Acta Math., 177 (1996), 163–224

Local connectivity of some Julia sets containing a circle with an irrational rotation

by

CARSTEN LUNDE PETERSEN

Roskilde University
Roskilde, Denmark

The Fatou set F_R for a rational map $R: \overline{\mathbf{C}} \to \overline{\mathbf{C}}$ is the set of points $z \in \overline{\mathbf{C}}$ possessing a neighbourhood on which the family of iterates $\{R^n\}_{n \geqslant 0}$ is normal (in the sense of Montel). The Julia set $J_R = \overline{\mathbf{C}} - F_R$ is the complement of the Fatou set. (The monographs [CG], [Be], [St] provide introductions to the theory of iteration of rational maps.)

Let $\theta \in]0,1[-\mathbf{Q}$ be an irrational number and write it as a continued fraction

$$\theta = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{a_4 + \cfrac{1}{\ddots}}}}}$$

where $a_n \in \mathbf{N}$ for each $n \geqslant 1$. The number θ is termed of constant type, or equivalently, is termed Diophantine of exponent 2, if the sequence $\{a_n\}_{n \in \mathbf{N}}$ is bounded.

For $\theta \in [0,1]$ define $\lambda_\theta = \exp(i2\pi\theta)$ and $P_\theta(z) := \lambda_\theta z + z^2$. Moreover, let J_{P_θ} denote the Julia set of P_θ. The polynomial P_θ has a Siegel disc around the (indifferent) fixed point 0, if and only if it is locally linearizable. That is, if there exists a local change of coordinates $\phi: (\mathbf{C}, 0) \to (\mathbf{C}, 0)$ with $\phi \circ P_\theta = \lambda_\theta \cdot \phi$. It is well known that P_θ has a Siegel disc around 0 for every θ of constant type (see e.g. [Si]).

THEOREM A. *For every θ of constant type the Julia set J_{P_θ} is locally connected and has zero Lebesgue measure.*

The proof uses in an essential way a model J_θ of J_{P_θ}. The model J_θ was constructed in 1986 and proved to be quasi-conformally equivalent to J_{P_θ} in 1987 (see [Do] for the

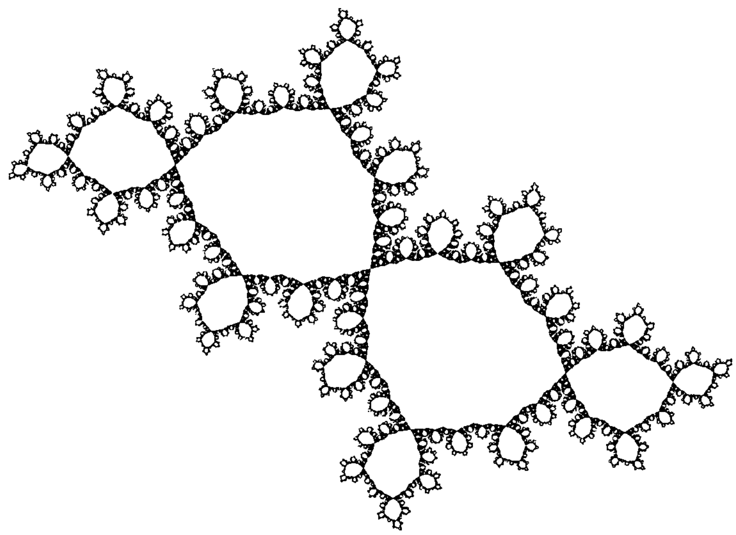

Fig. 1. The Julia set J_{P_θ} for $\theta=\frac{1}{2}(\sqrt{5}-1)$

particular result, and e.g. the monograph [LV] for the theory of quasi-conformal maps of the plane). Let us briefly discuss the model J_θ, as it is essential in the proof. Consider the degree-three Blaschke function:

$$f_0(z)=z^2\frac{z-3}{1-3z}.$$

Its restriction $f_0\colon \mathbf{S}^1\to\mathbf{S}^1$ is an analytic circle homeomorphism with 1 as a fixed critical point, an inflection point of order three. In particular, f_0 has (Poincaré) rotation number 0. For each irrational rotation number $\theta\in[0,1]$ there exists a unique $\varrho_\theta\in\mathbf{S}^1$ such that the restriction of $f_\theta:=\varrho_\theta\cdot f_0$ to $\mathbf{S}^1$ has rotation number θ. We let J_{f_θ} denote the Julia set of f_θ.

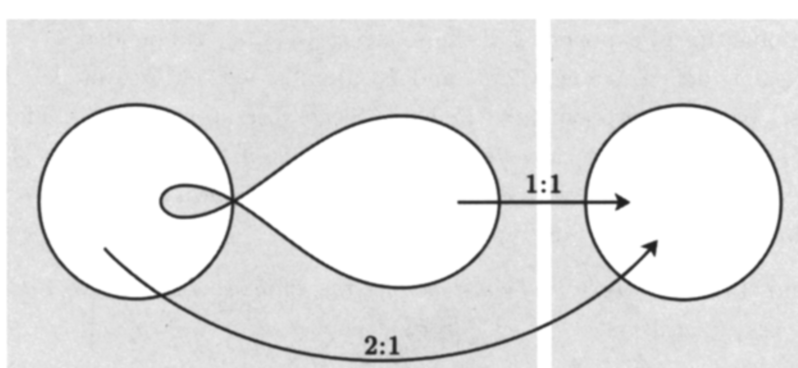

Fig. 2. The basic dynamics of f_θ

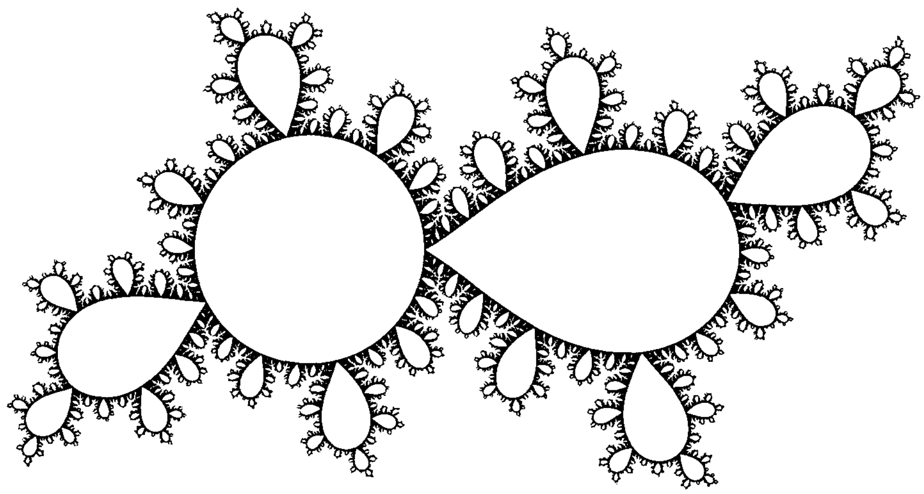

Fig. 3. The "Julia" set J_θ for $\theta=\frac{1}{2}(\sqrt{5}-1)$

Each f_θ commutes with $\tau(z)=1/\bar{z}$ (reflection in the unit circle). Thus all dynamical properties of f_θ are symmetric with respect to $\mathbf{S}^1$. In particular, the Julia set J_{f_θ} is symmetric. Moreover, the points 0 and ∞ are super-attractive (critical) fixed points with simply-connected immediate basins. Let U_0 be the connected component of $f_0^{-1}(\mathbf{D})=f_\theta^{-1}(\mathbf{D})$ contained in the complement of $\mathbf{D}$. The immediate basin of ∞, $\Lambda_0(\infty)$, is contained in $\overline{\mathbf{C}}-(\overline{\mathbf{D}}\cup\overline{U}_0)$.

For each irrational θ there exists a homeomorphism (unique up to postcomposition by a rigid rotation) $h_\theta\colon \mathbf{S}^1\to\mathbf{S}^1$ conjugating f_θ to the rigid rotation $R_\theta(z)=\lambda_\theta z$ on $\mathbf{S}^1$ (see [Yo1]). Let $H_\theta\colon\overline{\mathbf{D}}\to\overline{\mathbf{D}}$ denote a homeomorphism extending h_θ. We shall suppose H_θ quasi-conformal if h_θ is quasi-symmetric (quasi-symmetric means that any two neighbouring intervals of the same length have images whose lengths are uniformly comparable).

Definition. For each irrational θ we shall define a new degree-two branched, but non-holomorphic, covering map $F_\theta\colon\overline{\mathbf{C}}\to\overline{\mathbf{C}}$ by

$$F_\theta(z)=\begin{cases} f_\theta(z) & \text{if and only if } |z|\geqslant 1,\\ H_\theta^{-1}\circ R_\theta\circ H_\theta(z) & \text{if and only if } |z|\leqslant 1,\end{cases}$$

and an F_θ-invariant "Julia" set $J_\theta=J_{f_\theta}-\bigcup_{n\geqslant 0} f_\theta^{-n}(\mathbf{D})$. See Figure 3.

THEOREM (Douady, Shishikura, Ghys, ..., 1986). *If h_θ is quasi-symmetric, there exists a quasi-conformal homeomorphism $\phi_\theta\colon\overline{\mathbf{C}}\to\overline{\mathbf{C}}$ conjugating (the then quasi-regular map) F_θ to the polynomial P_θ. The homeomorphism ϕ_θ maps $\mathbf{D}$ onto a Siegel disc Δ_θ*

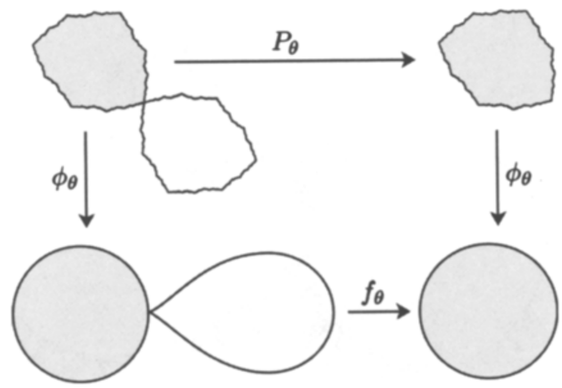

Fig. 4. The conjugation ϕ_θ

around 0 *for* P_θ, *and maps* J_θ *onto the Julia set* J_{P_θ}. *Furthermore,* ϕ_θ *can be chosen to be conformal on the immediate basin of* ∞ (*see also Figure* 4).

The above meta-theorem was given a real content a year later by M. Herman, who used inequalities, a priori real bounds, obtained by Świątec to prove the following.

THEOREM (Świątec–Herman, [He], 1987). *An analytic circle homeomorphism with irrational rotation number* θ *and with one* (*double*) *critical point is quasi-symmetrically conjugate to the rigid rotation* R_θ *if and only if* θ *is of constant type.*

Proof. See [He]. □

One asked if the above would help in proving Theorem A. The answer is yes and is the main concern of this paper. Note that Theorem A would follow if we knew that J_θ is locally connected and has Lebesgue measure 0 whenever θ is of constant type (quasi-conformal homeomorphisms map Lebesgue null sets to Lebesgue null sets). We can actually prove more than this.

THEOREM B. *For any* $\theta \in]0,1[- \mathbf{Q}$ *the Julia set* J_{f_θ} *and the set* J_θ *are locally connected.*

THEOREM C. *For every* θ *of constant type the Lebesgue measure of* J_θ *is zero.*

Theorem B gives rise to the question: Suppose that P_θ has a Siegel disc whose boundary is a Jordan curve containing the critical point. Does this imply that J_{P_θ} is locally connected?

Another interesting question is: does there exist θ for which the full Julia set J_{f_θ} has positive measure?

The main ingredients in proving the Herman–Świątec Theorem are the Świątec a priori real bounds (inequalities) for the ratios of closest returns of the critical point (here 1) to itself. We shall state later the precise statement of the Świątec a priori real bounds, when we have introduced the points of closest return. The main ingredient in the present proof that J_θ and J_{f_θ} are locally connected is a dynamically defined geometric construction, a "puzzle", which permits to transmit the Świątec a priori real bounds to complex bounds for the Julia sets. The puzzle is inspired by Yoccoz puzzles (see [Hu] for quadratic polynomials) and Branner–Hubbard puzzles (see [BH] for cubic polynomials). Knowing the "classical" puzzle constructions by Branner–Hubbard and Yoccoz, there were several mental obstacles to overcome in order to arrive at this new type of puzzle and in controlling it. One has to accept that the critical point chops up puzzle pieces giving puzzle pieces containing the critical point on the boundary. Moreover, one has to turn this phenomena into a "friend". Secondly, when estimating the size of puzzle pieces, one has to give up completely the central idea in "classical puzzles" that some annuli defined by differences of puzzle pieces map properly to each other. Thus killing the foundations of the central Grötzsch argument in proving divergence of nests. The replacement is ideas which permit to control lengths of boundaries of puzzle pieces. In implementing these ideas, we use essentially the "realness" of f_θ, i.e. that J_θ contains the unit circle. The first consequence of the "realness" of f_θ is that we can draw arcs of finite Euclidean length in J_θ. The second is that the Świątec a priori bounds hold. These say that the closest returns of the critical point to itself essentially come geometrically. The third is that we can transform the angular contraction for inverse branches around the critical point into a hyperbolic contraction on appropriate domains.

The structure of the rest of this paper is as follows. §0 contains the red thread of the proof of spreading local connectivity from the critical point to all of the sets J_θ and J_{P_θ}, together with some additional results, interesting in their own right. Moreover, it introduces the notation used in subsequent sections. §1 is essentially self-contained. It introduces the "puzzle pieces" containing the critical point on their boundary. Moreover, the results needed to prove local connectivity at the critical point are stated. §2 contains the proofs of the statements of §1 together with the necessary technical machinery to do so. It has §1 as prerequisite. §3 spreads local connectivity from 1 to all of J_θ and proves the theorem on zero measure. Finally §4 shows how to spread local connectivity also to all of J_{f_θ}.

Added in revision. C. T. McMullen has proved, using the results of this paper, that the Hausdorff dimension of J_{P_θ} is strictly less than two whenever θ is of constant type, thus improving the measure statement of Theorem A. Moreover, he proves that the Siegel disc for P_θ is self-similar about the critical point, whenever θ is a quadratic irrational (such as the golden mean) (see the manuscript [Mc]). M. Lyubich has proved that J_θ has Lebesgue measure zero for every irrational θ, thus improving Theorem C. This result would also follow by slightly changing the proof of Theorem C given in this paper. The proof by Lyubich is outlined in the preprint [Ya] by M. Yampolsky, which also outlines an alternative proof of Theorems A and B above.

Acknowledgements. The author would like to thank Institut des Hautes Études Scientifiques for its hospitality and support during the birth and writing of this paper. Especially, I would like to thank Marie-Claude Vergne for her drawings. Moreover, I would like to thank Tan Lei, Marguerite Flexor and Dierk Schleicher for helpful conversations and suggestions. Also, I would like to express my gratitude to Jean-Christophe Yoccoz for introducing me to the results of Świątec and to Adrien Douady for his moral support.

0. Strategy of the proof of local connectivity and further results

The definitions and structures we are about to discuss depend on $\theta\in]0,1[-\mathbf{Q}$. We shall however only use an additional index θ in our definitions when we want to stress the dependence on θ. Thus the dependence on θ is always to be assumed, if not stated explicitly otherwise.

The point ∞ is a super-attractive fixed point for each f_θ and F_θ. The corresponding immediate basin $\Lambda_0(\infty)$ is simply-connected. Let $\psi=\psi_\theta\colon\Lambda_0(\infty)\to\overline{\mathbf{C}}-\overline{\mathbf{D}}$ denote the Riemann map conjugating f_θ on $\Lambda_0(\infty)$ to $z\mapsto z^2$ on $\overline{\mathbf{C}}-\overline{\mathbf{D}}$. The image by ψ of the line $\{re^{i2\pi\eta}|r>1\}$ for $\eta\in[0,1]$ shall be called the η external ray and be denoted R_η. The ray R_η lands if and only if ψ has a continuous extension along $\{re^{i2\pi\eta}|r>1\}$ to $e^{i2\pi\eta}$. The impression of the η prime end is the set of accumulation points for sequences $\{\psi(z_n)\}_{n\geqslant 0}$, with z_n converging to $e^{i2\pi\eta}$. In particular, the impression of the η prime end is a singleton if and only if ψ extends continuously to $e^{i2\pi\eta}$.

THEOREM 1.3. *For each $\theta\in]0,1[-\mathbf{Q}$ the critical point $1\in J_{f_\theta}$ is in the impression of precisely two prime ends of the immediate basin of ∞ for f_θ. The impressions of these two prime ends equal $\{1\}$. In particular, there are precisely two external rays landing on* 1.

The proof shall be given in §§1 and 2.

First we describe an abstract topologial model J_θ^{abs} for J_θ and a model dynamics F_θ^{abs} on J_θ^{abs}. Secondly we discuss the proof of spreading local connectivity. Before however let us mention another topological model known as the pinched disc model. The pinched disc model is well described by K. Keller in [Ke]. Our work implies that for any irrational θ the corresponding pinched disc model described by Keller is homeomorphic to J_θ. The pinched disc model is locally connected and thus not homeomorphic to J_{P_θ}

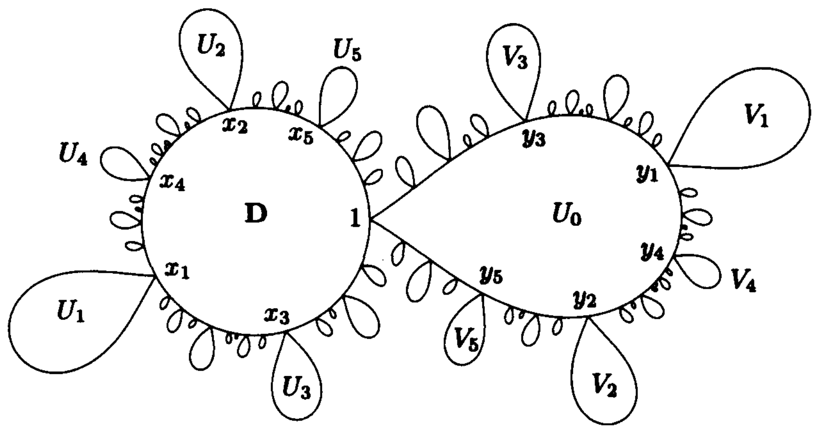

Fig. 5. The initial 34 first-generation drops with parent either $\mathbf{S}^1$ or ∂U_0

when J_{P_θ} is not locally connected, e.g. when P_θ is not linearizable on any neighbourhood of 0. The construction of J_θ^{abs} and F_θ^{abs} follows on the next couple of pages.

Define a subset $J_\theta^{\text{skeleton}}=\bigcup_{n\geqslant 0} F_\theta^{-n}(\mathbf{S}^1)=J_\theta\cap\bigcup_{n\geqslant 0} f_\theta^{-n}(\mathbf{S}^1)$. The set $J_\theta^{\text{skeleton}}$ (or even the set $J_{f_\theta}^{\text{skeleton}}=\bigcup_{n\geqslant 0} f_\theta^{-n}(\mathbf{S}^1)$) naturally decomposes into a countable union of Jordan curves, with two such curves having at most one common point. Moreover, $J_\theta^{\text{skeleton}}$ is dense in $J_\theta=J_{f_\theta}-\bigcup_{n\geqslant 0} f_\theta^{-n}(\mathbf{D})$, because $\partial(\bigcup_{k=0}^{n} f_\theta^{-k}(\mathbf{D}))=F_\theta^{-n}(\mathbf{S}^1)$ and $J_{f_\theta}^{\text{skeleton}}$ is dense in J_{f_θ}.

LEMMA 0.1. *Let $n\geqslant 0$ and let ω be a connected component of $F_\theta^{-n}(\bar{U}_0)$. Then the restriction $F_\theta^n=f_\theta^n:\omega\to\bar{U}_0$ is a diffeomorphism.*

Proof. The map f_θ is a branched covering map. Moreover, the set $\bar{U}_0$ is simply-connected and does not intersect the forward orbits of critical points. □

For ω and n as in the lemma we shall say that ω is a (closed) *n-drop* or just a drop, if n is understood. We shall say that the interior of ω is an (open) n-drop. Moreover, we define the *root* z of ω to be the boundary point given by $\{z\}=f_\theta^{-n}(1)\cap\partial\omega$. Then the relation root of drop defines a bijection between $F_\theta^{-n}(1)$ and the set of n-drops, $n\geqslant 0$.

LEMMA 0.2. *Let ω be an n-drop for some $n\geqslant 0$ and let z be the root of ω. Then either $z\in\mathbf{S}^1$ or z belongs to the boundary of an n'-drop ω' with $0\leqslant n'<n$.*

Proof. Let $0\leqslant k\leqslant n$ be minimal with the property $F_\theta^k(z)\in\mathbf{S}^1$. If $z\notin\mathbf{S}^1$ then $k>0$ and $F_\theta^{k-1}(z)\in\partial U_0-\{1\}$, because $F_\theta^{-1}(\mathbf{S}^1)=\mathbf{S}^1\cup\partial U_0$. Let $n'=k-1<n$ and let ω' be the closed n'-drop containing z. Then n' and ω' satisfies the conclusion of the lemma. □

For ω as in the lemma above we say that $\mathbf{S}^1$ and ω' respectively is the parent (drop) of ω. More generally we shall define generations as follows: The two discs $\bar{\mathbf{D}}$ and $\bar{U}_0$

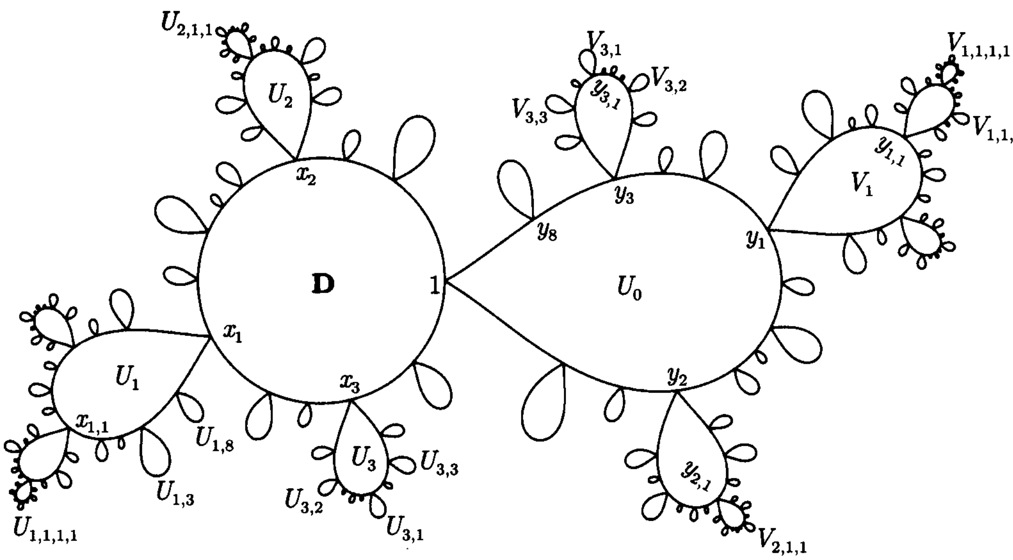

Fig. 6. Addresses of some drops and their roots

form generation zero. The drops of first generation are the drops ω with root $z \in \mathbf{S}^1 \cup \partial U_0$. A drop ω and its root are of generation $m \geqslant 2$ precisely if the root belongs to the boundary of a drop $\omega' \neq \omega$ of the $(m-1)$st generation.

Let $x_j = f_\theta^{-j}(1) \cap \mathbf{S}^1$, $\forall j \in \mathbf{Z}$, and let $y_j \in \partial U_0$ be given by $f_\theta(x_j) = f_\theta(y_j)$ for $n \geqslant 1$. For $s \geqslant 1$ let U_s, V_s be the open s-drops with roots x_s and y_s respectively. The first generation drops and their respective roots are precisely the drops U_s, V_s and roots x_s, y_s, for $s \geqslant 1$. See Figure 5.

