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Equivalent norms on Lipschitz-type spaces of holomorphic functions

by

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1. Introduction and results

A continuous function $\omega: [0, +\infty) \to \mathbf{R}$ with $\omega(0)=0$ will be called a *majorant* if $\omega(t)$ is increasing and $\omega(t)/t$ is nonincreasing for $t>0$. If, in addition, there is a constant $C(\omega)>0$ such that

$$\int_0^\delta \frac{\omega(t)}{t}\,dt + \delta \int_\delta^\infty \frac{\omega(t)}{t^2}\,dt \leqslant C(\omega)\cdot\omega(\delta)$$

whenever $0<\delta<1$, then we say that ω is a *regular majorant.* Given a majorant ω and a compact set $E\subset\mathbf{C}$, the (*Lipschitz-type*) space $\Lambda_\omega(E)$ consists, by definition, of the functions $f: E\to\mathbf{C}$ satisfying

$$\|f\|_{\Lambda_\omega(E)} \stackrel{\text{def}}{=} \sup\left\{\frac{|f(z_1)-f(z_2)|}{\omega(|z_1-z_2|)} : z_1, z_2 \in E,\, z_1 \neq z_2\right\} < \infty.$$

Now let $\mathbf{D}$ denote the unit disk $\{|z|<1\}$, $\mathbf{T}$ its boundary and $\overline{\mathbf{D}} \stackrel{\text{def}}{=} \mathbf{D}\cup\mathbf{T}$. Further, let A stand for the algebra of holomorphic functions on $\mathbf{D}$ that are continuous up to $\mathbf{T}$. We shall be concerned with the space

$$A_\omega \stackrel{\text{def}}{=} A\cap\Lambda_\omega(\overline{\mathbf{D}}),$$

which in fact coincides with $A\cap\Lambda_\omega(\mathbf{T})$ (for regular majorants, this last statement follows from Lemma 4 below; for the general case, see [T]).

The purpose of this paper is to characterize the functions of class A_ω in terms of their moduli (the ω's involved are assumed to be regular majorants). To this end, we

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introduce several equivalent norms on A_ω, each of them depending only on the modulus of the function in question.

In order to state the results, we require some more notation. With each point $z\in\mathbf{D}$ we associate the Poisson kernel

$$P_z(\zeta)\stackrel{\text{def}}{=}\frac{1-|z|^2}{|\zeta-z|^2},\quad \zeta\in\mathbf{T},$$

and the harmonic measure $d\mu_z\stackrel{\text{def}}{=}P_z\,dm$; here and throughout, dm is the normalized arc length measure on $\mathbf{T}$. The Poisson integral of a function $f\in C(\mathbf{T})$ is defined, as usual, by

$$\mathcal{P}f(z)\stackrel{\text{def}}{=}\int_{\mathbf{T}}f(\zeta)\,d\mu_z(\zeta),\quad z\in\mathbf{D},$$

and by

$$\mathcal{P}f(\zeta)\stackrel{\text{def}}{=}f(\zeta),\quad \zeta\in\mathbf{T}.$$

In case f lives on a larger set containing $\mathbf{T}$, $\mathcal{P}f$ will stand for $\mathcal{P}(f|_{\mathbf{T}})$. Finally, given two nonnegative (possibly infinite) quantities X and Y, we write $X\asymp Y$ to mean that there exists a constant $C>0$ such that

$$C^{-1}X\leqslant Y\leqslant CX;$$

the constants in this paper are allowed to depend only on ω. We begin with the following simple result.

THEOREM 1. *If $f\in A$ and if both ω and ω^2 are regular majorants, then*

$$\|f\|_{\Lambda_\omega(\bar{\mathbf{D}})}\asymp\sup_{z\in\mathbf{D}}\frac{1}{\omega(1-|z|)}\{\mathcal{P}(|f|^2)(z)-|f(z)|^2\}^{1/2}. \tag{1.1}$$

The norm appearing on the right-hand side of (1.1) can be viewed as an analogue of the so-called Garsia norm on the space BMO (see [G, Chapter VI] or [K, Chapter X]). The proof of Theorem 1 is fairly elementary (see §3 below), but the point is that the assumptions imposed on ω are somewhat too restrictive. For example, among the majorants of the form $\omega(t)=t^\alpha$, only those with $0<\alpha<\frac{1}{2}$ satisfy the hypotheses of the theorem. Moreover, letting $f(z)=z$ and $\omega(t)=t^\alpha$ with $\frac{1}{2}<\alpha<1$, one observes that $f\in A_\omega$, whereas the right-hand side of (1.1) is infinite. Thus Theorem 1 becomes false if one drops the assumption on ω^2.

This apparent defect is dispensed with in the next theorem, which applies to an arbitrary regular majorant ω (and, in particular, to $\omega(t)=t^\alpha$ with $0<\alpha<1$). We find it quite spectacular that dealing with higher degrees of smoothness makes it so much harder (see §3) to introduce the sought-after modulus-dependent norms.

THEOREM 2. *Let $f\in A$, and set*

$$\varphi(z)\stackrel{\text{def}}{=}|f(z)|,\quad z\in\overline{\mathbf{D}}.$$

Given a majorant ω, consider the quantities

$$N_1(f)\stackrel{\text{def}}{=}\|\varphi\|_{\Lambda_\omega(\mathbf{T})}+\sup_{z\in\mathbf{D}}\frac{\mathcal{P}\varphi(z)-\varphi(z)}{\omega(1-|z|)},$$
$$N_2(f)\stackrel{\text{def}}{=}\|\varphi\|_{\Lambda_\omega(\mathbf{T})}+\|\varphi\|_{\omega,\mathrm{rad}},$$

where

$$\|\varphi\|_{\omega,\mathrm{rad}}\stackrel{\text{def}}{=}\sup\left\{\frac{|\varphi(\zeta)-\varphi(r\zeta)|}{\omega(1-r)}:\zeta\in\mathbf{T},\,0\leqslant r<1\right\},$$

and

$$N_3(f)\stackrel{\text{def}}{=}\|\varphi\|_{\Lambda_\omega(\overline{\mathbf{D}})}.$$

If ω is regular, then

$$\|f\|_{\Lambda_\omega(\overline{\mathbf{D}})}\asymp N_1(f)\asymp N_2(f)\asymp N_3(f), \tag{1.2}$$

the constants involved being either numerical or dependent only on ω.

In particular, taking the endpoints of (1.2), one obtains

$$\|f\|_{\Lambda_\omega(\overline{\mathbf{D}})}\asymp\||f|\|_{\Lambda_\omega(\overline{\mathbf{D}})},$$

which looks tantalizingly (and perhaps deceptively) simple. However, the author knows no simple proof of this fact.

Further, Theorem 2 enables us to derive exhaustive information on the canonical (inner-outer) factorization of A_ω-functions. Let us recall that, given a nonnegative function φ on $\mathbf{T}$ with $\log\varphi\in L^1(\mathbf{T},dm)$, the corresponding *outer function* is defined by

$$\mathcal{O}_\varphi(z)\stackrel{\text{def}}{=}\exp\left(\int_{\mathbf{T}}\frac{\zeta+z}{\zeta-z}\log\varphi(\zeta)\,dm(\zeta)\right),\quad z\in\mathbf{D}.$$

If $\varphi\in C(\mathbf{T})$, then $\mathcal{O}_\varphi\in H^\infty$ and $\lim_{r\to1^-}|\mathcal{O}_\varphi(r\zeta)|=\varphi(\zeta)$ everywhere on $\mathbf{T}$ (cf. [G, Chapter II]). Recall also that a function $\theta\in H^\infty$ is called *inner* if $\lim_{r\to1^-}|\theta(r\zeta)|=1$ for m-almost all $\zeta\in\mathbf{T}$.

We proceed by pointing out two corollaries of Theorem 2. One of these characterizes the outer functions lying in A_ω in terms of their moduli on the boundary, while the other one describes the interaction between inner and outer factors.

COROLLARY 1. *Let ω be a regular majorant. Suppose that $\varphi\in\Lambda_\omega(\mathbf{T})$, $\varphi\geqslant 0$, and* $\log\varphi\in L^1(\mathbf{T},dm)$. *Consider the extension of φ into $\mathbf{D}$ given by*

$$\varphi(z)\stackrel{\mathrm{def}}{=}\exp\left(\int_{\mathbf{T}}\log\varphi(\zeta)\,d\mu_z(\zeta)\right),\quad z\in\mathbf{D}.$$

The following are equivalent.

(i) $\mathcal{O}_\varphi\in A_\omega$.

(ii) $\mathcal{P}\varphi(z)-\varphi(z)=O(\omega(1-|z|))$, $z\in\mathbf{D}$.

(iii) $\varphi(\zeta)-\varphi(r\zeta)=O(\omega(1-r))$ *as* $r\to 1^-$, *uniformly for all* $\zeta\in\mathbf{T}$.

