(z)$ are also separated at the degenerating points z . This is one reason why we employ the space $\overline{\mathcal{M}}_{0,q}$.

Next we prepare notation and formulate Theorem 4, from which we derive both Theorem 1 and Theorem 2. Then we shall discuss farther details of the proofs of our theorems.

Remark 1.2.1. When we consider the special case that f is a transcendental meromorphic function on \mathbf{C} and a_i are distinct rational functions on \mathbf{C} , the proof becomes simpler than that of the general case. One reason for this is that the field \mathfrak{K} is contained in the field of rational functions on \mathbf{C} , and hence the transcendental degree of the field

extension \mathfrak{K}/\mathbb{C} is equal to or less than one. Especially, we need neither algebraic geometry nor the moduli space of stable curves. This case was treated in [Y2]. In the present paper, we freely use the language of algebraic geometry.

1.3. Notation

In this paper, we assume that all arcs on a Riemann surface are piecewise analytic, i.e., every arc is parametrized by a continuous map $\alpha(t)$ on the interval $[0, 1]$ with the following property: There is a sequence

$$0 = t_0 < t_1 < \dots < t_n = 1$$

such that the restriction of $\alpha(t)$ to each closed interval $[t_i, t_{i+1}]$ is regular and analytic. In particular, we assume that all Jordan domains are bounded by piecewise analytic Jordan curves.

The following fact is an easy consequence of the identity theorem for analytic functions. See also [Mi, Theorem 1].

LEMMA 1. *The intersection of two arcs on a Riemann surface consists of at most a finite number of points or subarcs of the original arcs.*

Let \mathcal{F} be a Riemann surface. We say that F is a *finite domain* of \mathcal{F} when

- (1) F is a compactly contained, connected open subset of \mathcal{F} ;
- (2) ∂F is a finite union of arcs, which are piecewise analytic by our convention;
- (3) F and $\mathcal{F} \setminus \bar{F}$ have the same boundary.

Here we denote by \bar{F} the closure of F and by ∂F the boundary of F . Then a finite domain F is compact if and only if \mathcal{F} is compact and $F = \mathcal{F}$.

Let $\bigcup_{\lambda \in \Lambda} \Delta_\lambda$ be a triangulation of \mathcal{F} where all edges are piecewise analytic Jordan arcs. Let $F \subset \mathcal{F}$ be a finite domain, and let $(\gamma_i)_i$ be a finite set of arcs with $\partial F = \bigcup_i \gamma_i$. By applying Lemma 1 for arcs γ_i and edges of triangles Δ_λ , and passing to a suitable subdivision of the triangulation, we may assume that each arc γ_i is a finite union of edges of triangles Δ_λ . Since \bar{F} is compact, there is a finite set $\Lambda' \subset \Lambda$ such that $\bar{F} = \bigcup_{\lambda \in \Lambda'} \Delta_\lambda$. This gives a triangulation of \bar{F} by a finite number of triangles. In this triangulation of \bar{F} , some edges may belong to only one triangle; such edges form the boundary ∂F because of the condition (3) above. Using the triangulation of \bar{F} , we define the *characteristic* $\varrho(F)$ of F by

$$-[\text{number of interior vertices}] + [\text{number of interior edges}] - [\text{number of triangles}].$$

Then it is well known that this definition is independent of the choice of the triangulation. This characteristic is normalized such that $\varrho(\text{disc}) = -1$ as usual in Ahlfors's theory. We also put $\varrho^+(F) = \max\{0, \varrho(F)\}$.

Let Ω be a compactly contained open subset of \mathcal{F} . Let f and a be meromorphic functions on \mathcal{F} . Assume that $f \neq a$. Put

$$\bar{n}(a, f, \Omega) = \text{card}\{z \in \Omega : f(z) = a(z)\}.$$

Let M be a smooth complex algebraic variety, and let ω be a smooth $(1, 1)$ -form on M . Let $g: \mathcal{F} \rightarrow M$ be a holomorphic map. We put

$$A(g, \Omega, \omega) = \int_{\Omega} g^* \omega.$$

Let γ be an arc on \mathcal{F} and let ω_M be a Kähler form on M . We denote by

$$l(g, \gamma, \omega_M)$$

the length of the arc $g|_{\gamma}: \gamma \rightarrow M$ with respect to the associated Kähler metric of ω_M . Let $Z \subset M$ be an effective divisor such that $g(\mathcal{F}) \not\subset \text{supp } Z$. We put

$$n(g, Z, \Omega) = \sum_{x \in \Omega} \text{ord}_x g^* Z$$

and

$$\bar{n}(g, Z, \Omega) = \sum_{x \in \Omega} \min\{1, \text{ord}_x g^* Z\} = \text{card}(\Omega \cap \text{supp } g^{-1}(Z)).$$

Let \mathcal{F}' be a Riemann surface and let $\pi: \mathcal{F}' \rightarrow \mathcal{F}$ be a proper, surjective holomorphic map. We denote by $\text{ram } \pi$ the ramification divisor of π , which is a divisor on \mathcal{F}' . Put

$$\text{disc}(\pi, \Omega) = \sum_{x \in \pi^{-1}(\Omega)} \text{ord}_x(\text{ram } \pi).$$

1.4. Nevanlinna theory

Let Y be a Riemann surface with a proper, surjective holomorphic map $\pi: Y \rightarrow \mathbf{C}$. Let M be a smooth projective variety. Let $g: Y \rightarrow M$ be a holomorphic map. Let $Z \subset M$ be an effective divisor such that $g(Y) \not\subset \text{supp } Z$, and let ω be a smooth $(1, 1)$ -form on M . For $r > 1$, we put

$$\begin{aligned} N(r, g, Z) &= \frac{1}{\deg \pi} \int_1^r \frac{n(g, Z, Y(t))}{t} dt, \\ \bar{N}(r, g, Z) &= \frac{1}{\deg \pi} \int_1^r \frac{\bar{n}(g, Z, Y(t))}{t} dt, \\ T(r, g, \omega) &= \frac{1}{\deg \pi} \int_1^r \frac{A(g, Y(t), \omega)}{t} dt \end{aligned}$$

and

$$N_{\text{ram } \pi}(r) = \frac{1}{\deg \pi} \int_1^r \frac{\text{disc}(\pi, \mathbf{C}(t))}{t} dt.$$

Here $\mathbf{C}(t) = \{z \in \mathbf{C} : |z| < t\}$ and $Y(t) = \pi^{-1}(\mathbf{C}(t))$.

Let E be a line bundle on M . Let $\|\cdot\|_1$ and $\|\cdot\|_2$ be two Hermitian metrics on E . Let ω_1 and ω_2 be the curvature forms of $\|\cdot\|_1$ and $\|\cdot\|_2$, respectively. Then we have

$$T(r, g, \omega_1) = T(r, g, \omega_2) + O(1) \quad \text{when } r \rightarrow \infty,$$

which follows by Jensen's formula (cf. [NoO, p. 180], [LC, IV.2.1]). Therefore we define the characteristic function $T(r, g, E)$ by

$$T(r, g, E) = T(r, g, \omega_1) + O(1),$$

which is well-defined up to a bounded function.

Let f and a be meromorphic functions on Y such that $f \neq a$. Then we put

$$\bar{N}(r, a, f) = \frac{1}{\deg \pi} \int_1^r \frac{\bar{n}(a, f, Y(t))}{t} dt.$$

We denote by $\omega_{\mathbf{P}^1}$ the Fubini–Study form on the projective line \mathbf{P}^1 , i.e.,

$$\omega_{\mathbf{P}^1} = \frac{1}{(1+|w|^2)^2} \frac{\sqrt{-1}}{2\pi} dw \wedge d\bar{w}.$$

We define the spherical characteristic function by

$$T(r, f) = T(r, f, \omega_{\mathbf{P}^1}) = \frac{1}{\deg \pi} \int_1^r \int_{Y(t)} f^* \omega_{\mathbf{P}^1} \frac{dt}{t}.$$

Then it is well known that this function $T(r, f)$ is equal to the usual Nevanlinna characteristic function of f up to a bounded term in r (cf. the Shimizu–Ahlfors theorem).

1.5. The moduli space of stable curves

Our references are [Kn], [Ke], [FP] and [Ma]. In this subsection, we always assume $q \geq 3$.

Definition 1.5.1. A q -pointed stable curve of genus 0 (or simply q -pointed stable curve) is a connected reduced curve C of (arithmetic) genus 0 with q distinct marked points (s_1, \dots, s_q) provided that

- (1) each irreducible component of C is isomorphic to the projective line \mathbf{P}^1 ;
- (2) C is a tree of \mathbf{P}^1 with at worst ordinary double points;
- (3) s_i is a smooth point of C for $i=1, \dots, q$;
- (4) each irreducible component of C has at least three special points, which are either the marked points, or the nodes where the component meets the other components.

Let $C=(C, s_1, \dots, s_q)$ and $C'=(C', s'_1, \dots, s'_q)$ be two q -pointed stable curves. We say that C and C' are isomorphic if there exists an isomorphism $\tau: C \rightarrow C'$ such that $\tau(s_i)=s'_i$ for all $i=1, \dots, q$.

We use the following notation:

$\overline{\mathcal{M}}_{0,q}$: the moduli space of q -pointed stable curves of genus 0, where $\overline{\mathcal{M}}_{0,q}$ is a smooth projective variety;

$\overline{\mathcal{U}}_{0,q} \xrightarrow{\varpi_q} \overline{\mathcal{M}}_{0,q}$: the universal curve, where $\overline{\mathcal{U}}_{0,q}$ is a smooth projective variety and ϖ_q is a proper flat morphism;

$\sigma_1, \dots, \sigma_q$: the universal sections of ϖ_q , where $\sigma_i(\overline{\mathcal{M}}_{0,q}) \cap \sigma_j(\overline{\mathcal{M}}_{0,q}) = \emptyset$ for $i \neq j$;

\mathcal{D}_q : the divisor on $\overline{\mathcal{U}}_{0,q}$ determined by $\sum_{i=1}^q \sigma_i(\overline{\mathcal{M}}_{0,q})$;

\mathcal{C}_x : the fiber $\varpi_q^{-1}(x)$ over $x \in \overline{\mathcal{M}}_{0,q}$;

$K_{\overline{\mathcal{U}}_{0,q}/\overline{\mathcal{M}}_{0,q}}$: the line bundle on $\overline{\mathcal{U}}_{0,q}$ associated to the relative dualizing sheaf of the morphism $\varpi_q: \overline{\mathcal{U}}_{0,q} \rightarrow \overline{\mathcal{M}}_{0,q}$;

K_q : the line bundle $K_{\overline{\mathcal{U}}_{0,q}/\overline{\mathcal{M}}_{0,q}}(\mathcal{D}_q)$;

ω_q : a fixed Kähler form on $\overline{\mathcal{U}}_{0,q}$;

η_q : a fixed Kähler form on $\overline{\mathcal{M}}_{0,q}$;

\varkappa_q : the curvature form of a fixed smooth Hermitian metric on K_q ;

(q) : the set $\{1, \dots, q\}$;

$\mathcal{I} = \mathcal{I}^q$: the set $\{(i, j, k, l) : 1 \leq i < j < k < l \leq q\}$;

$\mathcal{J} = \mathcal{J}^q$: the set $\{(i, j, k) : 1 \leq i < j < k \leq q\}$.

Remark 1.5.2. By definition, the family $\varpi_q: \overline{\mathcal{U}}_{0,q} \rightarrow \overline{\mathcal{M}}_{0,q}$ with the q distinct sections $\sigma_1, \dots, \sigma_q$ has the following two properties:

(1) For a point $x \in \overline{\mathcal{M}}_{0,q}$, the q -pointed fiber $\mathcal{C}_x = (\mathcal{C}_x, \sigma_1(x), \dots, \sigma_q(x))$ is a q -pointed stable curve.

(2) Let $C=(C, s_1, \dots, s_q)$ be an arbitrary q -pointed stable curve. Then there exists a unique point $x \in \overline{\mathcal{M}}_{0,q}$ such that C and \mathcal{C}_x are isomorphic as q -pointed stable curves.

A family of q -pointed stable curves is a proper flat morphism of schemes $p: X \rightarrow M$ with q sections τ_1, \dots, τ_q such that the geometric fiber $X_{\bar{m}}$ together with q marked points $\tau_1(\bar{m}), \dots, \tau_q(\bar{m})$ is a q -pointed stable curve for all $m \in M$. Then the assertion (2) of the above remark is generalized to the following moduli-space property: If $p: X \rightarrow M$ with sections τ_1, \dots, τ_q is a family of q -pointed stable curves, then there exists a unique morphism $M \rightarrow \overline{\mathcal{M}}_{0,q}$ such that p and τ_1, \dots, τ_q are induced from ϖ_q and $\sigma_1, \dots, \sigma_q$, respectively, by the base change to M (cf. [Ma, III.3.1 (a)]). Note that the complex structure of $\overline{\mathcal{M}}_{0,q}$ is uniquely determined by this property.

In this paper, the corresponding moduli-space property in the analytic category is only asserted over $\mathcal{M}_{0,q}$, which is a Zariski-open subset of $\overline{\mathcal{M}}_{0,q}$, via the argument below

(cf. (1.5.9)).

The space $\mathcal{M}_{0,q}$. Two q -tuples $s=(s_1, \dots, s_q)$ and $s'=(s'_1, \dots, s'_q)$ of points on \mathbf{P}^1 are said to be isomorphic if and only if there exists an isomorphism τ of \mathbf{P}^1 such that $s'_i=\tau(s_i)$ for all $i=1, \dots, q$. We denote by $\mathcal{M}_{0,q}$ the space of q -tuples of *distinct* points on \mathbf{P}^1 modulo isomorphism. Then $\mathcal{M}_{0,q}$ is isomorphic to

$$\mathcal{P}_q = \underbrace{(\mathbf{P}^1 \setminus \{0, 1, \infty\}) \times \dots \times (\mathbf{P}^1 \setminus \{0, 1, \infty\})}_{q-3 \text{ factors}} \setminus [\text{diagonals}].$$

Here note that an isomorphism of \mathbf{P}^1 is uniquely determined by its action on three distinct points. Then $\overline{\mathcal{M}}_{0,q}$ gives a compactification of $\mathcal{M}_{0,q}$ by the natural inclusion $\mathcal{M}_{0,q} \subset \overline{\mathcal{M}}_{0,q}$, because q -distinct points on \mathbf{P}^1 naturally determine a q -pointed stable curve whose underlying curve is non-singular. Put $\mathcal{L}_q = \overline{\mathcal{M}}_{0,q} \setminus \mathcal{M}_{0,q}$, which is a divisor on $\overline{\mathcal{M}}_{0,q}$ (cf. [Kn, 2.7]) and called the degenerate locus.

Remark 1.5.3. (1) We have $\mathcal{M}_{0,q} = \{x \in \overline{\mathcal{M}}_{0,q} : \mathcal{C}_x \simeq \mathbf{P}^1\}$.

(2) For $i=1, \dots, q$, we define a holomorphic map $p_i: \mathcal{P}_q \rightarrow \mathbf{P}^1$ as follows: For $i=1, \dots, q-3$, let p_i be the obvious map coming from the projection to the i th factor. Put $p_{q-2} \equiv 0, p_{q-1} \equiv 1$ and $p_q \equiv \infty$. Put

$$\bar{p}_i = (\text{id}_{\mathcal{P}_q}, p_i): \mathcal{P}_q \rightarrow \mathcal{P}_q \times \mathbf{P}^1.$$

Then \bar{p}_i is a section of the first projection $\mathcal{P}_q \times \mathbf{P}^1 \rightarrow \mathcal{P}_q$. Put $\mathcal{U}_{0,q} = \varpi_q^{-1}(\mathcal{M}_{0,q})$. For $i=1, \dots, q$, let $\sigma'_i: \mathcal{M}_{0,q} \rightarrow \mathcal{U}_{0,q}$ be the restriction of σ_i . Then there exist isomorphisms $\psi: \mathcal{M}_{0,q} \rightarrow \mathcal{P}_q$ and $\psi': \mathcal{U}_{0,q} \rightarrow \mathcal{P}_q \times \mathbf{P}^1$ which fit into the following commutative diagram of holomorphic maps:

$$\begin{array}{ccc} \mathcal{U}_{0,q} & \xrightarrow{\psi'} & \mathcal{P}_q \times \mathbf{P}^1 \\ \varpi_q \downarrow & & \downarrow \text{1st projection} \\ \mathcal{M}_{0,q} & \xrightarrow{\psi} & \mathcal{P}_q. \end{array} \tag{1.5.4}$$

Here $\psi' \circ \sigma'_i = \bar{p}_i \circ \psi$ for $i=1, \dots, q$. Hence the family $\varpi_q: \mathcal{U}_{0,q} \rightarrow \mathcal{M}_{0,q}$ with q sections σ'_i is isomorphic to the family $\mathcal{P}_q \times \mathbf{P}^1 \rightarrow \mathcal{P}_q$ with q sections \bar{p}_i .

The dual graph Γ_x . Let $x \in \overline{\mathcal{M}}_{0,q}$ be a point. Then $(\mathcal{C}_x, \sigma_1(x), \dots, \sigma_q(x))$ is a q -pointed stable curve. Let Γ_x be the associated graph, i.e., each element v of the set of vertices $\text{vert}(\Gamma_x)$ corresponds to the irreducible component C_v of \mathcal{C}_x , and two vertices v and v' are adjacent if and only if C_v and $C_{v'}$ meet transversally at the node $C_v \cap C_{v'} \in \mathcal{C}_x$. Then Γ_x is a tree.

The classification maps cl_a and $\text{cl}_{(f,a)}$. Let $\pi: \mathcal{F}' \rightarrow \mathcal{F}$ be a proper, surjective holomorphic map of Riemann surfaces \mathcal{F}' and \mathcal{F} . Let f be a meromorphic function on \mathcal{F}' ,

and let a_1, \dots, a_q be distinct meromorphic functions on \mathcal{F} . Then we have the classification maps

$$\mathcal{F} \xrightarrow{\text{cl}_a} \overline{\mathcal{M}}_{0,q} \quad \text{and} \quad \mathcal{F}' \xrightarrow{\text{cl}_{(f,a)}} \overline{\mathcal{U}}_{0,q},$$

which fit into the following commutative diagram of holomorphic maps:

$$\begin{array}{ccc} \mathcal{F}' & \xrightarrow{\text{cl}_{(f,a)}} & \overline{\mathcal{U}}_{0,q} \\ \pi \downarrow & & \downarrow \varpi_q \\ \mathcal{F} & \xrightarrow{\text{cl}_a} & \overline{\mathcal{M}}_{0,q}. \end{array} \quad (1.5.5)$$

These classification maps are defined as follows: Put

$$U = \{z \in \mathcal{F} : a_1(z), \dots, a_q(z) \text{ are all distinct}\} \subset \mathcal{F},$$

which is a dense open subset of \mathcal{F} . We first define the restrictions

$$\text{cl}_a|_U: U \longrightarrow \overline{\mathcal{M}}_{0,q} \quad \text{and} \quad \text{cl}_{(f,a)}|_{\pi^{-1}(U)}: \pi^{-1}(U) \longrightarrow \overline{\mathcal{U}}_{0,q}.$$

For $z \in U$, let $\text{cl}_a(z) \in \overline{\mathcal{M}}_{0,q}$ be the unique point such that two q -pointed stable curves

$$(\mathbf{P}^1, a_1(z), \dots, a_q(z)) \quad \text{and} \quad (\mathcal{C}_{\text{cl}_a(z)}, \sigma_1(\text{cl}_a(z)), \dots, \sigma_q(\text{cl}_a(z)))$$

are isomorphic (cf. Remark 1.5.2). Then there exists an isomorphism $\tau: \mathbf{P}^1 \rightarrow \mathcal{C}_{\text{cl}_a(z)}$ such that

$$\tau(a_i(z)) = \sigma_i(\text{cl}_a(z)) \quad \text{for all } i = 1, \dots, q. \quad (1.5.6)$$

For $y \in \pi^{-1}(z)$, put

$$\text{cl}_{(f,a)}(y) = \tau(f(y)) \in \mathcal{C}_{\text{cl}_a(z)}. \quad (1.5.7)$$

Next, we define the holomorphic maps

$$\text{cl}_a: \mathcal{F} \longrightarrow \overline{\mathcal{M}}_{0,q} \quad \text{and} \quad \text{cl}_{(f,a)}: \mathcal{F}' \longrightarrow \overline{\mathcal{U}}_{0,q}$$

by the unique holomorphic extensions of $\text{cl}_a|_U$ and $\text{cl}_{(f,a)}|_{\pi^{-1}(U)}$, respectively.

Remark 1.5.8. In view of (1.5.4), we may write

$$p_i \circ \psi \circ \text{cl}_a(z) = \frac{a_i(z) - a_{q-2}(z)}{a_i(z) - a_q(z)} \frac{a_{q-1}(z) - a_q(z)}{a_{q-1}(z) - a_{q-2}(z)}, \quad i = 1, \dots, q-3, \quad (1.5.9)$$

and

$$s \circ \psi' \circ \text{cl}_{(f,a)}(y) = \frac{f(y) - a_{q-2}(z)}{f(y) - a_q(z)} \frac{a_{q-1}(z) - a_q(z)}{a_{q-1}(z) - a_{q-2}(z)} \quad (1.5.10)$$

for $z \in U$ and $y \in \pi^{-1}(z)$. Here $s: \mathcal{P}_q \times \mathbf{P}^1 \rightarrow \mathbf{P}^1$ is the second projection. These equations (1.5.9) and (1.5.10) easily follow from the fact that the two $(q+1)$ -tuples of points on \mathbf{P}^1 ,

$$(f(y), a_1(z), \dots, a_q(z))$$

and

$$(s \circ \psi' \circ \text{cl}_{(f,a)}(y), p_1 \circ \psi \circ \text{cl}_a(z), \dots, p_{q-3} \circ \psi \circ \text{cl}_a(z), 0, 1, \infty),$$

are isomorphic for $z \in U$ and $y \in \pi^{-1}(z)$.

Contraction. Let $p: X \rightarrow M$ with sections $\tau_1, \dots, \tau_{q+1}$ be a family of $(q+1)$ -pointed stable curves. Then we say that a family of q -pointed stable curves $p': X' \rightarrow M$ with sections τ'_1, \dots, τ'_q is a contraction of $p: X \rightarrow M$ obtained by forgetting the section τ_{q+1} if there is a commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{c} & X' \\ p \downarrow & & \downarrow p' \\ M & \xlongequal{\quad} & M \end{array}$$

satisfying the following two conditions:

(1) $c \circ \tau_i = \tau'_i$ for all $i=1, \dots, q$;

(2) Consider the induced morphism $c_{\bar{m}}$ on the geometric fiber $X_{\bar{m}}$. Let $E \subset X_{\bar{m}}$ be the irreducible component such that $\tau_{q+1}(\bar{m}) \in E$. If the number of the special points on E , which are either the marked points or the nodes, is at least four, then $c_{\bar{m}}: X_{\bar{m}} \rightarrow X'_{\bar{m}}$ is an isomorphism. Otherwise, $c_{\bar{m}}$ contracts E to a point $x = c_{\bar{m}}(E) \in X'_{\bar{m}}$, and the restriction $c_{\bar{m}}: X_{\bar{m}} \setminus E \rightarrow X'_{\bar{m}} \setminus x$ is an isomorphism.

The above definition of the contraction is slightly different from that of [Kn], but we can easily check that they are equivalent (cf. [Ke, p. 547]). We have the following fundamental result: For any family of $(q+1)$ -pointed stable curves, there exists up to a unique isomorphism exactly one contraction (cf. [Kn, 2.1]).

Let S be a subset of (q) such that $q' = \text{card } S \geq 3$. Consider the universal family of q -pointed stable curves $\varpi_q: \overline{\mathcal{U}}_{0,q} \rightarrow \overline{\mathcal{M}}_{0,q}$ with the universal sections $\sigma_i, i \in (q)$. Then by forgetting all the sections except those marked in S , we get a family of q' -pointed stable curves $(\varpi_q)': \overline{\mathcal{U}}'_{0,q} \rightarrow \overline{\mathcal{M}}_{0,q}$ with sections $\sigma'_i, i \in S$, as a contraction. By the moduli-space property of $\overline{\mathcal{M}}_{0,q'}$, we have the morphism $u: \overline{\mathcal{M}}_{0,q} \rightarrow \overline{\mathcal{M}}_{0,q'}$ such that $(\varpi_q)'$ and σ'_i are induced from $\varpi_{q'}$ and $\hat{\sigma}_i$, respectively, by the base change to $\overline{\mathcal{M}}_{0,q}$. Here $\hat{\sigma}_i$ are the universal sections of $\varpi_{q'}: \overline{\mathcal{U}}_{0,q'} \rightarrow \overline{\mathcal{M}}_{0,q'}$, which are assumed to be labeled by the set S .

Therefore we have the commutative diagram of holomorphic maps,

$$\begin{array}{ccccc}
 \overline{\mathcal{U}}_{0,q} & \xrightarrow{c} & \overline{\mathcal{U}}'_{0,q} & \xrightarrow{u'} & \overline{\mathcal{U}}_{0,q'} \\
 \varpi_q \downarrow & & (\varpi_q)' \downarrow & & \downarrow \varpi_{q'} \\
 \overline{\mathcal{M}}_{0,q} & \xlongequal{\quad} & \overline{\mathcal{M}}_{0,q} & \xrightarrow{u} & \overline{\mathcal{M}}_{0,q'}
 \end{array} \tag{1.5.11}$$

where $u' \circ c \circ \sigma_i = \hat{\sigma}_i \circ u$ for all $i \in S$.

The contraction map φ_α . For $\alpha = (i, j, k) \in \mathcal{I}$, we denote by $\varphi_\alpha = \varphi_\alpha^{(q)}$ the morphism

$$\varphi_\alpha: \overline{\mathcal{U}}_{0,q} \rightarrow \mathbf{P}^1$$

uniquely characterized by the following conditions:

- (1) $\varphi_\alpha \circ \sigma_i \equiv 0$, $\varphi_\alpha \circ \sigma_j \equiv 1$ and $\varphi_\alpha \circ \sigma_k \equiv \infty$ (on $\overline{\mathcal{M}}_{0,q}$);
- (2) the restriction $\varphi_\alpha|_{\mathcal{C}_x}: \mathcal{C}_x \rightarrow \mathbf{P}^1$ is an isomorphism for all $x \in \overline{\mathcal{M}}_{0,q}$.

To obtain this φ_α , we observe the following. By forgetting all the markings except i, j and k , we get the following commutative diagram of holomorphic maps (cf. (1.5.11)):

$$\begin{array}{ccccc}
 \overline{\mathcal{U}}_{0,q} & \xrightarrow{c} & \overline{\mathcal{U}}'_{0,q} & \xrightarrow{u'} & \overline{\mathcal{U}}_{0,3} \\
 \varpi_q \downarrow & & (\varpi_q)' \downarrow & & \downarrow \varpi_3 \\
 \overline{\mathcal{M}}_{0,q} & \xlongequal{\quad} & \overline{\mathcal{M}}_{0,q} & \xrightarrow{u} & \overline{\mathcal{M}}_{0,3}
 \end{array}$$

Put $t = u' \circ c$. Note that $\overline{\mathcal{M}}_{0,3}$ is isomorphic to a point and $\overline{\mathcal{U}}_{0,3} \simeq \mathbf{P}^1$. We normalize the three universal sections of ϖ_3 as 0, 1 and ∞ . Then $t \circ \sigma_i \equiv 0$, $t \circ \sigma_j \equiv 1$ and $t \circ \sigma_k \equiv \infty$. Put $\varphi_\alpha = t$.