More generally we shall label the drops and roots of all generations by finite but arbitrarily long tuples with positive integers as entries. See Figure 6. A label should be thought of as an address: Suppose that ω is a drop of generation $m \geqslant 1$ with root x. They will be labelled by a common m-tuple $(s_1, \ldots, s_m) \in \mathbf{N}^m$, where $(s_1, \ldots, s_{m-1}) \in \mathbf{N}^{m-1}$ is the address of the parent and the sum $n = \sum_{i=1}^m s_i$ is the number of iterates it takes to map ω onto U_0 and x onto 1. Another way to view the last entry is to apply $F_\theta^{s_1 + \ldots + s_{m-1}}$ to ω and its parent, thus mapping the parent onto U_0 and ω onto V_{s_m}. It turns out to be convenient to denote drops descending directly to $\mathbf{S}^1$ by $U_{s_1, \ldots, s_m}$ and drops descending to U_0 by $V_{s_1, \ldots, s_m}$. Moreover, we let $x_{s_1, \ldots, s_m}$ and $y_{s_1, \ldots, s_m}$ denote the respective roots. To complete the picture we let ε denote the empty sequence and define $U_\varepsilon = \mathbf{D}$ and $V_\varepsilon = U_0$. In this way there is a natural bijection between drops and labels. Finally let us note that $F_\theta^{s_1}(x_{s_1, \ldots, s_{m+1}}) = F_\theta^{s_1}(y_{s_1, \ldots, s_{m+1}}) = y_{s_2, \ldots, s_{m+1}}$.

We define limbs and sublimbs $X^{\text{skeleton}}_{s_1, \ldots, s_m}$, $Y^{\text{skeleton}}_{s_1, \ldots, s_m}$ of $J_\theta^{\text{skeleton}}$, $(s_1, \ldots, s_m) \in \mathbf{N}^m$, $m \geqslant 0$, as the union of $U_{s_1, \ldots, s_m}$ with all its descendents, and $V_{s_1, \ldots, s_m}$ with all its descendents respectively:

$$X^{\text{skeleton}}_{s_1,\dots,s_m} = \partial U_{s_1,\dots,s_m} \cup \bigcup_{m'\geqslant 1} \bigcup_{(t_1,\dots,t_m)\in \mathbf{N}^{m'}} \partial U_{s_1,\dots,s_m,t_1,\dots,t_{m'}}$$

$$Y^{\text{skeleton}}_{s_1,\dots,s_m} = \partial V_{s_1,\dots,s_m} \cup \bigcup_{m'\geqslant 1} \bigcup_{(t_1,\dots,t_m)\in \mathbf{N}^{m'}} \partial V_{s_1,\dots,s_m,t_1,\dots,t_{m'}}.$$

Then

$$X^{\text{skeleton}}_{\varepsilon} \cap Y^{\text{skeleton}}_{\varepsilon} = \{1\} \quad \text{and} \quad J^{\text{skeleton}}_{\theta} = X^{\text{skeleton}}_{\varepsilon} \cup Y^{\text{skeleton}}_{\varepsilon}. \tag{1}$$

Moreover, apart from this, any two limbs of $J^{\text{skeleton}}_{\theta}$ are either disjoint or contained one in the other.

The model set J^{abs}_{θ} and the model dynamics $F^{\text{abs}}_{\theta}: J^{\text{abs}}_{\theta} \to J^{\text{abs}}_{\theta}$ are defined as follows: Let $\{\hat{x}_{\underline{s}}, \hat{y}_{\underline{s}}\}$, $\underline{s}\in\mathbf{N}^{\mathbf{N}}$, be a family of distinct ideal points, i.e. points not already in $J^{\text{skeleton}}_{\theta}$. Define $J^{\text{abs}}_{\theta} = J^{\text{skeleton}}_{\theta} \cup \bigcup_{\underline{s}\in\mathbf{N}^{\mathbf{N}}} \{\hat{x}_{\underline{s}}, \hat{y}_{\underline{s}}\}$. Moreover, define $F^{\text{abs}}_{\theta} = F_\theta$ on $J^{\text{skeleton}}_{\theta}$ and for $\underline{s} = (s_1, s_2, \dots, s_n, \dots)\in\mathbf{N}^{\mathbf{N}}$,

$$F^{\text{abs}}_{\theta}(\hat{x}_{\underline{s}}) = F^{\text{abs}}_{\theta}(\hat{y}_{\underline{s}}) = \begin{cases} \hat{x}_{s_1-1,s_2,\dots,s_n,\dots} & \text{if and only if } s_1 > 1, \\ \hat{y}_{s_2,s_3,\dots,s_n,\dots} & \text{if and only if } s_1 = 1. \end{cases}$$

Define abstract limbs

$$X^{\text{abs}}_{s_1,\dots,s_m} = X^{\text{skeleton}}_{s_1,\dots,s_m} \cup \bigcup_{\underline{t}\in\mathbf{N}^{\mathbf{N}}} \hat{x}_{s_1,\dots,s_m,\underline{t}},$$

$$Y^{\text{abs}}_{s_1,\dots,s_m} = Y^{\text{skeleton}}_{s_1,\dots,s_m} \cup \bigcup_{\underline{t}\in\mathbf{N}^{\mathbf{N}}} \hat{y}_{s_1,\dots,s_m,\underline{t}},$$

for all $m\geqslant 0$ and for all $(s_1,\dots,s_m)\in\mathbf{N}^m$. We shall say that $x_{s_1,\dots,s_m}, y_{s_1,\dots,s_m}$ are the roots of the respective (abstract) limbs.

We topologize the set J^{abs}_{θ} as follows: Define the nested sequences $\{X^{\text{abs}}_{s_1,\dots,s_m}\}_{m\geqslant 1}$ and $\{Y^{\text{abs}}_{s_1,\dots,s_m}\}_{m\geqslant 1}$ to be neighbourhood bases of $\hat{x}_{\underline{s}}$ and $\hat{y}_{\underline{s}}$ respectively. In order to define a neighbourhood basis for any point in $J^{\text{skeleton}}_{\theta} \subset J^{\text{abs}}_{\theta}$ also, we first do so by defining a neighbourhood basis for any point in $\mathbf{S}^1 - \{1\}$ and then pull these back by F^{abs}_{θ}, thus making this map automatically continuous. Given $z\in\mathbf{S}^1-\{1\}$, take as element of a neighbourhood basis at z any arc $I\in\mathbf{S}^1$ containing z as an interior point together with all limbs X^{abs}_{s} with root $x_s\in I$, $s>0$. Our proof of local connectivity of J_θ implies that J_θ is homeomorphic to J^{abs}_{θ} by a homeomorphism which conjugates dynamics.

Recall Theorem 1.3 and let R_+, R_- be the external rays of J_θ (and J_{f_θ}) landing on the critical point 1. Let $\Pi_\theta \subset \mathbf{C}$ denote the closed subset containing U_0 and bounded by the arc $R_+ \cup \{1\} \cup R_-$. For $n\geqslant 0$ and Ω a connected component of $F^{-n}_{\theta}(\Pi_\theta)$, the restriction

$$F^n_\theta = f^n_\theta : \Omega \to \Pi_\theta \tag{2}$$

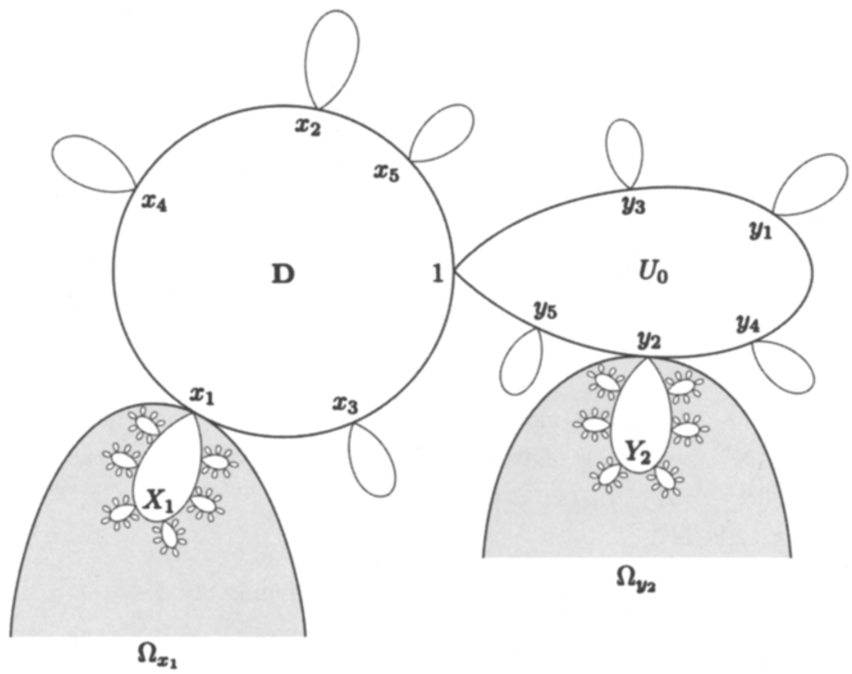

Fig. 7. Sketch of some wakes

is a diffeomorphism. The proof is identical with that of Lemma 0.1. For Ω and n as above we say that Ω is a (closed) n-*wake* or just a wake if n is understood. See Figure 7. We shall say that the interior of Ω is an (open) n-wake. Any two n-wakes are trivially disjoint, being preimages of the same set by a covering map. If Ω is an n-wake and Ω' is an n'-wake with $0 \leqslant n' < n$. Then either $\Omega' \cap \Omega = \varnothing$ or $\Omega \subset \Omega'$, because external rays do not cross.

An n-wake contains a central n-drop, whose root x is also called the root of Ω. It is the meeting point of the two external rays bounding the wake. This defines a one-to-one correspondence between roots of n-drops and n-wakes. The notions of generation and address is naturally carried over to wakes. To distinguish wakes descending to $\mathbf{S}^1$ and ∂U_0 we shall denote by $\Omega_{x,s_1,\ldots,s_m}$ and $\Omega_{y,s_1,\ldots,s_m}$ the wakes with central drops $U_{s_1,\ldots,s_m}$ and $V_{s_1,\ldots,s_m}$ respectively. Define $X_{s_1,\ldots,s_m} = \overline{X^{\text{skeleton}}_{s_1,\ldots,s_m}} = J_\theta \cap \Omega_{x,s_1,\ldots,s_m}$ and $Y_{s_1,\ldots,s_m} = \overline{Y^{\text{skeleton}}_{s_1,\ldots,s_m}} = J_\theta \cap \Omega_{y,s_1,\ldots,s_m}$, where the later equalities follows from $J_\theta^{\text{skeleton}}$ being dense in J_θ. Each limb is mapped diffeomorphically onto Y_ε by (2) (and Y_ε is mapped homeomorphically onto J_θ by F_θ).

THEOREM 3.7. *For each* $\theta \in]0,1[- \mathbf{Q}$ *the Euclidean diameter of the principal limbs* X_s *and* Y_s *tends to* 0 *as* $s \to \infty$.

The proof of this theorem shall be given in §3. We obtain immediately some corollaries. We shall use X_0 as a synonym for Y_ε.

COROLLARY 0.3. *For each $\theta \in]0,1[-\mathbf{Q}$ there are no ghost limbs of $\mathbf{D}$ and U_0 in J_θ. That is,*

$$J_\theta = \mathbf{S}^1 \cup \bigcup_{j\geqslant 0} X_j = \mathbf{S}^1 \cup \partial U_0 \cup \bigcup_{j\geqslant 1} (X_j \cup Y_j).$$

Proof. It suffices to prove the first equality sign, as $Y_\varepsilon = X_0$ maps homeomorphically onto J_θ. Any point in $z \in J_\theta = \overline{J_\theta^{\text{skeleton}}}$ is accumulated by the proper limbs. This is possible only if z is already in $\mathbf{S}^1$ or in one of the proper limbs X_j, because the size of the limb X_j tends to 0 as j tends to ∞, and each limb touches $\mathbf{S}^1$. □

THEOREM 0.4. *Let $\theta \in]0,1[-\mathbf{Q}$ be arbitrary. Any point of $\mathbf{S}^1$, and more generally any point of $J_\theta^{\text{skeleton}} = \bigcup_{n\geqslant 0} F_\theta^{-n}(\mathbf{S}^1)$, has a fundamental system of open connected neighbourhoods in J_θ.*

Proof. Let us first prove the corollary for any $z \in \mathbf{S}^1 - \{x_s\}_{s\geqslant 0}$. Let $\varepsilon > 0$ be given. We shall find an open connected neighbourhood ϖ of z in J_θ with $\varpi \subset \mathbf{D}_\varepsilon(z)$, where $\mathbf{D}_\varepsilon(z)$ is the Euclidean disc of center z and radius ε. Let $s_0 \geqslant 0$ be such that the Euclidean diameters $\operatorname{diam}_E(X_s) \leqslant \frac{1}{2}\varepsilon$ for all $s \geqslant s_0$. Let $z_1, z_2 \in \mathbf{S}^1 - \{x_s\}_{s\geqslant 0} \cap \mathbf{D}_{\varepsilon/2}(z)$ be points bounding an open subarc $]z_1, z_2[$ of $\mathbf{S}^1 - \bigcup_{j=0}^{s_0} x_j$ with $z \in]z_1, z_2[\subset \mathbf{D}_{\varepsilon/2}(z)$. Define

$$\varpi =]z_1, z_2[\cup \bigcup_{s, x_s \in]z_1, z_2[} X_s.$$

Then ϖ is the required neighbourhood. The above works in particular for the critical value v. Thus we can construct a fundamental system of connected neighbourhoods of 1 in J_θ from the system around v, as J_θ is invariant F_θ. By the same argument we prove local connectivity for the remaining points, first of $\mathbf{S}^1$ and secondly of $J_\theta^{\text{skeleton}}$. □

Let $E_\theta = J_\theta - J_\theta^{\text{skeleton}}$. The set E_θ is readily seen to be F_θ-invariant and to contain all the repelling periodic points for F_θ. In order to complete the proof of local connectivity of J_θ we need to produce a fundamental system of connected neighbourhoods for each point in E_θ. We shall introduce some notation in order to facilitate this discussion. This notation is inspired by the "puzzle" notation of Branner and Hubbard [BH].

Definition 0.5. For each $\underline{s} = (s_1, s_2, ..., s_m, ...) \in \mathbf{N}^{\mathbf{N}}$ define $\mathcal{X}_{\underline{s}}$, $\mathcal{Y}_{\underline{s}}$ to be the nested sequences of compact connected sets, $\mathcal{X}_{\underline{s}} = \{X_{s_1,...,s_m}\}_{m\geqslant 1}$ and $\mathcal{Y}_{\underline{s}} = \{Y_{s_1,...,s_m}\}_{m\geqslant 1}$. We call each such sequence a *Nest*.

We have $X_{s_1,...,s_m} \subset X_{s_1,...,s_{m-1}}$, $Y_{s_1,...,s_m} \subset Y_{s_1,...,s_{m-1}}$ for any m-tuple $(s_1, ..., s_m)$, because it holds already for the corresponding wakes.

Definition 0.6. We define the Core of a Nest $\mathcal{Y}_{\underline{s}}$ to be the set

$$\mathrm{Core}(\mathcal{Y}_{\underline{s}}) = \bigcap_{m\geqslant 1} Y_{s_1,\ldots,s_m} \subset E_\theta.$$

And likewise for $\mathrm{Core}(\mathcal{X}_{\underline{s}})$. The Core of a Nest is always non-empty, as it is the intersection of a nested sequence of compact non-empty sets. We shall say that the Core of the Nest $\mathcal{Y}_{\underline{s}}$ ($\mathcal{X}_{\underline{s}}$) is *trivial* if and only if it is a one-point set.

If $\mathrm{Core}(\mathcal{Y}_{\underline{s}})=\{z\}$, for some $z\in E_\theta$, then $\mathcal{Y}_{\underline{s}}$ is a neighbourhood basis of compact connected neighbourhoods of z in J_θ and likewise for $\mathrm{Core}(\mathcal{X}_{\underline{s}})$.

Let us recall that for any $s\geqslant 1$ the map F_θ^s maps the limb $X_{s,s_1,\ldots,s_m}$ homeomorphically (even diffeomorphically) onto the limb $Y_{s_1,\ldots,s_m}$ for every $(s_1,\ldots,s_m)\in\mathbf{N}^m$, $m\geqslant 0$. Thus for the question of triviality of Cores, it suffices to consider only Nest $\mathcal{Y}_{\underline{s}}$, $\underline{s}\in\mathbf{N}^{\mathbf{N}}$.

PROPOSITION 0.7. *For each $\theta\in\,]0,1[-\mathbf{Q}$ the following two statements are equivalent:*

(1) *The set J_θ is locally connected.*

(2) *For all $\underline{s}\in\mathbf{N}^{\mathbf{N}}$, $\mathrm{Core}(\mathcal{Y}_{\theta,\underline{s}})$ is trivial.*

Proof. Let $\theta\in\,]0,1[-\mathbf{Q}$ be given.

$(2)\Rightarrow(1)$. It suffices to show that any $z\in E_\theta$ has a fundamental system of connected neighbourhoods in J_θ, because of Theorem 0.4. Thus $(2)\Rightarrow(1)$ follows from the two remarks preceeding this proposition.

$(1)\Rightarrow(2)$. We shall actually prove the equivalent, non-(2) implies non-(1). Suppose that $\mathrm{Core}(\mathcal{Y}_{\underline{s}})$ is non-trivial for some $\underline{s}\in\mathbf{N}^{\mathbf{N}}$ (this case suffices by the remark preceding this proposition). For each $m\geqslant 1$ let Ω_m be the $(s_1+\ldots+s_m)$-wake with root $y_{s_1,\ldots,s_m}$ and let $\eta_m^+,\eta_m^-\in\mathbf{T}=\mathbf{R}/\mathbf{Z}$ be the arguments of the two external rays bounding Ω_m. Moreover, let $\delta_m\subset\mathbf{T}$ be the interval of arguments of external rays in Ω_m. Then an external ray of argument $\eta\in\mathbf{T}$ accumulates $Y_{s_1,\ldots,s_m}$ if and only if $\eta\in\delta_m$. Moreover, $2^{s_0+\cdots+s_m}\cdot l(\delta_m)=\frac{1}{2}$. We deduce that exactly 1 external ray accumulates $\mathrm{Core}(\mathcal{Y}_{\underline{s}})$. On the other hand if $J_\theta=\partial\Lambda_0(\infty)$ is locally connected, then any point $z\in J_\theta$ is the landing point of at least one ray and any external ray lands. Thus non-(2) and (1) (logical and) lead to a contradiction. This completes the proof. □

THEOREM 3.25. *For each $\theta\in\,]0,1[-\mathbf{Q}$ the $\mathrm{Core}(\mathcal{Y}_{\underline{s}})$ is trivial for any $\underline{s}\in\mathbf{N}^{\mathbf{N}}$. In particular, J_θ is locally connected for each irrational θ.*

Before we open the final discussion leading to local connectivity of the Julia sets J_{f_θ}, let us discuss a side result, which is interesting in its own right. It identifies for instance large compact hyperbolic subsets of J_θ.

Definition 0.8. Define a map of first return from the collection of first generation sublimbs of Y_ε onto Y_ε,

$$\mathcal{F}_\theta\colon \bigcup_{s\geqslant 1} Y_s \to Y_\varepsilon \quad \text{by} \quad \mathcal{F}_{\theta|Y_s} = F_\theta^s = f_\theta^s.$$

The map is infinite-to-1, but for each $s\geqslant 1$ it is the restriction of a univalent map from a neighbourhood of Y_s to a neighbourhood of Y_ε. Moreover, $\mathcal{F}_\theta$ leaves the set $EY_\theta := E_\theta \cap Y_\varepsilon$ invariant and carries all the essential dynamics of F_θ on EY_θ. Let λ denote both the hyperbolic metric on $\mathbf{C}-\overline{\mathbf{D}}$ and its coefficient function.

THEOREM 3.26. *For all $\theta\in]0,1[-\mathbf{Q}$ and for all $z\in EY_\theta$ we have*

$$\|D_z\mathcal{F}_\theta\|_\lambda = \frac{\lambda(\mathcal{F}_\theta(z))}{\lambda(z)}|\mathcal{F}_\theta'(z)| > 1 \quad \textit{and} \quad \|D_z\mathcal{F}_\theta^m\|_\lambda \underset{m\to\infty}{\longrightarrow} \infty. \tag{1}$$

Moreover, if θ is of constant type there exists $M>1$ such that

$$\|D_z\mathcal{F}_\theta\|_\lambda \geqslant M \quad \textit{for all } z\in EY_\theta. \tag{2}$$

The shift $\sigma\colon \mathbf{N}^\mathbf{N}\to\mathbf{N}^\mathbf{N}$ is the map which forgets the first entry and shifts all other entries one to the left, that is, $\sigma((s_1,s_2,\dots,s_m,\dots))\mapsto(s_2,\dots,s_{m-1},\dots)$. We define for any $\underline{s}=(s_1,\dots,s_m,\dots)\in\mathbf{N}^\mathbf{N}$,

$$\mathcal{F}_\theta(\mathcal{Y}_{\underline{s}}) := \{\mathcal{F}_\theta(Y_{s_1,\dots,s_m})\}_{m\geqslant 2} = \{F_\theta^{s_1}(Y_{s_1,\dots,s_m})\}_{m\geqslant 2} = \mathcal{Y}_{\sigma(\underline{s})}.$$

We see immediately that

$$\mathrm{Core}(\mathcal{Y}_{\sigma(\underline{s})}) = f_\theta^{s_1}(\mathrm{Core}(\mathcal{Y}_{\underline{s}})) = \mathcal{F}_\theta(\mathrm{Core}(\mathcal{Y}_{\underline{s}})),$$

as $f_\theta^{s_1}$ is holomorphic. In particular, the property of having trivial Core is invariant under σ.

The map $\mathrm{dist}(\cdot,\cdot)\colon \mathbf{N}^\mathbf{N}\to\mathbf{N}^\mathbf{N}$ given by $\mathrm{dist}(\underline{s},\underline{t})=\sum_{m\geqslant 1}\delta(s_j,t_j)/2^j$ is a metric on $\mathbf{N}^\mathbf{N}$, which makes the space complete but not compact. For $\{s_1<s_2<\dots<s_m\}\subset\mathbf{N}$ let $\Sigma_{s_1,\dots,s_m}=\{s_1,\dots,s_m\}^\mathbf{N}$. Then $\Sigma_{s_1,\dots,s_m}$ is a shift-invariant Cantor subset of $\mathbf{N}^\mathbf{N}$.

The following corollary of Proposition 0.7 shows that we can use symbolic dynamics to try to understand the dynamics of $\mathcal{F}_\theta$ on EY_θ, and thus the dynamics of F_θ on E_θ.

COROLLARY 0.9 (of Proposition 0.7 and Theorem 3.25). *Define a map $\Psi_\theta\colon \mathbf{N}^\mathbf{N}\to EY_\theta$ by*

$$\mathrm{Core}(\mathcal{Y}_{\underline{s}}) = \{\Psi_\theta(\underline{s})\}.$$

The map Ψ_θ is a homeomorphism which conjugates the shift map $\sigma\colon\mathbf{N}^\mathbf{N}\to\mathbf{N}^\mathbf{N}$ to the map $\mathcal{F}_\theta\colon EY_\theta\to EY_\theta$.

COROLLARY 0.10. *For any $\theta\in]0,1[-\mathbf{Q}$ and for any $\Sigma_{s_1,\dots,s_m}$ the set $\Psi_\theta(\Sigma_{s_1,\dots,s_m})$ is an $\mathcal{F}_\theta$-invariant hyperbolic Cantor set on which the dynamics of $\mathcal{F}_\theta$ is conjugate to the one-sided shift on m symbols.*

COROLLARY 0.11. *For any $\theta\in]0,1[-\mathbf{Q}$ of constant type there exists a constant $L=L(\theta)>1$ such that $L<|\mu|$ for μ the multiplier of any repelling periodic orbit for P_θ.*

Proof. Given $\theta\in]0,1[-\mathbf{Q}$ of constant type let M be as in Theorem 3.26 (2). Moreover, let $\phi_\theta\colon\overline{\mathbf{C}}\to\overline{\mathbf{C}}$ be a quasi-conformal homeomorphism conjugating F_θ to P_θ, and let $K>1$ be the constant of quasi-conformality of ϕ_θ. Then the constant $L=M^{1/K}$ works, because of the following two remarks.

(1) Any repelling periodic orbit for F_θ intersects EY_θ.

