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Rearrangements of functions, saddle points and uncountable families of steady configurations for a vortex

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1. Introduction

Kelvin [13] studied steady planar flows of an ideal fluid, confined in a bounded region with solid boundaries. He supposed the vorticity to have a given value in a region of fixed area, outside which the flow was irrotational, and considered possible configurations of the vortex. In the case of a circular region, he observed that infinitely many steady configurations exist, and one can easily verify that any radially symmetric configuration is steady. Less obviously, he claimed that in a dumb-bell shaped region, infinitely many steady configurations can be obtained by dividing the vorticity between the ends of the dumb-bell in an arbitrary proportion, and seeking the configuration that maximises the kinetic energy subject to this restriction.

In this paper we prove two results on the existence of infinitely many steady configurations for a vortex, based on Kelvin's principle of stationary kinetic energy. We admit flows in which the vorticity may be nonconstant in the vortex core (the region of non-zero vorticity), and seek steady flows in which the vorticity is a rearrangement of a given function; this more general formulation is based on a theory of 3-dimensional vortex rings proposed by Benjamin [1]; additionally, we prescribe the circulations of the velocity around the connected components of the boundary. We first show that in a bounded planar region of arbitrary shape, for a given non-negative function ζ_0, the kinetic energies of steady ideal fluid flows whose vorticities are

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rearrangements of ζ_0, and having the prescribed circulations, realise all values between a maximum and a minimum. This result is proved using a saddle-point theorem for a convex functional relative to a set having convex sections. Our second result applies to a simply connected region comprising two cavities connected by a constricted aperture, generalising Kelvin's dumb-bell. We prove the existence of steady flows for which the vorticity vanishes in the aperture, and for which the restriction of the vorticity to each cavity is a rearrangement of a prescribed non-negative function. This is proved by a maximisation argument, and yields a disconnected vortex core that avoids the boundary of the region. By assigning varying proportions of the vorticity to the two cavities, uncountably many steady flows are obtained, all of whose vorticities are rearrangements of one given function.

The main abstract results of the paper are Theorems 3.2 and 3.3 in Section 3, and the applications to fluids are given in Section 4.

2. Measures and rearrangements

When $(\Omega, \mathcal{M}, \mu)$ is a (positive) measure space and $1 \leqslant p \leqslant \infty$, then $L^p(\mu)$ will denote the space of real functions on Ω that are p-integrable with respect to μ. If $\Omega \subset \mathbf{R}^N$ is Lebesgue measurable, we denote by $L^p(\Omega)$ the L^p-space with respect to Lebesgue measure. Whenever we refer to Sobolev spaces, we intend that Lebesgue measure is used in their definition. We denote the Lebesgue measure of a set A by $|A|$.

A measure space $(\Omega, \mathcal{M}, \mu)$ is called a *measure interval* if $\omega = \mu(\Omega)$ is finite and positive, and there exists a bijection $\chi : \Omega \to [0, \omega]$, such that for $A \subset \Omega$ we have $A \in \mathcal{M}$ if and only if $\chi(A)$ is Lebesgue measurable, and for all $A \in \mathcal{M}$ we have $\mu(A) = |\chi(A)|$. It is well-known that any Lebesgue measurable subset $\Omega \subset \mathbf{R}^N$, together with any nontrivial finite positive measure on Ω that is absolutely continuous with respect to Lebesgue measure, is a measure interval; this can be deduced, for example, from a very general result in Royden [10], p. 270, Theorem 9. Measure intervals are a subclass of the finite separable nonatomic measure spaces studied in [3].

Two measures on the same set are called *equivalent* if each is absolutely continuous with respect to the other.

Let $(\Omega, \mathcal{M}, \mu)$ and $(\Omega', \mathcal{M}', \mu')$ be (positive) measure spaces with $\mu(\Omega) = \mu'(\Omega')$. Real measurable functions f on Ω and g on Ω' are *rearrangements* of each other if

$$\mu(f^{-1}[\beta, \infty)) = \mu'(g^{-1}[\beta, \infty)), \quad \forall \beta \in \mathbf{R};$$

if additionally $1 \leqslant p \leqslant \infty$ and $f \in L^p(\mu)$ then it follows that $g \in L^p(\mu')$ and $\|f\|_p = \|g\|_p$.

If $\omega=\mu(\Omega)$ is finite and positive, then every real measurable function f on Ω has an *increasing rearrangement* f^* defined on $[0,\omega]$, and for $1\leq p\leq\infty$ the inequality

$$\|f^*-g^*\|_p\leq\|f-g\|_p,\quad \forall f,g\in L^p(\mu) \tag{1}$$

is well-known. A particularly neat proof of (1), for $1\leq p<\infty$ and non-negative f and g, can be found in Crowe et al. [6].

The convexity assertion in the following lemma is due to Ryff [12] (who considered the case $p=1$) although it can be deduced (for $1<p<\infty$) from previous work of Brown [2] and Ryff [11]; it was subsequently rediscovered by Migliaccio [9] and the author [3]. The compactness assertion is easily proved.

LEMMA 2.1. *Let $(\Omega,\mathcal{M},\mu)$ be a measure interval, let $1\leq p<\infty$, let $f_0\in L^p(\mu)$, and $\mathcal{F}$ be the set of rearrangements of f_0 on Ω. Then the weak closure $\bar{\mathcal{F}}$ of $\mathcal{F}$ in $L^p(\mu)$ is convex (thus $\bar{\mathcal{F}}$ is the closed convex hull of $\mathcal{F}$), and $\bar{\mathcal{F}}$ is weakly compact.*

The next lemma is due to the author [5], Lemma 2.15.

LEMMA 2.2. *Let Ω be a nonempty open set in $\mathbf{R}^N$, let μ be a finite measure on Ω equivalent to Lebesgue measure, and let*

$$\mathcal{L}=\sum_{1\leq|\alpha|\leq m}a^\alpha(x)D^\alpha$$

define an m-th order linear partial differential operator, where the a^α are finite-valued measurable functions on Ω and there is no 0-th order term. Let $1\leq p<\infty$, let q be the conjugate exponent of p, let $f_0\in L^p(\mu)$ be non-negative, let $\mathcal{F}$ be the set of rearrangements of f_0 on Ω (with respect to μ), and let $\bar{\mathcal{F}}$ denote the weak closure of $\mathcal{F}$. Suppose there exist $\hat{f}\in\bar{\mathcal{F}}$ and $\psi\in L^q(\mu)\cap W^{m,1}_{\mathrm{loc}}(\Omega)$ such that

(i) *$\mathcal{L}\psi\geq\hat{f}$ almost everywhere in Ω, and*

(ii) *$\int_\Omega f\psi\,d\mu\leq\int_\Omega\hat{f}\psi\,d\mu$ for all $f\in\bar{\mathcal{F}}$.*

Then $\hat{f}\in\mathcal{F}$, and there exists an increasing function φ such that

$$\hat{f}=\varphi\circ\psi$$

almost everywhere in Ω.

