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Monstrous moonshine of higher weight

by

CHONGYING DONG and GEOFFREY MASON

University of California, Santa Cruz, CA, U.S.A.

University of California, Santa Cruz, CA, U.S.A.

1. Introduction

Suppose that V is a vertex operator algebra [B1], [FLM]. One of the basic problems is that of determining the so-called n-point correlation functions associated to V. There is a recursive procedure whereby n-point functions determine $(n+1)$-point functions [Z], so that understanding 1-point functions becomes important. In this paper we will study the 1-point functions on the torus associated with the moonshine module, which is of interest not only as an example of the general problem but because of connections with the monster simple group $\mathbf{M}$.

First we recall the definition of a 1-point function. Let the decomposition of V into homogeneous spaces be given by

$$V=\bigoplus_{n\geqslant n_0} V_n. \tag{1.1}$$

Each $v\in V$ is associated to a vertex operator

$$Y(v,z)=\sum_{n\in\mathbf{Z}} v(n)z^{-n-1} \tag{1.2}$$

with $v(n)\in\operatorname{End} V$. If v is homogeneous of weight k, that is, $v\in V_k$, we write $\operatorname{wt} v=k$. The *zero mode* of v is defined for homogeneous v to be the component operator

$$o(v)=v(\operatorname{wt} v-1), \tag{1.3}$$

and one knows that $o(v)$ induces an endomorphism of each homogeneous space, that is,

$$o(v)\colon V_n\to V_n. \tag{1.4}$$

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The 1-*point function determined by* v is then essentially the graded trace of $o(v)$ on V. More precisely, if V has central charge c we define the 1-point function (on the torus) via

$$Z(v,q)=Z(v,\tau)=\mathrm{tr}|_V o(v)q^{L(0)-c/24}=q^{-c/24}\sum_{n\geqslant n_0}(\mathrm{tr}|_{V_n}o(v))q^n. \tag{1.5}$$

Here, $L(0)$ is the usual degree operator and q may be taken either as an indeterminate or, less formally, to be $e^{2\pi i\tau}$ with τ in the upper half-plane $\mathfrak{h}$. If g is an automorphism of V we define

$$Z(v,g,q)=Z(v,g,\tau)=q^{-c/24}\sum_{n\geqslant n_0}(\mathrm{tr}|_{V_n}o(v)g)q^n. \tag{1.6}$$

These functions can be extended linearly to all $v\in V$ by defining $Z(v,g,q)=\sum_i Z(v_i,g,q)$ if $v=\sum_i v_i$ is the decomposition of v into homogeneous components. In this way we obtain the space of 1-point functions associated to V, namely the functions $Z(v,q)$ for $v\in V$.

In order to state our results efficiently we need some notation concerning modular forms. We denote by $\mathcal{F}$ the $\mathbf{C}$-linear space spanned by those (meromorphic) modular forms $f(\tau)$ of level 1 and integral weight $k\geqslant 0$ which satisfy

(i) $f(\tau)$ is holomorphic in $\mathfrak{h}$;

(ii) $f(\tau)$ has Fourier expansion of the form

$$f(\tau)=\sum_{n=-1}^{\infty}a_nq^n,\quad a_0=0. \tag{1.7}$$

Thus $f(\tau)$ has a pole of order at most 1 at infinity and constant 0. Let $\mathcal{M}$ be the space of holomorphic modular forms of level 1, and $\mathcal{S}$ the space of cusp forms of level 1. Thus we have $\mathcal{S}=\mathcal{F}\cap\mathcal{M}$.

Among the elements of $\mathcal{M}$ are the Eisenstein series $E_k(\tau)$ for even $k\geqslant 4$. We normalize them as in [DLM], namely

$$E_k(\tau)=\frac{-B_k}{k!}+\frac{2}{(k-1)!}\sum_{n=1}^{\infty}\sigma_{k-1}(n)q^n \tag{1.8}$$

with B_k the kth Bernoulli number defined by

$$\frac{t}{e^t-1}=\sum_{k=0}^{\infty}B_k\frac{t^k}{k!}. \tag{1.9}$$

If $\mathcal{M}_k$ is the space of forms $f(\tau)\in\mathcal{M}$ of weight k then there is a differential operator $\partial\colon\mathcal{M}_k\to\mathcal{M}_{k+2}$ defined via

$$\partial=\partial_k\colon f(\tau)\mapsto\frac{1}{2\pi i}\frac{d}{d\tau}f(\tau)+kE_2(\tau)f(\tau). \tag{1.10}$$

Here, $E_2(\tau)$ is again defined by (1.8), though E_2 is not a modular form.

By a ∂-*ideal* we mean an ideal $\mathcal{I}$ in the commutative algebra $\mathcal{M}$ which also satisfies $\partial(\mathcal{I})\subset\mathcal{I}$.

THEOREM 1. *Let $V^\natural$ be the moonshine module. The space of* 1*-point functions associated to $V^\natural$ is precisely the linear space $\mathcal{F}$ defined above.*

As we will explain in due course, it is a consequence of results in [Z] (see also [DLM]) that all 1-point functions associated to vectors $v\in V^\natural$ lie in $\mathcal{F}$. The new result here is therefore an existence result: for each $f(\tau)\in\mathcal{F}$ there is a $v\in V^\natural$ such that $Z(v,\tau)=f(\tau)$.

Recall next that $V^\natural$ is a direct sum of irreducible highest weight modules $M(c,k)$ for the Virasoro algebra Vir. Here, $c=24$ and for $k>0$, $M(c,k)$ is the Verma module generated by a highest weight vector $v\in V_k^\natural$. Thus $L(n)v=0$ for all $n>0$, where $L(n)$ are the usual generators for Vir, and $L(0)v=kv$.

The proof of Theorem 1 is facilitated by the next result.

PROPOSITION 2. *Let $v\in V_k^\natural$ be a highest weight vector of positive weight k. Then the following hold:*

(a) *$Z(v,\tau)$ is a cusp form of weight k, possibly* 0;

(b) *The space of* 1*-point functions consisting of all $Z(w,\tau)$ for w in the highest weight module for* Vir *generated by v is the ∂-ideal generated by $Z(v,\tau)$.*

While Proposition 2 actually holds for a wide class of vertex operator algebras, our final result is more closely tied to the structure of $V^\natural$. It gives us a large set of highest weight vectors (for the Virasoro algebra) to which we can usefully apply the preceding proposition.

First recall that to each λ in the Leech lattice Λ there is a corresponding element e^λ in the group algebra $\mathbf{C}[\Lambda]$ and an element, also denoted e^λ, in the vertex operator algebra V_Λ associated to Λ. See [B1], [FLM] and §4 below for more details. The relation of V_Λ to $V^\natural$ shows that $e^\lambda+e^{-\lambda}$ can be considered as an element of both vertex operator algebras.

THEOREM 3. *Let $v(\lambda)=e^\lambda+e^{-\lambda}$ be as above and considered as an element of $V^\natural$. Then $v(\lambda)$ is a highest weight vector of weight $k=\frac{1}{2}\langle\lambda,\lambda\rangle$, and if $0\neq\lambda\in 2\Lambda$ then*

$$Z(v(\lambda),\tau)=\eta(\tau)^{12}\big\{\big(\tfrac{1}{2}\Theta_1(\tau)\big)^{\langle\lambda,\lambda\rangle-12}+\big(\tfrac{1}{2}\Theta_2(\tau)\big)^{\langle\lambda,\lambda\rangle-12}-\big(\tfrac{1}{2}\Theta_3(\tau)\big)^{\langle\lambda,\lambda\rangle-12}\big\}. \quad (1.11)$$

In (1.11), $\eta(\tau)$ is the Dedekind eta-function, and $\Theta_1,\Theta_2,\Theta_3$ are the usual Jacobi theta-functions (see, for example, [Ch, p. 69]).