(2) The homeomorphism ϕ_θ preserves repelling periodic points, and moreover, if μ_F and μ_P are multipliers of corresponding repelling orbits then $d_\lambda(\mu_F,\mu_P)\leqslant\log K$, where $d_\lambda(\cdot,\cdot)$ denotes distance with respect to the hyperbolic metric on $\mathbf{C}-\overline{\mathbf{D}}$. □

For $K\subset\mathbf{C}$ a compact connected subset define Hull(K) to be the set K union the bounded connected components of $\mathbf{C}-K$.

LEMMA 0.12. *Let $\Omega\subset\mathbf{C}$ be any wake. Then*

$$\Omega\cap J_\theta\subset\Omega\cap J_{f_\theta}\subset\Omega-\Lambda_0(\infty)=\mathrm{Hull}(\Omega\cap J_\theta).$$

In particular, $\mathrm{diam}_e(\Omega\cap J_\theta)=\mathrm{diam}_e(\Omega\cap J_{f_\theta})$. *Moreover, the "limb" of J_{f_θ}, $\Omega\cap J_{f_\theta}$, is connected.*

Proof. The only non-trivial verification is the equal sign: $\Omega-\Lambda_0(\infty)=\mathrm{Hull}(\Omega\cap J_\theta)$. However, this follows from $\partial\Lambda_0(\infty)\subseteq J_\theta$ and the definition of wakes. □

COROLLARY 0.13 (of Proposition 0.7 and Theorem 3.25). *Any point in $E_\theta\cup\tau(E_\theta)$ has a fundamental system of connected neighbourhoods in J_{f_θ}, and thus so has also any point in*

$$\bigcup_{n\geqslant 0} f_\theta^{-n}(E_\theta\cup\tau(E_\theta)).$$

For $s\geqslant 0$ let Ω_s be the s-wake with root x_s. Define for $s\geqslant 0$ limbs of $\mathbf{S}^1$ in J_{f_θ} by $X_{+s}=J_{f_\theta}\cap\Omega_s$ and $X_{-s}=\tau(X_{+s})$ (the indices should be read plus s and minus s).

COROLLARY 0.14 (of Theorem 3.7). *The Euclidean diameters of the limbs X_{+s} and X_{-s} tend to 0 as $n\to\infty$. Moreover, $\mathbf{S}^1$ has no ghost limbs, i.e. $J_{f_\theta}=\mathbf{S}^1\cup\bigcup_{s\geqslant 0}(X_{+s}\cup X_{-s})$, and any point of $\bigcup_{n\geqslant 0} f_\theta^{-n}(\mathbf{S}^1)$ has a fundamental system of connected neighbourhoods in J_{f_θ}.*

Let $Z_\theta = J_{f_\theta} - \bigcup_{n\geqslant 0} f^{-n}(J_\theta \cup \tau(J_\theta))$. Rename U_0 to U_+ and define $U_- = \tau(U_+)$. Another caracterization of Z_θ is that it is the set consisting of those points $z \in J_{f_\theta}$ whose forward orbit passes infinitely often through alternately U_0 and $\tau(U_0)$.

The two corollaries above prove that J_{f_θ} is locally connected at any of its points except those in Z_θ. As a last theorem on local connectivity we present

THEOREM 4.1. *For all $\theta \in]0,1[-\mathbf{Q}$ any point of Z_θ has a fundamental system of connected neighbourhoods in J_{f_θ}, and thus J_{f_θ} is locally connected.*

1. Local connectivity at the critical point 1

A family of Jordan curves. Recall that $F_\theta = f_\theta$ on $\overline{\mathbf{C}} - \mathbf{D}$. They shall thus be used synonymously on this domain. Let β_θ be the unique repelling fixed point for F_θ in $\mathbf{C} - \overline{\mathbf{D}}$. Recall that $x_j = F_\theta^{-j}(1) \cap \mathbf{S}^1$ for each $j \in \mathbf{Z}$, that $y_j \in \partial U_0$ is given by $F_\theta(x_j) = F_\theta(y_j)$ for each $j \geqslant 1$, and moreover that $J_\theta^{\text{skeleton}} = \bigcup_{n\geqslant 0} F_\theta^{-n}(\mathbf{S}^1)$.

For z_1, z_2 both in $\mathbf{S}^1$ the symbols $\lceil z_1, z_2 \rceil$ and $\rceil z_1, z_2 \lceil$ denote the shorter, closed and open subarc respectively of $\mathbf{S}^1$ bounded by z_1 and z_2, if not stated explicitly otherwise. We shall furthermore use the same notation for subarcs of ∂U_0.

We shall construct a family of Jordan curves with nice properties. Each Jordan curve Γ in the family shall possess the following five fundamental properties:

(1) $\Gamma \cap J_\theta = \Gamma \cap J_{f_\theta}$ is a connected subset of $J_\theta^{\text{skeleton}} \cup \bigcup_{n\geqslant 0} F_\theta^{-n}(\beta_\theta)$.

(2) $\Gamma = (\Gamma \cap J_\theta) \cup (\Gamma \cap \Lambda_0(\infty)) \subset \mathbf{C} - (\mathbf{D} \cup U_0)$.

(3) $\Gamma \cap \mathbf{S}^1$ and $\Gamma \cap \partial U_0$ are non-trivial arcs of the form $\lceil 1, x_m \rceil$ and $\lceil 1, y_l \rceil$ respectively for some $m, l \geqslant 1$.

(4) $l_e(\Gamma) < \infty$, where $l_e(\cdot)$ denotes the Euclidean curve length.

(5) $\mathrm{Ind}_\Gamma(0) = 0$.

(See also Figure 8 and the subsection "An initial curve".)

THEOREM 1.1. *There exists a family of Jordan curves, $\{\Gamma_k\}_{k\geqslant 0}$, such that each curve has the five fundamental properties stated above and, moreover,*

$$l_e(\Gamma_k) \to 0 \quad \textit{as } k \to \infty.$$

For any Jordan curve $\gamma \in \mathbf{C}$ let $D(\gamma)$ denote the closure of the bounded connected component of $\mathbf{C} - \gamma$.

COROLLARY 1.2. *For each $\theta \in]0,1[-\mathbf{Q}$ there exists a fundamental system of connected neighbourhoods of 1 in both J_θ and J_{f_θ}.*

Proof. We construct, using the family $\{\Gamma_k\}_{k\geqslant 0}$, a neighbourhood basis of connected neighbourhoods of 1 in J_θ and in J_{f_θ} as follows. For each k let Ξ_k be the union of

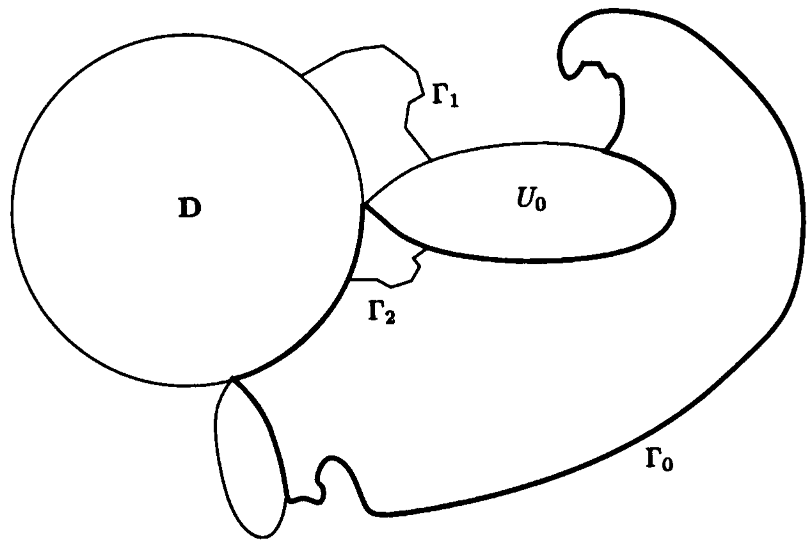

Fig. 8. The first three curves of the family $\{\Gamma_k\}_{k\geqslant 0}$

$f_\theta(D(\Gamma_k))$ with its reflection in $\mathbf{S}^1$. Then $\Xi_k\cap J_\theta$ and $\Xi_k\cap J_{f_\theta}$ are connected neighbourhoods of $v=f_\theta(1)$ in J_θ and J_{f_θ} respectively, because of properties (1) and (3) and because both J_θ and J_{f_θ} are connected. The diameter of Ξ_k tends to 0 as $k\to\infty$, because f_θ is continuous and $\operatorname{diam}(D(\Gamma_k))\to 0$. Hence the sequences $\{\Xi_k\cap J_\theta\}_{k\geqslant 0}$ and $\{\Xi_k\cap J_{f_\theta}\}_{k\geqslant 0}$ form neighbourhood bases of v in J_θ and J_{f_θ} respectively. Consequently the sequences of preimages $\{f_\theta^{-1}(\Xi_k\cap J_\theta)\}_{k\geqslant 0}$ and $\{f_\theta^{-1}(\Xi_k\cap J_{f_\theta})\}_{k\geqslant 0}$ form neighbourhood bases with connected neighbourhoods of 1 in J_θ and J_{f_θ} respectively. □

We obtain as an immediate corollary

THEOREM 1.3. *For each $\theta\in\,]0,1[\,-\mathbf{Q}$ the critical point $1\in J_{f_\theta}$ is in the impression of precisely two prime ends of the immediate basin of ∞ for f_θ. The impressions of these two prime ends equal $\{1\}$. In particular, there are precisely two external rays landing on 1.*

An initial curve. Denote by v the critical value $x_{-1}=f_\theta(1)\in\mathbf{S}^1$ and recall that y_1 is the preimage of 1 in ∂U_0. We shall suppose that $0<\theta<\frac{1}{2}$, so that also $0<t(\theta)<\frac{1}{2}$. Then v is in the upper half-plane and x_1 is in the lower half-plane. The other cases, $\frac{1}{2}<\theta<1$, can be obtained using, for instance, the symmetry under conjugation by complex conjugation.

Let $\varkappa_0$ be the closed subarc of ∂U_0 mapping homeomorphically to the subarc $\lceil 1,v\rceil\subset\mathbf{S}^1$ in the upper half-plane and let γ_0 be the closure of the complementary subarc of ∂U_0. The arcs $\varkappa_0,\gamma_0$ are thought of as starting at 1 and ending at y_1. Define $\varkappa_n$ and γ_n inductively as the arcs which start at the common endpoint of $\varkappa_{n-1}$ and γ_{n-1} and

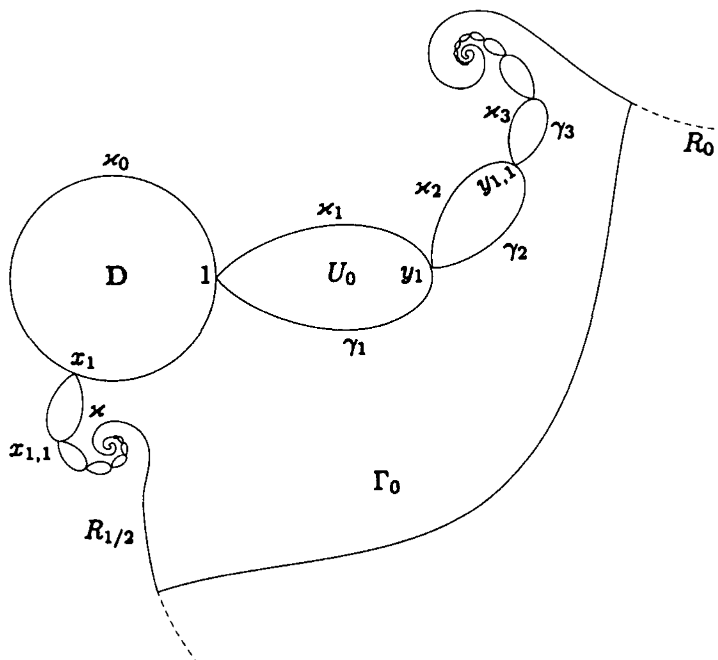

Fig. 9. The arcs γ and $\varkappa$ with some of their constituents

which map homeomorphically to $\varkappa_{n-1}$ and γ_{n-1} respectively by f_θ (see Figure 9). Let γ denote the arc $\gamma_0\gamma_1 \ldots \gamma_n \ldots$, i.e. γ_0 followed by γ_1 etc., and let $\varkappa$ denote the arc which is the preimage of $\varkappa_0\varkappa_1 \ldots \varkappa_n \ldots$ starting at $x_1 = F_\theta^{-1}(1)$.

THEOREM (Sullivan, Douady, Hubbard, Yin). *Let R be a rational map and let C_R denote the closure of the post-critical set union possible rotation domains for R. Suppose that γ: $]-\infty, 0] \to \overline{\mathbf{C}} - C_R$ is a curve with $R^n(\gamma(t)) = \gamma(t+1)$ for all $t \leqslant -1$. Then $\lim_{t\to-\infty} \gamma(t)$ exists and is a repelling or parabolic n-periodic point β for R. Moreover, if β is parabolic then its multiplier is an n-th root of unity.*

Proof. See [TY]. □

We make the arcs γ and $\varkappa$ closed by adding the points β_θ and β'_θ respectively, where β'_θ at the end of $\varkappa$ is a preimage of the repelling fixed point β_θ. Join the two arcs by the lower subarc of $\mathbf{S}^1$ between the two root points 1 and x_1. Also join the two arcs by following γ by the segment of the external ray of external argument 0 from β_θ to equipotential level 1, say. Next follow the equipotential curve at level 1 in the clockwise direction to the external ray of external argument $\frac{1}{2}$. Finally follow the later external ray into the endpoint β'_θ of $\varkappa$. We call the Jordan curve just constructed Γ_0. Evidently

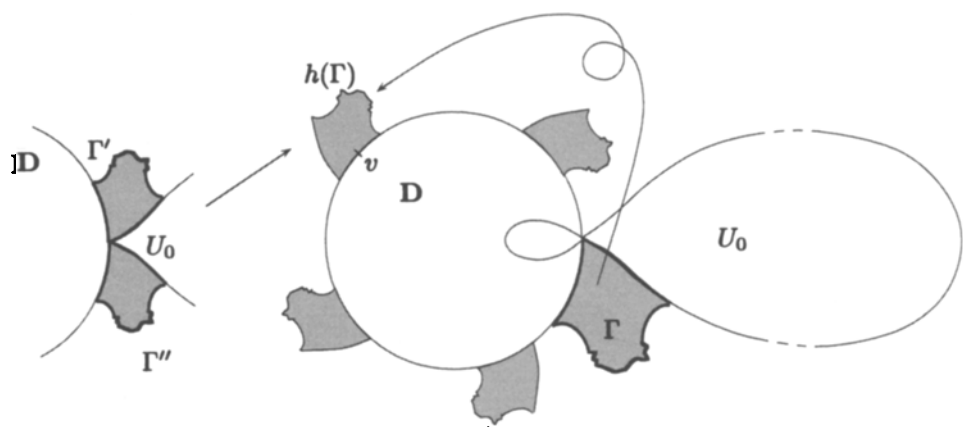

Fig. 10. Iterating Γ backwards until it hits the critical value v

$\Gamma_0 \cap J_\theta = \Gamma_0 \cap J_{f_\theta}$ is connected.

LEMMA 1.4. *The arc Γ_0 has the five fundamental properties* (1) *through* (5).

Proof. The only non-trivial verification is (4). However, this follows from the fact that the point β_θ is repelling. □

A binary tree $\mathcal{T}_\theta$ of Jordan curves. Let $\theta \in]0,1[-\mathbf{Q}$. We shall construct a binary tree $\mathcal{T}_\theta$ of Jordan curves Γ possessing the five fundamental properties (1) to (5) above. The root of the tree $\mathcal{T}_\theta$ is the curve Γ_0 constructed above. The two children of any $\Gamma \in \mathcal{T}_\theta$ shall be lifts of Γ to some appropriate iterate of f_θ. The motivation for creating the tree $\mathcal{T}_\theta$ is that we shall find the sequence $\{\Gamma_k\}_{k\geqslant 0}$ of Theorem 1.1 as a descending path in $\mathcal{T}_\theta$.

Moving from one Jordan curve to the next. Let Γ be a Jordan curve satisfying the five fundamental properties (1) through (5) above and with $I := \Gamma \cap \mathbf{S}^1 = [1, x_m]$. We move from Γ to anyone of its two children $\Gamma' \in \mathcal{T}_\theta$ as follows. If I does not contain the critical value v, then there is a unique inverse branch of f_θ defined on $D(\Gamma)$ and mapping I to some subarc of $\mathbf{S}^1$. If also the inverse image of I does not contain v we may continue to find a unique branch of f_θ^{-2} on $D(\Gamma)$ mapping I to some subarc of $\mathbf{S}^1$, and so on. We may continue this until we have obtained a branch h of $f_\theta^{-(j-1)}$ for some $j \geqslant 1$ defined on $D(\Gamma)$ and mapping I to some subarc of $\mathbf{S}^1$ containing v in the interior (θ is irrational).

Here we have to make a choice. The preimage of $h(\Gamma)$ by F_θ can be viewed as two Jordan curves with 1 as a common point. Each of the two choices for Γ' satisfies the fundamental properties (1) through (5) above, because Γ does so and they are lifts of Γ to F_θ^j (and to f_θ^j). See Figure 10.

Let g denote the composition of the final choice of inverse branch of F_θ with h. We will call g a move. The map $g: D(\Gamma) \to D(\Gamma')$ is a homeomorphism with $f_\theta^j \circ g = \mathrm{Id}$

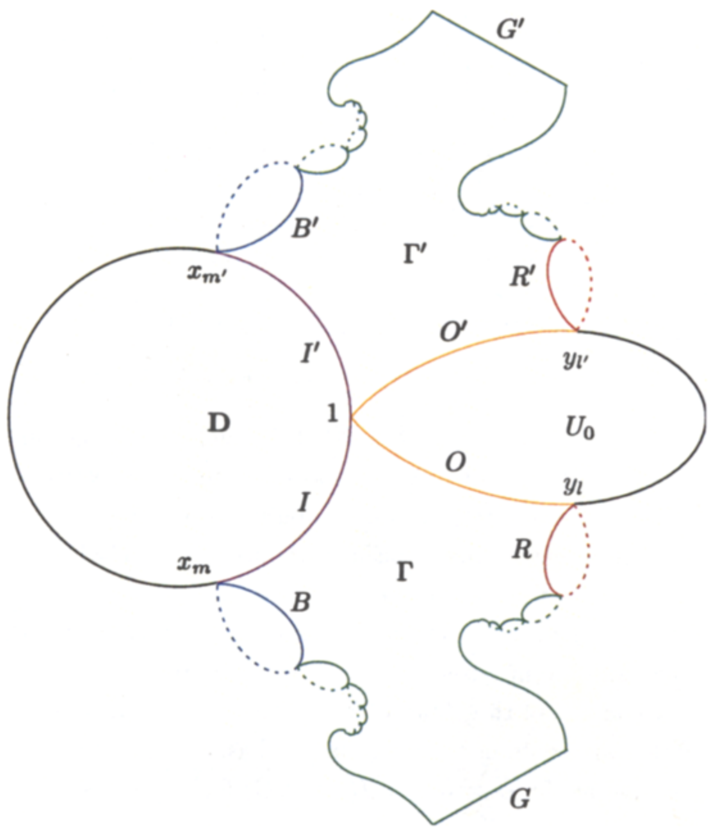

Fig. 11. The colouring of the Jordan curves Γ

on $D(\Gamma)$. It is easily checked that $I'=\Gamma'\cap\mathbf{S}^1=[1,x_{m'}]$ and $O'=\Gamma'\cap\partial U_0=[1,y_{l'}]$ with $\{m',l'\}=\{j,m+j\}$. The long composition h of inverse branches of f_θ is univalent on a domain containing $D(\Gamma)$ in its interior and g is a local diffeomorphism at each point of $D(\Gamma)$ except at the point $x_{-j}\in I=\mathbf{S}^1\cap\Gamma$, which is mapped to 1. Even though g is defined on all of $D(\Gamma)$ we shall often just write $g:\Gamma\to\Gamma'$. We note also that x_{-j} is the first return of 1 into I.

One of the two choices for Γ' has I' above 1, the other choice has I' below 1. This leads us to distinguish the following two types of moves $g:\Gamma\to\Gamma'$.

(1) The move g is called a *Gain* if I and I' are on the same side of 1, i.e. either both above or both below 1.

(2) The move g is called a *Swap* if I and I' are on different sides of 1.

The binary tree $\mathcal{T}_\theta$ of Jordan curves is constructed inductively with Γ_0 as root and using the above two moves. Moreover, for $k\geqslant 1$ we let $\mathcal{T}_{\theta,k}$ denote the union of the 2^k subtrees of $\mathcal{T}_\theta$ for which the root points are the curves k moves down from Γ_0.

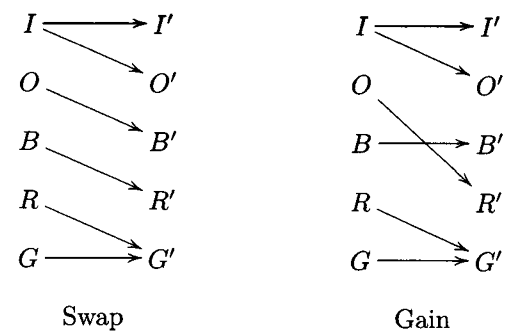

Fig. 12. The dynamics of colours under the two kinds of moves

Colouring the Jordan curves Γ. Recall that U_j and V_j are the unique connected components of $F_\theta^{-j}(U_0)$ (open j-drops) with roots x_j and y_j respectively, $j \geqslant 1$.

Any curve $\Gamma \in \mathcal{T}_\theta$ naturally falls into five subarcs, two of which, $I = \Gamma \cap \mathbf{S}^1 = [1, x_m]$ and $O = \Gamma \cap \partial U_0 = [1, y_l]$, have already been introduced. Let $B = \Gamma \cap \partial U_m$, $R = \Gamma \cap \partial V_l$, and let G be the closure of the complementary subarc of Γ left out by the others. When making drawings the reader is invited to colour the different subarcs of Γ, B(lue), G(reen), R(ed) and O(range) and invent a colour for I. The careful reader will have observed that the arc G actually consists of three parts, one part at either end contained in the Julia set and the middle part contained entirely in the basin of infinity. To emphasize the colouring we shall also at times write $\Gamma(I, B, G, R, O)$ for Γ (see also Figure 11).

Moving the colours. Let $g: \Gamma(I, B, G, R, O) \to \Gamma'(I', B', G', R', O')$ be a move between Jordan arcs satisfying (1) through (3). Then always $g(I) = I' \cup O'$ and $g(R \cup G) = G'$; whereas $g(O) = B'$, $g(B) = R'$ if g is a Swap; and $g(O) = R'$, $g(B) = B'$ if g is a Gain. See Figure 12.

A good subtree. As mentioned above the family $\{\Gamma_k\}_{k \geqslant 0}$ in Theorem 1.1 shall be found as a descending path in $\mathcal{T}_\theta$. To facilitate the choice of a good path we shall consider especially certain branches of $\mathcal{T}_\theta$. More precisely we shall consider such paths in $\mathcal{T}_\theta$ for which the sequence of moves does not contain consecutive Gains. We may illustrate this by the flowchart Figure 13.

We call a sequence of moves *admissible* if it complies with the flowchart. We let $\mathcal{G}_\theta^*$ denote the subtree of $\mathcal{T}_\theta$ consisting of those Jordan curves Γ obtained from Γ_0 by an admissible sequence of moves. Furthermore, we let $\mathcal{G}_{\theta,k}^* = \mathcal{G}_\theta^* \cap \mathcal{T}_{\theta,k}$ for $k \geqslant 0$.

We shall at the end of this section (Bounding G(reen)) describe how to choose the sequence $\{\Gamma_k\}_{k \geqslant 0}$ which satisfies the statement of Theorem 1.1. We are however already in position to control all but the length of the G part of each $\Gamma \in \mathcal{G}_\theta^*$. Let λ denote the hyperbolic metric on $\mathbf{C} - \overline{\mathbf{D}}$.

