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Algebraicity of holomorphic mappings between real algebraic sets in $\mathbf{C}^n$

by

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Contents

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0. Introduction

A subset $A \subset \mathbf{C}^N$ is a *real algebraic set* if it is defined by the vanishing of real-valued polynomials in $2N$ real variables; we shall always assume that A is irreducible. By A_{reg} we mean the regular points of A (see e.g. [HP] or [BCR]). Recall that A_{reg} is a real submanifold of $\mathbf{C}^N$, all points of which have the same dimension. We write $\dim A = \dim_{\mathbf{R}} A$ for the dimension of the real submanifold A_{reg}. A germ of a holomorphic function f at a point $p_0 \in \mathbf{C}^N$ is called *algebraic* if it satisfies a polynomial equation of the form

$$a_K(Z) f^K(Z) + \ldots + a_1(Z) f(Z) + a_0(Z) \equiv 0,$$

where the $a_j(Z)$ are holomorphic polynomials in N complex variables with $a_K(Z) \not\equiv 0$. A real-analytic submanifold in $\mathbf{C}^N$ is called *holomorphically degenerate* at $p_0 \in M$ if there exists a germ at p_0 of a holomorphic vector field, with holomorphic coefficients, tangent to M near p_0, but not vanishing identically on M; otherwise, we say that M is *holomorphically nondegenerate* at p_0 (see §1). In this paper, we shall give conditions under which a germ of a holomorphic map in $\mathbf{C}^N$, mapping an irreducible real algebraic set A into another of the same dimension, is actually algebraic. We shall now describe our main results.

THEOREM 1. *Let $A \subset \mathbf{C}^N$ be an irreducible real algebraic set, and p_0 a point in $\overline{A_{\text{reg}}}$, the closure of A_{reg} in $\mathbf{C}^N$. Suppose that the following two conditions hold.*

(1) *A is holomorphically nondegenerate at every point of some nonempty relatively open subset of A_{reg}.*

(2) *If f is a germ, at a point in A, of a holomorphic algebraic function in $\mathbf{C}^N$ such that the restriction of f to A is real-valued, then f is constant.*

Then if H is a holomorphic map from an open neighborhood in $\mathbf{C}^N$ of p_0 into $\mathbf{C}^N$, with $\operatorname{Jac} H \not\equiv 0$, *and mapping A into another real algebraic set A' with* $\dim A' = \dim A$, *necessarily the map H is algebraic.*

We shall show that the conditions (1) and (2) of Theorem 1 are essentially necessary by giving a converse to Theorem 1. For this, we need the following definitions. If M is a real submanifold of $\mathbf{C}^N$ and $p \in M$, let T_pM be its real tangent space at p, and let J denote the anti-involution of the standard complex structure of $\mathbf{C}^N$. We say that M is *CR* (for Cauchy–Riemann) at p if $\dim_{\mathbf{R}}(T_qM \cap JT_qM)$ is constant for q in a neighborhood of p in M. If M is CR at p, then $\dim_{\mathbf{R}} T_pM \cap JT_pM = 2n$ is even and n is called the *CR dimension* of M at p. We shall say that an algebraic manifold $M \subset \mathbf{C}^N$ is *homogeneous* if it is given by the vanishing of $N - \dim M$ real-valued polynomials, whose differentials are linearly independent at 0, and which are homogeneous with respect to some set of weights (see §3.6).

THEOREM 2. *Let $A \subset \mathbf{C}^N$ be an irreducible real algebraic set, and let* (1) *and* (2) *be the conditions of Theorem* 1. *Consider the following property.*

(3) *For every $p_0 \in A_{\mathrm{reg}}$ at which A is CR there exists a germ of a nonalgebraic biholomorphism H of $\mathbf{C}^N$ at p_0 mapping A into itself with $H(p_0)=p_0$.*

If (1) *does not hold then* (3) *holds. If* (1) *holds, but* (2) *does not hold, let f be a nonconstant holomorphic function whose restriction to A is real-valued. If f vanishes identically on A, then* (3) *holds. If f does not vanish identically on A, but A is a homogeneous CR submanifold of $\mathbf{C}^N$, then* (3) *still holds.*

We shall give another version of conditions (1) and (2) of Theorem 1, which will give a reformulation of Theorems 1 and 2. For a CR submanifold M of $\mathbf{C}^N$, we say that M is *minimal* at $p_0 \in M$ if there is no germ of a CR submanifold in $\mathbf{C}^N$ through p_0 with the same CR dimension as M at p_0, and properly contained in M. A CR submanifold is called *generic* at p if

$$T_pM + JT_pM = T_p\mathbf{C}^N, \tag{0.1}$$

where $T_p\mathbf{C}^N$ is the real tangent space of $\mathbf{C}^N$. (See §1.1 for more details and equivalent formulations.)

For an irreducible real algebraic subset A of $\mathbf{C}^N$, we let A_{CR} be the subset of points in A_{reg} at which A is CR. The following contains Theorems 1 and 2.

THEOREM 3. *Let $A \subset \mathbf{C}^N$ be an irreducible real algebraic set, and let* (1), (2) *and* (3) *be the conditions of Theorems* 1 *and* 2. *Consider also the following conditions.*

(i) *There exists $p \in A_{\mathrm{CR}}$ at which A is holomorphically nondegenerate.*

(ii) *There exists $p \in A_{\mathrm{CR}}$ at which A is generic.*

(iii) *There exists $p \in A_{\mathrm{CR}}$ at which A is minimal.*

Then condition (i) *is equivalent to condition* (1), *and conditions* (ii) *and* (iii) *together are equivalent to condition* (2). *In particular,* (i), (ii) *and* (iii) *together imply the conclusion of Theorem* 1. *If either* (i) *or* (ii) *does not hold, then* (3) *must hold. If* (iii) *does not hold, and A is a homogeneous CR manifold, then* (3) *must also hold.*

Note that conditions (i), (ii) and (iii) of Theorem 3 are all independent of each other. The following is a corollary of Theorems 1–3.

COROLLARY. *Let $M \subset \mathbf{C}^N$ be a connected real algebraic, holomorphically nondegenerate, generic submanifold. Assume that there exists $p \in M$, such that M is minimal at p. Suppose that A' is a real algebraic set in $\mathbf{C}^{N'}$ such that $\dim_{\mathbf{R}} A' = \dim_{\mathbf{R}} M$ and that H is a holomorphic mapping from an open neighborhood in $\mathbf{C}^N$ of a point $p_0 \in M$ satisfying $H(M) \subset A'$ such that the rank of H is equal to N at some point. Then H is algebraic.*

If M is a real-analytic CR submanifold of $\mathbf{C}^N$ and $p_0 \in M$ (with M not necessarily minimal at p_0), then by Nagano's theorem [N] there exists a real-analytic minimal CR submanifold of M through p_0 of minimum possible dimension (and the same CR dimension as M) contained in M. Such a manifold is called the *CR orbit* of p_0. We call the germ of the smallest complex-analytic manifold of $\mathbf{C}^N$ containing the CR orbit the *intrinsic complexification* of this orbit.

Note that if $\mathcal{V} \subset \mathbf{C}^N$ is a complex algebraic set, i.e. defined by the vanishing of holomorphic polynomials, then one can define the notion of an algebraic holomorphic function on an open subset of $\mathcal{V}_{\mathrm{reg}}$ (see §3.1).

For CR submanifolds which are nowhere minimal, we have the following.

THEOREM 4. *Let M be a real algebraic CR submanifold of $\mathbf{C}^N$ and $p_0 \in M$. Then the CR orbit of p_0 is a real algebraic submanifold of M and its intrinsic complexification, X, is a complex algebraic submanifold of $\mathbf{C}^N$. For any germ H of a biholomorphism at p_0 of $\mathbf{C}^N$ into itself mapping M into another real algebraic manifold of the same dimension as that of M, the restriction of H to X is algebraic.*

The algebraicity of the mapping in Theorem 4 follows from Theorem 1, after it is shown, in the first part of the theorem, that the CR orbits are algebraic. (See Theorem 2.2.1.) We mention here that the algebraic analog of the Frobenius or Nagano theorem does not hold, since the integral curves of a vector field with algebraic coefficients need not be algebraic. It is therefore surprising that the CR orbits of an algebraic CR manifold are algebraic. In §3.1 we formulate and prove Theorem 3.1.2, a more general result containing Theorems 1 and 4, which also applies to points in an algebraic set A at which A is not necessarily CR or even regular, and which, in some cases, yields algebraicity on a larger submanifold than the one obtained in Theorem 4. (See Example 3.1.5.)

Note that if a germ of a holomorphic function is algebraic, it extends as a (multi-valued) holomorphic function in all of $\mathbf{C}^N$ outside a proper complex algebraic subset. This may be viewed as one of the motivations for proving algebraicity of functions and mappings.

We give here a brief history of some previous work on the algebraicity of holomorphic mappings between real algebraic sets. Early in this century Poincaré [P] proved that if a biholomorphism defined in an open set in $\mathbf{C}^2$ maps an open piece of a sphere into another, it is necessarily a rational map. This result was extended by Tanaka [Ta] to spheres in higher dimensions. Webster [W1] proved a far-reaching result for algebraic, Levi-nondegenerate real hypersurfaces in $\mathbf{C}^N$; he proved that any biholomorphism mapping such a hypersurface into another is algebraic. Later, Webster's result was extended in some cases to Levi-nondegenerate hypersurfaces in complex spaces of different dimensions (see e.g. Webster [W2], Forstnerič [Fo], Huang [H] and their references). See also Bedford–Bell [BB] for other results related to this work. We refer the reader in addition to the work of Tumanov and Henkin [TH] and Tumanov [Tu2] which contain results on mappings of higher-codimensional quadratic manifolds. See also related results of Sharipov and Sukhov [SS] using Levi-form criteria; some of these results are special cases of the present work.

It should be perhaps mentioned that the algebraicity results here are deduced from local analyticity in contrast with the general "G.A.G.A. principle" of Serre [Ser], which deals with the algebraicity of global analytic objects.

The results and techniques in the papers mentioned above have been applied to other questions concerning mappings between hypersurfaces and manifolds of higher codimension. We mention here, for instance, the classification of ellipsoids in $\mathbf{C}^N$ proved in [W1] (see also [W3] for related problems). We refer also to the regularity results for CR mappings, proved in Huang [H], as well as the recent joint work of Huang with the first and third authors [BHR]. Applications of the results and techniques of the present paper to CR automorphisms of real-analytic manifolds of higher codimension and other related questions will be given in a forthcoming paper of the authors [BER].

In [BR3], the first and third authors proved that for real algebraic hypersurfaces in $\mathbf{C}^N$, $N>1$, holomorphic nondegeneracy is a necessary and sufficient condition for algebraicity of all biholomorphisms between such hypersurfaces. It should be noted that any real smooth hypersurface $M\subset\mathbf{C}^N$ is CR at all its points, and if such an M is real-analytic and holomorphically nondegenerate (and $N>1$), it is minimal at all points outside a proper analytic subset of M. Hence, the main result of [BR3] is contained in Theorem 3 above. (In fact the proofs given in this paper are, for the case of a hypersurface, slightly simplified from that in [BR3], see [BR4].) It is easy to check that in $\mathbf{C}$, any real algebraic hypersurface (i.e. curve) is holomorphically nondegenerate, but never minimal at any point. In fact, by the (algebraic) implicit function theorem, such a curve is locally algebraically equivalent to the real line, which is a homogeneous algebraic set in the sense of Theorem 3. The conclusion of Theorem 3 agrees with the observation that, for instance, the mapping $Z\mapsto e^Z$ maps the real line into itself.

The definition of holomorphic degeneracy was first introduced by Stanton [St1] for the case of a hypersurface. It is proved in [BR3] (see also [St2]) that if M is a connected real-analytic hypersurface, then M is holomorphically degenerate at one point if and only if M is holomorphically degenerate at all points. This condition is also equivalent to the condition that M is nowhere essentially finite (see §1). In higher codimension we show in this paper that holomorphic degeneracy propagates at all CR points (see §1.2). The definition of minimality given here was first introduced by Tumanov [Tu1].

For real-analytic CR manifolds minimality is equivalent (by Nagano's theorem [N]) to the finite type condition of Bloom–Graham [BG] (see also [BR1]). Both formulations, i.e. minimality and finite type, are used in this paper.

The main technical novelty of this work is the use of a sequence of sets, called here the *Segre sets* attached to every point in a real-analytic CR manifold. For M algebraic, the Segre sets are (pieces of) complex algebraic varieties. Another result of this paper, of independent interest, is a new characterization of minimality (or finite type) in terms of Segre sets (see Theorem 2.2.1). In fact, it is shown that the largest Segre set attached to a point $p_0 \in M$ is the intrinsic complexification of the CR orbit of p_0. This in particular proves the algebraicity of the CR orbit when M is algebraic. The first Segre set of a point coincides with the so-called Segre surface introduced by Segre [Seg] and used in the work of Webster [W1], Diederich–Webster [DW], Diederich–Fornaess [DF] and others. Our subsequent Segre sets are all unions of Segre surfaces. The difficulty in the present context arises from the fact that the real algebraic sets considered can be of real codimension greater than one. Indeed, in the codimension one case, i.e. hypersurface, the Segre sets we construct reduce to either the classical Segre surfaces or to all of $\mathbf{C}^N$.

The paper is organized as follows. In §1.1 we recall some of the basic definitions concerned with real-analytic manifolds in $\mathbf{C}^N$ and their CR structures. The other subsections of §1 are devoted to proving the main properties of holomorphic nondegeneracy, which are crucial for the proofs of the results of this paper. In §2 we introduce the notion of Segre sets, as described above; their basic properties, including the characterization of finite type and the algebraicity of the CR orbits, are given in Theorem 2.2.1. In §3 we prove the main results of this paper, of which Theorems 1–4 are consequences. For the proof of the most inclusive result, Theorem 3.1.2, a general lemma on propagation of algebraicity, which may be new, is needed; it is proved in §3.2. The actual proofs of Theorems 1–4 are given in §3.6. Examples are given throughout the paper.

1. Holomorphic nondegeneracy of real-analytic sets

1.1. Preliminaries on real submanifolds of $\mathbf{C}^N$

Let M be a real-analytic submanifold of $\mathbf{C}^N$ of codimension d and $p_0 \in M$. Then M near p_0 is given by $\varrho_j(Z,\bar{Z})=0$, $j=1,...,d$, where the ϱ_j are real-analytic, real-valued functions satisfying

$$d\varrho_1(Z,\bar{Z})\wedge...\wedge d\varrho_d(Z,\bar{Z})\neq 0$$

for Z near p_0. It can be easily checked that the manifold M is CR at p_0 if, in addition, the rank of $(\partial\varrho_1(Z,\bar{Z}),...,\partial\varrho_d(Z,\bar{Z}))$ is constant for Z near p_0, where $\partial f=\sum_j(\partial f/\partial Z_j)\,dZ_j$.

Also, M is generic at p_0 if the stronger condition

$$\partial\varrho_1(Z,\overline{Z})\wedge\ldots\wedge\partial\varrho_d(Z,\overline{Z})\neq 0 \tag{1.1.1}$$

holds for Z near p_0.

For $p\in M$, we denote by T_pM the real tangent space of M at p and by $\mathbf{C}T_pM$ its complexification. We denote by $T_p^{0,1}M$ the complex subspace of $\mathbf{C}T_pM$ consisting of all anti-holomorphic vectors tangent to M at p, and by $T_p^cM=\operatorname{Re}T_p^{0,1}M$ the complex tangent space of M at p considered as a real subspace of T_pM. If M is CR, then $\dim_{\mathbf{C}}T_p^{0,1}M$ and $\dim_{\mathbf{R}}T_p^cM$ are constant, i.e. independent of p, and we denote by $T^{0,1}M$ and T^cM the associated bundles. The *CR dimension* of M is then

$$\operatorname{CRdim} M=\dim_{\mathbf{C}}T_p^{0,1}M=\tfrac{1}{2}\dim_{\mathbf{R}}T_p^cM.$$

If M is generic, then $\dim_{\mathbf{C}}T_p^{0,1}M=N-d$ for all p. If M is CR, then by Nagano's theorem [N] M is the disjoint union of real-analytic submanifolds, called the *CR orbits* of M. The tangent space of such a submanifold at every point consists of the restrictions to that point of the Lie algebra generated by the sections of T^cM. Hence M is of finite type (in the sense of Bloom–Graham [BG]) or minimal at p (as defined in the introduction) if the codimension of the CR orbit through p is 0, i.e. if the Lie algebra generated by the sections of T^cM spans the tangent space of M at p.

Note that if M is a real-analytic submanifold of $\mathbf{C}^N$ then there is a proper real-analytic subvariety V of M such that $M\setminus V$ is a CR manifold. If M is CR at p_0 then we may find local coordinates $Z=(Z',Z'')$ such that near p_0, M is generic in the subspace $Z''=0$. Hence, any real-analytic CR manifold M is a generic manifold in a complex holomorphic submanifold $\mathcal{X}$ of $\mathbf{C}^N$, here called the *intrinsic complexification* of M. We call $\dim_{\mathbf{C}}\mathcal{X}-\operatorname{CRdim}M$ the *CR codimension* of M. Hence, if M is a generic submanifold of $\mathbf{C}^N$ of codimension d its CR dimension is $N-d$ and its CR codimension is d. In view of the observation above, we shall restrict most of our analysis to that of generic submanifolds of $\mathbf{C}^N$.

For a CR manifold M, we define its *Hörmander numbers* at $p_0\in M$ as follows. We let $E_0=T_{p_0}^cM$ and μ_1 be the smallest integer $\geqslant 2$ such that the sections of T^cM and their commutators of lengths $\leqslant\mu_1$ evaluated at p_0 span a subspace E_1 of $T_{p_0}M$ strictly bigger than E_0. The *multiplicity* of the first Hörmander number μ_1 is then $l_1=\dim_{\mathbf{R}}E_1-\dim_{\mathbf{R}}E_0$. Similarly, we define μ_2 as the smallest integer such that the sections of of T^cM and their commutators of lengths $\leqslant\mu_2$ evaluated at p_0 span a subspace E_2 of $T_{p_0}M$ strictly bigger than E_1, and we let $l_2=\dim_{\mathbf{R}}E_2-\dim_{\mathbf{R}}E_1$ be the multiplicity of μ_2. We continue inductively to find integers $2\leqslant\mu_1<\mu_2<\ldots<\mu_s$, and subspaces $T_{p_0}^cM=E_0\subsetneq E_1\subset$

$\ldots\subsetneq E_s\subset T_{p_0}M$, where E_s is the subspace spanned by the Lie algebra of the sections of T^cM evaluated at p_0. The multiplicity l_j of each μ_j is defined in the obvious way as above. It is convenient to denote by $m_1\leqslant m_2\leqslant\ldots\leqslant m_r$ the *Hörmander numbers with multiplicity* by taking $m_1=m_2=\ldots=m_{l_1}=\mu_1$, and so on. Note that if M is generic, then $r=d$ if and only if M is of finite type at p_0. More generally, if M is CR, then r coincides with the CR codimension of M if and only if M is of finite type at p_0.

Now suppose that M is a real-analytic generic submanifold of codimension d in $\mathbf{C}^N$ and $\varrho(Z,\bar{Z})=(\varrho_1(Z,\bar{Z}),\ldots,\varrho_d(Z,\bar{Z}))$ is a defining function for M near $p_0\in M$. We write $N=n+d$. We define the germ of an analytic subset $\mathcal{V}_{p_0}\subset\mathbf{C}^N$ through p_0 by

$$\mathcal{V}_{p_0}=\{Z:\varrho(Z,\zeta)=0 \text{ for all } \zeta \text{ near } \bar{p}_0 \text{ with } \varrho(p_0,\zeta)=0\}. \tag{1.1.2}$$

Note in fact that $\mathcal{V}_{p_0}\subset M$. Then M is called *essentially finite* at p_0 if $\mathcal{V}_{p_0}=\{p_0\}$.

Recall that by the use of the implicit function theorem (see [CM], [BJT], [BR2]) we can find holomorphic coordinates (z,w), $z\in\mathbf{C}^n$, $w\in\mathbf{C}^d$, vanishing at p_0 such that near p_0,

$$\varrho(Z,\bar{Z})=\operatorname{Im} w-\phi(z,\bar{z},\operatorname{Re} w),$$

where $\phi(z,\bar{z},s)=(\phi_1(z,\bar{z},s),\ldots,\phi_d(z,\bar{z},s))$ are real-valued real-analytic functions in $\mathbf{R}^{2n+d}$ extending as holomorphic functions $\phi(z,\chi,\sigma)$ in $\mathbf{C}^{2n+d}$ with

$$\phi(z,0,\sigma)\equiv\phi(0,\chi,\sigma)\equiv 0.$$

Hence, solving in w or $\bar{w}$ we can write the equation of M as

$$w=Q(z,\bar{z},\bar{w}) \quad\text{or}\quad \bar{w}=\bar{Q}(\bar{z},z,w), \tag{1.1.3}$$

where $Q(z,\chi,\tau)$ is holomorphic in a neighborhood of 0 in $\mathbf{C}^{2n+d}$, valued in $\mathbf{C}^d$ and satisfies

$$Q(z,0,\tau)\equiv Q(0,\chi,\tau)\equiv\tau. \tag{1.1.4}$$

It follows from the reality of the ϱ_j and (1.1.3) that the following identity holds for all $z,\chi,w\in C^{2n+d}$ near the origin:

$$Q(z,\chi,\bar{Q}(\chi,z,w))\equiv w. \tag{1.1.5}$$

Coordinates (z,w) satisfying the above properties are called *normal coordinates* at p_0.