If $\Lambda_n=\{\lambda\in\Lambda\,|\,\langle\lambda,\lambda\rangle=2n\}$ then $\Lambda_2=0$, so if $0\neq\lambda\in2\Lambda$ then $\frac{1}{2}\langle\lambda,\lambda\rangle=4m$ with $m\geqslant2$. If $m=2$ then $Z(v(\lambda),\tau)$ is a cusp form of level 1 and weight 8 by Proposition 2, and hence must be 0. Then (1.11) reduces to the identity $\Theta_1(\tau)^4+\Theta_2(\tau)^4-\Theta_3(\tau)^4=0$, which is well known in the theory of elliptic functions (loc. cit.). If $m\geqslant3$ then one can check that $Z(v(\lambda),\tau)\neq0$ (for example by looking at the coefficient of q in the Fourier expansion), so

$Z(v(\lambda),\tau)$ is a non-zero cusp form of level 1 and weight $4m=12,16,20,\ldots$. One knows (see, for example, [S]) that the cusp forms of level 1 and weights 12, 16, 20 are unique up to scalar (as are those of weight 18, 22 and 26), and given by $\Delta(\tau)$, $\Delta(\tau)E_4(\tau)$, $\Delta(\tau)E_8(\tau)$ respectively, where $\Delta(\tau)=\eta(\tau)^{24}$ is the discriminant. Once we know that $\Delta(\tau)$ can be realized as a 1-point function $Z(v,\tau)$ for some highest weight vector v, the fact that $\mathcal{S}=\mathcal{M}\Delta(\tau)$ (loc. cit.) together with Proposition 2 then shows that every $f(\tau)\in\mathcal{S}$ can be so realized. This in turn reduces the proof of Theorem 1 to dealing with forms which have a pole at infinity.

Our discussion so far has not taken into account the automorphisms g of $V^\natural$ (that is, elements of the monster). There are some general results, which follow from [DLM], which imply that if $v\in V^\natural$ is homogeneous of weight k with respect to a certain operator $L[0]$, then $Z(v,g,\tau)$ is a modular form of weight k for each $g\in\mathbf{M}$. Moreover, the level is the same as that for the McKay–Thompson series $Z(\mathbf{1},g,\tau)$ described in [CN] and proved in [B2]. We describe the precise subgroup of $\mathrm{SL}(2,\mathbf{Z})$ which fixes $Z(v,g,\tau)$ in Theorem 6.1.

Group theorists may be disappointed to learn that if we fix v so that all $Z(v,g,\tau)$ are modular forms of weight k then in general the Fourier coefficients of the forms (for varying g) do *not* define characters, or even generalized characters. This is so even if $Z(v,1,\tau)$ has integer coefficients. This does not mean, however, that these higher weight McKay–Thompson series are of no arithmetic interest. If we combine our results with some calculations of Harada and Lang [HL], for example, we find that for each of the weights $k=12,16,20$ there is a *unique* vector v in the moonshine module $V^\natural$ with the following properties:

(a) v is a highest weight vector for Vir which lies in $V_k^\natural$ and is monster-invariant;

(b) The 1-point function $Z(v,\tau)=q+\ldots$ is the unique normalized cusp form of level 1 and weight k.

Such a v may be obtained by averaging the vector $v(\lambda)$ of Theorem 3 over the monster ($\lambda\in 2\Lambda_m$, $m=3,4$ or 5). The unicity of such v makes them entirely analogous to the vacuum vector $\mathbf{1}$, and it is likely that the trace functions $Z(v,g,\tau)$ are of particular interest in these cases.

We can understand the representation-theoretic meaning of the functions $Z(v,g,\tau)$ as follows: since v is monster-invariant then each g commutes with the zero mode $o(v)$ and its semi-simple part $o(v)_s$ with regard to its action on the homogeneous space $V_n^\natural$. Thus if $o(v)_s$ has distinct eigenvalues $\lambda_1,\ldots,\lambda_t$ on $V_n^\natural$, the corresponding eigenspaces $V_{n,1}^\natural,\ldots,V_{n,t}^\natural$ are monster modules and the $(n-1)$th Fourier coefficient of $Z(v,g,\tau)$ is equal to $\sum_{i=1}^t \lambda_i\,\mathrm{tr}_{V_{n,i}^\natural}\,g$.

We complete our discussion with two conjectures:

(A) For each cusp form $f(\tau)\in\mathcal{S}$ of weight k there is a (monster-invariant) highest weight vector $v\in V_k^\natural$ with $Z(v,\tau)=f(\tau)$;

(B) If $Z(v,\tau)$ is a cusp form then so is $Z(v,g,\tau)$ for each monster element g.

The paper is organized as follows: In §2 we review the required results from the theory of vertex operator algebras and prove Proposition 2. In §3 we reduce the proof of Theorem 1 to that of Theorem 3, which is proved by lengthy calculation in §4. In §5 we give an equivariant version of formula (1.11), that is, we calculate $Z(v(\lambda),g,\tau)$ for various elements $g\in\mathbf{M}$, namely those that lie in the centralizer of a central involution, and in §6 we describe the invariance group of $Z(v,g,\tau)$ in $\mathrm{SL}(2,\mathbf{Z})$.

Background from the theory of elliptic functions and modular forms can be found in [Ch] and [S], for example.

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2. Proof of Proposition 2

We start by recalling some results from [Z]. If V is a vertex operator algebra as in (1.1) then there is a second VOA structure $(V,Y[\cdot,\cdot])$ defined on V with vertex operator $Y[v,z]$. The two VOAs are related by a change of variables and have the same vacuum vector $\mathbf{1}$ and central charge c. The conformal vectors are distinct, however, and we denote the standard Virasoro generators for the second VOA by $L[n]$. The relation between the $L(n)$ and $L[n]$ (cf. [Z]) shows that both Virasoro algebras have the *same* highest weight vectors v.

A most important identity for us is the following (cf. [Z] and [DLM, equation (5.8)]): if $w\in V$ then

$$Z(L[-2]w,\tau)=\partial Z(w,\tau)+\sum_{l=2}^{\infty}E_{2l}(\tau)Z(L[2l-2]w,\tau) \tag{2.1}$$

where we are using the notation of §1. We should emphasize that it is a consequence of the main results of [Z] and [DLM] that if v is homogeneous of weight k with respect to $L[0]$, where we are taking $V=V^\natural$ to be the moonshine module, then the trace function $Z(v,\tau)$ is indeed a meromorphic modular form of level 1 which lies in the space $\mathcal{F}$ defined in (1.7).

It is also shown in [Z] (cf. [DLM, equation (5.1)]) that the following holds:

$$Z(L[-1]w,\tau)=0 \quad \text{for all } w\in V. \tag{2.2}$$

We turn to the proof of Proposition 2, beginning with part (a), which is elementary. Namely, from the creation axiom

$$\lim_{z\to 0}Y(v,z)\mathbf{1}=v,$$

we get $v(n)\mathbf{1}=0$ if $n\geqslant 0$. So if $v\in V_k$ with $k>0$ then $o(v)\mathbf{1}=0$, in which case we see that

$$Z(v,\tau)=q^{-1}\sum_{n=2}^{\infty}\mathrm{tr}|_{V_n}o(v)q^n$$

is a modular form of level 1, holomorphic in $\mathfrak{h}$ with a zero of order at least 1 at ∞. So indeed $Z(v,\tau)$ is a cusp form, as asserted in Proposition 2 (a).

We turn to the proof of (b) of Proposition 2, which is established by a systematic use of equations (2.1) and (2.2). Let $v\in V_k$ be a highest weight vector. By a *descendant* of v we will mean a vector of the form $L[n_1]\dots L[n_t]v$ with each $n_i\leqslant 0$, or any linear combination of such vectors; we write $v\to w$ if w is a descendant of v.

Let $I=\langle Z(w,\tau)\,|\,v\to w\rangle$ be the linear span of the indicated forms, and let J be the ∂-ideal generated by $Z(v,\tau)$. We must prove that $I=J$.

First we show that $I\subset J$. We do this by proving by induction on $\mathrm{wt}[w]$ (the weight of w, homogeneous with respect to the second Virasoro algebra) that $Z(w,\tau)\in J$. Because $L[-1]$ and $L[-2]$ generate $L[-n]$ for all $n>0$, we may take w in the form $w=L[n_1]\dots L[n_t]v$ with each $n_i=-1$ or -2. If $n_1=-1$ then $Z(w,\tau)=0$ by (2.2), so we may take $n_1=-2$. So $w=L[-2]x$ where $x=L[n_2]\dots L[n_t]v$ has weight equal to $\mathrm{wt}[w]-2$.

By (2.1) we have

$$Z(w,\tau)=\partial Z(x,\tau)+\sum_{l=2}^{\infty}E_{2l}(\tau)Z(L[2l-2]x,\tau). \tag{2.3}$$

Since $v\to x$ and $v\to L[2k-2]x$, induction tells us that $Z(x,\tau)$ and $Z(L[2l-2]x,\tau)$ both lie in J, whence so does the right-hand side of (2.3) since J is a ∂-ideal. So indeed $Z(w,\tau)$ lies in J.