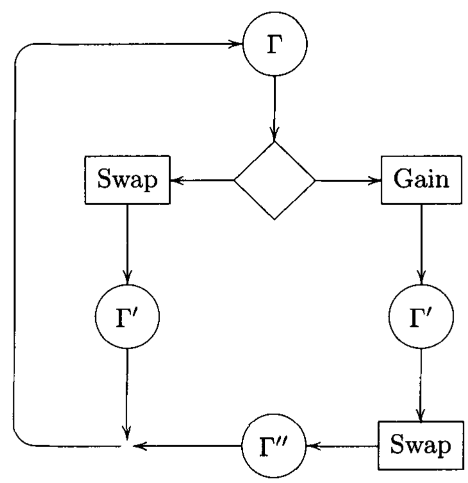

Fig. 13. Flowchart for defining $\mathcal{G}_\theta^*$

PROPOSITION 1.5. *For each* $\theta\in\,]0,1[-\mathbf{Q}$ *there exist constants* $K_{O,\theta}, K_{B,\theta}, K_{R,\theta}>0$ *and* $L_{R,\theta}>0$ *such that any Jordan curve* $\Gamma(I,B,G,R,O)\in\mathcal{G}_\theta^*$ *satisfies*:

(1) $l_e(O)\leqslant K_{O,\theta}\cdot l_e(I)$,

(2) $l_e(B)\leqslant K_{B,\theta}\cdot l_e(I)$,

(3) $l_e(R)\leqslant K_{R,\theta}\cdot l_e(I)$,

(4) $l_\lambda(R)\leqslant L_{R,\theta}$.

Here $l_e(\cdot)$ denotes Euclidean length and $l_\lambda(\cdot)$ denotes length with respect to the hyperbolic metric λ on $\mathbf{C}-\overline{\mathbf{D}}$. We shall postpone the proof of Proposition 1.5 to the next chapter. That chapter is devoted to proving a universal version of this proposition. The proof essentially consists in obtaining complex bounds from the Świątec a priori real bounds.

We shall study the endpoints x_m of I and y_l of O. This leads us to discuss first and closest return.

Moments and points of closest return. Let $\theta\in\,]0,1[-\mathbf{Q}$. The nth convergent of θ is the rational number p_n/q_n obtained by truncating the continued fraction expansion of θ at level $n-1$, i.e.

$$\frac{p_n}{q_n}=\cfrac{1}{a_1+\cfrac{1}{a_2+\cfrac{1}{\ddots+\cfrac{1}{a_{n-1}}}}}.$$

Defining $q_0=p_1=0$ and $q_1=p_0=1$ gives the recurrence formulas $p_{n+1}=a_np_n+p_{n-1}$ and

$q_{n+1}=a_nq_n+q_{n-1}$. We are however not interested in the p_n.

The integers q_n are called moments of closest return and the integers αq_n+q_{n-1}, $0<\alpha\leqslant a_n$, are called moments of first return for orbits of $f_\theta\colon \mathbf{S}^1\to\mathbf{S}^1$ and $f_\theta^{-1}\colon \mathbf{S}^1\to\mathbf{S}^1$ ($\alpha=0$, a_n are also closest returns). The corresponding points $x_{\pm q_n}$ and $x_{\mp(\alpha q_n+q_{n-1})}$ are called points of closest and first return respectively of 1 to itself under $f_\theta^{\pm 1}$. Note that as usual in this paper the backward iterates of 1 in $\mathbf{S}^1$ have positive indices, whereas the forward iterates have negative indices.

Let $\tilde{v}$ denote the logarithm of the critical value v in $]0,i2\pi[$. For $j\in\mathbf{Z}-\{-1\}$ let $\tilde{x}_j$ be the logarithm of x_j in $]\tilde{v}-2\pi i,\tilde{v}[$.

LEMMA 1.6. *Let $<$ denote the natural ordering on $i\mathbf{R}$. Suppose that n is even and $0<\theta<\frac{1}{2}$ (the case n odd or $\frac{1}{2}<\theta<1$ is analogous, but with all inequalities reversed). Then, if $a_n\neq 1$:*

$$\tilde{x}_{-(q_n-q_{n-1})}<\tilde{x}_{q_{n-1}}<\tilde{x}_{q_n+q_{n-1}}<\ldots<\tilde{x}_{q_{n+1}-q_n}<\tilde{x}_{-q_n}$$
$$<\tilde{x}_{a_nq_n+q_{n-1}}=\tilde{x}_{q_{n+1}}<0<\tilde{x}_{q_n}<\tilde{x}_{-q_{n-1}},$$

and if $a_n=1$:

$$\tilde{x}_{-(q_n-q_{n-1})}<\tilde{x}_{q_{n-1}}<\tilde{x}_{-q_n}<\tilde{x}_{q_n+q_{n-1}}=\tilde{x}_{q_{n+1}}<0<\tilde{x}_{q_n}<\tilde{x}_{-q_{n-1}}.$$

Proof. For the rigid rotations, $R_\theta(z)=z\cdot e^{i2\pi\theta}$, the above is a standard result, thus it follows from the Poincaré semiconjugation theorem for circle homeomorphisms. □

LEMMA 1.7. *The largest subarc of $\mathbf{S}^1$ around 1 which is mapped diffeomorphically into $\mathbf{S}^1$ by $f_\theta^{-(q_n-1)}$ is the arc bounded by $x_{-(q_n-q_{n-1})}$ and $x_{-q_{n-1}}$.*

Proof. Follows from the previous lemma, because $q_n=a_{n-1}q_{n-1}+q_{n-2}$ and the first return of 1 under f_θ into the subarc of the lemma is the point x_{-q_n}. □

Tracing the endpoints of I and O. In the sequel we shall focus on the combinatorics of the end points of the subarcs I and O of the Jordan curve $\Gamma(I,B,G,R,O)$. For this reason it will be convenient to introduce $\Gamma(x_m,y_l)$ as a synonym for Γ, where x_m and y_l are given by $I=\lceil 1,x_m\rceil$ and $O=\lceil 1,y_l\rceil$. (Note that the points x_m and y_l alone do not specify the curve $\Gamma(x_m,y_l)$ uniquely.)

LEMMA 1.8. *For each $\Gamma(x_m,y_l)\in\mathcal{T}_\theta$ there exist $n\geqslant 1$ and $0\leqslant\alpha\leqslant a_n$ such that*

$$\{m,l\}=\{q_n,\alpha q_n+q_{n-1}\}\quad (\textit{equal as sets}).$$

Note that $a_nq_n+q_{n-1}=0q_{n+2}+q_{n+1}$, and hence the numbers n and α are not unique when $\alpha=0$ or $\alpha=a_n$.

COMPLEMENT TO LEMMA 1.8. *Let* $g:\Gamma(x_m, y_l)\to\Gamma'(x_{m'}, y_{l'})$ *be a move and suppose that* $m=\alpha q_n+q_{n-1}$, $n\geqslant 1$ *and* $0\leqslant\alpha<a_n$. *Then* $f_\theta^{q_n}\circ g=\mathrm{Id}$. *Moreover,*

(1) $(m', l')=((\alpha+1)q_n+q_{n-1}, q_n)$ *if* g *is a Gain,*

(2) $(m', l')=(q_n, (\alpha+1)q_n+q_{n-1})$ *if* g *is a Swap.*

(Note that here we have equality of ordered pairs.)

Proof. We prove the lemma by induction on the number of moves it takes to produce $\Gamma(x_m, y_l)$ from $\Gamma_0(x_1, y_1)$. In doing so we shall simultaneously prove the complement. As $\Gamma_0(x_m, y_l)$ has $m=l=1=q_1=q_1+q_0$ the induction basis is okay. Assume next that any Jordan curve $\Gamma\in\mathcal{T}_\theta$ which is at most $k\geqslant 0$ moves down from Γ_0 satisfies the statement of the lemma, and let $\Gamma(x_m, y_l)\in\mathcal{T}_\theta$ be any such curve. Write $m=\alpha q_n+q_{n-1}$ with $n\geqslant 1$, $0\leqslant\alpha<a_n$, and let $g:\Gamma(x_m, y_l)\to\Gamma'(x_{m'}, y_{l'})$ be any of the two moves from Γ. Then $f_\theta^{q_n}\circ g=\mathrm{Id}$ by Lemma 1.6 and the definition of moves. It follows that $\{m', l'\}=\{q_n, (\alpha+1)q_n+q_{n-1}\}$. This proves the lemma and moreover the complement, because $g(x_m)=x_{m'}$ if and only if g is a Gain. □

LEMMA 1.9. *Suppose that* $\Gamma, \Gamma'\in\mathcal{T}_\theta$ *and that* $g:\Gamma(x_m, y_l)\to\Gamma'(x_{m'}, y_{l'})$ *is a move. Then there exists* $n\geqslant 2$ *such that* $f_\theta^{q_n}\circ g=\mathrm{Id}$, *and moreover,*

$$m=\alpha q_n+q_{n-1},\quad 0\leqslant\alpha<a_n,$$
$$l=\beta q_{n-1}+q_{n-2},\quad 0\leqslant\beta\leqslant a_{n-1}.$$

Proof. Let $n\geqslant 2$ and $0\leqslant\alpha<a_n$ be given by $m=\alpha q_n+q_{n-1}$. It follows from Lemma 1.8 and its complement that $f_\theta^{q_n}\circ g=\mathrm{Id}$. If $0<\alpha$, then $l=q_n=a_{n-1}q_{n-1}+q_{n-2}$ by Lemma 1.8. And if $0=\alpha$, then $m=q_{n-1}$, and Lemma 1.8 implies $l=\beta q_{n-1}+q_{n-2}$, with $0\leqslant\beta\leqslant a_{n-1}$. Thus in either case the lemma follows. □

PROPOSITION 1.10. *Let* $\Gamma\in\mathcal{T}_\theta$ *be arbitrary and let* Γ_1 *and* Γ_2 *be the arcs obtained by the two moves from* Γ. *Then*

$$D(f_\theta(\Gamma_1))=D(f_\theta(\Gamma_2))\subset D(f_\theta(\Gamma)).$$

Proof. The first equality sign follows from the definition of moves. For each $\Gamma\in\mathcal{T}_\theta$ let $h_\Gamma: D(\Gamma)\to\mathbf{C}$ be the long composition of inverse branches of f_θ with $v\in h_\Gamma(I)\subset\mathbf{S}^1$, where I equals $\mathbf{S}^1\cap\Gamma$. We shall prove the following equivalent formulation of the proposition: $\forall\Gamma\in\mathcal{T}_\theta$,

$$h_\Gamma(\Gamma)\subset D(f_\theta(\Gamma)). \tag{1}$$

Moreover, (1) is equivalent to $J_\theta\cap h_\Gamma(\Gamma)=h_\Gamma(\Gamma\cap J_\theta)\subset D(f_\theta(\Gamma))$, because external rays do not cross and f_θ maps the equipotential curve at level $p>0$ in $\Lambda_0(\infty)$ to the equipotential curve at level $2p$. We divide the curves $\Gamma \in \mathcal{T}_\theta$ into two complementary classes. The first class consists of all curves Γ of the form $\Gamma=\Gamma(x_{q_n}, y_l)$ for some $n\geqslant 1$ and some $l=\beta q_n+q_{n-1}$, $0\leqslant\beta\leqslant a_n$. The second class consists of all curves Γ of the form $\Gamma=\Gamma(x_m, y_{q_n})$ for some $n\geqslant 1$ and some $m=\alpha q_n+q_{n-1}$, $0<\alpha<a_n$ (recall Lemma 1.8).

Suppose first that $\Gamma=\Gamma(I,B,G,R,O)$ belongs to the first class. If $0\leqslant\beta<a_n$, then

$$h_\Gamma(I)=f_\theta(\lceil x_{q_{n+1}+q_n}, x_{q_{n+1}}\rceil)\subset f_\theta(\rceil x_{q_n}, x_{\beta q_n+q_{n-1}}\lceil)\subset f_\theta(I\cup O).$$

The fundamental curve properties (1) through (3) then implies that $h_\Gamma(\Gamma\cap J_\theta)\subset D(f_\theta(\Gamma))$, and thus (1) holds.

For $\beta=a_n$ we have to look a little bit further.

CLAIM. *For any* $\Gamma(x_m,y_l)=\Gamma(I,B,G,R,O)\in\mathcal{T}_\theta$, *with* $m=\alpha q_n+q_{n-1}$, *for some* $n\geqslant 1$, $0\leqslant\alpha<a_n$, *we have* $\lceil x_m, x_{m,q_{n-1}}\rceil\subset B$, *where*

$$\lceil x_m, x_{m,q_{n-1}}\rceil = f_\theta^{-m}(\lceil 1, y_{q_{n-1}}\rceil)\cap\partial U_m.$$

Proof of the claim. Note at first that it suffices to observe that the claim holds for Γ_0 and to prove that the claim holds for those Γ obtained by a Swap from their predecessor, because a Gain preserves B (see the subsection "Moving the colours"). Thus we can suppose that $m=q_{n-1}$ and that the last move $g'\colon\Gamma'(x_{m'},y_{l'})\to\Gamma(x_{q_{n-1}},y_l)$ is a Swap. Then $l'=\beta q_{n-2}+q_{n-3}$ for some $0\leqslant\beta\leqslant a_{n-2}$. The claim then follows because g' maps O' to B and $\lceil 1, y_{q_{n-1}}\rceil\subseteq O'$ by Lemma 1.9.

Suppose that Γ is in the first class with $\beta=a_n$, so that $\Gamma=\Gamma(x_{q_n}, y_{q_{n+1}})$. Let the last move in obtaining Γ be $g'\colon\Gamma'(x_{m'},y_{l'})\to\Gamma(x_{q_n},y_{q_{n+1}})$. It follows from Lemma 1.8 with its complement that g' is a Swap and $m'=(a_n-1)q_n+q_{n-1}=q_{n+1}-q_n$. Let

$$B'=\lceil x_{q_{n+1}-q_n}, x_{q_{n+1}-q_n,k}\rceil=\Gamma'\cap\partial U_{q_{n+1}-q_n}.$$

Then

$$\lceil x_{q_{n+1}-q_n}, x_{q_{n+1}-q_n,q_{n-1}}\rceil\subset B' \quad\text{and}\quad \lceil y_{q_{n+1}}, y_{q_{n+1},q_{n-1}}\rceil\subset\lceil y_{q_{n+1}}, y_{q_{n+1},k}\lceil = g'(B')=R$$

by the claim, and hence

$$h_\Gamma(I)=f_\theta(\lceil x_{q_{n+1}+q_n}, x_{q_{n+1}}\rceil)\subset f_\theta(\rceil x_{q_n}, x_{q_{n+1}}\rceil)\subset f_\theta(I\cup O)$$

and

$$h_\Gamma(O)=f_\theta(\lceil y_{q_{n+1}}, y_{q_{n+1},q_{n+1}}\rceil)\subset f_\theta(\lceil y_{q_{n+1}}, y_{q_{n+1},k}\lceil)=f_\theta(R).$$

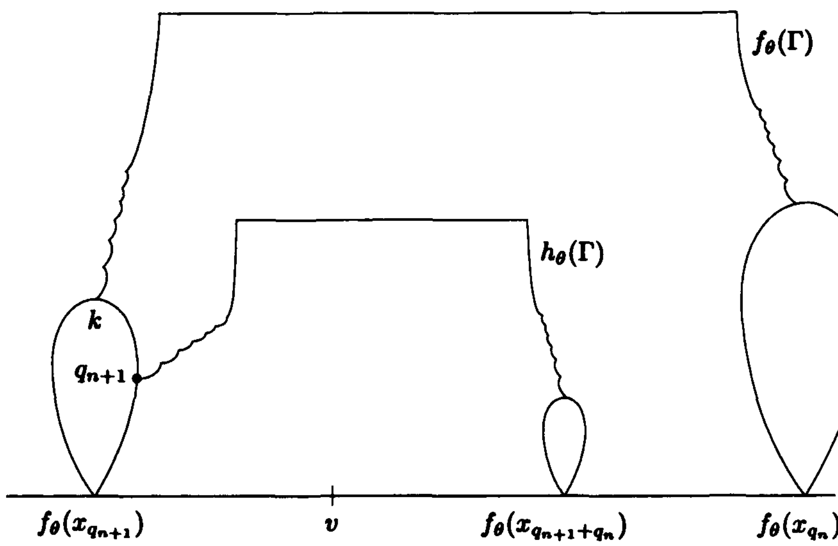

Fig. 14. The "worst" case $\beta=a_n$

The fundamental curve properties (1) through (3) then imply that $h_\Gamma(\Gamma\cap J_\theta)\subset D(f_\theta(\Gamma))$ (see also Figure 14), and thus (1) holds.

This proves that (1) holds for any Γ in the first class. Suppose next that Γ belongs to the second class, i.e. $\Gamma(x_{\alpha q_n+q_{n-1}}, y_{q_n})$ for some $0<\alpha<a_n$. It follows from the complement to Lemma 1.8 that there exists a $\Gamma'=\Gamma'(x_{q_{n-1}}, y_l)\in\mathcal{T}_\theta$ such that Γ is obtained from Γ' by α consecutive Gains. Moreover, each Gain is a local inverse of $f_\theta^{q_n}$ mapping 1 to y_{q_n}. Let $g'\colon\Gamma'(x_{q_{n-1}}, y_{l'})\to\Gamma''(x_{q_n+q_{n-1}}, y_{q_n})$ be the Gain of Γ'. Then $h_{\Gamma'}=f_\theta\circ g'$ and $D(g'(\Gamma'))\subset D(\Gamma')$, because Γ' belongs to the first class and hence satisfies (1) by the above. But then the Gain of Γ'' coincides with the restriction of g' to $D(\Gamma'')$. It then follows by induction that $D(\Gamma)\subset D(\Gamma')$ and that the Gain g of Γ coincides with the restriction of g' to $D(\Gamma)$, because $\alpha<a_n$ implies that g is also an inverse branch of $f_\theta^{q_n}$ mapping 1 to y_{q_n}. But then Γ satisfies (1). This completes the proof. □

Bounding G(reen). We shall bound G and complete the proof of Theorem 1.1 (assuming Proposition 1.5) before we end this section. Let $W_1=F_\theta^{-1}(\mathbf{C}-\overline{\mathbf{D}})\subset\mathbf{C}-\overline{\mathbf{D}}$ so that $f_\theta=F_\theta\colon W_1\to\mathbf{C}-\overline{\mathbf{D}}$ is a degree-two covering. In particular, it is infinitesimally expanding with respect to the hyperbolic metric λ on $\mathbf{C}-\overline{\mathbf{D}}$. Fix $\theta\in]0,1[-\mathbf{Q}$ and let $h_0\colon\Gamma_0\to\mathbf{C}$ be the long composition of inverse branches of f_θ in the definition of moves from Γ_0. Recall that h_0 extends to a diffeomorphism from a neighbourhood of $D(\Gamma_0)$ onto a neighbourhood of $D(h_0(\Gamma_0))$. Consider the half-line l from 0 to ∞ through the critical value v. Let α be the first intersection outside $\overline{\mathbf{D}}$ of l with $h_0(\Gamma_0)$. Let $[v,\alpha]$ denote the closed line segment from v to α. The segment $[v,\alpha]$ cuts $D(h_0(\Gamma_0))$ into two

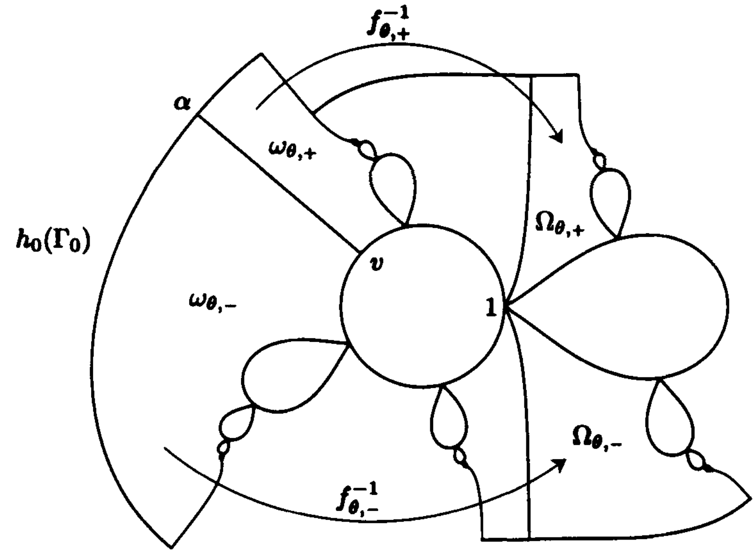

Fig. 15. The strongly contracting inverse branches

closed pieces (we include $[v,\alpha]$ in both pieces and orient $[v,\alpha]$ outwards). Let $\widehat{\omega}_{\theta,+}$ be the piece containing the points in $h_0(D(\Gamma_0))$ immediately to the right of $[v,\alpha]$ and let $\widehat{\omega}_{\theta,-}$ be the piece containing the points in $D(h_0(\Gamma_0))$ immediately to the left of $[v,\alpha]$. Finally let $\omega_{\theta,\pm}=\widehat{\omega}_{\theta,\pm}-\mathbf{S}^1$. Let $\Omega_{\theta,+}$, $\Omega_{\theta,-}$ be the connected components of $f_\theta^{-1}(\omega_{\theta,+})$ and $f_\theta^{-1}(\omega_{\theta,-})$ respectively, having a non-trivial boundary arc in common with U_0. See Figure 15. Let $f^{-1}_{\theta,+}, f^{-1}_{\theta,-}: D(h_0(\Gamma_0))\to\mathbf{C}$ denote the inverse branches of f_θ with $f^{-1}_{\theta,+}(\omega_{\theta,+})=\Omega_{\theta,+}$ and $f^{-1}_{\theta,-}(\omega_{\theta,-})=\Omega_{\theta,-}$ respectively.

LEMMA 1.11. *Let $\theta\in]0,1[-\mathbf{Q}$. On $\Omega_{\theta,\pm}$ the map f_θ is strongly infinitesimally expanding with respect to the hyperbolic metric λ on $\mathbf{C}-\overline{\mathbf{D}}$. That is, there exists $0<2\varepsilon(\theta)<1$ such that*

$$\|D_z f_\theta\|_\lambda=\frac{\lambda(f_\theta(z))}{\lambda(z)}|f'_\theta(z)|\geqslant\frac{1}{1-2\varepsilon}\quad\forall z\in\Omega_{\theta,\pm},$$

and consequently the corresponding local inverse branches are strongly contracting,

$$\|D_z f^{-1}_{\theta,\pm}\|_\lambda\leqslant(1-2\varepsilon)\quad\forall z\in\omega_{\theta,\pm}. \tag{1}$$

Moreover, $\liminf\|D_z f_\theta\|_\lambda=\frac{3}{2}$*, where* inf *is over* $\mathbf{D}_r(1)\cap\Omega_{\theta,\pm}$ *and* lim *is for* $r\to0$.

Proof. On W_1 the map f_θ is expanding with respect to λ and moreover $\|D_z f_\theta\|_\lambda\to\frac{3}{2}$, when $z\to1$ in $f_\theta^{-1}(]v,\alpha])\cap W_1$. □

Let Γ_1 and Γ_2 be the curves obtained by one and two Swaps respectively from Γ_0. (This choice is not essential but convenient.) Let $\Gamma(I,B,G,R,O)$ be any curve in the subtree of $\mathcal{T}_\theta$ with root Γ_2. Let h be the long composition of inverse branches of f_θ on

Γ with $v \in h(I)$. It is easy to check that $h(I) \subset h_2(I_2) \subset (h_0(I_0))^\circ$. It then follows that $D(h(\Gamma)) \subset D(h_0(\Gamma_0))$. In particular we have $h(R \cup G) \subset \omega_{\theta,+} \cup \omega_{\theta,-}$. Alternatively we can appeal to Proposition 1.10, whose proof however requires a little more work.

We are now ready to choose the sequence $\{\Gamma_k\}_{k \geqslant 0} \subset \mathcal{G}_\theta^*$ for Theorem 1.1. The first three curves, Γ_0, Γ_1 and Γ_2, have already been chosen above. We shall choose the sequence as a descending path in $\mathcal{G}_\theta^*$. Thus we need only specify how the decision between a Swap and a Gain is taken at the top of the flowchart (Figure 13). Suppose that $\Gamma_k(I_k, B_k, G_k, R_k, O_k)$, $k \geqslant 2$, has already been chosen. Let h_k be the long composition of univalent inverse branches of f_θ defined on some neighbourhood of $D(\Gamma_k)$ and with $v \in h_k(I_k)$. Now at least half of the λ-length of $h_k(R_k \cup G_k)$ is in either $\omega_{\theta,+}$ or $\omega_{\theta,-}$. If in $\omega_{\theta,+}$ we choose $f_{\theta,+}^{-1}$ as final inverse branch of f_θ on $D(h_k(\Gamma_k))$, and if in $\omega_{\theta,-}$ we choose the other branch $f_{\theta,-}^{-1}$. If the obtained move is a Swap, then we have chosen the Swap, and if it is a Gain we have chosen the Gain. If both $\omega_{\theta,+}$ and $\omega_{\theta,-}$ contain at least half the λ-length, we choose the Swap.