(iv) $\varphi\in\Lambda_\omega(\overline{\mathbf{D}})$.

To prove Corollary 1, one merely notices that $\mathcal{O}_\varphi\in A$, $|\mathcal{O}_\varphi|=\varphi$ on $\overline{\mathbf{D}}$, and applies Theorem 2 to $f=\mathcal{O}_\varphi$. Writing out the conditions $N_j(\mathcal{O}_\varphi)<\infty$ with $j=1,2,3$, one arrives at (ii), (iii) and (iv), respectively.

The next consequence of Theorem 2 will be stated now and derived in §3 below.

COROLLARY 2. *Let ω be a regular majorant. Suppose that $F\in A$ and θ is an inner function. The following are equivalent.*

(i) $F\theta\in A_\omega$.

(ii) $F\in A_\omega$ *and*

$$F(z)=O\left(\frac{\omega(1-|z|)}{1-|\theta(z)|}\right),\quad z\in\mathbf{D}.\tag{1.3}$$

(iii) $F\in A_\omega$ *and*

$$F(\zeta)=O\left(\inf_{0\leqslant r<1}\frac{\omega(1-r)}{1-|\theta(r\zeta)|}\right),\quad \zeta\in\mathbf{T}.\tag{1.4}$$

In connection with Corollaries 1 and 2, we would like to mention some previous results on the canonical factorization of A_ω-functions. In the special case where $\omega(t)=t^\alpha$, $0<\alpha<1$ (let us denote the corresponding spaces Λ_ω and A_ω by Λ^α and A^α), Corollary 1 was recently obtained by the author [D3]. On the other hand, in [D1] and [D2] the author established the A^α-version of Corollary 2, part (i) $\Leftrightarrow$ (ii); a simpler proof was suggested later by E. M. Dyn'kin [Dyn]. In fact, the present paper reflects the author's attempt to develop a unified approach to his earlier results and to reveal the underlying mechanism (now it seems to be described by the new-born Theorem 2). Our current techniques are different from those of [D1] and [D2]; they are largely inspired by [Dyn], the main ingredient being the so-called pseudoanalytic extension.

An alternative study of the inner-outer factorization of A_ω-functions is due to N. A. Shirokov (see [Sh1] and [Sh2]). We cite some of his results as Theorems A and B below, so that the reader might compare them with our Corollaries 1 and 2, respectively. The ω involved is assumed to be a regular majorant.

THEOREM A. *Suppose $\varphi\in\Lambda_\omega(\mathbf{T})$ is a nonnegative function with $\int\log\varphi\,dm>-\infty$. For $z\in\mathbf{D}$, put*

$$M(z)\stackrel{\text{def}}{=}\max\{\varphi(\zeta):\zeta\in\mathbf{T},\,|\zeta-z|\leqslant 2(1-|z|)\}$$

and consider the set

$$\Omega\stackrel{\text{def}}{=}\{z\in\mathbf{D}:M(z)\geqslant\omega(1-|z|)\}.$$

In order that $\mathcal{O}_\varphi\in A_\omega$, it is necessary and sufficient that

$$\sup_{z\in\Omega}\int\left|\log\frac{M(z)}{\varphi(\zeta)}\right|d\mu_z(\zeta)<\infty.$$

THEOREM B. *Suppose that $F\in A$ and θ is an inner function. Denote by σ the singular support of θ (i.e. the smallest closed subset E of $\mathbf{T}$ such that θ is analytic across $\mathbf{T}\setminus E$). In order that $F\theta\in A_\omega$, it is necessary and sufficient that $F\in A_\omega$, $F|_\sigma=0$ and*

$$F(\zeta)=O(\omega(|\theta'(\zeta)|^{-1})),\quad \zeta\in\mathbf{T}\setminus\sigma.$$

The Λ^α-version of Theorem A is contained in [Sh2, Chapter II]; for generic regular ω's, the proof is essentially the same. Theorem B can be found in [Sh1], along with an earlier (and more cumbersome) version of Theorem A. When suitably modified, Theorems A and B remain valid for higher orders of smoothness, e.g. for Λ^α-spaces with $\alpha>1$ (see [Sh2]).

We remark that both results and techniques of [Sh1], [Sh2] seem to be quite different from ours.

As a final application of Theorem 2, we use it to give a new proof of the following (essentially known) result, sometimes called the Havin–Shamoyan–Carleson–Jacobs embedding theorem.

THEOREM 3. *Let ω be a majorant such that $\psi(t)\stackrel{\text{def}}{=}\omega(\sqrt{t})$ is a regular majorant. Suppose that $\varphi\in\Lambda_\omega(\mathbf{T})$, $\varphi\geqslant 0$, and $\log\varphi\in L^1(\mathbf{T},dm)$. Then $\mathcal{O}_\varphi\in A_\psi$ and*

$$\|\mathcal{O}_\varphi\|_{\Lambda_\psi(\bar{\mathbf{D}})}\leqslant\text{const}\cdot\|\varphi\|_{\Lambda_\omega(\mathbf{T})}\left(1+\int_{\mathbf{T}}\log\frac{\|\varphi\|_\infty}{\varphi(\zeta)}\,dm(\zeta)\right),\tag{1.5}$$

where $\|\varphi\|_\infty\stackrel{\text{def}}{=}\max_{\mathbf{T}}\varphi$ and const *depends only on ω.*

This theorem was proved, in a somewhat more general setting, by V. P. Havin [H]; the estimate (1.5) was not, however, stated explicitly. It was also shown in [H] that, under the hypotheses of the theorem, the conclusion $\mathcal{O}_\varphi\in A_\psi$ could not, in general, be improved. The Λ^α-version of Theorem 3, with $0<\alpha\leqslant 1$, had been obtained earlier by

Havin and Shamoyan [HS] and, independently, by Carleson and Jacobs (unpublished). Alternative proofs of Theorem 3 (except possibly for the estimate (1.5)) and of its Λ^α-version with $\alpha>1$ can be found in [Sh1] and [Sh2, Chapter II].

Our proof of Theorem 3, based on an application of the norm N_2 from Theorem 2, is shorter than the original one; in addition, it yields (1.5). It is actually intended to demonstrate the strength of our current approach.

Besides, it is interesting to compare Theorems 2 and 3. While Theorem 3 tells us that, given an outer function f, the inclusion

$$|f| \in \Lambda_\omega(\mathbf{T}) \tag{1.6}$$

implies merely that $f \in A_\psi$ with $\psi(t)=\omega(\sqrt{t})$ (and, in fact, nothing more can be expected), Theorem 2 provides the "correct" point of view, where one replaces (1.6) with the condition $|f| \in \Lambda_\omega(\overline{\mathbf{D}})$ and arrives at the (seemingly) natural conclusion that $f \in A_\omega$.

This said, the author believes that the current results might also lead to further progress in some other directions. For instance, they might prove useful in connection with the peak sets for the algebra A_ω; a complete characterization of such sets (even in the A^α-case) seems to be a long-standing open problem.

The rest of the paper is organized as follows. In §2, we collect a number of auxiliary (and mostly elementary) facts about majorants and A_ω-spaces, to be employed later on. In §3, we prove Theorems 1, 2 and Corollary 2. Finally, §4 contains the deduction of the Havin–Shamoyan–Carleson–Jacobs embedding theorem.

Throughout, we use the following notation. For $z \in \mathbf{C} \setminus \{0\}$, we set $\tilde{z} \stackrel{\text{def}}{=} z/|z|$ and $z^* \stackrel{\text{def}}{=} 1/\bar{z}$; in case $z=0$, it is understood that $\tilde{z}=1$ and $z^*=\infty$. Further, we put

$$\varrho_z \stackrel{\text{def}}{=} \operatorname{dist}(z, \mathbf{T}) = \big||z|-1\big|, \quad z \in \mathbf{C}.$$

The region $\mathbf{C} \setminus \overline{\mathbf{D}}$ is denoted by $\mathbf{D}_-$. By C we denote any absolute positive constant, and by $C(\omega)$ any positive constant depending on ω. The values of these constants may vary with each occurrence (even within a single calculation).

2. Preliminaries

Our first lemma is essentially borrowed from [H]. For the reader's convenience, we include a short proof.

LEMMA 1. *Let ω be a majorant. Then ω_1, defined by*

$$\omega_1(\delta) \stackrel{\text{def}}{=} \frac{1}{\delta} \int_0^\delta \omega(t)\, dt, \quad \delta > 0, \tag{2.1}$$

is a majorant of class C^1 satisfying

$$\tfrac{1}{2}\omega(\delta) \leqslant \omega_1(\delta) \leqslant \omega(\delta) \tag{2.2}$$

and

$$0 \leqslant \omega_1'(\delta) \leqslant \frac{\omega_1(\delta)}{\delta}. \tag{2.3}$$

If ω is regular, then ω_1 is also regular.