Next we show that I is also a ∂-ideal. Since $Z(v,\tau)$ is in I it follows from this that $J\subset I$ and hence that $I=J$, as required.

Let $r\geqslant 1$ with $v\to w$ and consider the vector $x=L[-2]L[-1]^{2r}w$. If $2l-2<2r$ then $L[2l-2]L[-1]^{2r}w$ can be written as a linear combination of vectors of the shape $L[-1]u$ for some u. Thus (2.2) tells us that $Z(L[2l-2]L[-1]^{2r}w,\tau)=0$ if $2l-2<2r$. Now by (2.1) we get

$$Z(x,\tau)=\sum_{l=r+1}^{\infty}E_{2l}(\tau)Z(L[2l-2]L[-1]^{2r}w,\tau). \tag{2.4}$$

Assuming that w is homogeneous with respect to the second Virasoro algebra, it follows in the same way that $Z(L[2r]L[-1]^{2r}w,\tau)$ is a non-zero multiple of $Z(w,\tau)$. If $l>r+1$ then $L[2l-2]L[-1]^{2r}w$ has weight less than that of w, while if also $v=w$ then $L[2l-2]L[-1]^{2r}v=0$. Thus (2.4) now reads

$$Z(x,\tau)=\alpha E_{2r+2}(\tau)Z(w,\tau)+\sum_{l=r+2}^{\infty}E_{2l}(\tau)Z(u_l,\tau) \tag{2.5}$$

where $v \to u_l$, $\mathrm{wt}[u_l] < \mathrm{wt}[w]$ and α is a non-zero scalar. From (2.5) and what we have said it follows by induction on $\mathrm{wt}[w]$ that $E_{2r+2}(\tau)Z(w,\tau)$ lies in I whenever $r \geqslant 1$. Since the forms $E_{2r+2}(\tau)$ generate the space $\mathcal{M}$ of modular forms (in fact, $E_4(\tau)$ and $E_6(\tau)$ suffice), it follows that I is an ideal in $\mathcal{M}$. But then (2.1) shows that $\partial Z(w,\tau)$ lies in I whenever $v \to w$, so I is a ∂-ideal. This completes the proof of Proposition 2 (b).

3. Trace functions with a pole

In this section we prove

PROPOSITION 3.1. *Let k be a non-negative integer. Then the trace function $Z(L[-2]^k\mathbf{1},\tau)$ is non-zero, and more precisely has a q-expansion of form $\varepsilon q^{-1}+\ldots$ where $(-1)^k\varepsilon>0$.*

Set $w=L[-2]^k\mathbf{1}$. Note that the truth of the proposition shows that $Z(w,\tau)$ is a form of level 1 and weight $2k$ which is non-zero with a pole at ∞. If we have two such trace functions of the same weight and the same residue at ∞ then they differ by a cusp form. So together with Proposition 2, this reduces the proof of Theorem 1 to showing that $\Delta(\tau)$, say, can be realized as a trace function. As we have pointed out in §1, this is implicit in the statement of Theorem 3.

We turn to the proof of Proposition 3.1, using induction on k. The case $k=0$ is obvious. Set $x=L[-2]^{k-1}\mathbf{1}$, so that $w=L[-2]x$. By (2.1) and (1.10) we get

$$Z(w,\tau)=q\frac{d}{dq}Z(x,\tau)+\sum_{l=1}^{\infty}E_{2l}(\tau)Z(L[2l-2]x,\tau). \tag{3.1}$$

Now by another induction argument using the Virasoro relations, we easily find that if $l \geqslant 1$ then there is an identity of the form

$$L[2l-2]x=n_l L[-2]^{k-l}\mathbf{1} \tag{3.2}$$

where n_l is positive and the right-hand side is interpreted as 0 if $l>k$.

From (1.8), the q-expansion of $E_{2l}(\tau)$ begins

$$-\frac{B_{2l}}{(2l)!}+\ldots,$$

and it is easily seen from (1.9) that we have

$$(-1)^{l+1}B_{2l}>0. \tag{3.3}$$

By induction we have $Z(L[-2]^r\mathbf{1},\tau)=\varepsilon(r)q^{-1}+\ldots$ with $(-1)^r\varepsilon(r)>0$ for $0\leqslant r<k$. It follows that the coefficient of q^{-1} on the right-hand side of (3.1) is equal to

$$-\varepsilon(k-1)-\sum_{l=1}^{k}\frac{B_{2l}}{(2l)!}n_l\varepsilon(k-l)$$
$$=(-1)^k\left\{(-1)^{k-1}\varepsilon(k-1)+\sum_{l=1}^{k}(-1)^{l+1}\frac{B_{2l}}{(2l)!}n_l(-1)^{k-l}\varepsilon(k-l)\right\}.$$

From what we have said, the sum of the terms in the braces is positive, so Proposition 3.1 is proved. □

4. Proof of Theorem 3

We have reduced the proof of Theorem 1 to that of Theorem 3, which we carry out in this section.

We first take over *en bloc* the notation of [FLM] with regard to the lattice VOA V_Λ and associated vertex operators, where Λ is the Leech lattice. In particular, $\mathfrak{h}=\mathbf{C}\otimes_{\mathbf{Z}}\Lambda$; $\hat{\mathfrak{h}}_{\mathbf{Z}}$ is the corresponding Heisenberg algebra; $M(1)$ is the associated irreducible induced module for $\hat{\mathfrak{h}}_{\mathbf{Z}}$ such that the canonical central element of $\hat{\mathfrak{h}}_{\mathbf{Z}}$ acts as 1; $V_\Lambda=M(1)\otimes\mathbf{C}[\Lambda]$;

$$Y(e^\alpha,z)=E^-(-\alpha,z)E^+(-\alpha,z)e_\alpha z^\alpha$$

is the vertex operator associated to $\alpha\in\Lambda$ where

$$E^\pm(\alpha,z)=\exp\left(\sum_{n\in\mathbf{N}}\frac{\alpha(\pm n)}{\pm n}z^{\mp n}\right)$$

for $\alpha\in\mathfrak{h}$ and e_α acts on $\mathbf{C}[\Lambda]$ by

$$e_\alpha\colon e^\beta\mapsto\varepsilon(\alpha,\beta)e^{\alpha+\beta}$$

where $\varepsilon(\cdot,\cdot)$ is a bilinear 2-cocycle of Λ with values in $\{\pm1\}$; t is the automorphism of V_Λ of order 2 induced from the (-1)-isometry of Λ such that $te^\alpha=e^{-\alpha}$; t acts on $M(1)$ by $t(\beta_1(-n_1)\ldots\beta_k(-n_k))=(-1)^k\beta_1(-n_1)\ldots\beta_k(-n_k)$ for $\beta_i\in\mathfrak{h}$ and $n_i>0$.

For a t-stable subspace W of V_Λ we define $W^\pm$ to be the eigenspaces of t with eigenvalues ±1. We start by considering the action of $Y(e^\alpha+e^{-\alpha},z)$ on V_Λ^+. Thus V_Λ^+ is spanned by elements of the form

$$v\otimes e^\beta+tv\otimes e^{-\beta},\tag{4.1}$$

and we have

$$\begin{aligned} Y(v(\alpha),z)(v\otimes e^{\beta}+tv\otimes e^{-\beta}) = {} & z^{\langle\alpha,\beta\rangle}E^{-}(-\alpha,z)E^{+}(-\alpha,z)v\otimes\varepsilon(\alpha,\beta)e^{\alpha+\beta} \\ & +z^{-\langle\alpha,\beta\rangle}E^{-}(-\alpha,z)E^{+}(-\alpha,z)tv\otimes\varepsilon(\alpha,-\beta)e^{\alpha-\beta} \\ & +z^{-\langle\alpha,\beta\rangle}E^{-}(\alpha,z)E^{+}(\alpha,z)v\otimes\varepsilon(-\alpha,\beta)e^{-\alpha+\beta} \\ & +z^{\langle\alpha,\beta\rangle}E^{-}(\alpha,z)E^{+}(\alpha,z)tv\otimes\varepsilon(-\alpha,-\beta)e^{-\alpha-\beta}. \end{aligned} \tag{4.2}$$

From this we see that non-zero contributions to the trace on V_Λ^+ can arise only when $\alpha\in 2\Lambda$, and more precisely when $\alpha=\pm 2\beta$ in (4.2).

