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On the asymptotic geometry of abelian-by-cyclic groups

by

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1. Introduction

Gromov's polynomial growth theorem [Gr1] states that the property of having polynomial growth characterizes virtually nilpotent groups among all finitely generated groups.

Gromov's theorem inspired the more general problem (see e.g. [GH]) of understanding to what extent the asymptotic geometry of a finitely generated solvable group determines its algebraic structure—in short, are solvable groups quasi-isometrically rigid? In general they are not: very recently A. Dioubina [D] has found a solvable group which is quasi-isometric to a group which is not virtually solvable; these groups are finitely generated but not finitely presentable. In the opposite direction, the first steps in identifying quasi-isometrically rigid solvable groups which are not virtually nilpotent were taken for a special class of examples, the solvable Baumslag–Solitar groups, in [FM1] and [FM2].

The goal of the present paper is to show that a much broader class of solvable groups, the class of finitely presented, nonpolycyclic, abelian-by-cyclic groups, is characterized among all finitely generated groups by its quasi-isometry type. We also give a complete quasi-isometry classification of the groups in this class; such a classification for nilpotent groups remains a major open question. Motivated by these results, we offer a conjectural picture of quasi-isometric classification and rigidity for polycyclic abelian-by-cyclic groups in §10.1.

The proofs of these results lead one naturally from a geometry-of-groups problem to the theory of dynamical systems via the asymptotic behavior of certain flows and their associated foliations.

1.1. The abelian-by-cyclic group Γ_M

A group Γ is *abelian-by-cyclic* if there is an exact sequence

$$1 \to A \to \Gamma \to Z \to 1$$

where A is an abelian group and Z is an infinite cyclic group. If Γ is finitely generated, then A is a finitely generated module over the group ring $\mathbf{Z}[Z]$, although A may not be finitely generated as a group.

By a result of Bieri and Strebel [BS1], the class of finitely presented, torsion-free, abelian-by-cyclic groups may be described in another way. Consider an $(n\times n)$-matrix M with integral entries and $\det M\neq 0$. Let Γ_M be the ascending HNN extension of $\mathbf{Z}^n$ given by the monomorphism ϕ_M with matrix M. Then Γ_M has a finite presentation

$$\langle t, a_1, ..., a_n \mid [a_i, a_j]=1, ta_it^{-1}=\phi_M(a_i), i,j=1,...,n\rangle,$$

where $\phi_M(a_i)$ is the word $a_1^{m_1}...a_n^{m_n}$, and the vector $(m_1, ..., m_n)$ is the ith column of the matrix M. Such groups Γ_M are precisely the class of finitely presented, torsion-free, abelian-by-cyclic groups (see [BS1] for a proof involving a precursor of the Bieri–Neumann–Strebel invariant, or [FM2] for a proof using trees). The group Γ_M is polycyclic if and only if $|\det M|=1$; this is easy to see directly, and also follows from [BS2].

1.2. Statement of results

The first main theorem in this paper gives a classification of all finitely presented, nonpolycyclic, abelian-by-cyclic groups up to quasi-isometry. It is easy to see that any such group has a torsion-free subgroup of finite index, and so is commensurable (hence quasi-isometric) to some Γ_M. The classification of these groups is actually quite delicate—the standard quasi-isometry invariants (ends, growth, isoperimetric inequalities, etc.) do not distinguish any of these groups from each other, except that the size of the matrix M can be detected by large-scale cohomological invariants of Γ_M.

Given $M\in \mathrm{GL}(n,\mathbf{R})$, the *absolute Jordan form* of M is the matrix obtained from the Jordan form for M over $\mathbf{C}$ by replacing each diagonal entry with its absolute value, and rearranging the Jordan blocks in some canonical order.

THEOREM 1.1 (classification theorem). *Let M_1 and M_2 be integral matrices with $|\det M_i|>1$ for $i=1,2$. Then Γ_{M_1} is quasi-isometric to Γ_{M_2} if and only if there are positive integers r_1, r_2 such that $M_1^{r_1}$ and $M_2^{r_2}$ have the same absolute Jordan form.*

Remark. Theorem 1.1 generalizes the main result of [FM1], which is the case when M_1, M_2 are positive (1×1)-matrices; in that case the result of [FM1] says even more, namely that Γ_{M_1} and Γ_{M_2} are quasi-isometric if and only if they are commensurable. When $n\geqslant 2$, however, it is not hard to find $(n\times n)$-matrices M_1, M_2 such that $\Gamma_{M_1}, \Gamma_{M_2}$ are quasi-isometric but not commensurable. Polycyclic examples are given in [BG], and the same ideas may be used to produce nonpolycyclic examples.

The following theorem shows that the algebraic property of being a finitely presented, nonpolycyclic, abelian-by-cyclic group is in fact a large-scale geometric property.

THEOREM 1.2 (quasi-isometric rigidity). *Let $\Gamma=\Gamma_M$ be a finitely presented abelian-by-cyclic group, determined by an integer $(n\times n)$-matrix M with $|\det M|>1$. Let G be any finitely generated group quasi-isometric to Γ. Then there is a finite normal subgroup $K\subset G$ such that G/K is abstractly commensurable to Γ_N, for some integer $(n\times n)$-matrix N with $|\det N|>1$.*

Remark. Theorem 1.2 generalizes the main result of [FM2], which covers the case when M is a positive (1×1)-matrix. The latter result was given a new proof in [MSW], and in §9 we follow the methods of [MSW] in proving Theorem 1.2.

Remark. The "finitely presented" hypothesis in Theorem 1.2 cannot be weakened to "finitely generated". Dioubina shows [D] that the wreath product $\mathbf{Z}\,\mathrm{wr}\,\mathbf{Z}$, an abelian-by-cyclic group of the form $\mathbf{Z}[\mathbf{Z}]$-by-$\mathbf{Z}$, is quasi-isometric to the wreath product $(\mathbf{Z}\oplus F)\,\mathrm{wr}\,\mathbf{Z}$ whenever F is a finite group. But $(\mathbf{Z}\oplus F)\,\mathrm{wr}\,\mathbf{Z}$ has no nontrivial finite normal subgroups, and when F is nonabelian it is not abstractly commensurable to an abelian-by-cyclic group.

One of the key technical results used to prove Theorem 1.1 is the following theorem, which we believe is of independent interest. It describes a rigidity phenomenon for 1-parameter subgroups of $\mathrm{GL}(n,\mathbf{R})$ which generalizes work of Benardete [Be] (see also [W]).

A 1-parameter subgroup M^t of $\mathrm{GL}(n,\mathbf{R})$ determines a 1-parameter family of quadratic forms $Q_M(t)=(M^{-t})^T(M^{-t})$ on $\mathbf{R}^n$, where the superscript ^{T} denotes transpose. Each $Q_M(t)$ determines a norm $\|\cdot\|_{M,t}$ and a distance function $d_{M,t}$ on $\mathbf{R}^n$.

THEOREM 5.11 (1-parameter subgroup rigidity). *Let M^t, N^t be 1-parameter subgroups of $\mathrm{GL}(n,\mathbf{R})$ such that $M=M^1$ and $N=N^1$ have no eigenvalues on the unit circle. If there exists a bijection $f\colon\mathbf{R}^n\to\mathbf{R}^n$ and constants $K\geqslant 1$, $C\geqslant 0$ such that for each $t\in\mathbf{R}$ and $p,q\in\mathbf{R}^n$,*

$$-C+\frac{1}{K}\cdot d_{M,t}(p,q)\leqslant d_{N,t}(f(p),f(q))\leqslant K\cdot d_{M,t}(p,q)+C,$$

then M and N have the same absolute Jordan form.

The proof of Theorem 5.11 is given in §6 and shows that in fact f is a homeomorphism with a reasonably high degree of regularity; see Proposition 6.3.

Remark. The case of Theorem 5.11 when f is the identity map follows from a theorem of D. Benardete [Be]. See also D. Witte [W]. Benardete's theorem determines precisely when two 1-parameter subgroups of $\mathrm{GL}(n,\mathbf{R})$ diverge, and it applies as well to matrices with eigenvalues on the unit circle.

1.3. Homogeneous spaces

Using coarse topological and geometrical methods, we reduce the study of quasi-isometries of Γ_M to that of a certain Lie group G_M.

After squaring M if necessary, we can assume that $\det M>0$ and that M lies on a 1-parameter subgroup M^t of $\mathrm{GL}(n,\mathbf{R})$. The group Γ_M is a cocompact subgroup of the solvable Lie group $G_M=\mathbf{R}^n\rtimes_M\mathbf{R}$, where $\mathbf{R}$ acts on $\mathbf{R}^n$ by the 1-parameter subgroup M^t. The group Γ_M is discrete in G_M if and only if $\det M=1$. See §4 for details.

The groups G_M, with their left-invariant metrics, give a rich and familiar collection of examples, including: all real hyperbolic spaces, when M is a constant times the identity; many negatively curved homogeneous spaces, when M has all eigenvalues >1 in absolute value; and 3-dimensional SOLV-geometry, when M is a hyperbolic (2×2)-matrix of determinant 1. The negatively curved examples associated to a real diagonal matrix with all eigenvalues >1 were studied by Pansu [P1] (and later Gromov [Gr2]), who analyzed their quasi-isometric geometry using the idea of "conformal dimension".

We should mention also the result of Heintze [He] that the class of connected, negatively curved homogeneous spaces consists precisely of those spaces of the form $N\rtimes\mathbf{R}$ where N is a nilpotent Lie group, and the action of $\mathbf{R}$ on the Lie algebra has all eigenvalues strictly outside the unit circle.

1.4. Outline of proofs

After preliminary sections, §3 on linear algebra, and §4 on the solvable Lie group G_M, the proof of Theorem 1.1 can be divided into three main parts: §§ 5 and 6 on the dynamics of G_M; §7 on quasi-isometries of Γ_M via coarse topology; and §8 on finding the integers, where the pieces of the proof are put together. The proof of Theorem 1.2 is contained in §9 on quasi-isometric rigidity. Finally we pose some conjectures and problems in §10.

§§ 5 *and* 6. *Dynamics of* G_M. In these two sections we classify the Lie groups G_M up to *horizontal-respecting* quasi-isometry, that is, up to quasi-isometries $\phi: G_M\to G_N$ which take each set of the form $\mathbf{R}^m\times\{t\}$ to a set of the form $\mathbf{R}^n\times\{h(t)\}$ for some function h called the *induced time change.*

THEOREM 5.2′ (horizontal-respecting quasi-isometries: special case). *Let* M,N *lie on* 1*-parameter subgroups* M^t,N^t *of* $\mathrm{GL}(n,\mathbf{R})$, *and suppose that* $\det M,\det N>1$. *If there exists a horizontal-respecting quasi-isometry* $\phi: G_M\to G_N$, *then there exist real numbers* $r,s>0$ *so that* M^r *and* N^s *have the same absolute Jordan form.*

Remark. In the special case where M,N are diagonalizable with all eigenvalues >1, this can be extracted from work of Pansu [P1] *without* the assumption that ϕ is horizontal-respecting. This special case was later reconsidered by Gromov (see [Gr2, §7.C]), as an application of his "infδim"-invariant. Our statement and proof of Theorem 5.2 is inspired in part by the ideas of exponential growth rates built into the infδim-invariant (see also comments after Proposition 5.8).

In §5 we give a slightly more general version of this statement, Theorem 5.2.

The proof of Theorem 5.2 uses a certain dynamical system on G_M, the "vertical flow" which flows upward at unit speed along flow lines of the form $(\text{point})\times\mathbf{R}\subset\mathbf{R}^m\rtimes_M\mathbf{R}$. When M has no eigenvalues on the unit circle this is a hyperbolic or Anosov flow, and in general it is a partially hyperbolic flow. We prove Theorem 5.2 in several steps, using stronger and stronger dynamical properties of flows in G_M.

Step 1 (*foliations rigidity, Proposition* 5.4). Using the shadowing lemma from hyperbolic dynamics we show that ϕ coarsely respects three dynamically defined foliations of G_M and G_N: the weak stable, weak unstable, and center foliations. This, together with a result of Bridson–Gersten that depends in turn on work of Pansu (see Corollary 5.6), allows reduction to the case where M, N have no eigenvalues on the unit circle.

Step 2 (*time rigidity, Proposition* 5.8). We show that the induced time change map of ϕ is actually an *affine* map between the time parameters of G_M and G_N. After taking a real power of N and composing with a vertical translation, we can assume that ϕ preserves the time parameter, that is, $h(t)=t$.

Step 3 (1-*parameter subgroup rigidity, Theorem* 5.11). From Step 2, ϕ induces a quasi-isometry between corresponding level sets of the time parameter on G_M, G_N, which reduces the proof to Theorem 5.11, 1-parameter subgroup rigidity. The latter theorem is proved in §6, by studying rigidity properties of certain flags of foliations of $\mathbf{R}^n$ associated to the absolute Jordan form of $M\in\mathrm{GL}(n,\mathbf{R})$.

§7. *Quasi-isometries of* Γ_M *via coarse topology.* Given an integer matrix $M\in\mathrm{GL}(n,\mathbf{R})$ with $\det M>1$, we study the geometry of Γ_M by constructing a contractible metric cell complex X_M on which Γ_M acts freely, properly discontinuously and cocompactly by isometries, so that Γ_M is quasi-isometric to X_M. Topologically, X_M is a product of $\mathbf{R}^m$ with the homogeneous directed tree T_M with one edge entering and d edges leaving each vertex. Here $d=\det M$. Metrically, for every coherently oriented line l in T_M, the metric on X_M is such that $\mathbf{R}^m\times l$ is isometric to G_M.

The main result of this section, Proposition 7.1, says that a quasi-isometry f: $X_M\to X_N$ induces a quasi-isometry $\phi: G_M\to G_N$ which respects horizontal foliations. This is proved using coarse geometric and topological methods. This is precisely where the condition $\det M, \det N>1$ is essential for our proof, since it gives that the trees T_M, T_N have nontrivial branching, and this branching allows us to show that f "remembers" the branch points (see Step 2 of §7.2).

While this proof is in the spirit of [FM1], further complications arise in this more general case (see §7.2). Also, for other applications (e.g. [FM3], [MSW]), we shall derive Proposition 7.1 from a still more general result, Theorem 7.7, which applies to many graphs of groups whose vertex and edge groups are fundamental groups of aspherical manifolds of fixed dimension.

§8. *Finding the integers.* Given integer matrices $M, N \in \mathrm{GL}(n, \mathbf{R})$ with $|\det M| > 1$ and $|\det N| > 1$ such that Γ_M and Γ_N are quasi-isometric, a simple argument allows us to reduce to the case of positive determinant, and then the results of §§5–7 combine to show that there are positive real numbers r, s so that M^r and N^s have the same absolute Jordan form. We need to show that *integral* r, s exist. This is done by showing that a quasi-isometry $X_M \to X_N$ induces a bi-Lipschitz homeomorphism between certain self-similar Cantor sets attached to X_M and X_N. Applying a theorem of Cooper on bi-Lipschitz types of these Cantor sets allows us to conclude that $(\det M)^p = (\det N)^q$ for some integers $p, q \geqslant 1$, from which the desired conclusion follows.

§9. *Quasi-isometric rigidity.* To prove Theorem 1.2, we use the coarse topology results from §7 to show that a group quasi-isometric to some Γ_M admits a quasi-action on a tree of n-dimensional Euclidean spaces. We then use the results of [MSW] to convert this quasi-action into a true action on a tree whose edge and vertex stabilizers are finitely generated groups quasi-isometric to $\mathbf{Z}^n$. The proof is completed by invoking well-known quasi-isometry invariants, combined with a brief study of injective endomorphisms of virtually abelian groups.

§10. *Questions.* We make some conjectures concerning possible extensions of this work to the polycyclic case. Also, we state some problems on the quasi-isometry group of Γ_M.

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2. Preliminaries

This brief section reviews some basic material; see for example [GH].

Given $K \geqslant 1$, $C \geqslant 0$, a (K, C)-*quasi-isometry* between metric spaces is a map $f: X \to Y$ such that:

(1) For all $x, x' \in X$ we have

$$\frac{1}{K} \cdot d_X(x,x') - C \leqslant d_Y(f(x), f(x')) \leqslant K \cdot d_X(x,x') + C;$$

(2) For all $y \in Y$ we have $d_Y(y, f(X)) \leqslant C$.

If f satisfies (1) but not necessarily (2) then it is called a (K,C)-*quasi-isometric embedding*. If f satisfies only the right-hand inequality of (1) then f is (K,C)-*coarsely Lipschitz*, and if in addition $C=0$ then f is K-*Lipschitz*.

A *coarse inverse* of a quasi-isometry $f: X \to Y$ is a quasi-isometry $g: Y \to X$ such that, for some constant $C'>0$, we have $d(g \circ f(x), x) < C'$ and $d(f \circ g(y), y) < C'$ for all $x \in X$ and $y \in Y$. Every (K,C)-quasi-isometry $f: X \to Y$ has a (K,C')-coarse inverse $g: Y \to X$, where C' depends only on K, C: for each $y \in Y$ define $g(y)$ to be any point $x \in X$ such that $d(f(x), y) \leqslant C$.

A fundamental fact observed by Efremovich, by Milnor [Mi] and by Švarc, which we use repeatedly without mentioning, states that if a group G acts properly discontinuously and cocompactly by isometries on a proper geodesic metric space X, then G is finitely generated, and X is quasi-isometric to G equipped with the word metric.

Given a metric space X and $A, B \subset X$, we denote the *Hausdorff distance* by

$$d_{\mathcal{H}}(A,B) = \inf\{r \in [0,\infty] \mid A \subset N_r(B) \text{ and } B \subset N_r(A)\}.$$

The following lemma says that an ambient quasi-isometry induces a quasi-isometry between subspaces of a certain type. A map $\sigma: S \to X$ between geodesic metric spaces is *uniformly proper* if there is a function $\varrho: [0,\infty) \to [0,\infty)$ with $\lim_{t\to\infty} \varrho(t) = +\infty$, and constants $K \geqslant 1$, $C \geqslant 0$, such that for all $x, y \in S$ we have

$$\varrho(d_S(x,y)) \leqslant d_X(\sigma(x), \sigma(y)) \leqslant K \cdot d_S(x,y) + C.$$

The function ϱ and the constants K, C are called *uniformity data* for σ.

LEMMA 2.1. *Given geodesic metric spaces X, Y, S, T, a quasi-isometry $f: X \to Y$ and uniformly proper maps $\sigma: S \to X$ and $\tau: T \to Y$, suppose that $d_{\mathcal{H}}(f\sigma(S), \tau(T)) < \infty$. Then S, T are quasi-isometric. To be explicit, any function $g: S \to T$ such that $d_Y(f\sigma(x), \tau g(x))$ is uniformly bounded is a quasi-isometry; the quasi-isometry constants for g depend only on those for f, the uniformity data for σ and τ, and the bound for $d_Y(f\sigma(x), g\tau(x))$.*

Proof. Pick $K \geqslant 1$, $C \geqslant 0$ and $\varrho: [0,\infty) \to [0,\infty)$ such that f is a (K,C)-quasi-isometry, $d_Y(f\sigma(x), g\tau(x)) \leqslant C$ and ϱ, K, C are uniformity data for σ, τ.

Consider $x, y \in S$ such that $d_S(x,y) \leqslant 1$. We have $d_Y(f\sigma(x), f\sigma(y)) \leqslant K^2 + KC + C$, and so $d_Y(\tau g(x), \tau g(y)) \leqslant K^2 + KC + 3C$, from which it follows that $\varrho(d_T(g(x), g(y))) \leqslant$

$K^2+KC+3C$. Since $\lim_{t\to\infty}\varrho(t)=\infty$ we obtain a bound $d_T(g(x),g(y))\leqslant A$ depending only on K,C,ϱ. The usual "rubber-band" argument, using geodesics in S divided into subsegments of length 1 with a terminal subsegment of length $\leqslant 1$, suffices to prove that g is (K',C')-coarsely Lipschitz, with K',C' depending only on K,C,ϱ.

For any $\xi\in T$ there is a point $\bar g(\xi)\in S$ such that $d_Y(f\sigma\bar g(\xi),\tau(\xi))\leqslant C$. For any $\xi,\eta\in T$ with $d(\xi,\eta)\leqslant 1$ we have

$$d_Y(f\sigma\bar g(\xi),f\sigma\bar g(\eta))\leqslant d_Y(f\sigma\bar g(\xi),\tau(\xi))+d_Y(\tau(\xi),\tau(\eta))+d_Y(f\sigma\bar g(\eta),\tau(\eta))\leqslant K+3C,$$

and so $\varrho(d_S(\bar g(\xi),\bar g(\eta)))\leqslant d_X(\sigma\bar g(\xi),\sigma\bar g(\eta))\leqslant K^2+4KC$. As above we obtain an upper bound for $d_S(\bar g(\xi),\bar g(\eta))$, and the rubber-band argument shows that $\bar g$ is coarsely Lipschitz.