Proof. Clearly, $\omega_1 \in C^1$. Using directly the definition (2.1) and noting that

$$\frac{\omega(\delta)}{\delta} t \leqslant \omega(t) \leqslant \omega(\delta) \quad \text{for } 0<t<\delta,$$

one arrives at (2.2). Further, differentiating (2.1) gives

$$\omega_1'(\delta) = \frac{1}{\delta}(\omega(\delta)-\omega_1(\delta)).$$

In conjunction with (2.2), this leads to (2.3). Since

$$\left(\frac{\omega_1(\delta)}{\delta}\right)' = \frac{1}{\delta}\left(\omega_1'(\delta) - \frac{\omega_1(\delta)}{\delta}\right),$$

(2.3) just means that ω_1 is increasing and $\omega_1(\delta)/\delta$ is decreasing; thus ω_1 is indeed a majorant. Finally, the last statement of the lemma is an immediate consequence of (2.2). □

Lemma 2. *If ω is a regular majorant and $z \in \mathbf{D}$, then*

$$\int \frac{\omega(|\zeta-\tilde z|)}{|\zeta - z|^2}\, dm(\zeta) \leqslant C(\omega) \frac{\omega(\varrho_z)}{\varrho_z}$$

(*see* §1 *for the meaning of the symbols $\tilde z$ and ϱ_z*).

Proof. Once $\zeta \in \mathbf{T}$ is fixed, let $t \in (-\pi, \pi]$ be defined by $e^{it} = \zeta/\tilde z$. Observing that

$$|\zeta - \tilde z| \asymp |t| \quad \text{and} \quad |\zeta - z|^2 \asymp t^2 + \varrho_z^2$$

(the constants involved being numerical), we get

$$\int \frac{\omega(|\zeta-\tilde z|)}{|\zeta - z|^2}\, dm(\zeta) \leqslant C \int_0^\pi \frac{\omega(t)}{t^2+\varrho_z^2}\, dt = C\left(\int_0^{\varrho_z} + \int_{\varrho_z}^{\pi}\right). \tag{2.4}$$

Estimating the two integrals by using the inequalities

$$\frac{\omega(t)}{t^2+\varrho_z^2} \leqslant \frac{\omega(\varrho_z)}{\varrho_z^2} \quad (0<t<\varrho_z) \quad \text{and} \quad t^2+\varrho_z^2 \geqslant t^2,$$

we see that the right-hand side of (2.4) is bounded by

$$\frac{C}{\varrho_z}\left(\omega(\varrho_z) + \varrho_z \int_{\varrho_z}^{\pi} \frac{\omega(t)}{t^2}\, dt\right) \leqslant \frac{C(\omega)}{\varrho_z}\omega(\varrho_z).$$

□

LEMMA 3. *If ω is a regular majorant, $f\in\Lambda_\omega(\mathbf{T})$ and $z\in\mathbf{D}$, then*

$$|\mathcal{P}f(z)-f(\tilde z)|\leqslant C(\omega)\|f\|_{\Lambda_\omega(\mathbf{T})}\omega(\varrho_z).$$

Proof. We have

$$\begin{aligned}|\mathcal{P}f(z)-f(\tilde z)|&=\left|\int (f(\zeta)-f(\tilde z))\,d\mu_z(\zeta)\right|\\&\leqslant \|f\|_{\Lambda_\omega(\mathbf{T})}\int \omega(|\zeta-\tilde z|)\,d\mu_z(\zeta)\leqslant C(\omega)\|f\|_{\Lambda_\omega(\mathbf{T})}\omega(\varrho_z),\end{aligned}$$

where the last inequality relies on Lemma 2. □

LEMMA 4. *If ω is a regular majorant and $f\in\Lambda_\omega(\mathbf{T})$, then $\mathcal{P}f\in\Lambda_\omega(\overline{\mathbf{D}})$; moreover,*

$$\|\mathcal{P}f\|_{\Lambda_\omega(\overline{\mathbf{D}})}\leqslant C(\omega)\|f\|_{\Lambda_\omega(\mathbf{T})}. \tag{2.5}$$

In particular, for $f\in A$ one has

$$\|f\|_{\Lambda_\omega(\overline{\mathbf{D}})}\leqslant C(\omega)\|f\|_{\Lambda_\omega(\mathbf{T})}. \tag{2.6}$$

Proof. In order to verify (2.5), we let $z_1,z_2\in\mathbf{D}$ and look at the quantity

$$R(z_1,z_2)\stackrel{\text{def}}{=}\mathcal{P}f(z_1)-\mathcal{P}f(z_2).$$

Let us distinguish two cases.

Case 1: $\max(|z_1|,|z_2|)\leqslant\frac12$. It is not hard to see that, for $\zeta\in\mathbf{T}$,

$$|P_{z_1}(\zeta)-P_{z_2}(\zeta)|\leqslant C|z_1-z_2|\leqslant\frac{2C}{\omega(2)}\omega(|z_1-z_2|) \tag{2.7}$$

(the last inequality holds because $\omega(t)/t$ is nonincreasing). From (2.7) it easily follows that

$$|R(z_1,z_2)|\leqslant C\|f\|_{\Lambda_\omega(\mathbf{T})}\omega(|z_1-z_2|). \tag{2.8}$$

Case 2: $\max(|z_1|,|z_2|)>\frac12$. For $j=1,2$ write $z_j=r_j\zeta_j$, where $r_j=|z_j|$ and $\zeta_j\in\mathbf{T}$. Assume that $r_1\geqslant r_2$ (and hence $r_1>\frac12$). Further, for a fixed $r\in(0,1)$ set

$$g_r(z)\stackrel{\text{def}}{=}\mathcal{P}f(rz),\quad z\in\overline{\mathbf{D}}.$$

Of course, g_r is harmonic on $\mathbf{D}$ and continuous up to $\mathbf{T}$. Also, it is easily shown that $g_r|_\mathbf{T}\in\Lambda_\omega(\mathbf{T})$ and

$$\|g_r\|_{\Lambda_\omega(\mathbf{T})}\leqslant\|f\|_{\Lambda_\omega(\mathbf{T})}. \tag{2.9}$$

We have

$$R(z_1,z_2)=\{g_{r_1}(\zeta_1)-g_{r_1}(\zeta_2)\}+\{g_{r_1}(\zeta_2)-g_{r_2}(\zeta_2)\}\stackrel{\text{def}}{=}R_1+R_2.$$

In view of (2.9),

$$|R_1|\leqslant \|f\|_{\Lambda_\omega(\mathbf{T})}\,\omega(|\zeta_1-\zeta_2|). \tag{2.10}$$

On the other hand, since

$$g_{r_2}(\zeta_2)=g_{r_1}\Big(\frac{r_2}{r_1}\zeta_2\Big)=\mathcal{P}(g_{r_1}|\mathbf{T})\Big(\frac{r_2}{r_1}\zeta_2\Big),$$

Lemma 3 tells us, in combination with (2.9), that

$$|R_2|\leqslant C(\omega)\|g_{r_1}\|_{\Lambda_\omega(\mathbf{T})}\,\omega\Big(1-\frac{r_2}{r_1}\Big)\leqslant C(\omega)\|f\|_{\Lambda_\omega(\mathbf{T})}\,\omega\Big(1-\frac{r_2}{r_1}\Big). \tag{2.11}$$

Since $r_1>\frac{1}{2}$, it follows that

$$|\zeta_1-\zeta_2|\leqslant 4|z_1-z_2| \quad\text{and}\quad 1-\frac{r_2}{r_1}\leqslant 2|z_1-z_2|.$$

Using these inequalities to estimate the right-hand sides of (2.10) and (2.11), we eventually obtain

$$|R(z_1,z_2)|\leqslant C(\omega)\|f\|_{\Lambda_\omega(\mathbf{T})}\,\omega(|z_1-z_2|). \tag{2.12}$$

A juxtaposition of (2.8) and (2.12) now yields (2.5), which in turn reduces to (2.6) in case $f\in A$. □

Remark. In connection with (2.6), see also [T].

Before stating the next lemma, recall that the Cauchy–Riemann operator $\bar\partial$ is defined by

$$\bar\partial=\frac{\partial}{\partial\bar z}\stackrel{\text{def}}{=}\frac{1}{2}\Big(\frac{\partial}{\partial x}+i\frac{\partial}{\partial y}\Big),\qquad z=x+iy.$$

LEMMA 5. *If ω is a regular majorant, $f\in\Lambda_\omega(\mathbf{T})$ and $z\in\mathbf{D}_-$, then*

$$|\bar\partial(\mathcal{P}f(z^*))|\leqslant C(\omega)\|f\|_{\Lambda_\omega(\mathbf{T})}\frac{\omega(\varrho_z)}{\varrho_z}$$

(recall that $z^*=1/\bar z$ and note that ϱ_z now stands for $|z|-1$).