We now prove:

LEMMA 2.3. *Let $(\Omega, \mathcal{M}, \mu)$ be a measure interval, let $\omega = \mu(\Omega)$, let $1 \leq p < \infty$, let $f_0, g_0 \in L^p(\mu)$, and let $\mathcal{F}$ and $\mathcal{G}$ be the sets of rearrangements on Ω of f_0 and g_0, with weak closures $\bar{\mathcal{F}}$ and $\bar{\mathcal{G}}$.*

(i) *Let $f \in \mathcal{F}$. Then*

$$\inf_{g \in \mathcal{G}} \|f-g\|_p = \|f_0^* - g_0^*\|_p,$$

and the infimum is attained.

(ii) *Let $f \in \bar{\mathcal{F}}$. Then*

$$\inf_{g \in \bar{\mathcal{G}}} \|f-g\|_p \leq \| f_0^* - g_0^*\|_p,$$

and the infimum is attained.

Proof. There is no loss of generality in assuming that $\Omega = [0, \omega]$ and μ is Lebesgue measure.

To prove (i), all that is required in view of (1) is to construct $g \in \mathcal{G}$ satisfying $\|f-g\|_p = \|f_0^* - g_0^*\|_p$. By a result of Ryff [11], we can write $f = f_0^* \circ \sigma$ where $\sigma : \Omega \to \Omega$ is a measure-preserving transformation; that is, σ is measurable and $|\sigma^{-1}(A)| = |A|$ for every measurable set $A \subset \Omega$. We define $g = g_0^* \circ \sigma$. Then $g \in \mathcal{G}$, and $f-g$ is a rearrangement of $f_0^* - g_0^*$, hence $\|f-g\|_p = \|f_0^* - g_0^*\|_p$.

To prove (ii), observe that $\|f-g\|_p$ is a weakly lower-semicontinuous function of $g \in L^p(\mu)$, and $\bar{\mathcal{G}}$ is weakly compact, so the infimum is attained. The function $\delta : L^p(\mu) \to \mathbf{R}$, defined by

$$\delta(f) = \inf_{g \in \bar{\mathcal{G}}} \|f-g\|_p$$

is weakly lower-semicontinuous, being continuous and convex, and the inequality

$$\delta(f) \leq \|f_0^* - g_0^*\|_p \tag{2}$$

holds for all $f \in \mathcal{F}$ by (i), hence (2) holds for all $f \in \bar{\mathcal{F}}$. □

3. Saddle points relative to sets of rearrangements

Notation. When X is any topological vector space and $\Phi : X \to \mathbf{R}$ is Gâteaux differentiable at $u \in X$, the derivative is denoted $\partial\Phi(u)$. If additionally X is a product of two topological vector spaces, the derivatives with respect to the first and second variables are denoted $\partial_1\Phi(u)$ and $\partial_2\Phi(u)$. The bilinear pairing of X and its dual space X^* will be denoted $\langle \cdot\, , \cdot \rangle$.

THEOREM 3.1. *Let C_1 and C_2 be nonempty sequentially compact sets in topological vector spaces X_1 and X_2, and suppose C_2 is convex. Let $\Phi: X_1\times X_2\to \mathbf{R}$ be a Gâteaux differentiable sequentially continuous convex functional. Define*

$$\Phi_0(u) = \inf_{v\in C_2} \Phi(u, v), \quad u\in C_1.$$

Then

(i) *Φ_0 attains its supremum relative to C_1, and for each $u\in C_1$ the infimum in the definition of Φ_0 is attained.*

(ii) *If $\bar{u}$ is a maximiser for Φ_0 relative to C_1 and $\bar{v}$ is a minimiser for $\Phi(\bar{u}, \cdot\,)$ relative to C_2, then*

$$\bar{v} \text{ minimises } \langle \partial_2\Phi(\bar{u}, \bar{v}), \cdot \,\rangle \text{ relative to } C_2, \text{ and}$$
$$\bar{u} \text{ maximises } \langle \partial_1\Phi(\bar{u}, \bar{v}), \cdot \,\rangle \text{ relative to } C_1.$$

Remark. Theorem 3.1 does not give a saddle point of Φ in the classical sense. Instead it gives a point $(\bar{u}, \bar{v})$ such that $(\bar{u}, \bar{v})$ is a saddle point of $\langle \partial\Phi(\bar{u}, \bar{v}), \cdot \,\rangle$ relative to $C_1\times C_2$.

Proof of Theorem 3.1. It follows from sequential continuity and compactness that the infimum in the definition of Φ_0 is attained. Write $M=\sup \Phi_0(C_1)$, and let $\{u_n\}_{n=1}^{\infty}$ be a maximising sequence in C_1 for Φ_0. For each n let v_n be a minimiser for $\Phi(u_n, \cdot\,)$ relative to C_2. By sequential compactness we can replace $\{u_n\}_{n=1}^{\infty}$ and $\{v_n\}_{n=1}^{\infty}$ by subsequences such that $u_n\to u_0$ and $v_n\to v_0$, for some $u_0\in C_1$ and $v_0\in C_2$. We now have $\Phi(u_n, v_n)\to\Phi(u_0, v_0)$ and so $\Phi(u_0, v_0)=M$. We show that $\Phi_0(u_0)=M$. If $v\in C_2$ then $\Phi(u_n, v)\geq\Phi(u_n, v_n)$ for each n, hence letting $n\to\infty$ we have $\Phi(u_0, v)\geq\Phi(u_0, v_0)$. This shows that v_0 minimises $\Phi(u_0, \cdot\,)$ relative to C_2, so $\Phi_0(u_0)=\Phi(u_0, v_0)=M$. Now u_0 is the required maximiser for Φ_0.

Consider any maximiser $\bar{u}$ for Φ_0 relative to C_1, and let $\bar{v}$ be any minimiser for $\Phi(\bar{u}, \cdot\,)$ relative to C_2. For any $v\in C_2$ and $0<t<1$ we have $\bar{v}+t(v-\bar{v})\in C_2$ by convexity, so

$$t^{-1}(\Phi(\bar{u}, \bar{v}+t(v-\bar{v}))-\Phi(\bar{u}, \bar{v}))\geq 0.$$

Letting $t \to 0$ we obtain

$$\langle \partial_2\Phi(\bar{u}, \bar{v}), v - \bar{v} \rangle \geq 0$$

for every $v \in C_2$, so $\bar{v}$ minimises $\langle \partial_2\Phi(\bar{u}, \bar{v}), \cdot \rangle$ relative to C_2.