Definition 1.12. Define $\{\Gamma_k\}_{k \geqslant 0}$ to be the descending sequence in $\mathcal{G}_\theta^*$ chosen above.

LEMMA 1.13. *There exist constants $L_{G,\theta}, K_{G,\theta} > 0$ such that*

$$l_\lambda(G_k) \leqslant L_{G,\theta} \quad \text{and} \quad l_e(G_k) \leqslant K_{G,\theta} \cdot l_e(I_k)$$

for all $k \geqslant 0$.

Proof. We have $g_k(R_k \cup G_k) = G_{k+1}$ and thus $l_\lambda(G_{k+1}) \leqslant l_\lambda(G_k) + l_\lambda(R_k)$ for all k, as f_θ is expanding with respect to λ. Moreover, $l_\lambda(G_0) < \infty$ and by Proposition 1.5 (4) there exists a constant $L_{R,\theta}$ such that $l_\lambda(R_k) \leqslant L_{R,\theta}$ for all k. By construction and Lemma 1.11 (1) we thus have

$$\begin{aligned} l_\lambda(G_{k+1}) &\leqslant \tfrac{1}{2}(l_\lambda(G_k) + l_\lambda(R_k)) + (1-2\varepsilon)\tfrac{1}{2}(l_\lambda(G_k) + l_\lambda(R_k)) \\ &\leqslant (1-\varepsilon)(l_\lambda(G_k) + L_{R,\theta}) \end{aligned}$$

for at least every second k. Let $L' = 2L_{\theta,R}/\varepsilon$. If $l_\lambda(G_k) \geqslant L'$, then

$$l_\lambda(G_{k+2}) \leqslant (1-\varepsilon)(l_\lambda(G_k) + L_{R,\theta}) + L_{R,\theta} \leqslant l_\lambda(G_k) - \varepsilon \cdot L_{R,\theta}.$$

Thus $\limsup l_\lambda(G_k) \leqslant L' + L_R$. This proves the existence of an upper bound $L_{G,\theta}$ for $l_\lambda(G_k)$.

To prove the existence of $K_{G,\theta}$ note that G_k and B_k have a common endpoint and that B_k touches $\mathbf{S}^1$. Moreover, $l_e(B) \leqslant K_{B,\theta} \cdot l_e(I)$ by Proposition 1.5 (2). The weight function $\lambda(z)$ of the hyperbolic metric on $\mathbf{C} - \overline{\mathbf{D}}$ is asymptotic to $1/(|z|-1)$ when z approaches $\mathbf{S}^1$. Hence we also get the existence of $K_{G,\theta}$. □

Proof of Theorem 1.1. Let $\{\Gamma_k\}_{k\geqslant 0}\in\mathcal{G}_\theta^*$ be the sequence defined in Definition 1.12. Set $K_{\Gamma,\theta}=1+K_{B,\theta}+K_{G,\theta}+K_{R,\theta}+K_{O,\theta}$. Then $l_e(\Gamma_k)\leqslant K_{\Gamma,\theta}\cdot l_e(I_k)$. It follows from the definition of $\mathcal{G}_\theta^*$ and the fact that f_θ is conjugate to the rigid rotation R_θ on $\mathbf{S}^1$, that $l_e(I_k)\to 0$ when $k\to\infty$. This proves Theorem 1.1. Appealing to the Świątec a priori real bounds (see §2) we even obtain exponential convergence to zero. □

2. Complex bounds from real bounds

The Świątec a priori real bounds. The following theorem is often referred to as the Świątec a priori real bounds:

THEOREM (Świątec, Herman). *There exists a constant* $0<a<1$ *such that for all* $\theta\in\,]0,1[-\mathbf{Q}$ *the points of closest return under* $f_\theta^{-1}:\mathbf{S}^1\to\mathbf{S}^1$, x_{q_n}, $n\geqslant 1$, *satisfy*

$$a\leqslant\frac{|x_{q_{n+1}}-1|}{|x_{q_n}-1|}\leqslant\frac{1}{a}.$$

Proof. [Sw], [Yo2, §3, proposition, p. 6]. □

An initial version of this result in the case of rational θ and for n up to $\theta=p_n/q_n$ appeared in [Sw]. M. Herman [He] observed that Świątec's inequalities hold for all n, when θ is irrational. He then used the inequalitites to prove the Herman–Świątec conjugation theorem. The Świątec a priori real bounds are actually better than stated above. The constant a depends only on f_θ' and hence only on f_0. More precisely, it depends on macroscopic properties of f_θ, such as the order of the critical point and the total variation of $\log|f_\theta'|$ on $\mathbf{S}^1-J$, where J is an interval around the critical point c, such that f_θ has negative Schwarzian derivative on $J-\{c\}$. For more precise statements see the manuscript [Yo2].

Terminology 2.1. Let $\Gamma(x_m,y_l)\in\mathcal{T}_\theta$. We say that Γ has a *Fresh B(lue)* if $\Gamma=\Gamma_0$ or if the last move in obtaining Γ was a Swap. If the last move in obtaining Γ was a Gain, we distinguish two cases. First we note that $(m,l)=(\alpha q_n+q_{n-1},q_n)$, $0<\alpha\leqslant a_n$, for some n, by the complement to Lemma 1.8. Moreover, an easy induction argument on that same complement shows that Γ is obtained from a $\Gamma'(x_{q_n},y_l)$ by α consecutive Gains. If Γ' has a Fresh B, then Γ is said to have a *Young B(lue)*. If not, then Γ is said to have an old $B(lue)$.

We shall use a similar notion for $R(ed)$. Let $\Gamma(x_m,y_l)\in\mathcal{T}_\theta$. We say that Γ has a *Fresh R(ed)* if $\Gamma=\Gamma_0$ or if the last move in obtaining Γ was a Gain. If the last move in obtaining Γ was a Swap, then we distinguish three cases. It follows from Lemma 1.8 and its complement that $(m,l)=(q_n,(\alpha+1)q_n+q_{n-1})$ for some n and $0\leqslant\alpha<a_n$, and that the

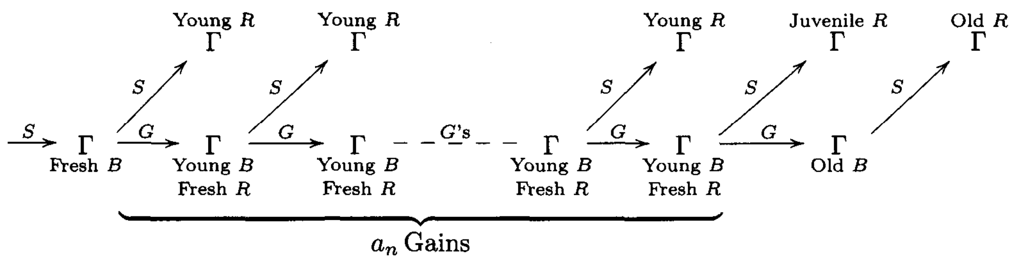

Fig. 16. A branch of $\mathcal{G}_\theta$ with Γ's up to old $B(lue)$ and old $R(ed)$ (G = Gain, S = Swap)

predecessor $\Gamma'(x_{m'}, y_{l'})$ of Γ has $m = \alpha q_n + q_{n-1}$. We say that Γ has a *Young* $R(ed)$ if Γ′ has a Fresh B or if Γ′ has a Young B and $0 < \alpha$. Moreover, we say that Γ has a *Juvenile* $R(ed)$ if Γ′ has a Young B, but $\alpha = 0$. If none of the above, then Γ is said to have an old $R(ed)$ (see also Figure 16).

The philosophy or motivation for the above terminology is that the O part of a curve $\Gamma \in \mathcal{T}_\theta$ is always newborn (and well controlled). A Fresh B or R is only one move away from the O stage (recall Figure 12). Moreover, a Fresh B belongs to a $\Gamma(x_{q_n}, y_l)$ for some n. An arc B or R is considered Young, if it is at most a_n moves away from the Fresh B stage. Finally the borderline Juvenile R is $a_n + 1$ moves away from the Fresh B stage. In Propositions 2.10 and 2.11 as well as in the proof of Theorem 2.2 we shall see how we can carry over the initial control of O to its close descendents.

Define $\mathcal{G}_\theta$ to be the subtree of $\mathcal{T}_\theta$ consisting of the set of Γ for which the descending path from Γ_0 to Γ does not pass any Γ′ with an old B. We note that trivially $\mathcal{G}_\theta^* \subseteq \mathcal{G}_\theta$. Moreover, we note that any $\Gamma \in \mathcal{G}_\theta$ has either a Fresh, a Young or a Juvenile R. Define $\mathcal{G}_{\theta,k} = \mathcal{G}_\theta \cap \mathcal{T}_{\theta,k}$.

THEOREM 2.2. *There exist universal constants $K_O, K_B, K_R > 0$ and $L_R > 0$, i.e. not depending on $\theta \in]0,1[- \mathbf{Q}$, such that*

$$\limsup_{\mathcal{T}_{\theta,k}} \frac{l_e(O)}{l_e(I)} \leqslant K_O, \tag{1}$$

$$\limsup_{\mathcal{G}_{\theta,k}} \frac{l_e(B)}{l_e(I)} \leqslant K_B, \tag{2}$$

$$\limsup_{\mathcal{G}_{\theta,k}} \frac{l_e(R)}{l_e(I)} \leqslant K_R, \tag{3}$$

$$\limsup_{\mathcal{G}_{\theta,k}} l_\lambda(R) \leqslant L_R. \tag{4}$$

We note that Proposition 1.5 is an immediate corollary of the above Theorem 2.2, because $\mathcal{G}_\theta^* \subset \mathcal{G}_\theta$. In the language of Sullivan [Su], the O, B and R of a $\Gamma \in \mathcal{G}_\theta$ are "beau", i.e. bounded (here in terms of I), and after a finite number of moves the bounds are universal. We shall express this by saying that the bounds are asymptotically universal. We remark also that it now easily follows that the bounds $L_{G,\theta}$ and $K_{G,\theta}$ are asymptotically universal. This chapter is devoted to proving the above theorem. We shall immediately prove the first statement.

Proof of Theorem 2.2 (1). Let $\Gamma(x_m, y_l) \in \mathcal{T}_{\theta,1}$ so that $\{m,l\} = \{q_n, \alpha q_n + q_{n-1}\}$ for some $n \geqslant 1$ and $0 \leqslant \alpha \leqslant a_n$, by Lemma 1.8. Thus $\lceil 0, q_n \rceil \subset I$ and $O \subset \lceil 0, q_{n-1} \rceil$ for some $n \geqslant 1$. As f_θ has a double critical point at 1 we obtain

$$\frac{l_e(O)}{l_e(I)} \leqslant \frac{l_e(\lceil 1, y_{q_{n-1}} \rceil)}{l_e(\lceil 1, x_{q_n} \rceil)} \quad \text{and} \quad \limsup_{\Gamma \in \mathcal{T}_{\theta,k}} \frac{l_e(O)}{l_e(I)} \leqslant \frac{1}{a},$$

where the latter comes from the Świątec a priori real bounds. □

Some hyperbolic geometry. Besides Terminology 2.3 the rest of this subsection are elementary facts about the hyperbolic geometry of some particular sets. The reader making a first reading of the paper is recommended to read the terminology and continue reading in the following subsection, "The complex bounds", thus using this subsection as a reference when needed.

Terminology 2.3. For $K \subsetneq \mathbf{S}^1$ an open subarc define

$$A_K = \overline{\mathbf{C}} - (\mathbf{S}^1 - K) \quad \text{and} \quad A_K^* = A_K - \{0, \infty\}.$$

Let λ_K denote both the hyperbolic metric and the coefficient function of the hyperbolic metric on A_K^*. Moreover, let $d_K(\cdot,\cdot)$ and $l_K(\cdot)$ denote the λ_K-distance and curve length functions respectively.

For $J \subsetneq \mathbf{S}^1$ an open subarc let $K = f_\theta(J)$. Define $W_J^* = f_\theta^{-1}(A_K^*) \subsetneq A_J^*$ so that the restriction $f_\theta: W_J^* \to A_K^*$ is a branched covering map of degree three, with 1 as only possible branch point. Let ϱ_J denote both the hyperbolic metric and the coefficient function of the hyperbolic metric on W_J^*.

Define $\mathbf{H}_\pm = \{z \mid \mathrm{Re}(z) \gtrless 0\}$. For $a \neq b \in \mathbf{C}$ let $[a,b]$ and $]a,b[$ denote the closed and open line segments from a to b respectively. For $\widetilde{K} \subsetneq i\mathbf{R}$ an open interval let $\mathbf{C}_{\widetilde{K}}$ denote the set $\mathbf{H}_- \cup \mathbf{H}_+ \cup \widetilde{K}$ and let $\delta_{\widetilde{K}}$ denote the hyperbolic metric on $\mathbf{C}_{\widetilde{K}}$. We shall often refer to $\mathbf{C}_{\widetilde{K}}$ as a doubly slit plane (with gab $\widetilde{K}$) when $\widetilde{K}$ is relatively compact in $i\mathbf{R}$.

LEMMA 2.4. *Suppose that $V \subset U \subset \mathbf{C}$ are hyperbolic subsets of $\mathbf{C}$, i.e. each carries a hyperbolic metric. Let λ_V and λ_U denote the coefficient functions of the respective hyperbolic metrics and let $d_U(\cdot,\cdot)$ denote the hyperbolic distance in U. Then*

$$\tanh\left(\tfrac{1}{2} d_U(z, \partial V)\right) \leqslant \frac{\lambda_U(z)}{\lambda_V(z)} < 1 \quad \forall z \in V.$$

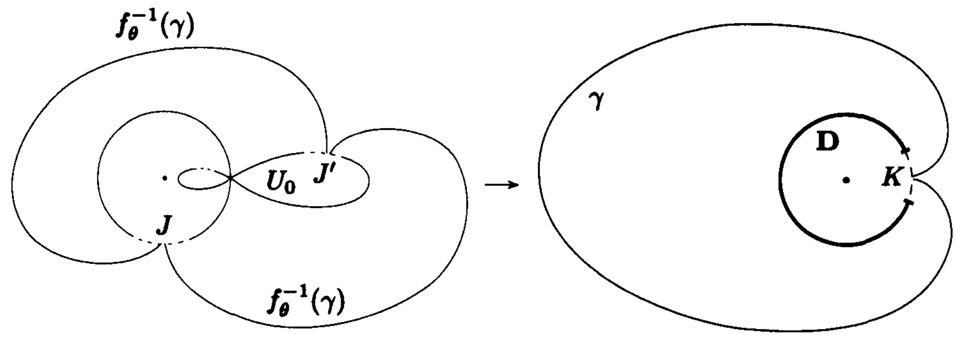

Fig. 17. The arc γ_K and its two lifts to f_θ outside $\mathbf{D}$

Proof. Let $z\in V$ be arbitrary. Lifting to a universal cover of U if necessary, we can suppose that $U=\mathbf{D}$ and $z=0$. Then the largest Euclidean disc centered at 0 and entirely contained in V has Euclidean radius $\tanh\left(\frac{1}{2}d_U(z,\partial V)\right)$, so the lemma follows. □

Given $K\subsetneq\mathbf{S}^1$ an open subarc let $e^{i\eta}$ denote the midpoint of K. Let $\gamma_K:[0,1]\to A_K^*$ denote the arc $\gamma_K(t)=\exp(i\pi(1-e^{i\pi t})+i\eta)$. Then γ_K is a closed arc beginning and ending at $e^{i\eta}\in K$ and with index 1 around 0. Let $[\gamma]_K$ be the homotopy class of γ_K in A_K^* through curves γ' with endpoints in K. See Figure 17. Define

$$E_K:=\inf_{\gamma'\in[\gamma]_K} l_K(\gamma') \quad\text{and}\quad F_K:=d_{\sigma_K}(K,\infty),$$

where σ_K denotes the hyperbolic metric on A_K (see also Figure 17).

There exist (continuous) decreasing functions E and F with $E_K=E(l_E(K))$ and $F_K=F(l_e(K))$, because multiplication by a constant η of norm 1 is a hyperbolic isometry between A_K, A_K^* and $A_{\eta K}$, $A_{\eta K}^*$ respectively. Our sole interest here is however that $E(l)\to\infty$ as $l\to 0$. The latter can be seen as follows: An elementary calculation shows that $F(l)=\log\left(\cot\left(\frac{1}{8}l\right)\right)$. Moreover, $E(l)>2F(l)$ for all $0<l\leqslant 2\pi$ as $A_K^*\subset A_K$. We shall use the function E through the following lemma.

LEMMA 2.5. *Let $K,J\subset\mathbf{S}^1$ and $J'\subset\partial U_0$ be open arcs such that f_θ maps J and J' diffeomorphically to K. Then*

$$d_{\varrho_J}(\mathbf{S}^1,J')=d_{\varrho_J}(J,J')=E(l_e(K)).$$

In particular, the distance $d_{\varrho_J}(\mathbf{S}^1,J')$ depends only on $l_e(K)$ and tends to ∞ as $l_e(K)\to 0$.

Proof. The first equality is because $J=\mathbf{S}^1\cap W_J^*$. The restriction $f_\theta: W_J^*\to A_K^*$ is a covering map and an isometry with respect to the hyperbolic metrics ϱ_J and λ_K, because $1\notin J$. Moreover, any curve in $[\gamma]_K$ has two lifts to f_θ joining J and J' in W_J^*, and any simple curve which joins J and J' in W_J^* maps by f_θ to a curve in the equivalence class $[\gamma]_K$ (see also Figure 17). □

LEMMA 2.6. *For $K \subsetneq \mathbf{S}^1$ an open arc and $t>0$ define*

$$\Omega_{K,t} = \{\gamma \subset A_K^* \mid \gamma \text{ a curve with } \gamma \cap K \neq \varnothing \text{ and } l_K(\gamma) \leqslant t\}.$$

The number $\sup\{l_e(\gamma) \mid \gamma \in \Omega_{K,t}\}$ depends only on $l = l_e(K)$. We denote it $S_{l,t}$. It satisfies

$$\lim_{l \to 0} \frac{S_{l,t}}{l} = \tfrac{1}{2}\sinh t \tag{1}$$

Proof. We shall first prove an analogous statement for doubly slit planes.

SUBLEMMA. *Let $\widetilde{K} \subsetneq i\mathbf{R}$ be an open relatively compact interval and let $\delta_{\widetilde{K}}$ denote the hyperbolic metric on $\mathbf{C}_{\widetilde{K}}$, the doubly slit plane with gab $\widetilde{K}$. Any curve $\widetilde{\gamma} \in \mathbf{C}_{\widetilde{K}}$ with $\widetilde{\gamma} \cap \widetilde{K} \neq \varnothing$ satisfies*

$$l_e(\widetilde{\gamma}) \leqslant \tfrac{1}{2} \cdot l_e(\widetilde{K}) \cdot \sinh(l_{\delta_{\widetilde{K}}}(\widetilde{\gamma})) \tag{2}$$

with equality if and only if $\widetilde{\gamma}$ is a horizontal line segment emanating from the midpoint of $\widetilde{K}$.

Proof of the sublemma. It suffices to consider $\widetilde{K}_2 =]-i, i[$, as the affine map $z \mapsto s + rz$ is both an Euclidean congruence and a hyperbolic isometry between $\mathbf{C}_2 = \mathbf{C}_{\widetilde{K}_2}$ and $\mathbf{C}_{s+r\widetilde{K}_2}$ for all $r>0$ and $s \in i\mathbf{R}$. We let δ_2 denote the hyperbolic metric on $\mathbf{C}_2$.

Let $\pi: \mathbf{H}_+ \to \mathbf{C}_2$ be the univalent map $\pi(z) = \frac{1}{2}(z - 1/z)$. Then $\pi([1, e^t]) = [0, \sinh t]$, which shows that the horizontal linesegment $\widetilde{\gamma}_t = [0, \sinh t]$ has δ_2-length t. This proves the optimality of (2). We shall prove that for any $x \in \mathbf{R}$,

$$\delta_2(x) \leqslant \delta_2(z) \quad \text{for all } z \text{ with } d_{\delta_2}(z, \widetilde{K}_2) \leqslant d_{\delta_2}(x, \widetilde{K}_2), \tag{3}$$

with equality if and only if $z = \pm x$. Let us first prove that $|x_1| < |x_2|$ implies $\delta_2(x_1) > \delta_2(x_2)$. This follows from the computation

$$\delta_2(\pi(s)) = \frac{1}{s|\pi'(s)|} = \frac{2}{|s + 1/s|} \quad \forall s > 0,$$

because π is an increasing homeomorphism of $\mathbf{R}_+$ onto $\mathbf{R}$ mapping 1 to 0, and $1/x$ is the coefficient of the hyperbolic metric on $\mathbf{H}_+$ at the point $x + iy$. Next we consider the automorphisms of $\mathbf{C}_2$,

$$H_r(z) = \frac{z + ir}{1 - irz}, \quad -1 < r < 1.$$

Each H_r, being a Möbius transformation, preserves the circles through its fixed points $\pm i$. We deduce that the arcs of circle between $\pm i$ are lines of equidistance to $\widetilde{K}_2$, which is itself such an arc. Let $x \in \mathbf{R}$ be arbitrary. We compute

$$\delta_2(x) = \delta_2(H_r(x)) \cdot |H_r'(x)| = \delta_2(H_r(x)) \cdot \frac{1 - r^2}{1 + r^2 x^2} < \delta_2(H_r(x)).$$

The computation shows that δ_2, when restricted to the arc of circle between $\pm i$ through x, attains its infimum only at x. This completes the proof of (3).

We shall prove the sublemma only for piecewise differentiable curves and leave the generalization to rectifiable curves to the reader, as we only need the piecewise differentiable case. Let $\tilde{\gamma}\subset \mathbf{C}_2$ be a piecewise differentiable curve with $\tilde{\gamma}\cap\widetilde{K}_2\neq\varnothing$ and $l_{\delta_2}(\tilde{\gamma})=t\leqslant\infty$. We can suppose that $t<\infty$, as (2) is void if $t=\infty$. It suffices to consider curves with one end point on $\widetilde{K}_2$, because $\sinh t_1+\sinh t_2<\sinh(t_1+t_2)$. Reparametrizing if necessary we can suppose that $\tilde{\gamma}\colon[0,t]\to\mathbf{C}_2$ is parametrized by hyperbolic curve length and starts on $\widetilde{K}_2$. Then $|\tilde{\gamma}'(s)|=1/\delta_2(\tilde{\gamma}(s))$ for all $0\leqslant s\leqslant t$ and

$$l_e(\tilde{\gamma})=\int_0^t|\tilde{\gamma}'(s)|\,ds=\int_0^t\frac{1}{\delta_2(\tilde{\gamma}(s))}\,ds\leqslant\int_0^t\frac{1}{\delta_2(s)}\,ds=l_e(\tilde{\gamma}_0)=\sinh t,$$

where the inequality comes from (3), and equality applies if and only if $\gamma=\tilde{\gamma}_t$. This proves the sublemma. □

To justify the definition of $S_{l,t}$ we note that the rigid rotations $z\mapsto\lambda z$, with $|\lambda|=1$, are both Euclidean isometries and hyperbolic isometries between A_K^* and $A_{\lambda K}^*$. We shall hence only consider the arcs $K_l\subsetneq\mathbf{S}^1$ of Euclidean length $0<l\leqslant 2\pi$ and with mid point 1. The lemma then follows from the sublemma by considering the Möbius transformation $H(z)=(1+z)/(1-z)$, which maps 1 to 0 and each $A_{K_l}^*$ univalently into a doubly slit plane. □

The complex bounds.

Definition 2.7. Let $\theta\in]0,1[-\mathbf{Q}$. Define $Q_0=\lceil 1,y_{q_1}\rceil$, $Q_1=\partial U_0-\mathring{Q}_0$ and, for $n\geqslant 2$, $Q_n=\lceil 1,y_{q_n}\rceil$. Moreover, define $K_m=\rceil x_{-q_m},x_{-q_{m+1}}\lceil\subset\mathbf{S}^1$ for $m\geqslant 1$. (For typographical reasons we shall not add an index θ to Q_n and K_m.)