Proof. We have

$$\begin{aligned}\bar\partial(\mathcal{P}f(z^*))&=\int f(\zeta)\frac{\partial}{\partial\bar z}P_{z^*}(\zeta)\,dm(\zeta)\\&=-\int f(\zeta)\frac{\zeta z^{*2}}{(\zeta-z^*)^2}\,dm(\zeta)=\int (f(\tilde z)-f(\zeta))\frac{\zeta z^{*2}}{(\zeta-z^*)^2}\,dm(\zeta),\end{aligned}$$

whence

$$|\bar{\partial}(\mathcal{P}f(z^*))| \leqslant \|f\|_{\Lambda_\omega(\mathbf{T})}|z^*|^2 \int \frac{\omega(|\zeta-\tilde{z}|)}{|\zeta-z^*|^2}\, dm(\zeta). \tag{2.13}$$

Applying Lemma 2, with z replaced by z^*, and noting that

$$\varrho_{z^*} = \frac{\varrho_z}{|z|} \leqslant \varrho_z,$$

we obtain

$$\int \frac{\omega(|\zeta-\tilde{z}|)}{|\zeta-z^*|^2}\, dm(\zeta) \leqslant C(\omega)|z| \frac{\omega(\varrho_z)}{\varrho_z}. \tag{2.14}$$

Substituting (2.14) into (2.13) yields the result. □

Remark. Lemma 5 remains valid if $\bar{\partial}$ is replaced by $\partial = \partial/\partial z$.

LEMMA 6. *Let ω be a majorant, and let $f \in A$. Consider the quantities*

$$M_0(f) \stackrel{\text{def}}{=} \sup_{z\in\mathbf{D}} |f'(z)| \frac{\varrho_z}{\omega(\varrho_z)},$$

$$M_1(f) \stackrel{\text{def}}{=} \sup_{z\in\mathbf{D}} \frac{1}{\omega(\varrho_z)} \int |f(\zeta)-f(z)|\, d\mu_z(\zeta),$$

$$M_2(f) \stackrel{\text{def}}{=} \sup_{z\in\mathbf{D}} \frac{1}{\omega(\varrho_z)} \left(\int |f(\zeta)-f(z)|^2\, d\mu_z(\zeta)\right)^{1/2}.$$

The following assertions hold true:

(i) *One always has*

$$M_0(f) \leqslant M_1(f) \leqslant M_2(f). \tag{2.15}$$

(ii) *If ω is regular, then*

$$(C(\omega))^{-1} M_1(f) \leqslant \|f\|_{\Lambda_\omega(\mathbf{T})} \leqslant C(\omega) M_0(f). \tag{2.16}$$

(iii) *If ω^2 is a regular majorant, then*

$$M_2(f) \leqslant C(\omega) \|f\|_{\Lambda_\omega(\mathbf{T})}. \tag{2.17}$$

Proof. (i) For $z \in \mathbf{D}$, we have

$$\varrho_z |f'(z)| = \left| \frac{1}{2\pi i} \int (f(\zeta)-f(z)) \frac{\varrho_z}{(\zeta-z)^2}\, d\zeta \right| \leqslant \int |f(\zeta)-f(z)|\, d\mu_z(\zeta),$$

which implies the first inequality in (2.15). The second one is obvious.

(ii) First let us prove the right-hand inequality in (2.16). Given $\zeta_1, \zeta_2 \in \mathbf{T}$, put $h \stackrel{\text{def}}{=} \frac{1}{2}|\zeta_1 - \zeta_2|$ and let I be the shorter subarc of $\mathbf{T}$ connecting ζ_1 and ζ_2. Set

$$\begin{aligned} \gamma_1 &\stackrel{\text{def}}{=} \{r\zeta_1 : r \in [1-h, 1]\}, \\ \gamma_2 &\stackrel{\text{def}}{=} \{(1-h)\zeta : \zeta \in I\}, \\ \gamma_3 &\stackrel{\text{def}}{=} \{r\zeta_2 : r \in [1-h, 1]\}, \end{aligned}$$

and $\gamma \stackrel{\text{def}}{=} \gamma_1 \cup \gamma_2 \cup \gamma_3$. When endowed with the appropriate orientation, γ becomes a path going from ζ_1 to ζ_2. We have

$$|f(\zeta_1) - f(\zeta_2)| = \left| \int_\gamma f'(z)\, dz \right| \leqslant \sum_{j=1}^{3} \int_{\gamma_j} |f'(z)|\, |dz|. \tag{2.18}$$

Using the estimate

$$|f'(z)| \leqslant M_0(f) \frac{\omega(\varrho_z)}{\varrho_z}$$

and the assumption

$$\int_0^\delta \frac{\omega(t)}{t}\, dt \leqslant C(\omega) \cdot \omega(\delta)$$

(this should be used when dealing with the integrals along γ_1 and γ_3), one deduces from (2.18) that

$$|f(\zeta_1) - f(\zeta_2)| \leqslant C(\omega) M_0(f) \omega(h).$$

Hence $\|f\|_{\Lambda_\omega(\mathbf{T})} \leqslant C(\omega) M_0(f)$.

To prove the rest of (2.16), we fix $\zeta \in \mathbf{T}$, $z \in \mathbf{D}$ and write

$$\begin{aligned} |f(\zeta) - f(z)| &\leqslant |f(\zeta) - f(\tilde{z})| + |f(\tilde{z}) - f(z)| \\ &\leqslant \|f\|_{\Lambda_\omega(\mathbf{T})} \omega(|\zeta - \tilde{z}|) + C(\omega) \|f\|_{\Lambda_\omega(\mathbf{T})} \omega(\varrho_z) \end{aligned} \tag{2.19}$$

(we have invoked Lemma 3 to estimate the second term). Integrating (2.19) against $d\mu_z(\zeta)$, while taking Lemma 2 into account, yields $M_1(f) \leqslant C(\omega) \|f\|_{\Lambda_\omega(\mathbf{T})}$.

(iii) Noting that the regularity of ω^2 implies the conclusion of Lemma 3, we still have (2.19) at our disposal. Rewriting it in the form

$$|f(\zeta) - f(z)|^2 \leqslant C(\omega) \|f\|^2_{\Lambda_\omega(\mathbf{T})} \{\omega^2(|\zeta - \tilde{z}|) + \omega^2(\varrho_z)\}, \tag{2.20}$$

then integrating (2.20) against $d\mu_z(\zeta)$, and finally using Lemma 2 (with ω replaced by ω^2), we arrive at (2.17). □

Remark. Of course, for $\omega(t) = t^\alpha$ $(0 < \alpha < 1)$, the equivalence of $M_0(f)$ and $\|f\|_{\Lambda_\omega(\mathbf{T})}$ is a well-known fact, due to Hardy and Littlewood.

The next lemma is a version of the remarkable E. M. Dyn'kin theorem on pseudo-analytic extensions (see [Dyn] for the Λ^α-case). Because of its crucial role in what follows, we include a complete proof of the version required.

LEMMA 7. *Let ω be a regular majorant and $f\in A$. In order that $f\in A_\omega$, it is necessary and sufficient that there exist a bounded function $F\in C^1(\mathbf{D}_-)$ such that*

$$\lim_{\substack{z\to\zeta\\ z\in\mathbf{D}_-}} F(z)=f(\zeta),\quad \zeta\in\mathbf{T}, \tag{2.21}$$

and

$$|\bar\partial F(z)|\leqslant \mathrm{const}\cdot\frac{\omega(\varrho_z)}{\varrho_z},\quad z\in\mathbf{D}_-. \tag{2.22}$$

Moreover,

$$\|f\|_{\Lambda_\omega(\mathbf{T})}\asymp \inf_F \sup\left\{\frac{\varrho_z}{\omega(\varrho_z)}|\bar\partial F(z)|: z\in\mathbf{D}_-\right\},$$

where F ranges over the bounded functions of class $C^1(\mathbf{D}_-)$ satisfying (2.21) *and* (2.22).

Proof. Given $f\in A_\omega$, set

$$F(z)=f(z^*),\quad z\in\mathbf{D}_-.$$

Clearly, F is bounded, smooth and satisfies (2.21). Further, using the estimate $M_0(f)\leqslant C(\omega)\|f\|_{\Lambda_\omega(\mathbf{T})}$ (see Lemma 6, parts (i) and (ii)), one gets

$$|\bar\partial F(z)|=|f'(z^*)|\cdot|z^*|^2\leqslant C(\omega)\|f\|_{\Lambda_\omega(\mathbf{T})}\frac{\omega(\varrho_{z^*})}{\varrho_{z^*}}|z^*|^2\leqslant C(\omega)\|f\|_{\Lambda_\omega(\mathbf{T})}\frac{\omega(\varrho_z)}{\varrho_z},$$

and so F satisfies (2.22) with const$\leqslant C(\omega)\|f\|_{\Lambda_\omega(\mathbf{T})}$.