For any $x\in S$, setting $\xi=g(x)\in T$, we have

$$d_Y(f\sigma(x),f\sigma\bar g(\xi))\leqslant d_Y(f\sigma(x),\tau g(x))+d_Y(\tau(\xi),f\sigma\bar g(\xi))\leqslant 2C.$$

It follows that $d_X(\sigma(x),\sigma\bar g(\xi))\leqslant 3KC$, and so

$$\varrho(d_S(x,\bar g g(x)))=\varrho(d_S(x,\bar g(\xi)))\leqslant 3KC,$$

yielding an upper bound for $d_S(x,\bar g g(x))$. Similarly, $d_Y(\xi,g\bar g(\xi))$ is bounded for all $\xi\in T$.

Knowing that $g:S\to T$ and $\bar g:T\to S$ are coarse Lipschitz maps which are coarse inverses of each other, it easily follows that g is a quasi-isometry, with quasi-isometry constants depending only on the coarse Lipschitz constants for g and $\bar g$, and on the coarse inverse constants for g, $\bar g$. □

3. Linear algebra

In this section we collect some basic results about canonical forms of matrices, and growth of vectors under the action of a matrix.

Let $\mathcal{M}(n,F)$ denote all $(n\times n)$-matrices over a field F, and let $\mathrm{GL}(n,F)$ be the group of invertible matrices. Let $\mathrm{GL}_0(n,\mathbf{R})$ be the identity component of $\mathrm{GL}(n,\mathbf{R})$, consisting of all matrices of positive determinant.

3.1. Jordan forms

A matrix $J\in\mathcal{M}(k,\mathbf{C})$ is a *Jordan block* if it has the form $J=J(k,\lambda)=\lambda\cdot\mathrm{Id}+N$ where $\lambda\in\mathbf{C}$ and $N_{ij}=\delta(i+1,j)$, so that N is the $(k\times k)$-matrix with 1's on the superdiagonal and 0's elsewhere.

A matrix $M \in \mathcal{M}(n, \mathbf{C})$ is in *Jordan form* if it is in block diagonal form

$$M = \begin{pmatrix} J_1 & 0 & \dots & 0 \\ 0 & J_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & J_I \end{pmatrix}$$

where each J_i is a Jordan block. Every matrix in $\mathcal{M}(n, \mathbf{C})$ is conjugate, via an invertible complex matrix, to a matrix in Jordan form, unique up to permutation of the Jordan blocks. When all eigenvalues are real, say that J_i has eigenvalue l_i, we resolve the nonuniqueness by requiring $l_1 \geqslant l_2 \geqslant \dots \geqslant l_I$, and for each $i=1, \dots, I-1$, if $l_i = l_{i+1}$ then $\mathrm{rk}(J_i) \geqslant \mathrm{rk}(J_{i+1})$.

A matrix $J \in \mathcal{M}(k, \mathbf{R})$ is a *real Jordan block* if it has one of the following two forms. The first form is an ordinary Jordan block $J(k, l)$ where $l \in \mathbf{R}$. The second form, which requires k to be even, has a (2×2)-block decomposition of the form

$$J = J(k, a, b) = \begin{pmatrix} Q(a,b) & \mathrm{Id} & \dots & 0 & 0 \\ 0 & Q(a,b) & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & Q(a,b) & \mathrm{Id} \\ 0 & 0 & \dots & 0 & Q(a,b) \end{pmatrix}$$

where Id is the identity, 0 is the 0-matrix,

$$Q(a,b) = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$$

and $b \neq 0$.

A matrix $M \in \mathcal{M}(n, \mathbf{R})$ is in *real Jordan form* if it is in block diagonal form as above where each block J_i is a real Jordan block. Every matrix in $\mathcal{M}(n, \mathbf{R})$ is conjugate, via an invertible real matrix, to a matrix in real Jordan form, unique up to permutation of blocks.

The *absolute Jordan form* of $M \in \mathcal{M}(n, \mathbf{R})$ is the matrix obtained from the Jordan form of M by replacing each diagonal entry λ by $l = |\lambda|$, and permuting the blocks to resolve the nonuniqueness. If M is invertible then the absolute Jordan form of M can be written in block diagonal form

$$\begin{pmatrix} J_M^+ & 0 & 0 \\ 0 & J_M^0 & 0 \\ 0 & 0 & J_M^- \end{pmatrix}$$

where the diagonal entries of J_M^+ are >1, of J_M^0 are $=1$, and of J_M^- are <1. We call J_M^+ the *expanding part* of the absolute Jordan form, J_M^0 the *unipotent part*, and J_M^- the *contracting part*. The block matrix

$$\begin{pmatrix} J_M^+ & 0 \\ 0 & J_M^- \end{pmatrix}$$

is called the *nonunipotent part*. Of course, one or more of these parts might be empty.

Note that the Jordan form of the real matrix $J(k,a,b)$ is

$$\begin{pmatrix} J\left(\frac{1}{2}k, a+bi\right) & 0 \\ 0 & J\left(\frac{1}{2}k, a-bi\right) \end{pmatrix},$$

and so the absolute Jordan form of $J(k,a,b)$ is

$$\begin{pmatrix} J\left(\frac{1}{2}k, \sqrt{a^2+b^2}\right) & 0 \\ 0 & J\left(\frac{1}{2}k, \sqrt{a^2+b^2}\right) \end{pmatrix}.$$

Given $M\in\mathcal{M}(n,\mathbf{R})$, this process may be applied block by block to the real Jordan form of M, and the blocks then permuted, to obtain the absolute Jordan form of M.

Let $\mathrm{GL}_\times(n,\mathbf{R})$ denote the set of all matrices in $\mathrm{GL}(n,\mathbf{R})$ lying on a 1-parameter subgroup of $\mathrm{GL}(n,\mathbf{R})$, so that $\mathrm{GL}_\times(n,\mathbf{R})\subset\mathrm{GL}_0(n,\mathbf{R})$. It is well known and easy to see, given a matrix $M\in\mathrm{GL}(n,\mathbf{R})$, that $M\in\mathrm{GL}_\times(n,\mathbf{R})$ if and only if the negative-eigenvalue Jordan blocks of M may be paired up so that the two blocks occuring in each pair are identical to each other, and this occurs if and only if M has a square root in $\mathrm{GL}(n,\mathbf{R})$. Thus, if M does not already lie on a 1-parameter subgroup then M^2 does. We are therefore free to replace a matrix by its square in order to land on a 1-parameter subgroup.

Given a 1-parameter subgroup $\varrho(t)$ of $\mathrm{GL}(n,\mathbf{R})$, if $M=\varrho(1)$ then we will often abuse notation and write $\varrho(t)=M^t$, despite the fact that M may not lie on a unique 1-parameter subgroup.

Given $A\in\mathcal{M}(n,\mathbf{R})$ in Jordan form—no $J(k,a,b)$-blocks—we say that $\varrho(t)=e^{At}$ is a 1-parameter *Jordan subgroup*. Notice that the matrices e^{At} are *not* themselves in Jordan form. For example when $A=J(n,l)=l\cdot\mathrm{Id}+N$ is a single $(n\times n)$-Jordan block then e^{At} is obtained by multiplying the scalar e^{lt} with the matrix

$$e^{N\cdot t}=\sum_{i=0}^{n}\frac{1}{i!}N^i\cdot t^i=\begin{pmatrix} 1 & t & \frac{t^2}{2!} & \cdots & \frac{t^{n-1}}{(n-1)!} & \frac{t^n}{n!} \\ & 1 & t & \cdots & \frac{t^{n-2}}{(n-2)!} & \frac{t^{n-1}}{(n-1)!} \\ & & 1 & \cdots & \frac{t^{n-3}}{(n-3)!} & \frac{t^{n-2}}{(n-2)!} \\ & & & \ddots & \vdots & \vdots \\ & & & & 1 & t \\ & & & & & 1 \end{pmatrix}. \tag{3.1}$$

Nevertheless, for any Jordan form matrix $J=l\cdot\mathrm{Id}+N$ with $l\in\mathbf{R}$, the Jordan form of e^J is $e^l\cdot\mathrm{Id}+N$.

Given a general 1-parameter subgroup $e^{\mu t}$ in $\mathrm{GL}(n,\mathbf{R})$, choose A so that $A^{-1}\mu A$ is in real Jordan form, and so $A^{-1}\mu A=\delta+\nu+\eta$ where δ is diagonal, ν is superdiagonal, and η is skew-symmetric. We then have

$$e^{\mu t}=(Ae^{(\delta+\nu)t}A^{-1})(Ae^{\eta t}A^{-1}).$$

Since η is skew-symmetric it follows that $e^{\eta t}$ is in the orthogonal group $\mathrm{O}(n,\mathbf{R})$. We have therefore proved (see [W] for this particular formulation)

PROPOSITION 3.1 (1-parameter real Jordan form). *Let M^t be a 1-parameter subgroup of $\mathrm{GL}(n,\mathbf{R})$. There exists a 1-parameter Jordan subgroup e^{Jt}, a matrix $A\in\mathrm{GL}(n,\mathbf{R})$ and a bounded 1-parameter subgroup P^t conjugate into the orthogonal group $\mathrm{O}(n,\mathbf{R})$, such that e^J is the absolute Jordan form of M, and letting $\overline{M}^t=A^{-1}e^{Jt}A$ we have*

$$M^t=\overline{M}^tP^t=P^t\overline{M}^t.$$

Remark. In [W] the subgroup $\overline{M}^t$ is called the *nonelliptic part* of M^t, and P^t is called the *elliptic part*. These two 1-parameter subgroups, which commute with each other, are uniquely determined by M^t.

3.2. Growth of vectors under a linear transformation

Consider a 1-parameter subgroup M^t of $\mathrm{GL}(n,\mathbf{R})$ with real Jordan form

$$M^t=(A^{-1}e^{Jt}A)P^t=\overline{M}^tP^t.$$

Let

$$0<\lambda_1<\ldots<\lambda_L$$

be the eigenvalues of $\overline{M}$. Let $V_l=\ker((\lambda_l\cdot\mathrm{Id}-\overline{M})^m)$ be the *root space* of the eigenvalue λ_l, where m is the multiplicity of λ_l. Let n_l be the *index of nilpotency* of $\overline{M}|V_l$, the smallest integer such that $V_l=\ker((\lambda_l\cdot\mathrm{Id}-\overline{M})^{n_l})$. For $i=0,\ldots,n_l-1$ let $V_{l,i}=\ker((\lambda_l\cdot\mathrm{Id}-\overline{M})^{i+1})$, so that $V_{l,0}$ is the eigenspace of λ_l and $V_{l,n_l-1}=V_l$. We thus have the *Jordan decomposition* of $\overline{M}$, which consists of the direct sum of root spaces

$$\mathbf{R}^n=V_1\oplus\ldots\oplus V_L$$

together with the *Jordan filtrations*

$$V_{l,0}\subset\ldots\subset V_{l,n_l-1}=V_l,\quad l=1,\ldots,L.$$

This decomposition is uniquely determined by $\overline{M}$, and hence by M.

PROPOSITION 3.2 (growth of vectors). *With the above notation, there exist constants $A, B>0$ with the following properties. Given $l=1, ..., L$ with $\lambda_l \geqslant 1$, we have:*

Exponential lower bound. *If $v \in V_l$ and $t \geqslant 0$ then*

$$\|M^t v\| \geqslant A \lambda_l^t \|v\|.$$

In fact, the same lower bound holds if $v \in V_l \oplus V_{l+1} \oplus ... \oplus V_L$.

Exponential–polynomial upper bound. *Given $i=0, ..., n_l-1$, if $v \in V_{l,i}$ and $t \geqslant 1$ then*

$$\|M^t v\| \leqslant B \lambda_l^t t^i \|v\|.$$

In fact, the same upper bound holds if $v \in (V_1 \oplus ... \oplus V_{l-1}) \oplus V_{l,i}$.

Exponential–polynomial lower bound. *Given $i=0, ..., n_l-1$, if $v \in V_{l,i} \setminus V_{l,i-1}$ then there exists $C_v > 0$ such that if $t \geqslant 1$ then*

$$\|M^t v\| \geqslant C_v \lambda_l^t t^i.$$

Given $l=1, ..., L$ with $\lambda_l \leqslant 0$, similar statements are true with negative values of t.

Proof. We start with the case when $M^t = e^{Jt}$ is a 1-parameter Jordan subgroup, and the proposition follows by examining each Jordan block (3.1).

The second case we consider is when M^t has all positive real eigenvalues. By Proposition 3.1 we have $M^t = A^{-1} e^{Jt} A$, and Proposition 3.2 follows immediately from the first case applied to e^{Jt}, together with the fact that A takes the Jordan decomposition of M^t to the Jordan decomposition of e^{Jt}.

In the general case, applying Proposition 3.1 we have $M^t = (A^{-1} e^{Jt} A) P^t = \overline{M}^t P^t$. We can the apply the second case to $\overline{M}^t = A^{-1} e^{Jt} A$. Since P^t commutes with $\overline{M}^t$ it follows that P^t preserves the Jordan decomposition of $\overline{M}^t$. Proposition 3.2 then follows from the boundedness of P^t. □

4. The solvable Lie group G_M

Recall that $\mathrm{GL}_\times(n, \mathbf{R})$ denotes those matrices in $\mathrm{GL}(n, \mathbf{R})$ which lie on a 1-parameter subgroup of $\mathrm{GL}(n, \mathbf{R})$. Also, each matrix in $\mathrm{GL}_\times(n, \mathbf{R})$ has positive determinant.

Given a matrix $M \in \mathrm{GL}_\times(n, \mathbf{R})$ lying on a 1-parameter subgroup M^t of $\mathrm{GL}(n, \mathbf{R})$, we associate a solvable Lie group denoted G_M. This is the semidirect product $G_M = \mathbf{R}^n \rtimes_M \mathbf{R}$ with multiplication defined by

$$(x, t) \cdot (y, s) = (x + M^t y, t + s)$$

for all $(x,t),(y,s)\in\mathbf{R}^n\times\mathbf{R}$. We will often identify $G_M=\mathbf{R}^n\rtimes_M\mathbf{R}$ with the underlying set $\mathbf{R}^n\times\mathbf{R}$.

Remark. Although the Lie group G_M depends on more than just the matrix $M=M^1$ itself—it depends on the entire 1-parameter subgroup M^t—we suppress this dependence in our notation $G_M=\mathbf{R}^n\rtimes_M\mathbf{R}$. This is justified by the fact that the quasi-isometry type of G_M depends only on M, not on the 1-parameter subgroup containing M (see the remark after Proposition 4.1). Henceforth, when we say something like "given $M\in \mathrm{GL}_\times(n,\mathbf{R})$ ...", we will either implicitly or explicitly choose a 1-parameter subgroup $M^t<\mathrm{GL}(n,\mathbf{R})$ with $M^1=M$, which in turn determines G_M.

If M has integer entries then there is a homomorphism $\Gamma_M\to G_M$ taking the commuting generators $a_1,\dots,a_n$ to the standard basis of the integer lattice $\mathbf{Z}^n\times 0\subset\mathbf{R}^n\times 0\subset \mathbf{R}^n\times\mathbf{R}$, and taking the stable letter t to the generator $(0,1)\in\mathbf{R}^n\times\mathbf{R}$. The relator $ta_it^{-1}=\phi_M(a_i)$ is checked by noting that

$$(0,1)\cdot(x,0)\cdot(0,-1)=(Mx,0),\quad \text{for all } x\in\mathbf{R}^n.$$

Cocompactness of the image of this homomorphism is obvious. To see that Γ_M embeds in G_M one checks that in the abelian-by-cyclic extension $1\to A\to\Gamma_M\to\mathbf{Z}\to 1$, the group A is identified with the nested union $\mathbf{Z}^n\cup M^{-1}(\mathbf{Z}^n)\cup M^{-2}(\mathbf{Z}^n)\cup\dots$ in $\mathbf{R}^n$. This also shows that discreteness of Γ_M in G_M is equivalent to $\det M=1$, which is equivalent to $\mathbf{Z}^n=M(\mathbf{Z}^n)$.

For the next several sections we will investigate the geometry of the solvable Lie group G_M. In this section we begin by showing that G_M and G_N are quasi-isometric if M,N have powers with the same absolute Jordan form. Later in §7 we will see that when M has integer entries, much of the geometry of Γ_M is reflected in the geometry of G_M.

We endow G_M with the left-invariant metric determined by taking the standard Euclidean metric at the identity of $G_M\approx\mathbf{R}^n\times\mathbf{R}=\mathbf{R}^{n+1}$. At a point $(x,t)\in\mathbf{R}^n\times\mathbf{R}\approx G_M$, the tangent space is identified with $\mathbf{R}^n\times\mathbf{R}$, and the Riemannian metric is given by the symmetric matrix

$$\begin{pmatrix} Q_M(t) & 0\\ 0 & 1\end{pmatrix}$$

where $Q_M(t)=(M^{-t})^TM^{-t}$. For each $t\in\mathbf{R}$, the identification $\mathbf{R}^n\approx\mathbf{R}^n\times t\subset G_M$ induces in $\mathbf{R}^n$ the metric determined by the quadratic form $Q_M(t)$. This metric has distance formula

$$d_{M,t}(x,y)=\|M^{-t}(x-y)\|.$$

Remarks. (1) When M is a (1×1)-matrix with entry $a>1$, the group G_M is isomorphic to $\mathrm{Aff}(\mathbf{R})$, the group of affine transformations of $\mathbf{R}$, and as a Riemannian manifold G_M is isometric to a scaled copy of the hyperbolic plane with constant sectional curvature depending on a.

(2) The eigenvalues of M are greater than 1 in absolute value if and only if all sectional curvatures of G_M are negative (see [He]).

PROPOSITION 4.1 (how the metric on G_M depends on choices). *Given 1-parameter subgroups M^t, N^t in $\mathrm{GL}(n,\mathbf{R})$, suppose that there exist real numbers $r,s>0$ such that M^r and N^s have the same absolute Jordan form. Then the metric spaces G_M and G_N are quasi-isometric. To be explicit there exists $A\in\mathrm{GL}(n,\mathbf{R})$ and $K\geqslant 1$ such that for each $t\in\mathbf{R}$, the map $v\mapsto A(v)$ is a K-bi-Lipschitz homeomorphism from the metric $d_{M,t}$ to the metric $d_{N,(s/r)\cdot t}$; it follows that the map from $G_M=\mathbf{R}^n\rtimes_M\mathbf{R}$ to $G_N=\mathbf{R}^n\rtimes_N\mathbf{R}$ given by*

$$(x,t)\mapsto\Big(Ax,\frac{s}{r}\cdot t\Big)$$

is a bi-Lipschitz homeomorphism from G_M to G_N, with bi-Lipschitz constant

$$\sup\Big\{K,\frac{s}{r},\frac{r}{s}\Big\}.$$

Remark. The absolute Jordan form of M^r is uniquely determined by M and r: it is the rth power of the absolute Jordan form of M. It follows in particular that the quasi-isometry type of G_M depends only on the matrix $M=M^1$, not on the choice of 1-parameter subgroup M^t.

Proof of Proposition 4.1. We proceed in cases.

Case 1. Assume that $N^t=e^{Jt}$ is the unique 1-parameter Jordan subgroup such that $N=e^J$ is conjugate to the absolute Jordan form of M. Applying Proposition 3.1 we have

$$M^t=(A^{-1}N^tA)P^t$$

where $A\in\mathrm{GL}(n,R)$ and the 1-parameter subgroup P^t is bounded.

Choose $t\in\mathbf{R}$ and $v\in\mathbf{R}^n$. We must show that the two numbers

$$\|M^{-t}v\|=\|P^{-t}(A^{-1}N^{-t}A)v\| \quad\text{and}\quad \|N^{-t}Av\|$$

have ratio bounded away from 0 and ∞, with bound independent of t,v. Setting $u=N^{-t}Av$, it suffices to show that $\|P^{-t}A^{-1}u\|$ and $\|u\|$ have bounded ratio. But this is clearly true, with a bound of

$$\big(\sup_t\|P^t\|\big)\cdot\max\Big\{\|A\|,\frac{1}{\|A\|}\Big\}$$

since the 1-parameter subgroup P^t is bounded.

Case 2. Assume that there exists $a>0$ such that $M^t=N^{at}$ for all t. Then the metrics $d_{M,t}$ and $d_{N,at}$ are identical.