If $Z=(z,w)$ are normal coordinates at p_0, then the analytic variety defined in (1.1.2) is given by

$$\mathcal{V}_{p_0}=\{(z,0):Q(z,\chi,0)=0 \text{ for all } \chi\in\mathbf{C}^N\}. \tag{1.1.6}$$

Remark 1.1.1. If the generic submanifold M is real algebraic, then after a holomorphic algebraic change of coordinates one can find normal coordinates (z,w) as above such that the function Q in (1.1.3) is algebraic holomorphic in a neighborhood of 0 in $\mathbf{C}^{2n+d}$, and hence $\mathcal{V}_{p_0}$ is a complex algebraic manifold. If M is a real algebraic CR submanifold, then its intrinsic complexification is a complex algebraic submanifold. Indeed, these are obtained by the use of the implicit function theorem, which preserves algebraicity. (See [BM] and [BR3] for more details.)

1.2. Holomorphic nondegeneracy and its propagation

A real-analytic submanifold M of $\mathbf{C}^N$ is called *holomorphically degenerate* at $p_0 \in M$ if there exists a vector field $X=\sum_{j=1}^N a_j(Z)\partial/\partial Z_j$ tangent to M where the $a_j(Z)$ are germs of holomorphic functions at p_0 not all vanishing identically on M. For CR submanifolds, we shall show that holomorphic nondegeneracy is in fact independent of the choice of the point p_0.

PROPOSITION 1.2.1. *Let M be a connected real-analytic CR submanifold of $\mathbf{C}^N$, and let $p_1, p_2 \in M$. Then M is holomorphically degenerate at p_1 if and only if it is holomorphically degenerate at p_2.*

Proof. Since, as observed in §1.1, every CR manifold is a generic submanifold of a complex manifold, it suffices to assume that M is a generic submanifold of $\mathbf{C}^N$. We shall be brief here, since the proof is very similar to that of the case where M is a hypersurface, i.e. $d=1$, given in [BR3]. We start with an arbitrary point $p_0 \in M$ and we choose normal coordinates (z,w) vanishing at p_0. We assume that M is given by (1.1.3) for (z,w) near 0. We write

$$\bar{Q}(\chi,z,w)=\sum_\alpha q_\alpha(z,w)\chi^\alpha, \tag{1.2.1}$$

for $|z|,|\chi|,|w|<\delta$. We shall assume that δ is chosen sufficiently small so that the right-hand side of (1.2.1) is absolutely convergent. Here q_α is a holomorphic function defined for $|z|,|w|<\delta$ valued in $\mathbf{C}^d$. We leave the proof of the following claim to the reader, since it is very similar to the case $d=1$ proved in [BR3]:

Let $(z^1,w^1)\in M$, with $|z^1|,|w^1|<\delta$. If X is a germ at (z^1,w^1) of a holomorphic vector field in $\mathbf{C}^N$, then X is tangent to M if and only if

$$X=\sum_{j=1}^n a_j(z,w)\frac{\partial}{\partial z_j} \quad \textit{and} \quad \sum_{j=1}^n a_j(z,w)q_{\alpha,z_j}(z,w)\equiv 0, \tag{1.2.2}$$

with a_j holomorphic in a neighborhood of (z^1,w^1), for all multi-indices α, and (z,w) in a neighborhood of (z^1,w^1), where the q_{α,z_j} are the derivatives with respect to z_j of the q_α given by (1.2.1).

As in [BR3], it easily follows by linear algebra from (1.2.2) that if M is holomorphically degenerate at a point (z^1, w^1) as above, then it is holomorphically degenerate at any point (z, w) in the local chart of normal coordinates. Proposition 1.2.1 then follows by the existence of normal coordinates at every point and the connectedness of M. □

In view of Proposition 1.2.1, if M is a connected CR manifold in $\mathbf{C}^N$ we shall say that M is *holomorphically nondegenerate* if it is holomorphically nondegenerate at some point, and hence at every point, of M.

1.3. The Levi number and essential finiteness

Let M be a real-analytic generic manifold in $\mathbf{C}^N$, $p_0 \in M$ and $\varrho(Z, \bar{Z})$ defining functions for M near p_0 as in (1.1.1). Without loss of generality, we may assume $p_0 = 0$. For p_1 close to 0 we define the manifold Σ_{p_1} by

$$\Sigma_{p_1} = \{\zeta \in \mathbf{C}^N : \varrho(p_1, \zeta) = 0\}.$$

(This is the complex conjugate of the classical Segre manifold.) Note that by (1.1.1), Σ_{p_1} is a germ of a smooth holomorphic manifold in $\mathbf{C}^N$ of codimension d. Let $L_1, \ldots, L_n$, $n = N - d$, given by $L_j = \sum_{k=1}^N a_{jk}(Z, \bar{Z}) \partial/\partial \bar{Z}_k$, be a basis of the CR vector fields on M near 0 with the a_{jk} real-analytic (i.e. a basis near 0 of the sections of the bundle $T^{0,1}M$). If $X_1, \ldots, X_n$ are the complex vector fields given by $X_j = \sum_{k=1}^N a_{jk}(p_1, \zeta) \partial/\partial \zeta_k$, $j = 1, \ldots, n$, then X_j is tangent to Σ_{p_1} and the X_j span the tangent space to Σ_{p_1} for $\zeta \in \Sigma_{p_1}$ in a neighborhood of 0, with $(p_1, \zeta) \mapsto a_{jk}(p_1, \zeta)$ holomorphic near $(0, 0)$ in $\mathbf{C}^{2N}$. For a multi-index $\alpha = (\alpha_1, \ldots, \alpha_n)$ and $j = 1, \ldots, d$, we define $c_{j\alpha}(Z, p_1, \zeta)$ in $\mathbf{C}\{Z, p_1, \zeta\}$, the ring of convergent power series in $3N$ complex variables, by

$$c_{j\alpha}(Z, p_1, \zeta) = X^\alpha \varrho_j(Z + p_1, \zeta), \quad j = 1, \ldots, d, \tag{1.3.1}$$

where $X^\alpha = X_1^{\alpha_1} \ldots X_n^{\alpha_n}$.

Note that since the X_j are tangent to Σ_{p_1}, we have $c_{j\alpha}(0, p_1, \zeta) = 0$ for all (p_1, ζ) near $(0, 0)$ and $\zeta \in \Sigma_{p_1}$. In particular, $c_{j\alpha}(0, p_1, \bar{p}_1) = 0$ for $p_1 \in M$ close to 0. It can be checked that M is essentially finite at p_1 if the functions $Z \mapsto c_{j\alpha}(Z, p_1, \bar{p}_1)$, $1 \leqslant j \leqslant d$, $\alpha \in \mathbf{Z}_+^n$, have only 0 as a common zero near the origin for p_1 fixed, small. (See [BR2] or [BHR] for a similar argument in the case of a hypersurface.)

For $1 \leqslant j \leqslant d$, $\alpha \in \mathbf{Z}_+^n$, let $V_{j\alpha}$ be the real-analytic $\mathbf{C}^N$-valued functions defined near 0 in $\mathbf{C}^N$ by

$$V_{j\alpha}(Z, \bar{Z}) = L^\alpha \varrho_{j,Z}(Z, \bar{Z}), \tag{1.3.2}$$

where $\varrho_{j,Z}$ denotes the gradient of ϱ_j with respect to Z and $L^\alpha = L_1^{\alpha_1} \dots L_1^{\alpha_n}$, where $L_1, \dots, L_n$ are as above.

In the sequel we shall say that a property holds *generically* on M if it holds in M outside a proper real-analytic subset.

If M is a generic real-analytic submanifold of $\mathbf{C}^N$ as above, we say that M is *k-nondegenerate* at $Z \in M$ if the linear span of the vectors $V_{j\alpha}(Z, \bar{Z})$, $1 \leqslant j \leqslant d$, $|\alpha| \leqslant k$, is all of $\mathbf{C}^N$. This definition is independent of the choice of the defining functions ϱ and the vector fields L_j.

We have the following proposition.

PROPOSITION 1.3.1. *Let M be a connected real-analytic generic manifold of codimension d in $\mathbf{C}^N$. Then the following conditions are equivalent.*

(i) *M is holomorphically nondegenerate.*

(ii) *There exists $p_1 \in M$ and $k > 0$ such that M is k-nondegenerate at p_1.*

(iii) *There exists V, a proper real-analytic subset of M and an integer $l = l(M)$, $1 \leqslant l(M) \leqslant N - d$, such that M is l-nondegenerate at every $p \in M \setminus V$.*

(iv) *There exists $p_1 \in M$ such that M is essentially finite at p_1.*

(v) *M is essentially finite at all points in a dense open subset of M.*

We shall call the number $l(M)$ given in (iii) above the *Levi number* of M.

Proof. We shall first prove the equivalence of (i), (ii) and (iii). It is clear that (iii) implies (ii). We shall now prove that (ii) implies (i). Assume that M is k-nondegenerate at p_1. We take normal coordinates (z, w) vanishing at p_1, so that M is given by (1.1.3) near $(z, w) = (0, 0)$. We can take for a basis of CR vector fields

$$L_j = \frac{\partial}{\partial \bar{z}_j} + \sum_{k=1}^{d} \bar{Q}_{k,\bar{z}_j}(\bar{z}, z, w) \frac{\partial}{\partial \bar{w}_k}, \quad j = 1, \dots, n, \tag{1.3.3}$$

so that the $V_{j\alpha}$ given by (1.3.2) become, with $Z = (z, w)$,

$$V_{j\alpha}(Z, \bar{Z}) = -\bar{Q}_{j,\bar{z}^\alpha Z}(\bar{z}, z, w). \tag{1.3.4}$$

The hypothesis (ii) implies that the vectors $V_{j\alpha}(0,0)$, $j = 1, \dots, d$, $|\alpha| \leqslant k$, span $\mathbf{C}^N$. By the normality of coordinates, this implies that the $q_{j\alpha,z}(0,0)$, $|\alpha| \leqslant k$, where the $q_{j\alpha}(z, w)$ are the components of the vector $q_\alpha(z, w)$ defined in (1.2.1), span $\mathbf{C}^n$. This implies, by linear algebra, that the $a_j(z, w)$ satisfying (1.2.2) in a neighborhood of 0 must vanish identically. Hence M is not holomorphically degenerate at 0, proving (i).

To show that (i) $\Rightarrow$ (iii), we shall need the following two lemmas, whose proofs are elementary and left to the reader.

LEMMA 1.3.2. *Let* $f_1(\chi), ..., f_d(\chi)$ *be* d *holomorphic functions defined in an open set* Ω *in* $\mathbf{C}^p$, *valued in* $\mathbf{C}^N$ *and generically linearly independent in* Ω. *If the* $\partial^\alpha f_j(\chi)$, $j=1, ..., d$, $\alpha \in \mathbf{Z}_+^p$, *span* $\mathbf{C}^N$ *generically in* Ω, *then the* $\partial^\alpha f_j(\chi)$, $j=1, ..., d$, $|\alpha| \leqslant N-d$, *also span* $\mathbf{C}^N$ *generically in* Ω.

LEMMA 1.3.3. *Let* (z, w) *be normal coordinates for* M *as above, and let* $h(\chi, z, w)$ *be a holomorphic function in* $2n+d$ *variables defined in a connected neighborhood in* $\mathbf{C}^{2n+d}$ *of* $z=z^1, w=w^1, \chi=\bar{z}^1$, *with* $(z^1, w^1) \in M$, *and assume that* $h(\bar{z}, z, w) \equiv 0$, *for* $(z, w) \in M$. *Then* $h \equiv 0$.

To prove that (i) $\Rightarrow$ (iii), we again take (z, w) to be normal coordinates around some point $p_0 \in M$. By the assumption (i) and (1.2.2), it follows that the $q_{j\alpha,z}(z, w)$, $j=1, ..., d$, all α, span $\mathbf{C}^n$ generically. Equivalently, by the normality of the coordinates, we obtain that the $\bar{Q}_{j,\bar{z}^\alpha Z}(0, z, w)$ generically span $\mathbf{C}^N$. We claim that the $\bar{Q}_{j,\bar{z}^\alpha Z}(\bar{z}, z, w)$ generically span $\mathbf{C}^N$ for $(z, w) \in M$. Indeed, if the $\bar{Q}_{j,\bar{z}^\alpha Z}(\bar{z}, z, w)$ do not span, then all $N \times N$ determinants $\Delta(\bar{z}, z, w)$ extracted from the components of these vectors vanish identically on M and hence, by Lemma 1.3.3, $\Delta(\chi, z, w) \equiv 0$ in $\mathbf{C}^{2n+d}$. In particular, $\Delta(0, z, w) \equiv 0$, which would contradict the fact that the $\bar{Q}_{j,\bar{z}^\alpha Z}(0, z, w)$ generically span $\mathbf{C}^N$. This proves the claim.

Now choose $(z^0, w^0) \in M$ so that $\Delta(0, z^0, w^0) \neq 0$ for some determinant Δ as above. We apply Lemma 1.3.2 with $f_j(\chi) = \bar{Q}_{j,Z}(\chi, z^0, w^0)$, $j=1, ..., d$, to conclude that there exists $l \leqslant N-d$ such that in the local chart (z, w), the $V_{j\alpha}(Z, \bar{Z})$ (see (1.3.4)) for $|\alpha| \leqslant l$ span $\mathbf{C}^N$ generically for $Z \in M$. Since this property is independent of the choice of local coordinates, condition (iii) follows from the connectedness of M. This completes the proof of the equivalence of (i), (ii) and (iii).

It remains to show that (i), (ii) and (iii) are equivalent to (iv) and (v). We show first that (iii) $\Rightarrow$ (iv). Let $p_1 \in M$ be any l-nondegenerate point, i.e., the span of $V_{j\alpha}(p_1, \bar{p}_1)$, $1 \leqslant j \leqslant d$, $|\alpha| \leqslant l$, is $\mathbf{C}^N$. On the other hand, it follows from (1.3.1) and (1.3.2) that

$$c_{j\alpha,Z}(0, p_1, \bar{p}_1) = V_{j\alpha}(p_1, \bar{p}_1). \tag{1.3.5}$$

Hence by the inverse mapping theorem the only common zero, near 0, of the functions $Z \mapsto c_{j\alpha}(Z, p_1, \bar{p}_1)$ is 0, which proves that M is essentially finite at p_1, hence (iv).

Next, assume that (v) holds. If the rank of the $V_{j\alpha}(Z, \bar{Z})$ were less than N generically on M, then at any point p_1 of maximal rank near 0 in M, in view of (1.3.5) and the implicit function theorem, there would exist a complex curve $Z(t)$ through 0 such that $c_{j\alpha}(Z(t), p_1, \bar{p}_1) = 0$ for all small t and all j, α. Hence M would not be essentially finite at p_1, contradicting (v), since p_1 can be chosen in an open dense set.

Since (v) $\Rightarrow$ (iv) is trivial, it remains only to show that (iv) $\Rightarrow$ (v). For this we need the following lemma.

LEMMA 1.3.4. *Let $\{f_j\}_{j\in J}$ be holomorphic in a neighborhood of 0 in $\mathbf{C}^N$. Suppose that $Z=0$ is an isolated zero of the functions $f_j(Z)-f_j(0)$, $j\in J$. Then there exists $\delta>0$ such that for $|Z_0|<\delta$, $Z=0$ is an isolated zero of the functions $f_j(Z+Z_0)-f_j(Z_0)$, $j\in J$.*

Proof. For $j\in J$, let $F_j(Z,\zeta)=f_j(Z)-f_j(\zeta)$, which is holomorphic near 0 in $\mathbf{C}^{2N}$. Let V be the variety of zeros of the F_j. We claim that there exists $\varepsilon>0$ and $\delta>0$ such that if $|\zeta_0|<\delta$, then the set $V\cap\{(Z,\zeta)\in\mathbf{C}^{2N}:|Z|<\varepsilon,\ \zeta=\zeta_0\}$ is discrete. Indeed, by assumption there exists $\varepsilon>0$ such that $V\cap\{|Z|=\varepsilon,\ \zeta=0\}=\varnothing$. Therefore by compactness, there exists δ, $0<\delta<\varepsilon$, such that $V\cap\{|Z|=\varepsilon,\ |\zeta|<\delta\}=\varnothing$. Hence for any $|\zeta_0|<\delta$, the set $V\cap\{|Z|<\varepsilon,\ \zeta=\zeta_0\}$ is discrete. Hence the zero $Z=\zeta_0$ of $F(Z,\zeta_0)$ is isolated, which completes the proof of the lemma. □

We may now prove that (iv) $\Rightarrow$ (v). Choose normal coordinates $Z=(z,w)$ around $p_1\in M$ at which M is essentially finite, and observe that if $p_0=(z^0,w^0)$ is in this local chart, we have

$$c_{j\alpha}(Z,p_0,\bar p_0)=-\bar Q_{j,\chi^\alpha}(\bar z^0,z^0,w^0)+\bar Q_{j,\chi^\alpha}(\bar z^0,z+z^0,w+w^0). \tag{1.3.6}$$

By Lemma 1.3.4, we conclude from (1.3.6) that M is essentially finite for any p_0 in a neighborhood of p_1. Property (v) follows by connectedness of M. This completes the proof of Proposition 1.3.1. □

1.4. Holomorphic nondegeneracy of real algebraic sets

Recall that if A is an irreducible, real algebraic subset of $\mathbf{C}^N$, we denote by A_{CR} the set of points of A_{reg} at which A_{reg} is CR. In this subsection we prove the following result.

PROPOSITION 1.4.1. *Let $A\subset\mathbf{C}^N$ be an irreducible real algebraic set and $p_1,p_2\in A_{\mathrm{CR}}$. Then A is holomorphically degenerate at p_1 if and only if it is holomorphically degenerate at p_2.*

We note that if A_{CR} is connected then the proposition follows immediately from Proposition 1.2.1. However, even if A is irreducible, A, A_{reg} and A_{CR} need not be connected.

Proof. It follows from the proof of Proposition 1.2.1 that if M is a real algebraic CR manifold, holomorphically degenerate at $p_0\in M$, then we can find a holomorphic vector field

$$X=\sum_{j=1}^N a_j(Z)\frac{\partial}{\partial Z_j} \tag{1.4.1}$$

tangent to M with $a_j(Z)$ algebraic holomorphic near p_0 and not all vanishing identically on M. Indeed, by Remark 1.1.1, we may assume that the functions $\bar{Q}$ and q_α in (1.2.1) are algebraic. Since the $a_j(z,w)$ in (1.2.2) are obtained by solving a linear system of equations, we can find a set of solutions which are algebraic.

Assume that A_{CR} is holomorphically degenerate at p_1. By the observation above, we can find X of the form (1.4.1), with the $a_j(Z)$ holomorphic algebraic, tangent to A near p_1. Since the $a_j(Z)$ are algebraic, they extend as multi-valued holomorphic functions to $\mathbf{C}^N \setminus V$, where V is a proper complex algebraic subvariety of $\mathbf{C}^N$ with $p_1 \notin V$. Hence $A \cap V$ is a proper real algebraic subvariety of A. Let U be a connected open neighborhood of p_2 in A_{CR} and let $p_3 \in U \setminus V$. (If $p_2 \notin V$, we may take $p_3 = p_2$.) If $d = \operatorname{codim}_{\mathbf{R}} A$, then by a classical theorem in real algebraic geometry [HP, Chapter 10], there exist real-valued polynomials $\varrho_1(Z,\bar{Z}), \ldots, \varrho_d(Z,\bar{Z})$ with

$$A = \{Z \in \mathbf{C}^N : \varrho_j(Z,\bar{Z}) = 0,\ j = 1, \ldots, d\}$$

and $d\varrho_1, \ldots, d\varrho_d$ generically linearly independent on A. Let $\mathcal{A}$ be the complexification of A, i.e. the irreducible complex algebraic set in $\mathbf{C}^{2N}$ given by $\mathcal{A} = \{(Z,\zeta) \in \mathbf{C}^{2N} : \varrho_j(Z,\zeta) = 0,\ j = 1, \ldots, d\}$, and let $\hat{V} = V \times \mathbf{C}^N_\zeta$. We identify $\mathbf{C}^N$ with a subset of $\mathbf{C}^{2N}$ by the diagonal mapping $Z \mapsto (Z,\bar{Z})$, so that A and V become subsets of $\mathcal{A}$ and $\hat{V}$, respectively. We claim that p_1 and p_3 (considered now as points $\mathcal{A}$) can be connected by a curve contained in $\mathcal{A}_{\mathrm{reg}} \setminus \hat{V}$. The claim follows from the fact that $\mathcal{A}_{\mathrm{reg}} \cap \hat{V}$ is a proper algebraic subvariety of $\mathcal{A}_{\mathrm{reg}}$ and hence its complement in $\mathcal{A}_{\mathrm{reg}}$ is connected, by the irreducibility of $\mathcal{A}$. We conclude that the holomorphic continuation of the vector field (1.4.1), thought of as a vector field in $\mathbf{C}^{2N}$, is tangent to $\mathcal{A}$ at every point along this curve, from which we conclude that A is holomorphically degenerate at p_3. We may now apply Proposition 1.2.1 to the CR manifold U to conclude that A is also holomorphically degenerate at p_2. $\square$

Remark 1.4.2. For a general real algebraic submanifold $M \subset \mathbf{C}^N$, not necessarily CR, it can happen that M is holomorphically degenerate at all CR points, but not holomorphically degenerate at points where M is not CR, as is illustrated by the following example. Let $M \subset \mathbf{C}^4$ be the manifold of dimension 5 given by

$$Z_3 = \bar{Z}_1^2, \quad \operatorname{Re} Z_4 = Z_1 \bar{Z}_2 + Z_2 \bar{Z}_1.$$

M is a CR manifold away from $Z_1 = Z_3 = 0$, and M and M_{CR} are connected. At the CR points the holomorphic vector fields tangent to M are all holomorphic multiples of the vector field $X = \partial/\partial Z_2 + 2Z_3^{1/2}\partial/\partial Z_4$. (Note that here $Z_3^{1/2} = \bar{Z}_1$ on M.) We conclude that there is no nontrivial germ of a holomorphic vector field tangent to M at a non-CR point of M.