For $\beta\in\Lambda$ we set

$$V(\beta)=M(1)\otimes(\mathbf{C}e^{\beta}+\mathbf{C}e^{-\beta})$$

which is t-stable. So the trace of $o(v(\lambda))$ on V_Λ^+ is equal to the trace of $o(v(\lambda))$ on $V(\alpha)^+$ where $\lambda=2\alpha\in\Lambda$, which we now assume. Clearly $v(\lambda)$ is a highest weight vector with weight $\frac{1}{2}\langle\lambda,\lambda\rangle$.

Note that $\varepsilon(\pm\lambda,\pm\alpha)=1$. It follows from (4.2) that only expressions of the form

$$z^{-2\langle\alpha,\alpha\rangle}(E^{-}(-2\alpha,z)E^{+}(-2\alpha,z)tv\otimes e^{\alpha}+E^{-}(2\alpha,z)E^{+}(2\alpha,z)v\otimes e^{-\alpha}) \tag{4.3}$$

contribute to the trace. Thus we are essentially reduced to computing the trace of the degree-zero operators of $E^{-}(-2\alpha,z)E^{+}(-2\alpha,z)$ and $E^{-}(2\alpha,z)E^{+}(2\alpha,z)$ on $M(1)$.

Let $A=\mathbf{C}\lambda$ and $\mathfrak{h}=A\perp B$ be an orthogonal direct sum. Then we have $M(1)=S(\hat{\mathfrak{h}}^{-})=S(\hat{A}^{-})\otimes S(\widehat{B}^{-})$. Let x be a formal variable and define $x^{N}\in(\operatorname{End}M(1))[x]$ such that $x^{N}(\alpha_1(-n_1)\dots\alpha_k(-n_k))=x^{k}\alpha_1(-n_1)\dots\alpha_k(-n_k)$ for $\alpha_i\in\mathfrak{h}$ and $n_i>0$. Set

$$E^{-}(\pm\lambda,z)E^{+}(\pm\lambda,z)=\sum_{n\in\mathbf{Z}}E^{\pm}(n)z^{-n}.$$

LEMMA 4.1. *We have*

$$\operatorname{tr}E^{\pm}(0)q^{L(0)}x^{N}|_{S(\hat{A}^{-})}=\frac{\exp\left(\sum_{n>0}-\langle\lambda,\lambda\rangle xq^{n}/n(1-xq^{n})\right)}{\prod_{n>0}(1-xq^{n})}. \tag{4.4}$$

Proof. Note that $S(\hat{A}^{-})$ has a basis

$$\{\lambda(-n)^{k_n}\dots\lambda(-1)^{k_1}\mid k_i\geqslant 0,\, n\geqslant 1\}.$$

In order to compute the trace it suffices to compute the coefficients of $\lambda(-n)^{k_n}\dots\lambda(-1)^{k_1}$ in $E(0)^{\pm}\lambda(-n)^{k_n}\dots\lambda(-1)^{k_1}$, that is, we need to compute the projection

$$P_{k_1,\dots,k_n}\colon E(0)^{\pm}\lambda(-n)^{k_n}\dots\lambda(-1)^{k_1}\to\mathbf{C}\lambda(-n)^{k_n}\dots\lambda(-1)^{k_1}.$$

Recall that

$$[\lambda(s),\lambda(t)]=s\langle\lambda,\lambda\rangle\delta_{s+t,0}$$

for $s,t\in\mathbf{Z}$. Then

$$\begin{aligned}
&P_{k_1,\dots,k_n}E(0)^{\pm}\lambda(-n)^{k_n}\dots\lambda(-1)^{k_1}\\
&\quad=\sum_{p_i\leqslant k_i}(-1)^{p_1+\dots+p_n}\frac{\lambda(-1)^{p_1}}{p_1!}\cdots\frac{\lambda(-n)^{p_n}}{n^{p_n}p_n!}\frac{\lambda(1)^{p_1}}{p_1!}\cdots\frac{\lambda(n)^{p_n}}{n^{p_n}p_n!}\lambda(-n)^{k_1}\dots\lambda(-1)^{k_1}\\
&\quad=\sum_{p_i\leqslant k_i}(-1)^{p_1+\dots+p_n}\left(\prod_{i=1}^{n}\frac{\langle\lambda,\lambda\rangle^{p_i}i^{p_i}k_i(k_i-1)\dots(k_i-p_i+1)}{(p_i!)^2i^{2p_i}}\right)\lambda(-n)^{k_1}\dots\lambda(-1)^{k_1}\\
&\quad=\sum_{p_i\leqslant k_i}\left(\prod_{i=1}^{n}\frac{\binom{k_i}{p_i}}{p_i!}\left(\frac{-\langle\lambda,\lambda\rangle}{i}\right)^{p_i}\right)\lambda(-n)^{k_1}\dots\lambda(-1)^{k_1}.
\end{aligned}$$

Thus

$$\begin{aligned}
\operatorname{tr}E^{\pm}(0)q^{L(0)}x^N|_{S(\hat{A}^-)}&=\sum_{n\geqslant 1}\sum_{k_i,p_i\geqslant 0}\left(\prod_{i=1}^{n}\frac{\binom{k_i}{p_i}}{p_i!}\left(\frac{-\langle\lambda,\lambda\rangle}{i}\right)^{p_i}\right)q^{k_1+2k_2+\dots+nk_n}x^{k_1+\dots+k_n}\\
&=\prod_{i\geqslant 1}\left(\sum_{k_i,p_i\geqslant 0}\frac{\binom{k_i}{p_i}}{p_i!}\left(\frac{-\langle\lambda,\lambda\rangle}{i}\right)^{p_i}q^{ik_i}x^{k_i}\right).
\end{aligned}$$

Note that if y is a formal variable and s is a non-negative integer then

$$\sum_{m\geqslant s}\binom{m}{s}y^m=y^s\sum_{m=0}^{\infty}\binom{s+m}{s}y^m=y^s\sum_{m=0}^{\infty}\binom{s+m}{m}y^m=\frac{y^s}{(1-y)^{1+s}}.$$

Then for any $p_i\geqslant 0$ we have

$$\sum_{k_i\geqslant 0}\binom{k_i}{p_i}(xq^i)^{k_i}=\frac{(xq^i)^{p_i}}{(1-xq^i)^{1+p_i}}.$$

Hence

$$\begin{aligned}
\operatorname{tr}E^{\pm}(0)q^{L(0)}x^N|_{S(\hat{A}^-)}&=\prod_{i\geqslant 1}\sum_{p_i\geqslant 0}\frac{1}{(1-xq^i)}\frac{1}{p_i!}\left(\frac{-\langle\lambda,\lambda\rangle xq^i}{i(1-xq^i)}\right)^{p_i}\\
&=\prod_{n\geqslant 1}\frac{1}{(1-xq^n)}\exp\left(\frac{-\langle\lambda,\lambda\rangle xq^n}{n(1-xq^n)}\right)\\
&=\prod_{m=1}^{\infty}\frac{1}{(1-xq^m)}\exp\left(\sum_{n=1}^{\infty}\frac{-\langle\lambda,\lambda\rangle xq^n}{n(1-xq^n)}\right),
\end{aligned}$$

as desired. □

LEMMA 4.2. *We have*

$$\operatorname{tr} E^{\pm}(0) q^{L(0)} x^N|_{M(1)} = \frac{\exp\left(\sum_{n>0} -\langle\lambda,\lambda\rangle x q^n / n(1-xq^n)\right)}{\prod_{n>0}(1-xq^n)^{24}}. \tag{4.5}$$

Proof. Since $M(1)=S(\hat{A}^-)\otimes S(\widehat{B}^-)$ and $E^{\pm}(0)$ commute with $\beta(n)$ for $\beta\in B$ and $n\in\mathbf{Z}$, we immediately have

$$\operatorname{tr} E^{\pm}(0) q^{L(0)} x^N|_{M(1)} = \operatorname{tr} E^{\pm}(0) q^{L(0)} x^N|_{S(\hat{A}^-)} \operatorname{tr} q^{L(0)} x^N|_{S(\widehat{B}^-)}$$

and also

$$\operatorname{tr} q^{L(0)} x^N|_{S(\widehat{B}^-)} = \frac{1}{\prod_{n>0}(1-xq^n)^{23}}.$$