General case. Applying Case 2 we may assume that $\det M=\det N$. Applying Case 1 twice we may go from G_M to G_{e^J} to G_N, where e^J is conjugate to the absolute Jordan form of M and of N. □

5. Dynamics of G_M, Part I: Horizontal-respecting quasi-isometries

In this section we begin studying the asymptotic geometry of the solvable Lie groups G_M associated to 1-parameter subgroups M^t of $\mathrm{GL}(n,\mathbf{R})$. As we saw in §4, the quasi-isometry type of G_M depends only on M, not on the choice of 1-parameter subgroup M^t passing through M; see the remark after Proposition 4.1. We therefore continue to suppress the choice of 1-parameter subgroup in our notation. Further, we do not restrict the determinant to be >1: the results of this section hold even when $\det M=1$.

5.1. Theorem 5.2 on horizontal-respecting quasi-isometries

Let X,Y be metric spaces. Let $\mathcal{F}$ be a decomposition of X, that is, a collection of disjoint subsets of X whose union is X. Let $\mathcal{G}$ be a decomposition of Y. Motivated by a foliation of a manifold, the elements of these decompositions are called *leaves* and the decomposition itself is called the *leaf space.* A quasi-isometry $\phi: X\to Y$ is said to *coarsely respect* the decompositions $\mathcal{F},\mathcal{G}$ if there exists a number $A\geqslant 0$ and a map of leaf spaces $h:\mathcal{F}\to\mathcal{G}$ such that for each leaf $L\in\mathcal{F}$ we have

$$d_{\mathcal{H}}(\phi(L),h(L))\leqslant A.$$

For example, consider the space G_M. The coordinate function $G_M\approx\mathbf{R}^n\times\mathbf{R}\to\mathbf{R}$ given by $(x,t)\mapsto t$ is called the *time function* of G_M. The level sets $P_t\approx\mathbf{R}^n\times t$ form the *horizontal foliation* of G_M, whose leaves are called *horizontal leaves* of G_M, and whose leaf space is $\mathbf{R}$. Notice that $d_{\mathcal{H}}(P_s,P_t)=|s-t|$, and so the time function induces an isometry between the horizontal leaf space equipped with the Hausdorff metric and $\mathbf{R}$.

Consider another matrix $N\in\mathrm{GL}_\times(n,\mathbf{R})$, and denote the horizontal leaves of G_N by P'_t.

Definition (*horizontal-respecting*). A quasi-isometry $\phi: G_M\to G_N$ is said to be *horizontal-respecting* if it coarsely respects the horizontal foliations of G_M, G_N. That is, there exists a function $h:\mathbf{R}\to\mathbf{R}$ and $A\geqslant 0$ such that $d_{\mathcal{H}}(\phi(P_t),P'_{h(t)})\leqslant A$ for all $t\in\mathbf{R}$.

The function $h: \mathbf{R} \to \mathbf{R}$ is called an *induced time change* for ϕ, with *Hausdorff constant* A.

If h, h' are two induced time changes for ϕ then $\sup_t |h(t)-h'(t)| \leqslant A+A' < \infty$, where A, A' are Hausdorff constants for h, h' respectively. Also, if $h: \mathbf{R} \to \mathbf{R}$ is an induced time change for ϕ with Hausdorff constant A, if $A' \geqslant 0$ and if $h': \mathbf{R} \to \mathbf{R}$ is any function satisfying $\sup_{t \in \mathbf{R}} |h(t)-h'(t)| \leqslant A'$, then h' is also an induced time change for ϕ, with Hausdorff constant $A+A'$.

LEMMA 5.1. *For each K, C, A there exists C' such that if $\phi: G_M \to G_N$ is a horizontal-respecting (K,C)-quasi-isometry, and $h: \mathbf{R} \to \mathbf{R}$ is an induced time change for ϕ with Hausdorff constant A, then h is a (K, C')-quasi-isometry of $\mathbf{R}$.*

Proof. We have $|h(t)-h(s)| \leqslant d_{\mathcal{H}}(P_{h(t)}, P_{h(s)}) + 2A \leqslant K|t-s| + C + 2A$. The reverse inequality is similar, and so h is a quasi-isometric embedding. Since ϕ is coarsely onto, an easy argument shows that h is coarsely onto. □

A (K, C')-quasi-isometry $h: \mathbf{R} \to \mathbf{R}$ induces a bijection of the two-point set $\mathrm{Ends}(\mathbf{R}) = \{-\infty, +\infty\}$: given $\eta_1, \eta_2 \in \mathrm{Ends}(\mathbf{R})$, we have $h(\eta_1) = \eta_2$ if and only if h takes every sequence that diverges to η_1 to a sequence that diverges to η_2. The following two properties of h are equivalent:

(1) h induces the identity on $\mathrm{Ends}(\mathbf{R})$;

(2) h is *coarsely increasing*, that is, there exists $L > 0$ such that if $t > s+L$ then $h(t) > h(s)$.

That (2) implies (1) is obvious. The other direction is true with any $L > 2C'K$, for if there existed $t \geqslant s+L$ with $h(t) \leqslant h(s)$, then since h induces the identity on $\mathrm{Ends}(\mathbf{R})$ there would exist $t' > t$ such that $|h(s)-h(t')| \leqslant C'$, but also $|h(s)-h(t')| \geqslant |s-t'|/K - C' \geqslant L/K - C' > C'$, a contradiction.

If $h: \mathbf{R} \to \mathbf{R}$ is an induced time change of a horizontal-respecting quasi-isometry $\phi: G_M \to G_N$, and if h satisfies the equivalent properties (1) and (2), then we say that ϕ *coarsely respects the transverse orientation* of the horizontal foliations.

Terminology (*time vs. height*). In some contexts the vertical parameter which we have been calling "time" will also be called *height*, as sometimes this terminology is more suggestive, for example in discussing horizontal foliations.

Here is the main result, whose proof will occupy the remainder of this section and the next section.

THEOREM 5.2 (horizontal-respecting quasi-isometries). *Let $\phi: G_M \to G_N$ be a quasi-isometry which coarsely respects the transversely oriented horizontal foliations of G_M*

and G_N. Then there exist real numbers $r,s>0$ so that M^r and N^s have the same absolute Jordan form.

Our proof of Theorem 5.2 proceeds in steps, following the outline given in the introduction.

5.2. Step 1a: Hyperbolic dynamics and the shadowing lemma

The Lie group G_M has a natural flow which fits into the theory of partially hyperbolic dynamical systems. From the dynamics we find that the flow has several invariant foliations, the "weak stable, weak unstable and center" foliations. In §§ 5.2, 5.3, by using the shadowing lemma [HPS, Lemma 7.A.2, p. 133], we prove that a horizontal-respecting quasi-isometry $G_M \to G_N$ also respects the dynamically defined foliations of G_M, G_N.

From this result we obtain the first piece of our rigidity theorem by showing that expanding, contracting and unipotent parts of the absolute Jordan forms of M and N have the same ranks respectively, and that the unipotent parts are identical.

5.2.1. *Dynamically defined foliations.* Let $M^t \in \mathrm{GL}(n, \mathbf{R})$ be a 1-parameter subgroup with real Jordan form $M^t = \overline{M}^t P^t$. Consider the Jordan decomposition of $\overline{M}$, and group the root spaces according to whether the corresponding eigenvalue is <1, $=1$ or >1 (alternatively, a logarithm which is <0, $=0$ or >0), to obtain a decomposition $\mathbf{R}^n = V^- \oplus V^0 \oplus V^+$.

Remark. It might happen that one or two of the factors V^-, V^0, V^+ is trivial, that is, 0-dimensional, for instance when all eigenvalues of M lie outside the unit circle.

Now consider the Lie group $G_M = \mathbf{R}^n \rtimes_M \mathbf{R}$ determined by a 1-parameter subgroup M^t. Define the *vertical flow* Φ on G_M to be

$$\Phi_t(x,s) = (x, s+t).$$

The tangent bundle TG_M has a Φ-invariant splitting

$$TG_M = E^s \oplus E^c \oplus E^u$$

defined as follows. The tangent space at each point $x \in G_M$ is identified with $\mathbf{R}^n \oplus \mathbf{R}$, and we take

$$E^s_x = V^- \oplus 0, \quad E^c_x = V^0 \oplus \mathbf{R}, \quad E^u_x = V^+ \oplus 0.$$

It is evident from the construction that each of the distributions $E^s \oplus E^c$, $E^u \oplus E^c$ and E^c is integrable, tangent to foliations denoted W^s, W^u and W^c. We call these foliations the (weak) *stable, unstable and center foliations* respectively. The stable and unstable foliations are transverse, and the intersection of any stable leaf with any unstable leaf is a center leaf.

Applying the exponential lower bound from Proposition 3.2, there exist constants $A>0$, $\lambda>1$ such that:

(1) If $v\in E^u$ then for $t\geqslant 0$ we have $\|D\Phi_t v\|\geqslant A\lambda^t\|v\|$, and for $t\leqslant 0$ we have $\|D\Phi_t v\|\leqslant (1/A)\lambda^t\|v\|$.

(2) If $v\in E^s$ then for $t\leqslant 0$ we have $\|D\Phi_t v\|\geqslant A\lambda^{-t}\|v\|$, and for $t\geqslant 0$ we have $\|D\Phi_t v\|\leqslant (1/A)\lambda^{-t}\|v\|$.

Also, applying the exponential–polynomial upper bound from Proposition 3.2, there exists $B>0$ and an integer $n\geqslant 1$ such that:

(3) If $v\in E^c$ then for $|t|\geqslant 1$ we have $\|D\Phi_t v\|\leqslant B|t|^n\|v\|$.

When we want to emphasize the dependence of the V's and E's on the 1-parameter subgroup M^t, we will append a subscript, e.g. V_M^+, E_M^s, etc.

5.2.2. *Shadowing lemma.* Consider a flow Φ on a metric space X. We write $x\cdot t$ as an abbreviation for $\Phi_t(x)$. Given $\varepsilon, T>0$, an *(ε,T)-pseudo-orbit* of Φ consists of a sequence of flow segments $(x_i\cdot[0,t_i])$, where the index i runs over an interval in $\mathbf{Z}$, such that $d_X(x_i\cdot t_i, x_{i+1})<\varepsilon$ and $t_i>T$ for all i.

LEMMA 5.3 (shadowing lemma). *Consider a 1-parameter subgroup M^t of* $\mathrm{GL}(n,\mathbf{R})$, *and let Φ be the vertical flow on G_M. For every $\varepsilon, T>0$ there exists $\delta,\varepsilon',T'>0$ such that every (ε,T)-pseudo-orbit of Φ is δ-*shadowed *by an (ε',T')-pseudo-orbit of Φ which is contained in some center leaf. That is, if $(x_i\cdot[0,t_i])$ is an (ε,T)-pseudo-orbit, then there is an (ε',T')-pseudo-orbit $(y_i\cdot[0,t_i])$ contained in some center leaf so that $d(x_i\cdot t, y_i\cdot t)<\delta$ for all i and all $t\in[0,t_i]$.*

Proof. By construction, the foliations W^s and W^u are coordinate foliations in $\mathbf{R}^{n+1}$; this shows that the flow Φ has a "global product structure" in the language of hyperbolic dynamical systems. The lemma now follows the proof of the shadowing lemma in [HPS, Lemma 7.A.2, p. 133]. A direct proof is also easy to work out, and is left to the reader. □

5.3. Step 1b: Foliations rigidity

The shadowing lemma implies further rigidity properties of horizontal-respecting quasi-isometries:

PROPOSITION 5.4 (foliations rigidity). *Suppose that $\phi: G_M\to G_N$ is a quasi-isometry which coarsely respects the horizontal foliations and their transverse orientations. Then*

ϕ also coarsely respects the weak unstable foliations W^u_M, W^u_N, the weak stable foliations W^s_M, W^s_N, and the center foliations W^c_M, W^c_N. In particular,

(1) $\dim(V^+_M)=\dim(V^+_N)$;

(2) $\dim(V^-_M)=\dim(V^-_N)$;

(3) $\dim(V^0_M)=\dim(V^0_N)$.

Remarks. (1) In the case where neither M nor N has any eigenvalue on the unit circle, the center foliations of both G_M and G_N are simply the foliations by vertical flow lines, and Proposition 5.4 says that ϕ respects these foliations. But in the general case, it is not true that ϕ always respects the foliations by vertical flow lines. For a simple counterexample, consider the (1×1)-matrix $M=N=(1)$, which gives $\Gamma_M=\Gamma_N=\mathbf{Z}^2$. There exist horizontal-respecting quasi-isometries of $\mathbf{R}^2=\mathbf{R}\times\mathbf{R}$ which do not respect the vertical foliation.

(2) If all eigenvalues of M and N are outside the unit circle, then both G_M and G_N are negatively curved, and the proposition follows from a standard fact: a quasi-geodesic in a negatively curved space X is Hausdorff-close to a geodesic (this was the approach taken in [FM1] in the case of a (1×1)-matrix M, where G_M is isometric to a scaled copy of the hyperbolic plane). This "fact" is unavailable when $X=G_M$ is not negatively curved, forcing us to study horizontal-respecting quasi-isometries via the shadowing lemma.

Before proving Proposition 5.4, we use it to obtain some pieces of our classification theorem. Since $\mathrm{rk}(J^-_M)=\dim(V^-_M)$, etc., we immediately have

COROLLARY 5.5. *If there is a quasi-isometry from G_M to G_N which coarsely respects the transversely oriented horizontal foliations, then $\mathrm{rk}(J^-_M)=\mathrm{rk}(J^-_N)$, $\mathrm{rk}(J^0_M)=\mathrm{rk}(J^0_N)$ and $\mathrm{rk}(J^+_M)=\mathrm{rk}(J^+_N)$.*

We also have

COROLLARY 5.6. *The unipotent blocks of the absolute Jordan forms of M and N are identical.*

Proof. Let L be some center leaf of G_M, of dimension k. From Proposition 5.4 it follows that $\phi(L)$ is Hausdorff-close to some center leaf L' of G_N, also of dimension k. By composition with nearest point projection (which moves points a uniformly bounded amount) we get an induced map $L\to L'$. By Lemma 2.1 this map is a quasi-isometry. By Proposition 4.1, L and L' are quasi-isometric to the nilpotent Lie groups $\mathbf{R}^{k-1}\rtimes_{J^0_M}\mathbf{R}$ and $\mathbf{R}^{k-1}\rtimes_{J^0_N}\mathbf{R}$ respectively. As Bridson and Gersten have shown [BG], Pansu's invariant [P2] may be used to prove that $J^0_M=J^0_N$. □

Proof of Proposition 5.4. We begin with

CLAIM 5.7. *For each vertical flow line* $\gamma=\Phi_{\mathbf{R}}(x)$ *in* G_M*, there exists a center leaf* τ_γ *in* G_N *such that* $\phi(\gamma)$ *is contained in the* α*-neighborhood of* τ_γ*, where the constant* $\alpha>0$ *does not depend on* γ*.*

Before proving the claim, we apply it to prove the proposition as follows.

Consider any two vertical flow lines γ_1, γ_2 in G_M. By the claim we have that $\phi(\gamma_1)$ and $\phi(\gamma_2)$ lie, respectively, in bounded neighborhoods of center leaves σ_1 and σ_2 of G_N. Since $h(t)\to\pm\infty$ as $t\to\pm\infty$, for each choice of sign $+$ or $-$ the following two statements are equivalent, and the second statement implies the third:

(1) The distance between the points $\gamma_1\cap P_t$ and $\gamma_2\cap P_t$ in P_t stays bounded as $t\to\pm\infty$.

(2) The distance between the points $\phi(\gamma_1)\cap P_{h(t)}$ and $\phi(\gamma_2)\cap P_{h(t)}$ in $P_{h(t)}$ stays bounded as $t\to\pm\infty$.

(3) The Hausdorff distance between the sets $\sigma_1\cap P_{h(t)}$ and $\sigma_2\cap P_{h(t)}$ in $P_{h(t)}$ stays bounded as $t\to\pm\infty$.

Using $-$ signs, the first statement is equivalent to saying that γ_1, γ_2 are contained in the same unstable leaf of G_M, and the third statement is equivalent to saying that σ_1, σ_2 are contained in the same unstable leaf of G_N. It follows that ϕ takes every unstable leaf of G_M into a bounded neighborhood of an unstable leaf of G_N. Applying the same argument to a coarse inverse $\bar{\phi}$ of ϕ gives the opposite inclusion. Since $d(\bar{\phi}\circ\phi, \mathrm{Id})<\infty$ it follows that the image under ϕ of any unstable leaf of G_M lies a bounded Hausdorff distance from an unstable leaf of G_N, that is, ϕ coarsely preserves the unstable foliations. A similar argument using $+$ signs shows that ϕ coarsely preserves stable foliations. By taking intersections of stable and unstable leaves it follows that ϕ coarsely preserves center foliations.

The final statements about dimensions follow from the fact that dimension is a quasi-isometry invariant for leaves of the foliations in question; see [Ge1] or [BW].

It remains to prove the claim. Applying Lemma 5.1, we have an induced time change $h: \mathbf{R}\to\mathbf{R}$ which is a (K, C')-quasi-isometry with Hausdorff constant A, where C' depends only on K, C, A. Furthermore by Lemma 5.1 and the comments following it, the map h is coarsely increasing: there exists $L=L(K,C,A)>0$ such that if $t\geqslant s+L$ then $h(t)>h(s)$.

We can furthermore increase L, depending only on K, C', A, so that

$$t'\geqslant t+L,\ x\in P_{t'},\ y\in P_t,\ \phi(x)\in P_{s'},\ \phi(y)\in P_s \quad\Rightarrow\quad s'\geqslant s+1. \tag{5.1}$$

In fact, taking $L>(C'+2A+1)K$ will do, for then we have

$$h(t')\geqslant h(t)+\frac{t'-t}{K}-C'\geqslant h(t')+\frac{L}{K}-C'\geqslant h(t)+2A+1$$

and, since $P_{s'}$ is A-Hausdorff-close to $P_{h(t')}$ and P_s is A-Hausdorff-close to $P_{h(t)}$, it follows that $s' \geqslant s+1$.

To prove the claim, we first show that $\phi(\gamma)$ is Hausdorff-close to some pseudo-orbit in G_N, and then we apply the shadowing lemma to show that the pseudo-orbit lies in a bounded neighborhood of some center leaf.

To be more precise, fix a point $x_0 \in \gamma$ and consider the sequence $x_i = \Phi_{i \cdot L}(x_0)$ for $i \in \mathbf{Z}$. Let $y_i = \phi(x_i)$, and let s_i be such that $y_i \in P_{s_i}$. From (5.1) it follows that $s_{i+1} \geqslant s_i + 1$. Let $t_i = s_{i+1} - s_i \geqslant 1$.

We claim that there exists $\varepsilon > 0$, depending ultimately only on K, C, so that $(y_i \cdot [0, t_i])$ is an $(\varepsilon, 1)$-pseudo-orbit; in other words, $d(y_i \cdot t_i, y_{i+1})$ is bounded. To see why, first note that

$$d(y_i \cdot t_i, y_i) = t_i = s_{i+1} - s_i \leqslant 2A + h(L \cdot (i+1)) - h(L \cdot i) \leqslant 2A + KL + C'$$

and then

$$d(y_i, y_{i+1}) \leqslant K \cdot d(x_i, x_{i+1}) + C \leqslant KL + C,$$

so we may take $\varepsilon = 2A + 2KL + C + C'$.

Applying the shadowing lemma, there exists β, ε', T' such that $(y_i \cdot [0, t_i])$ is β-Hausdorff-close to an (ε', T')-pseudo-orbit $(y_i' \cdot [0, t_i])$ contained in some center leaf of G_N. On the other hand, since every point of γ is within distance L of some x_i, it follows that $\phi(\gamma)$ is uniformly Hausdorff-close to $(y_i \cdot [0, t_i])$, and so it is also uniformly close to the pseudo-orbit $(y_i' \cdot [0, t_i])$. □

5.4. Step 2: Time rigidity

The main result of this subsection says that a horizontal-respecting quasi-isometry has an induced time change function which is affine.

PROPOSITION 5.8 (time rigidity). *Consider the Lie groups G_M, G_N where $M, N \in \mathrm{GL}_\times(n, \mathbf{R})$ each have an eigenvalue of absolute value greater than* 1. *Then there exists $m \in \mathbf{R}_+$ with the following properties. For all $K \geqslant 1$, $C, A \geqslant 0$ there exists $A' \geqslant 0$ such that if $\phi: G_M \to G_N$ is a (K, C)-quasi-isometry which coarsely respects horizontal foliations and their transverse orientations, with an induced time change of Hausdorff constant A, then there exists $b \in \mathbf{R}$ such that $h(t) = mt + b$ is an induced time change with Hausdorff constant A'. In fact, m can be computed as follows: Let α (resp. β) be the least eigenvalue greater than* 1 *of the absolute Jordan form of M (resp. N); the numbers α, β exist by the assumption on eigenvalues. Then $m = \log \alpha / \log \beta$.*

Remarks. (1) In the case of self-quasi-isometries of $\mathrm{Aff}(R) = G_{(e^1)} = \mathbf{H}^2$ which coarsely respect the horizontal foliation, this result is part of Proposition 5.3 of [FM1], where the conclusion is that the induced time change is a translation of $\mathbf{R}$.