2. The Segre sets of a real-analytic CR submanifold

2.1. Complexification of M, involution and projections

Let M denote a generic real-analytic submanifold in some neighborhood $U\subset\mathbf{C}^N$ of $p_0\in M$. Let $\varrho=(\varrho_1,\ldots,\varrho_d)$ be defining functions satisfying (1.1.1) and choose holomorphic coordinates $Z=(Z_1,\ldots,Z_N)$ vanishing at p_0. Embed $\mathbf{C}^N$ in $\mathbf{C}^{2N}=\mathbf{C}^N_Z\times\mathbf{C}^N_\zeta$ as the totally real plane $\{(Z,\zeta)\in\mathbf{C}^{2N}:\zeta=\bar Z\}$. Let us denote by pr_Z and pr_ζ the projections of $\mathbf{C}^{2N}$ onto $\mathbf{C}^N_Z$ and $\mathbf{C}^N_\zeta$, respectively. The natural anti-holomorphic involution $\sharp$ in $\mathbf{C}^{2N}$ defined by

$$^\sharp(Z,\zeta)=(\bar\zeta,\bar Z) \tag{2.1.1}$$

leaves the plane $\{(Z,\zeta):\zeta=\bar Z\}$ invariant. This involution induces the usual anti-holomorphic involution in $\mathbf{C}^N$ by

$$\mathbf{C}^N\ni Z\to\mathrm{pr}_\zeta(^\sharp\mathrm{pr}_Z^{-1}(Z))=\bar Z\in\mathbf{C}^N. \tag{2.1.2}$$

Given a set S in $\mathbf{C}^N_Z$ we denote by $^*S$ the set in $\mathbf{C}^N_\zeta$ defined by

$$^*S=\mathrm{pr}_\zeta(^\sharp\mathrm{pr}_Z^{-1}(S))=\{\zeta:\bar\zeta\in S\}. \tag{2.1.3}$$

By a slight abuse of notation, we use the same notation for the corresponding transformation taking sets in $\mathbf{C}^N_\zeta$ to sets in $\mathbf{C}^N_Z$. Note that if X is a complex-analytic set defined near Z^0 in some domain $\Omega\subset\mathbf{C}^N_Z$ by $h_1(Z)=\ldots=h_k(Z)=0$, then $^*X$ is the complex-analytic set in $^*\Omega\subset\mathbf{C}^N_\zeta$ defined near $\zeta^0=\bar Z^0$ by $\bar h_1(\zeta)=\ldots=\bar h_k(\zeta)=0$. Here, given a holomorphic function $h(Z)$ we use the notation $\bar h(Z)=\overline{h(\bar Z)}$. The transformation $*$ also preserves algebraicity of sets.

Denote by $\mathcal{M}\subset\mathbf{C}^{2N}$ the complexification of M given by

$$\mathcal{M}=\{(Z,\zeta)\in\mathbf{C}^{2N}:\varrho(Z,\zeta)=0\}. \tag{2.1.4}$$

This is a complex submanifold of codimension d in some neighborhood of 0 in $\mathbf{C}^{2N}$. We choose our neighborhood U in $\mathbf{C}^N$ so small that $U\times{}^*U\subset\mathbf{C}^{2N}$ is contained in the neighborhood where $\mathcal{M}$ is a manifold. Note that $\mathcal{M}$ is invariant under the involution $\sharp$ defined in (2.1.1). Indeed all the defining functions $\varrho(Z,\bar Z)$ for M are real-valued, which implies that the holomorphic extensions $\varrho(Z,\zeta)$ satisfy

$$\bar\varrho(Z,\zeta)=\varrho(\zeta,Z). \tag{2.1.5}$$

Thus, given $(Z,\zeta)\in\mathbf{C}^{2N}$ we have $\overline{\varrho(^\sharp(Z,\zeta))}=\overline{\varrho(\bar\zeta,\bar Z)}=\bar\varrho(\zeta,Z)=\varrho(Z,\zeta)$, so $^\sharp(Z,\zeta)\in\mathcal{M}$ if and only if $(Z,\zeta)\in\mathcal{M}$.

2.2. Definition of the Segre sets of M at p_0

We associate to M at p_0 a sequence of germs of sets $N_0, N_1, \dots, N_{j_0}$ at p_0 in $\mathbf{C}^N$—henceforth called *the Segre sets* of M at p_0 for reasons that will become apparent—defined as follows. Define $N_0=\{p_0\}$ and define the consecutive sets inductively (the number j_0 will be defined later) by

$$N_{j+1}=\mathrm{pr}_Z(\mathcal{M}\cap \mathrm{pr}_\zeta^{-1}({}^*N_j))=\mathrm{pr}_Z(\mathcal{M}\cap {}^\sharp\mathrm{pr}_Z^{-1}(N_j)). \tag{2.2.1}$$

Here, and in what follows, we abuse the notation slightly by identifying a germ N_j with some representative of it. These sets are, by definition, invariantly defined and they arise naturally in the study of mappings between submanifolds (see §3).

Let the defining functions ϱ and the holomorphic coordinates Z be as in §1.1. Then the sets N_j can be described as follows, as is easily verified. For odd $j=2k+1$ $(k=0,1,\dots)$, we have

$$\begin{aligned} N_{2k+1}=\{Z:\exists Z^1,\dots,Z^k,\zeta^1,\dots,\zeta^k\colon &\varrho(Z,\zeta^k)=\varrho(Z^k,\zeta^{k-1})=\dots=\varrho(Z^1,0)=0,\\ &\varrho(Z^k,\zeta^k)=\varrho(Z^{k-1},\zeta^{k-1})=\dots=\varrho(Z^1,\zeta^1)=0\}; \end{aligned} \tag{2.2.2}$$

note that for $k=0$ we have

$$N_1=\{Z:\varrho(Z,0)=0\}. \tag{2.2.3}$$

For even $j=2k$ $(k=1,2,\dots)$, we have

$$\begin{aligned} N_{2k}=\{Z:\exists Z^1,\dots,Z^{k-1},\zeta^1,\dots,\zeta^k\colon &\varrho(Z,\zeta^k)=\varrho(Z^{k-1},\zeta^{k-1})=\dots=\varrho(Z^1,\zeta^1)=0,\\ &\varrho(Z^{k-1},\zeta^k)=\varrho(Z^{k-2},\zeta^{k-1})=\dots=\varrho(0,\zeta^1)=0\}. \end{aligned} \tag{2.2.4}$$

For $k=1$, we have

$$N_2=\{Z:\exists\zeta^1\colon \varrho(Z,\zeta^1)=0,\ \varrho(0,\zeta^1)=0\}. \tag{2.2.5}$$

From (2.2.2) and (2.2.4) it is easy to deduce the inclusions

$$N_0\subset N_1\subset\dots\subset N_j\subset\dots. \tag{2.2.6}$$

When $d=1$ the set N_1 is the so-called Segre surface through 0 as introduced by Segre [S], and used by Webster [W1], Diederich–Webster [DW], Diederich–Fornaess [DF], Chern–Ji [CJ], and others. Here the set N_2 is the union of Segre manifolds through points ζ_1 such that $\bar\zeta_1$ belongs to the Segre surface through 0. Subsequent N_j's can be described similarily as unions of Segre manifolds.

In order to simplify the calculations, it is convenient to use normal coordinates $Z=(z,w)$ for M as in §1.1. Recall that M is assumed to be generic and of codimension d; we write $N=n+d$. If M is given by (1.1.3), it will be convenient to write

$$Q(z,\chi,\tau)=\tau+q(z,\chi,\tau), \tag{2.2.7}$$

where

$$q(z,0,\tau)\equiv q(0,\chi,\tau)\equiv 0. \tag{2.2.8}$$

In $\mathbf{C}^{2N}$, we choose coordinates (Z,ζ) with $Z=(z,w)$ and $\zeta=(\chi,\tau)$, where $z,\chi\in\mathbf{C}^n$ and $w,\tau\in\mathbf{C}^d$. Thus, in view of (1.1.3), the complex manifold $\mathcal{M}$ is defined by either of the equations

$$w=Q(z,\chi,\tau) \quad \text{or} \quad \tau=\overline{Q}(\chi,z,w). \tag{2.2.9}$$

In normal coordinates, we find that in the expression (2.2.2) for N_{2k+1} we can solve recursively for $w^1,\tau^1,w^2,\tau^2,\ldots,w^k,\tau^k$ and parametrize N_{2k+1} by

$$\mathbf{C}^{(2k+1)n}\ni(z,z^1,\ldots,z^k,\chi^1,\ldots,\chi^k)=\Lambda\mapsto(z,v^{2k+1}(\Lambda))\in\mathbf{C}^N, \tag{2.2.10}$$

where

$$v^{2k+1}(\Lambda)=\tau^k+q(z,\chi^k,\tau^k), \tag{2.2.11}$$

and recursively

$$\tau^l=w^l+\bar{q}(\chi^l,z^l,w^l) \quad \text{with } w^l=\begin{cases}\tau^{l-1}+q(z^l,\chi^{l-1},\tau^{l-1}), & l\geqslant 2,\\ 0, & l=1,\end{cases} \tag{2.2.12}$$

for $l=1,2,\ldots,k$; for $k=0$, we have $v^1\equiv 0$. Similarily, we can parametrize N_{2k} by

$$\mathbf{C}^{2kn}\ni(z,z^1,\ldots,z^{k-1},\chi^1,\ldots,\chi^k)=\Lambda\mapsto(z,v^{2k}(\Lambda))\in\mathbf{C}^N, \tag{2.2.13}$$

where

$$v^{2k}(\Lambda)=\tau^k+q(z,\chi^k,\tau^k), \tag{2.2.14}$$

and recursively

$$\tau^{l+1}=w^l+\bar{q}(\chi^{l+1},z^l,w^l) \quad \text{with } w^l=\tau^l+q(z^l,\chi^l,\tau^l), \tag{2.2.15}$$

for $l=1,\ldots,k-1$ and $\tau^1=0$. Define d_j to be the maximal rank of the mapping (2.2.10) or (2.2.13) (depending on whether j is odd or even) near $0\in\mathbf{C}^{jn}$. It is easy to see that $d_0=0$ and $d_1=n$. In view of (2.2.6), we have $d_0<d_1\leqslant d_2\leqslant d_3\leqslant\ldots$. We define the number $j_0\geqslant 1$ to be the greatest integer such that we have strict inequalities

$$d_0<d_1<\ldots<d_{j_0}. \tag{2.2.16}$$

Clearly, j_0 is a well-defined finite number because, for all j, we have $d_j\leqslant N=n+d$ and $d_{j_0}\geqslant n+j_0-1$ so that we have $j_0\leqslant d+1$. The d_j's stabilize for $j\geqslant j_0$, i.e. $d_{j_0}=d_{j_0+1}=d_{j_0+2}=\ldots$, by the definition of the Segre sets.

So far we have only considered generic submanifolds. If M is a real-analytic CR submanifold of $\mathbf{C}^N$, then M is generic as a submanifold of its intrinsic complexification $\mathcal{X}$ (see §1.1). If M is real algebraic then $\mathcal{X}$ is complex algebraic. The Segre sets of M at a point $p_0 \in M$ can be defined as subsets of $\mathbf{C}^N$ by the process described at the beginning of this subsection (i.e. by (2.2.1)) just as for generic submanifolds or they can be defined as subsets of $\mathcal{X}$ by identifying $\mathcal{X}$ near p_0 with $\mathbf{C}^K$ and considering M as a generic submanifold of $\mathbf{C}^K$. It is an easy exercise (left to the reader) to show that these definitions are equivalent (i.e. the latter sets are equal to the former when viewed as subsets of $\mathbf{C}^N$).

The main result in this section is the following. Let the Hörmander numbers, with multiplicity, be defined as in §1.1.

THEOREM 2.2.1. *Let M be a real-analytic CR submanifold in $\mathbf{C}^N$ of CR dimension n and of CR codimension d and $p_0 \in M$. Assume that there are r (finite) Hörmander numbers of M at p_0, counted with multiplicity. Then the following hold.*

(a) *There is a holomorphic manifold X of (complex) dimension $n+r$ through p_0 containing the maximal Segre set N_{j_0} of M at p_0 (or, more precisely, every sufficiently small representative of it) such that N_{j_0} contains a relatively open subset of X. In particular, the generic dimension d_{j_0} of N_{j_0} equals $n+r$.*

(b) *The intersection $M \cap X$ is the CR orbit of the point p_0 in M.*

(c) *If M is real algebraic then X is complex algebraic, i.e. X extends as an irreducible algebraic variety in $\mathbf{C}^N$.*

In particular, this theorem gives a new criterion for M to be of finite type (or minimal) at p_0. The following is an immediate consequence of the theorem.

COROLLARY 2.2.2. *Let M be a real-analytic CR submanifold in $\mathbf{C}^N$ of CR dimension n and of CR codimension d and $p_0 \in M$. Then M is minimal at p_0, if and only if the generic dimension d_{j_0} of the maximal Segre set N_{j_0} of M at p_0 is $n+d$. In particular, if M is generic, then M is minimal at p_0 if and only if $d_{j_0} = N$.*

Example 2.2.3. Let $M \subset \mathbf{C}^3$ be the generic submanifold defined by

$$\operatorname{Im} w_1 = |z|^2, \quad \operatorname{Im} w_2 = |z|^4.$$

Then M is of finite type at 0 with Hörmander numbers 2, 4. The Segre sets N_1 and N_2 at 0 are given by

$$N_1 = \{(z, w_1, w_2) : w_1 = 0, w_2 = 0\}, \tag{2.2.17}$$

$$N_2 = \{(z, w_1, w_2) : w_1 = 2iz\chi,\ w_2 = 2iz^2\chi^2,\ \chi \in \mathbf{C}\}. \tag{2.2.18}$$

Solving for χ in (2.1.18) we obtain in this way (outside the plane $\{z=0\}$)

$$N_2 = \left\{(z, w_1, w_2) : w_2 = -\tfrac{1}{2} i w_1^2\right\}.$$

Using the definition (2.2.1), we obtain

$$N_3 = \left\{(z, w_1, w_2) : w_2 = iw_1\left(\tfrac{1}{2}w_1 - 2z\chi\right),\, \chi \in \mathbf{C}\right\}.$$

We have $d_3 = 3$; N_3 contains $\mathbf{C}^3$ minus the planes $\{z=0\}$ and $\{w_1=0\}$.

Example 2.2.4. Consider $M \subset \mathbf{C}^3$ defined by

$$\operatorname{Im} w_1 = |z|^2, \quad \operatorname{Im} w_2 = \operatorname{Re} w_2 |z|^4.$$

Here 2 is the only Hörmander number at the origin. Again, N_1 is given by (2.2.17), and

$$N_2 = \{(z, w_1, w_2) : z \neq 0,\, w_2 = 0\} \cup \{0, 0, 0\}.$$

It is easy to see that subsequent Segre sets are equal to N_2. Thus, N_2 is the maximal Segre set of M at 0, $d_2 = 2$, and the intersection of (the closure of) N_2 with M equals the CR orbit of 0.

Let us also note that part (c) of Theorem 2.2.1 implies the following.

COROLLARY 2.2.5. *The CR orbits of a real algebraic CR manifold are algebraic.*

The theorem of Nagano ([N]) states that the integral manifolds of systems of vector fields, with real-analytic coefficients, are real-analytic. Thus, the CR orbits of a real-analytic CR manifold M are real-analytic submanifolds of M. However, in general the integral manifolds of systems of vector fields with real algebraic coefficients are not algebraic manifolds, as can be readily seen by examples. Hence, one cannot use Nagano's theorem to deduce that the orbits of an algebraic CR manifold are algebraic. Corollary 2.2.5 seems not to have been known before.

Before we prove Theorem 2.2.1 (in §2.5) we first discuss the homogeneous case because the proof of the theorem will essentially reduce to this case. We first consider the case where the CR dimension is 1 (§2.3) and then give the modifications needed to consider the general case (§2.4).

2.3. Homogeneous submanifolds of CR dimension 1

Let $\mu_1 \leqslant \ldots \leqslant \mu_N$ be N positive integers. For $t > 0$ and $Z = (Z_1, \ldots, Z_N) \in \mathbf{C}^N$, we let $\delta_t Z = (t^{\mu_1} Z_1, \ldots, t^{\mu_N} Z_N)$. A polynomial $P(Z, \bar{Z})$ is *weighted homogeneous of degree* m with respect to the weights $\mu_1, \ldots, \mu_N$ if $P(\delta_t Z, \delta_t \bar{Z}) = t^m P(Z, \bar{Z})$ for $t > 0$.

In this section and the next, we consider submanifolds M in $\mathbf{C}^N$, $N=n+d$, of the form

$$M:\begin{cases} w_1=\bar{w}_1+q_1(z,\bar{z}), \\ \vdots \\ w_j=\bar{w}_j+q_j(z,\bar{z},\bar{w}_1,\dots,\bar{w}_{j-1}), \\ \vdots \\ w_r=\bar{w}_r+q_r(z,\bar{z},\bar{w}_1,\dots,\bar{w}_{r-1}), \\ w_{r+1}=\bar{w}_{r+1}, \\ \vdots \\ w_d=\bar{w}_d, \end{cases} \tag{2.3.1}$$

where $0\leqslant r\leqslant d$ is an integer ($r=0$ corresponds to the canonically flat submanifold), and each q_j, for $j=1,\dots,r$, is a weighted homogeneous polynomial of degree m_j. The weight of each z_j is 1 and the weight of w_k, for $k=1,\dots,r$, is m_k. Since the defining equations of M are polynomials, we can, and we will, consider the sets $N_0,\dots,N_{j_0}$ attached to M at 0 as globally defined subsets of $\mathbf{C}^N$. Each N_j is contained in an irreducible complex algebraic variety of dimension d_j (here, an algebraic variety of dimension N is the whole space $\mathbf{C}^N$). The latter follows from the parametric definitions (2.2.10) and (2.2.13) of N_j and the algebraic implicit function theorem.

We let π_j, for $j=2,\dots,d+1$, be the projection $\pi_j\colon \mathbf{C}^{n+d}\mapsto\mathbf{C}^{n+j-1}$ defined by

$$\pi_j(z,w_1,\dots,w_d)=(z,w_1,\dots,w_{j-1}). \tag{2.3.2}$$

We define $M^j\subset\mathbf{C}^{n+j-1}$ to be $\pi_j(M)$. By the form (2.3.1) of M, it follows that each M^j is the CR manifold of codimension $j-1$ defined by the $j-1$ first equations of (2.3.1). Throughout this section and the next, we work under the assumption that M satifies the following.

CONDITION 2.3.1. *The CR manifold M^j, for $j=2,\dots,r+1$, is of finite type at* 0.

For clarity, we consider first the case where the CR dimension, n, is one, i.e. $z\in\mathbf{C}$. The rest of this section is devoted to this case. The purpose of the following proposition is to relate the integer j_0, defined in (2.2.16), to the integer r in Condition 2.3.1, and to give, by induction on j, parametrizations of a particular form of open pieces of $N_1,\dots,N_{j_0}$.

PROPOSITION 2.3.2. *Let M be of the form* (2.3.1) *with CR dimension $n=1$ and assume that M satisfies Condition* 2.3.1. *Let $N_0,N_1,\dots,N_{j_0}$ be the Segre sets of M at* 0, *and let $d_0,d_1,\dots,d_{j_0}$ be their generic dimensions. Then $j_0=r+1$ and $d_j=j$, for $0\leqslant j\leqslant r+1$. Furthermore, for each $j=0,\dots,r+1$, there is a proper complex algebraic variety $V_j\subset\mathbf{C}^j$ such that N_j satisfies*

$$\begin{gathered} N_j\cap((\mathbf{C}^j\setminus V_j)\times\mathbf{C}^{d-j+1}) \\ =\{(z,w_1,\dots,w_d)\in((\mathbf{C}^j\setminus V_j)\times\mathbf{C}^{d-j+1}):w_k=f_{jk}(z,w_1,\dots,w_{j-1}),\ k=j,\dots,d\}, \end{gathered} \tag{2.3.3}$$

where each f_{jk}, for $k=j,\ldots,r$, is a (multi-valued) algebraic function outside V_j and where $f_{jk}\equiv 0$ for $k=r+1,\ldots,d$. We write b_{jk} for the number of holomorphic disjoint branches of f_{jk} outside V_j.