Conversely, let $F\in C^1(\mathbf{D}_-)\cap L^\infty(\mathbf{D}_-)$ be a function with the properties (2.21) and (2.22), so that

$$B\stackrel{\text{def}}{=}\sup_{z\in\mathbf{D}_-}\frac{\varrho_z}{\omega(\varrho_z)}|\bar\partial F(z)|<\infty.$$

Fix $z\in\mathbf{D}$ and $R>1$. The Cauchy–Green formula, applied to the function that equals f on $\bar{\mathbf{D}}$ and F on $\mathbf{D}_-$, gives

$$f(z)=\frac{1}{2\pi i}\oint_{|\zeta|=R}\frac{F(\zeta)}{\zeta-z}\,d\zeta-\frac{1}{\pi}\iint_{1<|\zeta|<R}\frac{(\bar\partial F)(\zeta)}{\zeta-z}\,d\xi\,d\eta \tag{2.23}$$

(it is understood that $\zeta=\xi+i\eta$). Differentiating (2.23) and noting that the arising contour integral is $O(1/R)$ as $R\to\infty$, one obtains

$$\begin{aligned}|f'(z)|&=\frac{1}{\pi}\left|\iint_{|\zeta|>1}\frac{\bar\partial F(\zeta)}{(\zeta-z)^2}\,d\xi\,d\eta\right|\leqslant\frac{B}{\pi}\int_1^\infty\frac{\omega(r-1)}{r-1}r\,dr\int_0^{2\pi}\frac{dt}{|re^{it}-z|^2}\\&\leqslant 2B\int_1^\infty\frac{\omega(r-1)}{(r-1)(r-|z|)}\,dr=2B\int_0^\infty\frac{\omega(t)}{t(t+\varrho_z)}\,dt.\end{aligned}$$

The regularity assumption on ω implies that the last integral is $\leqslant C(\omega)\cdot\omega(\varrho_z)/\varrho_z$. Thus

$$M_0(f)\stackrel{\text{def}}{=}\sup_{z\in\mathbf{D}}\frac{\varrho_z}{\omega(\varrho_z)}|f'(z)|\leqslant C(\omega)B$$

and hence (in view of Lemma 6, part (ii)) $\|f\|_{\Lambda_\omega(\mathbf{T})}\leqslant C(\omega)B$. □

LEMMA 8. *Let $f\in A$ and θ be an inner function. If $f\theta\in A_\omega$, where ω is a regular majorant, then $f\in A_\omega$ and*

$$\|f\|_{\Lambda_\omega(\mathbf{T})}\leqslant C(\omega)\,\|f\theta\|_{\Lambda_\omega(\mathbf{T})}.$$

Proof. By Lemma 7, there exists a bounded function $G\in C^1(\mathbf{D}_-)$ such that

$$\lim_{\substack{z\to\zeta\\ z\in\mathbf{D}_-}} G(z)=f(\zeta)\theta(\zeta),\quad \zeta\in\mathbf{T},$$

and

$$|\bar\partial G(z)|\leqslant C(\omega)\,\|f\theta\|_{\Lambda_\omega(\mathbf{T})}\frac{\omega(\varrho_z)}{\varrho_z},\quad z\in\mathbf{D}_-.$$

It then follows that the function

$$F(z)\stackrel{\text{def}}{=}G(z)\,\overline{\theta(z^*)},\quad z\in\mathbf{D}_-,$$

is bounded, smooth and satisfies (2.21), while

$$|\bar\partial F(z)|=\bigl|\overline{\theta(z^*)}\,\bar\partial G(z)\bigr|\leqslant|\bar\partial G(z)|.$$

Another application of Lemma 7 completes the proof. □

Remark. Another proof of Lemma 8 can be found in [Sh1].

Finally, we require some modified versions of Lemmas 2 and 3 above.

LEMMA 9. *Let ω be a majorant. Given $0<\beta<1$, there exists a constant $C_1(\beta)>0$ such that*

$$\delta\int_\delta^1\frac{\omega(t)}{t^2}\,dt\leqslant C_1(\beta)\cdot\omega(\delta^\beta),\tag{2.24}$$

whenever $0<\delta<1$.

Proof. The integral on the left can be written as $\int_\delta^{\delta^\beta}+\int_{\delta^\beta}^1$. Using the inequalities

$$\omega(t)\leqslant\omega(\delta^\beta)\quad(0\leqslant t\leqslant\delta^\beta),$$
$$\frac{\omega(t)}{t}\leqslant\frac{\omega(\delta^\beta)}{\delta^\beta}\quad(t\geqslant\delta^\beta)$$

to estimate the two integrals, one eventually arrives at (2.24). □

LEMMA 10. *Let ω be a majorant. Given $0<\beta<1$, there is a constant $C_1(\beta)>0$ making the following statements true for any $z\in\mathbf{D}$ and $f\in\Lambda_\omega(\mathbf{T})$:*

(a) $\displaystyle\int\frac{\omega(|\zeta-\bar z|)}{|\zeta-z|^2}\,dm(\zeta)\leqslant C_1(\beta)\frac{\omega(\varrho_z^\beta)}{\varrho_z}$.

(b) $|\mathcal{P}f(z)-f(\tilde z)|\leqslant C_1(\beta)\,\|f\|_{\Lambda_\omega(\mathbf{T})}\,\omega(\varrho_z^\beta)$.

Proof. To verify (a), one recalls the relation (2.4) from the proof of Lemma 2. The term $\int_0^{\varrho_z}$ is handled as before, while $\int_{\varrho_z}^\pi$ is now estimated with the help of (2.24). To derive (b), one reproduces the proof of Lemma 3 (except for the very last step) and employs part (a). □

3. Proofs of Theorems 1, 2 and Corollary 2

Proof of Theorem 1. Under the hypotheses of the theorem, Lemmas 4 and 6 tell us that

$$\|f\|_{\Lambda_\omega(\bar{\mathbf{D}})} \asymp \|f\|_{\Lambda_\omega(\mathbf{T})} \asymp M_2(f),$$

where $M_2(f)$ is the quantity appearing in Lemma 6. It remains to notice that the right-hand side of (1.1) coincides with $M_2(f)$. □

Proof of Theorem 2. First let us remark that, in view of Lemma 1, there is no loss of generality in assuming that $\omega \in C^1(0,\infty)$ and

$$\omega'(t) \leqslant \frac{\omega(t)}{t}, \quad 0<t<\infty. \tag{3.1}$$

Otherwise, one should work with ω_1 instead of ω, where ω_1 is defined by (2.1). (In fact, once we assume that ω is smooth, (3.1) is just a restatement of the hypothesis that $\omega(t)/t$ is nonincreasing.)

This said, we proceed by estimating the norms involved. Clearly,

$$N_3(f) \leqslant \|f\|_{\Lambda_\omega(\bar{\mathbf{D}})}. \tag{3.2}$$

It is also obvious that

$$\max(\|\varphi\|_{\Lambda_\omega(\mathbf{T})}, \|\varphi\|_{\omega,\mathrm{rad}}) \leqslant N_3(f),$$

whence

$$N_2(f) \leqslant 2N_3(f). \tag{3.3}$$

Further, for $z \in \mathbf{D}$, one has

$$\frac{\mathcal{P}\varphi(z)-\varphi(z)}{\omega(\varrho_z)} \leqslant \frac{|\mathcal{P}\varphi(z)-\varphi(\tilde{z})|}{\omega(\varrho_z)} + \frac{|\varphi(\tilde{z})-\varphi(z)|}{\omega(\varrho_z)} \leqslant C(\omega)\|\varphi\|_{\Lambda_\omega(\mathbf{T})} + \|\varphi\|_{\omega,\mathrm{rad}},$$

where we have used Lemma 3 to estimate the first term. It follows that

$$N_1(f) \leqslant C(\omega) N_2(f). \tag{3.4}$$

Now that (3.2), (3.3) and (3.4) are established, it remains to prove that

$$\|f\|_{\Lambda_\omega(\bar{\mathbf{D}})} \leqslant C(\omega) N_1(f). \tag{3.5}$$

Set $K \stackrel{\text{def}}{=} \|\varphi\|_{\Lambda_\omega(\mathbf{T})}$ and

$$M \stackrel{\text{def}}{=} \sup_{z\in\mathbf{D}} \frac{\mathcal{P}\varphi(z)-\varphi(z)}{\omega(\varrho_z)},$$

so that $N_1(f)=K+M$. Let $f=g\theta$ be the canonical factorization of f (here g is outer and θ is inner), and let

$$h \stackrel{\text{def}}{=} f\theta = g\theta^2.$$

Note that $|h|=|g|=\varphi$ on $\mathbf{T}$. In view of Lemmas 4 and 8, we have

$$\|f\|_{\Lambda_\omega(\bar{\mathbf{D}})} \leqslant C_1(\omega)\|f\|_{\Lambda_\omega(\mathbf{T})} \leqslant C_2(\omega)\|h\|_{\Lambda_\omega(\mathbf{T})}.$$