(2) One of the delicate points in Gromov's development of the infδim invariant is the rescaling problem discussed at the beginning of §7.C_1 of [Gr2]: the rate of exponential growth changes when the parameter is rescaled. Time rigidity allows us to avoid the rescaling problem altogether, by showing that the time parameter is "natural" with respect to quasi-isometries.

Proof. This proof will define a sequence of constants which will depend on K, C, A and on the matrices M and N. We will indicate the dependence on K, C, A by writing, for example, $C_1 = C_1(K, C, A)$, but we will suppress the dependence on M, N. Although each constant in the sequence will depend on previous constants in the sequence, by induction it will ultimately depend only on K, C, A, M, N.

CLAIM 5.9. *For each fixed time t_0, and for each $t \leqslant t_0$, we have*

$$h(t) \geqslant m(t-t_0)+h(t_0)-C_1$$

for some $C_1 = C_1(K, C, A) \geqslant 0$.

Accepting this claim for the moment, we prove the proposition. The idea is simply that the conclusion of the claim, applied to both h and its coarse inverse $\bar{h}$, with $t_0 \to +\infty$, implies the proposition.

Let s be a time parameter for G_N. Let $\bar{\phi}: G_N \to G_M$ be a coarse inverse for ϕ, also a quasi-isometry which coarsely respects the horizontal foliations and their transverse orientations, and with an induced time change $\bar{h}(s)$. The constants for $\bar{\phi}$ and $\bar{h}$ depend only on K, C, A. The claim therefore applies as well to $\bar{h}$ and we obtain, for each fixed time s_0 and each $s \leqslant s_0$,

$$\bar{h}(s) \geqslant \frac{1}{m}(s-s_0)+\bar{h}(s_0)-C_2$$

for some $C_2 = C_2(K, C, A) \geqslant 0$.

It is clear that $\bar{h}$ is a coarse inverse for h, that is,

$$|\bar{h}(h(t))-t| \leqslant C_3, \quad |h(\bar{h}(s))-s| \leqslant C_3$$

for some $C_3 = C_3(K, C, A) \geqslant 0$.

Also, by Lemma 5.1 and the comments after it, the map h is coarsely increasing: there exists $L = L(K, C, A) \geqslant 0$ such that if $t' > t+L$ then $h(t') > h(t)$.

We reverse the inequality in the claim as follows. Fix t_0. Let $s_0 = t_0$. Consider for the moment some $t \leqslant t_0 - L$. Letting $s = h(t)$ it follows that $s \leqslant s_0$, and so we have

$$\bar{h}(h(t)) \geqslant \frac{1}{m}(h(t)-h(t_0))+\bar{h}(h(t_0))-C_2.$$

But $t+C_3 \geqslant \bar{h}(h(t))$ and $\bar{h}(h(t_0)) \geqslant t_0 - C_3$, and so we obtain

$$t \geqslant \frac{1}{m}(h(t)-h(t_0))+t_0-(2C_3+C_2),$$
$$h(t) \leqslant m(t-t_0)+h(t_0)+m(2C_3+C_2).$$

This has been derived only for $t \leqslant t_0 - L$, but for $t_0 - L \leqslant t \leqslant t_0$ we obtain a similar inequality with another constant in place of $m(2C_3+C_2)$. Therefore, for all $t \leqslant t_0$ we obtain

$$m(t-t_0)+h(t_0)-C_4 \leqslant h(t) \leqslant m(t-t_0)+h(t_0)+C_4$$

for some $C_4 = C_4(K, C, A)$. Note that this is true *for all* t_0, with C_4 independent of t_0.

In particular, taking $t_0 = 0$, for all $t \leqslant 0$ we obtain

$$mt+h(0)-C_4 \leqslant h(t) \leqslant mt+h(0)+C_4.$$

Now take any $t_1 \geqslant 0$, and since $0 \leqslant t_1$ we obtain

$$m(0-t_1)+h(t_1)-C_4 \leqslant h(0) \leqslant m(0-t_1)+h(t_1)+C_4$$

and so

$$mt_1+h(0)-C_4 \leqslant h(t_1) \leqslant mt_1+h(0)+C_4.$$

Taking $b = h(0)$, this proves that $mt+b$ is an induced time change for ϕ, with Hausdorff constant $A' = C_4 + A$.

Now we turn to the proof of Claim 5.9.

Let $M^t = \overline{M}^t Q^t$, $N^t = \overline{N}^t Q'^t$ be the real Jordan forms. Let U (resp. U') be the root space with eigenvalue 1 for $\overline{M}$ (resp. $\overline{N}$). Let W (resp. W') be the direct sum of root spaces with eigenvalue $\geqslant 1$ for $\overline{M}$ (resp. $\overline{N}$). Recall that α is the smallest eigenvalue >1 for $\overline{M}$, and β is the smallest eigenvalue >1 for $\overline{N}$. Let V be the direct sum of U and the eigenspace with eigenvalue α for $\overline{M}$. We have $U \subset V \subset W$; let $\mathcal{F}(U), \mathcal{F}(V), \mathcal{F}(W)$ be the corresponding foliations of $G_M \approx \mathbf{R}^n \times \mathbf{R}$ whose leaves are parallel to $U \times \mathbf{R}$, $V \times \mathbf{R}$, $W \times \mathbf{R}$ respectively. We also have $U' \subset W'$; let $\mathcal{F}(U'), \mathcal{F}(W')$ be the corresponding foliations of G_N.

Here is the idea for proving Claim 5.9. Each leaf of $\mathcal{F}(V)$ is foliated by leaves of $\mathcal{F}(U)$. Because V is the direct sum of U with the α-eigenspace of $\overline{M}$, it follows that as $t \to -\infty$ distinct leaves of $\mathcal{F}(U)$ in $\mathcal{F}(V)$ diverge from each other *exactly* as α^{-t}, measured in the time-t horizontal plane of G_M. This is a consequence of the exponential lower bound and the exponential-polynomial upper bound in Proposition 3.2; notice that it is critical here that V not be the direct sum of U with the *α-root space*, for then the exponential–polynomial upper bound would be at best α^{-t} times some polynomial, which would mess up the following calculations. Mapping over via the quasi-isometry $\phi: G_M \to G_N$, distinct leaves of $\mathcal{F}(U)$ in a single leaf of $\mathcal{F}(V)$ must (coarsely) map to distinct leaves of $\mathcal{F}(U')$ in a single leaf of $\mathcal{F}(W')$, which as $s \to -\infty$ diverge from each other *at least as fast as* β^{-s}, by the exponential lower bound. The time change map $t \mapsto h(t) = s$ therefore cannot grow slower than $s = (\log \alpha / \log \beta) \cdot t$ as $t \to -\infty$.

To make this precise, pick a leaf L_V of $\mathcal{F}(V)$ contained in some leaf L_W of $\mathcal{F}(W)$. We use the symbol γ to denote a general leaf of $\mathcal{F}(U)$, which we will typically take to be a subset of L_V. By Proposition 5.4, there exists a leaf $L_{W'}$ of $\mathcal{F}(W')$ such that

$$d_{\mathcal{H}}(f(L_W), L_{W'}) \leqslant C_5 = C_5(K, C, A),$$

and for each leaf γ of $\mathcal{F}(U)$ there exists a leaf γ' of $\mathcal{F}(U')$ such that

$$d_{\mathcal{H}}(f(\gamma), \gamma') \leqslant C_5.$$

Moreover, if $\gamma \subset L_V$ then $\gamma' \subset L_{W'}$, because $L_V \subset L_W$ and so γ' stays in a bounded neighborhood of $L_{W'}$, but any leaf of $\mathcal{F}(U')$ which is not a subset of $L_{W'}$ has points which are arbitrarily far from $L_{W'}$.

Let P_t be the horizontal subset of G_M at height $t \in \mathbf{R}$, and let d_t denote Hausdorff distance in P_t between closed subsets of P_t. Let P'_s be the horizontal subset of G_N at height $s \in \mathbf{R}$, and let d'_s denote Hausdorff distance in P'_s.

Since the Hausdorff distance in G_N between $\phi(P_t)$ and $P'_{h(t)}$ is at most A, the vertical projection from $\phi(P_t)$ to $P'_{h(t)}$ induces a quasi-isometry between P_t and $P'_{h(t)}$; the multiplicative constant of this quasi-isometry is K, and its additive constant depends only on K, C, A. It follows that there exists a "coarseness constant" $C_6 = C_6(K, C, A)$ so that for any t, and for any $x, y \in P_t$ with $d_t(x, y) \geqslant C_6$, if $x', y' \in P'_{h(t)}$ are the vertical projections of $\phi(x), \phi(y)$ then

$$\frac{1}{2K} d_t(x, y) \leqslant d'_{h(t)}(x', y') \leqslant 2K d_t(x, y). \tag{5.2}$$

To prove Claim 5.9, fix a time t_0 and let $s_0 = h(t_0)$. Let γ_1, γ_2 be two leaves of $\mathcal{F}(U)$ contained in L_V, and let γ'_i be the unique leaf of $\mathcal{F}(U')$ within bounded Hausdorff distance of $\phi(\gamma_i)$; this bound depends only on K, C, A, as shown in Proposition 5.4.

In G_M, apply the exponential lower bound and the exponential–polynomial upper bound of Proposition 3.2, so that for all $t \leqslant t_0$ we have

$$A \cdot \alpha^{-t+t_0} d_{t_0}(\gamma_1 \cap P_{t_0}, \gamma_2 \cap P_{t_0}) \leqslant d_t(\gamma_1 \cap P_t, \gamma_2 \cap P_t) \leqslant B \cdot \alpha^{-t+t_0} d_{t_0}(\gamma_1 \cap P_{t_0}, \gamma_2 \cap P_{t_0})$$

where A, B depend only on G_M (note that $t = t_0$ gives $A \leqslant 1 \leqslant B$).

We want the distance between γ_1 and γ_2 in P_t to be greater than the coarseness constant C_6, for each $t \leqslant t_0$, in order that property (5.2) may be applied. We therefore impose a condition on γ_1 and γ_2, namely that

$$d_{t_0}(\gamma_1 \cap P_{t_0}, \gamma_2 \cap P_{t_0}) \geqslant \frac{C_6}{A},$$

which implies, for all $t \leqslant t_0$, that

$$d_t(\gamma_1 \cap P_t, \gamma_2 \cap P_t) \geqslant C_6$$

and so

$$\frac{1}{2K} \cdot d_t(\gamma_1 \cap P_t, \gamma_2 \cap P_t) \leqslant d'_{h(t)}(\gamma'_1 \cap P'_{h(t)}, \gamma'_2 \cap P'_{h(t)}) \leqslant 2K \cdot d_t(\gamma_1 \cap P_t, \gamma_2 \cap P_t),$$

which implies

$$\begin{aligned} \frac{A}{2K} \alpha^{-t+t_0} d_{t_0}(\gamma_1 \cap P_{t_0}, \gamma_2 \cap P_{t_0}) &\leqslant d'_{h(t)}(\gamma'_1 \cap P'_{h(t)}, \gamma'_2 \cap P'_{h(t)}) \\ &\leqslant 2BK\alpha^{-t+t_0} d_{t_0}(\gamma_1 \cap P_{t_0}, \gamma_2 \cap P_{t_0}). \end{aligned}$$

Next, applying the exponential lower bound of Proposition 3.2 in G_N, for each $s \leqslant s_0$ we have

$$d'_s(\gamma'_1 \cap P'_s, \gamma'_2 \cap P'_s) \geqslant A \cdot \beta^{-s+s_0} d'_{s_0}(\gamma'_1 \cap P'_{s_0}, \gamma'_2 \cap P'_{s_0}).$$

Taking $s = h(t)$, and using the fact that $s_0 = h(t_0)$, this implies

$$\beta^{-h(t)+h(t_0)} d'_{h(t_0)}(\gamma'_1 \cap P'_{h(t_0)}, \gamma'_2 \cap P'_{h(t_0)}) \leqslant \frac{2BK}{A} \alpha^{-t+t_0} d_{t_0}(\gamma_1 \cap P_{t_0}, \gamma_2 \cap P_{t_0}).$$

Therefore,

$$\beta^{-h(t)+h(t_0)} d_{t_0}(\gamma_1 \cap P_{t_0}, \gamma_2 \cap P_{t_0}) \leqslant \frac{4BK^2}{A} \alpha^{-t+t_0} d_{t_0}(\gamma_1 \cap P_{t_0}, \gamma_2 \cap P_{t_0}).$$

Now divide both sides by $d_{t_0}(\gamma_1 \cap P_{t_0}, \gamma_2 \cap P_{t_0})$, and take logarithms, obtaining

$$(-h(t)+h(t_0)) \log(\beta) \leqslant \log\left(\frac{4BK^2}{A}\right) + (-t+t_0) \log(\alpha)$$

and so

$$h(t) \geqslant \frac{\log(\alpha)}{\log(\beta)} (t-t_0) + h(t_0) - \frac{\log(4BK^2/A)}{\log(\beta)},$$

proving Claim 5.9 and therefore completing the proof of Proposition 5.8. □

5.5. Interlude: The induced boundary map

The *upper boundary* $\partial^u G_M$ is defined to be the leaf space of the weak stable foliation; this leaf space is identified with V^+. The *lower boundary* $\partial_l G_M$ is the leaf space of the weak unstable foliation, identified with V^-. The *internal boundary* $\partial_{\text{int}} G_M$ is defined as

$$\partial_{\text{int}} G_M = \partial_l G_M \times \partial^u G_M = V^- \times V^+ \approx \mathbf{R}^n / V^0,$$

which is identified with the leaf space of the center foliation.

As a consequence of Proposition 5.4, a quasi-isometry $\phi: G_M \to G_L$ which respects the transversely oriented horizontal foliations induces a bijection

$$\partial_{\text{int}} \phi: \partial_{\text{int}} G_M \to \partial_{\text{int}} G_L$$

which preserves the factors, that is,

$$\partial_{\text{int}} \phi = \partial_l \phi \times \partial^u \phi: \partial_l G_M \times \partial^u G_M \to \partial_l G_L \times \partial^u G_L.$$

Recall the 1-parameter family of metrics $d_{M,t}$ on $\mathbf{R}^n$ given by the quadratic form $Q_{M,t} = (M^{-t})^T M^{-t}$. The internal boundary $\partial_{\text{int}} G_M$ is identified with $\mathbf{R}^n / V^0$ and with $V^- \times V^+$, and we consider two 1-parameter families of metrics.

First, regarding points of $\mathbf{R}^n / V^0$ as affine subspaces parallel to V^0, there is a 1-parameter family of Hausdorff metrics induced from $d_{M,t}$, which we denote $dh_{M,t}$. Second, restrict the action of M^t to the subspace $V^- \times V^+$ to get a 1-parameter subgroup of $\mathrm{GL}(V^- \times V^+)$, and by choosing a basis for $V^- \times V^+$ we obtain a 1-parameter subgroup $\widehat{M}^t$ of $\mathrm{GL}(k, \mathbf{R})$, where k is the dimension of $V^- \times V^+$. We obtain a 1-parameter family of metrics $d_{\widehat{M},t}$. There is a canonical identification $V^- \times V^+ \approx \mathbf{R}^n / V^0$, and with respect to this identification the metrics $d_{\widehat{M},t}$ and $dh_{M,t}$ are bi-Lipschitz-equivalent, with a uniform bi-Lipschitz constant independent of t.

Note that the absolute Jordan form of $\widehat{M}$ is identical with the nonunipotent part of the absolute Jordan form of M, and similarly for N.

LEMMA 5.10. *Given two 1-parameter subgroups M^t, N^t of $\mathrm{GL}(n, \mathbf{R})$, for all $K \geqslant 1$, $C, A \geqslant 0$, there exist $K' \geqslant 1$, $C' \geqslant 0$ with the following properties. If $\phi: G_M \to G_N$ is a (K, C)-quasi-isometry which coarsely respects the transversely oriented horizontal foliations, with Hausdorff constant A, then for every $t \in \mathbf{R}$ the induced bijection $\partial_{\text{int}} \phi$: $\partial_{\text{int}} G_M \to \partial_{\text{int}} G_N$ is a (K', C')-quasi-isometry from the metric $d_{\widehat{M},t}$ to the metric $d_{\widehat{N},h(t)}$.*

Proof. With what we know, the proof is mostly a matter of chasing through definitions.

The quasi-isometry ϕ is a bounded distance from a quasi-isometry $\psi: G_M \to G_N$ which takes the horizontal leaf P_t to the horizontal leaf $P'_{h(t)}$, and which simultaneously takes center leaves of G_M to center leaves of G_N. Now restrict the center foliations of G_M, G_N to $P_t, P'_{h(t)}$, and denote the respective leaf spaces as $Q_t, Q'_{h(t)}$.

In order to apply Lemma 2.1, consider each horizontal leaf P_t of G_M as a geodesic metric space with respect to the Riemannian metric induced by restriction from G_M. The inclusion map $P_t \hookrightarrow G_M$ is evidently $(1,0)$-coarsely Lipschitz, and it is uniformly proper, with a uniformity function $s(r)=a^r$ where $a>1$ is larger than the maximum of the absolute values of all eigenvalues of M and their multiplicative inverses. Note in particular that the coarse Lipschitz constants and the uniformity functions of the maps $P_t \hookrightarrow G_M$ depend only on K, C, A and on the matrix M, but not on t. Similar remarks apply to the inclusion map $P'_{h(t)} \hookrightarrow G_N$. Applying Lemma 2.1, restricting ψ to P_t results in a map $\psi_t: P_t \mapsto P'_{h(t)}$ which is a quasi-isometry. There is in turn an induced map $\theta_t: Q_t \mapsto Q'_{h(t)}$ which is a quasi-isometry with respect to the associated Hausdorff metric. The quasi-isometry constants of the maps ψ_t and θ_t depend only on K, C, A.

Now consider the coordinate identifications $G_M \approx \mathbf{R}^n \times \mathbf{R}$, $G_N \approx \mathbf{R}^n \times \mathbf{R}$. By construction of the left-invariant metrics, for each t the space P_t is identified with $\mathbf{R}^n \times t \approx \mathbf{R}^n$ with metric $d_{M,t}$, and the space $P'_{h(t)}$ is identified with $\mathbf{R}^n$ with metric $d_{N,h(t)}$, and so the maps $\psi_t: \mathbf{R}^n \to \mathbf{R}^n$ are uniform quasi-isometries from $d_{M,t}$ to $d_{N,h(t)}$ for all t. Also, Q_t is identified with $\mathbf{R}^n / V_M^0$ with the associated Hausdorff metric $dh_{M,t}$, and $Q'_{h(t)}$ is identified with $\mathbf{R}^n / V_N^0$ with the associated Hausdorff metric $dh_{N,h(t)}$, and so the maps $\theta_t: \mathbf{R}^n/V_M^0 \to \mathbf{R}^n/V_N^0$ are uniform quasi-isometries from $dh_{M,t}$ to $dh_{N,h(t)}$ for all t. This implies that $\theta_t: V_M^- \times V_M^+ \to V_N^- \times V_N^+$ is a quasi-isometry from $d_{\widehat{M},t}$ to $d_{\widehat{N},t}$ for all t. But for all t the map θ_t is identical to $\partial_{\text{int}}\phi: \partial_{\text{int}} G_M \to \partial_{\text{int}} G_N$, proving the lemma. □

5.6. Step 3: Reduction to Theorem 5.11 on 1-parameter subgroup rigidity

Assume the hypotheses of Theorem 5.2, namely that we have 1-parameter subgroups M^t, N^t, and a quasi-isometry $\phi: G_M \to G_N$ which coarsely respects the transversely oriented horizontal foliations. Applying Proposition 5.8, there is an induced time change of the form $h(t)=mt+b$ with $m>0$. Applying Proposition 4.1, there is a horizontal-respecting quasi-isometry $G_N \to G_{N^m}$ with an induced time change of the form $s \mapsto s/m$. By composition we obtain a horizontal-respecting quasi-isometry $G_M \to G_{N^m}$ with an induced time change of the form $t \mapsto t+b'$. Changing the coordinates in G_M by a translation of the time coordinate t, we have a horizontal-respecting quasi-isometry $G_M \to G_{N^m}$ for which the identity map $t \mapsto t$ is an induced time change. Applying Lemma 5.10, we obtain a bijection $\partial_{\text{int}}\phi: \mathbf{R}^n \to \mathbf{R}^n$ which, for each t, is a (K', C')-quasi-isometry from

$d_{\widehat{M},t}$ to $d_{\widehat{N^m},t}$.