The lemma now follows from Lemma 4.1. □

Set $f(q,x)=\operatorname{tr} E^{\pm}(0) q^{L(0)} x^N|_{M(1)}$. Then one can easily see that

$$\begin{aligned} \operatorname{tr} E^{\pm}(0) q^{L(0)}|_{M(1)^+} &= \tfrac{1}{2}(f(q,1)+f(q,-1)), \\ \operatorname{tr} E^{\pm}(0) q^{L(0)}|_{M(1)^-} &= \tfrac{1}{2}(f(q,1)-f(q,-1)). \end{aligned} \tag{4.6}$$

LEMMA 4.3. *The contribution of V_Λ^+ to $Z(v(\lambda),\tau)$ is*

$$q^{\langle\lambda,\lambda\rangle/8-1} \prod_{n=1}^{\infty} \frac{(1-q^n)^{24}}{(1-q^{2n})^{24}} \prod_{n=1}^{\infty} \frac{(1-q^{2n})^{2\langle\lambda,\lambda\rangle}}{(1-q^n)^{\langle\lambda,\lambda\rangle}} = \frac{\eta(2\tau)^{2\langle\lambda,\lambda\rangle-24}}{\eta(\tau)^{\langle\lambda,\lambda\rangle-24}}. \tag{4.7}$$

Proof. We have already seen that

$$\operatorname{tr} o(v(\lambda)) q^{L(0)}|_{V_\Lambda^+} = \operatorname{tr} o(v(\lambda)) q^{L(0)}|_{V(\alpha)^+}.$$

Clearly, $q^{L(0)}e^{\pm\alpha}=q^{\langle\lambda,\lambda\rangle/8}e^{\pm\alpha}$. From the proof of Lemma 4.1 we see that $E^{\pm}(0)$ have the same eigenvectors and the corresponding eigenvalues are also the same. It follows from (4.3), (4.5) and (4.6) that

$$\begin{aligned} \operatorname{tr} o(v(\lambda)) q^{L(0)}|_{V(\alpha)^+} &= q^{\langle\lambda,\lambda\rangle/8}(E^{\pm}(0) q^{L(0)}|_{M(1)^+} - \operatorname{tr} E^{\pm}(0) q^{L(0)}|_{M(1)^-}) \\ &= q^{\langle\lambda,\lambda\rangle/8} f(q,-1) = q^{\langle\lambda,\lambda\rangle/8} \frac{\exp\left(\sum_{n>0} \langle\lambda,\lambda\rangle q^n/n(1+q^n)\right)}{\prod_{n>0}(1+q^n)^{24}}. \end{aligned}$$

Next note that

$$\sum_{n>0} \frac{q^n}{n(1+q^n)} = \sum_{n=1}^{\infty} \frac{q^n}{n} \sum_{i=0}^{\infty} (-1)^i q^{in} = -\sum_{i=1}^{\infty} (-1)^i \sum_{n=1}^{\infty} \frac{q^{in}}{n} = \sum_{n=1}^{\infty} (-1)^n \log(1-q^n).$$

So $\operatorname{tr} o(v(\lambda))q^{L(0)}|_{V(\alpha)^+}$ may be written as

$$q^{\langle\lambda,\lambda\rangle/8}\prod_{n=1}^{\infty}(1+q^n)^{-24}\prod_{n=1}^{\infty}(1-q^n)^{(-1)^n\langle\lambda,\lambda\rangle}. \tag{4.8}$$

If now we incorporate the grade shift of $q^{-c/24}=q^{-1}$, the lemma follows from (4.8). $\square$

At this point, recall [Ch] the Jacobi theta-functions Θ_i, $i=1,2,3$, considered as functions of τ, that is, with the "other" variable set equal to 0:

$$\Theta_1(\tau)=2q^{1/8}\prod_{n=1}^{\infty}(1-q^n)(1+q^n)^2=2\frac{\eta(2\tau)^2}{\eta(\tau)}, \tag{4.9}$$

$$\Theta_2(\tau)=\prod_{n=1}^{\infty}(1-q^n)(1-q^{n-1/2})^2=\frac{\eta\left(\frac{1}{2}\tau\right)^2}{\eta(\tau)}, \tag{4.10}$$

$$\Theta_3(\tau)=\prod_{n=1}^{\infty}(1-q^n)(1+q^{n-1/2})^2=\frac{\eta(\tau)^5}{\eta\left(\frac{1}{2}\tau\right)^2\eta(2\tau)^2}. \tag{4.11}$$

Combining (4.7) and (4.9) then yields

LEMMA 4.4. *The contribution of V_Λ^+ to $Z(v(\lambda),\tau)$ is equal to*

$$\eta(\tau)^{12}\left(\tfrac{1}{2}\Theta_1(\tau)\right)^{\langle\lambda,\lambda\rangle-12}.$$

Now let V_Λ^T be the t-twisted V_Λ-module (cf. [FLM]). Then the moonshine module $V^\natural$ is the direct sum of V_Λ^+ and $(V_\Lambda^T)^+$ where again $+$ refers to the fixed points of the action t on V_Λ^T. The space V_Λ^T can be described as

$$V_\Lambda^T=S(\hat{\mathfrak{h}}[-1]^-)\otimes T$$

where $\hat{\mathfrak{h}}[-1]=\sum_{n\in\mathbf{Z}}\mathfrak{h}\otimes t^{n+1/2}\oplus\mathbf{C}c$ is the (-1)-twisted Heisenberg algebra, $\hat{\mathfrak{h}}[-1]^-=\sum_{n>0}\mathfrak{h}\otimes t^{-n+1/2}$ and T is the 2^{12}-dimensional projective representation for Λ such that $2L$ acts on T trivially. The grading on V_Λ^T is the natural one together with an overall shift of $q^{3/2}$. Now t acts on T as multiplication by -1, and on $S(\hat{\mathfrak{h}}[-1]^-)$ by $t(\beta_1(-n_1)\dots\beta_k(-n_k))=(-1)^k\beta_1(-n_1)\dots\beta_k(-n_k)$ for $b_i\in\mathfrak{h}$ and positive $n_i\in\frac{1}{2}+\mathbf{Z}$. As before, for any t-stable subspace W of V_Λ^T, we denote by $W^\pm$ the eigenspaces of t with eigenvalues ±1. Then $(V_\Lambda^T)^+$ is the tensor product of T and $S(\hat{\mathfrak{h}}[-1]^-)^-$.

The twisted vertex operator $Y(e^\beta,z)$ for $\beta\in\Lambda$ on V_Λ^T is defined to be

$$Y(e^\beta,z)=2^{-\langle\beta,\beta\rangle}E_{1/2}^-(-\beta,z)E_{1/2}^+(-\beta,z)e_\beta z^{-\langle\beta,\beta\rangle/2}$$

where

$$E^{\pm}_{1/2}(h,z)=\exp\left(\sum_{n=0}^{\infty}\frac{h(\pm(n+\frac{1}{2}))}{\pm(n+\frac{1}{2})}z^{\mp(n+1/2)}\right)$$

for $h\in\mathfrak{h}$, and e_β acts on T. Because $\lambda\in 2\Lambda$ then e_λ and $e_{-\lambda}$ act trivially on T, and we see that

$$\begin{aligned}&Y(v(\lambda),z)\\&\quad=2^{-\langle\lambda,\lambda\rangle}E^{-}_{1/2}(-\lambda,z)E^{+}_{1/2}(-\lambda,z)z^{-\langle\beta,\beta\rangle/2}+2^{-\langle\lambda,\lambda\rangle}E^{-}_{1/2}(\lambda,z)E^{+}_{1/2}(\lambda,z)z^{-\langle\beta,\beta\rangle/2}\end{aligned}$$

on V^T_Λ. As before we set

$$E^{-}_{1/2}(\pm\lambda,z)E^{+}_{1/2}(\pm\lambda,z)=\sum_{n\in\mathbf{Z}+1/2}E^{\pm}_{1/2}(n)z^{-n}.$$

Then the contribution of $(V^T_\Lambda)^+$ to $Z(v(\lambda),\tau)$ is equal to

$$q^{-1}2^{12-\langle\lambda,\lambda\rangle}\operatorname{tr}(E^{+}_{1/2}(0)+E^{-}_{1/2}(0))q^{L(0)}|_{S(\hat{\mathfrak{h}}[-1]^-)^-}.$$

For a formal variable x we define the operator $x^N\in(\operatorname{End}S(\hat{\mathfrak{h}}[-1]^-))[x]$ as before, so that $x^N(\beta_1(-n_1)\dots\beta_k(-n_k))=x^k\beta_1(-n_1)\dots\beta_k(-n_k)$ for $b_i\in\mathfrak{h}$ and positive $n_i\in\frac{1}{2}+\mathbf{Z}$. Set

$$g(q,x)=q^{3/2}\exp\left(\sum_{n=0}^{\infty}\frac{-\langle\lambda,\lambda\rangle xq^{n+1/2}}{(n+\frac{1}{2})(1-xq^{n+1/2})}\right)\prod_{n>0}(1-xq^{n-1/2})^{-24}.$$

By a proof not essentially different to that of Lemmas 4.1 and 4.2 we find

LEMMA 4.5. *The traces* $\operatorname{tr}E^{+}_{1/2}(0)q^{L(0)}x^N|_{S(\hat{\mathfrak{h}}[-1]^-)}$ *and* $\operatorname{tr}E^{-}_{1/2}(0)q^{L(0)}x^N|_{S(\hat{\mathfrak{h}}[-1]^-)}$ *are the same and equal to* $g(q,x)$.