Thus, to verify (3.5) it would be sufficient to show that

$$\|h\|_{\Lambda_\omega(\mathbf{T})} \leqslant C(\omega)(K+M).$$

By Lemma 7, this last task will be accomplished if we construct an appropriate "pseudo-analytic extension" for h, i.e. a bounded function $H \in C^1(\mathbf{D}_-)$ satisfying

$$\lim_{\substack{z\to\zeta \\ z\in\mathbf{D}_-}} H(z) = h(\zeta), \quad \zeta\in\mathbf{T}, \tag{3.6}$$

and

$$|\bar{\partial} H(z)| \leqslant C(\omega)(K+M)\frac{\omega(\varrho_z)}{\varrho_z}, \quad z\in\mathbf{D}_-. \tag{3.7}$$

We begin with some preparations. Recalling the definition of M and using the obvious inequalities

$$|g(z)|\cdot|\theta(z)| \leqslant \tfrac{1}{2}|g(z)|(1+|\theta(z)|^2) \leqslant \tfrac{1}{2}\mathcal{P}\varphi(z)+\tfrac{1}{2}|g(z)|\cdot|\theta(z)|^2, \quad z\in\mathbf{D},$$

we obtain

$$\begin{aligned} M\omega(\varrho_z) &\geqslant \mathcal{P}\varphi(z)-\varphi(z) = \mathcal{P}\varphi(z)-|g(z)|\cdot|\theta(z)| \\ &\geqslant \tfrac{1}{2}\mathcal{P}\varphi(z)-\tfrac{1}{2}|g(z)|\cdot|\theta(z)|^2 = \tfrac{1}{2}\{\mathcal{P}\varphi(z)-|h(z)|\}, \quad z\in\mathbf{D}. \end{aligned}$$

Thus

$$\mathcal{P}\varphi(z)-|h(z)| \leqslant 2M\omega(\varrho_z), \quad z\in\mathbf{D}. \tag{3.8}$$

Further, let $h_1 \stackrel{\text{def}}{=} \theta\sqrt{g}$ (here $\sqrt{g}$ is the outer function with modulus $\sqrt{\varphi}$ on $\mathbf{T}$) and note that the left-hand side of (3.8) equals

$$\mathcal{P}|h_1|^2(z)-|h_1(z)|^2 = \int |h_1(\zeta)-h_1(z)|^2\,d\mu_z(\zeta).$$

This enables us to rewrite (3.8) in the form

$$\int |h_1(\zeta)-h_1(z)|^2\,d\mu_z(\zeta) \leqslant 2M\omega(\varrho_z), \quad z\in\mathbf{D},$$

and hence to conclude, by virtue of Lemma 6, part (i) (where we currently replace f by h_1 and ω by $\sqrt{\omega}$), that

$$|h_1'(z)| \leqslant \sqrt{2M}\,\frac{\sqrt{\omega(\varrho_z)}}{\varrho_z}, \quad z\in\mathbf{D}. \tag{3.9}$$

Throughout the rest of the proof, z will denote a point in $\mathbf{D}_-$. Consider the sets

$$E_1 \stackrel{\text{def}}{=} \{z\in\mathbf{D}_- : |h(z^*)| \leqslant M\omega(\varrho_z)\},$$
$$E_2 \stackrel{\text{def}}{=} \{z\in\mathbf{D}_- : |h(z^*)| \geqslant 2M\omega(\varrho_z)\}$$

and the functions $H_{1,2}$ defined on $\mathbf{D}_-$ by

$$H_1(z) \stackrel{\text{def}}{=} h(z^*) \quad (=h_1^2(z^*)),$$
$$H_2(z) \stackrel{\text{def}}{=} \psi^2(z)/\overline{h(z^*)},$$

where $\psi(z)\stackrel{\text{def}}{=}\mathcal{P}\varphi(z^*)$. (Strictly speaking, H_2 lives on $\{z: h(z^*)\neq 0\}$, but that does not really matter.)

Claim 1. For $z\in E_1$,

$$|\bar\partial H_1(z)| \leqslant 2^{3/2} M\,\frac{\omega(\varrho_z)}{\varrho_z}. \tag{3.10}$$

Indeed, we have

$$|\bar\partial H_1(z)| = |2h_1(z^*)\,\bar\partial h_1(z^*)| = 2|h_1(z^*)|\cdot|h_1'(z^*)|\cdot|z^*|^2. \tag{3.11}$$

Recalling the definition of E_1, we get

$$|h_1(z^*)| \leqslant \sqrt{M\omega(\varrho_z)}, \quad z\in E_1, \tag{3.12}$$

while (3.9) gives

$$|h_1'(z^*)| \leqslant \sqrt{2M}\,\frac{\sqrt{\omega(\varrho_{z^*})}}{\varrho_{z^*}} \leqslant \sqrt{2M}\,\frac{\sqrt{\omega(\varrho_z)}}{\varrho_z}\,|z|, \quad z\in\mathbf{D}_-. \tag{3.13}$$

Substituting (3.12) and (3.13) into (3.11) yields (3.10).

Claim 2. For $z\in E_2$,

$$|\bar\partial H_2(z)| \leqslant C(\omega) K\,\frac{\omega(\varrho_z)}{\varrho_z}. \tag{3.14}$$

Indeed, since the function $z\mapsto 1/\overline{h(z^*)}$ is holomorphic on E_2, one has

$$\bar\partial H_2(z) = 2\,\frac{\psi(z)}{\overline{h(z^*)}}\,\bar\partial\psi(z). \tag{3.15}$$

Further,

$$\left|\frac{\psi(z)}{h(z^*)}\right| = \frac{\psi(z)-|h(z^*)|}{|h(z^*)|}+1. \tag{3.16}$$

Since

$$\psi(z)-|h(z^*)| = \mathcal{P}\varphi(z^*)-|h(z^*)| \leqslant 2M\omega(\varrho_{z^*}) \leqslant 2M\omega(\varrho_z)$$

(here we have employed (3.8)) and

$$|h(z^*)| \geqslant 2M\omega(\varrho_z) \quad \text{for } z\in E_2,$$

(3.16) shows that

$$\left|\frac{\psi(z)}{h(z^*)}\right| \leqslant 2, \quad z\in E_2. \tag{3.17}$$

Besides, Lemma 5 tells us that

$$|\bar{\partial}\psi(z)| \leqslant C(\omega)K\frac{\omega(\varrho_z)}{\varrho_z}. \tag{3.18}$$

Now a juxtaposition of (3.15), (3.17) and (3.18) yields (3.14).

The two claims having been verified, we also observe that

$$\lim_{\substack{z\to\zeta \\ z\in E_1}} H_1(z) = h(\zeta), \quad \zeta\in \mathbf{T}\cap \operatorname{clos} E_1, \tag{3.19}$$

(this is obvious) and

$$\lim_{\substack{z\to\zeta \\ z\in E_2}} H_2(z) = h(\zeta), \quad \zeta\in \mathbf{T}\cap \operatorname{clos} E_2. \tag{3.20}$$

(If $h(\zeta)\neq 0$, then (3.20) just follows from the fact that $\psi|_{\mathbf{T}}=|h|$; in case $h(\zeta)=0$, one should also make use of (3.17).)

Finally, we are going to construct the sought-after pseudoanalytic extension $H(z)$, satisfying (3.6) and (3.7), by welding the two "partial extensions" H_1 and H_2 together. To this end, we pick a nondecreasing function $\chi\in C^1[0,\infty)$ such that

$$\chi(t) = \begin{cases} 0 & \text{for } 0\leqslant t\leqslant 1, \\ 1 & \text{for } 2\leqslant t<\infty, \end{cases}$$

and set, for $z\in\mathbf{D}_-$,

$$H(z) \stackrel{\text{def}}{=} H_1(z)\left\{1-\chi\left(\frac{|h(z^*)|}{M\omega(\varrho_z)}\right)\right\}+H_2(z)\chi\left(\frac{|h(z^*)|}{M\omega(\varrho_z)}\right). \tag{3.21}$$

Clearly, $H|_{E_1}=H_1$ and $H|_{E_2}=H_2$. Consider the region

$$E_3 \stackrel{\text{def}}{=} \mathbf{D}_-\setminus(E_1\cup E_2).$$

In view of Claims 1 and 2, accompanied with relations (3.19) and (3.20), it now suffices to show that H is bounded on $\mathbf{D}_-$ and satisfies

$$\lim_{\substack{z\to\zeta\\ z\in E_3}} H(z)=h(\zeta),\quad \zeta\in\mathbf{T}\cap\operatorname{clos}E_3, \tag{3.22}$$

and

$$|\bar\partial H(z)|\leqslant C(\omega)(K+M)\frac{\omega(\varrho_z)}{\varrho_z},\quad z\in E_3. \tag{3.23}$$

For $z\in E_3$, we have

$$M\omega(\varrho_z)<|h(z^*)|<2M\omega(\varrho_z), \tag{3.24}$$

which enables us to repeat (with some very minor modifications, affecting only the numerical constants involved) all the estimates appearing in the proofs of Claims 1 and 2 above. In particular, we get

$$\left|\frac{\psi(z)}{h(z^*)}\right|\leqslant 3,\quad z\in E_3, \tag{3.25}$$

which is established in the same way as (3.17).