The contraction map ϕ_β . By forgetting the marking σ_q , $q \geq 4$, we have the morphism $u_q: \overline{\mathcal{M}}_{0,q} \rightarrow \overline{\mathcal{M}}_{0,q-1}$ (cf. (1.5.11)). There exists an isomorphism $\iota_q: \overline{\mathcal{M}}_{0,q} \rightarrow \overline{\mathcal{U}}_{0,q-1}$ which fits into the following commutative diagram of holomorphic maps (cf. [Ma, III.3.3 (b)]):

$$\begin{array}{ccc}
 \overline{\mathcal{M}}_{0,q} & \xrightarrow{\iota_q} & \overline{\mathcal{U}}_{0,q-1} \\
 u_q \downarrow & & \downarrow \varpi_{q-1} \\
 \overline{\mathcal{M}}_{0,q-1} & \xlongequal{\quad} & \overline{\mathcal{M}}_{0,q-1}
 \end{array} \tag{1.5.12}$$

For $l < q$, put $u_{q,l} = u_{l+1} \circ \dots \circ u_q: \overline{\mathcal{M}}_{0,q} \rightarrow \overline{\mathcal{M}}_{0,l}$. Put $u_{q,q} = \text{id}_{\overline{\mathcal{M}}_{0,q}}$. For $\beta = (i, j, k, l) \in \mathcal{I}$, we define $\phi_\beta: \overline{\mathcal{M}}_{0,q} \rightarrow \mathbf{P}^1$ by the composition of the morphisms

$$\overline{\mathcal{M}}_{0,q} \xrightarrow{u_{q,l}} \overline{\mathcal{M}}_{0,l} \xrightarrow{\iota_l} \overline{\mathcal{U}}_{0,l-1} \xrightarrow{\varphi_{(i,j,k)}^{(l-1)}} \mathbf{P}^1.$$

1.6. Outline of the proofs

The proof of Theorem 2 is similar to that of Theorem 1 (actually easier). So we only consider the case of Theorem 1. We first formulate the following theorem.

THEOREM 3. *Let Y, B and π be the same as in Theorem 1, and let $q \geq 3$ be an integer. Consider the following commutative diagram of holomorphic maps, where g is non-constant:*

$$\begin{array}{ccc}
 Y & \xrightarrow{g} & \overline{\mathcal{W}}_{0,q} \\
 \pi \downarrow & & \downarrow \varpi_q \\
 B & \xrightarrow{b} & \overline{\mathcal{M}}_{0,q}.
 \end{array} \tag{1.6.1}$$

Assume the non-degeneracy condition that $g(Y) \not\subset \text{supp } \mathcal{D}_q \cup \varpi_q^{-1}(\text{supp } \mathcal{L}_q)$. Then for all $\varepsilon > 0$, there exists a positive constant $C(\varepsilon) > 0$ such that

$$\begin{aligned}
 T(r, g, \varkappa_q) &\leq \bar{N}(r, g, \mathcal{D}_q) + N_{\text{ram } \pi_Y}(r) + \varepsilon T(r, g, \omega_q) \\
 &+ C(\varepsilon)(T(r, b, \eta_q) + N_{\text{ram } \pi_B}(r)) \quad ||.
 \end{aligned} \tag{1.6.2}$$

In §2, we derive Theorem 1 from Theorem 3, applying to the case that $g = \text{cl}_{(f,a)}$ and $b = \text{cl}_a$. Using the Schwarz inequality, we prove Theorem 3 from Theorem 4 below.

Definition 1.6.3. Let $q \geq 3$ be an integer.

(1) A *q-hol-quintet* is an object $(\mathcal{F}, \mathcal{R}, \pi, g, b)$ where \mathcal{F} and \mathcal{R} are Riemann surfaces with a proper, surjective holomorphic map $\pi: \mathcal{F} \rightarrow \mathcal{R}$, and g and b are holomorphic maps which fit into the commutative diagram

$$\begin{array}{ccc}
 \mathcal{F} & \xrightarrow{g} & \overline{\mathcal{W}}_{0,q} \\
 \pi \downarrow & & \downarrow \varpi_q \\
 \mathcal{R} & \xrightarrow{b} & \overline{\mathcal{M}}_{0,q}.
 \end{array} \tag{1.6.4}$$

We say that a *q-hol-quintet* $(\mathcal{F}, \mathcal{R}, \pi, g, b)$ is *non-degenerate* if $b(\mathcal{R}) \not\subset \text{supp } \mathcal{L}_q$ and if the meromorphic functions $\varphi_\alpha \circ g$ on \mathcal{F} are non-constant for all $\alpha \in \mathcal{I}$.

(2) A *specified q-hol-quintet* is an object $(\mathcal{F}, \mathcal{R}, \pi, g, b, F, R)$ where $(\mathcal{F}, \mathcal{R}, \pi, g, b)$ is a *q-hol-quintet*, $R \subset \mathcal{R}$ is a finite domain and $F = \pi^{-1}(R)$. We say that a specified *q-hol-quintet* is *non-degenerate* if the *q-hol-quintet* $(\mathcal{F}, \mathcal{R}, \pi, g, b)$ is non-degenerate.

THEOREM 4. *Let $q \geq 3$ be an integer. For all $\varepsilon > 0$, there exists a positive constant $C(q, \varepsilon) > 0$ with the following property: Let $(\mathcal{F}, \mathcal{R}, \pi, g, b, F, R)$ be a non-degenerate specified *q-hol-quintet*. Then we have*

$$\begin{aligned}
 A(g, F, \varkappa_q) &\leq \bar{n}(g, \mathcal{D}_q, F) + \text{disc}(\pi, R) + \varepsilon A(g, F, \omega_q) \\
 &+ C(q, \varepsilon)(\text{deg } \pi)(A(b, R, \eta_q) + \bar{n}(b, \mathcal{L}_q, R) + \varrho^+(R) + l(g, \partial F, \omega_q)).
 \end{aligned} \tag{1.6.5}$$

The most important part of this paper is the proof of Theorem 4. The proof naturally divides into the following three steps.

Step 1. We prove a local version of our theorem, which roughly reads as follows: For each point $x \in \overline{\mathcal{M}}_{0,q}$, there exists an open neighborhood V_x of x such that if a non-degenerate specified q -hol-quintet satisfies the condition $b(R) \subset V_x$, then our theorem is valid. For the precise statement, see Lemma 11. To prove this, we use a lemma from [Y2], which is an application of Ahlfors's theory (cf. Lemma 8). For each vertex $v \in \Gamma_x$, we attach a contraction map $\varphi_{\langle v \rangle}: \overline{\mathcal{U}}_{0,q} \rightarrow \mathbf{P}^1$, $\langle v \rangle \in \mathcal{J}$. This contraction map $\varphi_{\langle v \rangle}$ has the properties that the restriction to the component C_v is an isomorphism and that the restrictions to the other components $C_{v'}$ are constant maps. Applying Lemma 8 to $\psi = \varphi_{\langle v \rangle} \circ g$ and $\zeta = \varphi_{\langle v' \rangle} \circ g$, where v and v' are adjacent vertices, we obtain some sort of "difference" of the usual second main theorem of Ahlfors. Adding these "differences" over all the edges of Γ_x , we obtain (a modification of) the usual second main theorem of Ahlfors. Applying Rouché's theorem (Lemma 9), we get the local version of our theorem. This method is similar to that of [Y2]. The major differences are that instead of the tree constructed in [Y2, §8], we use the tree Γ_x , and instead of the combinatorial lemma [Y2, Lemma 4], we use a geometric lemma (cf. Lemma 10).

Step 2. By a finite union of curves γ , we divide \mathbf{P}^1 into a finite number of Jordan domains D_k , $1 \leq k \leq K$. This division of \mathbf{P}^1 gives the division of $(\mathbf{P}^1)^{\mathcal{J}}$ in the form of the open subsets

$$\prod_{i \in \mathcal{J}} D_{k_i}, \quad 1 \leq k_i \leq K. \quad (1.6.6)$$

Put $\Phi = (\phi_i)_{i \in \mathcal{J}}: \overline{\mathcal{M}}_{0,q} \rightarrow (\mathbf{P}^1)^{\mathcal{J}}$. We consider connected components R' of the pullback of the open subsets (1.6.6) by the composition of the morphisms

$$R \xrightarrow{b} \overline{\mathcal{M}}_{0,q} \xrightarrow{\Phi} (\mathbf{P}^1)^{\mathcal{J}}.$$

Then R is divided into a finite number of the finite domains R' . We assume that the Jordan domains D_k are very small. Then using the facts that $\overline{\mathcal{M}}_{0,q}$ is compact and that Φ is an injection (cf. Lemma 12), we conclude the following: For every R' , there exists a point $x \in \overline{\mathcal{M}}_{0,q}$ such that $b(R') \subset V_x$.

Step 3. We apply the local version of the theorem for each finite domain R' and add over all these finite domains to conclude our theorem. Here we need to estimate extra error terms coming from

- (1) the sum of the lengths $l(g, \partial' \pi^{-1}(R'), \omega_q)$ over all R' , where $\partial' \pi^{-1}(R')$ is the part of the boundary $\partial \pi^{-1}(R')$ which lies in the interior of F ;
- (2) the sum of $\varrho^+(R')$ over all R' .

See Lemma 13 for these estimates. Here we only point out the idea of the method of the first estimate. Take a slightly smaller Jordan domain $D'_k \subset D_k$ for each k . Then we obtain slightly smaller finite domains $R'' \subset R'$ in the same manner for R' but from the Jordan domains D'_k . We use the length–area principle to find a finite domain \tilde{R} with $R'' \subset \tilde{R} \subset R'$ such that the length $l(g, \partial' \pi^{-1}(\tilde{R}), \omega_q)$ is small enough provided that the area $A(g, \pi^{-1}(R''), \omega_q)$ is sufficiently large. We replace $\{R'\}$ by $\{\tilde{R}\}$ to conclude the estimate.

1.7. Remarks

(1) A part of Theorem 3 can be generalized as follows. For a smooth algebraic variety X , we denote by K_X the canonical bundle of X . Given a morphism $p: X \rightarrow M$ between smooth algebraic varieties, we define the relative canonical bundle $K_{X/M}$ to be $K_X - p^*K_M$. The relative canonical bundle $K_{X/M}$ is a line bundle on X . Note that the line bundle $K_{\overline{\mathcal{W}}_{0,q}/\overline{\mathcal{M}}_{0,q}}$ defined in §1.5 is equal to the relative canonical bundle of $\varpi_q: \overline{\mathcal{W}}_{0,q} \rightarrow \overline{\mathcal{M}}_{0,q}$ in this sense, so our notations do not contradict.

COROLLARY 3. *Let X and M be smooth projective varieties over \mathbf{C} . Let $p: X \rightarrow M$ be a surjective morphism where the relative dimension of X over M is equal to one. Let $D \subset X$ be a reduced divisor on X . Let L and E be ample line bundles on X and M , respectively. Let Y, B and π be the same as in Theorem 1. Consider the following commutative diagram of holomorphic maps, where g is non-constant:*

$$\begin{array}{ccc} Y & \xrightarrow{g} & X \\ \pi \downarrow & & \downarrow p \\ B & \xrightarrow{b} & M. \end{array}$$

Assume that the image $b(B)$ is Zariski dense in M and that $g(Y) \not\subset \text{supp } D$. Then for all $\varepsilon > 0$, there exists a positive constant $C(\varepsilon) > 0$ such that

$$\begin{aligned} T(r, g, K_{X/M}(D)) &\leq \bar{N}(r, g, D) + N_{\text{ram } \pi_Y}(r) + \varepsilon T(r, g, L) \\ &\quad + C(\varepsilon)(T(r, b, E) + N_{\text{ram } \pi_B}(r)) \|. \end{aligned} \tag{1.7.1}$$

We shall prove this corollary as part of the derivation of Corollary 2.

(2) Consider the case $B = \mathbf{C}$ and $\pi_B = \text{id}_{\mathbf{C}}$ in the corollary above. A consequence of the general second fundamental conjecture is that the inequality

$$T(r, g, K_X(D)) \leq \bar{N}(r, g, D) + N_{\text{ram } \pi_Y}(r) + \varepsilon T(r, g, L) \tag{1.7.2}$$

holds for all $\varepsilon > 0$ and for all suitably non-degenerate g . Since we have

$$T(r, g, K_X(D)) = T(r, g, K_{X/M}(D)) + T(r, b, K_M) + O(1),$$

the inequality (1.7.1) (and hence (1.6.2)) is a weak form of (1.7.2).

The paper is organized as follows. In §2, we prepare some lemmas and derive Theorem 1 from Theorem 3, and Theorem 3 from Theorem 4. The proof of Theorem 4 begins in §3. This section is a preliminary for the proof including some lemmas from [Y2] and a review of Ahlfors's theory, which will be used in the proof. In §§4 and 5, we prove Lemmas 11 and 13, respectively. The proof of Theorem 4 ends at §6. In §7, we prove Corollaries 2 and 3 together with some generalization of Theorem 1. In §8, we prove Theorem 2 from Theorem 4. This proof is similar to that of Theorem 1. In §9, we introduce some notations from [V1] and [V3], and prove the height inequality for curves over function fields (Theorem 5).

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This paper is an expanded and largely rewritten version of [Y1].

2. Derivations of Theorem 1 from Theorem 3, and Theorem 3 from Theorem 4

2.1. Basic estimates in Nevanlinna theory

Let Y be a Riemann surface with a proper, surjective holomorphic map $\pi: Y \rightarrow \mathbf{C}$. Let X be a smooth projective variety, and let $g: Y \rightarrow X$ be a holomorphic map.

2.1.1. *The Nevanlinna inequality.* For an effective divisor $Z \subset X$ with $g(Y) \not\subset \text{supp } Z$, we have the Nevanlinna inequality⁽¹⁾

$$N(r, g, Z) \leq T(r, g, [Z]) + O(1), \tag{2.1.1}$$

where $[Z]$ is the associated line bundle for the divisor Z . This estimate follows directly from the first main theorem in Nevanlinna theory. When $Y = \mathbf{C}$ and $\pi = \text{id}_{\mathbf{C}}$, the first main theorem and the Nevanlinna inequality (2.1.1) is contained in [NoO, 5.2.18]. The first main theorem for a general Y and an ample divisor Z is contained in [No1, (3.5)], from which the general case easily follows because an arbitrary divisor Z can be written as the difference of two ample divisors. See also [LC, IV 2.3].

⁽¹⁾ In this paper, we use big and little “oh” notation for asymptotic statements as $r \rightarrow \infty$.

2.1.2. *Functorial properties.* Let M be a smooth projective variety, and let $p: X \rightarrow M$ be a morphism. Let L_1 and L_2 be line bundles on X , and let E be a line bundle on M . In this paper, we often use the following functorial properties of the characteristic function:

$$\begin{aligned} T(r, g, L_1 + L_2) &= T(r, g, L_1) + T(r, g, L_2) + O(1), \\ T(r, g, p^*E) &= T(r, p \circ g, E) + O(1) \end{aligned}$$

for every holomorphic map $g: Y \rightarrow X$. We can easily check these properties by the definition.

2.1.3. *Growth estimates of the characteristic function.* The lemmas in this subsection may be found somewhere in the literature, but in lack of precise references, we provide proofs. (For the case $Y = \mathbf{C}$ and $\pi = \text{id}_{\mathbf{C}}$, see also [NoO, 5.2.29, 6.1.5].)

LEMMA 2. *Let X, M, p, Y and g be as above. Let L be a line bundle on X , and let E be an ample line bundle on M . Assume that $\dim X = \dim p(X)$ and that $g(Y)$ is Zariski dense in X . Then there is a positive constant C , which only depends on X, M, p, L and E , such that*

$$|T(r, g, L)| \leq CT(r, p \circ g, E) + O(1).$$

Proof. There is an ample line bundle L' on X such that both $L' - L$ and $L' + L$ are ample. Since the characteristic function of an ample line bundle is bounded from below, we have

$$-T(r, g, L') \leq T(r, g, L) + O(1) \leq T(r, g, L') + O(1),$$

which yields $|T(r, g, L)| \leq T(r, g, L') + O(1)$. Therefore we have reduced our proof to the case that L is ample.

Observe that the line bundle p^*E is big. Hence, by Kodaira's lemma (cf. [KM, 2.60]), we may take positive integers k and m such that $H^0(X, m(p^*E) - kL) \neq 0$. Let F be a divisor on X which corresponds to this non-zero global section. Since $N(r, g, F) \geq 0$ for $r > 1$, the estimate (2.1.1) yields

$$0 \leq T(r, g, [F]) + O(1) = mT(r, g, p^*E) - kT(r, g, L) + O(1),$$

and hence $T(r, g, L) \leq (m/k)T(r, g, p^*E) + O(1)$. Using the functorial property of the characteristic function, we conclude the lemma. (Put $C = m/k$.) \square

LEMMA 3. *Let X be a smooth projective variety, and let $g: Y \rightarrow X$ be a holomorphic map. For a function $\psi: \mathbf{R}_{>0} \rightarrow \mathbf{R}$ with $\psi(r) \geq 1$, the following four conditions are equivalent:*

- (1) *There exists an ample line bundle L on X such that $T(r, g, L) = O(\psi(r))$.*

(2) For all line bundles L on X , we have $T(r, g, L) = O(\psi(r))$.

(3) For all smooth $(1, 1)$ -forms Ω on X , we have $T(r, g, \Omega) = O(\psi(r))$.

(4) Let $W \subset X$ be the Zariski closure of $g(Y)$, and let $\mathbf{C}(W)$ be the rational function field of W . Then we have $T(r, v \circ g) = O(\psi(r))$ for all $v \in \mathbf{C}(W)$, where $v \circ g$ is a meromorphic function on Y .

Remark 2.1.2. If g satisfies one of the above equivalent conditions, we say that the order of the growth of g is bounded by $\psi(r)$.

Proof. First, we shall prove the equivalence of the conditions (1), (2) and (3). Observe that the implications (3) \Rightarrow (2) and (2) \Rightarrow (1) are trivial from the definition. For (1) \Rightarrow (3), let Ω' be the curvature form of a Hermitian metric on L . Since L is ample, we may assume that Ω' is positive. For a smooth $(1, 1)$ -form Ω , there is a positive constant C such that $-C\Omega' < \Omega < C\Omega'$ because X is compact. Hence we have

$$|T(r, g, \Omega)| \leq CT(r, g, \Omega') = CT(r, g, L) + O(1) = O(\psi(r)),$$

where we may include the term $O(1)$ in $O(\psi(r))$ because $\psi(r) \geq 1$. Hence we conclude that the conditions (1), (2) and (3) are equivalent.

Next we shall prove the equivalence of (2) and (4). Since our assertion is trivial for a constant map g , we only consider the case that g is non-constant.

(4) \Rightarrow (2). Let $v_1, \dots, v_d \in \mathbf{C}(W)$ be a transcendence basis of the field extension $\mathbf{C}(W)/\mathbf{C}$, where $d = \dim W$. Put

$$P = \underbrace{\mathbf{P}^1 \times \dots \times \mathbf{P}^1}_{d \text{ factors}} \quad \text{and} \quad v = (v_1, \dots, v_d): W \dashrightarrow P.$$

By Hironaka's theorem, we may take a blowing-up $\widetilde{W} \rightarrow W$, where \widetilde{W} is smooth, such that the rational map $\tilde{v}: \widetilde{W} \dashrightarrow P$ induced from v is regular at every point of \widetilde{W} . Then \tilde{v} is a generically finite map. Let $\tilde{g}: Y \rightarrow \widetilde{W}$ be the holomorphic map such that $u \circ \tilde{g} = g$, where $u: \widetilde{W} \rightarrow X$ is the composition of the morphism $\widetilde{W} \rightarrow W$ and the closed immersion $W \rightarrow X$. We denote by \mathcal{L} the hyperplane section bundle on \mathbf{P}^1 , which is the unique line bundle of degree one. Put $E = \text{pr}_1^* \mathcal{L} + \dots + \text{pr}_d^* \mathcal{L}$, where $\text{pr}_i: P \rightarrow \mathbf{P}^1$ is the i th projection for $i = 1, \dots, d$. Then E is an ample line bundle on P . Since \tilde{v} is generically finite, we may apply Lemma 2 to get

$$T(r, g, L) = T(r, \tilde{g}, u^*L) + O(1) = O(T(r, \tilde{v} \circ \tilde{g}, E))$$

for all line bundles L on X . Here we note that $T(r, \tilde{v} \circ \tilde{g}, E) \rightarrow \infty$ as $r \rightarrow \infty$, because $\tilde{v} \circ \tilde{g}$ is non-constant. Observe that we have the estimate

$$T(r, \tilde{v} \circ \tilde{g}, E) = T(r, v \circ g, E) + O(1) = \sum_{i=1}^d T(r, v_i \circ g, \mathcal{L}) + O(1) = \sum_{i=1}^d T(r, v_i \circ g) + O(1),$$

because $\omega_{\mathbf{P}^1}$ is the curvature form of the Fubini–Study metric on \mathcal{L} . Hence by (4), we get

$$T(r, g, L) = O(\psi(r))$$

for all line bundles L on X . This proves (2).

(2) \Rightarrow (4). Let $v \in \mathbf{C}(W)$. By Hironaka’s theorem, we may take a blowing-up $\widetilde{W} \rightarrow W$, where \widetilde{W} is smooth, such that the induced rational map $\tilde{v}: \widetilde{W} \dashrightarrow \mathbf{P}^1$ is regular at every point of \widetilde{W} . Let $\tilde{g}: Y \rightarrow \widetilde{W}$ be the holomorphic map such that $u \circ \tilde{g} = g$, where $u: \widetilde{W} \rightarrow X$ is the composition of the morphism $\widetilde{W} \rightarrow W$ and the closed immersion $W \rightarrow X$. Let L be an ample line bundle on X . Apply Lemma 2 to get

$$T(r, \tilde{g}, \tilde{v}^* \mathcal{L}) = O(T(r, u \circ \tilde{g}, L)),$$

where \mathcal{L} is the hyperplane section bundle on \mathbf{P}^1 . Since we have

$$T(r, \tilde{g}, \tilde{v}^* \mathcal{L}) = T(r, \tilde{v} \circ \tilde{g}, \mathcal{L}) + O(1) = T(r, v \circ g) + O(1),$$

we obtain $T(r, v \circ g) = O(T(r, u \circ \tilde{g}, L))$. By (2), we have

$$T(r, u \circ \tilde{g}, L) = T(r, g, L) + O(1) = O(\psi(r)),$$

and hence $T(r, v \circ g) = O(\psi(r))$. This proves (4) and concludes the proof of the lemma. \square

LEMMA 4. *Let X and M be smooth projective varieties, let $p: X \rightarrow M$ be a morphism and let $g: Y \rightarrow X$ be a holomorphic map. Let $D \subset X$ be a divisor such that $p(\text{supp } D) \neq M$. Assume that $p \circ g(Y) \not\subset p(\text{supp } D)$ and that the order of the growth of $p \circ g$ is bounded by $\psi(r)$ for some $\psi: \mathbf{R}_{>0} \rightarrow \mathbf{R}_{\geq 1}$. Then we have*

$$T(r, g, [D]) = O(\psi(r)).$$

Proof. There is an effective divisor Z on M such that $p(\text{supp } D) \subset \text{supp } Z$ and $p \circ g(Y) \not\subset \text{supp } Z$. We may take a positive integer m such that the divisors $m(p^*Z) - D$ and $m(p^*Z) + D$ are effective. By (2.1.1), we have

$$0 \leq T(r, g, [m(p^*Z) - D]) + O(1) = mT(r, p \circ g, [Z]) - T(r, g, [D]) + O(1)$$

and

$$0 \leq T(r, g, [m(p^*Z) + D]) + O(1) = mT(r, p \circ g, [Z]) + T(r, g, [D]) + O(1).$$

Hence we get $T(r, g, [D]) = O(\psi(r))$ by Lemma 3. This proves our lemma. \square

2.2. An algebraic lemma

LEMMA 5. *There exist a line bundle E on $\overline{\mathcal{M}}_{0,q}$ and a divisor Ξ on $\overline{\mathcal{U}}_{0,q}$ such that $\varpi_q(\text{supp } \Xi) \subset \text{supp } \mathcal{L}_q$ and*

$$(q-2)\varphi_{(1,2,3)}^*\mathcal{L} = K_q + \varpi_q^*E + [\Xi].$$

Proof. Let $x \in \mathcal{M}_{0,q}$, and observe that the restriction $K_q|_{\mathcal{C}_x}$ is isomorphic to $K_{\mathcal{C}_x}(\sum_{i=1}^q \sigma_i(x))$, where $K_{\mathcal{C}_x}$ is the canonical line bundle on $\mathcal{C}_x \simeq \mathbf{P}^1$. Hence we have $\text{deg } K_q|_{\mathcal{C}_x} = q-2$ because $\text{deg } K_{\mathcal{C}_x} = -2$. Since the restriction $\varphi_{(1,2,3)}|_{\mathcal{C}_x}: \mathcal{C}_x \rightarrow \mathbf{P}^1$ is an isomorphism, we conclude that $(q-2)(\varphi_{(1,2,3)}^*\mathcal{L})|_{\mathcal{C}_x}$ and $K_q|_{\mathcal{C}_x}$ are isomorphic. Here we note that $\text{deg}(\varphi_{(1,2,3)}^*\mathcal{L})|_{\mathcal{C}_x} = 1$. Put $L = (q-2)\varphi_{(1,2,3)}^*\mathcal{L} - K_q$. Then we deduce that the restriction $L|_{\mathcal{C}_x}$ is the trivial line bundle for every $x \in \mathcal{M}_{0,q}$.

Since $\varpi_q^{-1}(\mathcal{M}_{0,q}) \rightarrow \mathcal{M}_{0,q}$ is a \mathbf{P}^1 -bundle, we conclude that there exists a line bundle E_0 on $\mathcal{M}_{0,q}$ such that the restriction $L|_{\varpi_q^{-1}(\mathcal{M}_{0,q})}$ is isomorphic to $\varpi_q^*E_0$ ([Har, Chapter II, Exercise 7.9]). Let E be an extension of E_0 to $\overline{\mathcal{M}}_{0,q}$. Put $L' = L - \varpi_q^*E$. Then $L'|_{\varpi_q^{-1}(\mathcal{M}_{0,q})}$ is the trivial line bundle on $\varpi_q^{-1}(\mathcal{M}_{0,q})$. Hence there exists a divisor Ξ on $\overline{\mathcal{U}}_{0,q}$ such that $\varpi_q(\text{supp } \Xi) \subset \text{supp } \mathcal{L}_q$ and $L' = [\Xi]$. This proves our lemma. \square

2.3. Theorem 3 implies Theorem 1

We only consider the case $q \geq 3$ because Theorem 1 is trivial for $q < 3$. Let f, a_1, \dots, a_q be the functions in Theorem 1 with the conditions that f is non-constant, that the functions a_i are distinct and that $f \neq a_i \circ \pi$ for $i=1, \dots, q$.