PROPOSITION 2.8. *For each $\theta\in]0,1[-\mathbf{Q}$ there exist positive constants $L_{d,\theta}$, $d=1,2,3,\ldots$, such that for all n,*

$$l_{K_{n+d}}(Q_n)\leqslant L_{d,\theta}.$$

COMPLEMENT TO PROPOSITION 2.8. *There exist (explicit) universal constants $L,M>0$, i.e. not depending on θ, such that*

$$\limsup_{n\to\infty} l_{K_{n+d}}(Q_n)\leqslant L+(d+2)M. \tag{1}$$

For $d\geqslant 1$ we define $L_d=L+(d+2)M$.

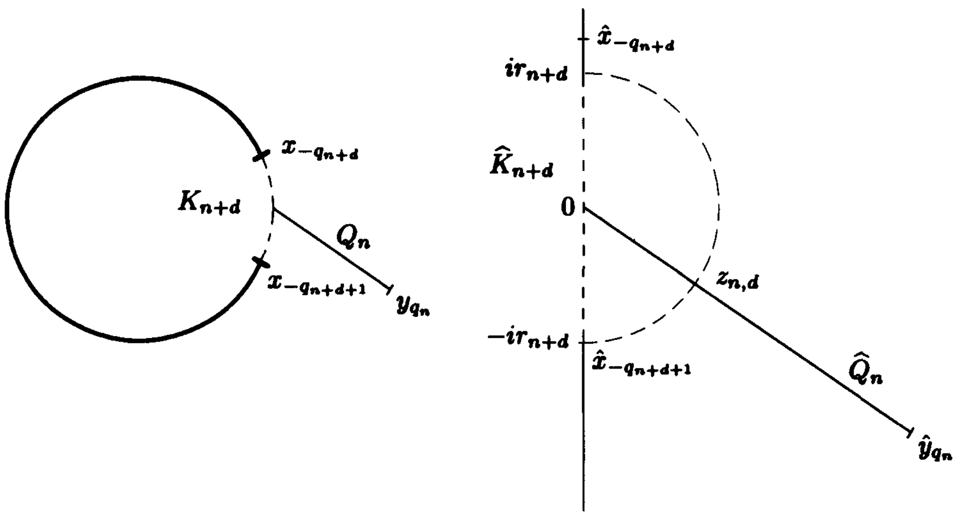

Fig. 18. The (bounded) geometry of Q_n relative to K_{n+d}

Sketch of proof. When n tends to ∞, the set $A^*_{K_{n+d}}$ looks more and more like a doubly slit plane with gab K_{n+d}. Moreover, the arc Q_n looks more and more like a line segment sticking out of K_{n+d} making an angle $\pm 30°$ with the horizontal, because f_θ has a double critical point at 1 and preserves the unit circle. Finally the Świątec a priori real bounds imply that the described configuration has bounded geometry independent of θ. The proof we shall give renders the above sketch into a proof, essentially by changing coordinates so that we get actual slit planes and actual line segments. See Figure 18.

Proof. Let $\tilde{v}$ be a logarithm of v and define $F_{\tilde{v}}(z) = \tilde{v} - \pi z^3$. Define a univalent parameter $\Phi: \mathbf{D} \to V \subset \mathbf{C}$ with $\Phi(0) = 1$ and $\Phi'(0) > 0$ such that the following diagram is commutative:

$$\begin{array}{ccc} \mathbf{D} & \xrightarrow{F_{\tilde{v}}} & \mathbf{D}_\pi(\tilde{v}) \\ {\scriptstyle\Phi}\downarrow & & \downarrow{\scriptstyle\exp} \\ V & \xrightarrow{f_\theta} & V'. \end{array} \tag{2}$$

Note that exp is univalent on $\mathbf{D}_\pi(\tilde{v})$ and let $V' = \exp(\mathbf{D}_\pi(\tilde{v}))$. Define V to be the preimage of V' under f_θ. Then V is simply-connected and the restriction $f_\theta: V \to V'$ is a branched triple cover, branched only above $\tilde{v}$. Define $\Phi: \mathbf{D} \to V$ as the Riemann map with $\Phi(0) = 1$ and $\Phi'(0) > 0$. It is easy to verify that Φ satisfies (2).

Define $\hat{x}_j = \Phi^{-1}(x_j)$ for $j \in \mathbf{Z} - \{-1\}$ and $\hat{y}_j = \Phi^{-1}(y_j)$ for $j \geqslant 1$. The quotients of $\hat{x}_j$ over $\hat{y}_j$ for $j \geqslant 1$ are all some third root of unity, as they have the same image under $F_{\tilde{v}}$.

For all n and d we have $l_{K_{n+d}}(Q_n) < \infty$ and thus it suffices to prove the complement of the lemma. Fix $\theta\in]0,1[-\mathbf{Q}$ and $d\geqslant 1$. For $n\geqslant 2$ let

$$\widehat{Q}_n=\Phi^{-1}(Q_n)=[0,\hat{y}_{q_n}] \quad\text{and}\quad \widehat{K}_{n+d}=\Phi^{-1}(K_{n+d})=]\hat{x}_{-q_{n+d}},\hat{x}_{-q_{n+d+1}}[.$$

We shall first find explicit universal constants $L,M>0$ such that

$$\limsup_{n\to\infty} l_{\delta_{\widehat{K}_{n+d}}}(\widehat{Q}_n)\leqslant L+(d+2)M, \tag{3}$$

where $\delta_{\widehat{K}_{n+d}}$ is the hyperbolic metric on the doubly slit plane with gab $\widehat{K}_{n+d}$. Second and finally we shall prove that (3) implies (1) with the same constants.

Let $r_{n+d}=\min\{|\hat{x}_{-q_{n+d}}|,|\hat{x}_{-q_{n+d+1}}|\}$. Let $z_{n,d}\in[0,\hat{y}_{q_n}]$ be given by $|z_{n,d}|=r_{n+d}$. The hyperbolic length L of the segment $[0,z_{n,d}]$ measured in the doubly slit plane with gab $]-ir_{n+d},ir_{n+d}[$ does not depend on n or d, because the geometry is fixed so that the only thing which changes is the scale. As $]-ir_{n+d},ir_{n+d}[\subsetneq\widehat{K}_{n+d}$ we also have $l_{\delta_{\widehat{K}_{n+d}}}([0,z_{n,d}])<L$. Moreover, $l_{\delta_{\widehat{K}_{n+d}}}([z_{n,d},\hat{y}_{q_n}])\leqslant l_{\mathbf{H}_+}([z_{n,d},\hat{y}_{q_n}])$ because $\mathbf{H}_+\subset\mathbf{C}_{\widehat{K}_{n+d}}$. Combining the Świątec a priori real bounds with the univalence of Φ we obtain

$$\liminf\frac{|z_{n,d}|}{|\hat{y}_{q_n}|}=\liminf\frac{r_{n+d}}{|\hat{y}_{q_n}|}\geqslant\liminf\frac{\min\{|\hat{x}_{q_{n+d+1}}|,|\hat{x}_{q_{n+d+2}}|\}}{|\hat{y}_{q_n}|}\geqslant a^{d+2}. \tag{4}$$

Hence we obtain by direct computation $\limsup l_{\mathbf{H}_+}([z_{n,d},\hat{y}_{q_n}])\leqslant(2/\sqrt{3})(d+2)\log 1/a$. This proves (3) with $M=(2/\sqrt{3})\log 1/a$. Moreover, the bound does not depend on θ, because the Świątec a priori real bounds which imply (4) do not depend on θ, and L is universal.

It follows from (3) that the hyperbolic distance in the slit planes $d_{\delta_{\widehat{K}_{n+d}}}(\widehat{Q}_n,\partial\mathbf{D})$ diverges to ∞ as $n\to\infty$ and d is fixed. It then follows from Lemma 2.4 that we may replace $\delta_{\widehat{K}_{n+d}}$ in (3) with the hyperbolic metrics on the slit disc $\mathbf{D}_{\widehat{K}_{n+d}}=\mathbf{D}\cap\mathbf{C}_{\widehat{K}_{n+d}}$. Finally this remark proves that (3) implies (1) and thus the complement, because Φ maps $\mathbf{D}_{\widehat{K}_{n+d}}$ univalently into $A^*_{K_{n+d}}$ and in particular is a hyperbolic contraction. □

LEMMA 2.9. *Suppose that $N\geqslant 1$ and $J,K\subset\mathbf{S}^1$ are any pair of subarcs such that the restriction $f_\theta^N: J\to K$ is a diffeomorphism. Then any univalent branch $G: U\to W_J^*$ of f_θ^{-N} defined on a subset $U\subset A_K^*$ satisfies*

$$\|D_zG\|_{\lambda_J,\lambda_K}<\|D_zG\|_{\varrho_J,\lambda_K}=\frac{\varrho_J(G(z))}{\lambda_K(z)}|G'(z)|\leqslant 1\quad\forall z\in U,$$

with equality if and only if $N=1$. That is, G is infinitesimally contracting or possibly a local isometry with respect to the involved metrics.

Proof. We have $\varrho_J(z)>\lambda_J(z)$ for all $z\in W_J^*$. This proves the first inequality. Suppose first that $N=1$. Then the restriction $f_\theta: W_J^*\to A_K^*$ is a covering map of degree three, because the assumption that f_θ maps J diffeomorphically to K implies that $v\notin K$. In particular, the restriction is a local isometry for the respective hyperbolic metrics ϱ_J and λ_K. This proves the case $N=1$. We shall prove the general case by induction. So suppose that (1) holds for $N-1\geqslant 1$, that f_θ^N maps J diffeomorphically to K and that $G: U\to W_J^*$ is a univalent branch of f_θ^{-N} defined on a subset $U\subset A_K^*$. Define $J_1=K_1=f_\theta(J)$, $G_1=f_\theta\circ G$ and $U_1=G_1(U)$. Then $G_1: U\to U_1$ is a univalent branch of f_θ^{N-1} which by the induction hypotheses satisfies

$$\|D_zG_1\|_{\lambda_{J_1},\lambda_K}<\|D_zG_1\|_{\varrho_{J_1},\lambda_K}\leqslant 1 \quad \forall z\in U.$$

Let $G_2: U_1\to W_J^*$ be the inverse branch of f_θ on U_1 with $G=G_2\circ G_1$. Then we proved above that

$$\|D_zG_2\|_{\varrho_J,\lambda_{K_1}}=1 \quad \forall z\in U_1.$$

Since $J_1=K_1$ we can conclude by the chain rule. □

Recall Terminology 2.1 on Fresh and Young B.

PROPOSITION 2.10. *Suppose that $\Gamma(x_{q_n},y_l)\in\mathcal{T}_{\theta,1}$ has a Fresh B. Let J be the open arc $J=\rceil 1,x_{q_n-q_{n-1}}\lceil\subset\mathbf{S}^1$. Then*

$$l_J(B)\leqslant l_{K_{n-1}}(Q_{n-2})\leqslant L_{1,\theta},$$

where K_{n-1} and Q_{n-2} are as in Definition 2.7 and $L_{1,\theta}$ is as in Proposition 2.8.

Proof. The arc J is mapped diffeomorphically onto $\rceil x_{-q_n},x_{-q_{n-1}}\lceil=K_{n-1}$ by $f_\theta^{q_n}$. Let $g':\Gamma'\to\Gamma$ be the final Swap in obtaining Γ (Γ has a Fresh B). Then $f^{q_n}\circ g'=\mathrm{Id}$ and

$$\Gamma'(x_{m'},y_{\alpha' q_{n-1}+q_{n-2}})\xrightarrow[\text{Swap}]{g'}\Gamma(x_{q_n},y_l),$$

$$(Q_{n-2}\supseteq)\,O'\xrightarrow[\text{Swap}]{g'}B,$$

for some $0\leqslant\alpha'\leqslant a_{n-1}$, by Lemma 1.9. Moreover, g' can be defined univalently in a neighbourhood $U\subset A_K^*$ of O'. Thus by Lemma 2.9 and Proposition 2.8,

$$l_J(B)\leqslant l_{K_{n-1}}(O')\leqslant l_{K_{n-1}}(Q_{n-2})\leqslant L_{1,\theta}. \qquad \square$$

PROPOSITION 2.11. *Suppose that $\Gamma(x_{\alpha q_n+q_{n-1}},y_l)\in\mathcal{T}_\theta$, $0<\alpha\leqslant a_n$, has a Young B. Let J be the open subarc $J=\rceil 1,x_{(\alpha-1)q_n+q_{n-1}}\lceil\subset\mathbf{S}^1$. Then*

$$l_J(B)\leqslant l_{K_n}(Q_{n-3})\leqslant L_{3,\theta},$$

where K_n and Q_{n-3} are as in Definition 2.7 and $L_{3,\theta}$ is as in Proposition 2.8.

Proof. The open arc $J\subset\mathbf{S}^1$ is mapped diffeomorphically onto

$$K=\rceil x_{-(\alpha q_n+q_{n-1})},x_{-q_n}\lceil$$

by $f_\theta^{(\alpha q_n+q_{n-1})}$. Let $G:\Gamma''\to\Gamma$ be the long composition of the final α consecutive Gains in constructing Γ and let $g':\Gamma'\to\Gamma''$ be the final Swap in constructing Γ'' (Γ has a Young B). We assume here that $\Gamma''\neq\Gamma_0$. In case $\Gamma''=\Gamma_0$, one should replace g' by the branch of f_θ^{-1} mapping Q_0 to B_0 and Q_{n-3} by Q_0. The details are left to the reader. Then $f^{q_{n-1}}\circ g'=\mathrm{Id}$ and

$$\Gamma'(x_{m'},y_{\alpha' q_{n-2}+q_{n-3}})\xrightarrow[\text{Swap}]{g'}\Gamma''(x_{q_{n-1}},y_{l''})\xrightarrow[\alpha\text{ Gains}]{G}\Gamma(x_{\alpha q_n+q_{n-1}},y_l),$$

$$(Q_{n-3}\supseteq)\,O'\xrightarrow[\text{Swap}]{g'}B''\xrightarrow[\alpha\text{ Gains}]{G}B,$$

where $0\leqslant\alpha'\leqslant a_{n-2}$, by Lemma 1.9. Moreover, $f_\theta^{(\alpha q_n+q_{n-1})}\circ G\circ g'=\mathrm{Id}$, $G\circ g'$ can be defined univalently in a neighbourhood $U\subset A_K^*$ of O' and $K\supseteq K_n=\,]x_{-q_{n+1}},x_{-q_n}[$. Thus

$$l_J(B)\leqslant l_K(O')\leqslant l_{K_n}(Q_{n-3})\leqslant L_{3,\theta},$$

where the first inequality comes from Lemma 2.9 and the last two inequalities come from Proposition 2.8. □

Controlling Fresh and Young B(lue).

Proof of Theorem 2.2 (2). Let $\Gamma(I,B,G,R,O)\in\mathcal{G}_\theta$ be arbitrary. The Jordan curve Γ has either a Fresh or Young B by definition of $\mathcal{G}_\theta$. In particular, Γ satisfies the hypotheses of either Lemma 2.10 or Lemma 2.11. We shall treat the two cases separately.

Suppose that $\Gamma=\Gamma(x_{q_n},y_l)$ has a Fresh B so that $l_{J_n}(B)\leqslant l_{K_{n-1}}(Q_{n-2})$, where $J_n=\,]1,x_{q_n-q_{n-1}}[\,\subset\mathbf{S}^1$. As $l_e(J_n)\to0$, when $n\to\infty$, and $\limsup l_{K_{n-1}}(Q_{n-2})\leqslant L_1$ we obtain from Lemma 2.6 that

$$\limsup\frac{l_e(B)}{l_e(J_n)}\leqslant\tfrac12\sinh(L_1),\tag{1}$$

where sup is over curves $\Gamma(x_{q_n},y_l)$ with a Fresh B and lim is for $n\to\infty$. Moreover,

$$[1,x_{q_n}]=I\subset\bar J_n=[1,x_{q_n-q_{n-1}}]\subset[1,x_{q_{n-2}}]$$

so that

$$\limsup\frac{l_e(J_n)}{l_e(I)}\leqslant a^{-2}\tag{2}$$

by the Świątec a priori real bounds, and where lim sup has the same sense as above. Combining (1) and (2) we obtain

$$\limsup_{\text{Fresh}}\frac{l_e(B)}{l_e(I)}\leqslant\frac{1}{2a^2}\sinh(L_1).\tag{3}$$

This takes care of the case of Γ with a Fresh B. For Γ with a Young B one obtains

$$\limsup_{\text{Young}} \frac{l_e(B)}{l_e(I)} \leqslant \frac{1}{2a^2} \sinh(L_3) \tag{4}$$

by copying the above arguments. The details are left to the reader as an exercise. This completes the proof of Proposition 2.2 (2). □

Uniform bounds for Fresh, Young and Juvenile R.

Proof of Theorem 2.2 (4). We shall prove the following more precise estimates:

$$\limsup_{\text{Fresh}} l_\lambda(R) \leqslant L_1, \tag{1}$$

$$\limsup_{\text{Young}} l_\lambda(R) \leqslant L_3, \tag{2}$$

$$\limsup_{\text{Juvenile}} l_\lambda(R) \leqslant L_5. \tag{3}$$

Here sup is over $\Gamma \in \mathcal{T}_{\theta,k}$ with respectively a Fresh, Young or Juvenile R, and lim is for k tending to ∞.

Let $\Gamma(x_m, y_l) = \Gamma(I, B, G, R, O) \in \mathcal{T}_\theta$. We shall consider separately the three different cases (see Terminology 2.1 for the definitions of Fresh, Young and Juvenile). The relevant final moves in the three cases are for Fresh R,

$$\begin{gathered} \Gamma'(x_{\alpha q_n + q_{n-1}}, y_{l'}) \xrightarrow[\text{Gain}]{g'} \Gamma(x_{(\alpha+1)q_n+q_{n-1}}, y_{q_n}), \\ (Q_{n-2} \supseteq)\, O' \xrightarrow[\text{Gain}]{g'} R, \end{gathered} \tag{1'}$$

where $0 \leqslant \alpha < a_n$ and $f_\theta^{q_n} \circ g' = \mathrm{Id}$, so that $l' = \beta q_{n-1} + q_{n-2}$ with $0 \leqslant \beta \leqslant a_{n-1}$, by Lemma 1.9.

For Young R we have $(m, l) = (q_n, (\alpha+1)q_n + q_{n-1})$ for some n and $0 \leqslant \alpha < a_n$, and

$$\begin{gathered} \Gamma'(x_{m'}, y_{l'}) \xrightarrow[\text{Swap}]{g'} \Gamma''(x_{q_{n-1}}, y_{l''}) \xrightarrow[\alpha\ \text{Gains}]{G} \Gamma'''(x_{\alpha q_n + q_{n-1}}, y_{l'''}) \xrightarrow[\text{Swap}]{g'''} \Gamma(x_m, y_l), \\ (Q_{n-3} \supseteq)\, O' \xrightarrow[\text{Swap}]{g'} B'' \xrightarrow[\alpha\ \text{Gains}]{G} B''' \xrightarrow[\text{Swap}]{g'''} R, \end{gathered} \tag{2'}$$

where $f_\theta^{q_{n-1}} \circ g' = \mathrm{Id}$, so that $l' = \beta q_{n-2} + q_{n-3}$ for some $0 \leqslant \beta \leqslant a_{n-2}$. Here the indices of the marked points $(x_{m'}, y_{l'})$ of Γ' follow from Lemma 1.9 and from Lemma 1.8 with its complement for the others.

Finally for Juvenile R we have $(m, l) = (q_n, q_n + q_{n-1})$ for some n, and

$$\begin{gathered} \Gamma'(x_{m'}, y_{l'}) \xrightarrow[\text{Swap}]{g'} \Gamma''(x_{q_{n-3}}, y_{l''}) \xrightarrow[a_{n-2}\ \text{Gains}]{G} \Gamma'''(x_{q_{n-1}}, y_{q_{n-2}}) \xrightarrow[\text{Swap}]{g'''} \Gamma(x_m, y_l), \\ (Q_{n-5} \supseteq)\, O' \xrightarrow[\text{Swap}]{g'} B'' \xrightarrow[\alpha\ \text{Gains}]{G} B''' \xrightarrow[\text{Swap}]{g'''} R, \end{gathered} \tag{3'}$$

where $f_\theta^{q_{n-3}}\circ g'=\mathrm{Id}$, so that $l'=\beta q_{n-4}+q_{n-5}$ for some $0\leqslant\beta\leqslant a_{n-4}$. As above the indices of the marked points for Γ' follow from Lemma 1.9 and from Lemma 1.8 with its complement for the others.

Passing from (i′) to (i) is essentially the same for $i=1,2$ and 3. We shall hence only go through the details for $i=3$, the Juvenile (and worst) case. Let $J_n=\rceil 1, x_{q_{n-1}}\lceil$ and $J_n'=\rceil 1, y_{q_{n-1}}\lceil$. Then J_n is mapped onto the arc $K_n'=\rceil x_{-(q_n+q_{n-1})}, x_{-q_n}\lceil\supseteq K_n$ diffeomorphically by $f_\theta^{(q_n+q_{n-1})}$ and $f_\theta^{(q_n+q_{n-1})}\circ(g'''\circ G\circ g')=\mathrm{Id}$. Let Z_n be given by $f_\theta(J_n)=f_\theta(J_n')=Z_n$, and let $\varrho_n=\varrho_{J_n}$ denote the hyperbolic metric on $W_{J_n}^*$. Then we get from Lemma 2.9,

$$l_{\varrho_n}(R)=l_{\varrho_n}(g'''\circ G\circ g'(O'))\leqslant l_{K_n'}(O')\leqslant l_{K_n}(Q_{n-5})\leqslant L_{5,\theta}. \tag{4}$$

Let η_n denote the hyperbolic metric on $W_{J_n}^*-\overline{\mathbf{D}}$. Then

$$\begin{aligned} l_\lambda(R)<l_{\eta_n}(R)&\leqslant l_{\varrho_n}(R)\cdot\coth\left(\tfrac12 d_{\varrho_n}(R,J_n)\right)\\ &\leqslant l_{K_n}(Q_{n-5})\cdot\coth\left(\tfrac12 d_{\varrho_n}(R,J_n)\right)\end{aligned} \tag{5}$$

by Lemma 2.4. Moreover, from Lemma 2.5 and (4) we find that

$$d_{\varrho_n}(R,J_n)\geqslant E(l_e(Z_n))-L_{\theta,5}\underset{n\to\infty}{\longrightarrow}\infty. \tag{6}$$

Finally (3) follows from (5) combined with the complement of Proposition 2.8 and (6). The reader is encouraged to fill in the details for (1) and (2) to complete the proof of Theorem 2.2 (4). □

Proof of Theorem 2.2 (3). Combine Theorem 2.2, (4) and (1) with the fact that the hyperbolic metric λ on $\mathbf{C}-\overline{\mathbf{D}}$ is asymptotic to $1/(|z|-1)$ as $|z|\to 1$. □

As above the careful reader will have noticed that we only used that Γ had a Fresh, a Young or a Juvenile R in the above proof. Hence we might replace $\mathcal{G}_\theta$ in Theorem 2.2, (3) and (4) by the set of $\Gamma\in\mathcal{T}_\theta$ with a Fresh, a Young or a Juvenile R.

3. Local connectivity of J_θ

We introduce the notation $t_0=1$ and $t_n=q_n+q_{n-1}$ for $n\geqslant 1$. Moreover, we shall use the extended notation $\Gamma(x_m, y_l, x_{m,k})$ for the Jordan curve $\Gamma(I,B,G,R,O)\in\mathcal{T}_\theta$, where x_m and y_l are the "free" endpoints of I and O as before and $x_{m,k}$ is the "free" endpoint of B.

Estimating the sizes of limbs. We let $\mathcal{F}_\theta=\{\Gamma_k(x_{m_k}, y_{l_k})\}_{k\geqslant 0}\subset\mathcal{G}_\theta^*$ be the descending sequence of Jordan curves defined in Definition 1.12.