Proof. Clearly, the first statement of the proposition follows from the last one. Thus, it suffices to prove that, for each $j=0,\ldots,r+1$, there is a proper algebraic variety V_j such that (2.3.3) holds. The proof of this is by induction on j.

Since $N_0=\{0\}$ and $N_1=\{(z,w):w=0\}$, (2.3.3) holds for $j=0,1$ with $V_0=V_1=\varnothing$. We assume that there are $V_0,\ldots,V_{l-1}$ such that (2.3.3) holds for $j=0,\ldots,l-1$. By (2.2.1), we have

$$N_l=\{(z,w):\exists(\chi,\tau)\in {}^*N_{l-1},(z,w,\chi,\tau)\in\mathcal{M}\}. \tag{2.3.4}$$

ASSERTION 2.3.3. *The set of points $(z,w_1,\ldots,w_{l-1})\in\mathbf{C}^l$ such that there exists*

$$(w_l,\ldots,w_d)\in\mathbf{C}^{d-l+1} \quad \textit{and} \quad (\chi,\tau)\in{}^*(N_{l-1}\cap(V_{l-1}\times\mathbf{C}^{d-l+2}))$$

with the property that $(z,w,\chi,\tau)\in\mathcal{M}$ is contained in a proper algebraic variety $A_l\subset\mathbf{C}^l$.

Proof of Assertion 2.3.3. Let S be the set of points $(z,w_1,\ldots,w_{l-1})\in\mathbf{C}^l$ described in the assertion. Then $(z,w_1,\ldots,w_{l-1})\in\mathbf{C}^l$ is in S if

$$\tau_j=w_j+\bar q_j(\chi,z,w_1,\ldots,w_{j-1}),\quad j=1,\ldots,l-1, \tag{2.3.5}$$

for some $(\chi,\tau_1,\ldots,\tau_{l-1})\in{}^*(\pi_l(N_{l-1})\cap(V_{l-1}\times\mathbf{C}))$. (Recall the two equivalent sets of defining equations, (2.2.9), for $\mathcal{M}$. The operation $*$ here is taken in $\mathbf{C}^l$, i.e. mapping sets in $\mathbf{C}^l_{(z,w_1,\ldots,w_{l-1})}$ to $\mathbf{C}^l_{(\chi,\tau_1,\ldots,\tau_{l-1})}$.) We claim that the set S is contained in a proper algebraic variety $A_l\subset\mathbf{C}^l$. To see this, note first that (2.3.3) (which, by the induction hypothesis, holds for N_{l-1}) implies that $\pi_l(N_{l-1})$ is contained in a proper irreducible algebraic variety in $\mathbf{C}^l$. Let $P_1(\chi,\tau_1,\ldots,\tau_{l-2})$ be a (nontrivial) polynomial that vanishes on ${}^*V_{l-1}\subset\mathbf{C}^{l-1}$, and let $P_2(\chi,\tau_1,\ldots,\tau_{l-1})$ be a (nontrivial) irreducible polynomial that vanishes on ${}^*\pi_l(N_{l-1})$. Thus, if $(z,w_1,\ldots,w_{l-1})\in S$ then there exists a $\chi\in\mathbf{C}$ such that

$$\begin{aligned}\widetilde P_1(\chi,z,w_1,\ldots,w_{l-2})&:=P_1(\chi,w_1+\bar q_1(\chi,z),\ldots,w_{l-2}+\bar q_{l-2}(\chi,z,w_1,\ldots,w_{l-3}))=0,\\ \widetilde P_2(\chi,z,w_1,\ldots,w_{l-1})&:=P_2(\chi,w_1+\bar q_1(\chi,z),\ldots,w_{l-1}+\bar q_{l-1}(\chi,z,w_1,\ldots,w_{l-2}))=0,\end{aligned} \tag{2.3.6}$$

i.e. $\widetilde R(z,w_1,\ldots,w_{l-1})=0$ if we denote by $\widetilde R$ the resultant of $\widetilde P_1$ and $\widetilde P_2$ as polynomials in χ. The proof will be complete (with $A_l=\widetilde R^{-1}(0)$) if we can show that $\widetilde R$ is not identically 0, i.e. $\widetilde P_1$ and $\widetilde P_2$ have no common factors (it is easy to see that neither $\widetilde P_1$ nor $\widetilde P_2$ is identically 0). Note that, for arbitrary $\tau_1,\ldots,\tau_{l-1}$, we have (cf. (2.2.9))

$$\widetilde P_2(\chi,z,\tau_1+q_1(z,\chi),\ldots,\tau_{l-1}+q_{l-1}(z,\chi,\tau_1,\ldots,\tau_{l-2}))=P_2(\chi,\tau_1,\ldots,\tau_{l-1}). \tag{2.3.7}$$

It follows from this that $\widetilde{P}_2$ is irreducible (since P_2 is irreducible). Thus, $\widetilde{P}_1$ and $\widetilde{P}_2$ cannot have any common factors because $\widetilde{P}_2$ itself is the only nontrivial factor of $\widetilde{P}_2$ and, by the form (2.3.3) of N_{l-1}, $\widetilde{P}_2$ is not independent of w_{l-1}. This completes the proof of Assertion 2.3.3. □

We proceed with the proof of Proposition 2.3.2. Let us denote by $B_l \subset \mathbf{C}^{l-1}$ the proper algebraic variety with the property that $(z, w_1, ..., w_{l-2}) \in \mathbf{C}^{l-1} \setminus B_l$ implies that the polynomial $\widetilde{P}_1(X, z, w_1, ..., w_{l-2})$ defined by (2.3.6), considered as a polynomial in X, has the maximal number of distinct roots. Let $C_l \subset \mathbf{C}^l$ denote the union of A_l and $B_l \times \mathbf{C}$. For $(z, w_1, ..., w_{l-2})$ fixed, let $\Omega(z, w_1, ..., w_{l-2}) \subset \mathbf{C}$ be the domain obtained by removing from $\mathbf{C}$ the roots in X of the polynomial equation

$$\widetilde{P}_1(X, z, w_1, ..., w_{l-2}) = 0. \tag{2.3.8}$$

In view of Assertion 2.3.3 and the inductive hypothesis that (2.3.3) holds for N_{l-1}, it follows from (2.3.4) that

$$\begin{aligned} N_l \cap ((\mathbf{C}^l \setminus C_l) \times \mathbf{C}^{d-l+1}) = \{&(z, w_1, ..., w_d) \in ((\mathbf{C}^j \setminus C_l) \times \mathbf{C}^{d-j+1}) : \\ \exists \chi \in \Omega(z, w_1, ..., w_{l-2}) \subset \mathbf{C},\ &w_k = g_{lk}(\chi, z, w_1, ..., w_{k-1}),\ k = l-1, ..., d\}, \end{aligned} \tag{2.3.9}$$

where

$$\begin{aligned} &g_{lk}(\chi, z, w_1, ..., w_{k-1}) \\ &\quad = \bar{f}_{l-1,k}(\chi, w_1 + \bar{q}_1(\chi, z), ..., w_{l-2} + \bar{q}_{l-2}(\chi, z, w_1, ..., w_{l-3})) \\ &\qquad + q_k(z, \chi, w_1 + \bar{q}_1(\chi, z), ..., w_{k-1} + \bar{q}_{k-1}(\chi, z, w_1, ..., w_{k-2})), \end{aligned} \tag{2.3.10}$$

for $k = l-1, ..., d$. Note that each g_{lk}, for $k = l-1, ..., r$, is a (multi-valued) algebraic function such that all branches are holomorphic in a neighborhood of every point (χ, z, w) considered in (2.3.9), and $g_{lk} \equiv 0$ for $k = r+1, ..., d$.

Now, suppose that $g_{l,l-1}(\chi, z, w_1, ..., w_{l-2})$ actually depends on χ, i.e.

$$\frac{\partial g_{l,l-1}}{\partial \chi}(\chi, z, w_1, ..., w_{l-2}) \not\equiv 0. \tag{2.3.11}$$

Then, for each $(\chi^0, z^0, w_1^0, ..., w_{l-2}^0, w_{l-1}^0)$ such that one branch g of $g_{l,l-1}$ is holomorphic near $(\chi^0, z^0, w_1^0, ..., w_{l-2}^0)$ with

$$\frac{\partial g}{\partial \chi}(\chi^0, z^0, w_1^0, ..., w_{l-2}^0) \neq 0 \tag{2.3.12}$$

and

$$w_{l-1}^0 = g(\chi^0, z^0, w_1^0, ..., w_{l-2}^0), \tag{2.3.13}$$

we may apply the (algebraic) implicit function theorem and deduce that there is a holomorphic branch $\theta(z, w_1, ..., w_{l-1})$ of an algebraic function near $(z^0, w_1^0, ..., w_{l-1}^0)$ such that

$$w_{l-1} - g(\theta(z, w_1, ..., w_{l-1}), z, w_1, ..., w_{l-2}) \equiv 0. \tag{2.3.14}$$

Since $g_{l,l-1}$ is an algebraic function, which in particular means that any two choices of branches g at (possibly different) points $(\chi^0, z^0, w_1^0, ..., w_{l-2}^0)$ can be connected via a path in $(\chi, z, w_1, ..., w_{l-2})$-space avoiding the singularities of $g_{l,l-1}$ and also avoiding the zeros of $\partial g_{l,l-1}/\partial\chi$, it follows that any solution θ of (2.3.14) near a point $(z^0, w_1^0, ..., w_{l-1}^0)$ can be analytically continued to any other solution near a (possibly different) point. Thus, all solutions θ are branches of the same algebraic function, and we denote that algebraic function by θ_l. As a consequence, there is an irreducible polynomial $R_l(X, z, w_1, ..., w_{l-1})$ such that $X = \theta_l(z, w_1, ..., w_{l-1})$ is its root. Let $D_l \subset \mathbf{C}^l$ be the zero locus of the discriminant of R_l as a polynomial in X. Outside $(C_l \cup D_l) \times \mathbf{C}^{d-l+1} \subset \mathbf{C}^{d+1}$, we can, by solving for $\chi = \theta_l(z, w_1, ..., w_{l-1})$ in the equation

$$w_{l-1} = g_{l,l-1}(\chi, z, w_1, ..., w_{l-2}), \tag{2.3.15}$$

describe N_l as the (multi-sheeted) graph

$$w_k = f_{lk}(z, w_1, ..., w_{l-1}) := g_{lk}(\theta_l(z, w_1, ..., w_{l-1}), w_1, ..., w_{k-1}), \tag{2.3.16}$$

for $k = l, ..., d$. Clearly, we have $f_{lk} \equiv 0$ for $k = r+1, ..., d$. By taking V_l to be the union of $C_l \cup D_l$ and the proper algebraic variety consisting of points where any two distinct branches of f_{lk} coincide (for some $k = l, ..., d$), we have completed the proof of the inductive step for $j = l$ under the assumption that $g_{l,l-1}(\chi, z, w_1, ..., w_{l-2})$ actually depends on χ.

Now, we complete the proof of the proposition by showing that Condition 2.3.1 forces (2.3.11) to hold as long as $l-1 \leqslant r$. Assume, in order to reach a contradiction, that $g_{l,l-1}(\chi, z, w_1, ..., w_{l-2})$ does not depend on χ. It is easy to verify from the form (2.3.1) of M that the sets $\pi_k(N_j)$, for $j = 0, ..., k$, are the Segre sets of M^k at 0. Let us denote these sets by $N_j(M^k)$. Now, note that if we pick $(z^0, w_1^0, ..., w_{l-1}^0) \in M^l$ then

$$(\bar{z}^0, w_1^0 + \bar{q}_1(\bar{z}^0, z^0), ..., w_{l-1}^0 + \bar{q}_{l-1}(\bar{z}^0, z^0, w_1^0, ..., w_{l-2}^0)) = (\bar{z}^0, \bar{w}_1^0, ..., \bar{w}_{l-1}^0). \tag{2.3.17}$$

Thus, if we pick the point $(z^0, w_1^0, ..., w_{l-1}^0) \in M^l$ such that it is not on the algebraic variety C_l (which is possible since the generic real submanifold M^l cannot be contained in a proper algebraic variety; $C_l \cap M^l$ is a proper real algebraic subset of M^l) then, by construction of C_l, the point

$$(\bar{z}^0, w_1^0 + \bar{q}_1(\bar{z}^0, z^0), ..., w_{l-2}^0 + \bar{q}_{l-2}(\bar{z}^0, z^0, w_1^0, ..., w_{l-3}^0)) = (\bar{z}^0, \bar{w}_1^0, ..., \bar{w}_{l-2}^0) \tag{2.3.18}$$

is not in $^*\pi_l(V_{l-1})$. By the induction hypothesis, $\pi_l(N_{l-1})=N_{l-1}(M^l)$ consists of a $b_{l-1,l-1}$-sheeted graph (each sheet, disjoint from the other, corresponds to a branch of $f_{l-1,l-1}$) above a neighborhood of the point $(z^0, w_1^0, ..., w_{l-2}^0)$. Since $g_{l,l-1}$ is assumed independent of χ, we can, in view of (2.3.18), take $\chi=\bar{z}$ in the defining equation

$$w_{l-1}=g_{l,l-1}(\chi, z, w_1, ..., w_{l-2}) \tag{2.3.19}$$

for $N_l(M^l)$, near the point $(z^0, w_1^0, ..., w_{l-2}^0)$. From the definition (2.3.10) of $g_{l,l-1}$ and (2.3.18) it follows that $N_l(M^l)$ also consists of a b-sheeted graph, with $b\leqslant b_{l-1,l-1}$ (each sheet corresponds to a choice of branch of $\bar{f}_{l-1,l-1}$ at $(\bar{z}^0, \bar{w}_1^0, ..., \bar{w}_{l-2}^0)$), above a neighborhood of the point $(z^0, w_1^0, ..., w_{l-2}^0)$. Since $N_{l-1}(M^l)\subset N_l(M^l)$, we must have $b=b_{l-1,l-1}$ and, moreover, for each branch $f_{l-1,l-1}^k$ there is possibly another branch $f_{l-1,l-1}^{k'}$ such that for every $(z, w_1, ..., w_{l-2})$ the following holds:

$$\begin{aligned} &f_{l-1,l-1}^k(z, w_1, ..., w_{l-2}) \\ &\quad=\bar{f}_{l-1,l-1}^{k'}(\bar{z}, w_1+\bar{q}_1(\bar{z}, z), ..., w_{l-2}+\bar{q}_{l-2}(\bar{z}, z, w_1, ..., w_{l-3})) \\ &\qquad+q_{l-1}(z, \bar{z}, w_1+\bar{q}_1(\bar{z}, z), ..., w_{l-2}+\bar{q}_{l-2}(\bar{z}, z, w_1, ..., w_{l-3})). \end{aligned} \tag{2.3.20}$$

Since all the sheets of the graphs are disjoint, the mapping $k\to k'$ is a permutation. We average over k and k', restrict to points $(z, w_1, ..., w_{l-2})\in M^{l-1}$, and obtain, by (2.3.18) and (2.3.20),

$$\begin{aligned} &\frac{1}{b_{l-1,l-1}}\sum_{k=1}^{b_{l-1,l-1}} f_{l-1,l-1}^k(z, w_1, ..., w_{l-2}) \\ &=\frac{1}{b_{l-1,l-1}}\sum_{k'=1}^{b_{l-1,l-1}} \bar{f}_{l-1,l-1}^{k'}(\bar{z}, \bar{w}_1, ..., \bar{w}_{l-2})+q_{l-1}(z, \bar{z}, \bar{w}_1, ..., \bar{w}_{l-2}). \end{aligned} \tag{2.3.21}$$

Let us denote by f the holomorphic function near $(z^0, w_1^0, ..., w_{l-2}^0)$ defined by

$$f(z, w_1, ..., w_{l-2})=\frac{1}{b_{l-1,l-1}}\sum_{k=1}^{b_{l-1,l-1}} f_{l-1,l-1}^k(z, w_1, ..., w_{l-2}), \tag{2.3.22}$$

and by $K\subset\mathbf{C}^l$ the CR manifold of CR dimension 1 defined near

$$(z^0, w_1^0, ..., w_{l-2}^0, f(z^0, w_1^0, ..., w_{l-2}^0))$$

by

$$K:=\{(z, w_1, ..., w_{l-1}):(z, w_1, ..., w_{l-2})\in M^{l-1},\ w_{l-1}=f(z, w_1, ..., w_{l-2})\}. \tag{2.3.23}$$

The equation (2.3.21) immediately implies that $K\subset M^l$. By Condition 2.3.1, M^l is of finite type near 0. Note that, by the form (2.3.1) of M, the condition that M^l is of finite type at a point is only a condition on $(z, w_1, ..., w_{l-2})$ (i.e. not on w_{l-1}). Thus, by picking the point $(z^0, w_1^0, ..., w_{l-2}^0)\in M^{l-1}$ sufficiently close to 0 (which is possible since, as we mentioned above, $C_l\cap M^l$ is a proper real algebraic subset of M^l), we reach the desired contradiction. This completes the proof of Proposition 2.3.2. □

2.4. Homogeneous submanifolds of arbitrary CR dimension

We prove here the analog of Proposition 2.3.2 for arbitrary CR dimension n.

PROPOSITION 2.4.1. *Let M be of the form* (2.3.1) *and assume that M satisfies Condition* 2.3.1. *Let $N_0, N_1, ..., N_{j_0}$ be the Segre sets of M at* 0. *Then, for each $j=1, ..., j_0$, there is a partition of the set $\{1, 2, ..., r\}$ into $I_j=\{i_1, i_2, ..., i_{a_j}\}$ and $K_j=\{k_1, k_2, ..., k_{b_j}\}$ such that*

$$\varnothing = I_1 \subsetneqq I_2 \subsetneqq I_3 \subsetneqq ... \subsetneqq I_{j_0} = \{1, 2, ..., r\}, \tag{2.4.1}$$

and there is a proper algebraic variety $V_j \subset \mathbf{C}^{n+a_j}$ such that N_j satisfies

$$\begin{aligned} N_j \cap ((\mathbf{C}^{n+a_j} \setminus V_j) \times \mathbf{C}^{b_j} \times \mathbf{C}^{d-r}) = \{(z, w_1, ..., w_d) :& \\ w_{k_\mu} = f_{jk_\mu}(z, w_{i_1}, ..., w_{i_{a_j}}),\ \mu = 1, ..., b_j;\ w_k = 0,\ k = r+1, ..., d\}&. \end{aligned} \tag{2.4.2}$$

Here $(z, w_{i_1}, ..., w_{i_{a_j}}) \in \mathbf{C}^{n+a_j}$ and $(w_{k_1}, ..., w_{k_{b_j}}) \in \mathbf{C}^{b_j}$. Each f_{jk_μ}, for $k=1, ..., b_j$, is a (multi-valued) algebraic function with b_{jk_μ} holomorphic, disjoint branches outside V_j and, moreover, each f_{jk_μ} is independent of w_{i_ν} for all $i_\nu > k_\mu$.