We consider the classification maps cl_a and $\text{cl}_{(f,a)}$.

First we estimate the characteristic functions $T(r, \text{cl}_a, \eta_q)$ and $T(r, \text{cl}_{(f,a)}, \omega_q)$. Put

$$\psi(r) = \max \left\{ 1, \sum_{i=1}^q T(r, a_i) \right\},$$

which satisfies $\psi(r) \geq 1$. Then we have

$$\psi(r) = \sum_{i=1}^q T(r, a_i) + o(T(r, f)). \tag{2.3.1}$$

Let \mathfrak{K}_B be the field of all meromorphic functions on B . Let $W \subset \overline{\mathcal{M}}_{0,q}$ be the Zariski closure of the image $\text{cl}_a(B)$, and let $\mathbf{C}(W)$ be the rational function field of W . Then cl_a defines the natural injection $\iota: \mathbf{C}(W) \rightarrow \mathfrak{K}_B$ by the pullback of rational functions on W . Let $\mathbf{C}(a_1, \dots, a_q) \subset \mathfrak{K}_B$ be the subfield generated by the meromorphic functions a_1, \dots, a_q .

Then by the definition of cl_a , we have $\iota(\mathbf{C}(W)) \subset \mathbf{C}(a_1, \dots, a_q)$ (cf. (1.5.9)). Hence the order of the growth of cl_a is bounded by $\psi(r)$. Apply Lemma 3 to get

$$T(r, \text{cl}_a, \eta_q) = O(\psi(r)). \quad (2.3.2)$$

Similarly, using the field \mathfrak{K}_Y of meromorphic functions on Y , we observe that the order of the growth of $\text{cl}_{(f,a)}$ is bounded by $T(r, f) + \psi(r)$ (cf. (1.5.10)). Hence we get the estimate

$$T(r, \text{cl}_{(f,a)}, \omega_q) = O(T(r, f) + \psi(r)),$$

i.e.,

$$T(r, \text{cl}_{(f,a)}, \omega_q) \leq QT(r, f) + O(\psi(r)) \quad (2.3.3)$$

with a positive constant Q , which may depend on f, a_i and (fixed) ω_q .

Now we take an arbitrary positive constant ε , and apply Theorem 3 to the case $g = \text{cl}_{(f,a)}$, $b = \text{cl}_a$ and ε . The non-degeneracy condition of Theorem 3 easily follows from the assumptions that a_i are distinct and that $f \neq a_i \circ \pi$ for $i = 1, \dots, q$. Using (2.3.2) and (2.3.3), and replacing ε with ε/Q , we get

$$\begin{aligned} T(r, \text{cl}_{(f,a)}, \varkappa_q) &\leq \bar{N}(r, \text{cl}_{(f,a)}, \mathcal{D}_q) + N_{\text{ram } \pi_Y}(r) + \varepsilon T(r, f) \\ &\quad + O_\varepsilon(\psi(r) + N_{\text{ram } \pi_B}(r)), \end{aligned} \quad (2.3.4)$$

where we use the notation O_ε in place of O so as to better indicate that the constant used to define the symbol O depends on ε . To complete the proof, we need to estimate the terms of (2.3.4).

CLAIM. *The following inequalities hold:*

$$\bar{N}(r, \text{cl}_{(f,a)}, \mathcal{D}_q) \leq \sum_{i=1}^q \bar{N}(r, a_i \circ \pi, f) + O(\psi(r)), \quad (2.3.5)$$

$$(q-2)T(r, f) = T(r, \text{cl}_{(f,a)}, \varkappa_q) + O(\psi(r)). \quad (2.3.6)$$

Proof. We first prove (2.3.5). Put

$$U = \{z \in B : a_1(z), \dots, a_q(z) \text{ are all distinct}\}.$$

Then by the definition of the classification map, we have $\text{cl}_a(U) \subset \mathcal{M}_{0,q}$. For $z \in U$ and $y \in \pi^{-1}(z)$, we have $\text{cl}_{(f,a)}(y) \in \mathcal{D}_q$ if and only if $f(y) = a_i(z)$ for some $i \in (q)$ (cf. (1.5.6) and (1.5.7)). Hence we have

$$\{y \in Y : \text{cl}_{(f,a)}(y) \in \mathcal{D}_q\} \subset \{y \in Y : f(y) = a_i \circ \pi(y) \text{ for some } i \in (q)\} \cup \pi^{-1}(B \setminus U).$$

This implies that

$$\bar{n}(\text{cl}_{(f,a)}, \mathcal{D}_q, Y(r)) \leq \sum_{i=1}^q \bar{n}(a_i \circ \pi, f, Y(r)) + (\deg \pi) \sum_{i=1}^q \sum_{\substack{j=1 \\ j \neq i}}^q \bar{n}(a_i, a_j, B(r))$$

and

$$\bar{N}(r, \text{cl}_{(f,a)}, \mathcal{D}_q) \leq \sum_{i=1}^q \bar{N}(r, a_i \circ \pi, f) + \sum_{i=1}^q \sum_{\substack{j=1 \\ j \neq i}}^q \bar{N}(r, a_i, a_j), \quad r > 1.$$

Since we have

$$\sum_{i=1}^q \sum_{\substack{j=1 \\ j \neq i}}^q \bar{N}(r, a_i, a_j) = O(\psi(r)),$$

we get (2.3.5).

Next we prove (2.3.6). Since $\omega_{\mathbf{P}^1}$ is the curvature form of the Fubini–Study metric on \mathcal{L} , Lemma 5 implies the equality

$$(q-2)T(r, \varphi_{(1,2,3)} \circ \text{cl}_{(f,a)}) = T(r, \text{cl}_{(f,a)}, \kappa_q) + T(r, \text{cl}_a, E) + T(r, \text{cl}_{(f,a)}, [\Xi]) + O(1). \tag{2.3.7}$$

Here we used the functorial property of T , namely

$$\begin{aligned} T(r, \text{cl}_{(f,a)}, \varphi_{(1,2,3)}^* \mathcal{L}) &= T(r, \varphi_{(1,2,3)} \circ \text{cl}_{(f,a)}) + O(1), \\ T(r, \text{cl}_{(f,a)}, \varpi_q^* E) &= T(r, \text{cl}_a, E) + O(1). \end{aligned}$$

Since for $z \in \pi^{-1}(U)$, the two 4-tuples of points on \mathbf{P}^1 ,

$$(f(z), a_1 \circ \pi(z), a_2 \circ \pi(z), a_3 \circ \pi(z)) \quad \text{and} \quad (\varphi_{(1,2,3)} \circ \text{cl}_{(f,a)}(z), 0, 1, \infty),$$

are isomorphic (cf. (1.5.6) and (1.5.7)), we have

$$\varphi_{(1,2,3)} \circ \text{cl}_{(f,a)}(z) = \frac{f(z) - a_1 \circ \pi(z)}{f(z) - a_3 \circ \pi(z)} \frac{a_2 \circ \pi(z) - a_3 \circ \pi(z)}{a_2 \circ \pi(z) - a_1 \circ \pi(z)}.$$

Hence we get

$$T(r, \varphi_{(1,2,3)} \circ \text{cl}_{(f,a)}) = T(r, f) + O(\psi(r)). \tag{2.3.8}$$

By $\varpi_q(\text{supp } \Xi) \subset \text{supp } \mathcal{L}_q$, we may apply Lemma 4 to get

$$T(r, \text{cl}_{(f,a)}, [\Xi]) = O(\psi(r)). \tag{2.3.9}$$

Using (2.3.7), (2.3.8), (2.3.9) and the estimate

$$T(r, \text{cl}_a, E) = O(\psi(r)) \tag{2.3.10}$$

(cf. Lemma 3), we get our inequality (2.3.6) and conclude the proof of our claim. \square

Using (2.3.1), (2.3.4) and the above claim, we get our Theorem 1.

2.4. Theorem 4 implies Theorem 3

Let Y, B, π, g and b be the objects in Theorem 3 with the conditions that g is non-constant and that $g(Y) \not\subset \text{supp } \mathcal{D}_q \cup \varpi_q^{-1}(\text{supp } \mathcal{Z}_q)$. Put $\psi(r) = \max\{1, T(r, b, \eta_q)\}$. Then $\psi(r) \geq 1$, and

$$\psi(r) = T(r, b, \eta_q) + o(T(r, g, \omega_q)). \tag{2.4.1}$$

Observe that the order of the growth of b is bounded by $\psi(r)$ since η_q is positive.

First, we consider the case that $\varphi_\alpha \circ g$ is constant for some $\alpha \in \mathcal{J}$. By Lemma 5, which is obviously valid when $(1, 2, 3) \in \mathcal{J}$ is replaced by α , we can prove

$$(q-2)T(r, \varphi_\alpha \circ g) = T(r, g, \varkappa_q) + T(r, b, E') + T(r, g, [\Xi']) + O(1), \tag{2.4.2}$$

where E' is a line bundle on $\overline{\mathcal{M}}_{0,q}$ and Ξ' is a divisor on $\overline{\mathcal{U}}_{0,q}$ with $\varpi_q(\text{supp } \Xi') \subset \mathcal{Z}_q$. By Lemmas 3 and 4, we have

$$-T(r, b, E') = O(\psi(r)) \quad \text{and} \quad -T(r, g, [\Xi']) = O(\psi(r)), \tag{2.4.3}$$

respectively, where we note that $g(Y) \not\subset \varpi_q^{-1}(\mathcal{Z}_q)$. Using (2.4.2), (2.4.3) and the assumption that $\varphi_\alpha \circ g$ is constant, we conclude that $T(r, g, \varkappa_q) = O(\psi(r))$. This proves Theorem 3 in our case, because all terms on the right-hand side of (1.6.2) are non-negative for $r > 1$.

Next we consider the case that $\varphi_\alpha \circ g$ is non-constant for every $\alpha \in \mathcal{J}$. For $r > 0$, decompose $B(r)$ into connected components $B_1(r), \dots, B_{u_r}(r)$ and put

$$\lambda_i = (Y, B, \pi, g, b, Y_i(r), B_i(r)),$$

where $Y_i(r) = \pi^{-1}(B_i(r))$. Then λ_i is a non-degenerate specified q -hol-quintet for $i = 1, \dots, u_r$. We apply Theorem 4 to each λ_i and add over $i = 1, \dots, u_r$ to obtain

$$\begin{aligned} A(g, Y(r), \varkappa_q) &\leq \bar{n}(g, \mathcal{D}_q, Y(r)) + \text{disc}(\pi, B(r)) + \varepsilon A(g, Y(r), \omega_q) \\ &\quad + C(q, \varepsilon)(\deg \pi) \left(A(b, B(r), \eta_q) + \bar{n}(b, \mathcal{Z}_q, B(r)) \right. \\ &\quad \left. + \sum_{i=1}^{u_r} \varrho^+(B_i(r)) + l(g, \partial Y(r), \omega_q) \right) \end{aligned}$$

for all $\varepsilon > 0$. Here $C(q, \varepsilon)$ is the constant which appears in Theorem 4. We integrate the inequality and put

$$L(r) = \frac{1}{\deg \pi_Y} \int_1^r \frac{l(g, \partial Y(t), \omega_q)}{t} dt \quad \text{and} \quad J(r) = \frac{1}{\deg \pi_B} \int_1^r \frac{\sum_{i=1}^{u_t} \varrho^+(B_i(t))}{t} dt.$$

Then we get

$$\begin{aligned} T(r, g, \varkappa_q) \leq & \bar{N}(r, g, \mathcal{D}_q) + N_{\text{ram } \pi_Y}(r) - N_{\text{ram } \pi_B}(r) + \varepsilon T(r, g, \omega_q) \\ & + C(q, \varepsilon)(T(r, b, \eta_q) + \bar{N}(r, b, \mathcal{L}_q) + J(r) + (\deg \pi)L(r)), \quad r > 1, \end{aligned} \quad (2.4.4)$$

for all $\varepsilon > 0$. Here we note that $\text{ram } \pi_Y = \pi^*(\text{ram } \pi_B) + \text{ram } \pi$, and hence we have

$$\text{disc}(\pi_Y, \mathbf{C}(r)) = (\deg \pi) \text{disc}(\pi_B, \mathbf{C}(r)) + \text{disc}(\pi, B(r))$$

and

$$N_{\text{ram } \pi_Y}(r) - N_{\text{ram } \pi_B}(r) = \frac{1}{\deg \pi_Y} \int_1^r \frac{\text{disc}(\pi, B(t))}{t} dt. \quad (2.4.5)$$

CLAIM. *The following inequalities hold:*

$$J(r) \leq N_{\text{ram } \pi_B}(r) \quad \text{for } r > 1, \quad (2.4.6)$$

$$L(r) = o(T(r, g, \omega_q)) \quad ||. \quad (2.4.7)$$

Proof. We first prove (2.4.6). We apply Hurwitz's formula to the proper covering map $\pi_B|_{B_i(r)}: B_i(r) \rightarrow \mathbf{C}(r)$ to get

$$\varrho(B_i(r)) = (\deg \pi_B|_{B_i(r)}) \varrho(\mathbf{C}(r)) + \text{disc}(\pi_B|_{B_i(r)}, \mathbf{C}(r)).$$

Since $\varrho(\mathbf{C}(r)) = -1$ and $\varrho(B_i(r)) \geq -1$, we have

$$\varrho^+(B_i(r)) \leq \text{disc}(\pi_B|_{B_i(r)}, \mathbf{C}(r)).$$

Hence we have $\sum_{i=1}^{u_r} \varrho^+(B_i(r)) \leq \text{disc}(\pi_B, \mathbf{C}(r))$, and so (2.4.6).

Next we prove (2.4.7). In this proof, we denote the covering map $\pi_Y: Y \rightarrow \mathbf{C}$ by p to avoid the confusion with the classical constant π . Put $g^* \omega_q = \frac{1}{2} \sqrt{-1} G^2 dp \wedge d\bar{p}$, where G is a C^∞ -function on $Y \setminus \{z \in Y: p'(z) = 0\}$ with $G \geq 0$. Then we have

$$l(r) := l(g, \partial Y(r), \omega_q) = \int_{\partial Y(r)} G r d \arg p$$

and

$$A(r) := A(g, Y(r), \omega_q) = \int_0^r \int_{\partial Y(t)} G^2 t d \arg p dt.$$

Put $e = \deg p$. Using the Schwarz inequality, we have the following estimates for $r > 1$:

$$\begin{aligned} L(r) &= \frac{1}{e} \int_1^r l(t) \frac{dt}{t} \\ &= \frac{1}{e} \int_1^r \int_{\partial Y(t)} G t \, d \arg p \frac{dt}{t} \\ &\leq \frac{1}{e} \left(\int_1^r \int_{\partial Y(t)} d \arg p \frac{dt}{t} \right)^{1/2} \left(\int_1^r \int_{\partial Y(t)} G^2 t^2 \, d \arg p \frac{dt}{t} \right)^{1/2} \\ &= \frac{1}{e} (2\pi e \log r)^{1/2} (A(r) - A(1))^{1/2} \\ &\leq \frac{1}{e} (2\pi e \log r)^{1/2} \left(e r \frac{d}{dr} T(r) \right)^{1/2} \\ &= (2\pi r \log r)^{1/2} \left(\frac{d}{dr} T(r) \right)^{1/2}. \end{aligned}$$

Here we put $T(r) = T(r, g, \omega_q)$. Take $r_0 > 1$ such that $T(r_0) > 1$. Let E be a subset of $[r_0, \infty)$ defined by

$$r \in E \quad \text{if and only if} \quad L(r) \geq T(r)^{1/2} \log T(r).$$

Then we have

$$\begin{aligned} \int_E d \log \log r &= \int_E \frac{1}{r \log r} \, dr \leq 2\pi \int_E \frac{(dT/dr)(r)}{L(r)^2} \, dr \\ &\leq 2\pi \int_{r_0}^{\infty} \frac{(dT/dr)(r)}{T(r)(\log T(r))^2} \, dr = \frac{2\pi}{\log T(r_0)}. \end{aligned}$$

Hence outside the set E with $\int_E d \log \log r < \infty$, we have

$$L(r) \leq T(r)^{1/2} \log T(r) = o(T(r)),$$

which proves our claim. □

By the assumption $b(B) \not\subset \text{supp } \mathcal{Z}_q$, the Nevanlinna inequality (cf. (2.1.1)) yields $\bar{N}(r, b, \mathcal{Z}_q) \leq T(r, b, [\mathcal{Z}_q]) + O(1)$. Thus we have

$$\bar{N}(r, b, \mathcal{Z}_q) = O(\psi(r))$$

(cf. Lemma 3). Hence using (2.4.1), (2.4.4) and the above claim, and adjusting the constant $C(\varepsilon)$, we obtain Theorem 3.

3. Preliminaries for the proof of Theorem 4

3.1. A property of finite domains

LEMMA 6. *Let \mathcal{F} and \mathcal{F}_0 be Riemann surfaces. Let $F \subset \mathcal{F}$ and $F_0 \subset \mathcal{F}_0$ be finite domains. Let $\zeta: \mathcal{F} \rightarrow \mathcal{F}_0$ be a holomorphic map. Then $F \cap \zeta^{-1}(F_0)$ is a finite disjoint union of finite domains of \mathcal{F} .*

Proof. We may take a finite domain $F' \subset \mathcal{F}$ such that $F \subset F'$, and such that the branch points of ζ do not exist on $\partial F'$. Let $(\sigma_i)_i$ be a finite set of arcs on \mathcal{F}_0 such that $\bigcup_i \sigma_i = \partial F_0$. Observe that $\zeta^{-1}(\sigma_i) \cap F'$ consists of a union of arcs γ which are divided into the following three classes:

- (1) γ with $\zeta(\gamma) = \sigma_i$;
- (2) one of the end points of γ is a branch point of ζ ;
- (3) one of the end points of γ is contained in $\partial F'$.

Since \bar{F}' is compact, the numbers of arcs γ of the classes (1) and (2) are finite. We apply Lemma 1 for $\zeta(\partial F')$ and σ_i to deduce that the number of arcs γ of the class (3) is finite. Hence we conclude that $\zeta^{-1}(\partial F_0) \cap F'$ is a finite union of arcs. We apply Lemma 1 for $\zeta^{-1}(\partial F_0) \cap F'$ and ∂F to conclude that $\zeta^{-1}(\partial F_0) \cap F$ is a finite union of arcs.

Therefore we deduce that $F \cap \zeta^{-1}(F_0)$ consists of a finite number of connected components J , and that the boundary of each J is a finite union of arcs.

Now note that the condition $\partial J = \partial(\mathcal{F} \setminus \bar{J})$ comes from the corresponding conditions for F and F_0 . Hence each J is a finite domain. This proves our assertion. \square

The proofs of the lemmas stated in the rest of this section can be found in [Y2].⁽²⁾

3.2. Topology

Let \mathcal{F} be a Riemann surface. Let Ω and G be two open subsets in \mathcal{F} . We define two subsets $\mathcal{I}(G, \Omega)$ and $\mathcal{P}(G, \Omega)$ of the set of connected components of $G \cap \Omega$ in the following manner. Let G' be a connected component of $G \cap \Omega$. Then G' is contained in $\mathcal{I}(G, \Omega)$ if and only if G' is compactly contained in Ω , and otherwise G' is contained in $\mathcal{P}(G, \Omega)$. Then a connected component G' in $\mathcal{I}(G, \Omega)$ is also a connected component of G . The letters \mathcal{I} and \mathcal{P} refer to islands and peninsulas, respectively, in Ahlfors's theory of covering surfaces.

Let ζ be a non-constant meromorphic function on $\bar{\Omega} \subset \mathcal{F}$, where Ω is a domain of \mathcal{F} . Let E be a domain in \mathbf{P}^1 . We consider the following condition for $\zeta: \bar{\Omega} \rightarrow \mathbf{P}^1$ and E :

$$\text{If } a \in \bar{\Omega} \text{ is a branch point of } \zeta, \text{ then } \zeta(a) \notin \partial E. \quad (3.2.1)$$

⁽²⁾ Though our definition of a finite domain is slightly different from that in [Y2], the proofs are valid without any changes.

LEMMA 7. ([Y2, Lemma 1]) *Assume that a finite number of disjoint simple closed curves $\gamma_i, i=1, \dots, p$, divide \mathbf{P}^1 into connected domains D_1, \dots, D_{p+1} . Let ζ be a non-constant meromorphic function on $\bar{\Omega}$, where Ω is a finite domain of a Riemann surface \mathcal{F} . Assume that the condition (3.2.1) is satisfied for ζ and $D_i, 1 \leq i \leq p+1$. Put $\mathcal{A} = \bigcup_{i=1}^{p+1} \mathcal{I}(\zeta^{-1}(D_i), \Omega)$ and $\mathcal{B} = \bigcup_{i=1}^{p+1} \mathcal{P}(\zeta^{-1}(D_i), \Omega)$. Then we have*

$$\varrho^+(\Omega) \geq \sum_{A \in \mathcal{A}} \varrho(A) + \sum_{B \in \mathcal{B}} \varrho^+(B).$$

Remark 3.2.2. By Lemma 6, the right-hand side of the inequality above is a finite sum.

3.3. Review of Ahlfors's theory

Recall that we denote by $\omega_{\mathbf{P}^1}$ the Fubini–Study form on the projective line \mathbf{P}^1 . Let Ω_0 be a finite domain of \mathbf{P}^1 . Let \mathcal{F} be a Riemann surface, let $\Omega \subset \mathcal{F}$ be a finite domain and let ζ be a non-constant meromorphic function on $\bar{\Omega}$. Assume that $\zeta(\bar{\Omega}) \subset \bar{\Omega}_0$. Then we may consider $\zeta: \Omega \rightarrow \Omega_0$ as a covering surface in the sense of [Ne2, p. 323]. We call $\zeta^{-1}(\Omega_0) \cap \partial\Omega$ the relative boundary and $l(\zeta, \zeta^{-1}(\Omega_0) \cap \partial\Omega, \omega_{\mathbf{P}^1})$ the length of the relative boundary. Let $D \subset \Omega_0$ be a domain which is bounded by a finite union of arcs. We call

$$S_D = \frac{A(\zeta, \zeta^{-1}(D) \cap \Omega, \omega_{\mathbf{P}^1})}{\int_D \omega_{\mathbf{P}^1}}$$

the mean sheet number of ζ over D , and S_{Ω_0} the mean sheet number of ζ .

In the following two theorems, we assume that $\partial\Omega_0$ consists of a finite disjoint union of regular, analytic Jordan curves. We denote by S and L the mean sheet number and the length of the relative boundary of the covering $\zeta: \Omega \rightarrow \Omega_0$, respectively.

COVERING THEOREM 1. ([Ne2, p. 328]) *There exists a positive constant $h = h(\Omega_0) > 0$ which is independent of D, Ω and ζ , such that*

$$|S - S_D| \leq \frac{h}{\int_D \omega_{\mathbf{P}^1}} L. \tag{3.3.1}$$

Consider ζ as the covering map of the closed surfaces $\zeta: \bar{\Omega} \rightarrow \bar{\Omega}_0$. Put

$$S(\partial\Omega_0) = \frac{l(\zeta, \zeta^{-1}(\partial\Omega_0), \omega_{\mathbf{P}^1})}{\text{length of } \partial\Omega_0 \text{ with respect to the Fubini–Study metric}}.$$

COVERING THEOREM 2. ([Ne2, p. 331, Remark]) *There exists a positive constant $h=h(\Omega_0)>0$ which is independent of Ω and ζ , such that*

$$|S-S(\partial\Omega_0)| \leq hL. \quad (3.3.2)$$

Note that a regular, analytic Jordan curve is *regular* in the sense of [Ne2, p. 326] (cf. [Hay, Lemma 5.1]). The main theorem ([Ne2, p. 332]) of Ahlfors's theory was used to prove the following lemma. An *analytic Jordan domain* $E \subset \mathbf{P}^1$ is a Jordan domain whose boundary ∂E is regular and analytic.

LEMMA 8. ([Y2, Lemma 2]) *Let E^\dagger be an analytic Jordan domain in \mathbf{P}^1 , or \mathbf{P}^1 itself. Let $E_1, \dots, E_p, E_\infty$ be analytic Jordan domains in \mathbf{P}^1 . Assume that the closures \bar{E}_j of E_j , $j=1, \dots, p, \infty$, are mutually disjoint. Then there exists a positive constant $h>0$ which only depends on $E_1, \dots, E_p, E_\infty$, with the following property: Let Ω be a finite domain of a Riemann surface \mathcal{F} , and let ψ and ζ be two non-constant meromorphic functions on $\bar{\Omega}$. Assume that*

$$\zeta(\psi^{-1}(\mathbf{P}^1 \setminus E^\dagger) \cap \bar{\Omega}) \subset E_\infty \quad (3.3.3)$$

and that ζ and E_j satisfy the condition (3.2.1) for $j=1, \dots, p, \infty$.

Put

$$\begin{aligned} \mathcal{G}^I &= \mathcal{I}(\psi^{-1}(E^\dagger), \Omega), & \mathcal{G}^P &= \mathcal{P}(\psi^{-1}(E^\dagger), \Omega), \\ \mathcal{G}_j^I &= \mathcal{I}(\zeta^{-1}(E_j), \Omega), & \mathcal{G}_j^P &= \mathcal{P}(\zeta^{-1}(E_j), \Omega) \quad \text{for } j=1, \dots, p, \\ \mathcal{G}_\infty^I &= \mathcal{I}(\zeta^{-1}(E_\infty), \Omega \cap \psi^{-1}(E^\dagger)). \end{aligned}$$

Then we have the inequality

$$\begin{aligned} \vartheta(\zeta, \psi) + \sum_{G \in \mathcal{G}^I} \varrho(G) + \sum_{G \in \mathcal{G}^P} \varrho^+(G) - \sum_{j=1}^p \sum_{G \in \mathcal{G}_j^I} \varrho(G) - \sum_{j=1}^p \sum_{G \in \mathcal{G}_j^P} \varrho^+(G) - \sum_{G \in \mathcal{G}_\infty^I} \varrho(G) \\ \geq (p-1)A(\zeta, \Omega, \omega_{\mathbf{P}^1}) - hl(\zeta, \partial\Omega, \omega_{\mathbf{P}^1}), \end{aligned} \quad (3.3.4)$$

where $\vartheta(\zeta, \psi)$ is the number of connected components G in \mathcal{G}^I such that $\zeta(G) \subset E_\infty$.

Remark 3.3.5. (1) By Lemma 6, the left-hand side of the inequality (3.3.4) is a finite sum.

(2) Since we have $\int_{\mathbf{P}^1} \omega_{\mathbf{P}^1} = 1$, the term $A(\zeta, \Omega, \omega_{\mathbf{P}^1})$ is equal to the mean sheet number of the covering $\zeta: \Omega \rightarrow \mathbf{P}^1$. Also, since \mathbf{P}^1 is compact, the term $l(\zeta, \partial\Omega, \omega_{\mathbf{P}^1})$ is equal to the length of the relative boundary of the covering ζ .