Now apply the following theorem (with N in place of N^m), which will be proved in the next section:

THEOREM 5.11 (1-parameter subgroup rigidity). *Let M^t, N^t be 1-parameter subgroups of $\mathrm{GL}(n, \mathbf{R})$ such that $M=M^1$ and $N=N^1$ have no eigenvalues on the unit circle. If there exists a bijection $f: \mathbf{R}^n \to \mathbf{R}^n$ and constants $K \geqslant 1$, $C \geqslant 0$ such that for each $t \in \mathbf{R}$ and $p, q \in \mathbf{R}^n$ we have*

$$-C+\frac{1}{K} d_{M,t}(p,q) \leqslant d_{N,t}(f(p),f(q)) \leqslant K d_{M,t}(p,q)+C$$

then M and N have the same absolute Jordan form.

Returning to the previous discussion, this theorem allows us to conclude that $\widehat{M}$ and $\widehat{N}^m$ have the same absolute Jordan form, and so the nonunipotent parts of the absolute Jordan forms of M, N^m are identical. We have already proved in Corollary 5.6 that the unipotent parts are identical, and so M and N^m have the same absolute Jordan forms, finishing the proof of Theorem 5.2. □

6. Dynamics of G_M, Part II: 1-parameter subgroup rigidity

In this section we give a proof of Theorem 5.11.

Let M^t, N^t be 1-parameter subgroups of $\mathrm{GL}(n, \mathbf{R})$ with no eigenvalues on the unit circle. Let $M^t = \overline{M}^t P^t$, $N^t = \overline{N}^t Q^t$ be the real Jordan forms, so that $\overline{M}$ and $\overline{N}$ have all positive eigenvalues, none equal to 1. Let $f: \mathbf{R}^n \to \mathbf{R}^n$ be a bijection which satisfies

$$-C+\frac{1}{K} d_{M,t}(p,q) \leqslant d_{N,t}(f(p),f(q)) \leqslant K d_{M,t}(p,q)+C \tag{6.1}$$

for all $t \in \mathbf{R}$, $p, q \in \mathbf{R}^n$.

The bijection $f: \mathbf{R}^n \to \mathbf{R}^n$ must in fact be a homeomorphism. To see why, for each $p \in \mathbf{R}^n$, $R>0$, $T>0$ let

$$F_{p,R}(T) = \{q \in \mathbf{R}^n \mid d_{M,t}(p,q) < R \text{ for all } t \in (-T,T)\}.$$

In other words, $F_{p,R}(T)$ is the intersection of open balls of radius R about p in each of the metrics $d_{M,t}$, for $t \in (-T,T)$. Since the eigenvalues of $\overline{M}$ are all positive real numbers, none equal to 1, it follows from Proposition 3.2 that for each $p \in \mathbf{R}^n$ and each $R>0$ the collection of sets $F_{p,R}(T)$ as T ranges in $(0,\infty)$ is a neighborhood basis for p, in the standard topology on $\mathbf{R}^n$. We define a similar neighborhood basis using matrix N, denoted $G_{p,R}(T)$. Since $f(F_{p,R}(T)) \subseteq G_{f(p),KR+C}(T)$ for each $p \in \mathbf{R}^n$, $R>0$, $T>0$, it follows that f is continuous. The same argument applies to f^{-1}, and so f is a homeomorphism.

The idea of the proof of Theorem 5.11 is to show that f respects certain "flags of foliations" which are closely related to the Jordan decompositions of $\mathbf{R}^n$ with respect to M^t and N^t. We begin by setting up the notation needed to define and study these foliations.

Definition (flags of foliations). If V is a vector subspace of $\mathbf{R}^n$, define a foliation $\mathcal{F}(V)$ of $\mathbf{R}^n$ whose leaves are the affine subspaces of $\mathbf{R}^n$ parallel to V. Given a flag of subspaces $V_1 \subset \ldots \subset V_r$, it follows that if $1 \leqslant i < j \leqslant r$ then each leaf of $\mathcal{F}(V_i)$ is contained in some leaf of $\mathcal{F}(V_j)$; we denote this relation by saying that $\mathcal{F}(V_1) \prec \ldots \prec \mathcal{F}(V_r)$ is a *flag of foliations* of $\mathbf{R}^n$.

Recall the root space decompositions of $\mathbf{R}^n$ with respect to $\overline{M}$ and $\overline{N}$. We denote the eigenvalues of $\overline{M}$ and $\overline{N}$ by

$$0 < \mu_m^- < \ldots < \mu_1^- < 1 < \mu_1^+ < \ldots < \mu_r^+$$

and

$$0 < \nu_n^- < \ldots < \nu_1^- < 1 < \nu_1^+ < \ldots < \nu_s^+$$

respectively. The corresponding root space decompositions are denoted

$$V_m^- \oplus \ldots \oplus V_1^- \oplus V_1^+ \oplus \ldots \oplus V_r^+$$

and

$$W_n^- \oplus \ldots \oplus W_1^- \oplus W_1^+ \oplus \ldots \oplus W_s^+.$$

As in §4 we set

$$V^- = V_m^- \oplus \ldots \oplus V_1^-, \qquad V^+ = V_1^+ \oplus \ldots \oplus V_r^+,$$
$$W^- = W_n^- \oplus \ldots \oplus W_1^-, \quad W^+ = W_1^+ \oplus \ldots \oplus W_s^+.$$

Define the *root space flags*

$$U_i^- = V_i^- \oplus \ldots \oplus V_1^-, \qquad i = 1, \ldots, m,$$
$$U_j^+ = V_1^+ \oplus \ldots \oplus V_j^+, \qquad j = 1, \ldots, r,$$
$$Y_i^- = W_i^- \oplus \ldots \oplus W_1^-, \quad i = 1, \ldots, n,$$
$$Y_j^+ = W_1^+ \oplus \ldots \oplus W_j^+, \quad j = 1, \ldots, s,$$

and by convention we take U_0^-, U_0^+, Y_0^-, Y_0^+ each to be the trivial subspace. Associated to the root space flags we have *root space foliation flags*

$$\mathcal{F}(U_1^-) \prec \ldots \prec \mathcal{F}(U_m^-) = \mathcal{F}(V^-),$$
$$\mathcal{F}(U_1^+) \prec \ldots \prec \mathcal{F}(U_r^+) = \mathcal{F}(V^+),$$
$$\mathcal{F}(Y_1^-) \prec \ldots \prec \mathcal{F}(Y_n^-) = \mathcal{F}(W^-),$$
$$\mathcal{F}(Y_1^+) \prec \ldots \prec \mathcal{F}(Y_s^+) = \mathcal{F}(W^+).$$

Step 1: *f respects contracting and expanding foliations.* First we show that

$$f(\mathcal{F}(V^-))=\mathcal{F}(W^-) \quad \text{and} \quad f(\mathcal{F}(V^+))=\mathcal{F}(W^+).$$

Given $p,q\in\mathbf{R}^n$ we have the following chain of equivalences:

(1) p,q are in the same leaf of $\mathcal{F}(V^+)$;

(2) $d_{M,t}(p,q)=\|M^{-t}(p-q)\|\to 0$ as $t\to+\infty$;

(3) $d_{M,t}(p,q)$ is bounded for $t\in[0,+\infty)$;

(4) $d_{N,t}(f(p),f(q))$ is bounded for $t\in[0,+\infty)$;

(5) $d_{N,t}(f(p),f(q))=\|N^{-t}(f(p)-f(q))\|\to 0$ as $t\to+\infty$;

(6) $f(p),f(q)$ are in the same leaf of $\mathcal{F}(W^+)$.

The equivalence of (1)–(3) follows from Proposition 3.2, and similarly for (4)–(6). The equivalence of (3) and (4) follows from (6.1). This shows $f(\mathcal{F}(V^+))=\mathcal{F}(W^+)$. A similar argument with $t\in(-\infty,0]$ shows $f(\mathcal{F}(V^-))=\mathcal{F}(W^-)$.

Step 2: *f respects root space foliation flags.* Next we show

CLAIM 6.1. *$f:\mathbf{R}^n\to\mathbf{R}^n$ respects the root space foliation flags, and corresponding root spaces have the same eigenvalues. More precisely we have*:

(1) $r=s$;

(2) $\mu_j^+=\nu_j^+$ *for* $j=1,\dots,r$;

(3) $f(\mathcal{F}(U_j^+))=\mathcal{F}(Y_j^+)$ *for* $j=1,\dots,r$;

(4) $m=n$;

(5) $\mu_i^-=\nu_i^-$ *for* $i=1,\dots,m$;

(6) $f(\mathcal{F}(U_i^-))=\mathcal{F}(Y_i^-)$ *for* $i=1,\dots,m$.

It follows that M,N have the same eigenvalues with the same multiplicities.

We give the proof of (1), (2), (3); the proof of (4), (5), (6) is similar.

We know by Step 1 that $f(\mathcal{F}(V^+))=\mathcal{F}(W^+)$. Consider points p,q in the same leaf of $\mathcal{F}(V^+)$, so that $f(p),f(q)$ are in the same leaf of $\mathcal{F}(W^+)$. From Proposition 3.2 it follows that as $t\to-\infty$ both of the quantities $d_{M,t}(p,q)$ and $d_{N,t}(f(p),f(q))$ approach $+\infty$. It follows that for sufficiently large t, in the inequality (6.1) we can absorb the additive constant C, yielding

$$\frac{1}{K+1}d_{M,t}(p,q)\leqslant d_{N,t}(f(p),f(q))\leqslant(K+1)d_{M,t}(p,q). \tag{6.2}$$

Define displacement vectors $v=p-q$, $w=f(p)-f(q)$. Taking natural logarithms, dividing by t, and taking lim sup, we have

$$\begin{gathered}\limsup_{t\to-\infty}\frac{\log(d_{M,t}(p,q))}{t}=\limsup_{t\to-\infty}\frac{\log(d_{N,t}(f(p),f(q)))}{t},\\ \limsup_{t\to+\infty}\frac{\log\|M^tv\|}{t}=\limsup_{t\to+\infty}\frac{\log\|N^tw\|}{t}.\end{gathered} \tag{6.3}$$

To evaluate these limits, let $I(p,q)=I(v)$ be the unique integer such that

$$v \in U^+_{I(v)} - U^+_{I(v)-1}$$

or, equivalently, the unique integer such that p,q are in the same leaf of $\mathcal{F}(U^+_{I(p,q)})$ but not in the same leaf of $\mathcal{F}(U^+_{I(p,q)-1})$ (recall the convention that $U^+_0=0$, and so $I(p,q)=0$ if and only if $p=q$). Define $J(f(p),f(q))=J(w)$ similarly by

$$w \in Y^+_{J(w)} - Y^+_{J(w)-1}.$$

Applying Proposition 3.2 we have

$$\limsup_{t\to+\infty} \frac{\log \|M^t v\|}{t} = \mu^+_{I(v)}, \quad \limsup_{t\to+\infty} \frac{\log \|N^t w\|}{t} = \nu^+_{J(w)},$$

and so by (6.3) we have

$$\mu^+_{I(p,q)} = \mu^+_{I(v)} = \nu^+_{J(w)} = \nu^+_{J(f(p),f(q))}.$$

Since f is a bijection from each leaf of $\mathcal{F}(V^+)$ to some leaf of $\mathcal{F}(W^+)$, items (1) and (2) of Claim 6.1 now follow, and it also follows that

$$I(p,q) = J(f(p),f(q))$$

for all p,q contained in the same leaf of $\mathcal{F}(V^+)$.

We now prove item (3) of Claim 6.1 by induction on j. If p,q are in the same leaf of $\mathcal{F}(U^+_1)$ then $I(p,q)=1$ and so $J(p,q)=1$, which implies that $f(p),f(q)$ are in the same leaf of $\mathcal{F}(Y^+_1)$. A similar argument with f^{-1} proves that $f(\mathcal{F}(U^+_1))=\mathcal{F}(Y^+_1)$, proving the base step of the induction. Now assume that $f(\mathcal{F}(U^+_j))=\mathcal{F}(Y^+_j)$, and suppose that p,q are in the same leaf of $\mathcal{F}(U^+_{j+1})$. There are two cases to consider. If p,q lie in the same leaf of $\mathcal{F}(U^+_j)$ then by the induction hypothesis $f(p),f(q)$ lie in the same leaf of $\mathcal{F}(Y^+_j)$, in particular they lie in the same leaf of $\mathcal{F}(Y^+_{j+1})$. If p,q do not lie in the same leaf of $\mathcal{F}(U^+_j)$ then $I(p,q)=j+1$ and so $J(f(p),f(q))=j+1$, and thus $f(p),f(q)$ lie on the same leaf of $\mathcal{F}(Y^+_{j+1})$. A similar argument with f^{-1} shows that $f(\mathcal{F}(U^+_{j+1}))=Y^+_{j+1}$, completing the induction.

As mentioned earlier, (4)–(6) are proved similarly, completing the proof of Claim 6.1.

Step 3: *f respects Jordan foliation flags.* From Step 2, for each fixed $j=1,...,r$ the matrices M,N have μ^+_j-root spaces V^+_j, W^+_j respectively. As part of their root space flags we have

$$U^+_j = U^+_{j-1} \oplus V^+_j,$$
$$Y^+_j = Y^+_{j-1} \oplus W^+_j.$$

Let c_j be the index of nilpotency of $\mu_j \cdot I - M$, and let d_j be the index of nilpotency of $\mu_j \cdot I - N$. Then we have Jordan filtrations

$$V_{j,0}^+ \subset \ldots \subset V_{j,c_j}^+ = V_j^+,$$
$$W_{j,0}^+ \subset \ldots \subset W_{j,d_j}^+ = W_j^+,$$

and we set $U_{j,k}^+ = U_{j-1}^+ \oplus V_{j,k}^+$ and $Y_{j,k}^+ = Y_{j-1}^+ \oplus W_{j,k}^+$, yielding subspace flags

$$U_{j-1}^+ \subset U_{j,0}^+ \subset \ldots \subset U_{j,c_j-1}^+ = U_j^+,$$
$$Y_{j-1}^+ \subset Y_{j,0}^+ \subset \ldots \subset Y_{j,d_j-1}^+ = Y_j^+.$$

Corresponding to these subspace flags are foliation flags,

$$\mathcal{F}(U_{j-1}^+) \prec \mathcal{F}(U_{j,0}^+) \prec \ldots \prec \mathcal{F}(U_{j,c_j-1}^+) = \mathcal{F}(U_j^+),$$
$$\mathcal{F}(Y_{j-1}^+) \prec \mathcal{F}(Y_{j,0}^+) \prec \ldots \prec \mathcal{F}(Y_{j,d_j-1}^+) = \mathcal{F}(Y_j^+),$$

called the *expanding Jordan foliation flags* associated to the corresponding root space foliations $\mathcal{F}(U_j^+), \mathcal{F}(Y_j^+)$ respectively. The *contracting Jordan foliation flags* associated to each root space foliation $\mathcal{F}(U_i^-), \mathcal{F}(Y_i^-)$ are similarly defined.

CLAIM 6.2. *$f: \mathbf{R}^n \to \mathbf{R}^n$ respects the Jordan foliation flags associated to corresponding root space foliations. More precisely, for each $j=1, \ldots, r$ we have*

(1) *$c_j = d_j$,*

(2) *$f(\mathcal{F}(U_{j,k}^+)) = \mathcal{F}(Y_{j,k}^+)$ for $k=0, \ldots, c_j-1$,*

and similarly for the contracting Jordan foliation flags.

From this claim, for each $j=1, \ldots, r$ it immediately follows that $\overline{M}, \overline{N}$ have the same Jordan blocks with eigenvalue μ_j^+, and so the expanding parts of the Jordan forms for $\overline{M}, \overline{N}$ are identical; similarly for the contracting parts. Since M, N have no eigenvalues on the unit circle, it now follows that M, N have the same absolute Jordan forms, completing the proof of Theorem 5.11.

Proof of Claim 6.2. Consider $p, q \in \mathbf{R}^n$ in the same leaf of $\mathcal{F}(U_j^+)$ but not in the same leaf of $\mathcal{F}(U_{j-1}^+)$, so that $f(p), f(q)$ are in the same leaf of $\mathcal{F}(Y_j^+)$ but not in the same leaf of $\mathcal{F}(Y_{j-1}^+)$. Define displacement vectors $v = p - q$, $w = f(p) - f(q)$, so that $v \in U_j^+ - U_{j-1}^+$ and $w \in Y_j^+ - Y_{j-1}^+$. We know that

$$\limsup_{t \to +\infty} \frac{\log \|M^t v\|}{t} = \frac{\log \|N^t w\|}{t} = \mu_j^+.$$

We also know that (6.2) is true for t sufficiently close to $-\infty$, and so for t sufficiently close to $+\infty$ we have

$$\frac{1}{K+1}\|M^t v\| \leqslant \|N^t w\| \leqslant (K+1)\|M^t v\|. \tag{6.4}$$

By induction on $k=0,1,\dots$, we shall prove that $v\in U^+_{j,k}$ if and only if $w\in Y^+_{j,k}$, or equivalently that $f(\mathcal{F}(U^+_{j,k}))=\mathcal{F}(Y^+_{j,k})$.

For the basis step $k=0$, divide the inequality (6.4) by μ^t to obtain, for all t sufficiently close to $+\infty$,

$$\frac{1}{K+1}\cdot\frac{\|M^t v\|}{\mu^t} \leqslant \frac{\|N^t w\|}{\mu^t} \leqslant (K+1)\frac{\|M^t v\|}{\mu^t}. \tag{6.5}$$

By the exponential lower bound and the exponential–polynomial upper bound of Proposition 3.2, the quantity $\|M^t v\|/\mu^t$ is bounded for $t\geqslant 0$ if and only if $v\in U^+_{j,0}$; and the quantity $\|N^t w\|/\mu^t$ is bounded on $t\geqslant 0$ if and only if $w\in Y^+_{j,0}$. By (6.5), however, the boundedness of these two quantities on $t\geqslant 0$ are equivalent.

For the induction step, assume that $f(\mathcal{F}(U^+_{j,k-1}))=\mathcal{F}(Y^+_{j,k-1})$, that is, $v\in U^+_{j,k-1}$ if and only if $w\in Y^+_{j,k-1}$. We must prove that $v\in U^+_{j,k}-U^+_{j,k-1}$ if and only if $w\in Y^+_{j,k}-Y^+_{j,k-1}$. From (6.4), for t sufficiently close to $+\infty$ we have

$$\frac{1}{K+1}\cdot\frac{\|M^t v\|}{\mu^t t^k} \leqslant \frac{\|N^t w\|}{\mu^t t^k} \leqslant (K+1)\frac{\|M^t v\|}{\mu^t t^k} \tag{6.6}$$

and

$$\frac{1}{K+1}\cdot\frac{\|M^t v\|}{\mu^t t^{k-1}} \leqslant \frac{\|N^t w\|}{\mu^t t^{k-1}} \leqslant (K+1)\frac{\|M^t v\|}{\mu^t t^{k-1}}. \tag{6.7}$$

By the exponential–polynomial upper and lower bounds of Proposition 3.2, the following two statements are equivalent:

(1) $v\in U^+_{j,k}-U^+_{j,k-1}$.

(2) For $t\geqslant 0$, the quantity $\|M^t v\|/\mu^t t^k$ is bounded, but the quantity $\|M^t v\|/\mu^t t^{k-1}$ is not bounded.

Similarly, the following two statements are equivalent:

(3) $w\in Y^+_{j,k}-Y^+_{j,k-1}$.

(4) For $t\geqslant 0$, the quantity $\|N^t w\|/\mu^t t^k$ is bounded, but the quantity $\|N^t w\|/\mu^t t^{k-1}$ is not bounded.

But by inequalities (6.6) and (6.7), statements (2) and (4) are equivalent, and so statements (1) and (3) are equivalent, completing the inductive proof of item (2) of Claim 6.2 for all $k\geqslant 0$.

The foliation flag $\mathcal{F}(U^+_{j,0})\prec...\prec\mathcal{F}(U^+_{j,k})\prec...$ must terminate at $\mathcal{F}(U^+_j)$ for the same value of k for which the flag $\mathcal{F}(Y^+_{j,0})\prec...\prec\mathcal{F}(Y^+_{j,k})\prec...$ terminates at $\mathcal{F}(Y^+_j)$, proving that $c_j=d_j$, and completing the proof of Claim 6.2. □

Our proof of Theorem 5.11 actually provides for some regularity of f. We record the statement here, although it is not used at all in this paper.

PROPOSITION 6.3 (regularity). *With the assumptions as in Theorem* 5.11, *f is a homeomorphism which respects the contracting and expanding root space foliation flags of $\overline{M},\overline{N}$, and for each corresponding pair of root space foliations, f also respects the associated Jordan foliation flags.*

Remark. Even stronger regularity properties should hold. For instance, f should satisfy Lipschitz conditions in directions parallel to a root space, by arguments similar to the results of [FM1]. Understanding what happens transverse to root spaces will require new ideas.