Now let $u \in C_1$, and choose a minimiser v for $\Phi(u, \cdot)$ relative to C_2. Then using the convexity of Φ we have

$$\begin{aligned}\Phi(\bar{u}, \bar{v}) &\geq \Phi_0(u) = \Phi(u, v) \\ &\geq \Phi(\bar{u}, \bar{v}) + \langle \partial_1\Phi(\bar{u}, \bar{v}), u - \bar{u} \rangle + \langle \partial_2\Phi(\bar{u}, \bar{v}), v - \bar{v} \rangle \\ &\geq \Phi(\bar{u}, \bar{v}) + \langle \partial_1\Phi(\bar{u}, \bar{v}), u - \bar{u} \rangle,\end{aligned}$$

hence

$$\langle \partial_1\Phi(\bar{u}, \bar{v}), u - \bar{u} \rangle \leq 0$$

which shows that $\bar{u}$ maximises $\langle \partial_1\Phi(\bar{u}, \bar{v}), \cdot \rangle$ relative to C_1. □

We now prove three general results about boundary value problems, which are applied to planar vortices in Section 4. The results are applicable to a wide class of elliptic equations. For the remainder of this section we make the following hypotheses.

Hypotheses (H). Let $\Omega \subset \mathbf{R}^N$ be a nonempty open set, let μ be a finite positive measure on Ω equivalent to Lebesgue measure, and let $\omega = \mu(\Omega)$.

Let

$$\mathscr{L} = \sum_{1 \leq |\alpha| \leq m} a^\alpha(x) \mathscr{D}^\alpha$$

be an mth order linear partial differential operator with no 0th order term, where the a^α are finite-valued measurable functions on Ω.

Let $1 \leq p < \infty$, let q be the conjugate exponent of p, and suppose there exists a bounded linear operator $K : L^p(\mu) \to L^q(\mu)$ such that $Kv \in W^{m,1}_{\mathrm{loc}}(\Omega)$ for all $v \in L^p(\mu)$, and

$$\mathscr{L}Kv = v$$

almost everywhere in Ω, for every $v \in L^p(\mu)$.

Suppose that K is positive, in the sense that

$$\int_\Omega vKv \, d\mu \geq 0, \quad v \in L^p(\mu),$$

that K is symmetric, in the sense that

$$\int_\Omega uKv\,d\mu = \int_\Omega vKu\,d\mu, \quad u, v \in L^p(\mu),$$

and that K is compact.

Let $h \in L^p(\mu) \cap W^{m,1}_{\text{loc}}(\Omega)$ be a function satisfying

$$\mathscr{L}h \geqslant 0$$

almost everywhere in Ω, and let Φ be defined by

$$\Phi(v) = \frac{1}{2}\int_\Omega vKv\,d\mu + \int_\Omega hv\,d\mu, \quad v \in L^p(\mu).$$

THEOREM 3.2. *Let the hypotheses* (H) *be satisfied. Let $f_1, f_2 \in L^p(\mu)$ be nonnegative functions satisfying $f_1 f_2 = 0$ almost everywhere in Ω, and let the sets of rearrangements on Ω of f_1 and f_2 be $\mathscr{F}_1$ and $\mathscr{F}_2$ respectively, with weak closures $\bar{\mathscr{F}}_1$ and $\bar{\mathscr{F}}_2$. Define*

$$\Phi_0(u) = \inf_{v \in \bar{\mathscr{F}}_2} \Phi(u+v), \quad u \in \bar{\mathscr{F}}_1.$$

Then:

(i) *There exist $\bar{u} \in \bar{\mathscr{F}}_1$ and $\bar{v} \in \bar{\mathscr{F}}_2$ such that*

$$\sup_{u \in \bar{\mathscr{F}}_1} \Phi_0(u) = \Phi_0(\bar{u}) = \Phi(\bar{u}+\bar{v}).$$

(ii) *We have $\bar{u} \in \mathscr{F}_1$, $\bar{v} \in \mathscr{F}_2$, and $\bar{u}\bar{v} = 0$ almost everywhere in Ω (thus $\bar{u}+\bar{v}$ is a rearrangement of f_1+f_2).*

(iii) *For $\psi = K(\bar{u}+\bar{v})+h$ there exists a function φ such that*

$$\bar{u}+\bar{v} = \varphi \circ \psi$$

almost everywhere in Ω.

(iv) *Regarding φ there are three cases:*

(a) *If $f_1 = 0$ then φ is decreasing.*

(b) *If $f_2 = 0$ then φ is increasing.*

(c) *If f_1 and f_2 are both nonzero then there exist numbers $m_2 \leqslant m_1$, such that, apart from sets of zero measure,*

$$\bar{u}^{-1}(0, \infty) = \psi^{-1}(m_1, \infty),$$
$$\bar{v}^{-1}(0, \infty) = \psi^{-1}(-\infty, m_2).$$

Then φ is positive and decreasing on $(-\infty, m_2)$, $\varphi=0$ on $[m_2, m_1]$, and φ is positive and increasing on (m_1, ∞).

Proof. Since f_1 and f_2 are non-negative, so also are all elements of $\bar{\mathscr{F}}_1$ and $\bar{\mathscr{F}}_2$. Lemma 2.1 shows that $\bar{\mathscr{F}}_1$ and $\bar{\mathscr{F}}_2$ are convex and weakly compact, while the positivity and compactness of K ensure that Φ is convex and weakly sequentially continuous. Theorem 3.1 now applies. Consider any maximiser $\bar{u}\in\bar{\mathscr{F}}_1$ for Φ_0 and let $\bar{v}=\bar{\mathscr{F}}_2$ satisfy $\Phi(\bar{u}+\bar{v})=\Phi_0(\bar{u})$. Then

$$\langle\partial\Phi(\bar{u}+\bar{v}),f\rangle = \int_\Omega (K(\bar{u}+\bar{v})+h)f\,d\mu, \quad \forall f\in L^p(\mu).$$

So taking $\psi=K(\bar{u}+\bar{v})+h$ it follows from Theorem 3.1 that $f=\bar{u}$ maximises $\int_\Omega\psi f\,d\mu$ over $f\in\bar{\mathscr{F}}_1$ and $f=\bar{v}$ minimises $\int_\Omega\psi f\,d\mu$ over $f\in\bar{\mathscr{F}}_2$. This latter statement can be expressed as saying that $f=\bar{v}$ maximises $\int_\Omega(-\psi)f\,d\mu$ over $f\in\bar{\mathscr{F}}_2$. Moreover $\mathscr{L}\psi=\bar{u}+\bar{v}+\mathscr{L}h\geqslant\bar{u}$ and $(-\mathscr{L})(-\psi)=\bar{u}+\bar{v}+\mathscr{L}h\geqslant\bar{v}$ almost everywhere in Ω. It follows from Lemma 2.2 that $\bar{u}\in\mathscr{F}_1$, that $\bar{v}\in\mathscr{F}_2$ and that there exist increasing functions φ_1 and φ_2 such that $\bar{u}=\varphi_1\circ\psi$ and $\bar{v}=\varphi_2\circ(-\psi)$ almost everywhere in Ω. If $f_1=0$ or $f_2=0$ we are finished. Suppose therefore that f_1 and f_2 are nonzero. We can suppose φ_1 and φ_2 are non-negative and have domain $(-\infty,\infty)$; this may necessitate $+\infty$ being admitted to their ranges.