PROPOSITION AND DEFINITION 3.1. *For each $\theta\in]0,1[-\mathbf{Q}$ there exists a sequence of Jordan curves $\{\Sigma_n(x_{q_n},y_{t_n},x_{q_n,t_{n-1}})\}_{n\geqslant 1}\subset\mathcal{G}_\theta^*$ such that each Σ_n either belongs to $\mathcal{F}_\theta$ or is obtained from some $\Gamma_k\in\mathcal{F}_\theta$ by one, two or three Swaps.*

Proof. We are looking for curves of the form $\Sigma_n=\Sigma_n(x_{q_n},y_{t_n},x_{q_n,t_{n-1}})\in\mathcal{G}_\theta^*$. Such curves are obtained by two consecutive Swaps of a curve of the form $\Gamma'(x_{q_{n-2},y_{l'}})\in\mathcal{G}_\theta^*$, i.e.

$$\Gamma'(x_{q_{n-2}},y_{l'}) \xrightarrow[\text{Swap}]{g'} \Gamma''(x_{q_{n-1}},y_{t_{n-1}}) \xrightarrow[\text{Swap}]{g''} \Gamma(x_{q_n},y_{t_n},x_{q_n,t_{n-1}}).$$

Recall that each $\Gamma_{k+1}\in\mathcal{F}_\theta$ is obtained from Γ_k by a move $g_k:\Gamma_k\to\Gamma_{k+1}$. It follows that each number q_n, $n\geqslant 1$, appears at most once in the sequence $\{m_k\}_{k\geqslant 0}$.

Define

$$\Sigma_1(x_{q_1},y_{q_1+q_0},x_{q_1,t_0})=\Gamma_0(x_{q_1},y_{q_1}) \quad\text{and}\quad \Sigma_2(x_{q_2},y_{t_2},x_{q_2,t_1})=\Gamma_1(x_{q_2},y_{q_2+q_1}).$$

For $n\geqslant 3$ we define $\Sigma_n=\Sigma_n(x_{q_n},y_{t_n},x_{q_n,t_{n-1}})$ as follows: If there exists $k\geqslant 0$ with $\Gamma_k=\Gamma_k(x_{q_n},y_{t_n},x_{q_n,t_{n-1}})$, then we define $\Sigma_n=\Gamma_k$. If not, we look for a $k\geqslant 0$ with $\Gamma_k=\Gamma_k(x_{q_{n-1}},y_{t_{n-1}})$. If such a k exists define Σ_n to be the Swap of Γ_k. If such a k does not exist either, we look for a $k\geqslant 0$ with $\Gamma_k=\Gamma_k(x_{q_{n-2}},y_l)$ for some l. If such a k exists we define Σ_n as the curve obtained by two consecutive Swaps of Γ_k. Finally if such a k does not exist either, then there exists a $k\geqslant 0$ with $\Gamma_k=\Gamma_k(x_{q_{n-3}},y_l)$ for some l. Because the sequence $\{m_k\}_{k\geqslant 0}$ can jump over a q_n only if there exists $k\geqslant 0$ with $m_k=q_{n-1}$, the move g_k is a Gain and $a_n=1$, so that $m_{k+1}=q_{n+1}$. In the final case we let Σ_n be the curve obtained by three consecutive Swaps from Γ_k. The reader shall easily verify that with the above definition Σ_3 either equals Γ_2 or is obtained by a Swap from Γ_1, so that the final case in the definition occurs no earlier than for $n=4$. This completes the definition and the proposition. □

For each $n\geqslant 1$ and $0\leqslant j<q_{n+1}$ let

$$J_{n,j}=]x_{-q_n+j},x_{-q_{n+1}+q_n+j}[\subset\mathbf{S}^1,$$
$$J'_{n,j}=]y_{-q_n+j},y_{-q_{n+1}+q_n+j}[\subset\partial U_0,$$

so that $f_\theta(J_{n,j})=f_\theta(J'_{n,j})=J_{n,j-1}$ for $j>0$. Note that the arcs $J_{n,j}$ and $J'_{n,j}$ depend on θ.

LEMMA 3.2. *For each $\theta\in]0,1[-\mathbf{Q}$ the arc $J_{n,j}$ is mapped diffeomorphically onto $J_{n,0}$ by f_θ^j for every $n\geqslant 1$ and $0<j<q_{n+1}$.*

Proof. Evidently, $J_{n,0}=f_\theta^j(J_{n,j})$. Next we note that $x_{-q_{n+1}}$ is the first return of 1 into $J_{n,0}=]x_{-q_n},x_{-q_{n+1}+q_n}[\subset\mathbf{S}^1$ under f_θ. This proves that the restrictions f_θ^j: $J_{n,j}\to J_{n,0}$ are diffeomorphisms for each $0\leqslant j<q_{n+1}$, and so completes the proof. □

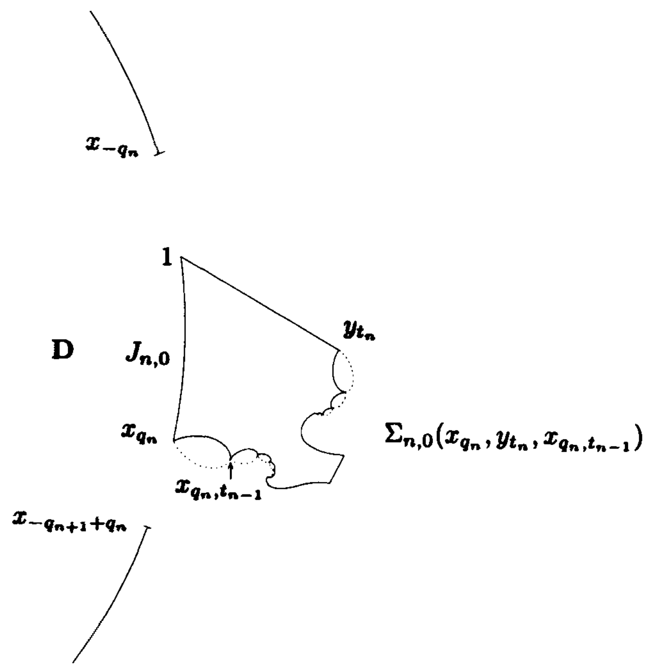

Fig. 19. The Jordan arc Σ_n relative to the gab $J_{n,0}$

LEMMA 3.3. *Given* $\theta \in]0,1[-\mathbf{Q}$ *write*

$$\Sigma_n(x_{q_n}, y_{t_n}, x_{q_n,t_{n-1}}) = \Sigma_n(I_n, B_n, G_n, R_n, O_n)$$

for each $n \geqslant 1$. *Then*

$$l_{J_{n,0}}(O_n) \leqslant L_{1,\theta}, \quad l_{J_{n,0}}(B_n) \leqslant L_{2,\theta} \quad \text{and} \quad l_\lambda(G_n) \leqslant L_{G,\theta} + 3L_{R,\theta}.$$

Moreover, the bounds are asymptotically universal.

Proof. See Figure 19. To obtain the estimate on O_n we note that $J_{n,0}$ contains $]x_{-q_n}, x_{-q_{n+1}}[= K_n$ and that $O_n \subset [1, y_{q_{n-1}}] = Q_{n-1}$. Thus the first estimate is immediate from Lemma 2.8. To obtain the estimate on B_n we note that $f_\theta^{q_n}$ maps $K' =]1, x_{-q_{n+1}+q_n}[\subset J_{n,0}$ diffeomorphically onto $]x_{-q_n}, x_{-q_{n+1}}[= K_n$ of Lemma 2.8. Let $g: \Gamma(I, B, G, R, O) \to \Sigma_n$ be the last move in obtaining Σ_n. Then $B_n = g(O)$, $f^{q_n} \circ g = \mathrm{Id}$ and g can be defined univalently in a neighbourhood $U \subset A^*_{K_n}$ of O. Moreover, O is a subarc of $[1, y_{q_{n-2}}] = Q_{n-2}$, because the free endpoint of B is $x_{q_n,t_{n-1}}$. Thus by combining Lemma 2.9 and Lemma 2.8 we obtain

$$l_{J_{n,0}}(B_n) < l_{K'}(B_n) \leqslant l_{K_n}(O) \leqslant l_{K_n}(Q_{n-2}) \leqslant L_{2,\theta}.$$

The estimate for $l_\lambda(G)$ comes from Σ_n being at most three admissible moves (Swaps) away from a Γ_k, for which we have $l_\lambda(G) \leqslant L_{G,\theta}$. Finally the bounds are easily seen to be asymptotically universal. □

LEMMA 3.4. *For each* $\theta \in]0,1[-\mathbf{Q}$ *the Euclidean length* $l_e(J_{n,j})$ *tends to* 0 *uniformly in* j *as* $n \to \infty$.

Proof. The restriction $f_\theta: \mathbf{S}^1 \to \mathbf{S}^1$ is conjugate to the rigid rotation R_θ according to Yoccoz [Yo1; there are no analytic Denjoy counterexamples]. Let $h: \mathbf{S}^1 \to \mathbf{S}^1$ be a conjugating homeomorphism, i.e. $h \circ R_\theta = f_\theta \circ h$. Then $l_e(h^{-1}(J_{n,j})) = l_e(h^{-1}(J_{n,0}))$ for all $n \geqslant 1$ and all $0 \leqslant j < q_{n+1}$. Moreover, h is uniformly continuous as $\mathbf{S}^1$ is compact and $l_e(h^{-1}(J_{n,0})) \to 0$ as $n \to \infty$. Thus the lemma follows. □

Definition 3.5. For each $n \geqslant 1$ rename Σ_n to $\Sigma_{n,0}$. Furthermore, for each $0 < j < q_{n+1}$ let $\Sigma_{n,j}$ and $\Delta_{n,j}$ be the unique lifts of $\Sigma_{n,0}$ to f_θ^j intersecting $\mathbf{S}^1$ and ∂U_0 respectively. Let $I_{n,0}$ be the I of $\Sigma_{n,0}$. Moreover, for $0 < j < q_{n+1}$ let $I_{n,j} = \Sigma_{n,j} \cap f_\theta^{-j}(I_{n,0})$.

LEMMA 3.6. *For each* $\theta \in]0,1[-\mathbf{Q}$ *there exist constants* $L_{\Sigma,\theta}, L_{\Delta,\theta} > 0$ *such that*

$$l_{J_{n,j}}(\Sigma_{n,j} - I_{n,j}) \leqslant L_{\Sigma,\theta} \quad \forall n \geqslant 1,\ 0 \leqslant j < q_{n+1}, \tag{1}$$

$$l_\lambda(\Delta_{n,j}) \leqslant L_{\Delta,\theta} \quad \forall n \geqslant 1,\ 0 < j < q_{n+1}. \tag{2}$$

Moreover, the constants $L_{\Sigma,\theta}$ *and* $L_{\Delta,\theta}$ *are asymptotically universal.*

Proof. Let $L_{\Sigma,\theta} = L_{2,\theta} + L_{G,\theta} + 4L_{R,\theta} + L_{1,\theta}$. Then $l_{J_{n,0}}(\Sigma_{n,0} - I_{n,0}) \leqslant L_{\Sigma,\theta}$ as it follows from Lemma 3.3. Let $\varrho_{n,j} = \varrho_{J_{n,j}}$ denote the hyperbolic metric on $W^*_{J_{n,j}} \subset A^*_{J_{n,j}}$. Let $I'_{n,j} = \Delta_{n,j} \cap f_\theta^{-j}(I_{n,0})$ for $0 < j < q_{n+1}$. Then $I'_{n,j} \subset J'_{n,j}$ is a $\varrho_{n,j}$-geodesic, because $J_{n,j-1} = f_\theta(J'_{n,j})$ is a $\lambda_{J_{n,j-1}}$-geodesic. In particular, $l_{\varrho_{n,j}}(\Delta_{n,j}) \leqslant 2 l_{\varrho_{n,j}}(\Delta_{n,j} - I'_{n,j})$. From Lemma 3.2 and Lemma 2.9 we obtain

$$\begin{aligned} l_{J_{n,j}}(\Sigma_{n,j} - I_{n,j}) < l_{\varrho_{n,j}}(\Sigma_{n,j} - I_{n,j}) \leqslant l_{J_{n,0}}(\Sigma_{n,0} - I_{n,0}) \leqslant L_{\Sigma,\theta}, \\ l_{\varrho_{n,j}}(\Delta_{n,j}) \leqslant 2 l_{\varrho_{n,j}}(\Delta_{n,j} - I'_{n,j}) \leqslant 2 l_{J_{n,0}}(\Sigma_{n,0} - I_{n,0}) \leqslant 2 L_{\Sigma,\theta}. \end{aligned} \tag{3}$$

The first line is identical with (1). Lemma 2.5 implies $d_{\varrho_{n,j}}(J_{n,j}, J'_{n,j}) = E(l_e(J_{n,j-1}))$, so that

$$d_{\varrho_{n,j}}(J_{n,j}, \Delta_{n,j}) \geqslant (E(l_e(J_{n,j-1})) - L_{\Sigma,\theta}) \to \infty \quad \text{as } n \to \infty,$$

by (3) and Lemma 3.4. Hence by Lemma 2.4, $\limsup_{n\to\infty} l_\lambda(\Delta_{n,j}) \leqslant 2L_{\Sigma,\theta}$, from which the existence of $L_{\Delta,\theta}$ as in (2) easily follows. Moreover, as $L_{1,\theta}, L_{2,\theta}, L_{R,\theta}$ and $L_{G,\theta}$ are asymptotically universal, so are $L_{\Sigma,\theta}$ and $L_{\Delta,\theta}$. □

THEOREM 3.7. *For each $\theta \in]0,1[-\mathbf{Q}$ the Euclidean diameter of the principal limbs X_s and Y_s, $s\geqslant 1$, tends to 0 as $s\to\infty$.*

Proof. It suffices to consider the limbs X_s, $s\geqslant 1$, as $f_\theta(Y_s)=f_\theta(X_s)$ for all $s\geqslant 1$. Let $\theta\in]0,1[-\mathbf{Q}$ be given. We shall first prove that

$$l_e(\Sigma_{n,j})\to 0 \quad \text{as } n\to\infty, \tag{1}$$

uniformly in $0\leqslant j<q_{n+1}$.

Combining Lemma 3.6 (1) with Lemma 3.4 and Lemma 2.6 we obtain

$$\limsup_{n\to\infty}\frac{l_e(\Sigma_{n,j}-I_{n,j})}{l_e(J_{n,j})}\leqslant 2\sinh\left(\tfrac{1}{2}L_{\Sigma,\theta}\right). \tag{3}$$

It follows that there exists a constant $K_{\Sigma,\theta}$ such that

$$l_e(\Sigma_{n,j})\leqslant K_{\Sigma,\theta}\cdot l_e(J_{n,j}) \quad \forall n\geqslant 1,\ 0\leqslant j<q_{n+1}, \tag{4}$$

as $l_e(\Sigma_{n,j})<\infty$ for all n,j and $I_{n,j}\subset J_{n,j}$. Combining (4) and Lemma 3.4 we obtain (1). Finally the theorem follows from (1), because $X_{\alpha q_{n+1}+q_n+j}\subset D(\Sigma_{n,j})$ for all $n\geqslant 1$, $0\leqslant j<q_{n+1}$ and $0<\alpha\leqslant a_{n+1}$. □

"The bridge across Lille Bælt". Suppose that η_1,η_2 are two curves with one or two common endpoint(s). We shall write $\eta_1\cdot\eta_2$ for the curve obtained by gluing the two arcs at their common endpoint(s).

PROPOSITION AND DEFINITION 3.8. *Let $\theta\in]0,1[-\mathbf{Q}$. There exist positive constants $L'_{d,\theta}$, $d=0,1,2,3,\ldots$, and for each pair $(l,l')=(\alpha q_n+q_{n-1},\beta q_{n+d}+q_{n+d-1})$ with $n\geqslant 2$ and $0\leqslant\alpha<a_n$, $0<\beta\leqslant a_{n+d}$, there exists an arc $\gamma_{l,l'}\in\bar{U}_0$ joining y_l to $y_{l'}$ and with*

$$l_\lambda(\gamma_{l,l'})\leqslant L'_{d,\theta},$$

where λ denotes the hyperbolic metric on $\mathbf{C}-\bar{\mathbf{D}}$. See Figure 20.

COMPLEMENT TO PROPOSITION 3.8. *There exist explicit and universal constants L', M, i.e. not depending on θ such that the same curves satisfy*

$$\limsup l_\lambda(\gamma_{l,l'})\leqslant\begin{cases} L'+(d+2)M & \text{for } d \text{ odd,}\\ (d+2)M & \text{for } d \text{ even.}\end{cases}$$

Proof. We shall define the curves $\gamma_{l,l'}$ so that they all have finite λ-length and then prove the complement, from which the proposition follows. Let $\Phi:\mathbf{D}\to V\subset\mathbf{C}$ be the univalent parameter with $\Phi(0)=1$, $\Phi'(0)>0$ and $\exp\circ(z\mapsto\tilde{v}-\pi z^3)=f_\theta\circ\Phi$ defined in the

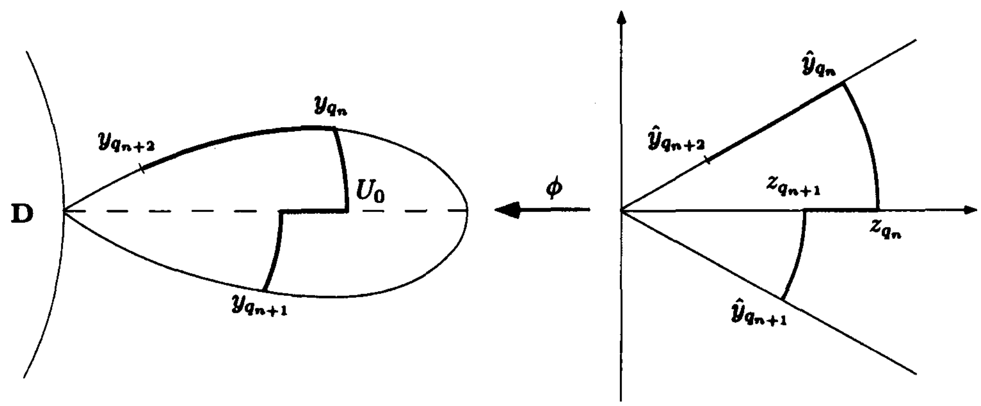

Fig. 20. The curves $\gamma_{l,l'}$ in $\bar{U}_0$

proof of Proposition 2.8 (here $\exp(\tilde{v})=v$). Define $\hat{y}_j=\Phi^{-1}(y_j)$ for $j\geqslant 1$, $z_j=|\hat{y}_j|\in\mathbf{R}_+$ and let C_j be the cirle with center 0 and radius z_j. Moreover, define

$$l_{\pm}=\{z\,|\,\mathrm{Arg}(z)=\pm\tfrac{1}{6}i\pi\}\cap\mathbf{D}$$

so that

$$\Phi(l_{\pm})=\partial U_0\cap\{z\,|\,\mathrm{Im}(z)\gtrless 0\}.$$

For $(l,l')=(\alpha q_n+q_{n-1},\beta q_{n+d}+q_{n+d-1})$ with $n\geqslant 2$, $0\leqslant\alpha<a_n$, $0<\beta\leqslant a_{n+d}$, define an arc $\widehat{\gamma}_{l,l'}$ as follows (see Figure 20): If d is even then y_l and $y_{l'}$ are in the same line segment l_+ or l_-. We define $\widehat{\gamma}_{l,l'}=[\hat{y}_l,\hat{y}_{l'}]$. For d odd y_l and $y_{l'}$ are in opposite line segments. Let $\lceil\hat{y}_l,z_l\rceil$ and $\lceil\hat{z}_{l'},\hat{y}_{l'}\rceil$ be the smaller subarcs of C_l and $C_{l'}$ respectively, between the respective points. Define $\widehat{\gamma}_{l,l'}=\lceil\hat{y}_l,z_l\rceil\cdot[z_l,z_{l'}]\cdot\lceil\hat{z}_{l'},\hat{y}_{l'}\rceil$. Let $\tilde{\lambda}$ denote the hyperbolic metric on $\mathbf{H}_+$ and let L' be the $\tilde{\lambda}$-distance between l_+ and l_-. Then the $\tilde{\lambda}$-length of any of the arcs $\lceil\hat{y}_j,z_j\rceil$ equals $\frac{1}{2}L'$. Moreover, $[\hat{y}_l,\hat{y}_{l'}]\subset[\hat{y}_{q_n-1},\hat{y}_{q_{n+d+1}}]$ if d is even and $[z_l,z_{l'}]\subset[z_{q_{n-1}},z_{q_{n+d+1}}]$. Hence we get by direct calculation

$$l_{\tilde{\lambda}}([\hat{y}_l,\hat{y}_{l'}])\leqslant l_{\tilde{\lambda}}([\hat{y}_{q_n-1},\hat{y}_{q_{n+d+1}}])=\frac{2}{\sqrt{3}}\log\frac{|\hat{y}_{q_{n-1}}|}{|\hat{y}_{q_{n+d+1}}|}\quad\text{for } d \text{ even},$$
$$l_{\tilde{\lambda}}([z_l,z_{l'}])\leqslant l_{\tilde{\lambda}}([z_{q_n-1},z_{q_{n+d+1}}])=\log\frac{z_{q_{n-1}}}{z_{q_{n+d+1}}}.$$

Combining the univalence of Φ with the Świątec a priori real bounds we obtain

$$\limsup\log\frac{z_{q_{n-1}}}{z_{q_{n+d+1}}}=\limsup\log\frac{|\hat{y}_{q_{n-1}}|}{|\hat{y}_{q_{n+d+1}}|}\leqslant(d+2)\log\frac{1}{a}.$$

This proves asymptotic bounds as in the complement, but for the curves $\widehat{\gamma}_{l,l'}$ in the metric $\tilde{\lambda}$. We define $\gamma_{l,l'}=\Phi(\widehat{\gamma}_{l,l'})$ and then appeal to Lemma 2.4 to carry over the asymptotic bounds for the curves $\widehat{\gamma}_{l,l'}$ to the asymptotic bounds of the complement for the curves $\gamma_{l,l'}$, as in the proof of Proposition 2.8. □

For $0<\eta<\frac{1}{2}\pi$ define

$$\widetilde{S}(\eta)=\{z\mid|\operatorname{Arg}(z)|\leqslant\eta\}\cup\{0\} \quad\text{and}\quad \tilde{l}(\pm\eta)=\{z\mid|\operatorname{Arg}(z)|=\pm\eta\}.$$

$\widetilde{S}_r(\eta)=\widetilde{S}(\eta)\cap\mathbf{D}_r$, where $\mathbf{D}_r$ is the Euclidean disc of center 0 and radius $r>0$. Moreover, let $S(\eta)=\exp(\widetilde{S}(\eta))$, $S_r(\eta)=\exp(\widetilde{S}_r(\eta))$ and $l(\pm\eta)=\exp(\tilde{l}(\pm\eta))$. Recall that Π_θ is the principal wake containing the limb $X_0=Y_\varepsilon$.

THEOREM 3.9. *Let $\theta\in]0,1[-\mathbf{Q}$ be of constant type. There exists an angle $0<\eta<\frac{1}{2}\pi$ and $r>0$ such that*

$$X_0\cap\exp(\mathbf{D}_r)\subset\Pi_\theta\cap\exp(\mathbf{D}_r)\subset S_r(\eta).$$

Moreover, the angle η depends only on $N=\limsup a_n$, whereas r depends on the number n_0 for which $a_n\leqslant N$ for all $n\geqslant n_0$.

COMPLEMENT TO THEOREM 3.9. *For θ as in the theorem there exists a constant $L_\theta>0$ such that*

$$\operatorname{diam}_\lambda(Y_s)\leqslant L_\theta \quad \forall s\geqslant 1.$$

Moreover, $\limsup\operatorname{diam}_\lambda(Y_s)\leqslant L(\limsup a_n)$, *where the constants $L(n)>0$, $n\geqslant1$, are independent of θ.*

Proof. We shall construct a "suspension bridge", which will prevent Π_θ and X_0 from coming too close to $\mathbf{S}^1$, and which will imply both the theorem and its complement. See Figure 21.