One can easily see that

$$\operatorname{tr}(E^{+}_{1/2}(0)+E^{-}_{1/2}(0))q^{L(0)}|_{S(\hat{\mathfrak{h}}[-1]^-)^-}=g(q,1)-g(q,-1).$$

Next,

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{xq^{n+1/2}}{(n+\frac{1}{2})(1-xq^{n+1/2})}&=\sum_{n=0}^{\infty}\frac{xq^{n+1/2}}{n+\frac{1}{2}}\sum_{i=0}^{\infty}x^iq^{i(n+1/2)}\\&=\sum_{i=1}^{\infty}x^i\sum_{n=0}^{\infty}\frac{q^{i(n+1/2)}}{n+\frac{1}{2}}=-\sum_{i=1}^{\infty}x^i\log\left(\frac{1-q^{i/2}}{1+q^{i/2}}\right).\end{aligned}$$

Then the contribution of $(V_\Lambda^T)^+$ to $Z(v(\lambda),\tau)$ is equal to

$$
\begin{aligned}
&2^{-\langle\lambda,\lambda\rangle+12}q^{1/2}\prod_{n=1}^{\infty}(1-q^{n-1/2})^{-24}\prod_{i=1}^{\infty}\left(\frac{1-q^{i/2}}{1+q^{i/2}}\right)^{\langle\lambda,\lambda\rangle}\\
&\quad-2^{-\langle\lambda,\lambda\rangle+12}q^{1/2}\prod_{n=1}^{\infty}(1+q^{n-1/2})^{-24}\prod_{i=1}^{\infty}\left(\frac{1-q^{i/2}}{1+q^{i/2}}\right)^{(-1)^i\langle\lambda,\lambda\rangle}\\
&=2^{-\langle\lambda,\lambda\rangle+12}q^{1/2}\prod_{n=1}^{\infty}\frac{(1-q^n)^{24}}{(1-q^{n/2})^{24}}\prod_{i=1}^{\infty}\frac{(1-q^{i/2})^{2\langle\lambda,\lambda\rangle}}{(1-q^i)^{\langle\lambda,\lambda\rangle}}\\
&\quad-2^{-\langle\lambda,\lambda\rangle+12}q^{1/2}\prod_{n=1}^{\infty}\frac{(1-q^{n/2})^{24}(1-q^{2n})^{24}}{(1-q^n)^{48}}\prod_{i=1}^{\infty}\frac{(1-q^i)^{5\langle\lambda,\lambda\rangle}}{(1-q^{2i})^{2\langle\lambda,\lambda\rangle}(1-q^{i/2})^{2\langle\lambda,\lambda\rangle}}\\
&=\eta(\tau)^{12}\left\{\left(\tfrac{1}{2}\Theta_2(\tau)\right)^{\langle\lambda,\lambda\rangle-12}-\left(\tfrac{1}{2}\Theta_3(\tau)\right)^{\langle\lambda,\lambda\rangle-12}\right\}.
\end{aligned}
$$

Thus we have proved

LEMMA 4.6. *The contribution of $(V_\Lambda^T)^+$ to $Z(v(\lambda),\tau)$ is equal to*

$$\eta(\tau)^{12}\left\{\left(\tfrac{1}{2}\Theta_2(\tau)\right)^{\langle\lambda,\lambda\rangle-12}-\left(\tfrac{1}{2}\Theta_3(\tau)\right)^{\langle\lambda,\lambda\rangle-12}\right\}.$$

Theorem 3 is an immediate consequence of Lemmas 4.4 and 4.6.

5. A generalization of Theorem 3

In this section we generalize Theorem 3 by computing explicitly the trace function $Z(v(\lambda),g,\tau)$ for certain automorphism g of the moonshine module. As before, Λ is the Leech lattice. To describe the result we first recall some facts about $\mathrm{Aut}(V^\natural)$, that is to say, the monster simple group $\mathbf{M}$.

The centralizer of an involution in $\mathbf{M}$ (of type $2B$) is a quotient of a group $\widehat{C}$, partially described by the short exact sequence

$$1\to Q\to\widehat{C}\to\mathrm{Aut}(\Lambda)\to 1$$

where $Q\cong 2_+^{1+24}$ is an extra-special group of type $+$ and order 2^{25}. For more information on this and other facts we use below, see [G] or [FLM]. The group $\widehat{C}$ acts on both $S(\hat{\mathfrak{h}}^-)$ and $S(\mathfrak{h}[-1]^-)$ through the natural action of $\mathrm{Aut}(\Lambda)$, that is, with kernel Q. It acts on $\mathbf{C}[\Lambda]$ with kernel the center $Z(Q)$ of Q, and on T with kernel a subgroup of $Z(\widehat{C})$ of order 2 distinct from $Z(Q)$. Then the quotient C of $\widehat{C}$ by the third subgroup of $Z(\widehat{C})$ of order 2 acts faithfully on $V^\natural$.

Let us fix $0 \neq \lambda \in 2\Lambda$, and let $H < \widehat{C}$ be the subgroup defined as

$$1 \to Q \to H \to (\operatorname{Aut}\Lambda)_\lambda \to 1$$

where $(\operatorname{Aut}\Lambda)_\lambda$ is the subgroup of $\operatorname{Aut}\Lambda$ which fixes λ. We will compute $Z(v(\lambda), h, \tau)$ for $h \in H$. The action of h on V_Λ is described by a pair (ξ, a) where $\xi \in \frac{1}{2}\Lambda/\Lambda$ and $a \in (\operatorname{Aut}\Lambda)_\lambda$; a acts in the natural manner, and ξ acts via

$$\xi\colon v \otimes e^\beta \mapsto e^{2\pi i \langle \xi, \beta \rangle} v \otimes e^\beta.$$

We let $-a$ denote the element $ta \in \operatorname{Aut}\Lambda$, and define a modified theta-function as

$$\theta_{\xi,-a}(\tau) = \sum_{\substack{\gamma \in \Lambda \\ a\gamma = -\gamma}} e^{2\pi i \langle \xi, \gamma \rangle} q^{\langle \gamma, \gamma \rangle / 2}. \tag{5.1}$$

(5.1) is a modification of the theta-series of the sublattice of Λ fixed by $-a$, and as such is a modular form of weight equal to one half the dimension of the $(-a)$-fixed sublattice. Finally, let $\eta_a(\tau)$ and $\eta_{-a}(\tau)$ be the "usual" eta-products associated to a and $-a$ (with regard to their action on Λ) (cf. [CN], [M]). We will establish

THEOREM 5.1. *Let $0 \neq \lambda = 2\alpha$, $\alpha \in \Lambda$, and let $h \in H$ be associated to (ξ, a) as above. Then we have*

$$\begin{aligned} Z(v(\lambda), h, \tau) = {} & e^{2\pi i \langle \xi, \alpha \rangle} \frac{\theta_{\xi,a}(\tau)}{\eta_{-a}(\tau)} \left(\frac{\Theta_1(\tau)}{2} \right)^{\langle \lambda, \lambda \rangle} \\ & + \operatorname{tr}_T(h) \left\{ \frac{\eta_a(\tau)}{\eta_a(\frac{1}{2}\tau)} \left(\frac{\Theta_2(\tau)}{2} \right)^{\langle \lambda, \lambda \rangle} - \frac{\eta_{-a}(\tau)}{\eta_{-a}(\frac{1}{2}\tau)} \left(\frac{\Theta_3(\tau)}{2} \right)^{\langle \lambda, \lambda \rangle} \right\}. \end{aligned} \tag{5.2}$$

The reader is invited to compare this result with Theorem 10.5.7 of [FLM], which deals with the case in which $v(\lambda)$ is replaced by the vacuum.