We next seek $m_2\leqslant m_1$ such that $\bar{u}^{-1}(0,\infty)=\psi^{-1}(m_1,\infty)$ and $\bar{v}^{-1}(0,\infty)=\psi^{-1}(-\infty,m_2)$ apart from sets of zero measure; it then follows that $\bar{u}^{-1}(0,\infty)$ and $\bar{v}^{-1}(0,\infty)$ are essentially disjoint, so $\bar{u}\bar{v}=0$ almost everywhere on Ω. Define $m_1=\inf\varphi_1^{-1}(0,\infty)$ and $m_2=-\inf\varphi_2^{-1}(0,\infty)$. Then $\psi^{-1}(m_1,\infty)\subset\bar{u}^{-1}(0,\infty)\subset\psi^{-1}[m_1,\infty)$ apart from sets of zero measure. But $\mu(\bar{u}^{-1}(0,\infty)\cap\psi^{-1}(m_1))=0$, since $\mathscr{L}\psi=0$ almost everywhere on $\psi^{-1}(m_1)$, by Lemma 7.7 of [7] for example, and $\mathscr{L}\psi=\bar{u}+\bar{v}+\mathscr{L}h\geqslant\bar{u}>0$ on $\bar{u}^{-1}(0,\infty)$. Therefore $\bar{u}^{-1}(0,\infty)=\psi^{-1}(m_1,\infty)$ and similarly $\bar{v}^{-1}(0,\infty)=\psi^{-1}(-\infty,m_2)$, apart from sets of zero measure. Consider the possibility that $m_2>m_1$. Then

$$\begin{aligned}\mu(\psi^{-1}(m_1,m_2)) &= \omega-\mu(\psi^{-1}(-\infty,m_1])-\mu(\psi^{-1}[m_2,\infty))\\ &=\mu(\psi^{-1}(m_1,\infty))+\mu(\psi^{-1}(-\infty,m_2))-\omega\\ &=\mu(\bar{u}^{-1}(0,\infty))+\mu((\bar{v}^{-1}(0,\infty))-\omega\\ &=\mu(f_1^{-1}(0,\infty))+\mu(f_2^{-1}(0,\infty))-\omega\leqslant 0,\end{aligned}$$

thus $\mu(\psi^{-1}(m_1,m_2))=0$. If we chose $m_1<m<m_2$ then we have $\bar{u}^{-1}(0,\infty)=\psi^{-1}(m,\infty)$ and $\bar{v}^{-1}(0,\infty)=\psi^{-1}(-\infty,m)$ apart from sets of zero measure. We can now redefine $m_1=m_2=m$ if $m_2>m_1$. So we can assume $m_2\leqslant m_1$. Define

$$\varphi(s)=\begin{cases}\varphi_1(s), & s>m_1\\ 0, & m_2\leq s\leq m_1\\ \varphi_2(-s), & s<m_2.\end{cases}$$

Then $\bar{u}+\bar{v}=\varphi\circ\psi$ almost everywhere in Ω. Finally, φ is decreasing on $(-\infty, m_2)$, zero on $[m_2, m_1]$ and increasing on (m_1, ∞). □

THEOREM 3.3. *Let the hypotheses* (H) *be satisfied. Let $f_0\in L^p(\mu)$ be non-negative, let $\mathscr{F}$ be the set of rearrangements of f_0 on Ω, and let a and A be respectively the infimum and supremum of Φ on $\mathscr{F}$.*

Then for each $\alpha\in[a, A]$ there exists $w\in\mathscr{F}$ satisfying

$$\Phi(w)=\alpha,$$

and such that, defining $\psi=Kw+h$, there exists a function φ for which

$$w=\varphi\circ\psi$$

almost everywhere in Ω.

Moreover, w and φ can be chosen to ensure the following:

(a) *If $\alpha=a$ then φ is decreasing.*

(b) *If $\alpha=A$ then φ is increasing.*

(c) *If $a<\alpha<A$ then there exist numbers m_1, m_2, with*

$$\operatorname{ess\,inf}\psi<m_2\leq m_1<\operatorname{ess\,sup}\psi,$$

such that φ is positive and decreasing on $(-\infty, m_2)$, $\varphi=0$ on $[m_2, m_1]$, and φ is positive and increasing on (m_1, ∞).

Proof. Let f denote some rearrangement on $[0, \omega]$ of f_0 (for example, we could take f to be the increasing rearrangement f_0^* of f_0). For $0\leq\lambda\leq\omega$ define

$$f_{1,\lambda}(s)=\begin{cases}f(s), & 0\leq s\leq\lambda,\\ 0, & \lambda<s\leq\omega,\end{cases}$$

$$f_{2,\lambda}(s)=f(s)-f_{1,\lambda}(s),\quad 0\leq s\leq\omega.$$

For $i=1,2$ let $\mathscr{F}_{i,\lambda}$ be the set of rearrangements of $f_{i,\lambda}$ on Ω and let $\bar{\mathscr{F}}_{i,\lambda}$ be the weak closure of $\mathscr{F}_{i,\lambda}$. For $u\in\bar{\mathscr{F}}_{1,\lambda}$ define

$$\Phi_\lambda(u) = \inf_{v \in \mathscr{F}_{2,\lambda}} \Phi(u+v)$$

and let

$$\sigma(\lambda) = \sup_{u \in \mathscr{F}_{1,\lambda}} \Phi_0(u).$$

Then Theorem 3.2 shows there exist $\bar{u}_\lambda \in \mathscr{F}_{1,\lambda}$ and $\bar{v}_\lambda \in \mathscr{F}_{2,\lambda}$ such that

$$\Phi(\bar{u}_\lambda+\bar{v}_\lambda) = \Phi_\lambda(\bar{u}_\lambda) = \sigma(\lambda),$$

such that

$$\bar{u}_\lambda+\bar{v}_\lambda = \varphi_\lambda \circ (K(\bar{u}_\lambda+\bar{v}_\lambda)+h)$$

almost everywhere in Ω for some function φ_λ, and such that $\bar{u}_\lambda \bar{v}_\lambda=0$ almost everywhere, so $\bar{u}_\lambda+\bar{v}_\lambda \in \mathscr{F}$. Since $\sigma(0)=a$ and $\sigma(\omega)=A$, to show that $\sigma[0,\omega]=[a,A]$ it will suffice to show that σ is a continuous function.