7. Quasi-isometries of Γ_M via coarse topology

Recall the notation for abelian-by-cyclic Lie groups: given $M\in \mathrm{GL}_\times(m,\mathbf{R})$, a 1-parameter subgroup $M^t\subset \mathrm{GL}(m,\mathbf{R})$ with $M^1=M$ determines a Lie group denoted $G_M=\mathbf{R}^m\rtimes_M\mathbf{R}$.

This entire section will be devoted to a proof of

PROPOSITION 7.1 (induced quasi-isometries of G_M). *Consider integral matrices*

$$M\in \mathrm{GL}_\times(m,\mathbf{R}),\quad N\in \mathrm{GL}_\times(n,\mathbf{R}),$$

and suppose that $\det M,\det N>1$. *If there exists a quasi-isometry $f:\Gamma_M\to\Gamma_N$ then $m=n$ and there exists a quasi-isometry $\phi: G_M\to G_N$ which coarsely respects horizontal foliations and their transverse orientations. Furthermore, all associated constants for ϕ depend only on those for f.*

7.1. A geometric model for Γ_M

Let $M\in \mathrm{GL}_\times(m,\mathbf{R})$ be an integral matrix lying on a 1-parameter subgroup M^t of $\mathrm{GL}(m,\mathbf{R})$ with $M^1=M$ and with associated Lie group G_M. We assume that $\det M>1$ and we denote $d=\det M$.

We start by constructing a contractible, $(m+1)$-dimensional metric complex X_M on which Γ_M acts properly discontinuously and cocompactly by isometries, and so the group Γ_M will be quasi-isometric to the geodesic metric space X_M.

The description of Γ_M as an ascending HNN extension shows that Γ_M is the fundamental group of the mapping torus of an injective endomorphism of the m-dimensional torus. Let X_M be the universal cover of this mapping torus. Topologically, there is a fibration

$$\begin{array}{ccc} \mathbf{R}^{n-1} & \longrightarrow & X_M \\ & & \downarrow \\ & & T_M \end{array}$$

where T_M is the homogeneous directed tree with one edge coming into each vertex and $d=\det M$ edges going out of each vertex. Hence X_M is a topological product $X_M \approx \mathbf{R}^{n-1} \times T_M$.

The action of Γ_M on X_M by deck transformations induces an action of Γ_M on T_M. This action is equivalent to the usual action of the HNN extension Γ_M on its Bass–Serre tree T_M.

Before constructing a metric on X_M, let us describe the essential properties of such a metric. These are best described by giving the isometry types of natural subcomplexes of X_M.

Definition (*doubled horoballs*). We define a *doubled G_M-horoball*, denoted by H_M, to be the metric space obtained by identifying two copies of $\{(x,t)\in G_M \mid t\geqslant 0\}$ along $\{(x,0)\in G_M\}$, endowed with the path metric.

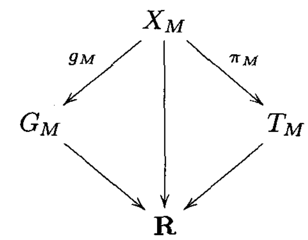
Definition (*hyperplanes in X_M*). Let $P_l=\pi_M^{-1}(l)$, where l is a bi-infinite line in the directed tree T_M. We call P_l a *hyperplane* in X_M. There are two cases to consider:

(1) l is coherently oriented in T_M. In this case P_l is isometric to G_M, and we call P_l a *coherent hyperplane* in X_M.

(2) l is not coherently oriented in T_M, and thus switches orientation precisely once. In this case P_l is isometric to H_M, and we call P_l an *incoherent hyperplane* in X_M.

This definition nearly determines a metric on X_M. To specify a metric on X_M, one proceeds as follows. Fix a path metric on T_M so that each edge has length 1. Fix a base vertex on T_M. These choices determine a unique height function $T_M \to \mathbf{R}$ taking the base vertex to the origin and taking each edge to a segment of length 1 via an orientation preserving isometry. We have also defined a height function $G_M \to \mathbf{R}$. Note that the height function on G_M was previously called the "time function"; we will use both terms.

The complex X_M is the fiber product of the two height functions $T_M \to \mathbf{R}$, $G_M \to \mathbf{R}$, as shown in the diagram



There are induced projections $g_M: X_M \to G_M$ and $\pi_M: X_M \to T_M$, and an induced height function $X_M \to \mathbf{R}$. There is a unique path metric on X_M so that each continuous cross section $G_M \to X_M$ of g_M is a path-isometric embedding; and hence each coherent hyperplane in X_M is an isometrically embedded copy of G_M.

Definition (horizontal leaf). A *horizontal leaf* L in X_M is a subset of the form $L = \pi_M^{-1}(v)$ where $v \in T_M$.

Note that the collection of horizontal leaves on X_M, equipped with the Hausdorff metric, forms a metric space which is isometric to T_M via the projection map $\pi_M: X_M \to T_M$.

Note that each hyperplane in X_M comes equipped with a foliation by horizontal leaves. For coherent hyperplanes P in X_M, which are isometric to G_M, the notion of horizontal leaf in P coincides with that of a horizontal leaf in G_M, given in §5.1.

7.2. Proof of Proposition 7.1 on induced quasi-isometries of G_M

Let M, N be as in the statement of the proposition.

We begin by showing that M and N have the same size. Suppose that $M \in \mathrm{GL}(m, \mathbf{R})$ and $N \in \mathrm{GL}(n, \mathbf{R})$. In §7.1 we constructed finite classifying spaces for Γ_M and Γ_N of dimensions $m+1, n+1$ respectively, and by Lemma 5.2 of [FM2] these numbers are the virtual cohomological dimensions of Γ_M, Γ_N. By a result of Block–Weinberger [BW] and Gersten [Ge1], virtual cohomological dimension is a quasi-isometry invariant for groups with finite classifying spaces. It follows that $m = n$.

Now Γ_M acts properly discontinuously, freely and cocompactly on X_M. This action is by isometries, because Γ_M acts on G_M, on T_M and on $\mathbf{R}$ by isometries, and the fiber product diagram is equivariant with respect to these actions. It follows that Γ_M in any word metric is quasi-isometric to X_M. Henceforth we will freely interchange Γ_M and X_M when discussing quasi-isometry type. The same discussion applies to Γ_N and X_N, and so the quasi-isometry $f: \Gamma_M \to \Gamma_N$ gives a quasi-isometry (perhaps with bigger constants) $f: X_M \to X_N$.

Proposition 7.1 generalizes the case when M and N are (1×1)-matrices, done in §4 and §5 of [FM1]. The proof here is more difficult, and the steps must be proved in different order. In Steps 1 and 2 we prove (in a more general context; see Theorem 7.7) that a quasi-isometry $X_M \to X_N$ coarsely respects hyperplanes and horizontal sets. We must, however, still distinguish between coherent and incoherent hyperplanes. This is easy in the (1×1)-case handled in [FM1], where G_M and G_N are (scaled versions of) $\mathbf{H}^2$, and a doubled $\mathbf{H}^2$-horoball is evidently not quasi-isometric to $\mathbf{H}^2$. In general we are unable to distinguish the quasi-isometry types of coherent and incoherent hyperplanes. To get around this, in Step 3, Proposition 7.11, we prove that there is no horizontal-respecting quasi-isometry between a coherent and an incoherent hyperplane.

Step 1: *Quasi-isometrically embedded hyperplanes are close to hyperplanes.* Given integral matrices $M, N \in \mathrm{GL}_\times(n, \mathbf{R})$, if $P = G_M$ or H_M, then for all $K \geqslant 1$, $C \geqslant 0$ there exists $A \geqslant 0$ such that if $\phi: P \to X_N$ is a (K, C)-quasi-isometric embedding then there is a unique hyperplane $Q \subset X_N$ with $d_{\mathcal{H}}(\phi(P), Q) \leqslant A$.

This was proved for (1×1)-matrices in [FM1]. Our proof of Step 1, while following the same outline as in the (1×1)-case, will actually apply in a much broader setting. The generalized versions of Steps 1 and 2, given in Theorem 7.3 and Theorem 7.7, are used for example in [FM3] to study surface-by-free groups, and also in [MSW] to prove quasi-isometric rigidity theorems for various "homogeneous" graphs of groups (see the remark after Theorem 7.7).

The generalization of Step 1 given in Theorem 7.3 will require moving from the category of quasi-isometric embeddings into the category of uniformly proper embeddings. After a fair amount of work to establish the new setting, we then quote some theorems of coarse algebraic topology and follow the proof of [FM1].

Consider a finite graph Γ of finitely generated groups; each edge e is oriented, with initial and final vertices $i(e), f(e)$. We say that Γ is *geometrically homogeneous* if each edge-to-vertex injection is a quasi-isometry with respect to the word metric, or equivalently, has finite index image. Ideally we would like to have a version of Step 1 for any geometrically homogeneous graph of groups in which each vertex and edge group is the fundamental group of a closed, aspherical n-manifold, or even more generally, an n-dimensional Poincaré duality group. This should come from a more careful reading of results in coarse algebraic topology such as [KK], but meanwhile we will use Theorems 7.5 and 7.6, which require us to impose additional assumptions on Γ.

Suppose that we have a category $\mathcal{C}$ of aspherical, closed, smooth manifolds such that $\mathcal{C}$ is closed under finite coverings and satisfies *smooth rigidity*, meaning that any homotopy equivalence between manifolds in $\mathcal{C}$ is homotopic to a diffeomorphism. Such categories include: the n-torus, $n \geqslant 1$; hyperbolic surfaces; all other irreducible, nonpositively curved, locally symmetric spaces, by Mostow's rigidity theorem [Mo2]; solvmanifolds, by earlier work of Mostow [Mo1]; nilmanifolds, by still earlier work of Malcev [Ma]; and various generalizations due to Farrell and Jones [FJ1], [FJ2].

We shall assume that Γ is a geometrically homogeneous graph of groups where each vertex group Γ_v is the fundamental group of a manifold M_v in the category $\mathcal{C}$. Construct a *graph of aspherical manifolds* M_Γ, with fundamental group $\pi_1\Gamma$, as follows. For each edge e, the two injections $\Gamma_e \to \Gamma_{i(e)}$, $\Gamma_e \to \Gamma_{t(e)}$ determine two finite covering spaces of M_v each of whose fundamental group is identified with Γ_e, and so we obtain a diffeomorphism between the two covering spaces; identify these covering spaces and let M_e be the resulting smooth manifold. We have smooth, finite covering maps $M_e \to M_{i(e)}$, $M_e \to M_{t(e)}$ inducing the corresponding edge-to-vertex group injections. Form M_Γ from the disjoint union

$$\Big(\bigcup_v M_v\Big) \cup \Big(\bigcup_e M_e \times e\Big)$$

by gluing $M_e \times i(e)$ to $M_{i(e)}$ and $M_e \times f(e)$ to $M_{f(e)}$ via the finite covering maps $M_e \to M_{i(e)}$ and $M_e \to M_{f(e)}$. From the construction of M_Γ we obtain a map $M_\Gamma \to \Gamma$ such that each fiber M_x, $x \in \Gamma$, is a manifold in the category $\mathcal{C}$.

Let X_Γ be the universal cover of M_Γ. There is a Γ-equivariant fiber bundle $X_\Gamma \to T_\Gamma$ over the Bass–Serre tree T_Γ of Γ whose fiber is a contractible n-manifold. Any geodesic metric on M_Γ lifts to a $\pi_1\Gamma$-equivariant geodesic metric on X_Γ. Smoothness allows us to impose additional geometric structure on X_Γ which we now describe.

A geodesic metric space is *proper* if closed balls are compact. A *bounded-geometry, metric simplicial complex* is a simplicial complex Σ equipped with a proper, geodesic metric such that for some constants $0 < C_1 < C_2$ each positive-dimensional simplex has diameter between C_1 and C_2, and for some constant $C > 0$ the link of each simplex has $\leqslant C$ simplices. A subset S of Σ is *rectifiable* if for any $p, q \in S$ there exists a path in S between p and q which is rectifiable in Σ, and which has the shortest Σ-length among all paths in S between p and q. The length of such a path defines a geodesic metric on S. A *D-homotopy* in Σ is a homotopy whose tracks all have diameter $\leqslant D$. The space Σ is *uniformly contractible* if there exists a function $\delta: [0,\infty) \to [0,\infty)$ such that for every bounded subset $S \subset \Sigma$, the inclusion map $S \hookrightarrow \Sigma$ is $\delta(\operatorname{diam}(S))$-homotopic to a constant map. More precisely we say that Σ is *δ-uniformly contractible.*

Let T be a bounded-geometry, metric simplicial tree, let X be a proper, geodesic metric space, and let $\pi: X \to T$ be a surjective map. Denote $X_A = \pi^{-1}(A)$ for each $A \subset T$. The map π is called a *metric fibration* if:

(1) X is a uniformly contractible, bounded-geometry, metric simplicial complex;

(2) For each subtree $T' \in T$, the subset $X_{T'}$ is a subcomplex of X and is rectifiable in X.

(3) For each $t \in T$ the subspace X_t is uniformly contractible and is a bounded-geometry, metric simplicial complex, with bounded geometry constants and uniform contractibility data independent of t;

(4) The map $\pi: X \to T$ is distance-nonincreasing;

(5) There is a homeomorphism $\Theta: X \to F \times T$ such that

(5a) for all $t \in T$, $\Theta(X_t) = F \times t$,

(5b) for all $x \in F$, the map

$$T \to x \times T \xrightarrow{\Theta^{-1}} X$$

is a locally isometric embedding,

(5c) there exists $K \geqslant 1$ such that for all edges e of T and $t \in e$, the retraction $r: e \to t$ induces a projection

$$X_e \xrightarrow{\Theta} F \times e \xrightarrow{\mathrm{Id} \times r} F \times t \xrightarrow{\Theta^{-1}} X_t$$

which is K-Lipschitz.

Each fiber X_t, $t \in T$, is called a *horizontal leaf* in X. If L is a bi-infinite line in T then X_L is called a *hyperplane* in X. Items (4) and (5b) combine to show that the map of item (5b) is an isometric embedding; the image $\Theta^{-1}(x \times T)$ is called a *vertical leaf* in X. For each subtree $T' \subset T$, the closest point retraction $r: T \to T'$ induces a map

$$X \xrightarrow{\Theta} F \times T \xrightarrow{\mathrm{Id} \times r} F \times T' \xrightarrow{\Theta^{-1}} X_{T'}$$

called *vertical projection* of X to $X_{T'}$.

Remark. Suppose that Γ is a graph of groups taken from a category $\mathcal{C}$ as above. Let M_Γ and $X_\Gamma \to T_\Gamma$ be as constructed above starting from Γ. Then elementary constructions produce a metric and a simplicial structure on M_Γ which lifts to a Γ-equivariant metric and simplicial structure on X such that $X \to T_\Gamma$ is a metric fibration. Item (1) follows by compactness of M_Γ.

Remark. The definition has some redundancy: item (1) is a formal consequence of item (3), as can be seen by elementary but mildly tedious arguments. But by the previous remark we may dispense with these arguments for the examples at hand.

The following lemma, applied to a bi-infinite line in T, gives good geometric properties for hyperplanes:

LEMMA 7.2. *If $\pi: X \to T$ is a metric fibration then there exist functions $\delta': [0, \infty) \to [0, \infty)$ and $\varrho: [0, \infty) \to [0, \infty)$ with $\lim_{t \to \infty} \varrho(t) = \infty$, such that for any subtree $T' \subset T$ we have*:

(1) *The embedding $X_{T'} \to X$ is ϱ-uniformly proper*;

(2) *The geodesic metric space $X_{T'}$ is δ'-uniformly contractible.*

Proof. To prove (1), consider $x,y\in X_{T'}$, let $D=d_X(x,y)$, and let $\gamma\colon[0,D]\to X$ be a geodesic connecting x and y. Let $N_D(T')$ be the D-neighborhood of T' in T so that $\gamma\subset X_{N_D(T')}$. Applying item (5) iteratively, projecting inward starting from the edges of $N_D(T')$ furthest from T', it follows that vertical projection $X_{N_D(T')}\to X_{T'}$ distorts any distance r by at worst K^Dr, and so $d_{X_{T'}}(x,y)\leqslant K^DD$.

To prove (2), suppose that $A\subset X_{T'}$ and $\operatorname{diam}_{X_{T'}}(A)\leqslant R$, so that A is R'-homotopic to a constant in X where R' depends on R but not on A. This homotopy may then be mapped back to $X_{T'}$ by vertical projection, distorting diameters of homotopy tracks by an amount bounded in terms of R' as we saw above. The result is an R''-homotopy of A to a constant in T', with R'' depending only on R and not on A. □

Here is our generalization of Step 1. It applies to any metric fibration of the form $X_\Gamma\to T_\Gamma$, where Γ is a finite, geometrically homogeneous graph of fundamental groups of manifolds in any of the categories $\mathcal{C}$ described earlier.

THEOREM 7.3. *Let $\pi\colon X\to T$ be a metric fibration whose fibers are contractible n-manifolds for some n. Let P be a contractible $(n+1)$-manifold which is a uniformly contractible, bounded-geometry, metric simplicial complex. Then for any uniformly proper embedding $\phi\colon P\to X$, there exists a unique hyperplane $Q\subset X$ such that $\phi(P)$ and Q have finite Hausdorff distance in X. The bound on Hausdorff distance depends only on the metric fibration data for π, the uniform contractibility data and bounded geometry data for P, and the uniform properness data for ϕ.*

Proof. Uniqueness of Q follows obviously from the fact that distinct hyperplanes in X have infinite Hausdorff distance.

For existence of Q we follow closely the proof of Proposition 4.1 of [FM1], concentrating on details needed to explicate the difference between the "quasi-isometric" setting of [FM1] and the present "uniformly proper" setting.

Using the bounded geometry of P, uniform contractibility of X, and uniform properness of ϕ, we may replace ϕ by a continuous, uniformly proper map, moving values of ϕ a bounded distance. Henceforth we shall assume that ϕ is continuous.

Pick a topologically proper embedding of T in an open disc D. For each component U of $D-T$, the frontier of U in D is a bi-infinite line $L(U)$ in T. There is a homeomorphism of pairs $(\overline{U},L(U))\approx(L(U)\times[0,\infty),L(U)\times 0)$.

Consider the topologically proper embedding

$$X\xrightarrow{\Theta}F\times T\hookrightarrow F\times D.$$

Note that $F\times D$ is a contractible $(n+2)$-manifold. For each component U of $D-T$ we have a homeomorphism

$$F\times\bar{U}\xrightarrow{\approx}F\times(L(U)\times[0,\infty))\xrightarrow{\approx}(F\times L(U))\times[0,\infty)\xrightarrow[\Theta\times\mathrm{Id}]{\approx}X_{L(U)}\times[0,\infty).$$

The frontier of this set in $F\times D$ is $F\times L(U)\approx X_{L(U)}$. Put a product metric and a product simplicial structure on $X_{L(U)}\times[0,\infty)$ and glue to $F\times L(U)$. Doing this for each U, we impose a proper, geodesic metric on $F\times D$ for which the inclusion $X\hookrightarrow F\times D$ is an isometric embedding.

The simplicial structure on $F\times D$ evidently has bounded geometry. Also, the metric space $F\times D$ is uniformly contractible. To see this, let $A\subset F\times D$ have diameter $\leqslant r$. If $A\cap X\neq\varnothing$ then homotoping along product lines of $X_{L(U)}\times[0,\infty)$ for each U we obtain an r-homotopy of A into $F\times T\approx X$, and then we use uniform contractibility of X. Whereas if $A\cap X=\varnothing$, then $A\subset F\times U\approx X_{L(U)}\times(0,\infty)$ for some component U of $D-T$; there is an r-homotopy of A into some $X_{L(U)}\times x$, and the latter is uniformly contractible by Lemma 7.2.

We now plug this setup into the coarse separation and packing methods of Farb–Schwartz [FS] and Schwartz [S]. We will use a generalization of the coarse separation theorem with more easily applied hypotheses, due to Kapovich–Kleiner [KK]. We denote the r-ball about a subset A of a metric space M by $B_r(A;M)$. In a metric space Z, a subset $U\subset Z$ is *deep in* Z if for each $r>0$ there exists $x\in U$ such that $B_r(x;Z)\subset U$. A subset $A\subset Z$ *coarsely separates* Z if for some $D>0$ there are at least two components of $Z-N_D(A;Z)$ which are deep in Z; the constant D is called a *coarse separation constant* for A. Note that if subsets A and B of Z have bounded Hausdorff distance from each other, then A coarsely separates Z if and only if B does.