Note that $\eta_{-a}(\tau)$ is a form of the same weight as $\theta_{\xi,a}(\tau)$ (loc. cit.), so that (5.2) is indeed a form of the same weight as $Z(v(\lambda), \tau)$, as expected. The proof of Theorem 5.1 is a modification of that of Theorem 3.

We begin with the appropriate modification of (4.2), concerning the action of $Y(v(\lambda), z)h$ on V_Λ^+. We have, setting $h = h(\xi, a)$,

$$\begin{aligned} Y(v(\lambda), z)h(v \otimes e^\beta + tv \otimes e^{-\beta}) &= Y(v(\lambda), z)h(\xi, 1)(a(v) \otimes e^{a(\beta)} + ta(v) \otimes e^{-a(\beta)}) \\ &= e^{2\pi i \langle \xi, a(\beta) \rangle} Y(v(\lambda), z)(a(v) \otimes e^{a(\beta)} + ta(v) \otimes e^{-a(\beta)}) \\ &= e^{2\pi i \langle \xi, a(\beta) \rangle} \{ z^{\langle \lambda, a(\beta) \rangle} E^-(-\lambda, z) E^+(-\lambda, z) a(v) \otimes e^{\lambda + a(\beta)} \\ &\quad + z^{-\langle \lambda, a(\beta) \rangle} E^-(-\lambda, z) E^+(-\lambda, z) ta(v) \otimes e^{\lambda - a(\beta)} \\ &\quad + z^{-\langle \lambda, a(\beta) \rangle} E^-(\lambda, z) E^+(\lambda, z) a(v) \otimes e^{-\lambda + a(\beta)} \\ &\quad + z^{\langle \lambda, a(\beta) \rangle} E^-(\lambda, z) E^+(\lambda, z) ta(v) \otimes e^{-\lambda - a(\beta)} \}. \end{aligned} \tag{5.3}$$

LEMMA 5.2. *We may take $\lambda-a(\beta)=\beta$ in* (5.3). *This holds if, and only if, $\alpha-\beta=\delta$ for some $\delta\in\Lambda$ satisfying $-a(\delta)=\delta$.*

Proof. We see from (5.3) that contributions to the trace of $o(v(\lambda))h$ on V_Λ^+ potentially only arise when $\lambda+\alpha(\beta)=\pm\beta$ or $\lambda-a(\beta)=\pm\beta$. If $\lambda=\pm(a(\beta)-\beta)$ then λ is both a commutator (that is, lies in $[a,\Lambda]$) and a fixed point of a (by hypothesis). This leads to the contradiction that $\lambda=0$, so in fact $\lambda+a(\beta)=-\beta$ or $\lambda-a(\beta)=\beta$. Since β and $-\beta$ are essentially interchangeable in (5.3), we may assume that indeed

$$\lambda-a(\beta)=\beta. \tag{5.4}$$

Applying a to (5.4) yields $\lambda-a^2(\beta)=a(\beta)=\lambda-\beta$, so that $a^2(\beta)=\beta$. This may be written as $(a+1)(a-1)\beta=0$. Set

$$a(\beta)-\beta=2\delta. \tag{5.5}$$

Hence $a(\delta)+\delta=0$, that is, 2δ lies in the sublattice of Λ fixed by $-a$. Moreover, (5.4) and (5.5) yield $\lambda-2\beta=2\delta$, so remembering that $\lambda=2\alpha$ we get

$$\alpha-\beta=\delta. \tag{5.6}$$

On the other hand, if (5.6) holds, application of a yields

$$\alpha-a(\beta)=-\delta, \tag{5.7}$$

and (5.6), (5.7) imply that $\lambda-a(\beta)=\beta$. □

From the lemma and (5.3) we see that only expressions of the form

$$e^{2\pi i\langle\xi,\alpha-\delta\rangle}z^{-\langle\lambda,\alpha-\delta\rangle}(E^-(-\lambda,z)E^+(-\lambda,z)ta(v)\otimes e^{\alpha-\delta}+E^-(\lambda,z)E^+(\lambda,z)a(v)\otimes e^{-\alpha+\delta})$$

contribute to the trace, where δ ranges over the $(-a)$-fixed sublattice of Λ.

We now follow the analysis of §4 which follows (4.3). Since a fixes λ, the contribution from $S(\hat{A}^-)$ is identical to that of (4.4). As for $S(\widehat{B}^-)$, the operators $E^\pm(0)$ are trivial, and we need to calculate

$$\operatorname{tr} q^{L(0)}ax^N|_{S(\widehat{B}^-)}. \tag{5.8}$$

If $x=1$ this is precisely $\eta_a(\tau)/\eta(\tau)$, by definition.(1) If $x=-1$ then ax^N is just the action of ta, and (5.8) is then $\eta_{-a}(\tau)\eta(2\tau)/\eta(\tau)$.

Combining (4.4) and the above, we obtain the analogue of Lemma 4.2, namely

(1) This takes into account the corresponding grade shift.

LEMMA 5.3. *We have for* $x = \pm 1$,

$$\operatorname{tr} E^{\pm}(0) a q^{L(0)} x^N|_{M(1)} = \exp\Bigl(\sum_{n>0} \frac{-\langle\lambda,\lambda\rangle x q^n}{n(1-xq^n)}\Bigr)\eta_{xa}(\tau)^{-1}. \tag{5.9}$$

Now use this, Lemma 5.2 and the proof of Lemma 4.3 to see that the contribution of V_Λ^+ to $Z(v(\lambda), h, \tau)$ is equal to

$$\sum_{\substack{\delta\in\Lambda \\ -a(\delta)=\delta}} e^{2\pi i\langle\xi,\alpha-\delta\rangle} q^{\langle\alpha-\delta,\alpha-\delta\rangle/2} \exp\Bigl(\sum_{n>0} \frac{\langle\lambda,\lambda\rangle q^n}{n(1+q^n)}\Bigr)\eta_{-a}(\tau)^{-1}. \tag{5.10}$$

Note that $\langle\alpha,\delta\rangle = 0$. Then (5.10) is equal to

$$e^{2\pi i\langle\xi,\alpha\rangle}\theta_{\xi,-a}(\tau)\bigl(\tfrac{1}{2}\Theta_1(\tau)\bigr)^{\langle\lambda,\lambda\rangle}\eta_{-a}(\tau)^{-1},$$

which is the first summand of (5.2).

The other two summands of (5.2) arise from the contribution of $(V_\Lambda^T)^+$ to the trace. The proofs are as before, and are easier than the part just completed as there is no theta-function to deal with. We leave details to the reader. This completes our discussion of Theorem 5.1.

6. The invariance group of $Z(v, g, \tau)$

We will determine the subgroup of $\Gamma = \mathrm{SL}(2, \mathbf{Z})$ which leaves $Z(v, g, \tau)$ invariant. More precisely, if v is homogeneous of weight k with respect to $L[0]$, so that $Z(v, g, \tau)$ is modular of weight k by [DLM], we will describe in Theorem 6.1 below the action of $\Gamma_0(n)$ on $Z(v, g, \tau)$, where n is the order of g. We recall that

$$\Gamma_0(N) = \left\{\begin{pmatrix} a & b \\ c & d\end{pmatrix} \in \Gamma \,\middle|\, c \equiv 0 \ (N)\right\},$$

$$\Gamma_1(N) = \left\{\begin{pmatrix} a & b \\ c & d\end{pmatrix} \in \Gamma_0(N) \,\middle|\, a \equiv d \equiv 1 \ (N)\right\}.$$

The case where $v = \mathbf{1}$ is the vacuum (and $k = 0$) is covered by results in [CN], [FLM] and [B2]. Precisely, one knows that there is a character ε_g of $\Gamma_0(n)$ such that

$$Z|\gamma(\mathbf{1}, g, \tau) := Z(\mathbf{1}, g, \gamma\tau) = \varepsilon_g(\gamma) Z(\mathbf{1}, g, \tau) \tag{6.1}$$

for $\gamma \in \Gamma_0(n)$. Moreover, $\ker \varepsilon_g \supset \Gamma_0(N)$ where $N = nh$, and h divides $\gcd(n, 24)$.