To prove continuity of σ let $\lambda, \xi \in [0,\omega]$ and define

$$\gamma(\lambda, \xi) = \left| \int_\lambda^\xi f^p \right|^{1/p},$$

so by (1) we have

$$\|f^*_{1,\lambda}-f^*_{1,\xi}\|_p \leqslant \gamma(\lambda, \xi).$$

Now $\bar{u}_\lambda \in \mathscr{F}_{1,\lambda}$ satisfies $\Phi_\lambda(\bar{u}_\lambda)=\sigma(\lambda)$; let us choose $u_\xi \in \mathscr{F}_{1,\xi}$ with $\|\bar{u}_\lambda-u_\xi\|_p \leqslant \gamma(\lambda, \xi)$. Now choose $v_\xi \in \bar{\mathscr{F}}_{2,\xi}$ with $\Phi(u_\xi+v_\xi)=\Phi_\xi(u_\xi)$, and by Lemma 2.3 choose $v_\lambda \in \bar{\mathscr{F}}_{2,\lambda}$ such that $\|v_\lambda-v_\xi\|_p \leqslant \gamma(\lambda, \xi)$. Write $w_\lambda=\bar{u}_\lambda+v_\lambda$ and $w_\xi=u_\xi+v_\xi$. Then

$$\begin{aligned}
\sigma(\xi) &\geqslant \Phi_\xi(u_\xi) = \Phi(w_\xi)\\
&= \Phi(w_\lambda)+\frac{1}{2}\int_\Omega (w_\xi-w_\lambda)\, K(w_\xi+w_\lambda)\, d\mu+\int_\Omega (w_\xi-w_\lambda)h\, d\mu\\
&\geqslant \Phi_\lambda(\bar{u}_\lambda)-\tfrac{1}{2}\|w_\xi-w_\lambda\|_p\, \|K\|\, \|w_\xi+w_\lambda\|_p-\|h\|_q\, \|w_\xi-w_\lambda\|_p\\
&\geqslant \sigma(\lambda)-2\gamma(\lambda, \xi)(\|K\|\, \|f_0\|_p+\|h\|_q).
\end{aligned}$$

The same inequality holds with λ, ξ interchanged, so

$$|\sigma(\gamma)-\sigma(\xi)| \leqslant 2\gamma(\lambda, \xi)(\|K\|\, \|f_0\|_p+\|h\|_q).$$

Since $\gamma(\lambda, \xi)\to 0$ as $\xi \to \lambda$ we deduce the continuity of σ.

Finally we consider the function φ. If $\lambda=0$ then $f_{1,\lambda}=0$, so φ is decreasing, and $\sigma(\lambda)=a$. If $\lambda=\omega$ then $f_{2,\lambda}=0$, so φ is increasing, and $\sigma(\lambda)=A$. If $a<\sigma(\lambda)<A$ then $f_{1,\lambda}$ and $f_{2,\lambda}$ must both be nonzero, then Theorem 3.2 shows $w=\bar{u}_\lambda+\bar{v}_\lambda$ and φ can be chosen so there exist numbers $m_2\leq m_1$ such that, writing $\psi=Kw+h$, we have $\bar{v}_\lambda^{-1}(0,\infty)=\psi^{-1}(-\infty,m_2)$, $\bar{u}_\lambda^{-1}(0,\infty)=\psi^{-1}(m_1,\infty)$ apart from sets of zero measure, φ is positive and decreasing on $(-\infty,m_2)$, φ is zero on $[m_2,m_1]$ and φ is positive and increasing on (m_1,∞). It follows that $m_2>\operatorname{ess\,inf}\psi$ and $m_1<\operatorname{ess\,sup}\psi$. □

THEOREM 3.4. *Let the hypotheses* (H) *be satisfied. Let* Ω_1,Ω_2 *be measurable subsets of* Ω. *For* $i=1,2$ *let* $f_i\in L^p(\mu)$ *be a non-negative function that vanishes on* $\Omega\setminus\Omega_i$, *and let* $\mathcal{F}_i$ *be the set of rearrangements of* f_i *on* Ω *that vanish on* $\Omega\setminus\Omega_i$.

Then $\Phi(u+v)$ *attains its supremum subject to* $(u,v)\in\mathcal{F}_1\times\mathcal{F}_2$. *If* $(\bar{u},\bar{v})$ *is a maximiser and* $\psi=K(\bar{u}+\bar{v})+h$, *then there exist increasing functions* φ_1,φ_2 *such that*

$$\bar{u}=\varphi_1\circ\psi \text{ almost everywhere in } \Omega_1, \text{ and}$$

$$\bar{v}=\varphi_2\circ\psi \text{ almost everywhere in } \Omega_2.$$

Proof. Let $\bar{\mathcal{F}}_1$ and $\bar{\mathcal{F}}_2$ be the weak closures in $L^p(\mu)$ of $\mathcal{F}_1$ and $\mathcal{F}_2$ respectively. Thus all elements of $\bar{\mathcal{F}}_1$ and $\bar{\mathcal{F}}_2$ are non-negative. Then by compactness $\Phi(u+v)$ attains its supremum subject to $(u,v)\in\bar{\mathcal{F}}_1\times\bar{\mathcal{F}}_2$. Let $(\bar{u},\bar{v})\in\bar{\mathcal{F}}_1\times\bar{\mathcal{F}}_2$ be a maximiser, and write $\psi=K(\bar{u}+\bar{v})+h$. Consider any $u\in\bar{\mathcal{F}}_1$ and $0<t\leq 1$. Then $(1-t)\bar{u}+tu\in\bar{\mathcal{F}}_1$ since $\bar{\mathcal{F}}_1$ is convex by Lemma 2.1, so

$$t^{-1}(\Phi((1-t)\,\bar{u}+tu+\bar{v})-\Phi(\bar{u}+\bar{v}))\leq 0.$$

Letting $t\to 0$ we obtain

$$\int_\Omega \psi(u-\bar{u})\,d\mu\leq 0.$$

Moreover $\mathcal{L}\psi=\bar{u}+\bar{v}+\mathcal{L}h\geq\bar{u}$. It now follows from Lemma 2.2 that $\bar{u}\in\mathcal{F}_1$ and that $\bar{u}=\varphi_1\circ\psi$ almost everywhere in Ω_1 for some increasing function φ_1. A similar argument shows that $\bar{v}\in\mathcal{F}_2$ and $\bar{v}=\varphi_2\circ\psi$ almost everywhere in Ω_2 for some increasing function φ_2. □

4. Application to fluid mechanics

We consider an ideal fluid (inviscid and incompressible) flowing without body forces in a bounded planar connected open region Ω, whose boundary is assumed to be a disjoint union of simple closed curves $C_0, \ldots, C_n$; we suppose that C_0 encloses Ω. If $\mathbf{u}$ denotes the fluid velocity field, then $\mathbf{u}$ must be tangential at the boundary.