(3) Consider the case $E^\dagger = \mathbf{P}^1$. Then the condition (3.3.3) is satisfied automatically. If Ω is non-compact, then $\mathcal{G}^I = \emptyset$ and $\mathcal{G}^P = \{\Omega\}$, and hence $\vartheta(\zeta, \psi) = 0$. On the other hand, if Ω is compact, then $\mathcal{G}^I = \{\Omega\}$ and $\mathcal{G}^P = \emptyset$. Since ζ is non-constant, we have $\zeta(\Omega) \not\subset E_\infty$, so $\vartheta(\zeta, \psi) = 0$. Hence we have $\vartheta(\zeta, \psi) = 0$ in both cases. Since $\varrho(\Omega) \leq \varrho^+(\Omega)$, we get

$$\begin{aligned} \varrho^+(\Omega) - \sum_{j=1}^p \sum_{G \in \mathcal{G}_j^I} \varrho(G) - \sum_{j=1}^p \sum_{G \in \mathcal{G}_j^P} \varrho^+(G) - \sum_{G \in \mathcal{G}_\infty^I} \varrho(G) \\ \geq (p-1)A(\zeta, \Omega, \omega_{\mathbf{P}^1}) - hl(\zeta, \partial\Omega, \omega_{\mathbf{P}^1}). \end{aligned} \tag{3.3.6}$$

Here we can write \mathcal{G}_∞^I as $\mathcal{I}(\zeta^{-1}(E_\infty), \Omega)$.

3.4. Rouché’s theorem

We denote by $\text{dist}(x, y)$ the distance between $x, y \in \mathbf{P}^1$ with respect to the Kähler metric associated to the Fubini–Study form $\omega_{\mathbf{P}^1}$.

LEMMA 9. ([Y2, Lemma 3]) *Let $E \subset \mathbf{P}^1$ be a Jordan domain, and let b be a point in E . Then there exists a positive constant $C = C(E, b) > 0$ with the following property: Let Ω be a finite domain in a Riemann surface \mathcal{F} , and let ζ be a meromorphic function on \mathcal{F} such that $\zeta(\Omega) = E$ and $\zeta(\partial\Omega) = \partial E$. Then for a meromorphic function α on \mathcal{F} such that $\text{dist}(\alpha(z), b) < C$ for all $z \in \bar{\Omega}$, there exists a point $z \in \Omega$ with $\zeta(z) = \alpha(z)$.*

4. Local value distribution

4.1. Notation

In this section, we work around a neighborhood of a point $x \in \bar{\mathcal{M}}_{0,q}$. This point x will be fixed in this section. We denote by $\text{edge}(\Gamma_x)$ the set of all edges of Γ_x , i.e.,

$$\text{edge}(\Gamma_x) = \{\{v, v'\} : v \text{ and } v' \text{ are adjacent vertices of } \Gamma_x\}.$$

Then $\text{edge}(\Gamma_x)$ is an empty set if and only if $x \in \mathcal{M}_{0,q}$. Let v and v' be distinct vertices of Γ_x . Since Γ_x is a tree, there exists a unique sequence of distinct vertices

$$v = v_0, v_1, \dots, v_r = v',$$

where v_{i-1} and v_i are adjacent for $i = 1, \dots, r$. We call this sequence the *path* joining v and v' .

4.1.1. Take a vertex $v \in \text{vert}(\Gamma_x)$. Recall that C_v is the irreducible component of \mathcal{C}_x which corresponds to $v \in \text{vert}(\Gamma_x)$. Put

$$\begin{aligned} P_v^m &= \{i \in (q) : \sigma_i(x) \in C_v\} && (m \text{ stands for "marked points"}), \\ P_v^n &= \{v' \in \text{vert}(\Gamma_x) : v' \text{ is adjacent with } v\} && (n \text{ stands for "nodes"}). \end{aligned}$$

Note that we have $\bigcup_{v \in \text{vert}(\Gamma_x)} P_v^m = (q)$ and $P_v^m \cap P_{v'}^m = \emptyset$ for $v \neq v'$ because marked points are smooth points of \mathcal{C}_x . Hence for each $i \in (q)$, there exists a unique vertex $v \in \text{vert}(\Gamma_x)$ such that $\sigma_i(x) \in C_v$. Put $P = (q) \amalg \text{vert}(\Gamma_x)$, $P_v = P_v^m \amalg P_v^n \subset P$ and $d_v = \text{card } P_v$.

4.1.2. Define $\varsigma: P_v \rightarrow C_v$ by the following rule. If $\tau \in P_v^m$, then $\varsigma(\tau) = \sigma_\tau(x)$; on the other hand, if $\tau \in P_v^n$, then $\varsigma(\tau) = C_v \cap C_\tau$. Then ς is an injection, and the image $\varsigma(P_v)$ is the set of the special points of C_v , which are either the marked points or the nodes. Hence P_v can be identified with the set of the special points on C_v by ς , so $d_v \geq 3$ (cf. Definition 1.5.1).

4.1.3. *Definition of $\varphi_{\langle v \rangle}$.* For each $v \in \text{vert}(\Gamma_x)$, there exists $\langle v \rangle \in \mathcal{J}$ with the following property: The restriction $\varphi_{\langle v \rangle}|_{C_v}: C_v \rightarrow \mathbf{P}^1$ is an isomorphism and the restrictions $\varphi_{\langle v \rangle}|_{C_{v'}}: C_{v'} \rightarrow \mathbf{P}^1$ are constant maps for all $v' \in \text{vert}(\Gamma_x) \setminus \{v\}$. To see this, we observe the following.

CLAIM. Let $C = (C, s_1, \dots, s_q)$ be a q -pointed stable curve, and let E be an irreducible component of C . Then there exists a subset $S \subset (q)$ with $\text{card } S = 3$ satisfying the following property: Consider the contraction $c: C \rightarrow \mathbf{P}^1$ obtained by forgetting the points s_j marked in $j \in (q) \setminus S$, where we note that the resultant 3-pointed stable curve is isomorphic to \mathbf{P}^1 . Then the restriction $c|_E: E \rightarrow \mathbf{P}^1$ is an isomorphism, and the restrictions $c|_{E'}$ are constant maps for the other components E' of C .

Proof. We shall prove this by induction on q . Note that the assertion is trivial for $q = 3$. Next we assume that the assertion is valid for $q - 1$, and consider the case for q where $q \geq 4$. We may take $j \in (q)$ such that the number of the special points on E other than s_j is at least three. (If there exists $j' \in (q)$ with $s_{j'} \notin E$, then put $j = j'$. Otherwise, we take arbitrary $j \in (q)$, where we note that $q \geq 4$.) Let $c': C \rightarrow C'$ be the contraction obtained by forgetting the point s_j , where the marked points on C' are assumed to be labeled by the set $(q) \setminus \{j\}$. Then by the property (2) in the definition of contraction (cf. §1.5), we conclude that the restriction $c'|_E: E \rightarrow C'$ is an injection and that $c'(E') \neq c'(E)$ for the other components $E' \subset C$.

Now by the induction hypothesis, there is a subset $S \subset (q) \setminus \{j\}$ with $\text{card } S = 3$ such that the contraction $c'': C' \rightarrow \mathbf{P}^1$ obtained by forgetting the points labeled by $(q) \setminus (S \cup \{j\})$ has the following property: The restriction $c''|_{c'(E)}: c'(E) \rightarrow \mathbf{P}^1$ is an isomorphism, and

the restrictions $c''|_{E'}$ to the other components E' of C' are constant maps. Put $c = c'' \circ c': C \rightarrow \mathbf{P}^1$, which is a contraction forgetting the points labeled by $(q) \setminus S$. Then c has the desired property. This proves our claim. \square

Now apply this claim to the case $C = \mathcal{C}_x$ and $E = C_v$ to get the subset $S \subset (q)$ and the contraction c . Put $\langle v \rangle = S$ (by ordering the elements of S). Then by the definition of $\varphi_{\langle v \rangle}$, we have $\varphi_{\langle v \rangle}|_{\mathcal{C}_x} = \psi \circ c$, where ψ is some automorphism of \mathbf{P}^1 . Hence $\varphi_{\langle v \rangle}$ has the desired property. This $\langle v \rangle$ will be fixed for each $v \in \text{vert}(\Gamma_x)$.

4.1.4. For $v \in \text{vert}(\Gamma_x)$ and $\tau \in P_v$, put $w_v(\tau) = \varphi_{\langle v \rangle} \circ \zeta(\tau) \in \mathbf{P}^1$. Then $w_v: P_v \rightarrow \mathbf{P}^1$ is an injection.

4.1.5. *Definitions of $\hat{\tau}_v$ and ι_v .* For $v \in \text{vert}(\Gamma_x)$, we define the map $\hat{\tau}_v: (q) \rightarrow P_v$ by the following rule. Take $i \in (q)$. If $i \in P_v^m$, then put $\hat{\tau}_v(i) = i \in P_v$. Otherwise, take the vertex $v' \in \text{vert}(\Gamma_x) \setminus \{v\}$ with $i \in P_{v'}^m$ and the path

$$v = v_0, v_1, \dots, v_r = v'$$

joining v and v' . Put $\hat{\tau}_v(i) = v_1 \in P_v$. Then we have

$$w_v(\hat{\tau}_v(i)) = \varphi_{\langle v \rangle} \circ \sigma_i(x) \quad \text{for all } i \in (q) \text{ and } v \in \text{vert}(\Gamma_x). \quad (4.1.1)$$

There exists a section $\iota_v: P_v \rightarrow (q)$ of $\hat{\tau}_v: (q) \rightarrow P_v$. This ι_v is defined by the following rule. For $i \in P_v^m$, put $\iota_v(i) = i \in (q)$. For a vertex $v' \in P_v^n$, take a maximal path

$$v, v', v_1, \dots, v_r \quad (4.1.2)$$

starting from the edge $\{v, v'\}$, i.e., there exists no path extending (4.1.2) to the right. Then we have $\text{card } P_{v_r}^n = 1$ (otherwise we can extend the path). By $d_{v_r} \geq 3$, there exists $i \in P_{v_r}^m$. Put $\iota_v(v') = i$. Then this ι_v is a section of $\hat{\tau}_v$, which will be fixed for each $v \in \text{vert}(\Gamma_x)$.

If v and v' are adjacent vertices of Γ_x , we have

$$\hat{\tau}_{v'}(\iota_v(v')) \neq v \quad (\text{as elements of } P_{v'}), \quad (4.1.3)$$

which easily follows from the definitions of the above objects.

4.1.6. For $v \in \text{vert}(\Gamma_x)$ and $\tau \in P_v$, put $\beta_{v,\tau} = \varphi_{\langle v \rangle} \circ \sigma_{\iota_v(\tau)}: \overline{\mathcal{M}}_{0,q} \rightarrow \mathbf{P}^1$. Then we have $\beta_{v,\tau}(x) = w_v(\tau) \in \mathbf{P}^1$, which follows from (4.1.1) and the fact that ι_v is a section of $\hat{\tau}_v$.

4.2. A geometric lemma

Recall that \mathcal{L} is the hyperplane section bundle on \mathbf{P}^1 .

LEMMA 10. *There exists a Zariski-open neighborhood $U_x \subset \overline{\mathcal{M}}_{0,q}$ of x such that*

$$\sum_{v \in \text{vert}(\Gamma_x)} (d_v - 2)\varphi_{(v)}^* \mathcal{L} = K_q \quad \text{on } \varpi_q^{-1}(U_x). \tag{4.2.1}$$

Proof. Put $M = \sum_{v \in \text{vert}(\Gamma_x)} (d_v - 2)\varphi_{(v)}^* \mathcal{L} - K_q$. For $y \in \overline{\mathcal{M}}_{0,q}$, let M_y be the restriction of M to \mathcal{C}_y . Note that C_v are isomorphic to \mathbf{P}^1 for all $v \in \text{vert}(\Gamma_x)$ and that the degrees of the restrictions $K_q|_{C_v}$ and $((d_v - 2)\varphi_{(v)}^* \mathcal{L})|_{C_v}$ are both equal to $d_v - 2$ (cf. [Ma, p. 202, (1.3)]). Hence $M_x|_{C_v}$ are the trivial line bundles on C_v for all $v \in \text{vert}(\Gamma_x)$. Since Γ_x is a tree, we conclude that M_x is the trivial line bundle on \mathcal{C}_x .

We apply the theorem of semi-continuity [Har, Chapter III, Theorem 12.8] to the flat morphism ϖ_q . Then we obtain a non-empty affine open neighborhood U_x of x such that

$$\dim H^0(\mathcal{C}_y, M_y) \leq 1 \quad \text{and} \quad \dim H^0(\mathcal{C}_y, M_y^{-1}) \leq 1 \tag{4.2.2}$$

for all $y \in U_x$. Put

$$Z = \{y \in U_x : \dim H^0(\mathcal{C}_y, M_y) = 1\}.$$

Again by the theorem of semi-continuity, we see that Z is a Zariski-closed subset of U_x . Take a point y from $U_x \setminus \mathcal{Z}_q$, which is a non-empty Zariski-open subset of U_x . Then \mathcal{C}_y is isomorphic to \mathbf{P}^1 , and hence the condition (4.2.2) implies that M_y is the trivial line bundle on \mathcal{C}_y . Hence $U_x \setminus \mathcal{Z}_q \subset Z$. This implies that $Z = U_x$.

Now by the theorem of Grauert [Har, Chapter III, Theorem 12.9], we have a section $s \in H^0(\varpi_q^{-1}(U_x), M)$ such that the restriction $s|_{\mathcal{C}_x}$ is equal to the section 1 of the trivial line bundle M_x , where we note that U_x is affine. Let D be the divisor on $\varpi_q^{-1}(U_x)$ defined by $s=0$. Since ϖ_q is a projective morphism, $\varpi_q(\text{supp } D)$ is a Zariski-closed subset of U_x , which does not contain x . Hence by replacing U_x by $U_x \setminus \varpi_q(\text{supp } D)$, we may assume that s is a nowhere vanishing section on $\varpi_q^{-1}(U_x)$. This implies that the restriction $M|_{\varpi_q^{-1}(U_x)}$ is the trivial line bundle, which proves our lemma. \square

4.3. The local version of the theorem

LEMMA 11. *Let Λ be a countable set of non-degenerate q -hol-quintets. Then for all $x \in \overline{\mathcal{M}}_{0,q}$, there exist an open neighborhood $V_x = V_x(\Lambda)$ of x and a positive constant $h_x = h_x(\Lambda) > 0$ with the following property: Let $(\mathcal{F}, \mathcal{R}, \pi, g, b) \in \Lambda$ be a q -hol-quintet contained*

in Λ . Let $R \subset \mathcal{R}$ be a finite domain such that $b(R) \subset V_x$. Put $F = \pi^{-1}(R)$. Then we have the inequality

$$A(g, F, \boldsymbol{x}_q) \leq \bar{n}(g, \mathcal{D}_q, F) + \text{disc}(\pi, R) + (\deg \pi) \varrho^+(R) + h_x l(g, \partial F, \omega_q) + h_x (\deg \pi) \bar{n}(b, \mathcal{L}_q, R). \tag{4.3.1}$$

Proof. For $(\mathcal{F}, \mathcal{R}, \pi, g, b) \in \Lambda$ and $\alpha \in \mathcal{J}$, put

$$g_\alpha = \varphi_\alpha \circ g,$$

which is a non-constant meromorphic function on \mathcal{F} . For $v \in \text{vert}(\Gamma_x)$ and $\tau \in P_v$, let E_τ^v be a small spherical disc in \mathbf{P}^1 centered at $w_v(\tau)$ such that

- (1) $\bar{E}_\tau^v \cap \bar{E}_{\tau'}^v = \emptyset$ for $\tau \neq \tau'$, where we note that $w_v(\tau) \neq w_v(\tau')$;
- (2) $g_\alpha: \mathcal{F} \rightarrow \mathbf{P}^1$ and E_τ^v satisfy the condition (3.2.1) for all $v \in \text{vert}(\Gamma_x)$, $\tau \in P_v$, $\alpha \in \mathcal{J}$ and $(\mathcal{F}, \mathcal{R}, \pi, g, b) \in \Lambda$, i.e., if $a \in \mathcal{F}$ is a branch point of g_α for some $(\mathcal{F}, \mathcal{R}, \pi, g, b) \in \Lambda$ and $\alpha \in \mathcal{J}$, then $g_\alpha(a) \notin \partial E_\tau^v$ for all $v \in \text{vert}(\Gamma_x)$ and $\tau \in P_v$.

Here in the second condition, we note that the set

$$\bigcup_{\substack{\alpha \in \mathcal{J} \\ (\mathcal{F}, \mathcal{R}, \pi, g, b) \in \Lambda}} \{g_\alpha(a) : a \text{ is a branch point of } g_\alpha: \mathcal{F} \rightarrow \mathbf{P}^1\}$$

is countable, because Λ and the set of branch points of g_α are countable.

For each $\{v, v'\} \in \text{edge}(\Gamma_x)$, put

$$D_{v,v'} = \varphi_{\langle v \rangle}^{-1}(\mathbf{P}^1 \setminus E_{v'}^v) \cap \varphi_{\langle v' \rangle}^{-1}(\mathbf{P}^1 \setminus E_v^{v'}),$$

which is a compact subset of $\overline{\mathcal{M}}_{0,q}$, because $\varphi_{\langle v \rangle}$ and $\varphi_{\langle v' \rangle}$ are proper maps. Note that $\mathcal{C}_x \setminus (C_v \cap C_{v'})$ consists of two connected components. The set $(\varphi_{\langle v \rangle}|_{\mathcal{C}_x})^{-1}(\mathbf{P}^1 \setminus E_{v'}^v)$ is contained in one component, and the set $(\varphi_{\langle v' \rangle}|_{\mathcal{C}_x})^{-1}(\mathbf{P}^1 \setminus E_v^{v'})$ is contained in the other component. Thus we have $\varpi_q^{-1}(x) \cap D_{v,v'} = \emptyset$. Hence the image $\varpi_q(D_{v,v'}) \subset \overline{\mathcal{M}}_{0,q}$ is a compact subset which does not contain the point x . Therefore, for all $\{v, v'\} \in \text{edge}(\Gamma_x)$, we conclude that there exists an open neighborhood $V_{v,v'}$ of x such that $\varpi_q^{-1}(V_{v,v'}) \cap D_{v,v'} = \emptyset$, i.e.,

$$\varphi_{\langle v \rangle}(\varphi_{\langle v' \rangle}^{-1}(\mathbf{P}^1 \setminus E_{v'}^v) \cap \varpi_q^{-1}(V_{v,v'})) \subset E_{v'}^v. \tag{4.3.2}$$

Let $V_x \subset \overline{\mathcal{M}}_{0,q}$ be an open neighborhood of x such that

- (1) $\bar{V}_x \subset U_x$ (cf. Lemma 10);
- (2) $\bar{V}_x \subset V_{v,v'}$ for all $\{v, v'\} \in \text{edge}(\Gamma_x)$;
- (3) $\text{dist}(w_v(\tau), \beta_{v,\tau}(y)) < C(E_\tau^v, w_v(\tau))$ for all $y \in \bar{V}_x$, $v \in \text{vert}(\Gamma_x)$ and $\tau \in P_v$, where we note that $\beta_{v,\tau}(x) = w_v(\tau)$ (cf. Lemma 9);
- (4) $\varphi_{\langle v \rangle} \circ \sigma_i(\bar{V}_x) \subset E_{\hat{\tau}_v(i)}^v$ for all $v \in \text{vert}(\Gamma_x)$ and $i \in (q)$, where we have $\varphi_{\langle v \rangle} \circ \sigma_i(x) = w_v(\hat{\tau}_v(i)) \in E_{\hat{\tau}_v(i)}^v$ (cf. (4.1.1)).

Let $\lambda=(\mathcal{F}, \mathcal{R}, \pi, g, b) \in \Lambda$. Let R be a finite domain of \mathcal{R} such that $b(R) \subset V_x$. Put $F = \pi^{-1}(R)$, which is a finite disjoint union of finite domains on \mathcal{F} . We shall derive the estimate (4.3.1) with the constant h_x which will be found below.

First we apply Lemma 8. For a vertex $v \in \text{vert}(\Gamma_x)$ and $\tau \in P_v$, put

$$\mathcal{G}_{v,\tau}^I = \mathcal{I}(g_{\langle v \rangle}^{-1}(E_\tau^v), F) \quad \text{and} \quad \mathcal{G}_{v,\tau}^P = \mathcal{P}(g_{\langle v \rangle}^{-1}(E_\tau^v), F).$$

We denote by $\mathcal{C}(F)$ the set of connected components of F . Let v_o be the unique vertex of Γ_x such that $\sigma_1(x) \in C_{v_o}$. For each vertex $v \in \text{vert} \Gamma_x \setminus \{v_o\}$, take the path joining v_o and v :

$$v_o = v_0, v_1, \dots, v_{r-1}, v_r = v.$$

We denote the vertex v_{r-1} by v^- , which is uniquely determined by the vertex v .

We first consider the vertex v_o . For each $H \in \mathcal{C}(F)$, we apply Lemma 8 (cf. (3.3.6)) to the case

$$\begin{aligned} \mathcal{F} &= \mathcal{F}, \quad \Omega = H, \quad \zeta = \psi = g_{\langle v_o \rangle}|_H, \quad E^\dagger = \mathbf{P}^1, \quad E_\infty = E_1^{v_o}, \\ \{E_j\}_{j=1}^p &= \{E_{v'}^{v_o}\}_{v' \in P_{v_o}^n} \cup \{E_i^{v_o}\}_{i \in P_{v_o}^m \setminus \{1\}}, \quad p = d_{v_o} - 1. \end{aligned}$$

Adding over all $H \in \mathcal{C}(F)$ and using the fact $\sum_{i \in P_{v_o}^m \setminus \{1\}} \sum_{G \in \mathcal{G}_{v_o,i}^P} \varrho^+(G) \geq 0$, we obtain the following: There exists a positive constant $h_{v_o} > 0$ which does not depend on the choices of $\lambda \in \Lambda$ and R , such that

$$\begin{aligned} \text{IE}(v_o): \quad & \sum_{H \in \mathcal{C}(F)} \varrho^+(H) - \sum_{v \in P_{v_o}^n} \left(\sum_{G \in \mathcal{G}_{v_o,v}^I} \varrho(G) + \sum_{G \in \mathcal{G}_{v_o,v}^P} \varrho^+(G) \right) - \sum_{i \in P_{v_o}^m} \sum_{G \in \mathcal{G}_{v_o,i}^I} \varrho(G) \\ & \geq (d_{v_o} - 2)A(g_{\langle v_o \rangle}, F, \omega_{\mathbf{P}^1}) - h_{v_o}l(g_{\langle v_o \rangle}, \partial F, \omega_{\mathbf{P}^1}). \end{aligned}$$

Next for a vertex $v \in \text{vert} \Gamma_x \setminus \{v_o\}$, we put

$$\mathcal{G}_v^I = \mathcal{I}(g_{\langle v \rangle}^{-1}(E_{v^-}^v), F \cap g_{\langle v^- \rangle}^{-1}(E_{v^-}^{v^-})).$$

For each $H \in \mathcal{C}(F)$, we apply Lemma 8 to the case

$$\begin{aligned} \mathcal{F} &= \mathcal{F}, \quad \Omega = H, \quad \zeta = g_{\langle v \rangle}|_H, \quad \psi = g_{\langle v^- \rangle}|_H, \quad E^\dagger = E_{v^-}^{v^-}, \quad E_\infty = E_{v^-}^v, \\ \{E_j\}_{j=1,\dots,p} &= \{E_{v'}^v\}_{v' \in P_v^n \setminus \{v^-\}} \cup \{E_i^v\}_{i \in P_v^m}, \quad p = d_v - 1, \end{aligned}$$

where the condition (3.3.3) follows from the property (4.3.2). Adding over all $H \in \mathcal{C}(F)$ and using the fact $\sum_{i \in P_v^m} \sum_{G \in \mathcal{G}_{v,i}^P} \varrho^+(G) \geq 0$, we obtain the following: There exists a

positive constant $h_v > 0$ which does not depend on the choices of $\lambda \in \Lambda$ and R , such that

$$\begin{aligned} \text{IE}(v): & \sum_{H \in \mathcal{C}(F)} \vartheta(g_{\langle v \rangle} |_H, g_{\langle v^- \rangle} |_H) + \sum_{G \in \mathcal{G}_{v^-,v}^I} \varrho(G) + \sum_{G \in \mathcal{G}_{v^-,v}^P} \varrho^+(G) \\ & - \sum_{v' \in P_v^n \setminus \{v^-\}} \left(\sum_{G \in \mathcal{G}_{v,v'}^I} \varrho(G) + \sum_{G \in \mathcal{G}_{v,v'}^P} \varrho^+(G) \right) - \sum_{i \in P_v^m} \sum_{G \in \mathcal{G}_{v,i}^I} \varrho(G) - \sum_{G \in \mathcal{G}_v^I} \varrho(G) \\ & \geq (d_v - 2)A(g_{\langle v \rangle}, F, \omega_{\mathbf{P}^1}) - h_v l(g_{\langle v \rangle}, \partial F, \omega_{\mathbf{P}^1}). \end{aligned}$$

Now, using the inequality $\text{IE}(v_o)$ for the vertex v_o and the inequalities $\text{IE}(v)$ for vertices $v \neq v_o$, we add the inequalities $\text{IE}(v)$ over all $v \in \text{vert}(\Gamma_x)$. Then we obtain

$$\begin{aligned} & \sum_{H \in \mathcal{C}(F)} \varrho^+(H) - \sum_{v \in \text{vert}(\Gamma_x)} \sum_{i \in P_v^m} \sum_{G \in \mathcal{G}_{v,i}^I} \varrho(G) \\ & + \sum_{v \in \text{vert}(\Gamma_x) \setminus \{v_o\}} \left(\sum_{H \in \mathcal{C}(F)} \vartheta(g_{\langle v \rangle} |_H, g_{\langle v^- \rangle} |_H) - \sum_{G \in \mathcal{G}_v^I} \varrho(G) \right) \tag{4.3.3} \\ & \geq \sum_{v \in \text{vert} \Gamma_x} (d_v - 2)A(g_{\langle v \rangle}, F, \omega_{\mathbf{P}^1}) - h' l(g, \partial F, \omega_q). \end{aligned}$$

Here we used the following two facts:

(1) There exists a positive constant $h' > 0$ which does not depend on the choices of $\lambda \in \Lambda$ and R such that

$$\sum_{v \in \text{vert}(\Gamma_x)} h_v l(g_{\langle v \rangle}, \partial F, \omega_{\mathbf{P}^1}) \leq h' l(g, \partial F, \omega_q).$$

(2) For a vertex $v \neq v_o$, the term

$$\sum_{G \in \mathcal{G}_{v^-,v}^I} \varrho(G) + \sum_{G \in \mathcal{G}_{v^-,v}^P} \varrho^+(G)$$

appears on the left-hand side of $\text{IE}(v)$, while the term

$$- \sum_{G \in \mathcal{G}_{v^-,v}^I} \varrho(G) - \sum_{G \in \mathcal{G}_{v^-,v}^P} \varrho^+(G)$$

appears on the left-hand side of $\text{IE}(v^-)$ because $v \in P_{v^-}^n$, and $v \neq (v^-)^-$ for $v^- \neq v_o$. Hence these terms are canceled by each other when we add inequalities over all $v \in \text{vert}(\Gamma_x)$.

Now we will estimate the terms on the left-hand side of (4.3.3).