Proof. We emphasize here those aspects of the proof which are different from that of Proposition 2.3.2. We proceed by induction on j. The statement of the proposition holds for $j=1$ with $V_1=\varnothing$ and each $f_{1k_\mu} \equiv 0$. Assume the statement holds for $j=1, ..., l-1$. Let us for simplicity denote the numbers a_{l-1} and b_{l-1} by a and b, respectively. The representation (2.3.4), with z in $\mathbf{C}^n$ rather than $\mathbf{C}$, still holds. Let $\Omega(z, w) \subset \mathbf{C}^n$ be the complement of the algebraic subset of χ such that, for fixed $(z, w) \in \mathbf{C}^{n+d}$,

$$(\chi, w_{i_1} + \bar{q}_{i_1}(\chi, z, w_1, ..., w_{i_1-1}), ..., w_{i_a} + \bar{q}_{i_a}(\chi, z, w_1, ..., w_{i_a-1})) \in {}^*V_{l-1}. \tag{2.4.3}$$

We describe a part $\widetilde{N}_l$ of N_l as follows

$$\begin{aligned} \widetilde{N}_l = \{(z, w) \in \mathbf{C}^{n+d} : \exists \chi \in \Omega(z, w) \subset \mathbf{C}^n,& \\ w_{k_\mu} = g_{lk_\mu}(\chi, z, w_1, ..., w_{k_\mu-1}),\ \mu = 1, ..., b;\ w_k = 0,\ k = r+1, ..., d\}&, \end{aligned} \tag{2.4.4}$$

where

$$\begin{aligned} &g_{lk_\mu}(\chi, z, w_1, ..., w_{k_\mu-1}) \\ &\quad = \bar{f}_{l-1,k_\mu}(\chi, w_{i_1} + \bar{q}_{i_1}(\chi, z, w_1, ..., w_{i_1-1}), ..., w_{i_a} + \bar{q}_{i_a}(\chi, z, w_1, ..., w_{i_a-1})) \\ &\qquad + q_{k_\mu}(z, \chi, w_1 + \bar{q}_1(\chi, z), ..., w_{k_\mu-1} + \bar{q}_{k_\mu-1}(\chi, z, w_1, ..., w_{k_\mu-2})). \end{aligned} \tag{2.4.5}$$

The fact that $w_k=0$ for $k=r+1, ..., d$ follows from (2.4.2) with $j=l-1$ and the form (2.3.1) of M. Note also that, by the induction hypothesis, f_{l-1,k_μ} is independent of

w_{i_ν} for $i_\nu > k_\mu$. Let $w' = (w_{i_1}, ..., w_{i_a})$ and $w'' = (w_{k_1}, ..., w_{k_b})$. Note that, for generic $(z, w'') \in \mathbf{C}^{n+b}$, the mapping from $\mathbf{C}^{n+a}$ into itself given by

$$(\chi, w') \mapsto (\chi, w_{i_1} + \bar{q}_{i_1}(\chi, z, w_1, ..., w_{i_1-1}), ..., w_{i_a} + \bar{q}_{i_a}(\chi, z, w_1, ..., w_{i_a-1})) \tag{2.4.6}$$

has generic rank $n+a$ (indeed, it has rank $n+a$ near the origin for $z=0$). Thus, the set of $w' \in \mathbf{C}^a$ for which (2.4.3) holds (with small z and w'' arbitrary) for all $\chi \in \mathbf{C}^n$ is a proper algebraic variety. Restricting (χ, w') to the complement of the set where (2.4.3) holds, we consider the mapping (2.4.6) with

$$w_{k_1} = g_{lk_1}(\chi, z, w_1, ..., w_{k_1-1}) \tag{2.4.7}$$

instead of w_{k_1} fixed. Again, one verifies that this mapping has generic rank $n+a$ for generic $(z, w_{k_2}, ..., w_{k_b})$ (e.g. with z small), and thus the set of w' for which (2.4.3) holds (with w_{k_1} given by (2.4.7)) for all χ is a proper algebraic variety. By proceeding inductively, substituting g_{lk_μ} for w_{k_μ} in the mapping (2.4.6), we find that we can take for $\widetilde{N}_l$ (for brevity, we write $w''' = (w_{r+1}, ..., w_d)$)

$$\begin{gathered}\widetilde{N}_l = \{(z, w', w'', w''') \in (\mathbf{C}^{n+a} \setminus C_l) \times \mathbf{C}^b \times \mathbf{C}^{d-r} : \\ \exists \chi \in \widetilde{\Omega}(z, w') \subset \mathbf{C}^n,\ w_{k_\mu} = \tilde{g}_{lk_\mu}(\chi, z, w'),\ \mu = 1, ..., b;\ w''' = 0\},\end{gathered} \tag{2.4.8}$$

where $C_l \subset \mathbf{C}^{n+a}$ is a proper algebraic variety, $\widetilde{\Omega}(z, w') \subset \mathbf{C}^n$ is the complement of a proper algebraic variety in $\mathbf{C}^n$, and where $\tilde{g}_{lk_1} = g_{lk_1}$ and subsequent $\tilde{g}_{lk_\mu}$ are obtained from g_{lk_μ} by substituting

$$w_{k_\gamma} = \tilde{g}_{lk_\gamma}(\chi, z, w'), \quad \gamma = 1, ..., \mu-1. \tag{2.4.9}$$

Thus, each $\tilde{g}_{lk_\mu}$ is a function only of those $w_{i_1}, ..., w_{i_\nu}$ for which $i_\nu < k_\mu$.

As in the proof of Proposition 2.3.2, we assume first that the map

$$\mathbf{C}^{2n+a} \ni (\chi, z, w') \mapsto (\tilde{g}_{lk_1}(\chi, z, w'), ..., \tilde{g}_{lk_b}(\chi, z, w')) =: G(\chi, z, w') \in \mathbf{C}^b \tag{2.4.10}$$

actually depends on χ, i.e.

$$G_\chi(\chi, z, w') := \frac{\partial G}{\partial \chi}(\chi, z, w') \not\equiv 0. \tag{2.4.11}$$

Denote by $m \geqslant 1$ the maximal rank of G_χ, and by $G' = (G^{t_1}, ..., G^{t_m})$ the m first components of G such that G'_χ has generic rank m (thus, the set $\{t_1, ..., t_m\}$ is a subset of K_{l-1}). Note that this does not necessarily need to be the first m components of G, but any component G^t, with $t_\alpha < t < t_{\alpha+1}$ for some $\alpha \in \{1, ..., m-1\}$, has then the property that

$$G^t_\chi(\chi, z, w') \equiv \sum_{j=1}^{\alpha} c_j(\chi, z, w') G^{t_j}_\chi(\chi, z, w'), \tag{2.4.12}$$

for some functions $c_1, ..., c_\alpha$. We may assume, by an algebraic change of coordinates in the χ-space if necessary, that $G'_{\chi'}$, where $\chi'=(\chi_1, ..., \chi_m)$, has generic rank m and that G is independent of the last coordinates $\chi'':=(\chi_{m+1}, ..., \chi_n)$. Now, solve for $\chi'=\theta_l(z, w', w_{t_1}, ..., w_{t_m})$ in the equations

$$w_{t_j}=G^{t_j}(\chi, z, w'), \quad j=1, ..., m. \tag{2.4.13}$$

The solution θ_l is a (multi-valued) algebraic function. By substituting

$$\chi'=\theta_l(z, w', w_{t_1}, ..., w_{t_m})$$

in the remaining equations for $\widetilde{N}_l$ (and remembering that, by the choice of m and χ', these equations are independent of χ'') we find, denoting by $K_l:=\{u_1, ..., u_{b-m}\}$ the complement of the set $\{t_1, ..., t_m\}$ in K_{l-1},

$$\begin{aligned} w_{u_j} &= \tilde{g}_{lu_j}(\theta_l(z, w', w_{t_1}, ..., w_{t_m}), \chi'', z, w') \\ &=: f_{lu_j}(z, w', w_{t_1}, ..., w_{t_m}), \quad j=1, ..., b-m. \end{aligned} \tag{2.4.14}$$

Since $\widetilde{N}_l$ is a dense open subset of N_l, the equations (2.4.14) imply that N_l is indeed of the form (2.4.2), with $K_l \subsetneqq K_{l-1}$ as defined above and $I_l=\{1, ..., r\}\setminus K_l$, and where we let $V_l\subset \mathbf{C}^{n+a+m}$ be a suitable proper algebraic variety containing the singularities of the algebraic functions f_{lu_j} $(j=1, ..., b-m)$. To finish the proof (under the assumption that the mapping G actually depends on χ), we need to show that each f_{lu_ν} is independent of w_t for $t>u_\nu$. Recall that

$$f_{lu_\nu}(z, w', w_{t_1}, ..., w_{t_m})=G^{u_\nu}(\theta_l(z, w', w_{t_1}, ..., w_{t_m}), \chi'', z, w'). \tag{2.4.15}$$

Let $1\leqslant\alpha<m-1$ be the number such that $t_\alpha<u_\nu<t_{\alpha+1}$ (unless there is such a number there is nothing to prove), and differentiate (2.4.15) with respect to w_t, where $t\geqslant u_\nu$. Using (2.4.12), we obtain (using vector notation; recall that $G^{u_\nu}(\xi, z, w')$ is independent of w_t)

$$f_{lu_\nu, w_t}=G^{u_\nu}_{\chi'}\theta_{l,w_t}=\sum_{j=1}^{\alpha} c_j G^{t_j}_{\chi'}\theta_{l,w_t}. \tag{2.4.16}$$

Now, by the definition of θ_l, $G^{t_j}(\theta_l(z, w', w_{t_1}, ..., w_{t_m}), \chi'', z, w')\equiv w_{t_j}$, and so $G^{t_j}_{\chi'}\theta_{l,w_t}=0$, if $t>t_j$. Thus, since $t>t_j$ for $j=1, ..., \alpha$, it follows from (2.4.16) that $f_{lu_\nu, w_t}=0$. This proves the induction hypothesis for $j=l$ under the assumption that the mapping G actually depends on χ.

As in the proof of Proposition 2.3.2, we are left to show that Condition 2.3.1 implies that G actually depends on χ as long as $I_{l-1}\subsetneqq\{1, 2, ..., r\}$. Assume, in order to reach a contradiction that G does not depend on χ. In particular then, the function $\tilde{g}_{lk_1}=g_{lk_1}$ does not depend on χ. Since, by the induction hypothesis, $g_{lk_1}(\chi,z,w')$ does not depend on w_j for $j\geqslant k_1$, we can consider the projection π_{k_1} and proceed exactly as in the conclusion of the proof of Proposition 2.3.2. We leave the straightforward verification to the reader. The proof of Proposition 2.4.1 is now complete. □

2.5. Proof of Theorem 2.2.1

By the remarks preceding the theorem, we may assume that M is generic throughout this proof. We start by proving (a). Since the Segre sets of M at p_0 are invariantly defined, we may choose any holomorphic coordinates near p_0. Let $m_1\leqslant\ldots\leqslant m_r$ be the Hörmander numbers of M at p_0. By [BR1, Theorem 2], there are holomorphic coordinates $(z,w)\in\mathbf{C}^n\times\mathbf{C}^d$ such that the equations of M near p_0 are given by

$$\begin{cases} w_j=\bar{w}_j+q_j(z,\bar{z},\bar{w}_1,\ldots,\bar{w}_{j-1})+R_j(z,\bar{z},\bar{w}), & j=1,\ldots,r,\\ w_k=\bar{w}_k+\sum_{l=r+1}^d f_{kl}(z,\bar{z},\bar{w})\bar{w}_l, & k=r+1,\ldots,d,\end{cases} \tag{2.5.1}$$

where, for $j=1,\ldots,r$, $q_j(z,\bar{z},\bar{w}_1,\ldots,\bar{w}_{j-1})$ is weighted homogeneous of degree m_j, $R_j(z,\bar{z},\bar{w})$ is a real-analytic function whose Taylor expansion at the origin consists of terms of weights at least m_j+1, and the f_{kl} are real-analytic functions that vanish at the origin. Here, z is assigned the weight 1, w_j the weight m_j for $j=1,\ldots,r$ and weight m_r+1 for $j=r+1,\ldots,d$. Moreover, the homogeneous manifold $M^0\subset\mathbf{C}^N$ defined by

$$\begin{cases} w_j=\bar{w}_j+q_j(z,\bar{z},\bar{w}_1,\ldots,\bar{w}_{j-1}), & j=1,\ldots,r,\\ w_k=\bar{w}_k, & k=r+1,\ldots,d,\end{cases} \tag{2.5.2}$$

satisfies Condition 2.3.1. For $\varepsilon>0$, we introduce the scaled coordinates $(\tilde{z},\widetilde{w})\in\mathbf{C}^{n+d}$ defined by

$$\begin{cases} z=z(\tilde{z};\varepsilon)=\varepsilon\tilde{z},\\ w_j=w_j(\widetilde{w};\varepsilon)=\varepsilon^{l_j}\widetilde{w}_j, \quad j=1,\ldots,d,\end{cases} \tag{2.5.3}$$

where $l_j=m_j$ for $j=1,\ldots,r$ and $l_k=m_r+1$ for $k=r+1,\ldots,d$. We write $\tilde{f}_{kl}$ for the function

$$\tilde{f}_{kl}(\tilde{z},\bar{\tilde{z}},\bar{\widetilde{w}};\varepsilon)=\frac{1}{\varepsilon}f_{kl}(z(\tilde{z};\varepsilon),\bar{z}(\tilde{z};\varepsilon),\bar{w}(\widetilde{w};\varepsilon)), \tag{2.5.4}$$

and similarly,

$$\widetilde{R}_j(\tilde{z},\bar{\tilde{z}},\bar{\widetilde{w}};\varepsilon)=\frac{1}{\varepsilon^{m_j+1}}R_j(z(\tilde{z};\varepsilon),\bar{z}(\tilde{z};\varepsilon),\bar{w}(\widetilde{w};\varepsilon)). \tag{2.5.5}$$

Note that both $\tilde{f}_{kl}(\tilde{z},\bar{\tilde{z}},\bar{\widetilde{w}};\varepsilon)$ and $\widetilde{R}_j(\tilde{z},\bar{\tilde{z}},\bar{\widetilde{w}};\varepsilon)$ are real-analytic functions of $(\tilde{z},\widetilde{w};\varepsilon)$ in a neighborhood of $(0,0;0)$. In the scaled coordinates, M is represented by the equations

$$\begin{cases} \widetilde{w}_j=\bar{\widetilde{w}}_j+q_j(\tilde{z},\bar{\tilde{z}},\bar{\widetilde{w}}_1,\ldots,\bar{\widetilde{w}}_{j-1})+\varepsilon\widetilde{R}_j(\tilde{z},\bar{\tilde{z}},\bar{\widetilde{w}};\varepsilon), & j=1,\ldots,r,\\ \widetilde{w}_k=\bar{\widetilde{w}}_k+\varepsilon\sum_{l=r+1}^d\tilde{f}_{kl}(\tilde{z},\bar{\tilde{z}},\bar{\widetilde{w}};\varepsilon)\bar{\widetilde{w}}_l, & k=r+1,\ldots,d.\end{cases} \tag{2.5.6}$$

Now, let $\tilde{v}^j(\tilde{\Lambda};\varepsilon)$ be the mapping $\mathbf{C}^{jn}\to\mathbf{C}^d$ described in §2.2, such that the Segre set N_j of M at p_0 is parametrized by

$$\mathbf{C}^{jn} \ni \tilde{\Lambda} \mapsto (\tilde{z}, \tilde{v}^j(\tilde{\Lambda};\varepsilon)) \in \mathbf{C}^N \tag{2.5.7}$$

in the scaled coordinates $(\tilde{z},\tilde{w})$ (cf. (2.2.10)–(2.2.12) and (2.2.13)–(2.2.15) to see how the map (2.5.7) is obtained from the defining equations (2.5.6)). Note that $\tilde{v}^j$ depends real-analytically on the small parameter ε. The generic dimension d_j of the Segre set N_j is the generic rank of the mapping (2.5.7) with $\varepsilon\neq 0$, and is in fact independent of ε. By the real-analytic dependence on ε there is a neighborhood I of $\varepsilon=0$ such that the generic rank of (2.5.7), for all $\varepsilon\in I\setminus\{0\}$, is at least the generic rank of (2.5.7) with $\varepsilon=0$. For $\varepsilon=0$ the mappings (2.5.7) parametrize the Segre sets N_j^0 of the homogeneous manifold M^0 defined by (2.5.2). By Proposition 2.4.1, applied to the Segre sets N_j^0 of M^0 at 0, we deduce that the generic dimension of the maximal Segre set of M^0 at 0 is $n+r$. Thus, $d_{j_0}\geqslant n+r$, where d_{j_0} is the generic dimension of the maximal Segre set of M at p_0. On the other hand, if we go back to the unscaled cordinates (z,w), we note from the construction of the Segre sets that each N_j is contained in the complex manifold $X=\{(z,w): w_{r+1}=\ldots=w_d=0\}$. Thus $d_{j_0}\leqslant n+r$, so that we obtain the desired equality $d_{j_0}=n+r$. This proves part (a) of the theorem.

It follows from (2.5.1) that the CR vector fields of M are all tangent to $M\cap X=\{(z,w)\in M: w_j=0, j=r+1,\ldots,d\}$. Thus, the local CR orbit of p_0 is contained in $M\cap X$. Also, since there are r Hörmander numbers, the CR orbit of p_0 has dimension $2n+r$. Since the dimension of $M\cap X$ is $2n+r$ as well, it follows that the local CR orbit of p_0 is $M\cap X$. This proves part (b) of the theorem.

Finally, to prove part (c) of the theorem we note that if M is real algebraic then each Segre set N_j is contained in a unique irreducible complex algebraic variety of dimension d_j. Since N_{j_0} contains a relatively open subset of X, this relatively open subset of X coincides with a relatively open subset of the unique algebraic variety containing N_{j_0}. Hence, X is complex algebraic. This completes the proof of Theorem 2.2.1. □

3. Algebraic properties of holomorphic mappings between real algebraic sets

3.1. A generalization of Theorems 1 and 4

We denote by $\mathcal{O}_N(p_0)$ the ring of germs of holomorphic functions in $\mathbf{C}^N$ at p_0, and by $\mathcal{A}_N(p_0)$ the subring of $\mathcal{O}_N(p_0)$ consisting of those germs that are also algebraic, i.e. those germs for which there is a nontrivial polynomial $P(Z,x)\in\mathbf{C}[Z,x]$ (with $Z\in\mathbf{C}^N$ and $x\in\mathbf{C}$)

such that any representative $f(Z)$ of the germ satisfies

$$P(Z, f(Z)) \equiv 0. \tag{3.1.1}$$

In particular, any function in $\mathcal{A}_N(p_0)$ extends as a possibly multi-valued holomorphic function in $\mathbf{C}^N \setminus V$, where V is a proper algebraic variety in $\mathbf{C}^N$. We refer the reader to e.g. [BR3, §1] for some elementary properties of algebraic holomorphic functions that will be used in this paper. If $U \subset \mathbf{C}^N$ is a domain we denote by $\mathcal{O}_N(U)$ the space of holomorphic functions in U.

If $X \subset \mathbf{C}^N$ is an algebraic variety with $\dim X = K$, $p_0 \in X_{\mathrm{reg}}$, and f is a holomorphic function on X defined near p_0 then we say that f is algebraic if, given algebraic coordinates

$$\mathbf{C}^K \ni \zeta \mapsto Z(\zeta) \in \mathbf{C}^N \tag{3.1.2}$$

on X near p_0 with $Z(0) = p_0$ (i.e. each component of (3.1.2) is in $\mathcal{A}_K(0)$), the function $h = f \circ Z$ is in $\mathcal{A}_K(0)$. The transitivity property of algebraic functions (e.g. [BM] or [BR3, Lemma 1.8 (iii)]) implies that this definition is independent of the choice of algebraic coordinates. If f is algebraic on X near p_0 and X is irreducible, then f extends as a possibly multi-valued holomorphic function on $X_{\mathrm{reg}} \setminus V$, where V is a proper algebraic subvariety of X_{reg}. (Note that X_{reg} is a connected manifold.) We denote by $\mathcal{O}_X(p_0)$ the ring of germs of holomorphic functions on X at p_0, and by $\mathcal{A}_X(p_0)$ the subring of germs that are algebraic.

Also, given two points $p_0 \in \mathbf{C}^N$ and $p_0' \in \mathbf{C}^{N'}$, we denote by $\mathrm{Hol}(p_0, p_0')$ the space of germs of holomorphic mappings at p_0 from $\mathbf{C}^N$ into $\mathbf{C}^{N'}$ taking p_0 to p_0'. We denote by $\mathrm{Alg}(p_0, p_0')$ the subspace of $\mathrm{Hol}(p_0, p_0')$ consisting of those germs for which each component of the mapping is algebraic. Similarly, given an algebraic variety X in $\mathbf{C}^N$ with $p_0 \in X_{\mathrm{reg}}$, we denote by $\mathrm{Hol}_X(p_0, p_0')$ the space of germs of holomorphic mappings at p_0 from X into $\mathbf{C}^{N'}$ taking p_0 to p_0', and by $\mathrm{Alg}_X(p_0, p_0')$ the subspace of germs with algebraic components.

Before we present the main results, we state the following lemma, whose proof is straightforward and left to the reader.

LEMMA 3.1.1. *Let M be a generic real-analytic submanifold in $\mathbf{C}^K$ and let $p_0 \in M$. Suppose that there is $h = (h_1, ..., h_q) \in (\mathcal{O}_K(p_0))^q$ such that the following holds.*

(i) *$h(p_0) = 0$ and $\partial h_1 \wedge ... \wedge \partial h_q \neq 0$ in a neighborhood of p_0.*

(ii) *$h|_M$ is valued in $\mathbf{R}^q$.*

Then $M \cap S_0$, where $S_0 = \{Z : h(Z) = 0\}$, is a generic submanifold of S_0 near p_0.

We are now in a position to formulate one of the main results in this paper.

THEOREM 3.1.2. *Let M be a real algebraic, holomorphically nondegenerate, CR submanifold in $\mathbf{C}^N$, let $\mathcal{V}\subset\mathbf{C}^N$ be the smallest complex algebraic variety containing M, and let $p_0\in\overline{M}$ be a regular point of $\mathcal{V}$, where $\overline{M}$ is the closure of M in $\mathbf{C}^N$. Assume that there is $h=(h_1,...,h_q)\in(\mathcal{A}_{\mathcal{V}}(p_0))^q$ satisfying the following.*

(i) *$h(p_0)=0$ and $\partial_{\mathcal{V}}h_1\wedge...\wedge\partial_{\mathcal{V}}h_q\neq 0$ in a neighborhood of p_0.*

(ii) *$h|_M$ is valued in $\mathbf{R}^q$.*

Let U be a sufficiently small neighborhood of p_0 in $\mathbf{C}^N$ and denote by S_c, for $c\in\mathbf{C}^q$ with $|c|$ small, the algebraic manifolds

$$S_c=\{Z\in\mathcal{V}\cap U: h(Z)=c\}. \tag{3.1.3}$$

Assume that the generic submanifold $M\cap S_{h(p)}$ is minimal at p for some $p\in M\cap U$. Then if A' is a real algebraic set in $\mathbf{C}^{N'}$ with $\dim_{\mathbf{R}} A'=\dim_{\mathbf{R}} M$, $p_0'\in A'$, and $H\in\mathrm{Hol}_{\mathcal{V}}(p_0,p_0')$ satisfies $H(M)\subset A'$, with generic rank equal to $\dim_{\mathbf{C}}\mathcal{V}$, there exists $\delta>0$ such that $H|_{S_c}$ is algebraic for every $|c|<\delta$.