Since H_1 is bounded on $\mathbf{D}_-$ and H_2 is bounded on $E_2\cup E_3$ (the latter follows from (3.17) and (3.25)), we conclude that H is bounded on $\mathbf{D}_-$. Further, for $z\in E_3$ one has

$$|H_1(z)|\leqslant 2M\omega(\varrho_z),\quad |H_2(z)|\leqslant CM\omega(\varrho_z) \tag{3.26}$$

and

$$|\bar\partial H_1(z)|\leqslant CM\frac{\omega(\varrho_z)}{\varrho_z},\quad |\bar\partial H_2(z)|\leqslant C(\omega)K\frac{\omega(\varrho_z)}{\varrho_z}. \tag{3.27}$$

(The inequalities for $|H_2(z)|$ and $|\bar\partial H_2(z)|$ rely on (3.25).)

From (3.24) it follows that $h(\zeta)=0$ whenever $\zeta\in\mathbf{T}\cap\operatorname{clos}E_3$, while (3.26) yields

$$\lim_{\substack{z\to\zeta\\ z\in E_3}} H(z)=0;$$

this proves (3.22).

Next, differentiating (3.21), we obtain

$$\begin{aligned}\bar\partial H(z)=\bar\partial H_1(z)\cdot\left\{1-\chi\left(\frac{|h(z^*)|}{M\omega(\varrho_z)}\right)\right\}+\bar\partial H_2(z)\cdot\chi\left(\frac{|h(z^*)|}{M\omega(\varrho_z)}\right)\\ +(H_2(z)-H_1(z))\cdot\chi'\left(\frac{|h(z^*)|}{M\omega(\varrho_z)}\right)\cdot\frac{1}{M}\bar\partial\left(\frac{|h(z^*)|}{\omega(\varrho_z)}\right).\end{aligned} \tag{3.28}$$

For $z\in E_3$, the first two terms on the right are bounded in modulus by

$$C(\omega)(K+M)\frac{\omega(\varrho_z)}{\varrho_z},$$

as is clear from (3.27). In order to estimate the third term in (3.28), we note that

$$|H_2(z)-H_1(z)|\leqslant CM\omega(\varrho_z), \quad z\in E_3 \tag{3.29}$$

(see (3.26)), while

$$|\chi'(t)|\leqslant C, \quad 0<t<\infty. \tag{3.30}$$

Finally,

$$\begin{aligned}\bar\partial\left(\frac{|h(z^*)|}{\omega(\varrho_z)}\right)&=\frac{1}{\omega(\varrho_z)}\bar\partial\big(h_1(z^*)\overline{h_1(z^*)}\big)+|h(z^*)|\bar\partial\left(\frac{1}{\omega(\varrho_z)}\right)\\&=-\frac{z^{*2}}{\omega(\varrho_z)}\overline{h_1(z^*)}\,h_1'(z^*)-\frac{1}{2}\cdot\frac{z}{|z|}|h(z^*)|\frac{\omega'(\varrho_z)}{\omega^2(\varrho_z)}.\end{aligned} \tag{3.31}$$

Recalling (3.1), (3.13) and the right-hand inequality in (3.24) (also used in the form $|h_1(z^*)|<\sqrt{2M\omega(\varrho_z)}$) we conclude from (3.31) that

$$\left|\bar\partial\left(\frac{|h(z^*)|}{\omega(\varrho_z)}\right)\right|\leqslant\frac{CM}{\varrho_z}, \quad z\in E_3. \tag{3.32}$$

Combining (3.29), (3.30) and (3.32), we see that the last term in (3.28) is bounded in modulus by $CM\omega(\varrho_z)/\varrho_z$ whenever $z\in E_3$. Eventually, we arrive at (3.23), and the proof is complete. □

Proof of Corollary 2. Assuming that the hypotheses of Corollary 2 are fulfilled, set $f\overset{\text{def}}{=}F\theta$. For $z\in\mathbf{D}$, one has the identity

$$\mathcal{P}|f|(z)-|f(z)|=\{\mathcal{P}|F|(z)-|F(z)|\}+\{|F(z)|(1-|\theta(z)|)\}. \tag{3.33}$$

Dividing (3.33) by $\omega(\varrho_z)$ and noting that both terms { } are nonnegative, one concludes that

$$N_1(f)\asymp N_1(F)+\sup_{z\in\mathbf{D}}|F(z)|\frac{1-|\theta(z)|}{\omega(1-|z|)}. \tag{3.34}$$

In view of Theorem 2, (3.34) proves the equivalence of conditions (i) and (ii) (see the statement of Corollary 2).

Further, set

$$S(F,\theta)\overset{\text{def}}{=}\sup\left\{|F(\zeta)|\frac{1-|\theta(r\zeta)|}{\omega(1-r)}:\zeta\in\mathbf{T},\,0\leqslant r<1\right\}.$$

The identity

$$|f(\zeta)|-|f(r\zeta)|=(|F(\zeta)|-|F(r\zeta)|)\,|\theta(r\zeta)|+|F(\zeta)|\,(1-|\theta(r\zeta)|),$$

where $\zeta\in\mathbf{T}$ and $0\leqslant r<1$, implies

$$N_2(f)\leqslant N_2(F)+S(F,\theta) \tag{3.35}$$

and

$$S(F,\theta)\leqslant N_2(f)+N_2(F). \tag{3.36}$$

From (3.34) and from the equivalence relation $N_1(\cdot)\asymp N_2(\cdot)$, it follows that

$$N_2(F)\leqslant C(\omega)N_2(f). \tag{3.37}$$

Taken together, (3.35), (3.36) and (3.37) yield

$$N_2(f)\asymp N_2(F)+S(F,\theta). \tag{3.38}$$

Finally, in virtue of Theorem 2, (3.38) proves the equivalence of conditions (i) and (iii). □

4. Deduction of the Havin–Shamoyan–Carleson–Jacobs embedding theorem

For the reader's convenience, we reproduce the theorem in question and then proceed with the proof.

THEOREM 3. *Let ω be a majorant such that $\psi(t)\stackrel{\text{def}}{=}\omega(\sqrt{t})$ is a regular majorant. Suppose that $\varphi\in\Lambda_\omega(\mathbf{T})$, $\varphi\geqslant 0$, and $\log\varphi\in L^1(\mathbf{T},dm)$. Then $\mathcal{O}_\varphi\in A_\psi$ and*

$$\|\mathcal{O}_\varphi\|_{\Lambda_\psi(\bar{\mathbf{D}})}\leqslant C(\omega)\,\|\varphi\|_{\Lambda_\omega(\mathbf{T})}\left(1+\int_{\mathbf{T}}\log\frac{\|\varphi\|_\infty}{\varphi(\zeta)}\,dm(\zeta)\right).$$

Proof. We may assume that $\min_{\zeta\in\mathbf{T}}\varphi(\zeta)=0$ (otherwise the theorem becomes almost trivial) and $\|\varphi\|_\infty=1$. Set

$$K\stackrel{\text{def}}{=}\|\varphi\|_{\Lambda_\omega(\mathbf{T})}\quad\text{and}\quad \mathcal{L}\stackrel{\text{def}}{=}\int_{\mathbf{T}}\log\frac{1}{\varphi(\zeta)}\,dm(\zeta).$$

Letting $\zeta_1,\zeta_2\in\mathbf{T}$ be such that $\varphi(\zeta_1)=1$, $\varphi(\zeta_2)=0$, and writing

$$1=\varphi(\zeta_1)-\varphi(\zeta_2)\leqslant K\omega(|\zeta_1-\zeta_2|)\leqslant K\omega(2),$$

we see that $K\geqslant 1/\omega(2)$.

We have to prove that

$$\|\mathcal{O}_\varphi\|_{\Lambda_\psi(\bar{\mathbf{D}})} \leqslant C(\omega) K(1+\mathcal{L}). \tag{4.1}$$

In virtue of Theorem 2, we can replace the left-hand side of (4.1) by $\|\varphi\|_{\Lambda_\psi(\mathbf{T})} + \|\varphi\|_{\psi,\mathrm{rad}}$, where φ is extended into $\mathbf{D}$ by the formula

$$\varphi(z) \stackrel{\mathrm{def}}{=} \exp\biggl(\int_{\mathbf{T}} \log\varphi\, d\mu_z\biggr) = |\mathcal{O}_\varphi(z)|.$$

Since $\omega(t) \leqslant \sqrt{2}\,\psi(t)$ for $0 \leqslant t \leqslant 2$, it follows that

$$\|\varphi\|_{\Lambda_\psi(\mathbf{T})} \leqslant \sqrt{2}\, K,$$

and so (4.1) is equivalent to the estimate

$$\|\varphi\|_{\psi,\mathrm{rad}} \leqslant C(\omega) K(1+\mathcal{L}).$$

The proof thus reduces to showing that, for $z \in \mathbf{D}$,

$$|\varphi(z) - \varphi(\tilde{z})| \leqslant C(\omega) K(1+\mathcal{L})\,\omega(\sqrt{\varrho_z}\,). \tag{4.2}$$

To this end, we fix a point $z \in \mathbf{D}$ and distinguish two cases.