CLAIM. *The following inequalities hold:*

$$\sum_{H \in \mathcal{C}(F)} \vartheta(g_{\langle v \rangle}|_H, g_{\langle v^- \rangle}|_H) - \sum_{G \in \mathcal{G}_v^I} \varrho(G) \leq 2(\deg \pi) \bar{n}(b, \mathcal{Z}_q, R) \quad \text{for all } v \neq v_o, \quad (4.3.4)$$

$$- \sum_{v \in \text{vert}(\Gamma_x)} \sum_{i \in P_v^m} \sum_{G \in \mathcal{G}_{v,i}^I} \varrho(G) \leq \bar{n}(g, \mathcal{D}_q, F) + q(\deg \pi) \bar{n}(b, \mathcal{Z}_q, R), \quad (4.3.5)$$

$$\sum_{H \in \mathcal{C}(F)} \varrho^+(H) \leq \text{disc}(\pi, R) + (\deg \pi) \varrho^+(R). \quad (4.3.6)$$

Proof of (4.3.4). For $H \in \mathcal{C}(F)$ and $\{v, v'\} \in \text{edge}(\Gamma_x)$, let $\vartheta'(v', v, H)$ denote the number of connected components G in $\mathcal{I}(g_{\langle v \rangle}^{-1}(E_{v'}^v), H)$ such that $g_{\langle v' \rangle}(G) \subset E_{v'}^{v'}$. Then we have

$$\vartheta'(v, v^-, H) = \vartheta(g_{\langle v \rangle}|_H, g_{\langle v^- \rangle}|_H)$$

and

$$- \sum_{G \in \mathcal{G}_v^I} \varrho(G) \leq \text{card } \mathcal{G}_v^I \leq \sum_{H \in \mathcal{C}(F)} \vartheta'(v^-, v, H).$$

Here we note that $G \in \mathcal{G}_v^I$ is non-compact because $g_{\langle v \rangle}$ is non-constant and $E_{v^-}^v$ is non-compact, hence $\varrho(G) \geq -1$. (By the definition, we have $g_{\langle v \rangle}(G) \subset E_{v^-}^v$.) Therefore to prove (4.3.4), it suffices to prove

$$\sum_{H \in \mathcal{C}(F)} \vartheta'(v', v, H) \leq (\deg \pi) \bar{n}(b, \mathcal{Z}_q, R) \quad (4.3.7)$$

for all $\{v, v'\} \in \text{edge}(\Gamma_x)$.

Take $G \in \mathcal{I}(g_{\langle v \rangle}^{-1}(E_{v'}^v), H)$ such that $g_{\langle v' \rangle}(G) \subset E_{v'}^{v'}$. Then by the definition of V_x , we may apply Lemma 9 to the case

$$\mathcal{F} = H, \quad E = E_{v'}^v, \quad \Omega = G, \quad \zeta = g_{\langle v \rangle} (= \varphi_{\langle v \rangle} \circ g), \quad \alpha = \beta_{v,v'} \circ b \circ \pi.$$

We conclude that there exists $z \in G$ such that

$$\varphi_{\langle v \rangle} \circ g(z) = \varphi_{\langle v \rangle} \circ \sigma_{\iota_v(v')} \circ b \circ \pi(z) \quad (4.3.8)$$

(note that $\beta_{v,v'} = \varphi_{\langle v \rangle} \circ \sigma_{\iota_v(v')}$). Now we shall prove $b \circ \pi(z) \in \text{supp } \mathcal{Z}_q$ by contradiction. Suppose $b \circ \pi(z) \notin \text{supp } \mathcal{Z}_q$. Then (4.3.8) implies

$$\varphi_{\langle v' \rangle} \circ g(z) = \varphi_{\langle v' \rangle} \circ \sigma_{\iota_v(v')} \circ b \circ \pi(z), \quad (4.3.9)$$

which follows from the facts that the restrictions $\varphi_{\langle v \rangle}|_{\mathcal{C}_y}$ and $\varphi_{\langle v' \rangle}|_{\mathcal{C}_y}$ give isomorphisms $\mathcal{C}_y \rightarrow \mathbf{P}^1$ for $y \in \overline{\mathcal{M}}_{0,q} \setminus \mathcal{Z}_q$ and that $\varpi_q \circ g(z) = b \circ \pi(z)$ (cf. (1.6.4)). By the assumption $g_{\langle v' \rangle}(G) \subset E_v^{v'}$, we have

$$\varphi_{\langle v' \rangle} \circ g(z) \in E_v^{v'}. \tag{4.3.10}$$

On the other hand, we have

$$\varphi_{\langle v' \rangle} \circ \sigma_{\iota_v(v')} \circ b \circ \pi(z) \notin E_v^{v'}. \tag{4.3.11}$$

To see this, we note that $\varphi_{\langle v' \rangle} \circ \sigma_{\iota_v(v')}(\overline{V}_x) \subset E_{\hat{\tau}_{v'}(\iota_v(v'))}^{v'}$ by the definition of V_x . Hence we have $\varphi_{\langle v' \rangle} \circ \sigma_{\iota_v(v')}(y) \notin E_v^{v'}$ for $y \in \overline{V}_x$ (cf. (4.1.3)). Since $b \circ \pi(z) \in V_x$, we get (4.3.11). The relations (4.3.9), (4.3.10) and (4.3.11) give a contradiction. Hence we have $b \circ \pi(z) \in \text{supp } \mathcal{Z}_q$. This proves (4.3.7) and (4.3.4). \square

Proof of (4.3.5). Let $G \in \mathcal{G}_{v,i}^I$, $i \in P_v^m$. Since $-\varrho(G) \leq 1$, we have

$$-\sum_{v \in \text{vert}(\Gamma_x)} \sum_{i \in P_v^m} \sum_{G \in \mathcal{G}_{v,i}^I} \varrho(G) \leq \sum_{v \in \text{vert}(\Gamma_x)} \sum_{i \in P_v^m} \text{card } \mathcal{G}_{v,i}^I.$$

By the definition of V_x , we may apply Lemma 9 to the case

$$E = E_i^v, \quad \Omega = G, \quad \zeta = \varphi_{\langle v \rangle} \circ g, \quad \alpha = \beta_{v,i} \circ b \circ \pi (= \varphi_{\langle v \rangle} \circ \sigma_i \circ b \circ \pi),$$

to conclude that there exists $z \in G$ such that

$$\varphi_{\langle v \rangle} \circ g(z) = \varphi_{\langle v \rangle} \circ \sigma_i \circ b \circ \pi(z).$$

This implies that either $g(z) = \sigma_i \circ b \circ \pi(z)$ or $b \circ \pi(z) \in \text{supp } \mathcal{Z}_q$. (Note that $\varphi_{\langle v \rangle}|_{\mathcal{C}_y}$ is an isomorphism for $y \in \overline{\mathcal{M}}_{0,q} \setminus \mathcal{Z}_q$.) Hence for $i \in P_v^m$, we have

$$\text{card } \mathcal{G}_{v,i}^I \leq \bar{n}(g, \mathcal{D}_{q,i}, F) + (\deg \pi) \bar{n}(b, \mathcal{Z}_q, R),$$

where we put $\mathcal{D}_{q,i} = \sigma_i(\overline{\mathcal{M}}_{0,q}) \subset \overline{\mathcal{U}}_{0,q}$. Since we have $\mathcal{D}_q = \sum_{i=1}^q \mathcal{D}_{q,i}$, $\mathcal{D}_{q,i} \cap \mathcal{D}_{q,i'} = \emptyset$ for $i \neq i'$, $\bigcup_{v \in \text{vert}(\Gamma_x)} P_v^m = (q)$ and $P_v^m \cap P_{v'}^m = \emptyset$ for $v \neq v'$, we obtain

$$\sum_{v \in \text{vert}(\Gamma_x)} \sum_{i \in P_v^m} \text{card } \mathcal{G}_{v,i}^I \leq \bar{n}(g, \mathcal{D}_q, F) + q(\deg \pi) \bar{n}(b, \mathcal{Z}_q, R).$$

This proves (4.3.5). \square

Proof of (4.3.6). For $H \in \mathcal{C}(F)$, the restriction $\pi|_H: H \rightarrow R$ is a proper map. Hence, by Hurwitz's formula, we have

$$\varrho(H) = (\deg \pi|_H) \varrho(R) + \text{disc}(\pi|_H, R).$$

Since $\varrho(R) \leq \varrho(H)$, we get

$$\varrho^+(H) \leq (\deg \pi|_H)\varrho^+(R) + \text{disc}(\pi|_H, R).$$

Adding over all $H \in \mathcal{C}(F)$, we obtain

$$\sum_{H \in \mathcal{C}(F)} \varrho^+(H) \leq \varrho^+(R) \sum_{H \in \mathcal{C}(F)} \deg \pi|_H + \sum_{H \in \mathcal{C}(F)} \text{disc}(\pi|_H, R) = \varrho^+(R) \deg \pi + \text{disc}(\pi, R).$$

This proves (4.3.6) and concludes our proof of the claim. □

Next we will estimate the first term on the right-hand side of (4.3.3).

Note that the Fubini–Study form $\omega_{\mathbf{P}^1}$ is the curvature form of the Fubini–Study metric on the hyperplane section bundle \mathcal{L} . Hence by Lemma 10, the restriction of the $(1, 1)$ -form

$$\sum_{v \in \text{vert}(\Gamma_x)} (d_v - 2)\varphi_{(v)}^* \omega_{\mathbf{P}^1} - \varkappa_q$$

to $\varpi_q^{-1}(U_x)$ is a curvature form of the trivial line bundle. Hence, there exists a C^∞ -function ξ on $\varpi_q^{-1}(U_x)$ such that

$$\sum_{v \in \text{vert}(\Gamma_x)} (d_v - 2)\varphi_{(v)}^* \omega_{\mathbf{P}^1} - \varkappa_q = dd^c \xi \quad \text{on } \varpi_q^{-1}(U_x). \tag{4.3.12}$$

By Stokes’s theorem, we have

$$|A(g, F, dd^c \xi)| = \left| \int_F g^* dd^c \xi \right| = \left| \int_{\partial F} g^* d^c \xi \right|. \tag{4.3.13}$$

There exists a positive constant $h'' > 0$ which does not depend on the choices of $\lambda \in \Lambda$ and R , such that

$$\left| \int_{\partial F} g^* d^c \xi \right| \leq h'' l(g, \partial F, \omega_q), \tag{4.3.14}$$

because the image $g(\bar{F})$ is contained in the compact set $\varpi_q^{-1}(\bar{V}_x)$. Hence using (4.3.12), (4.3.13) and (4.3.14), we get

$$\sum_{v \in \text{vert} \Gamma_x} (d_v - 2)A(g_{(v)}, F, \omega_{\mathbf{P}^1}) \geq A(g, F, \varkappa_q) - h'' l(g, \partial F, \omega_q). \tag{4.3.15}$$

Put $h_x = \max\{h' + h'', 2 \text{card}(\text{vert}(\Gamma_x)) + q - 2\}$, which is a positive constant independent of the choices of $\lambda \in \Lambda$ and R . Using (4.3.3)–(4.3.6) and (4.3.15), we obtain (4.3.1) and conclude the proof of Lemma 11. □

5. Lemmas for division and summation

5.1. An algebraic lemma

Put $\Phi = (\phi_i)_{i \in \mathcal{I}}: \overline{\mathcal{M}}_{0,q} \rightarrow (\mathbf{P}^1)^{\mathcal{I}}$.

LEMMA 12. Φ is an injection.

Proof. We prove this by induction on q . For $q=3$, our lemma is trivial because $\overline{\mathcal{M}}_{0,3}$ is isomorphic to a point. Suppose that our lemma is valid for all q' with $q' \leq q$, where $q \geq 3$. We shall prove our lemma for $q+1$.

Our lemma is equivalent to saying that for distinct points $x, y \in \overline{\mathcal{M}}_{0,q+1}$, there exists $i \in \mathcal{I}^{q+1}$ such that $\phi_i(x) \neq \phi_i(y)$. Let $u_{q+1}: \overline{\mathcal{M}}_{0,q+1} \rightarrow \overline{\mathcal{M}}_{0,q}$ be the morphism obtained by forgetting the marking σ_{q+1} (cf. (1.5.12)). In the case that $u_{q+1}(x)$ and $u_{q+1}(y)$ are distinct points in $\overline{\mathcal{M}}_{0,q}$, our lemma follows from the induction hypothesis.

In the other case, put $z = u_{q+1}(x)$. Using the isomorphism $\iota_{q+1}: \overline{\mathcal{M}}_{0,q+1} \rightarrow \overline{\mathcal{U}}_{0,q}$, the fiber $u_{q+1}^{-1}(z)$ is isomorphic to \mathcal{C}_z (cf. (1.5.12)).

We first consider the case when $\iota_{q+1}(x)$ is a smooth point of \mathcal{C}_z . Let $v \in \text{vert}(\Gamma_z)$ be the unique vertex such that $\iota_{q+1}(x) \in C_v$. Then since $\varphi_{\langle v \rangle}|_{C_v}: C_v \rightarrow \mathbf{P}^1$ is an isomorphism and $\varphi_{\langle v \rangle}|_{C_{v'}}$ is constant for $v' \in \text{vert}(\Gamma_x) \setminus \{v\}$, we have $\varphi_{\langle v \rangle}(\iota_{q+1}(x)) \neq \varphi_{\langle v \rangle}(\iota_{q+1}(y))$ as desired. (By definition, we may take $i \in \mathcal{I}^{q+1}$ with $\phi_i = \varphi_{\langle v \rangle} \circ \iota_{q+1}$.)

Next we consider the case when $\iota_{q+1}(x)$ is not a smooth point of \mathcal{C}_z . Then $\iota_{q+1}(x)$ is a node. There are adjacent vertices v and v' such that $\iota_{q+1}(x) = C_v \cap C_{v'}$. If $\varphi_{\langle v \rangle}(\iota_{q+1}(x)) \neq \varphi_{\langle v \rangle}(\iota_{q+1}(y))$, the proof is done. If $\varphi_{\langle v \rangle}(\iota_{q+1}(x)) = \varphi_{\langle v \rangle}(\iota_{q+1}(y))$, then we can easily see that $\varphi_{\langle v' \rangle}(\iota_{q+1}(x)) \neq \varphi_{\langle v' \rangle}(\iota_{q+1}(y))$, which proves our lemma for $q+1$. □

5.2. Estimates for summation

Let $\lambda = (\mathcal{I}, \mathcal{R}, \pi, g, b, F, R)$ be a specified q -hol-quintet. For $i \in \mathcal{I}$, put

$$b_i = \phi_i \circ b: \mathcal{R} \rightarrow \mathbf{P}^1$$

and

$$\mathcal{I}_\lambda = \{i \in \mathcal{I} : b_i \text{ is non-constant}\}.$$

Definition 5.2.1. We call \mathcal{I}_λ the *type* of the specified q -hol-quintet λ .

Let $\widehat{\mathcal{I}} \subset \mathcal{I}^q$ be a subset. Let $\mathfrak{D} = \{D_i\}_{i \in \widehat{\mathcal{I}}}$ be an $\widehat{\mathcal{I}}$ -tuple of Jordan domains $D_i \subset \mathbf{P}^1$. Let $\mathfrak{D}' = \{D'_i\}_{i \in \widehat{\mathcal{I}}}$ be another such tuple. We say that \mathfrak{D}' is compactly contained in \mathfrak{D} if all D'_i are compactly contained in D_i . We also write $\mathfrak{D}' \subset \mathfrak{D}$ if $D'_i \subset D_i$ for all $i \in \widehat{\mathcal{I}}$. Let

$\lambda=(\mathcal{F}, \mathcal{R}, \pi, g, b, F, R)$ be a specified q -hol-quintet of type $\widehat{\mathcal{F}}$. We consider the following condition for $\{b_i\}_{i \in \widehat{\mathcal{F}}}$ and $\{D_i\}_{i \in \widehat{\mathcal{F}}}$:

$$b_i|_{\bar{R}}: \bar{R} \rightarrow \mathbf{P}^1 \text{ and } D_i \text{ satisfy the condition (3.2.1) for all } i \in \widehat{\mathcal{F}}. \quad (5.2.2)$$

Put $R_{\mathcal{D}}=R \cap \bigcap_{i \in \widehat{\mathcal{F}}} b_i^{-1}(D_i)$ and $F_{\mathcal{D}}=\pi^{-1}(R_{\mathcal{D}})$. Then by Lemma 6, $R_{\mathcal{D}}$ (resp. $F_{\mathcal{D}}$) is a finite disjoint union of finite domains on \mathcal{R} (resp. \mathcal{F}), because Jordan domains D_i are finite domains (cf. §1.3).

LEMMA 13. (1) Let $\widehat{\mathcal{F}} \subset \mathcal{F}^q$ be a subset. Suppose that $\mathcal{D}'=\{D'_i\}_{i \in \widehat{\mathcal{F}}}$ is compactly contained in $\mathcal{D}=\{D_i\}_{i \in \widehat{\mathcal{F}}}$. Then for all $\varepsilon > 0$, there exists a positive constant $\mu_1=\mu_1(\varepsilon, \widehat{\mathcal{F}}, \mathcal{D}, \mathcal{D}')$ with the following property: Let $(\mathcal{F}, \mathcal{R}, \pi, g, b, F, R)$ be a specified q -hol-quintet of type $\widehat{\mathcal{F}}$ such that the inequality

$$A(g, F_{\mathcal{D}'}, \omega_q) > \mu_1(\deg \pi)(A(b, R, \eta_q) + l(g, \partial F, \omega_q)) \quad (5.2.3)$$

holds. Then there exists an $\widehat{\mathcal{F}}$ -tuple of Jordan domains $\mathcal{D}''=\{D''_i\}_{i \in \widehat{\mathcal{F}}}$ with $\mathcal{D}' \subset \mathcal{D}'' \subset \mathcal{D}$ such that we have the inequality

$$l(g, \partial F_{\mathcal{D}'}, \omega_q) \leq \varepsilon A(g, F_{\mathcal{D}'}, \omega_q) + l(g, \partial F, \omega_q).$$

Moreover, we may take \mathcal{D}'' such that $(b_i)_{i \in \widehat{\mathcal{F}}}$ and \mathcal{D}'' satisfy the condition (5.2.2).

(2) Let $\widehat{\mathcal{F}}, \mathcal{D}'$ and \mathcal{D} be the same as in (1). Then there exists a positive constant $\mu_2=\mu_2(\widehat{\mathcal{F}}, \mathcal{D}, \mathcal{D}') > 0$ with the following property: Let $(\mathcal{F}, \mathcal{R}, \pi, g, b, F, R)$ be a specified q -hol-quintet of type $\widehat{\mathcal{F}}$. Let \mathcal{D}'' be an $\widehat{\mathcal{F}}$ -tuple of Jordan domains such that

$$\mathcal{D}' \subset \mathcal{D}'' \subset \mathcal{D}. \quad (5.2.4)$$

Suppose that $(b_i)_{i \in \widehat{\mathcal{F}}}$ and \mathcal{D}'' satisfy the condition (5.2.2). Then we have

$$\sum_{G \in \mathcal{C}(R_{\mathcal{D}''})} \varrho^+(G) \leq \varrho^+(R) + \mu_2(A(b, R, \eta_q) + l(g, \partial F, \omega_q)).$$

Here we recall that $\mathcal{C}(R_{\mathcal{D}''})$ is the set of connected components of $R_{\mathcal{D}''}$.

Remark 5.2.5. If $\widehat{\mathcal{F}}=\emptyset$, then $R_{\mathcal{D}}=R$ and $F_{\mathcal{D}}=F$ for an $\widehat{\mathcal{F}}$ -tuple of Jordan domains \mathcal{D} . Hence the assertions of the lemma are trivial in this case. In the following, we consider the case $\widehat{\mathcal{F}} \neq \emptyset$.

Proof of Lemma 13 (1). For $i \in \widehat{\mathcal{F}}$, we fix a biholomorphic identification $\chi_i: D_i \xrightarrow{\sim} \Delta$. Put $D_i(r)=\chi_i^{-1}(\Delta(r))$ for $0 \leq r \leq 1$. Here $\Delta(r)=\{z \in \mathbf{C}: |z| < r\}$ and $\Delta=\Delta(1)$. Let $r_0 < 1$ be a constant such that $D'_i \subset D_i(r_0)$ for all $i \in \widehat{\mathcal{F}}$.

By replacing D_i by $D_i(s)$ and Δ by $\Delta(s)$ for $r_0 < s < 1$, we may assume that χ_i gives a biholomorphic map between neighborhoods of \bar{D}_i and $\bar{\Delta}$. In particular, we may assume that ∂D_i is regular and analytic for all $i \in \widehat{\mathcal{F}}$.

We fix a specified q -hol-quintet $\lambda = (\mathcal{F}, \mathcal{R}, \pi, g, b, F, R)$ of type $\widehat{\mathcal{F}}$. For $i \in \widehat{\mathcal{F}}$, put

$$\xi_i = b_i \circ \pi|_{\bar{F}}: \bar{F} \rightarrow \mathbf{P}^1, \quad F_i = \xi_i^{-1}(D_i) \cap F \quad \text{and} \quad \zeta_i = \chi_i \circ \xi_i|_{\bar{F}_i}: \bar{F}_i \rightarrow \bar{\Delta}.$$

For $0 < r \leq 1$, put

$$\gamma_i(r) = \xi_i^{-1}(\partial D_i(r)) \cap F \subset \bar{F}_i.$$

Let ω_E be the Euclidean form on $\Delta \subset \mathbf{C}$, which is a Kähler form. Put $S_i = A(\xi_i, F, \omega_{\mathbf{P}^1})$ and $L_i = l(\xi_i, \partial F, \omega_{\mathbf{P}^1})$, which are the mean sheet number and the length of the relative boundary of $\xi_i: F \rightarrow \mathbf{P}^1$, respectively.

CLAIM 1. *There exists a positive constant $Q_1 = Q_1(\widehat{\mathcal{F}}, \mathfrak{D}, \mathfrak{D}')$ which does not depend on the choice of λ , such that*

$$l(\zeta_i, \gamma_i(r), \omega_E) \leq Q_1(S_i + L_i) \quad \text{for } i \in \widehat{\mathcal{F}} \text{ and } r \in [r_0, 1]. \tag{5.2.6}$$

A proof of this claim will be given later.

Now we will find the constant μ_1 . We take a positive constant $Q_2 = Q_2(\widehat{\mathcal{F}})$ which does not depend on the choice of λ and satisfies the estimates

$$\sum_{i \in \widehat{\mathcal{F}}} S_i = (\deg \pi) \sum_{i \in \widehat{\mathcal{F}}} A(b_i, R, \omega_{\mathbf{P}^1}) \leq Q_2(\deg \pi) A(b, R, \eta_q) \tag{5.2.7}$$

and

$$\sum_{i \in \widehat{\mathcal{F}}} L_i = \sum_{i \in \widehat{\mathcal{F}}} l(\xi_i, \partial F, \omega_{\mathbf{P}^1}) \leq Q_2 l(g, \partial F, \omega_q) \leq Q_2(\deg \pi) l(g, \partial F, \omega_q). \tag{5.2.8}$$

(We note the trivial estimate $1 \leq \deg \pi$.) Let $\varepsilon > 0$ be an arbitrary positive constant. Put

$$\mu_1 = \frac{2Q_1Q_2}{\varepsilon^2(1-r_0)}. \tag{5.2.9}$$

Then μ_1 is a positive constant which only depends on ε , $\widehat{\mathcal{F}}$, \mathfrak{D} and \mathfrak{D}' , and does not depend on the choice of λ .

Next we will find \mathfrak{D}'' . For $r \in [0, 1]$, put

$$\begin{aligned} \mathfrak{D}(r) &= \{D_i(r)\}_{i \in \widehat{\mathcal{F}}}, & A(r) &= A(g, F_{\mathfrak{D}(r)}, \omega_q), \\ \partial' F_{\mathfrak{D}(r)} &= \partial F_{\mathfrak{D}(r)} \setminus (\partial F \cap \partial F_{\mathfrak{D}(r)}), & l(r) &= l(g, \partial' F_{\mathfrak{D}(r)}, \omega_q). \end{aligned}$$

Define a subset $E(\varepsilon) \subset [r_0, 1]$ by

$$r \in E(\varepsilon) \stackrel{\text{def}}{\iff} l(r) > \varepsilon A(r).$$

CLAIM 2. *Suppose that the inequality (5.2.3) holds for λ . Then the set $[r_0, 1] \setminus E(\varepsilon)$ is not a null set. Here a null set is a set of Lebesgue measure zero.*

Before proving this claim, we will complete the proof of (1) of our lemma. Note that the set

$$\{r \in [r_0, 1] : (b_i)_{i \in \widehat{\mathcal{F}}} \text{ and } \mathfrak{D}(r) \text{ do not satisfy the condition (5.2.2)}\}$$

is a finite set, and so a null set. Suppose that the condition (5.2.3) holds for λ . Then by Claim 2, we may take $r \in [r_0, 1]$ such that $(b_i)_{i \in \widehat{\mathcal{F}}}$ and $\mathfrak{D}(r)$ satisfy the condition (5.2.2), and such that the inequality

$$l(r) \leq \varepsilon A(r)$$

holds. Since $l(g, \partial F_{\mathfrak{D}(r)}, \omega_q) \leq l(r) + l(g, \partial F, \omega_q)$, we have

$$l(g, \partial F_{\mathfrak{D}(r)}, \omega_q) \leq \varepsilon A(g, F_{\mathfrak{D}(r)}, \omega_q) + l(g, \partial F, \omega_q).$$

Put $\mathfrak{D}'' = \mathfrak{D}(r)$, which proves (1) of our lemma. □

Now we prove the claims above to conclude the proof.

Proof of Claim 1. In this proof, we denote by \mathcal{Q} any positive constant which is independent of $i \in \widehat{\mathcal{F}}$, $r \in [r_0, 1]$ and the choice of λ .