To describe our generalization of this result, we need to recall some further results. Let $A_{\mathbf{M}}(\langle g\rangle)=N_{\mathbf{M}}(\langle g\rangle)/C_{\mathbf{M}}(\langle g\rangle)$ be the *automizer* of $\langle g\rangle$ in the monster $\mathbf{M}$. Then $A_{\mathbf{M}}(\langle g\rangle)$ is the group of automorphisms of $\langle g\rangle$ induced by conjugation in $\mathbf{M}$. As such, $A_{\mathbf{M}}(\langle g\rangle)$ has a canonical embedding

$$i_g\colon A_{\mathbf{M}}(\langle g\rangle)\to U_n \tag{6.2}$$

in which U_n is the group of units of $\mathbf{Z}/n\mathbf{Z}$, and $t\in N_{\mathbf{M}}(\langle g\rangle)$ satisfying $tgt^{-1}=g^d$ maps to d under i_g. From the character table of $\mathbf{M}$ [CC], we see that the following is true: $[U_n\colon \operatorname{im} i_g]\leqslant 2$, with equality if, and only if, g is *not* conjugate to g^{-1} in $\mathbf{M}$. In this case, $U_n=\operatorname{im} i_g\times\{\pm 1\}$.

Since $\Gamma_0(n)/\Gamma_1(n)$ is naturally isomorphic to U_n, we may define a subgroup Γ_g of $\Gamma_0(n)$ via the diagram (rows being short exact)

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \Gamma_1(n) & \longrightarrow & \Gamma_0(n) & \longrightarrow & U_n & \longrightarrow & 1\\
 & & \| & & \uparrow & & \uparrow & & \\
1 & \longrightarrow & \Gamma_1(n) & \longrightarrow & \Gamma_g & \longrightarrow & i_g(A_{\mathbf{M}}(\langle g\rangle)) & \longrightarrow & 1.
\end{array} \tag{6.3}$$

In (6.3),

$$\gamma=\begin{pmatrix} a & b\\ c & d\end{pmatrix}\in\Gamma_0(n)$$

maps to $d\in U_n$. From what we have said, we have $[\Gamma_0(n)\colon\Gamma_g]\leqslant 2$, and $\Gamma_0(n)=\Gamma_g\times\{\pm 1\}$ if we have equality.

Let χ range over the irreducible, complex characters of the normalizer $N_{\mathbf{M}}(\langle g\rangle)$ of $\langle g\rangle$ in $\mathbf{M}$. We will be particularly interested in those χ satisfying $C_{\mathbf{M}}(\langle g\rangle)\subset\ker\chi$. Such χ are 1-dimensional and induce characters

$$\chi\colon A_{\mathbf{M}}(\langle g\rangle)\to\mathbf{C}^*. \tag{6.4}$$

Using the lower row of (6.3), we can pull back χ to a character of Γ_g, also denoted by χ. If $[\Gamma_0(n)\colon\Gamma_g]=2$ then $\Gamma_0(n)=\Gamma_g\times\langle -I\rangle$ (where I is the identity (2×2)-matrix), and we then define a character χ_k ($k\in\mathbf{Z}$) of $\Gamma_0(n)$ so that its restriction to Γ_g is the earlier χ, and its value on $-I$ is $(-1)^k$. So in all cases we have defined characters χ_k of $\Gamma_0(n)$, with the convention that $\chi_k=\chi$ if $\Gamma_g=\Gamma_0(n)$.

We decompose $V^\natural$ into homogeneous subspaces $V^\natural_{[k]}$ with respect to the $L[0]$-operator. This commutes with the action of the monster $\mathbf{M}$, and we let $V^\natural_{[k],\chi}$ be the χ-isotypic subspaces of $V^\natural_{[k]}$ considered as an $N_{\mathbf{M}}(\langle g\rangle)$-module. We can now state our result:

THEOREM 6.1. *Fix $g \in \mathbf{M}$ and let the notation be as above. Suppose that $v \in V^\natural_{[k],\chi}$ for some simple character χ of $N_{\mathbf{M}}(\langle g \rangle)$. Then the following hold:*

(a) *If $C_{\mathbf{M}}(\langle g \rangle) \not\subset \ker \chi$ then $Z(v,g,\tau)=0$.*

(b) *If $C_{\mathbf{M}}(\langle g \rangle) \subset \ker \chi$ then*

$$Z|_k\gamma(v,g,\tau)=\varepsilon_g(\gamma)\overline{\chi_k(\gamma)}Z(v,g,\tau) \tag{6.5}$$

for $\gamma \in \Gamma_0(n)$.

Proof. We first prove (a). Since χ is a simple character of $N_{\mathbf{M}}(\langle g \rangle)$ and $C_{\mathbf{M}}(\langle g \rangle)$ is normal in $N_{\mathbf{M}}(\langle g \rangle)$, the assumption $C_{\mathbf{M}}(\langle g \rangle) \not\subset \ker \chi$ means that $C_{\mathbf{M}}(\langle g \rangle)$ does not leave v invariant if $0 \neq v \in V^\natural_{[k],\chi}$. Then v can be written as a linear combination $v=\sum_i v_i$ with each $v_i \in V^\natural_{[k],\chi}$ and $t_i v_i=\lambda_i v_i$ for each i, some $t_i \in C_{\mathbf{M}}(\langle g \rangle)$, and $1 \neq \lambda_i \in \mathbf{C}^*$.

We may thus assume that $v=v_i$, with $tv=\lambda v$ for some $t \in C_{\mathbf{M}}(\langle g \rangle)$ and some $1 \neq \lambda \in \mathbf{C}^*$. But then

$$\begin{aligned} Z(v,g,\tau) &= q^{-1}\sum_n (\mathrm{tr}|_{V^\natural_n} o(v)g)q^n = q^{-1}\sum_n (\mathrm{tr}|_{V^\natural_n} to(v)gt^{-1})q^n \\ &= q^{-1}\sum_n (\mathrm{tr}|_{V^\natural_n} o(tv)g)q^n = Z(tv,g,\tau)=\lambda Z(v,g,\tau). \end{aligned}$$

Since $\lambda \neq 1$, we get $Z(v,g,\tau)=0$, as required.

To prove (b) we need some results from [DLM], which we assume that the reader is familiar with. In particular, since g has order n then a matrix

$$\gamma=\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(n)$$

maps the $(1,g)$-conformal block to the $(1,g^d)$-conformal block. Since the trace functions $Z(v,g,\tau)$, $Z(v,g^d,\tau)$ span these conformal blocks, there is a scalar $\eta_g(\gamma)$, independent of v, such that

$$Z|\gamma(v,g,\tau)=\eta_g(\gamma)Z(v,g^d,\tau). \tag{6.6}$$

Here, if $v \in V^\natural_{[k]}$ then

$$Z|\gamma(v,g,\tau)=(c\tau+d)^{-k}Z(v,g,\gamma\tau). \tag{6.7}$$

Taking $v=\mathbf{1}$, $k=0$ in (6.6)–(6.7) and comparing with (6.1) then yields $\eta_g(\gamma)=\varepsilon_g(\gamma)$, that is,

$$Z|\gamma(v,g,\tau)=\varepsilon_g(\gamma)Z(v,g^d,\tau). \tag{6.8}$$

Suppose that $d \in i_g(A_{\mathbf{M}}(\langle g \rangle))$, that is, $\gamma \in \Gamma_g$. Then $g^d=tgt^{-1}$ for some $t \in N_{\mathbf{M}}(\langle g \rangle)$, and we calculate as before:

$$Z(v,g^d,\tau)=Z(v,tgt^{-1},\tau)=Z(t^{-1}v,g,\tau)=\chi(t^{-1})Z(v,g,\tau).$$

Then (6.8) reads

$$Z|\gamma(v,g,\tau)=\varepsilon_g(\gamma)\chi(t^{-1})Z(v,g,\tau). \tag{6.9}$$

By our conventions, $\chi(t^{-1})=\overline{\chi(\gamma)}$, so (6.9) is what we require.

Now assume that $\gamma\notin\Gamma_g$. From our earlier remarks, it suffices to take $\gamma=-I$. In this case $\gamma\in\Gamma_0(N)$, so $\varepsilon_g(\gamma)=1$, and (6.7) reads

$$Z|_k\gamma(v,g,\tau)=(-1)^k Z(v,g,\tau),$$

which is what (6.5) says in this case. The proof of the theorem is now complete. □

Remark 6.2. By Theorem 2 of [DM], each χ occurs in $V^\natural$, that is, given χ as above, there is a k such that $V_{[k],\chi}\neq 0$.

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Chongying Dong
Department of Mathematics
University of California
Santa Cruz, CA 95064
U.S.A.

Geoffrey Mason
Department of Mathematics
University of California
Santa Cruz, CA 95064
U.S.A.

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