Note that M is not required to be closed in Theorem 3.1.2. Since M is real algebraic, it is contained in a real algebraic set A of the same dimension in $\mathbf{C}^N$ such that A, in turn, is contained in the complex algebraic variety $\mathcal{V}$. Thus, the point $p_0\in\overline{M}$ is a point on the real algebraic set A, and the only thing required of p_0 is that it is a smooth point of $\mathcal{V}$; if e.g. M is generic then, of course, $\mathcal{V}$ is the whole space $\mathbf{C}^N$ and, hence, nothing at all is required of $p_0\in\overline{M}$. The point p_0 could be a singular point of A, a regular point where the CR dimension increases, or a point across which M extends as a CR manifold.

Specializing Theorem 3.1.2 to the case $q=0$ we obtain the following result.

COROLLARY 3.1.3. *Let M be a real algebraic, holomorphically nondegenerate, CR submanifold in $\mathbf{C}^N$, let $\mathcal{V}$ be the smallest complex algebraic variety that contains M, and let $p_0\in\overline{M}$ be a regular point of $\mathcal{V}$. Assume that there exists $p\in M$, such that M is minimal at p. If A' is a real algebraic set in $\mathbf{C}^{N'}$ such that $\dim_{\mathbf{R}} A'=\dim_{\mathbf{R}} M$, $p_0'\in A'$, and $H\in\mathrm{Hol}_{\mathcal{V}}(p_0,p_0')$ satisfies $H(M)\subset A'$ with generic rank equal to $\dim_{\mathbf{C}}\mathcal{V}$, then $H\in\mathrm{Alg}_{\mathcal{V}}(p_0,p_0')$.*

Specializing again in Corollary 3.1.3 to the case where M is generic, we obtain the corollary announced in the introduction.

Example 3.1.4. Consider the generic holomorphically nondegenerate submanifold $M\subset\mathbf{C}^4$ given by

$$\begin{cases} \operatorname{Im} w_1=|z|^2+\operatorname{Re} w_2|z|^2, \\ \operatorname{Im} w_2=\operatorname{Re} w_3|z|^4, \\ \operatorname{Im} w_3=0. \end{cases} \tag{3.1.4}$$

The function $h_1(z,w)=w_3$ is real on M, and $M\cap\{(z,w):w_3=c\}$ is clearly minimal near $(z,w_1,w_2)=(0,0,0)$ for all real $c\neq 0$. Thus, Theorem 3.1.2 implies that any holomorphic mapping $H:\mathbf{C}^4\to\mathbf{C}^{N'}$ near 0, generically of rank 4, such that $H(M)$ is contained in a 5-dimensional real algebraic subset of $\mathbf{C}^{N'}$ is algebraic on the leaves $\{w_3=c\}$, for all sufficiently small $c\in\mathbf{C}$. This result is optimal, because it is easy to verify that the mapping $H:\mathbf{C}^4\to\mathbf{C}^4$, defined by

$$H(z,w_1,w_2,w_3)=(ze^{iw_3},w_1,w_2,w_3),$$

is a biholomorphism near the origin, and maps M into itself. Moreover, H is algebraic on each $\{w_3=c\}$ but not in the whole space $\mathbf{C}^4$.

It is interesting to note that the only Hörmander number at 0 is 2, and that the maximal Segre set of M at 0 is $N_2=\{(z,w):w_2=w_3=0\}$. Thus, the dimension of the maximal Segre set at 0 is smaller than the dimension of the leaves on which H is algebraic. For generic points $p\in M$ though, the maximal Segre set of M at p coincides with one of these leaves.

Example 3.1.5. Consider the real algebraic subset $A\subset\mathbf{C}^4$ defined by

$$\begin{cases}(\operatorname{Im} w_1)^2=\operatorname{Re} w_2(|z_1|^2+|z_2|^2),\\ \operatorname{Im} w_2=0.\end{cases}\tag{3.1.5}$$

It is singular on $\{(z_1,z_2,w_1,w_2)=(0,0,s_1,s_2):s_1,s_2\in\mathbf{R}\}$, but outside that set it is a generic holomorphically nondegenerate manifold M. The function $h_1(z,w)=w_2$ is real on M, and $M\cap\{w_2=s_2\}$ is minimal everywhere for all real $s_2\neq 0$. Theorem 3.1.2 implies that any holomorphic mapping $H:\mathbf{C}^4\to\mathbf{C}^{N'}$ near 0, generically of rank 4, such that $H(M)$ is contained in a 6-dimensional real algebraic subset of $\mathbf{C}^{N'}$ is algebraic on the leaves $\{w_2=c\}$, for all sufficiently small $c\in\mathbf{C}$. Again, this result is optimal, because the biholomorphism

$$H(z_1,z_2,w_1,w_2)=(z_1e^{iw_2},z_2,w_1,w_2)$$

maps the set A into itself. This map is only algebraic on the leaves $\{w_2=c\}$ and not in the whole space.

Example 3.1.6. Consider the submanifold of $\mathbf{C}^4$ defined by

$$\begin{cases}\operatorname{Re} w_1=|z_1|^2,\\ \operatorname{Im} w_1=|z_2|^2,\\ \operatorname{Im} w_2=0.\end{cases}\tag{3.1.6}$$

Note that this submanifold is not generic (nor is it CR!) on the set $\{(z_1,z_2,w_1,w_2)=(0,0,0,s_2):s_2\in\mathbf{R}\}$. However, outside that set the manifold (3.1.6) is a generic holomorphically nondegenerate manifold M. The function $h(z,w)=w_2$ is real on M, and

$M\cap\{w_2=s_2\}$ is generically minimal for all $s_2\in\mathbf{R}$. As above, Theorem 3.1.2 implies that any holomorphic mapping $H:\mathbf{C}^4\to\mathbf{C}^{N'}$ near 0, generically of rank 4, such that $H(M)$ is contained in a 5-dimensional real algebraic subset of $\mathbf{C}^{N'}$ is algebraic on the leaves $\{w_2=c\}$, for all sufficiently small $c\in\mathbf{C}$. We invite the reader to construct an example, e.g. similar to the ones considered above, to show that one cannot have a stronger conclusion.

We can also formulate a result that holds at most, but not necessarily all, points of the algebraic set.

THEOREM 3.1.7. *Let $A\subset\mathbf{C}^N$ be an irreducible, holomorphically nondegenerate, real algebraic set, and let $\mathcal{V}$ be a complex algebraic variety in $\mathbf{C}^N$ that contains A. Then either of the following holds, for all points $p\in A_{\mathrm{reg}}$ outside a proper real algebraic subset of A:*

(i) *There is $h\in\mathcal{A}_{\mathcal{V}}(p)$ such that h is not constant and $h|_A$ is real-valued.*

(ii) *All mappings $H\in\mathrm{Hol}_{\mathcal{V}}(p,p')$, where $p'\in\mathbf{C}^{N'}$ is arbitrary, such that the generic rank of H equals $\dim_{\mathbf{C}}\mathcal{V}$ and such that $H(A)$ is contained in a real algebraic set A', with $p'\in A'$ and $\dim_{\mathbf{R}}A=\dim_{\mathbf{R}}A'$, are algebraic in $\mathcal{V}$, i.e. $H\in\mathcal{A}_{\mathcal{V}}(p,p')$.*

Before we proceed with the proofs of Theorems 3.1.2 and 3.1.7 (§3.3 and §3.4), we need a result on "propagation of algebraicity" that we establish in the next subsection.

3.2. Propagation of algebraicity

We assume that we have an algebraic foliation of some domain in complex space, and a holomorphic function f whose restriction to a certain sufficiently large collection of the leaves is algebraic. We shall show that the restrictions of f to all leaves in the domain are also algebraic, provided that the domain has a nice "product structure" with respect to the foliation. This will be essential in the proof of Theorem 3.1.2. This result may already be known.

LEMMA 3.2.1. *Let $f(z,w)$ be a holomorphic function in $U\times V$, where $U\subset\mathbf{C}^a_z$ and $V\subset\mathbf{C}^b_w$ are domains. Assume that there is a subdomain $V_0\subset V$ and a nontrivial polynomial $P(z,X;w)\in\mathcal{O}_b(V_0)[z,X]$, i.e. P is a polynomial in $z=(z_1,...,z_a)$ and X with coefficients holomorphic in V_0, such that*

$$P(z,f(z,w);w)\equiv 0,\quad z\in U,\ w\in V_0. \tag{3.2.1}$$

Then there is a nontrivial polynomial $\widetilde{P}(z,X;w)\in\mathcal{O}_b(V)[z,X]$ such that

$$\widetilde{P}(z,f(z,w);w)\equiv 0,\quad z\in U,\ w\in V. \tag{3.2.2}$$

Proof. Pick any point $w^0 \in V_0$, and consider $P=P(z,X;w)$ as an element of $\mathcal{O}_b(w^0)[z,X]$. We order the monomials z^α by choosing a bijection $i\colon \mathbf{Z}_+^a \to \mathbf{Z}_+$, and write

$$P(z,X;w)=\sum_{k=0}^{p} p_k(z;w)X^k, \tag{3.2.3}$$

where each $p_k(z;w)\in\mathcal{O}_b(w^0)[z]$ is of the form

$$p_k(z;w)=\sum_{i(\alpha)\leqslant q_k} a_\alpha^k(w)z^\alpha, \tag{3.2.4}$$

with $a_\alpha^k\in\mathcal{O}_b(w^0)$. We choose $p,q_1,\ldots,q_p$ minimal such that P can be written in this form with the leading terms in (3.2.3) and (3.2.4) not identically 0. We may assume that the numbers p and q_p are minimal in the sense that if P' is another polynomial in $\mathcal{O}_b(w^0)[z,X]$, with corresponding numbers p' and $q'_{p'}$, such that (3.2.1) holds then $p\leqslant p'$ and if $p=p'$ then $q_p\leqslant q'_{p'}$. The polynomial P is then unique modulo multiplication by elements in $\mathcal{O}_b(w^0)$ in the sense that if P' is as above with $p'=p$ and $q'_{p'}=q_p$ then there are germs $c_1(w),c_2(w)\in\mathcal{O}_b(w^0)$, not identically 0, such that

$$c_1(w)P(z,X;w)\equiv c_2(w)P'(z,X;w). \tag{3.2.5}$$

Since p is minimal, the function $p_0(z;w)$ is not identically 0, and thus there is a coefficient $a_{\alpha_0}^0(w)$ which is not identically 0. The equation (3.2.1) can then be written in the form

$$Q(z,f(z,w);w)\equiv -a_{\alpha_0}^0(w)z^{\alpha_0}, \tag{3.2.6}$$

with $Q(z,X;w)\in\mathcal{O}_b(w^0)[z,X]$. Now, the uniqueness of P in the sense of (3.2.5) and the fact that $a_{\alpha_0}^0\not\equiv 0$ imply that the coefficients of $Q(z,X;w)$ satisfying (3.2.6) are actually unique. After dividing (3.2.6) by $-a_{\alpha_0}^0(w)$, we find $Q'(z,X;w)\in\mathcal{M}_b(w^0)[z,X]$ satisfying

$$Q'(z,f(z,w);w)\equiv z^{\alpha_0}, \tag{3.2.7}$$

where $\mathcal{M}_b(w^0)$ denotes the field of germs of meromorphic functions at w_0.

We order the set of indices (k,α), for $k=0,\ldots,p$ and $i(\alpha)\leqslant q_k$, minus the index $(0,\alpha_0)$ in some way, e.g. the "canonical" way induced by the ordering $i=i(\alpha)$. We hence obtain a bijection $(k,\alpha)\mapsto j(k,\alpha)$ from this set of indices to the set of numbers $\{1,2,\ldots,\mu\}$, where μ is the number of elements in this set of indices. We introduce the $\mathbf{C}^\mu$-valued functions $A(z;w)$ defined by letting the jth component be

$$A_j(z;w)=z^\alpha f(z,w)^k, \quad \text{for } j=j(k,\alpha), \tag{3.2.8}$$

and $b(w)$ defined by

$$b_j(w) = \frac{a_\alpha^k(w)}{-a_{\alpha_0}^0(w)}, \quad \text{for } j = j(k,\alpha). \tag{3.2.9}$$

Then (3.2.7) can be written

$$A(z;w)\cdot b(w) \equiv z^{\alpha_0}, \tag{3.2.10}$$

where $\cdot$ denotes the usual dot product of vectors in $\mathbf{C}^\mu$. Moreover, the vector-valued meromorphic function $b(w)$ is the unique meromorphic solution of (3.2.10). Consider the $(\mu\times\mu)$-matrix-valued holomorphic function $B(z_1, ..., z_\mu, w)$ (of $a\mu+b$ variables) defined by letting the matrix element $B_{ij}(z_1, ..., z_\mu; w)$, for $i, j = 1, ..., \mu$, be

$$B_{ij}(z_1, ..., z_\mu; w) = A_j(z_i; w). \tag{3.2.11}$$

We claim that the determinant of $B(z_1, ..., z_\mu; w)$ is not identically 0. Indeed, if it were, then we could find a vector-valued holomorphic function $c(w)$, not identically 0, such that $A(z;w)\cdot c(w) \equiv 0$, which would contradict the uniqueness of the solution $b(w)$ of (3.2.10). Thus, we can find fixed values $z_1^0, ..., z_\mu^0$ such that $\Delta(w)$, the determinant of $B(z_1^0, ..., z_\mu^0; w)$ as a function of w, is not identically 0. We can then solve for $b(w)$ as the unique solution of the system obtained from (3.2.10) after substituting successively $z_1^0, ..., z_\mu^0$ for z. Since the matrix $B(z_1^0, ..., z_\mu^0; w)$ has entries holomorphic in all of V, by Cramer's rule it follows that the solution $b(w)$ thus obtained is in $\mathcal{M}_b(V)$. Hence $Q'(z, X; w) \in \mathcal{M}_b(V)[z, X]$. After clearing denominators we obtain (3.2.2) from (3.2.7). This completes the proof of Lemma 3.2.1. □

3.3. Proof of Theorem 3.1.2

First, since all assumptions and conclusions in the theorem are related to $\mathcal{V}$, and $p_0 \in \mathcal{V}$ is a regular point of $\mathcal{V}$, it suffices to consider the case where $\mathcal{V} = \mathbf{C}^N$ and M is generic; we will assume this for the rest of the proof. By assumption (i) in the theorem, we can find algebraic coordinates $(u, v) \in \mathbf{C}^{N-q} \times \mathbf{C}^q$, vanishing at p_0, in a neighborhood U_1 of p_0 such that $h_j = v_j$ for $j = 1, ..., q$. We may assume $U_1 = A_1 \times B_1$, where $A_1 \subset \mathbf{C}^{N-q}$ and $B_1 \subset \mathbf{C}^q$. It follows from the assumptions that $M \cap S_{h(p)}$ is minimal at p for p outside a proper real algebraic subset of $M \cap U_1$. Similarly, M is $l(M)$-nondegenerate outside a proper real algebraic subset of M, where $l(M)$ is the Levi number defined in §1.3. Also, the mapping H attains its maximal rank outside a proper complex-analytic subset of $\mathbf{C}^N$ near p_0, and since M is generic it is not contained in any proper complex-analytic set. Thus, H attains its maximal rank at points on M outside a proper real-analytic subset of M. Finally, for each j, the jth Segre set $N_j(p)$ of M at p (defined in §2.2) has maximal generic dimension for p outside a proper real algebraic subset of M. Hence we can find $p_1 \in M \cap U_1$ such that

(a) $M \cap S_{h(p_1)}$ is minimal at p_1,

(b) H has rank N at p_1,

(c) M is $l(M)$-nondegenerate at p_1,

(d) for each j, the generic dimension d_j of $N_j(p_1)$ is maximal.

We will prove Theorem 3.1.2 by first showing that there is a neighborhood of p_1 in $\mathbf{C}^N$ such that $H|_{S_{h(p)}}$ is algebraic for every p in that neighborhood, and then applying Lemma 3.2.1 to deduce the full statement of the theorem. For this, we claim that we may assume that the target A' is contained in $\mathbf{C}^N$ and that H is a mapping into $\mathbf{C}^N$. Indeed, there is a neighborhood $U_2 \subset U_1$ of the point p_1 such that $Y' = H(U_2)$ is a complex submanifold of dimension N in $\mathbf{C}^{N'}$ through the point $p_1' = H(p_1)$. Since M is generic and H is a biholomorphism of U_2 onto Y', it follows that A' is a generic submanifold of Y' near p_1'. Denote by M' a piece of A' near p_1' and choose it such that M' is a generic submanifold of Y'. Then, M' is real algebraic and its intrinsic complexification $\mathcal{V}' \subset \mathbf{C}^{N'}$ is a complex algebraic manifold near p_1'. Since both Y' and $\mathcal{V}'$ contain M', and M' is generic in both manifolds, it follows that $Y' = \mathcal{V}'$. We can therefore choose algebraic coordinates in a neighborhood U_2' of p_1' in $\mathbf{C}^{N'}$, vanishing at p_1', such that $H = (\widehat{H}, 0)$ in these coordinates and $\widehat{H}$ maps $M \cap U_2$ into $M' \cap U_2' \subset Y' \cong \mathbf{C}^N$. In what follows, we assume that $\mathbf{C}^{N'} = \mathbf{C}^N$ and we take $\widehat{H}$ as our mapping H.

Let $(z, w) \in \mathbf{C}^n \times \mathbf{C}^d = \mathbf{C}^N$, where n is the CR dimension and d the codimension of M, be (algebraic) normal coordinates for M, vanishing at p_1, i.e. M is defined near p_1 by (1.1.3) and similarly for the target M' (denoting the function defining M' by Q'). We write $(z, w) = (z(u, v), w(u, v))$ to indicate the relationship between the local normal coordinates (z, w) near p_1 and the coordinates (u, v) in U_1. Thus, we can write the mapping H as $H = (f, g)$, where $f(z, w) \in \mathbf{C}^n$ and $g(z, w) \in \mathbf{C}^d$, such that

$$\bar{g} = \bar{Q}'(\bar{f}, f, g) \tag{3.3.1}$$

holds for points $(z, w) \in M$ near $p_1 = 0$. By complexifying, we obtain

$$\bar{g}(\chi, \tau) = \bar{Q}'(\bar{f}(\chi, \tau), f(z, w), g(z, w)), \tag{3.3.2}$$

for all $(z, w, \chi, \tau) \in \mathcal{M}$ near 0. We define the holomorphic vector fields $\mathcal{L}_j$ in $\mathbf{C}^{2N}$ tangent to $\mathcal{M}$ (and resulting from the complexification of the CR vector fields of M) by

$$\mathcal{L}_j = \frac{\partial}{\partial \chi_j} + \sum_{k=1}^{d} \bar{Q}_{k, \chi_j}(\chi, z, w) \frac{\partial}{\partial \tau_k}, \quad j = 1, \ldots, n. \tag{3.3.3}$$

We shall also need the following vector fields tangent to $\mathcal{M}$,

$$\begin{aligned}\tilde{\mathcal{L}}_j &= \frac{\partial}{\partial z_j}+\sum_{k=1}^{d} Q_{k,z_j}(z,\chi,\tau)\frac{\partial}{\partial w_k}, \quad j=1,\ldots,n,\\ \mathcal{T}_j &= \frac{\partial}{\partial w_j}+\sum_{k=1}^{d} \bar{Q}_{k,w_j}(\chi,z,w)\frac{\partial}{\partial \tau_k}, \quad j=1,\ldots,d,\\ V_j &= \tilde{\mathcal{L}}_j-\sum_{k=1}^{d} Q_{k,z_j}(z,\chi,\tau)\mathcal{T}_k, \quad j=1,\ldots,n.\end{aligned} \tag{3.3.4}$$

Note that the coefficients of all the vector fields given by (3.3.3) and (3.3.4) are algebraic functions of (z,w,χ,τ).

Assertion 3.3.1. *There is a neighborhood $U_3\subset U_2$ such that, for all $(z,w,\chi,\tau)\in \mathcal{M}\cap(U_3\times{}^*U_3)$ and all multi-indices $\gamma=(\gamma',\gamma'')$,*

$$\frac{\partial^{|\gamma|}f_j}{\partial z^{\gamma'}\partial w^{\gamma''}}(z,w)=\Psi_j^{\gamma}(\ldots,V^{\alpha^3}\mathcal{T}^{\alpha^2}\mathcal{L}^{\alpha^1}\bar{f}_k(\chi,\tau),\ldots,V^{\beta^3}\mathcal{T}^{\beta^2}\mathcal{L}^{\beta^1}\bar{g}_l(\chi,\tau),\ldots), \tag{3.3.5}$$

where $j,k=1,\ldots,n$, $l=1,\ldots,d$, $|\alpha^1|,|\beta^1|\leqslant l(M)$, $|\alpha^2|,|\beta^2|\leqslant|\gamma''|$, $|\alpha^3|,|\beta^3|\leqslant|\gamma'|$, and the Ψ_j^{γ} are algebraic holomorphic functions of their arguments.