For $0 < r \leq 1$ and $i \in \widehat{\mathcal{F}}$, put $F_i(r) = \xi_i^{-1}(D_i(r)) \cap F$ and

$$\xi_{i,r} = \xi_i|_{\overline{F_i(r)}} : \overline{F_i(r)} \rightarrow \overline{D_i(r)}.$$

Define the map $\psi_r : \overline{D_i(r)} \rightarrow \overline{D_i}$ by

$$\overline{D_i(r)} \ni z \mapsto \chi_i^{-1}\left(\frac{\chi_i(z)}{r}\right) \in \overline{D_i}.$$

Let $S_{i,r}$ be the mean sheet number and $L_{i,r}$ be the length of the relative boundary of the covering $\xi_{i,r} : \overline{F_i(r)} \rightarrow \overline{D_i(r)}$. Let $S'_{i,r}$ be the mean sheet number and $L'_{i,r}$ be the length of the relative boundary of the covering $\psi_r \circ \xi_{i,r} : \overline{F_i(r)} \rightarrow \overline{D_i}$. Since we have

$$\frac{1}{\mathcal{Q}} \psi_r^*(\omega_{\mathbf{P}^1}|_{\overline{D_i}}) < \omega_{\mathbf{P}^1}|_{\overline{D_i(r)}} < \mathcal{Q} \psi_r^*(\omega_{\mathbf{P}^1}|_{\overline{D_i}}) \quad \text{for } i \in \widehat{\mathcal{F}} \text{ and } r \in [r_0, 1], \tag{5.2.10}$$

we get

$$l(\xi_{i,r}, \gamma_i(r), \omega_{\mathbf{P}^1}) \leq \mathcal{Q} l(\psi_r \circ \xi_{i,r}, \gamma_i(r), \omega_{\mathbf{P}^1}) \quad \text{for } i \in \widehat{\mathcal{F}} \text{ and } r \in [r_0, 1]. \tag{5.2.11}$$

Here we note that $\gamma_i(r) \subset \partial F_i(r)$. Since $\psi_r \circ \xi_{i,r}(\gamma_i(r)) \subset \partial D_i$, we may apply Covering theorem 2 (cf. (3.3.2)) to the covering $\psi_r \circ \xi_{i,r}: \overline{F_i(r)} \rightarrow \overline{D_i}$ to get

$$l(\psi_r \circ \xi_{i,r}, \gamma_i(r), \omega_{\mathbf{P}^1}) \leq \mathcal{Q}(S'_{i,r} + L'_{i,r}) \quad \text{for } i \in \widehat{\mathcal{F}} \text{ and } r \in [r_0, 1]. \tag{5.2.12}$$

Here we note that ∂D_i is regular and analytic for $i \in \widehat{\mathcal{F}}$ by the assumption made in the beginning of the proof of this lemma. By (5.2.10), we have

$$S'_{i,r} \leq \mathcal{Q}S_{i,r} \text{ and } L'_{i,r} \leq \mathcal{Q}L_{i,r} \quad \text{for } i \in \widehat{\mathcal{F}} \text{ and } r \in [r_0, 1].$$

Hence combining with (5.2.11) and (5.2.12), we obtain

$$l(\xi_{i,r}, \gamma_i(r), \omega_{\mathbf{P}^1}) \leq \mathcal{Q}(S_{i,r} + L_{i,r}) \quad \text{for } i \in \widehat{\mathcal{F}} \text{ and } r \in [r_0, 1].$$

Since $\chi_i^* \omega_E \leq \mathcal{Q} \omega_{\mathbf{P}^1}|_{\overline{D_i}}$ and $\chi_i \circ \xi_{i,r} = \zeta_i|_{\overline{F_i(r)}}$, we have

$$l(\zeta_i, \gamma_i(r), \omega_E) \leq \mathcal{Q}l(\xi_{i,r}, \gamma_i(r), \omega_{\mathbf{P}^1}),$$

and hence

$$l(\zeta_i, \gamma_i(r), \omega_E) \leq \mathcal{Q}(S_{i,r} + L_{i,r}) \quad \text{for } i \in \widehat{\mathcal{F}} \text{ and } r \in [r_0, 1].$$

We have $S_{i,r} \leq \mathcal{Q}(S_i + L_i)$ for $r_0 \leq r \leq 1$ by Covering theorem 1 (cf. (3.3.1)). Using that $L_{i,r} \leq L_i$, we obtain

$$l(\zeta_i, \gamma_i(r), \omega_E) \leq \mathcal{Q}(S_i + L_i) \quad \text{for } i \in \widehat{\mathcal{F}} \text{ and } r \in [r_0, 1].$$

This proves our claim. □

Proof of Claim 2. We shall also denote the restriction $\zeta_i|_{F_{\mathcal{D}}}$ by ζ_i . For $H \in \mathcal{C}(F_{\mathcal{D}})$, we take a subset $I_H \subset \widehat{\mathcal{F}}$ with the following properties:

- (1) If $i \in I_H$ and $i' \in I_H$ are distinct, then $|\zeta_i|$ and $|\zeta_{i'}|$ are distinct functions on H ;
- (2) For all $i \in \widehat{\mathcal{F}}$ there exists $i' \in I_H$ such that $|\zeta_i|$ and $|\zeta_{i'}|$ are the same function on H .

For $H \in \mathcal{C}(F_{\mathcal{D}})$, $i \in I_H$ and $r \in [0, 1]$, put

$$\begin{aligned} \Omega_{H,i} &= \{z \in H : |\zeta_i(z)| > |\zeta_{i'}(z)| \text{ for all } i' \in I_H \setminus \{i\}\}, \\ \Omega_{H,i}(r) &= \{z \in \Omega_{H,i} : |\zeta_i(z)| < r\}, \\ \widehat{\gamma}_{H,i}(r) &= \overline{\Omega_{H,i}} \cap \gamma_i(r) \end{aligned}$$

and

$$l_{H,i}(r) = l(g, \widehat{\gamma}_{H,i}(r), \omega_q), \quad A_{H,i}(r) = A(g, \Omega_{H,i}(r), \omega_q).$$

Then by the above definitions, we have

$$A(r) = \sum_{H \in \mathcal{C}(F_{\mathfrak{D}})} \sum_{i \in I_H} A_{H,i}(r) \quad \text{and} \quad l(r) \leq \sum_{H \in \mathcal{C}(F_{\mathfrak{D}})} \sum_{i \in I_H} l_{H,i}(r). \quad (5.2.13)$$

To see these estimates, we observe that

$$\bigcup_{H \in \mathcal{C}(F_{\mathfrak{D}})} \bigcup_{i \in I_H} \Omega_{H,i}(r) \subset F_{\mathfrak{D}(r)} \subset \bigcup_{H \in \mathcal{C}(F_{\mathfrak{D}})} \bigcup_{i \in I_H} \overline{\Omega_{H,i}(r)} \quad (5.2.14)$$

(note that $\widehat{\mathcal{F}} \neq \emptyset$), where $\Omega_{H,i}(r) \cap \Omega_{H',i'}(r) \neq \emptyset$ if and only if $H = H'$ and $i = i'$. From this fact, we immediately obtain the first estimate. For the second estimate, we observe that $|\zeta_i(z)| = r$ on $z \in \bar{\Omega}_{H,i} \cap \partial' F_{\mathfrak{D}(r)}$, and hence $\bar{\Omega}_{H,i} \cap \partial' F_{\mathfrak{D}(r)} \subset \widehat{\gamma}_{H,i}(r)$. By (5.2.14), we have

$$\partial' F_{\mathfrak{D}(r)} \subset \bigcup_{H \in \mathcal{C}(F_{\mathfrak{D}})} \bigcup_{i \in I_H} (\bar{\Omega}_{H,i} \cap \partial' F_{\mathfrak{D}(r)}) \subset \bigcup_{H \in \mathcal{C}(F_{\mathfrak{D}})} \bigcup_{i \in I_H} \widehat{\gamma}_{H,i}(r).$$

Hence, we obtain the second estimate.

Now we will use the length–area principle. For $H \in \mathcal{C}(F_{\mathfrak{D}})$ and $i \in I_H$, put

$$g^*(\omega_q)|_{\bar{\Omega}_{H,i}} = \frac{1}{2} \sqrt{-1} G_{H,i} d\zeta_i \wedge d\bar{\zeta}_i,$$

where $G_{H,i}$ is a C^∞ -function on $\bar{\Omega}_{H,i} \setminus \{z \in \bar{\Omega}_{H,i} : \zeta'_i(z) = 0\}$ with $G_{H,i} \geq 0$. Then for $r \in (0, 1]$, we have

$$l_{H,i}(r) = \int_{\widehat{\gamma}_{H,i}(r)} \sqrt{G_{H,i}} r \, d \arg \zeta_i$$

and

$$A_{H,i}(r) = \int_0^r \left(\int_{\widehat{\gamma}_{H,i}(t)} G_{H,i} t \, d \arg \zeta_i \right) dt.$$

Using (5.2.6), (5.2.13) and the Schwarz inequality, we have

$$\begin{aligned} l(r)^2 &\leq \left(\sum_{H \in \mathcal{C}(F_{\mathfrak{D}})} \sum_{i \in I_H} l_{H,i}(r) \right)^2 \\ &= \left(\sum_{H \in \mathcal{C}(F_{\mathfrak{D}})} \sum_{i \in I_H} \int_{\widehat{\gamma}_{H,i}(r)} \sqrt{G_{H,i}} r \, d \arg \zeta_i \right)^2 \\ &\leq \left(\sum_{H \in \mathcal{C}(F_{\mathfrak{D}})} \sum_{i \in I_H} \int_{\widehat{\gamma}_{H,i}(r)} r \, d \arg \zeta_i \right) \sum_{H \in \mathcal{C}(F_{\mathfrak{D}})} \sum_{i \in I_H} \int_{\widehat{\gamma}_{H,i}(r)} G_{H,i} r \, d \arg \zeta_i \\ &= \left(\sum_{H \in \mathcal{C}(F_{\mathfrak{D}})} \sum_{i \in I_H} l(\zeta_i, \widehat{\gamma}_{H,i}(r), \omega_E) \right) \sum_{H \in \mathcal{C}(F_{\mathfrak{D}})} \sum_{i \in I_H} \frac{d}{dr} A_{H,i}(r) \end{aligned}$$

$$\begin{aligned} &\leq \left(\sum_{H \in \mathcal{C}(F_{\mathfrak{D}})} \sum_{i \in \widehat{\mathcal{F}}} l(\zeta_i, \gamma_i(r) \cap \overline{H}, \omega_E) \right) \frac{d}{dr} A(r) \\ &\leq Q_1 \sum_{i \in \widehat{\mathcal{F}}} (S_i + L_i) \frac{d}{dr} A(r) \end{aligned}$$

for a.e. $r \in [r_0, 1]$.

Now, suppose that the set $[r_0, 1] \setminus E(\varepsilon)$ is a null set. Then using (5.2.3) and (5.2.7)–(5.2.9), we have

$$\begin{aligned} 1 - r_0 &= \int_{E(\varepsilon)} dr \\ &\leq Q_1 \sum_{i \in \widehat{\mathcal{F}}} (S_i + L_i) \int_{E(\varepsilon)} \left(\frac{d}{dr} A(r) \right) \frac{1}{l(r)^2} dr \\ &\leq \frac{Q_1 \sum_{i \in \widehat{\mathcal{F}}} (S_i + L_i)}{\varepsilon^2} \int_{r_0}^1 \left(\frac{d}{dr} A(r) \right) \frac{1}{A(r)^2} dr \\ &\leq \frac{Q_1 \sum_{i \in \widehat{\mathcal{F}}} (S_i + L_i)}{\varepsilon^2 A(r_0)} \\ &\leq \frac{Q_1 Q_2}{\varepsilon^2 A(r_0)} (\deg \pi)(A(b, R, \eta_q) + l(g, \partial F, \omega_q)) \\ &\leq \frac{Q_1 Q_2}{\varepsilon^2 \mu_1} \frac{A(g, F_{\mathfrak{D}'}, \omega_q)}{A(r_0)} \\ &\leq \frac{1}{2}(1 - r_0), \end{aligned}$$

which is a contradiction because $r_0 < 1$. This proves our claim and concludes the proof of (1) of our lemma. \square

Proof of Lemma 13 (2). Let $\lambda = (\mathcal{F}, \mathcal{R}, \pi, g, b, F, R)$ be a specified q -hol-quintet of type $\widehat{\mathcal{F}}$, and let \mathfrak{D}'' be an $\widehat{\mathcal{F}}$ -tuple of Jordan domains which satisfies (5.2.4). We also assume the condition (5.2.2) for $(b_i)_{i \in \widehat{\mathcal{F}}}$ and \mathfrak{D}'' . In this proof, we denote by \mathcal{Q} any positive constant which only depends on \mathfrak{D} , \mathfrak{D}' and $\widehat{\mathcal{F}}$, and does not depend on the choices of λ and \mathfrak{D}'' . We shall prove

$$\sum_{G \in \mathcal{C}(R_{\mathfrak{D}''})} \varrho^+(G) \leq \varrho^+(R) + \mathcal{Q}(A(b, R, \eta_q) + l(g, \partial F, \omega_q)), \tag{5.2.15}$$

which proves our lemma.

For a subset $I \subset \widehat{\mathcal{F}}$, put

$$R_I = R \cap \bigcap_{i \in I} b_i^{-1}(D_i'') \quad \text{and} \quad F_I = \pi^{-1}(R_I).$$

If $I \neq \widehat{\mathcal{F}}$, take $i \in \widehat{\mathcal{F}}$ with $i \notin I$, and put

$$\mathcal{I}_{i,I} = \mathcal{I}(b_i^{-1}(D_i''), R_I), \quad \mathcal{I}'_{i,I} = \mathcal{I}(b_i^{-1}(\mathbf{P}^1 \setminus \bar{D}_i''), R_I) \quad \text{and} \quad \mathcal{P}_{i,I} = \mathcal{P}(b_i^{-1}(D_i''), R_I).$$

For $H \in \mathcal{C}(R_I)$, we apply Lemma 7 to the case $\Omega = H$, $\zeta = b_i$ and $\gamma_1 = \partial D_i''$ (cf. (5.2.2)). Adding over all $H \in \mathcal{C}(R_I)$, we obtain

$$\sum_{H \in \mathcal{C}(R_I)} \varrho^+(H) \geq \sum_{H \in \mathcal{I}_{i,I}} \varrho(H) + \sum_{H \in \mathcal{I}'_{i,I}} \varrho(H) + \sum_{H \in \mathcal{P}_{i,I}} \varrho^+(H). \tag{5.2.16}$$

Let $S_{D_i'}$ be the mean sheet number of $b_i: R \rightarrow \mathbf{P}^1$ over $D_i' \subset \mathbf{P}^1$. Then we have

$$\sum_{H \in \mathcal{I}_{i,I}} \varrho^+(H) - \sum_{H \in \mathcal{I}_{i,I}} \varrho(H) \leq \text{card}(\mathcal{I}_{i,I}) \leq S_{D_i'}$$

(cf. (5.2.4)). Using Covering theorem 1 (cf. (3.3.1)), we get

$$\sum_{H \in \mathcal{I}_{i,I}} \varrho^+(H) - \sum_{H \in \mathcal{I}_{i,I}} \varrho(H) < \mathcal{Q}(A(b_i, R, \omega_{\mathbf{P}^1}) + l(b_i, \partial R, \omega_{\mathbf{P}^1})). \tag{5.2.17}$$

Similarly, we have

$$- \sum_{H \in \mathcal{I}'_{i,I}} \varrho(H) \leq \text{card}(\mathcal{I}'_{i,I}) \leq S_{\mathbf{P}^1 \setminus \bar{D}_i} \leq \mathcal{Q}(A(b_i, R, \omega_{\mathbf{P}^1}) + l(b_i, \partial R, \omega_{\mathbf{P}^1})), \tag{5.2.18}$$

where $S_{\mathbf{P}^1 \setminus \bar{D}_i}$ is the mean sheet number of $b_i: R \rightarrow \mathbf{P}^1$ over $\mathbf{P}^1 \setminus \bar{D}_i \subset \mathbf{P}^1$. Put $I' = I \cup \{i\}$. By (5.2.16)–(5.2.18) and $\mathcal{I}_{i,I} \cup \mathcal{P}_{i,I} = \mathcal{C}(R_{I'})$, we get

$$\sum_{H \in \mathcal{C}(R_{I'})} \varrho^+(H) \leq \sum_{H \in \mathcal{C}(R_I)} \varrho^+(H) + \mathcal{Q}(A(b_i, R, \omega_{\mathbf{P}^1}) + l(b_i, \partial R, \omega_{\mathbf{P}^1})).$$

Using this estimate inductively, we have

$$\sum_{H \in \mathcal{C}(R_{\mathfrak{D}''})} \varrho^+(H) \leq \varrho^+(R) + \mathcal{Q} \sum_{i \in \widehat{\mathcal{F}}} (A(b_i, R, \omega_{\mathbf{P}^1}) + l(b_i, \partial R, \omega_{\mathbf{P}^1})),$$

where we note that $R_{\emptyset} = R$ and $R_{\widehat{\mathcal{F}}} = R_{\mathfrak{D}''}$. By the estimates

$$\sum_{i \in \widehat{\mathcal{F}}} (A(b_i, R, \omega_{\mathbf{P}^1}) + l(b_i, \partial R, \omega_{\mathbf{P}^1})) \leq \mathcal{Q}(A(b, R, \eta_q) + l(b, \partial R, \eta_q))$$

and

$$l(b, \partial R, \eta_q) \leq \mathcal{Q}l(g, \partial F, \omega_q),$$

we obtain (5.2.15), which proves (2) of our lemma. □

6. Conclusion of the proof of Theorem 4

6.1. A weak version of the theorem

We first prove the following result.

CLAIM. Let $\widehat{\mathcal{F}} \subset \mathcal{I}^q$, $q \geq 3$, be a subset. Let Λ be a countable set of non-degenerate specified q -hol-quintets of type $\widehat{\mathcal{F}}$. Then for all $\varepsilon > 0$, there exists a positive constant $C = C(\varepsilon, \widehat{\mathcal{F}}, \Lambda)$ such that

$$A(g, F, \varkappa_q) \leq \bar{n}(g, \mathcal{D}_q, F) + \text{disc}(\pi, R) + \varepsilon A(g, F, \omega_q) + C(\deg \pi)(A(b, R, \eta_q) + \bar{n}(b, \mathcal{L}_q, R) + \varrho^+(R) + l(g, \partial F, \omega_q)) \tag{6.1.1}$$

for all $(\mathcal{F}, \mathcal{R}, \pi, g, b, F, R) \in \Lambda$.

Proof. Recall that we denote by $\text{dist}(x, y)$ the distance between $x, y \in \mathbf{P}^1$ with respect to the Kähler metric associated to the Kähler form $\omega_{\mathbf{P}^1}$. Put

$$\Lambda' = \{(\mathcal{F}, \mathcal{R}, \pi, g, b) : (\mathcal{F}, \mathcal{R}, \pi, g, b, F, R) \in \Lambda\},$$

which is a countable set of non-degenerate q -hol-quintets. For a point $x \in \overline{\mathcal{M}}_{0,q}$ and for $r > 0$, put

$$W_x(r) = \{y \in \overline{\mathcal{M}}_{0,q} : \text{dist}(\phi_i(x), \phi_i(y)) < r \text{ for all } i \in \mathcal{I}\}.$$

By Lemma 12, we may take $r_x > 0$ such that $W_x(r_x) \subset V_x(\Lambda')$ (cf. Lemma 11). Consider the open covering

$$\overline{\mathcal{M}}_{0,q} = \bigcup_{x \in \overline{\mathcal{M}}_{0,q}} W_x(\frac{1}{2}r_x).$$

Since $\overline{\mathcal{M}}_{0,q}$ is compact, we may take a finite set \mathcal{S} of points $x \in \overline{\mathcal{M}}_{0,q}$ such that the open sets $W_x(\frac{1}{2}r_x)$ for these $x \in \mathcal{S}$ give a covering of $\overline{\mathcal{M}}_{0,q}$. Let r_0 be the minimum of $\frac{1}{2}r_x$ for $x \in \mathcal{S}$. Then for all $y \in \overline{\mathcal{M}}_{0,q}$, there exists $x \in \mathcal{S}$ such that

$$W_y(r_0) \subset W_x(r_x) \subset V_x(\Lambda'). \tag{6.1.2}$$

Next, take a finite union of arcs γ on \mathbf{P}^1 which has the following property:

(P) $\mathbf{P}^1 \setminus \gamma$ is a finite disjoint union of Jordan domains $D_\alpha(\gamma)$, $1 \leq \alpha \leq \overline{1}$, such that $\sup_{x,y \in D_\alpha(\gamma)} \text{dist}(x, y) < r_0$.

Let ε be an arbitrary positive constant. Take a positive integer J such that $J > 1/\varepsilon$, and take small deformations $\gamma_1, \dots, \gamma_J$ of γ with the following properties:

- (1) each γ_j , $1 \leq j \leq J$, also satisfies the property (P);
- (2) $\gamma_j \cap \gamma_k \cap \gamma_l = \emptyset$ for $1 \leq j < k < l \leq J$.

Then for each integer j with $1 \leq j \leq J$, we may take a small closed neighborhood δ_j of γ_j with the following property:

(P') $\mathbf{P}^1 \setminus \delta_j$ is a finite disjoint union of Jordan domains $D_1(\delta_j), \dots, D_{\mathfrak{T}}(\delta_j)$, where each $D_\alpha(\delta_j)$, $1 \leq \alpha \leq \mathfrak{T}$, is compactly contained in $D_\alpha(\gamma_j)$.

We also assume that

$$\delta_j \cap \delta_k \cap \delta_l = \emptyset \quad \text{for } 1 \leq j < k < l \leq J. \quad (6.1.3)$$

Put $\mathcal{T} = \{1, \dots, \mathfrak{T}\}^{\widehat{\mathcal{F}}}$. For $\beta = (\beta_i)_{i \in \widehat{\mathcal{F}}} \in \mathcal{T}$ and $1 \leq j \leq J$, put $\mathfrak{D}_{\beta,j} = \{D_{\beta_i}(\gamma_j)\}_{i \in \widehat{\mathcal{F}}}$ and $\mathfrak{D}'_{\beta,j} = \{D_{\beta_i}(\delta_j)\}_{i \in \widehat{\mathcal{F}}}$, which are $\widehat{\mathcal{F}}$ -tuples of Jordan domains. Then $\mathfrak{D}'_{\beta,j}$ is compactly contained in $\mathfrak{D}_{\beta,j}$.

We take a positive constant h such that

$$\begin{aligned} h_y(\Lambda') &< h \quad \text{for all } y \in \mathcal{S} \quad (\text{cf. Lemma 11}), \\ \kappa_q &< h\omega_q \quad \text{on } \overline{\mathcal{U}}_{0,q}, \\ 1 &< h. \end{aligned} \quad (6.1.4)$$

Note that h is independent of the choice of ε . We also take a positive constant μ such that

$$\mu > \mu_1(\varepsilon, \widehat{\mathcal{F}}, \mathfrak{D}_{\beta,j}, \mathfrak{D}'_{\beta,j}) \quad \text{and} \quad \mu > \mu_2(\widehat{\mathcal{F}}, \mathfrak{D}_{\beta,j}, \mathfrak{D}'_{\beta,j}) \quad (6.1.5)$$

for all $\beta \in \mathcal{T}$ and $1 \leq j \leq J$ (cf. Lemma 13).

Take $(\mathcal{F}, \mathcal{R}, \pi, g, b, F, R) \in \Lambda$. We consider the covering

$$\xi_i = b_i \circ \pi|_F: F \rightarrow \mathbf{P}^1 \quad \text{for } i \in \widehat{\mathcal{F}}.$$

Since we have (by (6.1.3))

$$\sum_{j=1}^J A(g, \xi_i^{-1}(\delta_j), \omega_q) \leq 2A(g, F, \omega_q)$$

for all $i \in \widehat{\mathcal{F}}$, we have

$$\sum_{j=1}^J \sum_{i \in \widehat{\mathcal{F}}} A(g, \xi_i^{-1}(\delta_j), \omega_q) \leq 2\theta A(g, F, \omega_q), \quad \theta = \text{card } \widehat{\mathcal{F}}.$$

Hence there exists j , $1 \leq j \leq J$, such that

$$\sum_{i \in \widehat{\mathcal{F}}} A(g, \xi_i^{-1}(\delta_j), \omega_q) \leq \frac{2\theta}{J} A(g, F, \omega_q) \leq 2\varepsilon\theta A(g, F, \omega_q). \quad (6.1.6)$$

For the rest of this proof, we fix this j .

Now we will find \mathfrak{D}''_β with $\mathfrak{D}'_{\beta,j} \subset \mathfrak{D}''_\beta \subset \mathfrak{D}_{\beta,j}$ such that the local version of (6.1.1) is valid on $R_{\mathfrak{D}''_\beta}$.

SUBCLAIM. For each $\beta \in \mathcal{T}$, there exists an $\widehat{\mathcal{J}}$ -tuple of Jordan domains \mathcal{D}''_β which satisfies $\mathcal{D}'_{\beta,j} \subset \mathcal{D}''_\beta \subset \mathcal{D}_{\beta,j}$ and the inequality

$$\begin{aligned} A(g, F_{\mathcal{D}''_\beta}, \kappa_q) &\leq \bar{n}(g, \mathcal{D}_q, F_{\mathcal{D}''_\beta}) + \text{disc}(\pi, R_{\mathcal{D}''_\beta}) \\ &\quad + (\deg \pi) \varrho^+(R) + h\mu(\deg \pi)(A(b, R, \eta_q) + l(g, \partial F, \omega_q)) \\ &\quad + \varepsilon h A(g, F_{\mathcal{D}''_\beta}, \omega_q) + hl(g, \partial F, \omega_q) + h(\deg \pi) \bar{n}(b, \mathcal{L}_q, R_{\mathcal{D}''_\beta}). \end{aligned} \tag{6.1.7}$$

Proof. We first consider the case

$$A(g, F_{\mathcal{D}'_{\beta,j}}, \omega_q) \leq \mu(\deg \pi)(A(b, R, \eta_q) + l(g, \partial F, \omega_q)).$$

Put $\mathcal{D}''_\beta = \mathcal{D}'_{\beta,j}$. Then using (6.1.4), we have

$$A(g, F_{\mathcal{D}''_\beta}, \kappa_q) \leq h A(g, F_{\mathcal{D}''_\beta}, \omega_q) \leq h\mu(\deg \pi)(A(b, R, \eta_q) + l(g, \partial F, \omega_q)).$$

Since all terms on the right-hand side of (6.1.7) are non-negative, we conclude our assertion in this case.