The vorticity field ζ is defined by

$$\operatorname{curl} \mathbf{u} = \zeta \mathbf{k}$$

where $\mathbf{k}$ is a fixed unit vector perpendicular to the plane of Ω. In an unsteady flow, the functions ζ at any two instants are always rearrangements of one another. Other conserved quantities are the kinetic energy

$$E = \frac{1}{2} \int_\Omega |\mathbf{u}|^2,$$

and the circulations

$$\Gamma_i = \int_{C_i} \mathbf{u} \cdot d\mathbf{s}, \quad i = 0, \ldots, n.$$

We assume that Ω lies in the xy-plane of right-handed cartesian coordinates xyz, that $\mathbf{k}$ is directed along the positive z-axis, and that, viewed from the point $(0, 0, 1)$, C_0 is described anticlockwise and $C_1, \ldots, C_n$ are described clockwise. It then follows from Green's theorem that

$$\int_\Omega \zeta = \sum_{i=0}^{n} \Gamma_i, \tag{3}$$

subject to suitable regularity assumptions. We shall fix a non-negative function ζ_0 on Ω, whose class of rearrangements on Ω we denote $\mathscr{F}$, and fix real numbers $\gamma_1, \ldots, \gamma_n$. We shall then consider flows for which $\zeta \in \mathscr{F}$ and $\Gamma_i = \gamma_i$, $i = 1, \ldots, n$ (Γ_0 being then determined by (3)), and will prove two results showing, provided ζ_0 is nonconstant, that among such flows there exist uncountably many steady ones.

The notion of a stream function proves valuable in what follows. An incompressible flow satisfies $\operatorname{div} \mathbf{u} = 0$ in Ω. Subject to suitable regularity assumptions, a *stream function* ψ then exists, satisfying

$$\mathbf{u} = \left(\frac{\partial \psi}{\partial y}, \frac{-\partial \psi}{\partial x} \right)$$

in Ω; the multiple connectedness of Ω presents no difficulty since $\mathbf{u}$ has zero flux across each C_i. Taking the curl we obtain

$$\zeta = -\Delta \psi.$$

Tangency of **u** on $\partial\Omega$ is equivalent to requiring that ψ is constant on each C_i, and the circulations are given by

$$\Gamma_i = -\int_{C_i} \nabla\psi \cdot \mathbf{n}\, ds, \quad i=0, \ldots, n,$$

where the unit normal **n** is drawn outwards from Ω. The kinetic energy is now given by

$$E = \frac{1}{2}\int_\Omega |\nabla\psi|^2. \tag{4}$$

When the values of ζ and $\Gamma_1, \ldots, \Gamma_n$ are given, then ψ is determined up to an additive constant, and therefore **u** is uniquely determined; we shall always normalise ψ so that $\psi=0$ on C_0. To find a convenient expression for ψ in terms of ζ and $\Gamma_1, \ldots, \Gamma_n$, we introduce the following notation. Let $C_0, \ldots, C_n$ be of class C^2, and let $\gamma_1, \ldots, \gamma_n \in \mathbf{R}$ be prescribed. In the Appendix we prove the existence of a unique function $h \in C^\infty(\Omega) \cap C^1(\bar{\Omega})$ satisfying

$$\left.\begin{aligned} &-\Delta h = 0 \text{ in } \Omega,\\ &h = 0 \text{ on } C_0,\\ &h \text{ is constant on } C_1, \ldots, C_n,\\ &-\int_{C_i} \nabla h \cdot \mathbf{n}\, ds = \gamma_i, \quad i = 1, \ldots, n. \end{aligned}\right\} \tag{5}$$

For $\zeta \in L^p(\Omega)$ we also prove in the Appendix the existence of a unique function $K\zeta \in W^{2,p}(\Omega)$ satisfying

$$\left.\begin{aligned} &-\Delta(K\zeta) = \zeta \text{ in } \Omega,\\ &K\zeta = 0 \text{ on } C_0,\\ &K\zeta \text{ is constant on } C_1, \ldots, C_n,\\ &\int_{C_i} \nabla(K\zeta) \cdot \mathbf{n}\, ds = 0, \quad i = 1, \ldots, n. \end{aligned}\right\} \tag{6}$$

Then $K: L^p(\Omega) \to L^q(\Omega)$ is a symmetric, compact, positive linear operator, and if $\zeta \in L^p(\Omega)$ then $K\zeta + h$ is the stream function for the flow with vorticity ζ and circulations $\gamma_1, \ldots, \gamma_n$. If we set $\Psi = K\zeta$, $\psi = K\zeta + h$ and apply the divergence theorem to (4) we obtain

$$E=\frac{1}{2}\int_\Omega|\nabla\psi|^2=\frac{1}{2}\int_\Omega|\nabla\Psi|^2+\int_\Omega\nabla\Psi\cdot\nabla h+\frac{1}{2}\int_\Omega|\nabla h|^2$$
$$=\frac{1}{2}\int_\Omega(\operatorname{div}(\Psi\nabla\Psi)-\Psi\Delta\Psi)+\int_\Omega(\operatorname{div}(h\nabla\Psi)-h\Delta\psi)+\frac{1}{2}\int_\Omega|\nabla h|^2$$
$$=\frac{1}{2}\sum_{i=0}^n\int_{C_i}\Psi\nabla\Psi\cdot\mathbf{n}-\frac{1}{2}\int_\Omega\Psi\Delta\Psi+\sum_{i=0}^n\int_{C_i}h\nabla\Psi\cdot\mathbf{n}-\int_\Omega h\Delta\Psi+\frac{1}{2}\int_\Omega|\nabla h|^2.$$

Taking into account (5) and (6) we now obtain

$$E=E(\zeta)=\frac{1}{2}\int_\Omega\zeta K\zeta+\int_\Omega h\zeta+\frac{1}{2}\int_\Omega|\nabla h|^2; \tag{7}$$

thus E has been expressed as a function of ζ, and the third integral in (7) is a constant.

To verify that a flow is steady, it will suffice to show that the stream function ψ satisfies

$$-\Delta\psi=\varphi(\psi)\quad\text{in }\Omega \tag{8}$$

for some function φ; the relationship between (8) and the Euler equations for an ideal fluid will be discussed at the end of the section. The following theorem is an immediate consequence of Theorem 3.3.

THEOREM 4.1. *Let Ω be a nonempty, bounded, connected, open set in $\mathbf{R}^2$, whose boundary is a disjoint union of simple closed curves $C_0,\ldots,C_n$ of class C^2, with C_0 enclosing Ω, let $2<p<\infty$, and let h, K, E be defined by (5), (6), (7), where $\gamma_1,\ldots,\gamma_n$ are prescribed real numbers. Let $\zeta_0\in L^p(\Omega)$ be non-negative, and let $\mathcal{F}$ be the set of all rearrangements of ζ_0 on Ω.*

Then for each α, $\inf E(\mathcal{F})\leq\alpha\leq\sup E(\mathcal{F})$, there exists $\zeta\in\mathcal{F}$ such that $E(\zeta)=\alpha$, and such that $\psi=K\zeta+h$ satisfies

$$-\Delta\psi=\varphi(\psi)$$

almost everywhere in Ω, for some function φ; that is, $\mathcal{F}$ contains an element representing the vorticity of a steady ideal fluid flow, with kinetic energy α and with circulations $\gamma_1,\ldots,\gamma_n$ around $C_1,\ldots,C_n$.