Proof. We apply the operators $\mathcal{L}_j$ to the identity (3.3.2), and use the fact that the matrix $\mathcal{L}\bar{f}$ at $(z,w,\chi,\tau)=(0,0,0,0)$ is invertible (since H is a biholomorphism at $p_1=0$) to deduce that there are algebraic functions F_j such that, for points on $\mathcal{M}$ near 0,

$$\bar{Q}'_{\chi_j}(\bar{f},f,g)=F_j(\mathcal{L}\bar{f},\mathcal{L}\bar{g}). \tag{3.3.6}$$

We repeat this procedure, using in the next step (3.3.6) instead of (3.3.2) and so on. Since H is a biholomorphism at p_1, M' is $l(M)$-nondegenerate at p_1'. Hence (see §1.3)

$$\operatorname{span}\{\bar{Q}'_{z,\chi^{\alpha}}(0,0,0):|\alpha|\leqslant l(M)\}=\mathbf{C}^n. \tag{3.3.7}$$

It follows from the algebraic implicit function theorem and from (3.3.7) that, for all $(z,w,\chi,\tau)\in\mathcal{M}$ near the origin,

$$f_j(z,w)=\Psi_j(\ldots,\mathcal{L}^{\alpha}\bar{f}_k(\chi,\tau),\ldots,\mathcal{L}^{\beta}\bar{g}_l(\chi,\tau),\ldots), \quad j=1,\ldots,n, \tag{3.3.8}$$

where $k=1,\ldots,n$, $l=1,\ldots,d$, $|\alpha|,|\beta|\leqslant l(M)$, and the Ψ_j are algebraic holomorphic functions of their arguments (cf. e.g. [BR4, Lemma 2.3]). Now, since $f(z,w)$ is a function of (z,w) only, we have, for any multi-index $\gamma=(\gamma',\gamma'')$,

$$V^{\gamma'}\mathcal{T}^{\gamma''}f(z,w)=\frac{\partial^{|\gamma|}f_j}{\partial z^{\gamma'}\partial w^{\gamma''}}(z,w). \tag{3.3.9}$$

The assertion follows if we apply $V^{\gamma'}\mathcal{T}^{\gamma''}$ to (3.3.8), which is possible since the V_j and $\mathcal{T}_l$ are tangent to $\mathcal{M}$. □

We now proceed with the proof of Theorem 3.1.2. From (3.3.2) we have

$$g_l(z,w)=Q_l'(f(z,w),\bar{f}(\chi,\tau),\bar{g}(\chi,\tau)) \tag{3.3.10}$$

for $(z,w,\chi,\tau)\in\mathcal{M}$ and $l=1,...,d$. If we apply $V^{\gamma'}\mathcal{T}^{\gamma''}$ to this equation we obtain

$$\begin{aligned}&\frac{\partial^{|\gamma|}g_l}{\partial z^{\gamma'}\partial w^{\gamma''}}(z,w)\\ &=\Phi_l^\gamma\left(...,\frac{\partial^{|\alpha^1|+|\alpha^2|}f}{\partial z^{\alpha^2}\partial w^{\alpha^1}}(z,w),...,V^{\beta^2}\mathcal{T}^{\beta^1}\bar{f}_j(\chi,\tau),...,V^{\mu^2}\mathcal{T}^{\mu^1}\bar{g}_k(\chi,\tau),...\right),\end{aligned} \tag{3.3.11}$$

where $j=1,...,n$, $k,l=1,...,d$, $|\alpha^1|,|\beta^1|,|\mu^1|\leqslant|\gamma''|$, $|\alpha^2|,|\beta^2|,|\mu^2|\leqslant|\gamma'|$, and where Φ_l^γ are algebraic holomorphic functions of their arguments. Using (3.3.5), we obtain

$$\frac{\partial^{|\gamma|}g_l}{\partial z^{\gamma'}\partial w^{\gamma''}}(z,w)=\Xi_l^\gamma(...,V^{\alpha^3}\mathcal{T}^{\alpha^2}\mathcal{L}^{\alpha^1}\bar{f}_j(\chi,\tau),...,V^{\beta^3}\mathcal{T}^{\beta^2}\mathcal{L}^{\beta^1}\bar{g}_k(\chi,\tau),...), \tag{3.3.12}$$

where $j=1,...,n$, $k,l=1,...,d$, $|\alpha^1|,|\beta^1|\leqslant l(M)$, $|\alpha^2|,|\beta^2|\leqslant|\gamma''|$, $|\alpha^3|,|\beta^3|\leqslant|\gamma'|$, and the Ξ_l^γ are holomorphic algebraic functions of their arguments. For notational brevity, we use the notation $Z=(z,w)$ and $\zeta=(\chi,\tau)$. If we denote by $(H_1(Z),...,H_N(Z))$ the components of H in an arbitrary algebraic coordinate system near the point $p_1'=H(p_1)$ then it follows from (3.3.5) and (3.3.12) that we have

$$\frac{\partial^{|\gamma|}H_k}{\partial Z^\gamma}(Z)=\Theta_k^\gamma\left(Z,\zeta,...,\frac{\partial^\alpha\overline{H}_j}{\partial\zeta^\alpha}(\zeta),...\right), \tag{3.3.13}$$

where $j,k=1,...,N$, γ arbitrary, $|\alpha|\leqslant|\gamma|+l(M)$, and Θ_k^γ are holomorphic algebraic functions of their arguments for $(Z,\zeta)\in\mathcal{M}$ near $(p_1,\bar{p}_1)$.

ASSERTION 3.3.2. *For $Z\in M$ near p_1, let $N_j(Z)$ denote the jth Segre set of M at Z and d_j the generic dimension of $N_j(p_1)$. For some j, $1\leqslant j\leqslant j_0-1$, let*

$$\mathbf{C}^{d_j}\times\mathbf{C}^N\ni(s,Z)\mapsto\zeta(s,Z)\in\mathbf{C}^N \tag{3.3.14}$$

be an algebraic map, holomorphic near $(0,p_1)$. Suppose that $s\mapsto\zeta(s,p_1)$ has generic rank d_j, and $\zeta(s,Z)\in{}^*N_j(Z)$ for $z\in M$. Then there is an algebraic map

$$\mathbf{C}^{d_{j+1}}\times\mathbf{C}^N\ni(t,Z)\mapsto(\Pi(t,Z),s(t))\in\mathbf{C}^N\times\mathbf{C}^{d_j}, \tag{3.3.15}$$

holomorphic near $(0,p_1)$, *such that* $(\Pi(t,Z),\zeta(s(t),Z))\in\mathcal{M}$, *the mapping* $t\mapsto\Pi(t,Z)$ *has generic rank* d_{j+1}, *and* $\Pi(t,Z)\in N_{j+1}(Z)$, *for all* $Z\in M$ *near* p_1.

Remark. If $\zeta(s,Z)$ is algebraic anti-holomorphic in Z rather than algebraic holomorphic then the same conclusion holds with "holomorphic" replaced by "anti-holomorphic".

Proof. Note first that by assumption (d) in the choice of p_1, d_j is also the generic dimension of $N_j(Z)$ for Z near p_1. We write the map $\zeta(s,Z)$ in the normal coordinates as $(\chi(s,Z),\tau(s,Z))$. For $Z\in M$ fixed near p_1 consider the map

$$\mathbf{C}^n\times\mathbf{C}^{d_j}\ni(z,s)\mapsto(z,Q(z,\chi(s,Z),\tau(s,Z)))\in\mathbf{C}^n\times\mathbf{C}^d=\mathbf{C}^N. \tag{3.3.16}$$

Note that $(z,Q(z,\chi(s,Z),\tau(s,Z)),\chi(s,Z),\tau(s,Z))\in\mathcal{M}$. Since $N_{j+1}(Z)$, for $Z\in M$, is defined as $\{(z,Q(z,\chi,\tau)):\exists(\chi,\tau)\in{}^*N_j(Z)\}$, and the mapping $s\mapsto\zeta(s,Z)\in{}^*N_j(Z)$ has rank d_j, which is also the generic dimension of ${}^*N_j(Z)$, it is easy to verify that the map (3.3.16) has generic rank d_{j+1}. Thus, by the rank theorem, there is an algebraic map

$$\mathbf{C}^{d_{j+1}-n}\ni t'\mapsto s(t')\in\mathbf{C}^{d_j} \tag{3.3.17}$$

such that

$$\mathbf{C}^n\times\mathbf{C}^{d_{j+1}-n}\ni(z,t')=t\mapsto(z,Q(z,\chi(s(t'),Z),\tau(s(t'),Z)))\in\mathbf{C}^N \tag{3.3.18}$$

has rank d_{j+1}. The proof of Assertion 3.3.2 follows by taking $t=(z,t')$, $s(t)=s(t')$ and

$$\Pi(t,Z)=(z,Q(z,\chi(s(t'),Z),\tau(s(t'),Z))). \tag{3.3.19}$$

□

Now, define the map $\Pi_0(Z)=Z$ and the map $\zeta_0(Z)=\bar{Z}$. The latter, thought of as a map $\mathbf{C}^0\times\mathbf{C}^N\mapsto\mathbf{C}^N$, satisfies the hypothesis of Assertion 3.3.2 above with $j=0$ (see the remark following the assertion). Thus, we get an algebraic map, anti-holomorphic in Z,

$$\mathbf{C}^{d_1}\times\mathbf{C}^N\ni(t,Z)\mapsto\Pi_1(t,Z)\in\mathbf{C}^N, \tag{3.3.20}$$

of rank d_1 in t for Z near $p_1\in M$, such that $(\Pi_1(t,Z),\bar{Z})\in\mathcal{M}$, and $\Pi_1(t,Z)\in N_1(Z)$ for $Z\in M$. Defining the map $\zeta_1(t,Z)$ by

$$\zeta_1(t,Z)=\overline{\Pi_1(\bar{t},Z)} \tag{3.3.21}$$

we obtain a map into ${}^*N_1(Z)$ that satisfies the hypothesis of the assertion with $j=1$ (this time the map is holomorphic in Z). Applying the assertion again and proceeding inductively, we obtain a sequence of algebraic maps

$$\Pi_0(Z),\Pi_1(t_1,Z),\dots,\Pi_{j_0}(t_{j_0},Z),\zeta_0(Z),\zeta_1(s_1,Z),\dots,\zeta_{j_0-1}(s_{j_0-1},Z)$$

(either holomorphic or anti-holomorphic in Z) with $t_j \in \mathbf{C}^{d_j}$, $s_j \in \mathbf{C}^{d_j}$, and accompanying maps $s_1(t_2), ..., s_{j_0-1}(t_{j_0})$ such that the maps $t_j \mapsto \Pi_j(t_j, Z)$ and $s_j \mapsto \zeta_j(s_j, Z)$ are of rank d_j, map into $N_j(Z)$ and $^*N_j(Z)$ respectively for $Z \in M$, and satisfy

$$(\Pi_{j+1}(t_{j+1}, Z), \zeta_j(s_j(t_{j+1}), Z)) \in \mathcal{M}, \tag{3.3.22}$$

for $j=0, ..., j_0-1$. Morever, we have the relation

$$\zeta_j(s_j, Z) = \overline{\Pi_j(\bar{s}_j, Z)}. \tag{3.3.23}$$

ASSERTION 3.3.3. *For each $j=1, ..., j_0$,*

$$\frac{\partial^{|\gamma|} H_k}{\partial Z^\gamma}(\Pi_j(t_j, Z)) = F^\gamma_{jk}\left(t_j, Z, \bar{Z}, ..., \frac{\partial^{|\alpha|} H_l}{\partial Z^\alpha}(Z), ..., \frac{\partial^{|\beta|} \bar{H}_l}{\partial \zeta^\beta}(\bar{Z}), ...\right) \tag{3.3.24}$$

holds for $Z \in M$ near p_1, where $k, l=1, ..., N$, $|\alpha|, |\beta| \leqslant |\gamma| + jl(M)$, and F^γ_{jk} are holomorphic algebraic functions of their arguments.

Proof. The proof is by induction on j. For $j=1$, we prove the statement by taking Z to be $\Pi_1(t_1, Z)$ and ζ to be $\zeta_0(Z) = \bar{Z}$ in (3.3.13) (using (3.3.22)). Assume now that (3.3.24) holds for $j=1, ..., i$ (with $i < j_0$). By (3.3.23) we have

$$\frac{\partial^{|\gamma|} \bar{H}_k}{\partial \zeta^\gamma}(\zeta_i(s_i, Z)) = \bar{F}^\gamma_{jk}\left(s_j, \bar{Z}, Z, ..., \frac{\partial^{|\alpha|} \bar{H}_l}{\partial \zeta^\alpha}(\bar{Z}), ..., \frac{\partial^{|\beta|} H_l}{\partial Z^\beta}(Z), ...\right). \tag{3.3.25}$$

Now (3.3.24) follows for $j=i+1$ from (3.3.25) by taking Z to be $\Pi_{i+1}(t_{i+1}, Z)$ and ζ to be $\zeta_i(s_i(t_{i+1}), Z)$ in (3.3.13). □

We now complete the proof of Theorem 3.1.2. For p near p_1 it follows from Corollary 2.2.2, since $M \cap S_{h(p)}$ is minimal, that the maximal Segre set $N_{j_0}(p)$ is contained in and contains an open piece of $S_{h(p)}$. Since M is generic, it is easy to see that $h(M)$ contains an open neighborhood of $c^1 = h(p_1)$ in $\mathbf{R}^q$. Thus, by the rank theorem, and using the coordinates (u, v) in U_1, there is a real algebraic injective map $\mathbf{R}^q \ni c \mapsto (u(c), c) \in M$, for c near c^1, which can be complexified to an algebraic injective map $v \mapsto (u(v), v)$, for v in a neighborhood of c^1 in $\mathbf{C}^q$. Now, let Z be the point $Z(c) = (z(u(c), c), w(u(c), c))$ where $c \in \mathbf{R}^q$ is some arbitrary point near c^1. Applying Assertion 3.3.3 with this choice of Z, $\gamma=0$ and $j=j_0$, we deduce that each component H_l is algebraic on S_c and satisfies there a polynomial equation with coefficients that depend real-analytically on c (we may take t_{j_0} as algebraic coordinates on S_c). In terms of the coordinates (u, v) with $\widetilde{H}(u, v) = H(Z(u, v))$, there are polynomials $P_l(u, X; c)$ in $(u, X) \in \mathbf{C}^{N-q} \times \mathbf{C}$, $l=1, ..., N$, with coefficients that are real-analytic functions in c, for c close to c^1, such that

$$P_l(u, \widetilde{H}_l(u, c); c) \equiv 0 \tag{3.3.26}$$

(u are also algebraic coordinates on S_c and it is easy to see that the algebraic change of coordinates $u=u(t_{j_0})$ on S_c depends real algebraically on c). Extending the coefficients of the polynomials to be complex-analytic functions of v in a neighborhood B_2 of c^1 in $\mathbf{C}^q$, we obtain polynomials $P_l(u,X;v)\in\mathcal{O}_q(B_2)[u,X]$ such that

$$P_l(u,\widetilde{H}_l(u,v);v)\equiv 0 \tag{3.3.27}$$

holds in $A_1\times B_2$. Since $\widetilde{H}_l(u,v)$ is holomorphic in $A_1\times B_1$, there is, by Lemma 3.2.1, possibly another polynomial $\widetilde{P}_l(u,X;v)\in\mathcal{O}_q(B_1)[u,X]$ such that

$$\widetilde{P}_l(u,\widetilde{H}_l(u,v);v)\equiv 0 \tag{3.3.28}$$

holds in $U_1=A_1\times B_1$. This completes the proof of Theorem 3.1.2 with $U=U_1$, and $\delta>0$ being any number such that the ball of radius δ centered at $v=0$ is contained in B_1 (recall that B_1 is a neighborhood of $v=0$). The proof of Theorem 3.1.2 is now complete. □

3.4. Proof of Theorem 3.1.7

Put $M=A_{\mathrm{reg}}$. First, note that if M is contained in a proper complex algebraic subvariety of $\mathcal{V}$ then (i) holds for all points $p\in M$. If M is not contained in a proper algebraic subvariety of $\mathcal{V}$ then M is a generic real algebraic submanifold of $\mathcal{V}_{\mathrm{reg}}$ at p, for all p outside some proper real algebraic subvariety of M. Thus, as in the proof of Theorem 3.1.2, we may assume that $\mathcal{V}=\mathbf{C}^N$ and that M is a generic holomorphically nondegenerate submanifold in $\mathbf{C}^N$. Let $p_0\in M$ be a point whose CR orbit has maximal dimension. If M is minimal at p_0 then (ii) holds with $p=p_0$, by Corollary 3.1.4. Moreover, if M is minimal at p_0 then M is minimal for p outside a real algebraic variety and therefore (ii) holds at such p. Thus, the theorem follows if we can show that M is minimal at p_0 unless (i) holds at p_0. The proof of Theorem 3.1.7 will then be completed by the following lemma.

LEMMA 3.4.1. *Let M be a generic real algebraic submanifold in $\mathbf{C}^N$, and let p_0 be a point in M with CR orbit of maximal dimension. Then M is minimal at p_0 if and only if there is no nonconstant $h\in\mathcal{A}_N(p_0)$ such that $h|_M$ is real-valued. More precisely, if the codimension of the local CR orbit of p_0 in M is q then there are $h_1,\ldots,h_q\in\mathcal{A}_N(p_0)$ such that $h_j|_M$ is real-valued for $j=1,\ldots,q$ and*

$$\partial h_1(p_0)\wedge\ldots\wedge\partial h_q(p_0)\neq 0. \tag{3.4.1}$$

Remark. Lemma 3.4.1 implies that the decomposition of M into CR orbits near p_0 is actually an algebraic foliation, because the CR orbit of a point p_1 near p_0 must equal

$\{p\in M: h_j(p)=h_j(p_1),\, j=1,\dots,q\}$. If Corollary 2.2.5 is viewed as an algebraic version of the Nagano theorem (for the special class of algebraic vector fields that arise in this situation; see the paragraph following Corollary 2.2.5) then this lemma is the algebraic version of the Frobenius theorem.

Proof of Lemma 3.4.1. Assume that there is a nonconstant $h\in\mathcal{A}_N(p_0)$ such that $h|_M$ is real. Then, by Lemma 3.1.1, $M\cap\{Z: h(Z)=h(p)\}$ is a CR submanifold for all $p\in M$ near p_0 such that $dh(p)\neq 0$. Since h is real on M all CR and anti-CR vector fields tangent to M annihilate h; hence the submanifold $M\cap\{Z: h(Z)=h(p)\}$ has the same CR dimension as M. Thus, M is not minimal at p. Since this is true for all $p\in M$ near p_0 outside a proper real algebraic subset, M is not minimal anywhere. This proves the "only if" part of the first statement of the lemma. The "if" part will follow from the more precise statement at the end of the lemma, which we shall now prove.