Case 1. $\varphi(\tilde{z}) \leqslant \varphi(z) + 2K\omega(\sqrt{\varrho_z}\,)$. We have then

$$-2K\omega(\sqrt{\varrho_z}\,) \leqslant \varphi(z) - \varphi(\tilde{z}) \leqslant \mathcal{P}\varphi(z) - \varphi(\tilde{z}). \tag{4.3}$$

Applying Lemma 10(b) with $\beta = \frac{1}{2}$ gives

$$|\mathcal{P}\varphi(z) - \varphi(\tilde{z})| \leqslant CK\omega(\sqrt{\varrho_z}\,). \tag{4.4}$$

Combining (4.3) and (4.4) yields

$$|\varphi(z) - \varphi(\tilde{z})| \leqslant CK\omega(\sqrt{\varrho_z}\,),$$

so that (4.2) holds true.

Case 2. $\varphi(\tilde{z}) > \varphi(z) + 2K\omega(\sqrt{\varrho_z}\,)$. Let $\eta > 0$ be defined by

$$\omega(\eta) = \frac{\varphi(\tilde{z})}{2K} \tag{4.5}$$

(since $K \geqslant 1/\omega(2)$ and $\varphi(\tilde{z}) \leqslant 1$, one sees that $\varphi(\tilde{z})/(2K)$ belongs to the range of ω). Further, set

$$I \stackrel{\text{def}}{=} \{\zeta \in \mathbf{T} : |\zeta - \tilde{z}| \leqslant \eta\},$$

$$J_1 \stackrel{\text{def}}{=} \int_I \log \varphi \, d\mu_z \quad \text{and} \quad J_2 \stackrel{\text{def}}{=} \int_{\mathbf{T} \setminus I} \log \varphi \, d\mu_z.$$

We have

$$0 \leqslant \varphi(\tilde{z}) - \varphi(z) = \varphi(\tilde{z}) - e^{J_1} e^{J_2} = R + S, \tag{4.6}$$

where

$$R \stackrel{\text{def}}{=} \varphi(\tilde{z}) - e^{J_1} \quad \text{and} \quad S \stackrel{\text{def}}{=} e^{J_1}(1 - e^{J_2}).$$

In order to estimate R, we write

$$J_1 = \int_I (\log \varphi(\zeta) - \log \varphi(\tilde{z})) \, d\mu_z(\zeta) + \mu_z(I) \log \varphi(\tilde{z}) = J_3 + \gamma \log \varphi(\tilde{z}),$$

where we put

$$J_3 \stackrel{\text{def}}{=} \int_I (\log \varphi(\zeta) - \log \varphi(\tilde{z})) \, d\mu_z(\zeta)$$

and $\gamma \stackrel{\text{def}}{=} \mu_z(I)$. Since $0 \leqslant \varphi(\tilde{z}) \leqslant 1$, we have

$$e^{J_1} = \varphi(\tilde{z})^\gamma e^{J_3} \geqslant \varphi(\tilde{z}) e^{J_3},$$

and so

$$R = \varphi(\tilde{z}) - e^{J_1} \leqslant \varphi(\tilde{z})(1 - e^{J_3}) \leqslant \varphi(\tilde{z})|J_3|. \tag{4.7}$$

Further, for $\zeta \in I$ one has

$$\varphi(\tilde{z}) - \varphi(\zeta) \leqslant K\omega(\eta) = \tfrac{1}{2}\varphi(\tilde{z}),$$

whence $\varphi(\zeta) \geqslant \frac{1}{2}\varphi(\tilde{z})$ and

$$|\log \varphi(\zeta) - \log \varphi(\tilde{z})| \leqslant \frac{|\varphi(\zeta) - \varphi(\tilde{z})|}{\min\{\varphi(\zeta), \varphi(\tilde{z})\}} \leqslant \frac{2}{\varphi(\tilde{z})} K\omega(|\zeta - \tilde{z}|).$$

This in turn implies

$$|J_3| \leqslant \frac{2K}{\varphi(\tilde{z})} \int_{\mathbf{T}} \omega(|\zeta - \tilde{z}|) \, d\mu_z(\zeta),$$

and an application of Lemma 10(a), with $\beta = \frac{1}{2}$, now shows that

$$|J_3| \leqslant \frac{2K}{\varphi(\tilde{z})} C\omega(\sqrt{\varrho_z}). \tag{4.8}$$

Substituting (4.8) into the right-hand side of (4.7) gives

$$R \leqslant CK\omega(\sqrt{\varrho_z}). \tag{4.9}$$

We proceed by estimating S, the second term in (4.6). To this end, we write

$$S = \varphi(\tilde{z})^\gamma e^{J_3}(1-e^{J_2}) \leqslant \varphi(\tilde{z})^\gamma e^{J_3}|J_2|. \tag{4.10}$$

Using (4.8) and the inequality

$$\varphi(\tilde{z}) > 2K\omega(\sqrt{\varrho_z}) \tag{4.11}$$

(recall the hypothesis of Case 2), we see that $|J_3| \leqslant C$ and hence

$$e^{J_3} \leqslant C. \tag{4.12}$$

Further, it is clear that

$$1-\gamma = \mu_z(\mathbf{T}\setminus I) = \mu_z\{\zeta \in \mathbf{T} : |\zeta - \tilde{z}| > \eta\}. \tag{4.13}$$

Observing also that $\eta > \sqrt{\varrho_z}$ (see (4.5) and (4.11)), we deduce from (4.13) that

$$1-\gamma \leqslant \mu_z\{\zeta \in \mathbf{T} : |\zeta - \tilde{z}| > \sqrt{\varrho_z}\} \leqslant C\sqrt{\varrho_z} \tag{4.14}$$

(the last inequality can be verified by a straightforward calculation). Using (4.11) and (4.14), we obtain

$$\log \frac{\varphi(\tilde{z})^\gamma}{\varphi(\tilde{z})} = (1-\gamma)\log\frac{1}{\varphi(\tilde{z})} \leqslant C\sqrt{\varrho_z}\log\frac{1}{2K\omega(\sqrt{\varrho_z})} \leqslant C,$$

where the final conclusion holds because $K \geqslant 1/\omega(2)$ and $\omega(t) \geqslant \frac{1}{2}t\omega(2)$ for $t \leqslant 2$. Thus,

$$\varphi(\tilde{z})^\gamma \leqslant C\varphi(\tilde{z}). \tag{4.15}$$

Substituting (4.12) and (4.15) into the right-hand side of (4.10) yields

$$S \leqslant C\varphi(\tilde{z})|J_2|. \tag{4.16}$$

Further, for $\zeta \in \mathbf{T}\setminus I$ one has $|\zeta - \tilde{z}| > \eta$ and hence $|\zeta - z| > \frac{1}{2}\eta$. Consequently,

$$|J_2| = \int_{\mathbf{T}\setminus I} \log\left(\frac{1}{\varphi(\zeta)}\right)\frac{1-|z|^2}{|\zeta - z|^2}\,dm(\zeta) \leqslant 8\frac{\varrho_z}{\eta^2}\int_{\mathbf{T}\setminus I}\log\frac{1}{\varphi(\zeta)}\,dm(\zeta) \leqslant 8\mathcal{L}\frac{\varrho_z}{\eta^2}. \tag{4.17}$$

Since $\eta > \sqrt{\varrho_z}$ and the function $t \mapsto \omega(t)/t^2$ is decreasing, one has

$$\frac{\omega(\eta)}{\eta^2} \leqslant \frac{\omega(\sqrt{\varrho_z}\,)}{\varrho_z}. \tag{4.18}$$

Taking (4.5) into account, one can rewrite (4.18) as

$$\frac{\varrho_z}{\eta^2} \leqslant \frac{2K}{\varphi(\tilde{z})}\omega(\sqrt{\varrho_z}\,). \tag{4.19}$$

Combining (4.19) with (4.17) yields

$$|J_2| \leqslant 16K\mathcal{L}\frac{\omega(\sqrt{\varrho_z}\,)}{\varphi(\tilde{z})},$$

which in turn implies, together with (4.16), that

$$S \leqslant CK\mathcal{L}\omega(\sqrt{\varrho_z}\,). \tag{4.20}$$

Eventually, a juxtaposition of (4.6), (4.9) and (4.20) enables us to conclude that

$$0 \leqslant \varphi(\tilde{z}) - \varphi(z) \leqslant CK(1+\mathcal{L})\omega(\sqrt{\varrho_z}\,),$$

and so (4.2) holds true. □

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