Moreover, the choice of ζ and φ can be made such that if $\alpha=\inf E(\mathcal{F})$ then φ is decreasing, if $\alpha=\sup E(\mathcal{F})$ then φ is increasing, and if $\inf E(\mathcal{F})<\alpha<\sup E(\mathcal{F})$ then there exist numbers m_1,m_2 with $\inf\Psi<m_2\leq m_1<\sup\Psi$, such that φ is positive and decreasing on $(-\infty,m_2)$, $\varphi=0$ on $[m_2,m_1]$, and φ is positive and increasing on (m_1,∞).

It follows from the remarks after Corollary 3.4 of [5] that the minimiser is unique; if ζ_0 is nonconstant, then $\inf E(\mathscr{F})<\sup E(\mathscr{F})$ and so uncountably many steady solutions are obtained. The existence of steady flows having maximum or minimum energy can also be deduced easily from previous work [3, 5, 8]. Consider now the particular case when the boundary of Ω comprises just one simple closed curve C_0; then $h=0$. Then let ψ be the stream function corresponding to a non-maximising solution constructed by Theorem 4.1; thus $\inf\psi=0$ and is attained only on C_0. Then for some number $m>0$, the function φ occurring in (8) is positive on $(0,m)$, from which it follows that the vorticity is positive at all points sufficiently close to C_0.

Results on multiple solutions for rearrangement problems have previously been given in [5], where an appropriate form of the Mountain Pass lemma was proved. Fluid flow in a dumb-bell shaped region Ω was studied there, and for suitably chosen ζ_0, four steady configurations were given, one being the minimiser, two being local maximisers and the fourth being constructed by the Mountain Pass lemma. All the solutions constructed in [5] satisfy equation (8) in Ω for a monotonic φ, in contrast with Theorem 4.1 of the present paper.

We next turn to the existence of continuum-many steady configurations in a dumb-bell shaped region. The vortex cores of the solutions we construct avoid the boundary of the region, but equation (8) is only satisfied locally.

THEOREM 4.2. *Let $2<p<\infty$, let $b>a>0$, let $\omega>2\pi b^2$, let $0<\lambda<\frac{1}{2}$, let $f_0\in L^p[0,\omega]$ be positive almost everywhere on $[0,\pi a^2]$ and zero almost everywhere on $[\pi a^2,\omega]$, and suppose V is a closed triangle in $\mathbf{R}^2$. Then there exists $\delta>0$ such that the following holds: Assume*

(i) *that Ω is the planar region enclosed by a simple closed curve C_0 of class C^2, that Ω has the V-exterior cone property, that Ω countains open discs Ω_1 and Ω_2 of radius b, such that $\bar{\Omega}_1\cap\bar{\Omega}_2=\varnothing$, that every point of $\Omega_0=\Omega\setminus(\Omega_1\cup\Omega_2)$ lies within distance δ of C_0, and that $|\Omega|=\omega$,*

(ii) *that K and E are defined by* (6) *and* (7) *with $h=0$ and $n=0$,*

(iii) *that $\beta\in(\lambda,\frac{1}{2})$ is fixed, that*

$$f_1(s)=\begin{cases} f_0(s), & 0<s<\beta\pi a^2,\\ 0, & \beta\pi a^2<s<\omega,\end{cases}$$

$$f_2=f_0-f_1,$$

and that for $i=1,2$, $\mathscr{F}_i$ consists of the set of all rearrangements of f_i on Ω that vanish throughout $\Omega\setminus\Omega_i$.

Then $\Phi(\zeta_1+\zeta_2)$ attains its supremum subject to $(\zeta_1, \zeta_2) \in \mathscr{F}_1 \times \mathscr{F}_2$. If $(\bar{\zeta}_1, \bar{\zeta}_2)$ is a maximiser, if $\zeta = \bar{\zeta}_1 + \bar{\zeta}_2$, and if $\psi = K\zeta$, then ζ is a rearrangement of f_0 and there exist increasing functions φ_1 and φ_2 such that

$$-\Delta\psi = \varphi_i \circ \psi$$

almost everywhere in $\Omega_i \cup \Omega_0$, for $i=1,2$.

Proof. Let ξ be the lesser of $\int_0^{\lambda\pi a^2} f_0$ and $\int_{\pi a^2/2}^{\pi a^2} f_0$, and let D_1 and D_2 be the discs of radius a concentric with Ω_1 and Ω_2 respectively. By a weak compactness argument in conjunction with the Maximum principle, it follows that there is an $\eta>0$, depending only on a, b, ξ and $\|f_0\|_p$, such that for all non-negative $v \in L^p(\Omega_i)$ satisfying $\|v\|_1 \geq \xi$ and $\|v\|_p \leq \|f_0\|_p$, the solution $u \in H_0^1(\Omega_i)$ of $-\Delta u = v$ satisfies $u \geq \eta$ in D_i. Then by the Maximum principle we have $K(\zeta_1+\zeta_2) \geq \eta$ in $D_1 \cup D_2$ for all $\zeta_1 \in \mathscr{F}_1$ and $\zeta_2 \in \mathscr{F}_2$. By Theorems 8.16 and 8.27 of [7] there are constants $\alpha = \alpha(p, V) > 0$ and $c = c(p, V, \omega)$ such that if $v \in L^p(\Omega)$, if $x \in \Omega$, if $x_0 \in C_0$ and $|x-x_0|<1$ then $|Kv(x)| \leq c|x-x_0|^\alpha \|v\|_p$. We choose $\delta \in (0,1)$ such that $c\delta^\alpha \|f_0\|_p < \eta$.

Now consider a maximiser $(\bar{\zeta}_1, \bar{\zeta}_2)$ for $\Phi(\zeta_1+\zeta_2)$ relative to $(\zeta_1, \zeta_2) \in \mathscr{F}_1 \times \mathscr{F}_2$, let $\zeta = \bar{\zeta}_1 + \bar{\zeta}_2$ and let $\psi = K\zeta$. By Theorem 3.4 there exists such a maximiser, and there exist increasing functions φ_1 and φ_2 such that $\bar{\zeta}_i = \varphi_i \circ \psi$ almost everywhere in Ω_i for $i=1,2$. Now

$$|\{x \in \Omega_i \mid \psi(x) \geq \eta\}| \geq \pi a^2 > |\{x \in \Omega_i \mid \bar{\zeta}_i(x) > 0\}|$$

so since $\bar{\zeta}_i$ is essentially an increasing function of ψ on Ω_i, we deduce that

$$\{x \in \Omega_i \mid \bar{\zeta}_i(x) > 0\} \subset \{x \in \Omega_i \mid \psi(x) \geq \eta\}$$

apart from a set of measure zero. Hence we can assume $\varphi_i(s)=0$ for all $s<\eta$. By the choice of δ, we have $\psi(x)<\eta$ for all $x \in \Omega_0$, and $\bar{\zeta}_i(x)=0$ for all $x \in \Omega_0$. So $\zeta(x) = \bar{\zeta}_i(x) = \varphi_i(\psi(x))$ for all $x \in \Omega_i \cup \Omega_0$, for $i=1,2$. □

For the solutions constructed in Theorem 4.2, the vortex core avoids C_0. For in $\Omega_i \cup \Omega_0$, the vorticity ζ is an increasing function of the stream function ψ, and the area of the vortex core is less than $|\Omega_i|$, so the vortex core avoids the set where ψ attains its minimum, which is C_0. Since $\bar{\Omega}_1 \cap \bar{\Omega}_2 = \emptyset$, the vortex core is disconnected. Uncountably many solutions may be constructed on the same Ω, by varying β in $(\lambda, \frac{1}{2})$.