Next we consider the case

$$A(g, F_{\mathcal{D}'_{\beta,j}}, \omega_q) > \mu(\deg \pi)(A(b, R, \eta_q) + l(g, \partial F, \omega_q)).$$

Let \mathcal{D}''_β be the $\widehat{\mathcal{J}}$ -tuple of Jordan domains obtained in Lemma 13 (1) (cf. (6.1.5)). By the property (P) of γ_j , we see that $b(R_{\mathcal{D}''_\beta}) \subset W_{b(z)}(r_0)$ for $z \in R_{\mathcal{D}''_\beta}$. Hence by (6.1.2), we have $b(R_{\mathcal{D}''_\beta}) \subset V_x$ for some $x \in \mathcal{S}$. Hence we may apply Lemma 11 for each connected component $G \in \mathcal{C}(R_{\mathcal{D}''_\beta})$ to get

$$\begin{aligned} A(g, \pi^{-1}(G), \kappa_q) &\leq \bar{n}(g, \mathcal{D}_q, \pi^{-1}(G)) + \text{disc}(\pi, G) + (\deg \pi) \varrho^+(G) \\ &\quad + hl(g, \partial \pi^{-1}(G), \omega_q) + h(\deg \pi) \bar{n}(b, \mathcal{L}_q, G). \end{aligned}$$

Adding over all $G \in \mathcal{C}(R_{\mathcal{D}''_\beta})$ and using the estimates of Lemma 13 (1) and (2), we conclude our assertion. \square

Since $F = \bigcup_{\beta \in \mathcal{T}} F_{\mathcal{D}''_\beta} \cup \bigcup_{i \in \widehat{\mathcal{J}}} \xi_i^{-1}(\delta_j)$ and $F_{\mathcal{D}''_\beta} \cap F_{\mathcal{D}''_{\beta'}} = \emptyset$ for $\beta' \neq \beta$, we have

$$\begin{aligned} A(g, F, \kappa_q) &\leq \sum_{\beta \in \mathcal{T}} A(g, F_{\mathcal{D}''_\beta}, \kappa_q) + h \sum_{i \in \widehat{\mathcal{J}}} A(g, \xi_i^{-1}(\delta_j), \omega_q) \\ &\leq \sum_{\beta \in \mathcal{T}} A(g, F_{\mathcal{D}''_\beta}, \kappa_q) + 2\theta h \varepsilon A(g, F, \omega_q) \end{aligned} \tag{6.1.8}$$

(cf. (6.1.6)). Adding the inequalities (6.1.7) over all $\beta \in \mathcal{T}$, and using the estimate (6.1.8) above and $\text{card } \mathcal{T} = \mathfrak{T}^\theta$, we get

$$\begin{aligned} A(g, F, \varkappa_q) &\leq \bar{n}(g, \mathcal{D}_q, F) + \text{disc}(\pi, R) + (2\theta + 1)h\varepsilon A(g, F, \omega_q) \\ &\quad + \mathfrak{T}^\theta(\deg \pi)\varrho^+(R) + h\mu\mathfrak{T}^\theta(\deg \pi)A(b, R, \eta_q) \\ &\quad + h\mathfrak{T}^\theta(\mu \deg \pi + 1)l(g, \partial F, \omega_q) + h(\deg \pi)\bar{n}(b, \mathcal{L}_q, R). \end{aligned}$$

Note that the constants h , μ , θ and \mathfrak{T} are independent of the choice of $\lambda \in \Lambda$. Using the facts that $\varepsilon > 0$ is arbitrary and that the constant $(2\theta + 1)h$ is independent of the choice of ε , we see that the term $(2\theta + 1)h\varepsilon$ is also an arbitrary positive number. This proves our claim. \square

6.2. The end of the proof

We prove our theorem by contradiction. Suppose that our theorem is not correct. Then there exist $q \geq 3$ and $\varepsilon > 0$ with the following property: For all positive integers k , there exists a non-degenerate specified q -hol-quintet

$$\lambda_k = (\mathcal{F}_k, \mathcal{R}_k, \pi_k, g_k, b_k, F_k, R_k)$$

such that

$$\begin{aligned} A(g_k, F_k, \varkappa_q) &> \bar{n}(g_k, \mathcal{D}_q, F_k) + \text{disc}(\pi_k, R_k) + \varepsilon A(g_k, F_k, \omega_q) \\ &\quad + k(\deg \pi_k)(A(b_k, R_k, \eta_q) + \bar{n}(b_k, \mathcal{L}_q, R_k) \\ &\quad + \varrho^+(R_k) + l(g_k, \partial F_k, \omega_q)). \end{aligned} \tag{6.2.1}$$

Put $\Lambda = \{\lambda_1, \lambda_2, \dots\}$. Replacing Λ by an infinite subset, we may assume that the types of λ_k are all the same $\widehat{\mathcal{F}} \subset \mathcal{F}^q$. Using the above claim and (6.2.1), we conclude that

$$kQ_k < C(\varepsilon, \widehat{\mathcal{F}}, \Lambda)Q_k$$

for all positive integers k , where we put

$$Q_k = (\deg \pi_k)(A(b_k, R_k, \eta_q) + \bar{n}(b_k, \mathcal{L}_q, R_k) + \varrho^+(R_k) + l(g_k, \partial F_k, \omega_q)).$$

But this is a contradiction, because we have $Q_k \geq 0$. Hence we obtain our theorem.

7. The proof of Corollary 2

7.1. Preliminaries

We start with the following lemma (see also [NoO, 6.1.5] for the case $Y = \mathbf{C}$).

LEMMA 14. *Let Y be a Riemann surface with a proper, surjective holomorphic map $\pi_Y: Y \rightarrow \mathbf{C}$. Let $F(x) = \sum_{i=0}^d a_i x^i$ be a polynomial in one variable with coefficients a_i in \mathfrak{K}_Y , where $d \geq 1$ and $a_d \neq 0$ as elements in \mathfrak{K}_Y . Assume that $f \in \mathfrak{K}_Y$ satisfies the functional equation $F(f) = 0$. Then there are positive constants C and r_0 such that*

$$T(r, f) \leq C \sum_{i=0}^d T(r, a_i) \quad \text{for } r > r_0.$$

Proof. If all a_i are constant functions, then f is also a constant function. Hence, our lemma is trivial in this case. In the following, we only consider the case that some a_i is non-constant.

Put $\psi(r) = \sum_{i=0}^d T(r, a_i)$. Then we see that $\psi(r) \rightarrow \infty$ as $r \rightarrow \infty$. Let $\mathcal{L} \subset \mathfrak{K}_Y$ be the smallest subfield containing both \mathbf{C} and all a_i . Then \mathcal{L} is a finitely generated field over \mathbf{C} . Hence, by Hironaka’s theorem, there exists a smooth projective variety M over \mathbf{C} such that the rational function field $\mathbf{C}(M)$ of M is isomorphic to \mathcal{L} . In the following, we fix one isomorphism $\iota: \mathbf{C}(M) \xrightarrow{\sim} \mathcal{L}$. Then we have the holomorphic map $b: Y \rightarrow M$, which has Zariski-dense image, such that $\iota \circ b = \iota(v)$ for all $v \in \mathbf{C}(M)$. Note that the order of the growth of b is bounded by $\psi(r)$, because $\mathcal{L} \subset \mathfrak{K}_Y^\psi$.

Take an irreducible polynomial $G(x) \in \mathcal{L}[x]$ over \mathcal{L} such that $G(f) = 0$. Let $\bar{G}(x) \in \mathbf{C}(M)[x]$ be the polynomial obtained by $G(x)$ and the isomorphism $\iota^{-1}: \mathcal{L} \rightarrow \mathbf{C}(M)$. We may take a smooth projective variety X and a generically finite map $p: X \rightarrow M$ such that the rational function field $\mathbf{C}(X)$ is isomorphic to the field $\mathbf{C}(M)[x]/(\bar{G}(x))$, via the inclusion $\mathbf{C}(M) \subset \mathbf{C}(X)$ given by p . Here we denote by $(\bar{G}(x))$ the ideal generated by $\bar{G}(x)$. Then we have a holomorphic map $g: Y \rightarrow X$ such that $p \circ g = b$ and $x \circ g = f$, where we consider x as a rational function on X . By Lemma 2, the order of the growth of g is bounded by $\psi(r)$. Hence by Lemma 3, we get

$$T(r, f) = T(r, x \circ g) = O(\psi(r)).$$

This proves our lemma. □

7.2. A generalization of Theorem 1

Let Y, B, π and ψ be the same as in Corollary 2. Then we may consider \mathfrak{K}_B as a subfield of \mathfrak{K}_Y by the natural inclusion defined by $\pi: Y \rightarrow B$.

COROLLARY 4. Let $F(x) \in \mathfrak{K}_B^\psi[x]$ be a polynomial in one variable with coefficients in \mathfrak{K}_B^ψ . Assume that the equation $F(x)=0$ has no multiple solutions in an algebraic closure of \mathfrak{K}_B^ψ . Let f be a non-constant meromorphic function on Y such that $F(f) \neq 0$ as elements in \mathfrak{K}_Y . Then for all $\varepsilon > 0$, there exists a positive constant $C(\varepsilon) > 0$ such that

$$(\deg F - 2 - \varepsilon)T(r, f) \leq \bar{N}(r, 0, F(f)) + N_{\text{ram } \pi_Y}(r) + C(\varepsilon)(N_{\text{ram } \pi_B}(r) + \psi(r)),$$

where we consider $F(f)$ as a meromorphic function on Y .

Remark 7.2.1. If we put $F(x) = (x - a_1) \dots (x - a_q)$ for distinct $a_1, \dots, a_q \in \mathfrak{K}_B^\psi$, then the corollary above implies Theorem 1. This is because we have

$$\bar{N}(r, 0, F(f)) = \sum_{i=1}^q \bar{N}(r, a_i \circ \pi, f) + O(\psi(r)). \tag{7.2.2}$$

Note that the condition $F(f) \neq 0$ is equivalent to $f \neq a_i \circ \pi$ for all $i = 1, \dots, q$.

Proof of Corollary 4. Let $\bar{\mathfrak{K}}_C$ be an algebraic closure of \mathfrak{K}_C . We consider the fields $\mathfrak{K}_B^\psi, \mathfrak{K}_B$ and \mathfrak{K}_Y as subfields of $\bar{\mathfrak{K}}_C$. Note that each element of $\bar{\mathfrak{K}}_C$ is algebraic over \mathfrak{K}_B , and hence naturally defines a multi-valued analytic function on B with at worst algebraic singularities. Similarly, each element of $\bar{\mathfrak{K}}_C$ naturally defines a multi-valued analytic function on Y with the same type of singularities. Let $\mathfrak{L} \subset \bar{\mathfrak{K}}_C$ be the splitting field of $F(x)$ over \mathfrak{K}_B^ψ . Then there exist $a_1, \dots, a_q, \beta \in \mathfrak{L}$ such that

$$F(x) = \beta(x - a_1) \dots (x - a_q), \tag{7.2.3}$$

where $q = \deg F(x)$. Since \mathfrak{L} is a finite separable extension of \mathfrak{K}_B^ψ , there is a primitive element $\alpha \in \mathfrak{L}$, i.e., $\mathfrak{L} = \mathfrak{K}_B^\psi(\alpha)$. Let $B' \xrightarrow{\pi'} B$ be the Riemann surface of the multi-valued function α on B . Then a_1, \dots, a_q are meromorphic functions on B' . Let $G(x) \in \mathfrak{K}_B^\psi[x]$ be an irreducible polynomial such that $G(\alpha) = 0$. Since the ramification points of π' are either poles of the coefficients of G or zeros of the discriminant of G , we have

$$N_{\text{ram } \pi_{B'}}(r) = N_{\text{ram } \pi_B}(r) + O(\psi(r)), \tag{7.2.4}$$

where $\pi_{B'} = \pi_B \circ \pi'$ (cf. (2.4.5)).

Next let $Y' \xrightarrow{\pi''} Y$ be the Riemann surface of the multi-valued function α on Y . By a similar reasoning as for (7.2.4), we have

$$N_{\text{ram } \pi_{Y'}}(r) = N_{\text{ram } \pi_Y}(r) + O(\psi(r)), \tag{7.2.5}$$

where $\pi_{Y'} = \pi_Y \circ \pi''$. By the constructions of B' and Y' , there exists a proper, surjective holomorphic map $\hat{\pi}: Y' \rightarrow B'$ such that $\pi' \circ \hat{\pi} = \pi \circ \pi''$. Apply Theorem 1 to the case $Y', B', f \circ \pi''$ and a_1, \dots, a_q . Using the estimate of Lemma 14,

$$T(r, a_i) = O(\psi(r)) \quad \text{for } i = 1, \dots, q,$$

we get

$$(q-2-\varepsilon)T(r, f) \leq \sum_{i=1}^q \bar{N}(r, a_i \circ \hat{\pi}, f \circ \pi'') + N_{\text{ram } \pi_{Y'}}(r) + O_\varepsilon(N_{\text{ram } \pi_{B'}}(r) + \psi(r)) \quad ||$$

for all $\varepsilon > 0$. Here we note that $T(r, f) = T(r, f \circ \pi'')$ and that a_1, \dots, a_q are distinct because $F(x) = 0$ has no multiple solutions. By (7.2.2) and (7.2.3), we have

$$\sum_{i=1}^q \bar{N}(r, a_i \circ \hat{\pi}, f \circ \pi'') = \bar{N}(r, 0, F(f \circ \pi'')) + O(\psi(r)) \leq \bar{N}(r, 0, F(f)) + O(\psi(r)).$$

Hence using (7.2.4) and (7.2.5), we conclude the proof. □

7.3. The proof of Corollary 3

We use the notation in Corollary 3. Let Ω be the curvature form of a Hermitian metric on E . Put $\psi(r) = \max\{1, T(r, b, \Omega)\}$. Then we have $\psi(r) \geq 1$, and

$$\psi(r) = T(r, b, E) + o(T(r, g, L)). \tag{7.3.1}$$

Note that the order of the growth of b is bounded by $\psi(r)$.

Let W be the Zariski closure of the image $g(Y)$. We first consider the case $W \neq X$. By Hironaka's theorem, there exists a blowing-up $\widetilde{W} \rightarrow W$ with a smooth \widetilde{W} . Let $\tilde{g}: Y \rightarrow \widetilde{W}$ be a holomorphic map such that $g = u \circ \tilde{g}$, where $u: \widetilde{W} \rightarrow X$ is the composition of the map $\widetilde{W} \rightarrow W$ and the closed immersion $W \rightarrow X$. Since the map $p \circ u: \widetilde{W} \rightarrow M$ is surjective and $\dim \widetilde{W} = \dim M$, we may apply Lemma 2 to conclude that the order of the growth of \tilde{g} is bounded by $\psi(r)$. Hence by Lemma 3, we have

$$T(r, g, K_{X/M}(D)) = T(r, \tilde{g}, u^*K_{X/M}(D)) + O(1) = O(\psi(r)).$$

This proves our corollary in the case $W \neq X$.

Next we consider the case $W = X$. By Hironaka's theorem, there exists a blowing-up $u: \tilde{X} \rightarrow X$ with a smooth, projective variety \tilde{X} such that a generically finite map $\alpha: \tilde{X} \rightarrow \mathbf{P}^1 \times M$ over M exists. Let $M_0 \subset M$ be a non-empty Zariski-open subset such that

the restriction $\alpha_0 = \alpha|_{\tilde{X}_0}: \tilde{X}_0 \rightarrow \mathbf{P}^1 \times M_0$ is finite and the restriction $u_0 = u|_{\tilde{X}_0}: \tilde{X}_0 \rightarrow X_0$ is an isomorphism, where $X_0 = p^{-1}(M_0)$ and $\tilde{X}_0 = (p \circ u)^{-1}(M_0)$. Put $F_0 = \text{ram } \alpha_0$, i.e., the ramification divisor of α_0 . Then F_0 is a divisor on \tilde{X}_0 . Let $H_0 \subset \mathbf{P}^1 \times M_0$ be the reduced divisor supported by $\alpha_0(\text{supp}(F_0 + u_0^* D_0))$, where $D_0 = D|_{X_0}$. Put

$$G_0 = ((\alpha_0 \circ u_0^{-1})^*(H_0))_{\text{red}} - D_0.$$

Then G_0 is an effective divisor on X_0 because D_0 is reduced. By the ramification formula, we have

$$u_0^* K_{X_0/M_0}(G_0 + D_0) = \alpha_0^*(K_{(\mathbf{P}^1 \times M_0)/M_0}(H_0)). \quad (7.3.2)$$

Here $K_{(\mathbf{P}^1 \times M_0)/M_0}$ is the relative canonical bundle of the second projection $\mathbf{P}^1 \times M_0 \rightarrow M_0$. Let $H \subset \mathbf{P}^1 \times M$ be the natural extension of H_0 , and let $G \subset X$ be the natural extension of G_0 . Then by (7.3.2), there exists a divisor $Z \subset \tilde{X}$ such that

$$p \circ u(\text{supp } Z) \subset M \setminus M_0 \quad (7.3.3)$$

and

$$u^* K_{X/M}(G + D) = \alpha^*(K_{(\mathbf{P}^1 \times M)/M}(H)) + [Z]. \quad (7.3.4)$$

Here $[Z]$ is the associated line bundle for Z . Let $\tilde{g}: Y \rightarrow \tilde{X}$ be the holomorphic map with $g = u \circ \tilde{g}$. By (7.3.3), we have

$$T(r, \tilde{g}, [Z]) = O(\psi(r))$$

(cf. Lemma 4). Hence by (7.3.4), we obtain

$$T(r, g, K_{X/M}(G + D)) = T(r, \alpha \circ \tilde{g}, K_{(\mathbf{P}^1 \times M)/M}(H)) + O(\psi(r)). \quad (7.3.5)$$

CLAIM. For all $\varepsilon > 0$, the following inequality holds:

$$\begin{aligned} T(r, \alpha \circ \tilde{g}, K_{(\mathbf{P}^1 \times M)/M}(H)) &\leq \bar{N}(r, \alpha \circ \tilde{g}, H) + N_{\text{ram } \pi_Y}(r) + \varepsilon T(r, g, L) \\ &\quad + O_\varepsilon(N_{\text{ram } \pi_B}(r) + \psi(r)) \|. \end{aligned}$$

Proof. Let μ be the generic point of M in the sense of scheme theory. Let \mathbf{P}_μ^1 be the generic fiber of the second projection $p': \mathbf{P}^1 \times M \rightarrow M$. Then \mathbf{P}_μ^1 is the projective line over the rational function field $\mathbf{C}(M)$ of M . Let $H_\mu \subset \mathbf{P}_\mu^1$ be the restriction of H . By a coordinate change of the first factor of $\mathbf{P}^1 \times M$, if necessary, we may assume that the divisor $(\infty) \subset \mathbf{P}_\mu^1$ is not a component of H_μ . Hence we may take a polynomial $F(x) \in \mathbf{C}(M)[x]$ such that H_μ is defined by $F(x) = 0$.

First, we consider $F(x)$ as a rational function on $\mathbf{P}^1 \times M$. Let $(F)_0 \subset \mathbf{P}^1 \times M$ be the divisor of zeros of $F(x)$. Then we have

$$\bar{N}(r, 0, F \circ \alpha \circ \tilde{g}) \leq \bar{N}(r, \alpha \circ \tilde{g}, (F)_0),$$

where $F \circ \alpha \circ \tilde{g}$ is a non-constant meromorphic function on Y because of the assumption $W = X$. Note that we have

$$\bar{N}(r, \alpha \circ \tilde{g}, (F)_0) = \bar{N}(r, \alpha \circ \tilde{g}, H) + O(\psi(r))$$

because $p'(\text{supp}((F)_0 - H)) \neq M$ (cf. Lemma 4). Hence we get

$$\bar{N}(r, 0, F \circ \alpha \circ \tilde{g}) \leq \bar{N}(r, \alpha \circ \tilde{g}, H) + O(\psi(r)). \tag{7.3.6}$$

Next, let $\hat{F}(x)$ be the polynomial over \mathfrak{K}_B^ψ obtained from $F(x)$ by the natural injection $\mathbf{C}(M) \rightarrow \mathfrak{K}_B^\psi$ defined by b (cf. Lemma 3). Let $\zeta: \mathbf{P}^1 \times M \rightarrow \mathbf{P}^1$ be the first projection, and put $\hat{g} = \zeta \circ \alpha \circ \tilde{g}: Y \rightarrow \mathbf{P}^1$. Then we have

$$F \circ \alpha \circ \tilde{g} = \hat{F}(\hat{g}).$$

Hence, using (7.3.6), we get

$$\bar{N}(r, 0, \hat{F}(\hat{g})) \leq \bar{N}(r, \alpha \circ \tilde{g}, H) + O(\psi(r)). \tag{7.3.7}$$

We apply Corollary 4 to obtain

$$(\deg \hat{F} - 2 - \varepsilon)T(r, \hat{g}) \leq \bar{N}(r, 0, \hat{F}(\hat{g})) + N_{\text{ram } \pi_Y}(r) + O_\varepsilon(N_{\text{ram } \pi_B}(r) + \psi(r)) \quad || \tag{7.3.8}$$

for all $\varepsilon > 0$. Here we note that $\hat{F}(x)$ has no multiple solutions because H is a reduced divisor.

Now since $((\deg F - 2)\zeta^*\mathcal{L})|_{\mathbf{P}^1} = K_{(\mathbf{P}^1 \times M)/M}(H)|_{\mathbf{P}^1}$, there exists a divisor P on $\mathbf{P}^1 \times M$ with $p'(\text{supp } P) \neq M$ such that

$$(\deg F - 2)\zeta^*\mathcal{L} = K_{(\mathbf{P}^1 \times M)/M}(H) + [P].$$

Hence we may apply Lemma 4 to get

$$(\deg \hat{F} - 2)T(r, \hat{g}, \mathcal{L}) = T(r, \alpha \circ \tilde{g}, K_{(\mathbf{P}^1 \times M)/M}(H)) + O(\psi(r)). \tag{7.3.9}$$

Note that we have $T(r, \hat{g}, \mathcal{L}) = T(r, \hat{g}) + O(1)$, because the Fubini–Study form $\omega_{\mathbf{P}^1}$ is the curvature form of the Fubini–Study metric on \mathcal{L} . Hence combining (7.3.7), (7.3.8) and (7.3.9), we get

$$\begin{aligned} T(r, \alpha \circ \tilde{g}, K_{(\mathbf{P}^1 \times M)/M}(H)) &\leq \bar{N}(r, \alpha \circ \tilde{g}, H) + N_{\text{ram } \pi_Y}(r) + \varepsilon T(r, \hat{g}) \\ &\quad + O_\varepsilon(N_{\text{ram } \pi_B}(r) + \psi(r)) \quad || \end{aligned}$$

for all $\varepsilon > 0$. Note that the order of the growth of g is bounded by $T(r, g, L)$. Considering $\zeta \circ \alpha$ as a rational function on X , we apply Lemma 3 to get

$$T(r, \hat{g}) = O(T(r, g, L)),$$

i.e., there is a positive constant Q independent of ε such that

$$T(r, \hat{g}) \leq QT(r, g, L) + O(1).$$

Hence, we obtain our claim. □

Now we go back to the proof of the corollary. Since we have

$$p \circ u(\text{supp}((\alpha^*H)_{\text{red}} - u^*(G+D))) \neq M,$$

we obtain (cf. Lemma 4)

$$\begin{aligned} \bar{N}(r, \alpha \circ \tilde{g}, H) &= \bar{N}(r, \tilde{g}, (\alpha^*H)_{\text{red}}) = \bar{N}(r, g, G+D) + O(\psi(r)) \\ &\leq \bar{N}(r, g, G) + \bar{N}(r, g, D) + O(\psi(r)). \end{aligned}$$

Hence combining this with (7.3.5) and the above claim, we get

$$\begin{aligned} T(r, g, K_{X/M}(G+D)) &\leq \bar{N}(r, g, G) + \bar{N}(r, g, D) + N_{\text{ram } \pi_Y}(r) + \varepsilon T(r, g, L) \\ &\quad + O_\varepsilon(N_{\text{ram } \pi_B}(r) + \psi(r)) \end{aligned}$$

for all $\varepsilon > 0$. Using (7.3.1), $\bar{N}(r, g, G) \leq T(r, g, [G]) + O(1)$ (cf. (2.1.1)) and

$$T(r, g, K_{X/M}(G+D)) = T(r, g, K_{X/M}(D)) + T(r, g, [G]) + O(1),$$

we get our corollary. □

7.4. The proof of Corollary 2

We use the notation in Corollary 2. Put $\Psi(r) = \max\{1, \psi(r)\}$. Then we have $\Psi(r) \geq 1$ for $r > 0$, and $\mathfrak{K}_B^\Psi = \mathfrak{K}_B^\psi$. Note that the estimate in Corollary 2 is easily derived from the corresponding estimate where ψ is replaced by Ψ . Let $\mathfrak{L} \subset \mathfrak{K}_B^\psi$ be the smallest subfield containing both \mathbf{C} and all the coefficients of $F(x, y)$. Note that \mathfrak{L} is a finitely generated field over \mathbf{C} . Hence there exists a smooth projective variety M over \mathbf{C} such that the rational function field $\mathbf{C}(M)$ of M is isomorphic to \mathfrak{L} . In the following, we fix one isomorphism $\iota: \mathbf{C}(M) \xrightarrow{\sim} \mathfrak{L}$. Then we have the holomorphic map $b: B \rightarrow M$, which has

Zariski-dense image, such that $v \circ b = \iota(v)$ for all $v \in \mathbf{C}(M)$. Note that the order of the growth of b is bounded by $\Psi(r)$.

Observe that f_1 is algebraic over \mathfrak{L} if and only if f_2 is algebraic over \mathfrak{L} . In the case when f_1 and f_2 are algebraic over \mathfrak{L} , we have

$$T(r, f_i) = O(\Psi(r)) \quad \text{for } i = 1, 2,$$

by Lemma 14. This proves our corollary in this case. Thus, in the following, we assume that both f_1 and f_2 are non-algebraic over \mathfrak{L} .

We denote by μ the generic point of M in the sense of scheme theory. Let $\bar{F}(x, y) \in \mathbf{C}(M)[x, y]$ be the polynomial obtained by $F(x, y)$ and the isomorphism $\iota^{-1}: \mathfrak{L} \rightarrow \mathbf{C}(M)$. Let Q be the quotient field of the ring $\mathbf{C}(M)[x, y]/(\bar{F}(x, y))$, where $(\bar{F}(x, y))$ is the ideal generated by $\bar{F}(x, y)$. We may take a smooth projective variety X and a surjective morphism $p: X \rightarrow M$ such that the rational function field $\mathbf{C}(M)(X_\mu)$ of the generic fiber X_μ of p (in the sense of scheme theory) is isomorphic to Q . Note that X_μ is a smooth, projective curve over the field $\mathbf{C}(M)$. Then the rational function field $\mathbf{C}(X)$ of X is also isomorphic to Q . Since the meromorphic functions f_1 and f_2 satisfy the functional equation $F(f_1, f_2) = 0$ and they are not algebraic over \mathfrak{L} , we get the holomorphic map $g: Y \rightarrow X$ such that b and g fit into the commutative diagram in Corollary 3, and such that $x \circ g = f_1$ and $y \circ g = f_2$. Here we consider x and y as rational functions on X . By the assumption that, for general $z \in B$, the polynomial $F_z(x, y)$ is irreducible and the equation $F_z(x, y) = 0$ defines an algebraic curve of (topological) genus greater than one, we see that the curve X_μ is geometrically connected and has genus greater than one. Hence the canonical bundle K_{X_μ} is ample. Let L be an ample line bundle on X .

CLAIM. *There is a positive constant C , which only depends on $p: X \rightarrow M$ and L , such that $T(r, g, L) \leq CT(r, g, K_{X/M}) + O(\Psi(r))$.*

Proof. There exists a positive integer m such that $mK_{X_\mu} - L|_{X_\mu}$ is very ample on X_μ . Hence we may take an effective divisor H on X such that $[H|_{X_\mu}] = mK_{X_\mu} - L|_{X_\mu}$ and $g(Y) \not\subset \text{supp } H$. Since the restriction $K_{X/M}|_{X_\mu}$ is isomorphic to K_{X_μ} , we see that the restriction $(mK_{X/M} - L - [H])|_{X_\mu}$ is the trivial line bundle on X_μ . Hence there exists a divisor G on X such that $p(\text{supp } G) \neq M$ and $mK_{X/M} - L - [H] = [G]$. Therefore we obtain

$$T(r, g, L) = mT(r, g, K_{X/M}) - T(r, g, [H]) - T(r, g, [G]) + O(1).$$

Since we have

$$-T(r, g, [H]) \leq O(1)$$

(cf. (2.1.1)) and

$$-T(r, g, [G]) = O(\Psi(r))$$

(cf. Lemma 4), we conclude our claim. (Put $C=m$.) \square

Now, applying Corollary 3 for the case $D=\emptyset$ and using the above claim, we get

$$T(r, g, L) \leq O_\varepsilon(N_{\text{ram } \pi_Y}(r) + N_{\text{ram } \pi_B}(r) + \Psi(r)) + \varepsilon T(r, g, L) \parallel$$

for all $\varepsilon > 0$. Letting $\varepsilon < 1$, we get

$$T(r, g, L) = O(N_{\text{ram } \pi_Y}(r) + \Psi(r)) \parallel, \quad (7.4.1)$$

where we note that $N_{\text{ram } \pi_B}(r) \leq N_{\text{ram } \pi_Y}(r)$ for $r > 1$ (cf. (2.4.5)). Using $x \circ g = f_1$ and $y \circ g = f_2$, we obtain

$$T(r, f_1) = O(T(r, g, L)) \quad \text{and} \quad T(r, f_2) = O(T(r, g, L)) \quad (7.4.2)$$

(cf. Lemma 3). By (7.4.1) and (7.4.2), we get our corollary. \square

8. The proof of Theorem 2

In this section, we prove Theorem 2. Our theorem is trivial for $q \leq 2$. Hence in the following, we consider the case $q \geq 3$. Let $\varepsilon > 0$ be a positive constant and let

$$Y, B, \pi, f, a_1, \dots, a_q \quad (8.0.1)$$

be the objects in Theorem 2, which will be fixed in the following. Consider the specified q -hol-quintet $\lambda = (Y, B, \pi, \text{cl}_{(f,a)}, \text{cl}_a, Y, B)$ defined by (8.0.1).