Here is an elementary consequence of the definitions:

LEMMA 7.4. *Let $f:X\to Y$ be a quasi-isometry between geodesic metric spaces. If $A\subset X$ coarsely separates X then $f(A)$ coarsely separates Y, with separation constant depending only on the quasi-isometry constants of f and the separation constant for A.*

Here is the version of the coarse separation theorem that we will use.

THEOREM 7.5 ([KK]). *Let P be a contractible $(n+1)$-manifold, Z a contractible $(n+2)$-manifold, and suppose that P,Z are uniformly contractible, bounded-geometry, metric simplicial complexes. Let $\Phi:P\to Z$ be a uniformly proper map. Then $\Phi(P)$ coarsely separates Z, with coarse separation constant D depending only on the uniform contractibility and bounded geometry data for P and Z and the uniform properness data for Φ. Moreover, if Φ is continuous then we may take $D=0$, that is, $Z-\Phi(P)$ has at least two components which are deep in Z.*

Remark. In fact there are exactly two components of $Z-N_D(\Phi(P);Z)$ which are deep in Z (see [KK]).

Following [FS] we have a corollary:

THEOREM 7.6 (packing theorem). *Let Q,P be contractible $(n+1)$-manifolds which are uniformly contractible, bounded-geometry, metric simplicial complexes. Let $\psi\colon Q\to P$ be a uniformly proper map. Then there exists $R>0$ such that $N_R(\psi(Q);P)=P$. The constant R depends only on the uniform contractibility data and bounded geometry data for Q,P and the uniform properness data for ψ.*

Proof. If no such R exists then the image of the map

$$Q\xrightarrow{\psi}P\hookrightarrow P\times\mathbf{R}$$

does not coarsely separate $P\times\mathbf{R}$, violating Theorem 7.5. □

Continuing with the proof of Theorem 7.3, compose the continuous, uniformly proper map $\phi\colon P\to X$ with the isometric embedding $X\to F\times D$ to obtain a continuous, uniformly proper map $\Phi\colon P\to F\times D$. By the coarse separation theorem it follows that $(F\times D)-\Phi(P)$ has at least two components which are deep in $F\times D$.

Now take the argument of [FM1, Step 1, pp. 426–427] and apply it verbatim, to produce a hyperplane $Q\subset X$ such that $Q\subset\Phi(P)$. Next take the argument of [FM1, Step 2, pp. 427–428] and apply it verbatim, replacing "quasi-isometric embeddings" with "uniformly proper maps" and using the packing theorem above, to show the existence of R' such that $\phi(P)\subset N_{R'}(Q;X)$, where R' depends only on the metric fibration data for π, the uniform contractibility and bounded geometry data for P, and the uniform properness data for ϕ.

This finishes the proof of Theorem 7.3 and of Step 1. □

Step 2: A quasi-isometry takes hyperplanes and horizontal leaves in X_M to hyperplanes and horizontal leaves in X_N. Consider integral matrices $M,N\in\mathrm{GL}_\times(n,\mathbf{R})$ with $\det M,\det N>1$, and let $f\colon X_M\to X_N$ be a quasi-isometric embedding. Then there is a constant $A\geqslant 0$, depending only on X_M,X_N and the quasi-isometry constants of f, such that:

(1) For each hyperplane $P\subset X_M$ there exists a unique hyperplane $Q\subset X_N$ such that $d_{\mathcal{H}}(f(P),Q)\leqslant A$;

(2) For each horizontal leaf L of X_M there exists a horizontal leaf L' of X_N such that $d_{\mathcal{H}}(f(L),L')\leqslant A$.

The proof of this step is the first place in our arguments where the assumption that $\det M,\det N>1$ is crucial. Again we will investigate this step in the general setting of metric fibrations over trees.

Consider a metric fibration $\pi: X \to T$. The tree T is *bushy* if there exists a constant β such that each point of T is within distance β of some vertex v such that $T-v$ has at least 3 unbounded components. Note that if M is an integer matrix in $\mathrm{GL}_{\times}(n, \mathbf{R})$, and if $X_M \to T_M$ is the associated metric fibration over the Bass–Serre tree T_M of the group Γ_M, then T_M is bushy if and only if $\det M > 1$. In fact, for any graph of finitely generated groups, the Bass–Serre tree is either bounded, quasi-isometric to a line or bushy, and the question of which alternative holds is easily decided by inspection of the graph of groups.

Here is our generalization of Step 2:

THEOREM 7.7. *Let $\pi: X \to T, \pi': X' \to T'$ be metric fibrations over β-bushy trees T, T' such that the fibers of π and π' are contractible n-manifolds for some n. Let $f: X \to X'$ be a quasi-isometry. Then there exists a constant A, depending only on the metric fibration data of π, π', the quasi-isometry data for f, and the constant β, such that:*

(1) *For each hyperplane $P \subset X$ there exists a unique hyperplane $Q \subset X'$ such that $d_{\mathcal{H}}(f(P), Q) \leqslant A$;*

(2) *For each horizontal leaf $L \subset X$ there is a horizontal leaf $L' \subset X'$ such that $d_{\mathcal{H}}(f(L), L') \leqslant A$.*

Remark. This result is used in [MSW] to prove quasi-isometric rigidity for fundamental groups of geometrically homogeneous graphs of groups whose vertex groups are fundamental groups of manifolds in a category $\mathcal{C}$ as above, as long as that class of groups is itself quasi-isometrically rigid. For example, quasi-isometric rigidity is proved for graphs of $\mathbf{Z}$'s, $\mathbf{Z}^n$'s, surface groups, lattices in semisimple Lie groups, nilpotent groups, etc.

Proof. To prove (1), by Lemma 7.2 the inclusion map $P \hookrightarrow X$ is uniformly proper and P is uniformly contractible, and clearly P is a contractible $(n+1)$-manifold. Composing with f we obtain a uniformly proper map $P \to X'$. Now apply Theorem 7.3.

The idea of the proof of (2) is that bushiness of the tree allows one to gain quasi-isometric control over horizontal leaves by considering them as "coarse intersections" of hyperplanes.

Definition (*coarse intersection*). A subset W of a metric space X is a *coarse intersection* of subsets $U, V \subset X$, denoted $W = U \cap_C V$, if there exists K_0 such that for every $K \geqslant K_0$ there exists $K' \geqslant 0$ so that

$$d_{\mathcal{H}}(\mathrm{Nbhd}_K(U) \cap \mathrm{Nbhd}_K(V), W) \leqslant K'.$$

Note that although such a set W may not exist, when it does exist then any two such sets are a bounded Hausdorff distance from each other.

The following fact is an elementary consequence of the definitions.

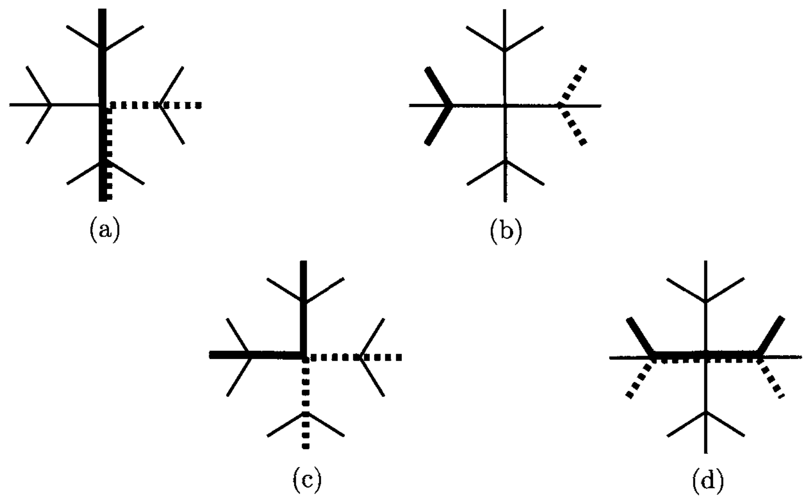


Fig. 1. Possible coarse intersections of distinct hyperplanes in X, projected to T. In (a), $P_1 \cap_C P_2 = P_1 \cap P_2$ is a half-plane. In (b)–(d), $P_1 \cap_C P_2$ is a horizontal leaf; $P_1 \cap P_2$ can be: (b) empty, (c) a horizontal leaf or (d) a finite strip of horizontal leaves.

LEMMA 7.8. *For any quasi-isometry $f: X \to Y$ of metric spaces, and $U, V \subset X$, if $U \cap_C V$ exists then $f(U \cap_C V)$ is a coarse intersection of $f(U), f(V)$, with constants depending only on the quasi-isometry constants for f and the coarse intersection constants for U and V.*

Consider now a metric fibration $\pi: X \to T$. A subset of X of the form $X_\sigma = \pi^{-1}(\sigma)$, where σ is an infinite ray in T, will be called a *half-plane* in X. The next lemma is an easy observation—see Figure 1.

LEMMA 7.9. *Let $\pi: X \to T$ be a metric fibration over a tree T. Let P_1 and P_2 be distinct hyperplanes in X. Then $P_1 \cap_C P_2$ exists and is a bounded Hausdorff distance from either a half-plane or a horizontal leaf in X. Moreover, $P_1 \cap_C P_2$ is a bounded Hausdorff distance from a half-plane if and only if $P_1 \cap P_2$ is a half-plane.*

We remark that $P_1 \cap_C P_2$ can be an arbitrarily large finite Hausdorff distance from a horizontal leaf; see Figure 1 (b), (d).

LEMMA 7.10. *Let $\pi: X \to T$, $\pi': X' \to T'$ be metric fibrations. Let $f: X \to X'$ be a quasi-isometry. Suppose that P_1 and P_2 are distinct hyperplanes in X which intersect in a half-plane. Then $f(P_1)$ and $f(P_2)$ are a uniformly bounded Hausdorff distance from distinct hyperplanes Q_1, Q_2 in X' which intersect in a half-plane in X'.*

Proof. By Theorem 7.3, there exists a constant A so that $f(P_i)$ is within Hausdorff

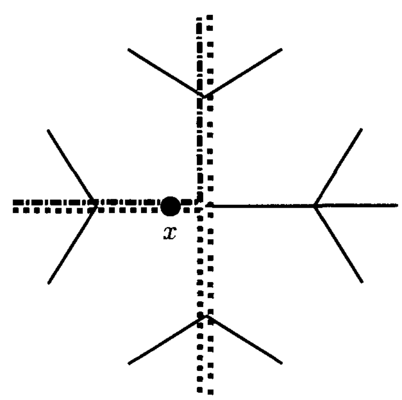


Fig. 2. Any point $x \in T$ is a bounded distance β from a vertex $v \in T$ that separates T into at least three unbounded components. The vertex v is the (coarse) intersection of three proper lines l_1, l_2, l_3 such that the pairwise intersections $l_1 \cap l_2, l_2 \cap l_3, l_3 \cap l_1$ are rays in T, any two of which have infinite Hausdorff distance. Moreover, $d(x, l_1 \cap l_2 \cap l_3) \leqslant \beta$.

distance A of a unique hyperplane Q_i in X'. Since P_1, P_2 are distinct they have infinite Hausdorff distance, so Q_1 and Q_2 have infinite Hausdorff distance, and hence $Q_1 \neq Q_2$.

By Lemma 7.9, it is enough to prove that $Q_1 \cap_C Q_2$ is not a bounded Hausdorff distance from a horizontal leaf in X'. If $Q_1 \cap_C Q_2$ is a bounded Hausdorff distance from a horizontal leaf, then since any horizontal leaf in Q_1 coarsely separates Q_1 it must be that $Q_1 \cap_C Q_2$ coarsely separates Q_1. But $P_1 \cap_C P_2$ does not coarsely separate P_1. This contradicts Lemma 7.4. □

We now prove Theorem 7.7. Consider the quasi-isometry $f: X \to X'$. Since T is bushy, any horizontal leaf L in X can be realized as a coarse intersection of three hyperplanes P_1, P_2, P_3 such that the pairwise intersections $P_1 \cap P_2$, $P_2 \cap P_3$, $P_3 \cap P_1$ form three half-planes, any two of which have infinite Hausdorff distance. Moreover,

$$d_{\mathcal{H}}(L, P_1 \cap P_2 \cap P_3) \leqslant \beta$$

where β is a bushiness constant for T (see Figure 2).

Consider the unique hyperplane Q_i which lies a Hausdorff distance of at most A from $f(P_i)$, $i=1,2,3$. By Lemma 7.10, the pairwise intersections $Q_1 \cap Q_2, Q_2 \cap Q_3, Q_3 \cap Q_1$ are all half-planes, any two of which have infinite Hausdorff distance. The following elementary fact about trees, applied to T', now shows that $Q_1 \cap Q_2 \cap Q_3$ is a horizontal leaf L' in X':

Fact about trees. Let l_1, l_2, l_3 be bi-infinite lines in a simplicial tree T', such that the pairwise intersections $l_1 \cap l_2, l_2 \cap l_3, l_3 \cap l_1$ are all infinite rays in T', any two of which have infinite Hausdorff distance. Then $l_1 \cap l_2 \cap l_3$ is a vertex of T'.

Since $L \subset N_\beta(P_i)$ it follows that

$$f(L) \subset N_{K\beta+C}(f(P_i)) \subset N_{K\beta+C+A}(Q_i), \quad i=1,2,3.$$

But clearly we have $\bigcap_{i=1}^{3} N_{K\beta+C+A}(Q_i) = N_{K\beta+C+A}(L')$.

To summarize, given a horizontal leaf L of X, we have found a horizontal leaf L' of X' such that $L \subset N_{A'}(L')$ where $A' = K\beta + C + A$. A similar argument using a coarse inverse for f provides the desired bound for $d_{\mathcal{H}}(f(L), L')$. This completes the proofs of Theorem 7.7 and of Step 2. □

Step 3: *A quasi-isometry takes coherent hyperplanes in X_M to coherent hyperplanes in X_N.* Let M, N be as in the statement of Proposition 7.1, and fix a quasi-isometry $f: X_M \to X_N$.

Let P be any coherent hyperplane in X_M. By Step 2 it follows that $f(P)$ is within a Hausdorff distance A from a unique hyperplane Q in X_N. By composing $f|P$ with vertical projection $X_N \to Q$ we obtain a map $\phi: P \to Q$. The inclusion maps $P \hookrightarrow X_M$ and $Q \hookrightarrow X_N$ are coarsely Lipschitz and uniformly proper; indeed they are isometric embeddings with respect to the induced path metrics on P, Q. By Lemma 2.1, ϕ is a quasi-isometry, with quasi-isometry constants depending only on those for f. By Step 2, f coarsely respects the horizontal foliations of X_M and X_N; vertical projection $X_N \to Q$ takes horizontal leaves to horizontal leaves, and so ϕ coarsely respects the horizontal foliations of P and Q, with a coarseness constant depending only on the quasi-isometry constants of f.

Since P is a coherent hyperplane it is isometric to G_M. Since Q is a hyperplane it is isometric to either G_N or H_N, and we now show that the second possibility cannot occur.

PROPOSITION 7.11. *Given matrices $M, N \in \mathrm{GL}_\times(n, \mathbf{R})$ with $\det M, \det N > 1$, there is no quasi-isometry $\phi: G_M \to H_N$ which coarsely respects horizontal foliations.*

Proof. The idea of the proof is to compare the growth types of the filling area functions for "quasi-vertical bigons" in G_M and in H_N. In G_M this growth type will be quadratic, while in H_N it will be exponential.

Let $H = G_M, H_M, G_N$ or H_N. There is a quotient map $H \to \mathbf{R}$ whose point pre-images give the horizontal foliation of H, and such that the Hausdorff distance between two horizontal leaves equals the distance between the corresponding points in $\mathbf{R}$. A path γ in H is said to be (K, C)-*quasi-vertical* if its projection to $\mathbf{R}$ is a (K, C)-quasi-geodesic. Define a (K, C)-*quasi-vertical bigon* in H to be a pair of (K, C)-quasi-vertical paths γ, γ' which begin and end at the same point.

If K, C are fixed, we define a filling area function $A(L)$ for (K, C)-quasi-vertical bigons in H. Given a (K, C)-quasi-vertical bigon γ, γ', its *filling area* is the infimal area of a Lipschitz map $D^2 \to H$ whose boundary is a reparameterization of the closed curve $\gamma^{-1} * \gamma'$; such a map $D^2 \to H$ is called a *filling disc* for $\gamma^{-1} * \gamma'$. For each $L \geqslant 0$ define $\mathcal{A}(L)$ to be the supremal filling area over all (K, C)-quasi-vertical bigons γ, γ' in H such that $\text{Length}(\gamma) + \text{Length}(\gamma') \leqslant L$.

Suppose that there is a quasi-isometry $\phi: G_M \to H_N$ which coarsely respects horizontal foliations. Let $\bar{\phi}: H_N \to G_M$ be a coarse inverse for ϕ, also coarsely respecting horizontal foliations. Clearly $\bar{\phi}$ takes any (K, C)-quasi-vertical bigon in H_N to a (K', C')-quasi-vertical bigon in G_M, distorting lengths by at worst an affine function; this affine function and the constants K', C' depend only on K, C, the quasi-isometry constants for ϕ, and the Hausdorff constant for the induced height function. Fill the resulting bigon in G_M as efficiently as possible, and map back to H_N via ϕ, distorting area by at worst an affine function which again has the same dependencies. We thereby obtain a filling of the original bigon in H_N. If $\mathcal{A}_1(L)$ denotes the filling area function for (K', C')-quasi-vertical bigons in G_M, and if $\mathcal{A}_2(L)$ denotes the filling area function for (K, C)-quasi-vertical bigons in H_N, it follows that the growth type of $\mathcal{A}_2(L)$ is dominated by the growth type of $\mathcal{A}_1(L)$, that is,

$$\mathcal{A}_2(L) \leqslant \alpha \cdot \mathcal{A}_1(\beta L + \delta) + \zeta$$

for some positive constants $\alpha, \beta, \delta, \zeta$ independent of L.

We shall, however, now show that $\mathcal{A}_1(L)$ has a quadratic upper bound while $\mathcal{A}_2(L)$ has an exponential lower bound, contradicting the above inequality.

Consider a (K', C')-quasi-vertical bigon γ, γ' in G_M. Applying the argument of Claim 5.7, there are center leaves τ, τ' in G_M and quasi-vertical paths $\varrho \subset \tau$, $\varrho' \subset \tau'$ which stay uniformly close to γ, γ' respectively. The initial points of ϱ, ϱ' are at a uniformly bounded distance, as are the terminal points, and it follows that ϱ' stays uniformly close to a quasi-vertical path $\varrho'' \subset \tau$. Connecting initial and terminal endpoints with short paths η, η' we thus obtain a closed curve $\varrho^{-1} * \eta * \varrho'' * \eta'$, contained in a center leaf of G_M, which stays uniformly close to $\gamma^{-1} * \gamma'$. Since center leaves of G_M are isometric to Euclidean space, in which the filling function is quadratic, it follows that $\mathcal{A}_1(L)$ has a quadratic upper bound.

To show that $\mathcal{A}_2(L)$ has an exponential lower bound, we now construct quasi-vertical bigons in H_N which can be filled only by discs of exponential area. In the case where N is a (1×1)-matrix such loops are given explicitly in [E, Chapter 7.4]; examples for general N are simple modifications of this example. To be explicit, choose an eigenvalue of N of absolute value $\alpha > 1$; such an eigenvalue exists because $\det N > 1$. Choose an affine subspace $A\subset \mathbf{R}^n$ parallel to the α-eigenspace of N. Consider the subspace $A\times\mathbf{R}\subset \mathbf{R}^n\times\mathbf{R}\approx G_N$.

For each fixed $L\geqslant 0$, choose two vertical segments g, g' in $A\times[0,\infty)$ whose upper endpoints are in $A\times L$ and whose lower endpoints are in $A\times 0$, and so that the distance in $A\times L$ between the upper endpoints, measured using the Riemannian metric on G_N, is equal to 1; it follows that the distance in $A\times 0$ between the lower endpoints, measured using the Riemannian metric on G_N, is within a constant multiple of α^L.

Now double this picture, in the doubled G_N-horoball H_N, to get a closed loop in H_N, that is: in one horoball go up g, across 1 unit, and down g', and then in the other horoball go up g', across 1 unit, and down g; let ϱ be the resulting closed curve in H_N. We have $\text{Length}(\varrho)=4L+2$. To see that the filling area of ϱ is exponential in L, note that any filling disc for ϱ must contain a path in $A\times 0$ connecting the lower endpoints of g, g', because $A\times 0$ separates the two halves of ϱ in H_N. This path has length exponential in L; and a neighborhood of this path in the filling disc has area exponential in L. □

Step 4: *A horizontal-respecting quasi-isometry preserves transverse orientation.* Let M, N and $f\colon \Gamma_M\to\Gamma_N$ be as in the statement of Proposition 7.1. By Step 3 there is a quasi-isometry $\phi\colon G_M\to G_N$, and by Step 2 ϕ coarsely respects the horizontal foliations of G_M and G_N. Suppose that ϕ reverses the transverse orientation. There is a quasi-isometry $G_N\to G_{N^{-1}}$ which coarsely respects horizontal foliations, *reversing* transverse orientations. Precomposing with $\phi\colon G_M\to G_N$ and applying Steps 1–3, we obtain a quasi-isometry $G_M\to G_{N^{-1}}$ which coarsely respects the transversely oriented horizontal foliations. Applying Theorem 5.2, it follows that M and N^{-1} have positive real powers with the same absolute Jordan form, and so these powers also have the same determinant. But each positive power of M has determinant >1, whereas every positive power of N^{-1} has determinant <1, a contradiction showing that ϕ must preserve the transverse orientation.