At this stage some remarks on regularity and on the Euler equations are in order.

Since $C_0, \ldots, C_n$ are of class C^2 and $\zeta \in L^p(\Omega)$ for some $p>2$, the stream function ψ is of class $W^{2,p}$, so the velocity $\mathbf{u}$ is of class $W^{1,p}$, and is therefore continuous on $\bar{\Omega}$. The steady Euler equations for an ideal fluid are

$$(\mathbf{u} \cdot \nabla)\mathbf{u} = -\nabla P, \tag{9}$$

$$\operatorname{div} \mathbf{u} = 0. \tag{10}$$

The equality of weak cross derivatives for ψ ensures that (10) is satisfied almost everywhere. To fulfill (9) we have to construct a pressure function P. Consider first the solutions constructed in Theorem 4.1. If φ is as in equation (8) and $F(s)=\int_0^s \varphi$ then it is easily verified that

$$-P(x) = \frac{1}{2}|\nabla\psi|^2 + F(\psi)$$

satisfies (9) formally. It can be shown that $F \circ \psi \in W^{1,p}_{\mathrm{loc}}$ and $\nabla(F \circ \psi) = (\varphi \circ \psi)\nabla\psi$ almost everywhere (the details for an increasing φ have been given during the proof of Lemma 9 of [4], and the φ considered here is a difference of increasing functions). The formal derivation of (9) is then justified for weak derivatives, and (9) holds almost everywhere. In the case of a solution constructed by Theorem 4.2, define $F_i(s)=\int_0^s \varphi_i$ and

$$-P(x) = \frac{1}{2}|\nabla\psi|^2 + F_i(\psi), \quad x \in \Omega_i \cup \Omega_0, \quad i = 1, 2.$$

Then $F_1(\psi)=0=F_2(\psi)$ in Ω_0 and $\bar{\Omega}_1 \cap \bar{\Omega}_2 = \varnothing$ so P is well defined and continuous, and (9) is satisfied almost everywhere.

Appendix

Since the boundary conditions in (5) and (6) of Section 4 are slightly unusual, we now give a detailed proof of the existence and uniqueness of solutions. Both (5) and (6) are special cases of the boundary value problem (11) considered below.

PROPOSITION. *Let Ω be a nonempty, bounded, connected open set in $\mathbf{R}^2$, whose boundary is a disjoint union of simple closed curves $C_0, \ldots, C_n$ of class C^2, and suppose C_0 encloses Ω. Let $2<p<\infty$, let $\gamma_1, \ldots, \gamma_n$ be real numbers, and let $v \in L^p(\Omega)$. Then there is exactly one function u satisfying*

$$\left.\begin{array}{l} u \in W^{2,p}(\Omega), \\ -\Delta u = v \text{ } in \text{ } \Omega, \\ u = 0 \text{ } on \text{ } C_0, \\ u \text{ } is \text{ } constant \text{ } on \text{ } C_i, \quad i=1,\dots,n, \\ -\int_{C_i} (\nabla u)\cdot \mathbf{n}\, ds = \gamma_i, \quad i=1,\dots,n. \end{array}\right\} \tag{11}$$

Proof. Let $\Omega_0, \dots, \Omega_n$ be the regions enclosed by $C_0, \dots, C_n$. Let

$$W = \{w \in H^1(\Omega) |\ w = 0 \text{ on } C_0 \text{ and } w \text{ is constant on } C,\ i=1,\dots,n\}$$

and for $w \in W$ let w_i denote the value of w on C_i, $i=1,\dots,n$. Define

$$J(w) = \frac{1}{2}\int_\Omega |\nabla w|^2 - \int_\Omega vw + \sum_{i=1}^n \gamma_i w_i, \quad w \in W.$$

Then the trace $H^1(\Omega) \to L^2(\partial\Omega)$ ensures that W is a closed linear subspace of $H^1(\Omega)$, and W comprises the restrictions to Ω of elements of $H_0^1(\Omega_0)$ that are constant on Ω_i, $i=1,\dots,n$. It now follows from Poincaré's inequality for $H_0^1(\Omega_0)$ that J is coercive on W. Moreover J is a smooth, strictly convex functional on W. It follows that J possesses exactly one critical point, the global minimiser.

The variational condition for an element $u \in W$ to be a critical point of J is that

$$\int_\Omega \nabla u \cdot \nabla w - \int_\Omega uw + \sum_{i=1}^n \gamma_i w_i = 0, \quad \forall w \in W. \tag{12}$$

Let Lipschitz functions $g^1, \dots, g^n \in W$ be chosen to satisfy the boundary conditions $g_i^j = \delta_{ij}$, $1 \leq i,j \leq n$. Then (12) is equivalent to

$$\int_\Omega \nabla u \cdot \nabla w - \int_\Omega uw = 0, \quad \forall w \in H_0^1(\Omega), \tag{13}$$

$$\int_\Omega \nabla u \cdot \nabla g^j - \int_\Omega vg^j + \gamma_j = 0, \quad j=1,\dots,n. \tag{14}$$

Now (13) is a variational formulation of $-\Delta w = v$, and in view of the regularity theory ([7], Theorem 9.15), (13) is equivalent to

$$\left.\begin{array}{l} -\Delta u = v \text{ in } \Omega, \\ u \in W^{2,p}(\Omega). \end{array}\right\} \tag{15}$$

Since $p>2$ we have the embedding $W^{2,p}(\Omega)\to C^1(\bar{\Omega})$. When (15) holds we can apply the Divergence theorem to write (14) in the form

$$\int_\Omega (-\Delta u-v)g^j+\sum_{i=1}^n \int_{C_i} g^j \nabla u\cdot \mathbf{n}+\gamma_j=0,\quad j=1,\ldots,n,$$

which reduces to

$$\int_{C_j} \nabla u\cdot \mathbf{n}+\gamma_j=0,\quad j=1,\ldots,n.$$

It follows that (11) holds if and only if u is a critical point of J, and therefore (11) has exactly one solution. □

The existence and uniqueness of h satisfying (5) follows by taking $v=0$; in this case, the regularity of harmonic functions and the embedding $W^{2,p}(\Omega)\to C^1(\bar{\Omega})$ show that $h\in C^\infty(\Omega)\cap C^1(\bar{\Omega})$. The existence and uniqueness of $K\zeta$ satisfying (6) is obtained by taking $v=\zeta$ and $\gamma_1=\ldots=\gamma_n=0$.

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