Put $\delta = \max_{1 \leq i \leq q} \deg a_i$ and

$$U = \{z \in B : a_1(z), \dots, a_q(z) \text{ are all distinct}\}.$$

Then U is a dense, open subset of B . For $(i, j, k) \in \mathcal{J}$ and for $z \in \pi^{-1}(U)$, the two 4-tuples of points on \mathbf{P}^1 ,

$$(f(z), a_i \circ \pi(z), a_j \circ \pi(z), a_k \circ \pi(z)) \quad \text{and} \quad (\varphi_{(i,j,k)} \circ \text{cl}_{(f,a)}(z), 0, 1, \infty),$$

are isomorphic (cf. (1.5.6) and (1.5.7)). Thus we have

$$\varphi_{(i,j,k)} \circ \text{cl}_{(f,a)}(z) = \frac{f(z) - a_i \circ \pi(z)}{f(z) - a_k \circ \pi(z)} \frac{a_j \circ \pi(z) - a_k \circ \pi(z)}{a_j \circ \pi(z) - a_i \circ \pi(z)},$$

which is a rational function on Y . Hence we get

$$|\deg(\varphi_{(i,j,k)} \circ \text{cl}_{(f,a)}) - \deg f| \leq 7\delta \deg \pi. \quad (8.0.2)$$

Also, for $(i, j, k, l) \in \mathcal{I}$ and for $z \in U$, the two 4-tuples of points on \mathbf{P}^1 ,

$$(a_l(z), a_i(z), a_j(z), a_k(z)) \quad \text{and} \quad (\phi_{(i,j,k,l)} \circ \text{cl}_a(z), 0, 1, \infty),$$

are isomorphic. Thus we have

$$\phi_{(i,j,k,l)} \circ \text{cl}_a(z) = \frac{a_l(z) - a_i(z)}{a_l(z) - a_k(z)} \frac{a_j(z) - a_k(z)}{a_j(z) - a_i(z)},$$

which is a rational function on B . Hence we get

$$\deg(\phi_{(i,j,k,l)} \circ \text{cl}_a) \leq 8\delta. \tag{8.0.3}$$

By the assumption that a_i are distinct, we conclude that

$$\text{cl}_a(B) \not\subset \text{supp } \mathcal{L}_q. \tag{8.0.4}$$

First, we consider the case that λ is non-degenerate. We apply Theorem 4 for the non-degenerate specified q -hol-quintet λ . Denoting by $C_1(q, \varepsilon)$ the constant $C(q, \varepsilon)$ obtained in Theorem 4, we get

$$\begin{aligned} \deg(\text{cl}_{(f,a)})^* K_q &\leq \bar{n}(\text{cl}_{(f,a)}, \mathcal{D}_q, Y) + \text{disc}(\pi, B) + \varepsilon A(\text{cl}_{(f,a)}, Y, \omega_q) \\ &\quad + C_1(q, \varepsilon)(\deg \pi)(A(\text{cl}_a, B, \eta_q) + \bar{n}(\text{cl}_a, \mathcal{L}_q, B) + \varrho^+(B)). \end{aligned} \tag{8.0.5}$$

Here we used the facts:

- (1) $A(\text{cl}_{(f,a)}, Y, \varkappa_q) = \deg(\text{cl}_{(f,a)})^* K_q$;
- (2) $\partial Y = \emptyset$ because Y is compact, and hence $l(\text{cl}_{(f,a)}, \partial Y, \omega_q) = 0$.

By the Riemann–Roch theorem and the Hurwitz theorem, we have

$$\varrho(B) = 2g(B) - 2 \quad \text{and} \quad \text{disc}(\pi, B) = (2g(Y) - 2) - (\deg \pi)(2g(B) - 2), \tag{8.0.6}$$

so

$$\varrho^+(B) \leq 2g(B) \quad \text{and} \quad \text{disc}(\pi, B) \leq 2g(Y) + 2 \deg \pi.$$

Hence by (8.0.5), we get

$$\begin{aligned} \deg(\text{cl}_{(f,a)})^* K_q &\leq \bar{n}(\text{cl}_{(f,a)}, \mathcal{D}_q, Y) + 2g(Y) + \varepsilon A(\text{cl}_{(f,a)}, Y, \omega_q) \\ &\quad + C_2(q, \varepsilon)(\deg \pi)(A(\text{cl}_a, B, \eta_q) + \bar{n}(\text{cl}_a, \mathcal{L}_q, B) + g(B) + 1), \end{aligned} \tag{8.0.7}$$

where we put $C_2(q, \varepsilon) = 2 \max\{C_1(q, \varepsilon), 2\}$.

CLAIM. *There exist positive constants Q_1, \dots, Q_5 which are independent of the choices of $\varepsilon > 0$ and of the objects in (8.0.1), such that*

$$A(\text{cl}_a, B, \eta_q) \leq Q_1 \delta, \tag{8.0.8}$$

$$\bar{n}(\text{cl}_a, \mathcal{Z}_q, B) \leq Q_2 \delta, \tag{8.0.9}$$

$$A(\text{cl}_{(f,a)}, Y, \omega_q) \leq Q_3 (\deg f + \delta \deg \pi), \tag{8.0.10}$$

$$\bar{n}(\text{cl}_{(f,a)}, \mathcal{D}_q, Y) \leq \sum_{i=1}^q \bar{n}(a_i \circ \pi, f, Y) + Q_4 \delta \deg \pi, \tag{8.0.11}$$

$$(q-2) \deg f \leq \deg(\text{cl}_{(f,a)})^* K_q + Q_5 \delta \deg \pi. \tag{8.0.12}$$

Proof of (8.0.8). For $i \in \mathcal{I}$, let $\text{pr}_i: (\mathbf{P}^1)^{\mathcal{I}} \rightarrow \mathbf{P}^1$ be the projection to the i th factor. Put

$$\bar{\mathcal{L}} = \sum_{i \in \mathcal{I}} \text{pr}_i^* \mathcal{L},$$

which is an ample line bundle on $(\mathbf{P}^1)^{\mathcal{I}}$. By Lemma 12, the line bundle $\Phi^* \bar{\mathcal{L}}$ is an ample line bundle on $\bar{\mathcal{M}}_{0,q}$. Hence there exists a curvature form ω' of $\Phi^* \bar{\mathcal{L}}$ that is a positive $(1,1)$ -form. Therefore there exists a positive constant Q'_1 such that $\eta_q < Q'_1 \omega'$. Using (8.0.3), we have

$$\begin{aligned} A(\text{cl}_a, B, \eta_q) &\leq Q'_1 A(\text{cl}_a, B, \omega') = Q'_1 \deg(\Phi \circ \text{cl}_a)^* \bar{\mathcal{L}} \\ &= Q'_1 \sum_{i \in \mathcal{I}} \deg(\phi_i \circ \text{cl}_a) \leq 8Q'_1 (\text{card } \mathcal{I}) \delta. \end{aligned}$$

Put $Q_1 = 8Q'_1 \text{card } \mathcal{I}$ to conclude the proof of (8.0.8). □

Proof of (8.0.9). There exists a positive integer Q'_2 such that $Q'_2 \Phi^* \bar{\mathcal{L}} - [\mathcal{Z}_q]$ is an ample line bundle on $\bar{\mathcal{M}}_{0,q}$. Hence using (8.0.3), we get

$$\bar{n}(\text{cl}_a, \mathcal{Z}_q, B) \leq \deg(\text{cl}_a)^* \mathcal{Z}_q \leq Q'_2 \deg(\Phi \circ \text{cl}_a)^* \bar{\mathcal{L}} \leq 8Q'_2 (\text{card } \mathcal{I}) \delta.$$

Put $Q_2 = 8Q'_2 \text{card } \mathcal{I}$ to conclude the proof of (8.0.9). □

Proof of (8.0.10). Using the isomorphism $\iota_{q+1}: \bar{\mathcal{M}}_{0,q+1} \rightarrow \bar{\mathcal{U}}_{0,q}$ (cf. (1.5.12)) and Lemma 12 for $\bar{\mathcal{M}}_{0,q+1}$, we see that the line bundle

$$P = \sum_{\alpha \in \mathcal{I}^q} \varphi_\alpha^* \mathcal{L} + \sum_{i \in \mathcal{I}^q} (\phi_i \circ \varpi_q)^* \mathcal{L}$$

is an ample line bundle on $\overline{\mathcal{W}}_{0,q}$. Let ω'' be a curvature form of P that is a positive $(1,1)$ -form. Then there exists a positive constant Q'_3 such that $\omega_q < Q'_3 \omega''$. Using (8.0.2) and (8.0.3), we get

$$\begin{aligned} A(\text{cl}_{(f,a)}, Y, \omega_q) &\leq Q'_3 A(\text{cl}_{(f,a)}, Y, \omega'') \\ &= Q'_3 \deg(\text{cl}_{(f,a)})^* P \\ &= Q'_3 \left(\sum_{\alpha \in \mathcal{J}^q} \deg(\varphi_\alpha \circ \text{cl}_{(f,a)}) + \sum_{i \in \mathcal{J}^q} \deg(\phi_i \circ \text{cl}_a \circ \pi) \right) \\ &\leq (Q'_3 \text{card } \mathcal{J}^q + 7Q'_3 \text{card } \mathcal{J}^q + 8Q'_3 \text{card } \mathcal{J}^q) (\deg f + \delta \deg \pi). \end{aligned}$$

Put $Q_3 = Q'_3 \text{card } \mathcal{J}^q + 7Q'_3 \text{card } \mathcal{J}^q + 8Q'_3 \text{card } \mathcal{J}^q$ to conclude the proof of (8.0.10). \square

Proof of (8.0.11) (cf. the proof of (2.3.5)). Put

$$U = \{z \in B : a_1(z), \dots, a_q(z) \text{ are all distinct}\}.$$

Then by the definition of the classification map, we have $\text{cl}_a(U) \subset \mathcal{M}_{0,q}$. For $z \in U$ and $y \in \pi^{-1}(z)$, we have $\text{cl}_{(f,a)}(y) \in \mathcal{D}_q$ if and only if $f(y) = a_i(z)$ for some $i \in (q)$ (cf. (1.5.6) and (1.5.7)). Hence we have

$$\{y \in Y : \text{cl}_{(f,a)}(y) \in \mathcal{D}_q\} \subset \{y \in Y : f(y) = a_i \circ \pi(y) \text{ for some } i \in (q)\} \cup \pi^{-1}(B \setminus U).$$

This implies that

$$\bar{n}(\text{cl}_{(f,a)}, \mathcal{D}_q, Y) \leq \sum_{i=1}^q \bar{n}(a_i \circ \pi, f, Y) + (\deg \pi) \sum_{i=1}^q \sum_{\substack{j=1 \\ j \neq i}}^q \bar{n}(a_i, a_j, B).$$

Since we have

$$\bar{n}(a_i, a_j, B) \leq 2\delta,$$

we get (8.0.11). (Put $Q_4 = 2q(q-1)$.) \square

Proof of (8.0.12) (cf. the proof of (2.3.6)). By Lemma 5, we have

$$(q-2) \deg(\varphi_{(1,2,3)} \circ \text{cl}_{(f,a)}) = \deg(\text{cl}_{(f,a)})^* K_q + (\deg \pi) \deg(\text{cl}_a)^* E + \deg(\text{cl}_{(f,a)})^* (\Xi), \quad (8.0.13)$$

where E and Ξ are obtained in the lemma. By $\varpi_q(\text{supp } \Xi) \subset \text{supp } \mathcal{Z}_q$, there exists a positive integer Q'_5 such that the divisor $Q'_5 \varpi_q^* \mathcal{Z}_q - \Xi$ is effective. Hence by (8.0.4) and by the proof of (8.0.9), we have

$$\deg(\text{cl}_{(f,a)})^* (\Xi) \leq Q'_5 (\deg \pi) \deg(\text{cl}_a)^* (\mathcal{Z}_q) \leq Q_2 Q'_5 \delta \deg \pi. \quad (8.0.14)$$

Since $\Phi^*\overline{\mathcal{L}}$ is ample, there exists a positive integer Q_5'' such that the line bundle $Q_5''\Phi^*\overline{\mathcal{L}} - E$ is ample. Using (8.0.3), we get

$$\deg(\text{cl}_a)^*E \leq Q_5'' \deg(\Phi \circ \text{cl}_a)^*\overline{\mathcal{L}} \leq 8Q_5''(\text{card } \mathcal{I})\delta. \quad (8.0.15)$$

Using (8.0.2), (8.0.13)–(8.0.15), and putting

$$Q_5 = Q_2Q_5' + 8Q_5'' \text{card } \mathcal{I} + 7(q-2),$$

we get our inequality (8.0.12) and conclude the proof of the claim. \square

Now using (8.0.7) and the above claim, we get

$$(q-2) \deg f \leq \sum_{i=1}^q \bar{n}(a_i \circ \pi, f, Y) + 2g(Y) + \varepsilon Q_3 \deg f \\ + (\varepsilon Q_3 + Q_4 + Q_5) \delta \deg \pi + C_2(q, \varepsilon) (\deg \pi) ((Q_1 + Q_2) \delta + g(B) + 1).$$

Put

$$C_3(q, \varepsilon) = \max\{\varepsilon Q_3 + Q_4 + Q_5 + C_2(q, \varepsilon)(Q_1 + Q_2), C_2(q, \varepsilon)\}.$$

Replacing ε by ε/Q_3 and putting $C(q, \varepsilon) = C_3(q, \varepsilon/Q_3)$, we obtain our theorem in the case that λ is non-degenerate.

Next we consider the case that λ is degenerate, i.e., there exists some $\alpha \in \mathcal{I}$ such that $\varphi_\alpha \circ \text{cl}_{(f, a)}$ is constant. Then by (8.0.2), we conclude that

$$\deg f \leq 7\delta \deg \pi.$$

Hence replacing $C(q, \varepsilon)$ by $\max\{C(q, \varepsilon), 7(q-2)\}$, we also get the theorem in the case that λ is degenerate. Here we note that all terms on the right-hand side of (1.1.4) are non-negative. This concludes the proof of Theorem 2.

9. The height inequality for curves over function fields

9.1. Notation

General references for this section are [L], [V1] and [V3]. See also [No2] for related results in higher-dimensional cases. Let k be a function field, i.e., the rational function field of a compact Riemann surface B . This B is uniquely determined by k (up to isomorphism), and called the model of k . We consider B as a smooth projective curve over \mathbf{C} . Let $S \subset B$ be a finite set of points, which will be fixed throughout.

Let X be a smooth projective curve over k , and let $D \subset X$ be an effective divisor. Let L be a line bundle on X . Following P. Vojta [V3], we define the functions

$$h_{L,k}(P), \quad N_{k,S}^{(1)}(D, P) \quad \text{and} \quad d_k(P)$$

for $P \in X(\bar{k})$ as follows.

First, take a model of X over B , i.e., a smooth variety \mathfrak{X} projective over B such that the generic fiber (in the sense of scheme theory) is isomorphic to X over k . To each $P \in X(\bar{k}) = \mathfrak{X}(\bar{k})$, we can associate the commutative diagram of holomorphic maps

$$\begin{array}{ccc} Y & \xrightarrow{f_P} & \mathfrak{X} \\ \pi \downarrow & & \downarrow p \\ B & \xlongequal{\quad} & B \end{array}$$

by taking the normalization of the Zariski closure of P in \mathfrak{X} . Here Y is the model of $k(P)$.

Let $\mathfrak{D} \subset \mathfrak{X}$ be an extension of $D \subset X$, and let \mathfrak{L} be an extension of L to \mathfrak{X} . Put

$$h_{\mathfrak{L},k}(P) = \frac{1}{\deg \pi} \deg f_P^* \mathfrak{L}$$

and

$$N_{k,S}^{(1)}(\mathfrak{D}, P) = \frac{1}{\deg \pi} \sum_{x \in Y \setminus \pi^{-1}(S)} \min\{1, \text{ord}_x f_P^* \mathfrak{D}\}, \quad P \in X(\bar{k}) \setminus D.$$

If we replace the models \mathfrak{X} , \mathfrak{D} and \mathfrak{L} by other models \mathfrak{X}' , \mathfrak{D}' and \mathfrak{L}' , we have

$$h_{\mathfrak{L},k}(P) = h_{\mathfrak{L}',k}(P) + O(1) \quad \text{and} \quad N_{k,S}^{(1)}(\mathfrak{D}, P) = N_{k,S}^{(1)}(\mathfrak{D}', P) + O(1),$$

where $O(1)$ are bounded terms independent of $P \in X(\bar{k})$. Then we define the functions $h_{L,k}(P)$ and $N_{k,S}^{(1)}(D, P)$ by

$$h_{L,k}(P) = h_{\mathfrak{L},k}(P) + O(1)$$

and

$$N_{k,S}^{(1)}(D, P) = N_{k,S}^{(1)}(\mathfrak{D}, P) + O(1), \quad P \in X(\bar{k}) \setminus D,$$

which are functions modulo bounded terms $O(1)$. Finally, put

$$d_k(P) = \frac{1}{\deg \pi} \text{disc}(\pi, B).$$

Then we have

$$d_k(P) = \frac{2g(Y)}{\deg \pi} + O(1)$$

(cf. (8.0.6)). The following facts are easy consequences of the above definitions:

(i) $N_{k,S}^{(1)}(D, P) \leq h_{[D],k}(P) + O(1)$, where $[D]$ is the associated line bundle.

(ii) $\bar{n}(f_P, \mathfrak{D}, Y) \leq (\deg \pi)(N_{k,S}^{(1)}(\mathfrak{D}, P) + \text{card } S)$.

(iii) Let \mathbf{P}_k^1 be the projective line over k . In the following, we always take $\mathbf{P}^1 \times B$ as a model of \mathbf{P}_k^1 over B . Then a point $P \in \mathbf{P}_k^1(\bar{k}) \setminus (\infty)$ corresponds to the rational function \hat{f}_P on Y obtained by the composition

$$\hat{f}_P: Y \xrightarrow{f_P} \mathbf{P}^1 \times B \xrightarrow{\text{1st proj}} \mathbf{P}^1.$$

Let \mathcal{L}_k be the hyperplane section bundle on \mathbf{P}_k^1 . Then we have

$$h_{\mathcal{L}_k,k}(P) = \frac{\deg \hat{f}_P}{\deg \pi} + O(1).$$

(iv) Let $k' \subset \bar{k}$ be a finite extension of k . Put $e = [k':k]$ and $X' = X \otimes_k k'$. Let B' be the model of k' . Let $b: B' \rightarrow B$ and $\hat{b}: X' \rightarrow X$ be the natural maps. Put $D' = \hat{b}^*D$, $L' = \hat{b}^*L$ and $S' = b^{-1}(S)$. Then using the natural identification $X'(\bar{k}) = X(\bar{k})$, we have

$$h_{L',k'}(P) = eh_{L,k}(P) + O(1), \quad N_{k',S'}^{(1)}(D', P) \leq eN_{k,S}^{(1)}(D, P) + O(1)$$

and

$$d_{k'}(P) \leq ed_k(P).$$

By these properties and Theorem 2, we obtain the following result.

LEMMA 15. *Let $D \subset \mathbf{P}_k^1$ be a reduced divisor and let $\varepsilon > 0$. Then we have*

$$h_{K_{\mathbf{P}_k^1(D),k}}(P) \leq N_{k,S}^{(1)}(D, P) + d_k(P) + \varepsilon h_{\mathcal{L}_k,k}(P) + O_\varepsilon(1)$$

for all $P \in \mathbf{P}_k^1(\bar{k}) \setminus D$. Here $O_\varepsilon(1)$ denotes a bounded term which depends on ε , but does not depend on $P \in \mathbf{P}_k^1(\bar{k})$.

Proof. We first prove the lemma for the special case that the divisor D has the form $D = (P_1) + \dots + (P_q)$ by k -rational points $P_i \in \mathbf{P}_k^1(k)$, $i = 1, \dots, q$. By a coordinate change of \mathbf{P}_k^1 , if necessary, we may assume that $P_i \neq \infty$ for all $i = 1, \dots, q$. By the property (iii) above, each P_i corresponds to the rational function \hat{f}_{P_i} on B because $k(P_i) = k$. Here B is the model of k . By the assumption that D is reduced, the points P_i are distinct. Hence the rational functions \hat{f}_{P_i} are distinct. Let $P \in \mathbf{P}_k^1(\bar{k}) \setminus D$, let Y be the model of $k(P)$ and let $\pi: Y \rightarrow B$ be the natural map. Since $h_{K_{\mathbf{P}_k^1(D),k}}(\infty) = O(1)$, it suffices to consider the case $P \neq \infty$. Then P corresponds to the rational function \hat{f}_P on Y . Because $P \notin \text{supp } D$, we have $\hat{f}_P \neq \hat{f}_{P_i} \circ \pi$ for $i = 1, \dots, q$. Apply Theorem 2 to get

$$(q-2-\varepsilon) \deg \hat{f}_P \leq \sum_{i=1}^q \bar{n}(\hat{f}_{P_i} \circ \pi, \hat{f}_P, Y) + 2g(Y) + O_\varepsilon(1) \deg \pi \quad (9.1.1)$$

for all $\varepsilon > 0$. Here we note that the functions \hat{f}_{P_i} and the Riemann surface B are fixed. Let $\mathfrak{D} \subset \mathbf{P}^1 \times B$ be the Zariski closure of $D \subset \mathbf{P}_k^1$ and let $f_P: Y \rightarrow \mathbf{P}^1 \times B$ be the associated holomorphic map for P . Then we have

$$\sum_{i=1}^q \bar{n}(\hat{f}_{P_i} \circ \pi, \hat{f}_P, Y) \leq \bar{n}(f_P, \mathfrak{D}, Y) + O(1) \deg \pi,$$

because \mathfrak{D} is the union of the graphs of \hat{f}_{P_i} . By the property (ii) above, we get

$$\sum_{i=1}^q \bar{n}(\hat{f}_{P_i} \circ \pi, \hat{f}_P, Y) \leq (\deg \pi)(N_{k,S}^{(1)}(D, P) + O(1)). \tag{9.1.2}$$

By (9.1.1), (9.1.2) and $K_{\mathbf{P}_k^1}(D) = (q-2)\mathcal{L}_k$, we get

$$h_{K_{\mathbf{P}_k^1}(D), k}(P) \leq N_{k,S}^{(1)}(D, P) + d_k(P) + \varepsilon h_{\mathcal{L}_k, k}(P) + O_\varepsilon(1) \quad \text{for all } \varepsilon > 0.$$

This proves the lemma for our special case.

Next we prove the general case. For a finite extension k' of k , we shall use the notation D' and S' in (iv) above.

Let $k' \subset \bar{k}$ be a finite extension of k such that the divisor $D' \subset \mathbf{P}_{k'}^1$ has the form $D' = (P_1) + \dots + (P_q)$ by k' -rational points $P_i \in \mathbf{P}_{k'}^1(k')$, $i = 1, \dots, q$. Then we have the natural identification $\mathbf{P}_k^1(\bar{k}) \setminus D = \mathbf{P}_{k'}^1(\bar{k}) \setminus D'$. For $P \in \mathbf{P}_k^1(\bar{k}) \setminus D$, we apply the special case above to obtain

$$h_{K_{\mathbf{P}_{k'}^1}(D'), k'}(P) \leq N_{k', S'}^{(1)}(D', P) + d_{k'}(P) + \varepsilon h_{\mathcal{L}_{k'}, k'}(P) + O_\varepsilon(1).$$

Using the property (iv) above, we conclude the proof. □

9.2. The height inequality

The following theorem proves Conjecture 2.3 in [V3] for the case of curves over function fields.

THEOREM 5. *Let k be a function field. Let X be a smooth projective curve over k , let D be a reduced divisor on X , let L be an ample line bundle on X and let $\varepsilon > 0$. Then we have*

$$h_{K_{X(D)}, k}(P) \leq N_{k,S}^{(1)}(D, P) + d_k(P) + \varepsilon h_{L, k}(P) + O_\varepsilon(1)$$

for all $P \in X(\bar{k}) \setminus D$.

Proof. Let $\alpha: X \rightarrow \mathbf{P}_k^1$ be a finite surjective map over k . Put $E = (\text{ram } \alpha)_{\text{red}} \subset X$. Let $H \subset \mathbf{P}_k^1$ be the reduced divisor supported by $\alpha(\text{supp } D \cup \text{supp } E)$. Then there exists

an effective divisor $G \subset X$ such that $(\alpha^*(H))_{\text{red}} = D + G$, since D is reduced. By the ramification formula, we have

$$K_X(D+G) = \alpha^*(K_{\mathbf{P}_k^1}(H)).$$

Then by Lemma 15 and the property (i) of the previous subsection, we have

$$\begin{aligned} h_{K_X(D+G),k}(P) &= h_{K_{\mathbf{P}_k^1}(H),k}(\alpha(P)) \\ &\leq N_{k,S}^{(1)}(H, \alpha(P)) + d_k(\alpha(P)) + \varepsilon h_{\mathcal{L}_k,k}(\alpha(P)) + O_\varepsilon(1) \\ &\leq N_{k,S}^{(1)}(D+G, P) + d_k(P) + \varepsilon h_{\alpha^*\mathcal{L}_k,k}(P) + O_\varepsilon(1) \\ &\leq N_{k,S}^{(1)}(D+G, P) + d_k(P) + \varepsilon C h_{L,k}(P) + O_\varepsilon(1) \\ &\leq N_{k,S}^{(1)}(D, P) + h_{[G],k}(P) + d_k(P) + \varepsilon C h_{L,k}(P) + O_\varepsilon(1) \end{aligned}$$

for all $P \in X(\bar{k}) \setminus (D+G)$. Here C is a positive integer such that the line bundle $CL - \alpha^*\mathcal{L}_k$ is ample; hence C is independent of P and ε . For the points $P \in \text{supp } G$, the values $h_{K_X(D)}(P)$ are bounded because $\text{supp } G$ consists of finite points. Hence, replacing ε by ε/C , we get

$$h_{K_X(D),k}(P) \leq N_{k,S}^{(1)}(D, P) + d_k(P) + \varepsilon h_{L,k}(P) + O_\varepsilon(1)$$

for all $P \in X(\bar{k}) \setminus D$. This proves our theorem. \square

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