This completes the proof of Proposition 7.1. □

Remark. Note in the proof of Proposition 7.1 that different choices of coherent hyperplanes in X_M yield different quasi-isometries ϕ. In some cases ϕ is well defined up to some constant A, that is, for any two choices of coherent hyperplane in X_M, the induced maps $\phi_1,\phi_2\colon G_M\to G_N$ satisfy $\sup_x d(\phi_1(x),\phi_2(x))\leqslant A$. This is true, for example, in the "centerless" case where M, N have no eigenvalues on the unit circle. In the general case, the best that can be said is that the map induced by ϕ from the center leaf space of G_M to the center leaf space of G_N is well defined up to a constant, with respect to the Hausdorff metrics on the center leaf spaces.

8. Finding the integers

In this section we prove Theorem 1.1. Let M, N be integral $(n \times n)$-matrices with $|\det M|, |\det N| > 1$. We must prove that Γ_M is quasi-isometric to Γ_N if and only if there exist positive integers a, b such that M^a and N^b have the same absolute Jordan form.

First we show that the groups Γ_{M^a} and Γ_M are quasi-isometric for any positive integer a, by showing that Γ_{M^a} is a subgroup of finite index in Γ_M, specifically of index a. To see why, consider the presentations

$$\Gamma_M = \langle \mathbf{Z}^n, t \mid t^{-1}xt = M(x), x \in \mathbf{Z}^n \rangle,$$
$$\Gamma_{M^a} = \langle \mathbf{Z}^n, s \mid s^{-1}xs = M^a(x), x \in \mathbf{Z}^n \rangle.$$

Define a homomorphism $\Gamma_M \mapsto \mathbf{Z}/a\mathbf{Z}$ by $\mathbf{Z}^n \mapsto 0$, $t \mapsto 1$. This homomorphism is onto, and its kernel is generated by $\mathbf{Z}^n, t^a$. This kernel is isomorphic to Γ_{M^a} under the injection $\Gamma_{M^a} \hookrightarrow \Gamma_M$ given by $x \mapsto x$, $s \mapsto t^a$.

Similarly, Γ_{N^b} is quasi-isometric to Γ_N for any positive integer b.

By squaring M, N if necessary, we may therefore assume that $\det M, \det N > 1$, and that M and N lie on 1-parameter subgroups; we continue with this assumption up through the end of the proof in §8.2. Choose 1-parameter subgroups M^t, N^t of $\mathrm{GL}(n, \mathbf{R})$ with $M = M^1$, $N = N^1$, let G_M, G_N be the associated Lie groups constructed in §4, and let X_M, X_N be the associated geodesic metric spaces constructed in §7. The group Γ_M is quasi-isometric to X_M, and Γ_N is quasi-isometric to X_N.

8.1. The first half of the classification

Assuming that M^a and N^b have the same absolute Jordan form, where a, b are positive integers, we must prove that Γ_M and Γ_N are quasi-isometric. We have shown above that Γ_{M^a} and Γ_M are quasi-isometric, and that Γ_{N^b} and Γ_N are quasi-isometric. Replacing M by M^a and N by N^b, we may therefore assume that M, N have the same absolute Jordan form. We shall prove that Γ_M, Γ_N are quasi-isometric by constructing a bi-Lipschitz homeomorphism between X_M and X_N.

Since the absolute Jordan forms of M, N are equal, it follows that $\det M = \det N$; let d be the common value. Applying Proposition 4.1, there is a bi-Lipschitz homeomorphism from $G_M = \mathbf{R}^n \rtimes_M \mathbf{R}$ to $G_N = \mathbf{R}^n \rtimes_M \mathbf{R}$ of the form $(x, t) \mapsto (Ax, t)$ for some $A \in \mathrm{GL}(n, \mathbf{R})$. In the fiber product description of X_M, X_N, the trees T_M and T_N may both be identified with the homogeneous, oriented tree T_d with one incoming and d outgoing edges at each vertex. The bi-Lipschitz homeomorphism $G_M \to G_N$ and the identity homeomorphism

$T_d \to T_d$ both respect the height functions, and so these two homeomorphisms combine to give the desired bi-Lipschitz homeomorphism $X_M \to X_N$.

8.2. Quasi-isometric implies that integral powers have the same absolute Jordan forms

Assuming that Γ_M, Γ_N are quasi-isometric, there is a quasi-isometry $f: X_M \to X_N$. Combining Proposition 7.1 and Theorem 5.2 gives $r \in \mathbf{R}_+$ such that M^r and N have the same absolute Jordan form. We must show that there exist $a, b \in \mathbf{Z}_+$ so that M^a and N^b have the same absolute Jordan form.

Since M^r and N have the same absolute Jordan form, listing the absolute values of the eigenvalues of M and N in increasing order we obtain

$$\mu_{-a} \leqslant ... \leqslant \mu_0 \leqslant 1 < \mu_1 =: \alpha_M \leqslant ... \leqslant \mu_b,$$
$$\nu_{-a} \leqslant ... \leqslant \nu_0 \leqslant 1 < \nu_1 =: \alpha_N \leqslant ... \leqslant \nu_b,$$

with $\mu_i^r = \nu_i$, $-a \leqslant i \leqslant b$. From this it follows that

$$\frac{\log \alpha_N}{\log \alpha_M} = r = \frac{\log \det N}{\log \det M}.$$

Let $\mathbf{Q}_M$ denote the set of coherent hyperplanes in X_M, and let h_M denote the height function on M. We define a metric on $\mathbf{Q}_M$ as follows: Given coherent hyperplanes P_1, P_2, let L denote the horizontal leaf $L = \partial(P_1 \cap P_2)$. Then we set

$$d_{\mathbf{Q}_M}(P_1, P_2) = (\det M)^{-h_M(L)}.$$

It is easy to check that this defines a metric on $\mathbf{Q}_M$, and since the tree T_M branches $m = \det M$ times as h_M increases by 1, the metric space $(\mathbf{Q}_M, d_{\mathbf{Q}_M})$ is isometric to the m-adic rational numbers in their usual metric of Hausdorff dimension 1. Similarly, attached to X_N is a metric space $(\mathbf{Q}_N, d_{\mathbf{Q}_N})$ isometric to the n-adic rational numbers, with $n = \det N$.

From Step 3 in the proof of Proposition 7.1 (see §7.2), the quasi-isometry $f: X_M \to X_N$ takes each coherent hyperplane in X_M to within a uniform Hausdorff distance of a unique coherent hyperplane in X_N, and hence induces a bijection $\psi: \mathbf{Q}_M \to \mathbf{Q}_N$. For each $l \in \mathbf{Q}_M$, setting $l' = \psi(l)$, there is an induced horizontal-respecting quasi-isometry $P_l \to P'_{l'}$, and by time rigidity (Proposition 5.8) this quasi-isometry has an induced time change of the form $t \mapsto mt + b$ where

$$m = \frac{\log \alpha_M}{\log \alpha_N} = \frac{1}{r}$$

and where b depends ostensibly on l. For another l_1, however, P_l and P_{l_1} coincide below some value of t, and so $t\mapsto mt+b$ is an induced time change for both $P_l\mapsto P'_{l'}$ and $P_{l_1}\mapsto P'_{l'_1}$, possibly with a larger coarseness constant (this argument is taken from Claim 6.3 on p. 436 of [FM1]). Therefore, there is a uniform induced time change $t\mapsto mt+b$ with b independent of l, and with a uniform Hausdorff constant A.

We now claim that ψ is a bi-Lipschitz homeomorphism. To this end, let $P_1, P_2 \in \mathbf{Q}_M$ be given. Let $L=\partial(P_1\cap P_2)$ and $L'=\partial(\psi(P_1)\cap\psi(P_2))$. Then

$$h_N(L')\geqslant m\cdot h_M(L)+b-A,$$

and so

$$\begin{aligned}\frac{d_{\mathbf{Q}_N}(\psi(P_1),\psi(P_2))}{d_{\mathbf{Q}_M}(P_1,P_2)}&=\frac{(\det N)^{-h_N(L')}}{(\det M)^{-h_M(L)}}\leqslant\frac{(\det N)^{-mh_M(L)-b+A}}{(\det M)^{-h_M(L)}}\\&=\frac{((\det N)^{\log\det M/\log\det N})^{-h_M(L)}(\det N)^{-b+A}}{(\det M)^{-h_M(L)}}=(\det N)^{-b+A},\end{aligned}$$

which is a constant not depending on P_1 or P_2. Hence ψ is Lipschitz. The same argument applied to ψ^{-1} shows that ψ is bi-Lipschitz.

Applying Cooper's theorem [FM1, Appendix, Corollary 10.11] on bi-Lipschitz homeomorphisms of Cantor sets, we obtain that there exist integers $a, b>0$ such that $(\det M)^a=(\det N)^b$. Since M^r and N have the same absolute Jordan form, we have

$$\frac{b}{a}=\frac{\log\det M}{\log\det N}=r,$$

and so $(M^r)^a=M^b$ and N^a have the same absolute Jordan form.

9. Quasi-isometric rigidity

In this section we prove Theorem 1.2 in a series of steps. Recall the hypotheses: M is an integer matrix in $\mathrm{GL}(n,\mathbf{R})$ with $|\det M|>1$, and G is a finitely generated group quasi-isometric to Γ_M. By squaring M if necessary we may assume that $M\in\mathrm{GL}_{\times}(n,\mathbf{R})$ and $\det M>1$, and therefore Γ_M is quasi-isometric to X_M. It follows that G is quasi-isometric to X_M.

Step 1. The action of G on itself by left multiplication can be conjugated by the quasi-isometry $G\to X_M$ to give a proper, cobounded quasi-action of G on X_M (see [FM2, Proposition 2.1]). Since $\det M>1$ we may apply Theorem 7.7, concluding that the quasi-action of G on X_M coarsely respects the fibers of the uniform metric fibration $X_M\to T_M$.

Step 2. Now we use the following result of [MSW]. Suppose that $\pi: X \to T$ is a uniform metric fibration over a bushy tree T. If G is a finitely presented group with a cobounded, proper quasi-action on X, and if the quasi-action coarsely respects the fibers, then G is the fundamental group of a graph of groups whose vertex and edge groups are quasi-isometric to a fiber $X_t = \pi^{-1}(t)$.

By Step 1, this result applies to the quasi-action of G on X_M, because G is quasi-isometric to the finitely presented group Γ_M, and so G is finitely presented. The fibers of the map $X_M \to T_M$ are isometric to $\mathbf{R}^n$, and it follows that G is the fundamental group of a graph of groups with each vertex and edge group quasi-isometric to $\mathbf{R}^n$.

Step 3. Any finitely generated group quasi-isometric to $\mathbf{R}^n$ is virtually $\mathbf{Z}^n$ (see [Ge2]), and so G is the fundamental group of a graph of groups whose vertex and edge groups are virtually $\mathbf{Z}^n$.

Step 4. Applying the argument in §5 of [FM2] to G gives that either G contains a noncyclic free group or G is an ascending HNN extension of the form

$$G = A_\phi = \langle A, t \mid tat^{-1} = \phi(a), \forall a \in A \rangle,$$

where A is virtually $\mathbf{Z}^n$ and $\phi: A \to A$ is an injective endomorphism. Since Γ_M is amenable, and since G is quasi-isometric to Γ_M, then G is amenable, and so G cannot contain a noncyclic free group. The second possibility must therefore occur: $G = A_\phi$ as above.

Step 5. Now we turn to an analysis of injective endomorphisms of virtually abelian groups. Suppose that A is a finitely generated, virtually abelian group. Any injective endomorphism of A has finite index image.

A subgroup $B \subset A$ is *characteristic for endomorphisms* if, for any injective endomorphism $\phi: A \to A$, we have $\phi(B) \subset B$.

Given a group A and $g \in A$, the centralizer of g in A is denoted $C_A(g)$. The *virtual center* of A, denoted $V(A)$, is the set of all $g \in A$ such that $[A : C_A(g)] < \infty$. This is a subgroup, because if $g, h \in V(A)$ then the subgroup $C_A(gh)$, which contains $C_A(g) \cap C_A(h)$, has finite index.

LEMMA 9.1 (some characteristic subgroups). *Let A be a finitely generated, virtually abelian group. Then the virtual center $V(A)$, its center $ZV(A)$, and its torsion subgroup $TZV(A)$, are all characteristic for endomorphisms of A. Moreover, $V(A)$ and $ZV(A)$ both have finite index in A, whereas $TZV(A)$ is finite.*

Lemma 9.1 is proved below.

Step 6. Consider the HNN extension $G = A_\phi$ above. Let $V(A)$, $ZV(A)$, $TZV(A)$ be as in Lemma 9.1, so that all these subgroups are taken into themselves by ϕ. Since

$TZV(A)$ is finite we in fact have $\phi(TZV(A))=TZV(A)$, and so $K=TZV(A)$ is a finite, normal subgroup of G.

Replacing G by G/K, we may assume that $TZV(A)$ is trivial, and it follows that $ZV(A)$ is torsion-free, abelian, and so is isomorphic to $\mathbf{Z}^n$. Since $\phi(ZV(A))\subset ZV(A)$, the action of ϕ on $ZV(A)$ is given by some $(n\times n)$-matrix of integers N. Thus, G/K has a finite-index subgroup isomorphic to Γ_N, finishing the proof of Theorem 1.2.

Proof of Lemma 9.1. To see $[A:V(A)]<\infty$, note that if B is any finite-index abelian subgroup of A then obviously $B\subset V(A)$.

Consider an endomorphism $\phi: A\to A$. We now show that $\phi(V(A))\subset V(A)$. Consider $g\in V(A)$, so that $[A:C_A(g)]<\infty$. It follows that $[\phi(A):C_{\phi(A)}(\phi(g))]<\infty$, and so $[A:C_{\phi(A)}(\phi(g))]<\infty$. But $C_{\phi(A)}(\phi(g))\subset C_A(\phi(g))$, and so $\phi(g)\in V(A)$.

Next we claim that $V(V(A))=V(A)$. To see why, note that if $g\in V(A)$ then we have $[A:C_G(g)]<\infty$, and so $[V(A):C_G(g)\cap V(A)]<\infty$. But $C_G(g)\cap V(A)\subset C_{V(A)}(g)$, and so $[V(A):C_{V(A)}(g)]<\infty$, i.e. $g\in V(V(A))$.

Next we claim that $[V(A):ZV(A)]<\infty$. In fact, if V is any finitely generated group which is its own virtual center, then $[V:ZV]<\infty$ (the converse is also true, trivially). To see why, let $g_1,\dots,g_k$ be a generating set for V. Since $V(V)=V$, each of the groups $C_V(g_1),\dots,C_V(g_k)$ has finite index in V. It follows that their intersection has finite index in V; but their intersection is precisely ZV.

Now we claim that $ZV(A)$ is characteristic for endomorphisms of $V(A)$ (and so is also characteristic for endomorphisms of A). In fact, if V is any finitely generated group whose center ZV has finite index, then ZV is characteristic for any injective endomorphism $\phi: V\to V$ whose image has finite index. To see why, note that $Z(\phi(V))=\phi(ZV)$, and so

$$[\phi(V):Z(\phi(V))]=[\phi(V):\phi(ZV)]=[V:ZV]<\infty.$$

Clearly $\phi(V)\cap ZV\subset Z(\phi(V))$, and so

$$[\phi(V):Z(\phi(V))]\leqslant[\phi(V):\phi(V)\cap ZV].$$

The quotient group V/ZV is finite, and the quotient homomorphism $V\to V/ZV$, when restricted to the subgroup $\phi(V)$, has kernel $\phi(V)\cap ZV$. It follows that

$$[\phi(V):\phi(V)\cap ZV]\leqslant|V/ZV|=[V:ZV]=[\phi(V):Z(\phi(V))].$$

All of the above inequalities are therefore equalities, and so

$$\phi(ZV)=Z(\phi(V))=\phi(V)\cap ZV,$$

which implies $\phi(ZV)\subset ZV$.

Finally, it is clear that for any finitely generated abelian group, the torsion subgroup is characteristic for injective endomorphisms.

10. Questions

10.1. Remarks on the polycyclic case

Given an integer matrix $M\in\mathrm{GL}(n,\mathbf{R})$, the group Γ_M is polycyclic if and only if $|\det M|=1$, and if $M\in\mathrm{GL}_\times(n,\mathbf{R})$ this occurs if and only if Γ_M is a cocompact *discrete* subgroup of G_M. In this case it follows that Γ_M is quasi-isometric to G_M, and the notion of horizontal-respecting quasi-isometry clearly transfers to Γ_M. The techniques of this paper do not provide a quasi-isometric classification in this case, but they do however yield the following partial result:

THEOREM 10.1. *If $M, N\in\mathrm{SL}(n,\mathbf{Z})$ lie on 1-parameter subgroups of $\mathrm{GL}(n,\mathbf{R})$, then there is a horizontal-respecting quasi-isometry $\Gamma_M\to\Gamma_N$ if and only if there is a horizontal-respecting quasi-isometry $G_M\to G_N$, and this occurs if and only if there are real numbers $a, b\neq 0$ such that M^a, N^b have the same absolute Jordan form.*

This raises the question: Is every quasi-isometry $\Gamma_M\to\Gamma_N$ horizontal-respecting? Equivalently, is every quasi-isometry $G_M\to G_N$ horizontal-respecting? The answer is obviously no, for example when M, N are identity matrices and G_M, G_N are Euclidean spaces. We conjecture, however:

CONJECTURE 10.2. *If $M, N\in\mathrm{SL}(n,\mathbf{Z})$ lie on 1-parameter subgroups of $\mathrm{GL}(n,\mathbf{R})$, and if M, N have no eigenvalues on the unit circle, then any quasi-isometry $G_M\to G_N$ is horizontal-respecting.*

Moreover, Theorem 10.1 and Conjecture 10.2 together would imply the following (see [FM4])

CONJECTURE 10.3. *Suppose that $M\in\mathrm{SL}(n,\mathbf{Z})$ has no eigenvalues on the unit circle. If G is any finitely generated group quasi-isometric to Γ_M, then there is a finite normal subgroup F of G so that G/F is abstractly commensurable to Γ_N, for some $N\in\mathrm{SL}(n,\mathbf{Z})$ with no eigenvalues on the unit circle.*

10.2. The quasi-isometry group of Γ_M

Given a finitely generated group G, the set of quasi-isometries from G to itself, modulo the identification of quasi-isometries which differ by a bounded amount, forms a group called the *quasi-isometry group* of G, denoted $\mathrm{QI}(G)$. Given a (1×1)-matrix $M=(m)$ with $m\geqslant 2$, the quasi-isometry group of the solvable Baumslag–Solitar group $\Gamma_M\approx\mathrm{BS}(1,m)$ was computed in [FM1]:

$$\mathrm{QI}(\mathrm{BS}(1,m))\approx\mathrm{Bilip}(\mathbf{R})\times\mathrm{Bilip}(\mathbf{Q}_m)$$

where $\mathbf{Q}_m$ is the metric space of m-adic rational numbers, and $\mathrm{Bilip}(X)$ denotes the group of bi-Lipschitz self maps of a metric space X.

PROBLEM 10.4. *Compute the quasi-isometry group of* Γ_M *in general.*

The strongest result we have on this problem so far is Proposition 6.3, but see the remarks after that proposition.

In [FM2] the computation of $\mathrm{QI}(\mathrm{BS}(1,m))$ was applied to prove quasi-isometric rigidity of $\mathrm{BS}(1,m)$, using techniques of Hinkkanen [Hi] and Tukia [T]. While quasi-isometric rigidity of $\mathrm{BS}(1,m)$ now has a completely different proof [MSW], which we have here generalized to Γ_M, one might still pursue:

PROBLEM 10.5. *Give a proof of quasi-isometric rigidity of* Γ_M*, generalizing the results of* [FM2].

This should lead to a deeper understanding of the geometry of Γ_M. For example, Tukia [T] characterizes subgroups of the quasi-conformal group of a sphere which are conjugate into the Möbius group. We have analogous results for lattices in 3-dimensional SOLV-geometry, and there should be generalizations to solvable Baumslag–Solitar groups and to Γ_M.

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