We choose algebraic normal coordinates $(z,w)\in\mathbf{C}^N$, vanishing at p_0, such that M is given by (1.1.3) near p_0. Denote by W_0 the CR orbit of $p_0=0$, and by X_0 its intrinsic complexification. By Theorem 2.2.1, $N_{j_0}(p_0)$, the maximal Segre set of M at p_0, is contained in and contains an open piece of X_0. The complex dimension of X_0 is d_{j_0}, the generic dimension of $N_{j_0}(p_0)$. Since the codimension of W_0 in M is q, the complex codimension of its intrinsic complexification X_0 is also q, i.e. $d_{j_0}=n+d-q$. Let $r=d-q$. By a linear change of the w coordinates, we may assume that the tangent plane of X_0 at 0 is $\{(z,w): w_{r+1}=\dots=w_d=0\}$. We decompose w as $(w',w'')\in\mathbf{C}^r\times\mathbf{C}^q=\mathbf{C}^d$. Note that at the point $\tilde p=(0,s)\in M$, where $s=(s',s'')\in\mathbf{R}^r\times\mathbf{R}^q$, $(\tilde z,\widetilde w)=(z,w-s)$ are normal coordinates vanishing at $\tilde p$ and M is given by

$$\widetilde w = Q(\tilde z,\bar{\tilde z},\bar{\widetilde w}+s)-s. \tag{3.4.2}$$

We denote by $W_{s''}$ the local CR orbit of $(0,0,s'')$, by $X_{s''}$ its intrinsic complexification, and by $N_{j_0}(s'')$ the maximal Segre set at $(0,0,s'')$. Since the CR orbit at p_0 has maximal dimension, all $W_{s''}$, $X_{s''}$ and $N_{j_0}(s'')$ have dimension $d_{j_0}=n+r$ for s'' near 0 in $\mathbf{R}^q$. Using the parametrizations (2.2.10), (2.2.13) and writing $\Lambda=(z,\Lambda')$, we can express $N_{j_0}(s'')$ in the coordinates (z,w) by

$$w = v^{j_0}(z,\Lambda';s''), \tag{3.4.3}$$

where $\Lambda'\in\mathbf{C}^{(j-1)n}$. Since the defining equations (3.4.2) of M at $(0,0,s'')$ depend algebraically on s'', it follows that $v^{j_0}(\,\cdot\,;s'')$ also does (cf. (2.2.10)–(2.2.12) and (2.2.13)–(2.2.15)). At a point $(z,\Lambda';0)$ where $\partial v^{j_0}/\partial\Lambda'$ has maximal rank $r=d_{j_0}-n$, we may assume (by a change of coordinates in the Λ'-space if necessary) that $\Lambda'=(\Lambda_1,\Lambda_2)\in\mathbf{C}^r\times\mathbf{C}^{(j-1)n-r}$, v^{j_0} is independent of Λ_2, and $\partial v^{j_0}/\partial\Lambda_1$ has rank r. Since the tangent plane of X_0 at 0 equals $\{w''=0\}$, it follows from the implicit function theorem that we can solve for Λ_1 in the first r equations of (3.4.3). We then substitute this into the last q equations and find that we can express $N_{j_0}(s'')$, for s'' close to 0, as a graph

$$w''=f(z,w';s'') \tag{3.4.4}$$

near some point $(z^1,w'^1,f(z^1,w'^1;s''))$ with $f(z,w';s'')$ holomorphic algebraic in a neighborhood of $(z^1,w^1;0)$. Now, since all the CR orbits near p_0 have the same dimension, it follows from the Frobenius theorem that they form a real-analytic foliation of a neighborhood of p_0 in M (as we have noted before, the Frobenius theorem does not imply that the orbits form a real *algebraic* foliation even though the vector fields are algebraic). Thus, there are q real-valued, real-analytic functions $k=(k_1,...,k_q)$ on M with linearly independent differentials near p_0 such that every local CR orbit near this point is of the form $\{(z,w)\in M: k(z,w)=c\}$ for some small $c\in\mathbf{R}^q$ (we may assume that $k(0)=0$). Since $(0,0,s'')\in W_{s''}$, we have

$$W_{s''}=\{(z,w',w''))\in M: k(z,w',w'')=k(0,0,s'')\}. \tag{3.4.5}$$

Clearly, these functions are CR and, hence, they extend, near 0 in $\mathbf{C}^N$, as holomorphic functions which we again denote by k. It follows that each $X_{s''}$, for real s'' close to 0, is given by

$$X_{s''}=\{(z,w',w''))\in \mathbf{C}^N: k(z,w',w'')=k(0,0,s'')\}. \tag{3.4.6}$$

Since the tangent plane of X_0 at 0 equals $\{w''=0\}$, it follows that there is a holomorphic function $g(z,w',s'')$ near $(0,0,0)$ with $g(0,0,s'')\equiv s''$ such that $X_{s''}$, for real s'' close to 0, is given by

$$w''=g(z,w',s''). \tag{3.4.7}$$

The maximal Segre set $N_{j_0}(s'')$ coincides with $X_{s''}$ on a dense open subset of the latter. Consequently, the algebraic representation (3.4.4) of $N_{j_0}(s'')$, which is valid near the point $(z^1,w'^1,f(z^1,w'^1;s''))$, implies that the holomorphic function $g(z,w',s'')$ in (3.4.7) is in fact algebraic. (The point (z^1,w'^1) can be taken arbitrarily close to 0.) Hence the algebraic function $f(z,w';s'')$ can be continued to an algebraic holomorphic function near $(0,0;0)$.

Now, as we noted above, we have the identity $f(0,0;s'')\equiv s''$ and hence

$$\frac{\partial f}{\partial s''}(0,0;0)=I. \tag{3.4.8}$$

Hence, we may solve the equation

$$w''=f(z,w';s'') \tag{3.4.9}$$

for s'' near the base point $(z,w',w'',s'')=(0,0,0,0)$. We obtain a $\mathbf{C}^q$-valued algebraic function $h(z,w',w'')$, holomorphic near $(0,0,0)$, satisfying

$$w'' \equiv f(z,w';h(z,w',w'')) \tag{3.4.10}$$

with $h(0,0,s'')=s''$. It follows that the restriction of $h(z,w',w'')$ to $X_{s''}$ is constant and equals s''. In particular, since the CR orbits $W_{s''}=M\cap X_{s''}$ (for $s''\in\mathbf{R}^q$ close to 0) cover a neighborhood of 0 in M, the restriction of h to M is valued in $\mathbf{R}^q$. Indeed, we have $h|_M=s''$ and, as a consequence, we also have

$$\partial h_1(0)\wedge\ldots\wedge\partial h_q(0)\neq 0. \tag{3.4.11}$$

The proof of Lemma 3.4.1 is complete. □

3.5. An example

Consider the 5-dimensional real algebraic submanifold $M\subset\mathbf{C}^4$ defined by

$$\operatorname{Re} Z_3=0, \quad \operatorname{Im} Z_3=|Z_1|^2, \quad \operatorname{Im} Z_4=|Z_2|^2. \tag{3.5.1}$$

On the set $\{(0,Z_2,0,X_4+i|Z_2|^2):Z_2\in\mathbf{C},\, X_4\in\mathbf{R}\}$, M is neither generic nor CR, but outside that set M is generic and holomorphically nondegenerate. The function $h_1(Z)=-iZ_3$ is real on M, but $M\cap\{Z:h_1(Z)=c\}$, for real $c>0$, is not minimal anywhere. Indeed, $M\cap\{Z:h_1(Z)=c\}$ is given by

$$|Z_1|^2=c, \quad Z_3=ic, \quad \operatorname{Im} Z_4=|Z_2|^2, \tag{3.5.2}$$

which is not minimal since it is a product of a circle and a 3-dimensional surface. We leave it to the reader to check that there is no germ at 0 of an algebraic holomorphic function h which is real on M and such that $\partial h(0)\wedge\partial h_1(0)\neq 0$.

Hence, we cannot apply Theorem 3.1.2 with $p_0=0$. However, a straightforward calculation reveals that the function

$$h_2(Z)=\frac{Z_1^2-iZ_3}{2Z_1} \tag{3.5.3}$$

is real on M, since $h_2(Z)|_M=\operatorname{Re} Z_1$. Near any point $p_1=(ir,0,ir^2,0)\in M$, with $r\in\mathbf{R}$, the leaves $\{Z:h_1(Z)=c_1,\, h_2(Z)=c_2\}$, for $c=(c_1,c_2)\in\mathbf{C}^2$ close to $(r^2,0)$, are equal to

$$\left\{Z:Z_1=c_2+\sqrt{c_2^2-c_1}\,,\, Z_3=ic_1\right\}, \tag{3.5.4}$$

where the square root is chosen so that $\sqrt{-1}=i$. Assume now that there is a holomorphic map $H: \mathbf{C}^4 \mapsto \mathbf{C}^{N'}$ near 0, generically of rank 4, such that $H(M)$ is contained in a 5-dimensional real algebraic subset of $\mathbf{C}^{N'}$. If we choose the point p_1 as above with $r \neq 0$ to be in the domain of definition of H then we may apply Theorem 3.1.2 in a neighborhood of p_1 since $M \cap \{Z : h_1(Z)=c_1,\, h_2(Z)=c_2\}$ is minimal for Z near p_1, and $c \in \mathbf{R}^2$ near $(r^2, 0)$. Theorem 3.1.2 implies that H is algebraic on the leaves

$$\{Z : h_1(Z)=c_1,\, h_2(Z)=c_2\}, \tag{3.5.5}$$

which are the same as the leaves defined by (3.5.4). More precisely, the proof of Theorem 3.1.2 implies that there are polynomials $P_l(Z_2, Z_4, X; Z_1, Z_3)$ in $(Z_2, Z_4, X) \in \mathbf{C}^3$ with coefficients that are holomorphic functions of (Z_1, Z_3) near (ir, ir^2) such that (with $H=(H_1, ..., H_{N'})$)

$$P_l(Z_2, Z_4, H_l(Z); Z_1, Z_3) \equiv 0 \tag{3.5.6}$$

holds for Z near $(ir, 0, ir^2, 0)$, for $l=1, ..., N'$. Since H is holomorphic in a neighborhood of 0, we can now apply Lemma 3.2.1 to conclude that H is algebraic on the leaves $\{Z : Z_1=Z_1^0,\, Z_3=Z_3^0\}$ for all (Z_1^0, Z_3^0) in a neighborhood of $(0,0)$. Note that, as mentioned above, we could not apply Theorem 3.1.2, as it is formulated, directly to this example at $p_0=0$.

It should be noted that there exists a nonalgebraic mapping H which is holomorphic outside $\{Z_1=0\}$, maps M into itself, has generically full rank, and which is algebraic on the leaves $\{Z : Z_1=Z_1^0,\, Z_3=Z_3^0\}$. Indeed, we may take

$$H(Z_1, Z_2, Z_3, Z_4) = (e^{ih_2(Z_1, Z_3)-iZ_3/2} Z_1, Z_2, e^{-iZ_3} Z_3, Z_4). \tag{3.5.14}$$

3.6. Proofs of Theorems 1 through 4

We begin by proving Theorem 1. Since $p_0 \in \overline{A_{\mathrm{reg}}}$ and A_{CR} is dense in A_{reg}, we can find a real algebraic CR submanifold M, as an open piece of A_{CR}, such that $p_0 \in \overline{M}$. Condition (1) and Proposition 1.4.1 imply that M is holomorphically nondegenerate, and condition (2), together with Lemma 3.4.1 imply that M is generic and minimal at some point. Now Theorem 1 follows from Theorem 3.1.2 by specializing in that theorem to $q=0$ and M generic.

For Theorem 4, we note first that Theorem 2.2.1 states that the CR orbits and their intrinsic complexifications are all algebraic. The rest of the proof of the theorem follows from Theorem 1, since any biholomorphism must map a CR orbit onto a CR orbit.

Now we shall prove Theorems 2 and 3. By Proposition 1.4.1, condition (1) of Theorem 1 is equivalent to condition (i) of Theorem 3. That (2) of Theorem 1 is equivalent to (i) and (ii) of Theorem 3 follows easily from Lemma 3.4.1.

PROPOSITION 3.6.1. *Let A be an irreducible real algebraic subset of $\mathbf{C}^N$. If either* (i) *or* (ii) *of Theorem* 3 *does not hold, then* (3) *of Theorem* 2 *holds.*

Proof. Assume first that (i) does not hold and let $p_0 \in A_{\mathrm{CR}}$. By Proposition 1.4.1, the definition of holomorphic degeneracy, and the observations in the proof of Proposition 1.4.1, there exists a nontrivial holomorphic vector field X of the form (1.4.1) tangent to A with coefficients algebraic holomorphic near p_0. Without loss of generality, we may assume $X(p_0)=0$. The proof now is essentially the same as that of the hypersurface case ([BR3, Proposition 3.5]). We take the complex flow of the vector field X or, if necessary, of fX, where f is a germ of a nonalgebraic holomorphic function at p_0 to find the desired germ of biholomorphism satisfying (3). See [BR3] for details.

Assume now that (ii) does not hold, and let $p_0 \in A_{\mathrm{CR}}$. Since A is not generic at p_0, there exists an algebraic holomorphic proper submanifold in $\mathbf{C}^N$ containing A_{CR}. After an algebraic holomorphic change of coordinates, we may assume that $p_0=0$ and that A is contained in the complex hyperplane $Z_N=0$ near 0. To prove that (3) holds, it suffices to take the mapping $H_j(Z)=Z_j$, $j=1, ..., N-1$, and $H_N(Z)=Z_N e^{Z_N}$. This proves Proposition 3.6.1. □

We now prove the last statement of Theorem 3. A homogeneous submanifold M of $\mathbf{C}^N$ of codimension d is given by

$$M=\{Z\in\mathbf{C}^N : \varrho_j(Z,\bar{Z})=0,\ j=1,...,d\},\tag{3.6.1}$$

where the ϱ_j are real-valued polynomials weighted homogeneous with respect to the weights $\nu_1\leqslant...\leqslant\nu_N$ (see §2.3). Let $r_1\leqslant...\leqslant r_d$ be the degrees of homogeneity of the polynomials $\varrho_1, ..., \varrho_d$, i.e., for $t>0$,

$$\varrho_j(t^{\nu_1}Z_1, ..., t^{\nu_N}Z_N)=t^{r_j}\varrho_j(Z,\bar{Z}),\quad j=1,...,d.\tag{3.6.2}$$

We also assume that

$$d\varrho_1(0)\wedge...\wedge d\varrho_d(0)\neq 0.\tag{3.6.3}$$

LEMMA 3.6.2. *Let M be a homogeneous generic submanifold of $\mathbf{C}^N$ which is not minimal at* 0. *Then there exists a holomorphic polynomial h in $\mathbf{C}^N$, with $h|_M$ nonconstant and real-valued.*

Proof. The homogeneous manifold M is generic (at 0 and hence at all points) if, in addition to (3.6.3), we have

$$\partial\varrho_1(0)\wedge...\wedge\partial\varrho_d(0)\neq 0.\tag{3.6.4}$$

The reader can easily check that if M is a generic homogeneous manifold of codimension d, after a linear holmorphic change of coordinates $Z=(z,w)$, M can be written in the form

$$w=Q(z,\bar{z},\bar{w}),\quad \text{with } Q_j(z,\bar{z},\bar{w})=\bar{w}_j+q_j(z,\bar{z},\bar{w}_1,...,\bar{w}_{j-1}),\tag{3.6.5}$$

$j=1, \dots, d$, with q_j a weighted homogeneous polynomial of weight r_j. Here Q is complex-valued and satisfies (1.1.5). After a further weighted homogeneous change of holomorphic coordinates, we may assume that the coordinates (z, w) are normal, i.e. (1.1.4) holds.

As in §2, we let M^k be the projection of M in $\mathbf{C}^{n+k-1}$, $k=2, \dots, d+1$. Each M^k is defined by the first $k-1$ equations in (3.6.5). If the hypersurface $M^2 \subset \mathbf{C}^{n+1}$ is not minimal at 0, then necessarily $q_1(z, \bar{z}) \equiv 0$, and we may take $h(z, w)=w_1$. If not, we let $l \leqslant d$ be the smallest integer for which M^l is minimal at 0, but M^{l+1} is not minimal at 0. Then the CR orbit W of 0 in the generic manifold M^{l+1} is a proper CR submanifold of M^{l+1} of CR dimension n. It must be a holomorphic graph over M^l in $\mathbf{C}^{n+l}$. That is, W is given by (3.6.5) for $1 \leqslant j \leqslant l-1$ and $w_{l+1}=f(z, w_1, \dots, w_{l-1})$. Since $W \subset M^{l+1}$ we must also have $\operatorname{Im} f(z, w_1, \dots, w_{l-1})|_M=(1/2i) q_l(z, \bar{z}, \bar{w}_1, \dots \bar{w}_{l-1})|_M$. The reader can check that this implies that $f(z, w_1, \dots, w_{l-1})$ is independent of z and is a weighted homogeneous holomorphic polynomial, and the function $h(z, w)=w_l-f(z, w_1, \dots, w_{l-1})$ satisfies the conclusion of the lemma. The following proposition concludes the proof of Theorem 3. □

PROPOSITION 3.6.3. *Let M be a homogeneous generic submanifold of $\mathbf{C}^N$ which is not minimal at* 0. *Then for any $p_0 \in M$, there exists a nonalgebraic holomorphic map H from $\mathbf{C}^N$ into itself with $H(p_0)=p_0$, $H(M) \subset M$, and* $\operatorname{Jac} H(p_0) \neq 0$.

Proof. By Lemma 3.6.2, there exists a nonconstant holomorphic polynomial h with $h|_M$ real. We may also assume $h(p_0)=0$. The reader can easily check that the map defined by

$$H_j(Z)=e^{\nu_j h(Z)} Z_j, \quad j=1, \dots, N,$$

satisfies the desired conclusion of the proposition. □

References

[BB] Bedford, E. & Bell, S., Extension of proper holomorphic mappings past the boundary. *Manuscripta Math.*, 50 (1985), 1–10.

[BCR] Bochnak, J., Coste, M. & Roy, M.-F., *Géometrie algébrique réelle.* Springer-Verlag, Berlin–Heidelberg, 1987.

[BER] Baouendi, M. S., Ebenfelt, P. & Rothschild, L. P., Infinitesimal CR automorphisms of real-analytic manifolds in complex space. To appear in *Comm. Anal. Geom.*

[BG] Bloom, T. & Graham, I., On type conditions for generic real submanifolds of $\mathbf{C}^n$. *Invent. Math.*, 40 (1977), 217–243.

[BHR] Baouendi, M. S., Huang, X. & Rothschild, L. P., Regularity of CR mappings between algebraic hypersurfaces. *Invent. Math.*, 125 (1996), 13–36.

[BJT] Baouendi, M. S., Jacobowitz, H. & Treves, F., On the analyticity of CR mappings. *Ann. of Math.*, 122 (1985), 365–400.

[BM] Bochner, S. & Martin, W. T., *Several Complex Variables.* Princeton Univ. Press, Princeton, NJ, 1948.

[BR1] Baouendi, M. S. & Rothschild, L. P., Normal forms for generic manifolds and holomorphic extension of CR functions. *J. Differential Geom.*, 25 (1987), 431–467.

[BR2] — Geometric properties of mappings between hypersurfaces in complex space. *J. Differential Geom.*, 31 (1990), 473–499.

[BR3] — Mappings of real algebraic hypersurfaces. *J. Amer. Math. Soc.*, 8 (1995), 997–1015.

[BR4] — Holomorphic mappings between algebraic hypersurfaces in complex space, in *Séminaire Équations aux Derivées Partielles*, 1994–95. École Polytechnique, Palaiseau, 1995.

[CJ] Chern, S. S. & Ji, S., Projective geometry and Riemann's mapping problem. *Math. Ann.*, 302 (1995), 581–600.

[CM] Chern, S. S. & Moser, J. K., Real hypersurfaces in complex manifolds. *Acta Math.*, 133 (1974), 219–271.

[DF] Diederich, K. & Fornaess, J. E., Proper holomorphic mappings between real-analytic pseudoconvex domains in $\mathbf{C}^n$. *Math. Ann.*, 282 (1988), 681–700.

[DW] Diederich, K. & Webster, S., A reflection principle for degenerate hypersurfaces. *Duke Math. J.*, 47 (1980), 835–843.

[F] Forstnerič, F., Extending proper holomorphic mappings of positive codimension. *Invent. Math.*, 95 (1989), 31–62.

[H] Huang, X., On the mapping problem for algebraic real hypersurfaces in the complex spaces of different dimensions. *Ann. Inst. Fourier* (*Grenoble*), 44 (1994), 433–463.

[HP] Hodge, W. H. D. & Pedoe, D., *Methods of Algebraic Geometry*, Vol. 2. Cambridge Univ. Press, Cambridge, 1952.

[N] Nagano, T., Linear differential systems with singularities and an application to transitive Lie algebras. *J. Math. Soc. Japan*, 18 (1966), 398–404.

[P] Poincaré, H., Les fonctions analytiques de deux variables et la représentation conforme. *Rend. Circ. Mat. Palermo*, 23 (1907), 185–220.

[Seg] Segre, B., Intorno al problema di Poincaré della rappresentazione pseudo-conforme. *Atti R. Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur.* (*6*), 13 (1931), 676–683.

[Ser] Serre, J.-P., Géometrie algébrique et géometrie analytique. *Ann. Inst. Fourier* (*Grenoble*), 6 (1955), 1–42.

[SS] Sharipov, R. & Sukhov, A., On CR-mappings between algebraic Cauchy–Riemann manifolds and separate algebraicity for holomorphic functions. Preprint, Université de Provence, Marseille, 1995.

[St1] STANTON, N., Infinitesimal CR automorphisms of rigid hypersurfaces. *Amer. J. Math.*, 117 (1995), 141–167.

[St2] — Infinitesimal CR automorphisms. *Amer. J. Math.*, 118 (1996), 209–233.

[Ta] TANAKA, N., On the pseudo-conformal geometry of hypersurfaces of the space of n complex variables. *J. Math. Soc. Japan*, 14 (1962), 397–429.

[TH] TUMANOV, A. E. & HENKIN, G. M., Local characterization of holomorphic automorphisms of Siegel domains. *Funktsional. Anal. i Prilozhen.*, 17 (1983), 49–61; English translation in *Functional Anal. Appl.*, 17 (1983).

[Tu1] TUMANOV, A. E., Extending CR functions on manifolds of finite type to a wedge. *Mat. Sb.* (*N.S.*), 136 (1988), 128–139.

[Tu2] — Finite-dimensionality of the group of CR automorphisms of a standard CR manifold, and proper holomorphic mappings of Siegel domains. *Izv. Akad. Nauk SSSR Ser. Mat.*, 52 (1988), 651–659; English translation in *Math. USSR-Izv.*, 32 (1989), 655–662.

[W1] WEBSTER, S., On the mapping problem for algebraic real hypersurfaces. *Invent. Math.*, 43 (1977), 53–68.

[W2] — On mapping an n-ball into an $(n+1)$-ball in complex space. *Pacific J. Math.*, 81 (1979), 267–272.

[W3] — Geometric and dynamical aspects of real submanifolds of complex space. To appear in *Proceedings of the International Congress of Mathematicians*, Zürich, 1994.

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