For each $n\geqslant1$ write $\Sigma_n=\Sigma_n(I_n,B_n,G_n,R_n,O_n)$ and let $P_n=R_n\cdot G_n$. For each $n\geqslant2$ define arcs as follows:

(1) $P_{n,\alpha}=f_\theta^{-(\alpha q_n+q_{n-1})}(P_n)\cap\Sigma_{n,\alpha q_n+q_{n-1}}$ for $0\leqslant\alpha<a_n$,

(2) $\gamma_{t_{n-1},t_n,\alpha}=f_\theta^{-(\alpha q_n+q_{n-1})}(\gamma_{t_{n-1},t_n})\cap\overline{U}_{\alpha q_n+q_{n-1}}$ for $0<\alpha\leqslant a_n$,

(3) $\gamma'_{t_{n-2},t_n}=f_\theta^{-q_{n-1}}(\gamma_{t_{n-2},t_n})\cap\overline{U}_{q_{n-1}}$,

(4) $\mathcal{P}_n=\gamma'_{t_{n-2},t_n}\cdot P_{n,0}\cdot\gamma_{t_{n-1},t_n,1}\cdot P_{n,1}\cdot\ldots\cdot\gamma_{t_{n-1},t_n,a_n-1}\cdot P_{n,a_n-1}\cdot\gamma_{t_{n-1},t_n,a_n}$,

(5) $\Xi_n=\mathcal{P}_{n-1}\cdot\mathcal{P}_n\cdot(-\mathcal{P}_{n+1})\cdot(-\gamma_{t_{n-1},t_{n+1}})$,

(6) $D_n=D(\Xi_n)$.

Finally define long arcs

(7) $\Delta_{\text{even}}=\mathcal{P}_2\cdot\mathcal{P}_4\cdot\ldots\cdot\mathcal{P}_{2n}\cdot\ldots$,

(8) $\Delta_{\text{odd}}=\mathcal{P}_3\cdot\mathcal{P}_5\cdot\ldots\cdot\mathcal{P}_{2n+1}\cdot\ldots$.

Then Δ_{even} and Δ_{odd} converges to 1 and can be made into closed arcs by adding the point 1 to each. Let R_+ and R_- be the two external rays landing on 1. Then

$$P_{n,\alpha}\cap R_+=P_{n,\alpha}\cap R_-=P_{n,\alpha}\cap X_0=\varnothing,$$

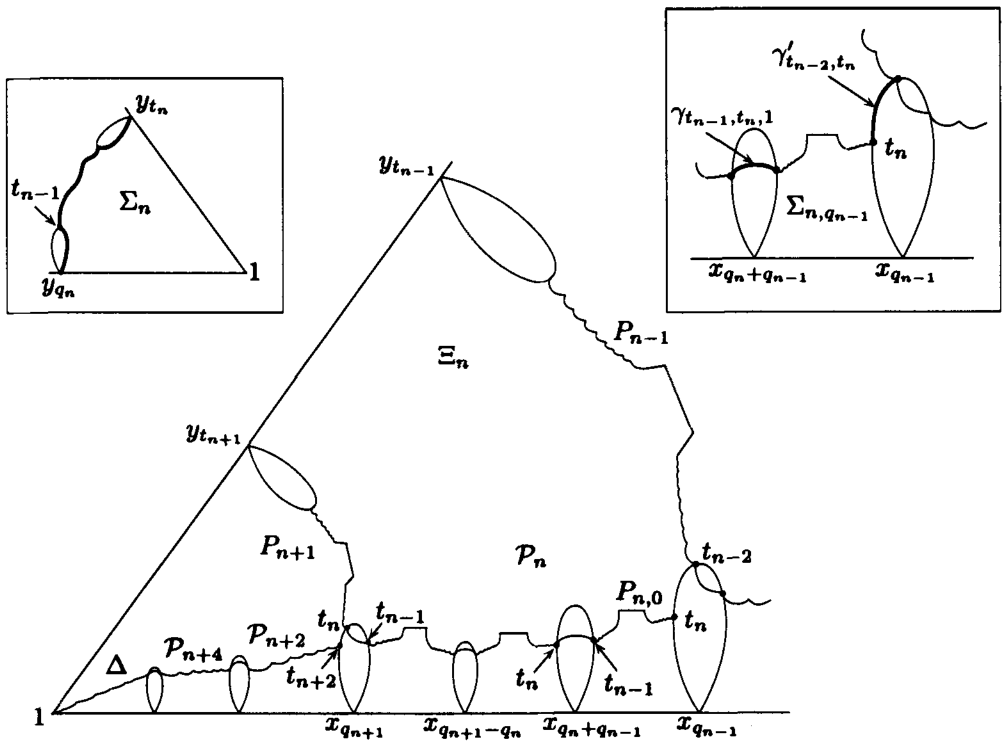

Fig. 21. The barriers Δ_{even} and Δ_{odd} separating the principal wake Π_θ from $\mathbf{D}$

because

$$J_\theta \cap \Sigma_{n,\alpha q_n+q_{n-1}} \subset X_{\alpha q_n+q_{n-1}} \cup X_{(\alpha+1)q_n+q_{n-1}} \cup [x_{\alpha q_n+q_{n-1}}, x_{(\alpha+1)q_n+q_{n-1}}].$$

$P_{n,\alpha} \subset \Sigma_{n,\alpha q_n+q_{n-1}}$ and external rays do not cross (see also Figure 21).

As f_θ^{-1} is contracting with respect to λ we obtain from Lemma 3.3 and Proposition 3.8 that

$$\begin{aligned} l_\lambda(\Xi_n) &\leqslant l_\lambda(P_{n-1}) + l_\lambda(\mathcal{P}_n) + l_\lambda(P_{n+1}) + l_\lambda(\gamma_{t_{n-1},t_{n+1}}) \\ &\leqslant 2(L'_2 + L_{G,\theta} + 4L_{R,\theta}) + a_n(L'_1 + L_{G,\theta} + 4L_{R,\theta}) \\ &= K_{2,\theta} + a_n \cdot K_{1,\theta}, \end{aligned}$$

where the constants $K_{1,\theta}, K_{2,\theta}$ are defined by the equality sign. The above curves can be constructed and the estimate on Ξ_n holds for all $\theta \in]0,1[- \mathbf{Q}$. We note however that the estimate depends on a_n.

Suppose that θ is of constant type. Let $N = \limsup a_n$ and let $n_0 \geqslant 2$ be minimal with $a_n \leqslant N$ for all $n \geqslant n_0$. Let us prove that there exists $0 < \eta < \frac{1}{2}\pi$ such that

$$\mathcal{P}_n \subset S(\eta) \quad \text{for all } n \geqslant n_0. \tag{1}$$

Let $\frac{1}{6}\pi \leqslant \eta_1 < \frac{1}{2}\pi$ be minimal with $U_0 \subset S(\eta_1)$. Next let $\eta_1 < \eta < \frac{1}{2}\pi$ be given by

$$\operatorname{dist}_{\tilde{\lambda}}(\tilde{l}(\eta_1), \tilde{l}(\eta)) = \tfrac{1}{2}(K_{2,\theta} + N \cdot K_{1,\theta}),$$

where $\tilde{\lambda}$ denotes the hyperbolic metric on $\mathbf{H}_+$. Then η satisfies (1).

Let $0 < r < \pi$ be small enough that

$$\exp(\mathbf{D}_r) \subset (D(\Sigma_{n_0,0}) \cup D(\Sigma_{n_0+1,0}) \cup U_0 \cup \mathbf{D}).$$

Then $X_0 \cap \exp(\mathbf{D}_r) \subset S_r(\eta)$, where η is as in (1). This proves the theorem.

To prove the complement we note first that for any $n \geqslant 2$, $0 < \alpha \leqslant a_n$ and $0 < j < q_n$ we have $Y_{\alpha q_n + q_{n-1} + j} \subset D(\Delta_{n,j})$. Moreover,

$$\operatorname{diam}_\lambda(D(\Delta_{n,j})) < \tfrac{1}{2} l_\lambda(\Delta_{n,j}) \leqslant L_{\Delta,\theta}$$

by Lemma 3.3. Thus we need only bound the λ-diameters of the limbs $Y_{\alpha q_n + q_{n-1}}$ for $n \geqslant 1$ and $0 < \alpha \leqslant a_n$. Each limb Y_s is a compact subset of $\mathbf{C} - \overline{\mathbf{D}}$. Thus we need only give a bound for the cases $n \geqslant n_0 + 2$, say. For any $n \geqslant n_0 + 2$ we have $Y_{\alpha q_n + q_{n-1}} \subset D_n \cup D_{n-2}$ and $\operatorname{diam}_\lambda(D_n \cup D_{n-2}) \leqslant K_{2,\theta} + N \cdot K_{1,\theta}$. This proves the existence of a bound in the remaining case. Finally the statement of asymptotic universality follows from the asymptotic universality of the bounding constants. □

We obtain as immediate corollary a proof of the last half of Theorem 3.26. (Recall Definition 0.8.)

THEOREM 3.26 (2). *Let $\theta \in]0,1[$ be of constant type. Then there exists a constant $M = M(\theta) > 1$ such that*

$$\|D_z F_\theta\|_\lambda \geqslant M \quad \textit{for all } z \in Y_\theta.$$

Proof. Given $\theta \in]0,1[- \mathbf{Q}$ of bounded type let $\frac{1}{6}\pi < \eta < \frac{1}{2}\pi$ and $r > 0$ be as in Theorem 3.9. Then $Y_\theta \subset (S_r(\eta) - \{0\}) \cup K$, where $K \subset \mathbf{C} - \overline{\mathbf{D}}$ is a compact subset. Thus there exists $M > 1$ such that $\varrho(z)/\lambda(z) \geqslant M$ for all $z \in Y_\theta$. As

$$\|D_z f_\theta\|_\lambda = \frac{\varrho(z)}{\lambda(z)} \cdot \|D_z f_\theta\|_{\lambda,\varrho} = \frac{\varrho(z)}{\lambda(z)} > 1 \quad \forall z \in W_1,$$

the theorem follows from the definition of F_θ. □

Lifting to the exponential. This subsection except Theorem 3.12 shall be used both in the subsequent subsection and in the subsection "Controlling the core of nests for all irrational θ". Let $J_0 =]\tilde{v} - i2\pi, \tilde{v}[$, where $\tilde{v}$ is the logarithm in $]0, i2\pi[$ of the critical value v. Moreover, let $\widetilde{U}_0$ be the connected component of $\exp^{-1}(U_0)$ with 0 on the boundary.

Recall that $f_\theta: W_1 \to \mathbf{C}-\overline{\mathbf{D}}$ is a degree-two covering map, where W_1 is the unbounded connected component of $f_\theta^{-1}(\mathbf{C}-\overline{\mathbf{D}})$. Let $\widetilde{W}_1 = \exp^{-1}(W_1)$ and let $G_\theta: \mathbf{H}_+ \to \widetilde{W}_1$ be a lift of exp to $f_\theta \circ \exp$. The map G_θ extends to a continuous map of $\overline{\mathbf{H}}_+$ onto $\overline{\widetilde{W}}_1$. The extended map G_θ is injective except on the set $\{\tilde{v}-i2\pi, \tilde{v}\}+i4\pi\mathbf{Z}$, which is mapped two-to-one onto $i2\pi\mathbf{Z}$. Choosing another lift, if necessary, we shall suppose that G_θ maps J_0 homeomorphically onto $\partial\widetilde{U}_0-\{0\}$. The map G_θ is an isometry with respect to the hyperbolic metrics $\tilde{\lambda}$ on $\mathbf{H}_+$ and $\tilde{\varrho}$ on $\widetilde{W}_1$. Let $\tilde{f}_\theta: \widetilde{W}_1 \to \mathbf{H}_+$ denote the inverse of G_θ and extend $\tilde{f}_\theta$ continuously to $\overline{\widetilde{W}}_1 - i2\pi\mathbf{Z}$. We note that $\overline{\widetilde{W}}_1$ does not depend on θ and that $\tilde{f}_{\theta_1}, \tilde{f}_{\theta_2}$ for $\theta_1, \theta_2 \in]0,1[-\mathbf{Q}$ differ only by an additive, imaginary constant, because f_{θ_1} and f_{θ_2} differ only by a multiplicative constant of modulus 1.

We shall write $\mathbf{C}_0$ for $\mathbf{C}_{J_0}$, the doubly slit plane with gab J_0. Moreover, we let δ_0 denote the hyperbolic metric on $\mathbf{C}_0$. For $0<T<\infty$ we define (see Figure 22)

$$\omega(T) = \{z \in \mathbf{H}_+ \mid 0 < d_{\delta_0}(J_0, z) \leqslant T\} \quad \text{and} \quad \Omega(T) = G_\theta(\omega(T)).$$

By the above remark the set $\Omega(T)$ does not depend on θ and the sets $\omega(T)=\omega_\theta(T)$ differ only by a purely imaginary translation. We remind the reader that for each $T>0$, the set $\omega(T)$ is bounded in $\mathbf{H}_+$ by an arc of circle through the endpoints of J_0. We call this arc of circle C_T. The angle between J_0 and C_T at any one of their common endpoints is in one-to-one correspondence with T. An elementary calculation shows that the angle $\frac{1}{2}\pi$ corresponds to the distance $T_C = \log(1+\sqrt{2})$. The arc C_{T_C} is a $\tilde{\lambda}$-geodesic. Moreover, each C_T is an arc of $\tilde{\lambda}$-equidistance from C_{T_C}. Thus the set $\omega(T)$ is a $\tilde{\lambda}$-convex subset of $\mathbf{H}_+$ if and only if $T \geqslant T_C$. Then also $\Omega(T)$ is a $\tilde{\varrho}$-convex subset of $\widetilde{W}_1$ if and only if $T \geqslant T_C$, as G_θ is an isometry.

LEMMA 3.10. *There exists an increasing function $M: \mathbf{R}_+ \to]0,1[$ such that*

$$\frac{\tilde{\lambda}(z)}{\tilde{\varrho}(z)} \leqslant M(T) \quad \forall z \in \Omega(T),$$

where $\tilde{\lambda}$ is the coefficient of the hyperbolic metric on $\mathbf{H}_+$ and $\tilde{\varrho}$ is the coefficient of the hyperbolic metric on $\widetilde{W}_1$. In particular, if $T \geqslant T_C$, then

$$\operatorname{diam}_{\tilde{\lambda}}(K) \leqslant M(T) \cdot \operatorname{diam}_{\tilde{\varrho}}(K)$$

for all compact subsets $K \subset \Omega(T)$.

Proof. The boundary of $\Omega(T)$ different from $\partial\widetilde{U}_0$ makes a non-zero angle with the imaginary axis at 0. □

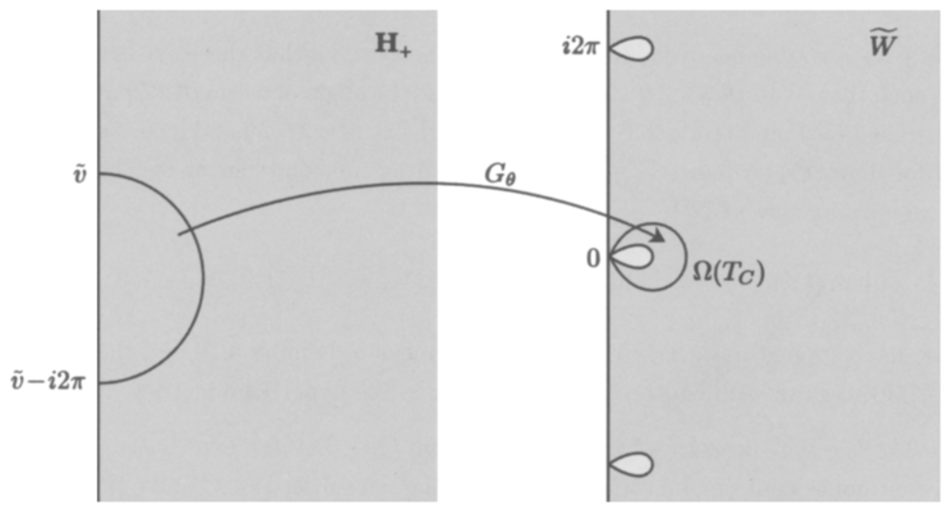

Fig. 22. The set $\widetilde{W}$ and $\Omega(T)$ with $T=T_C$

Let $\widetilde{X}_j$, $j\geqslant 0$, be the connected component of $\exp^{-1}(X_j)$ intersecting the segment J_0 and let $\widetilde{X}_0^*=\widetilde{X}_0-\{0\}$. Moreover, define $\widetilde{Y}_{s_1,\ldots,s_m}=\exp^{-1}(Y_{s_1,\ldots,s_m})\cap\widetilde{X}_0$ for each $(s_1,\ldots,s_m)\in\mathbf{N}^m$, $m\geqslant 1$, and $\tilde{y}_s=\widetilde{X}_0\cap\exp^{-1}(y_s)$ for $s\geqslant 1$. Define $W_0=\mathbf{C}-\overline{\mathbf{D}}$ and for $n\geqslant 2$ let W_n be the unbounded connected component of $f_\theta^{-1}(W_{n-1})$. Then each restriction $f_\theta\colon W_n\to W_{n-1}$ is a degree-two covering map. Define $\widetilde{W}_n=\exp^{-1}(W_n)$ for all $n\geqslant 1$ (the case $n=1$ has previously been taken care of).

Definition 3.11. Let $\theta\in\,]0,1[-\mathbf{Q}$. For each $s\geqslant 1$ let $G_{s,\theta}\colon\overline{\mathbf{H}}_+\to\overline{\widetilde{W}}_s\subseteq\overline{\widetilde{W}}_1$ be the lift of $\exp\colon\overline{\mathbf{H}}_+\to\mathbf{C}-\mathbf{D}$ to $f_\theta^s\circ\exp\colon\overline{\widetilde{W}}_s\to\mathbf{C}-\mathbf{D}$ with $G_{s,\theta}(0)=\tilde{y}_s$. We shall usually omit the index θ however. Moreover, for $(s_1,\ldots,s_m)\in\mathbf{N}^m$ we shall use the shorthand notation

$$G_{s_1,\ldots,s_m}=G_{s_1}\circ\ldots\circ G_{s_m}.$$

Each G_s is Lipschitz with constant 1 with respect to the hyperbolic metrics $\tilde{\lambda}$ and $\tilde{\varrho}$. Moreover, for all $m\geqslant 1$ and for all $(s_1,\ldots,s_m)\in\mathbf{N}^m$,

$$Y_{s_1,\ldots,s_m}=G_{s_1,\ldots,s_m}(X_0)=G_{s_1,\ldots,s_{m-1}}(Y_{s_m})=G_{s_1,\ldots,s_{m-2}}(Y_{s_{m-1},s_m}),$$

and $G_{s_1,\ldots,s_m}\colon\mathbf{H}_+\to\widetilde{W}_{s_1+\ldots+s_m}$ is biholomorphic, because it is a lift of the universal covering $\exp\colon\mathbf{H}_+\to\mathbf{C}-\overline{\mathbf{D}}$ to the universal covering $f_\theta^{s_1+\ldots+s_m}\circ\exp\colon\widetilde{W}_{s_1+\ldots+s_m}\to\mathbf{C}-\overline{\mathbf{D}}$.

THEOREM 3.12. *For any $\theta\in\,]0,1[-\mathbf{Q}$ of constant type the Core* Core$(\mathcal{Y}_{\underline{s},\theta})$ *is trivial for each $\underline{s}\in\mathbf{N}^{\mathbf{N}}$.*

Proof. We shall prove the following equivalent statement of the theorem: For every $\underline{s}=(s_1,\ldots,s_m,\ldots)\in\mathbf{N}^{\mathbf{N}}$ there exists $\tilde{z}_{\underline{s}}\in\mathbf{H}_+$ such that

$$\bigcap_{m\geqslant 1}\widetilde{Y}_{s_1,\ldots,s_m}=\{\tilde{z}_{\underline{s}}\}.$$

Let θ of constant type be given. It follows from Theorem 3.9 that there exists a constant $T \geqslant T_C$ such that $(\widetilde{X}_0 - \partial\widetilde{U}_0) \subset \Omega(T)$. This is because the angle between $\partial\Omega(T)$ at 0 and $i\mathbf{R}$ is determined by T and tends to 0 as T tends to ∞. Let $M = M(T)$ and let L be an upper bound for $\operatorname{diam}_{\tilde{\lambda}}(\widetilde{Y}_s) = \operatorname{diam}_{\lambda}(Y_s)$ for $s \geqslant 1$ as given by the complement to Theorem 3.9. Then we have for any $\underline{s} \in \mathbf{N}^{\mathbf{N}}$,

$$\operatorname{diam}_{\tilde{\lambda}}(\widetilde{Y}_{s_1,\ldots,s_m}) = \operatorname{diam}_{\tilde{\lambda}}(G_{s_1} \circ \ldots \circ G_{s_{m-1}}(Y_{s_m})) \leqslant L \cdot M^{(m-1)} \underset{m \to \infty}{\longrightarrow} 0.$$

Here we have applied iteratively the second statement of Lemma 3.10 and the fact that each G_s is Lipschitz with constant 1 with respect to the hyperbolic metrics $\tilde{\lambda}$ and $\tilde{\varrho}$. □

On the Lebesgue measure of J_θ for θ of constant type. Let $\tilde{J}_\theta = \exp^{-1}(J_\theta)$. Moreover, for δ a continuous conformal metric on a connected open subset $\mathcal{W} \subset \mathbf{C}$ and a point $z \in \mathcal{W}$, let $B_{\delta,R}(z) = \{w \in \mathcal{W} \mid d_\delta(w,z) \leqslant R\}$. If no metric is specified, then $W = \mathbf{C}$ and δ is the Euclidean metric. Finally for $\omega \subset \mathcal{W}$ a Borel-measurable subset let $\operatorname{Area}(\omega, \delta)$ denote the Area (infinite or not) of ω with respect to δ. We write however $\operatorname{mes}(\omega)$ for the Lebesgue measure of ω.

PROPOSITION 3.13. *Let $\theta \in]0,1[- \mathbf{Q}$ be of constant type. There exist $R = R_\theta > 0$ and $0 < \alpha = \alpha_\theta \leqslant 1$ such that for all $z \in \widetilde{X}_0^*$,*

$$\frac{\operatorname{Area}(\tilde{J}_\theta \cap B_{\tilde{\lambda},R}(z), \tilde{\lambda})}{\operatorname{Area}(B_{\tilde{\lambda},R}(z), \tilde{\lambda})} \leqslant 1 - \alpha.$$

Proof. Given $\theta \in]0,1[- \mathbf{Q}$ of constant type let $0 < \eta < \frac{1}{2}\pi$ and $r > 0$ be as in Proposition 3.9 so that $\widetilde{X}_0 \cap \overline{\mathbf{D}}_r \subset \widetilde{S}_r(\eta)$. Moreover, let $0 < \eta_1 \leqslant \frac{1}{6}\pi$ and $r_1 > 0$ be such that $S_{r_1}(\eta_1) \subset \widetilde{U}_0 \cup \{0\}$. Define $R = \operatorname{dist}_{\tilde{\lambda}}(\tilde{l}(-\eta), \tilde{l}(\eta_1))$. The function

$$\operatorname{relA}(z) = \frac{\operatorname{Area}(\tilde{J}_\theta \cap B_{\tilde{\lambda},R}(z), \tilde{\lambda})}{\operatorname{Area}(B_{\tilde{\lambda},R}(z), \tilde{\lambda})}$$

is a continuous function of $z \in \mathbf{H}_+$. Moreover, $\operatorname{relA}(z) < 1$ for all $z \in \widetilde{X}_0^*$, because J_θ and hence $\tilde{J}_\theta$ has empty interior. Thus it suffices to prove that

$$\limsup_{|z| \to 0} \operatorname{relA}(z) < 1, \quad z \in \widetilde{X}_0^*. \tag{1}$$

Let $R_1 = \frac{1}{2} \operatorname{dist}_{\tilde{\lambda}}(\tilde{l}(-\eta_1), \tilde{l}(\eta_1))$. Moreover, let $r_2 = \exp(-R) \cdot \min\{r, r_1\}$. Then for any $z \in \widetilde{S}_{r_2}(\eta)$ we have

$$B_{R_1,\tilde{\lambda}}(|z|) \subset B_{R,\tilde{\lambda}}(z) \cap \widetilde{U}_0 \subset \mathbf{H}_+ - \tilde{J}_\theta. \tag{2}$$

The number
$$\alpha = \frac{\mathrm{Area}(B_{R_1,\tilde{\lambda}}(|z|), \tilde{\lambda})}{B_{R,\tilde{\lambda}}(z)} > 0$$
does not depend on $z \in \widetilde{S}_{r_2}(\eta)$. Hence we have proved that the lim sup of (1) is bounded by $1-\alpha$. □

LEMMA 3.14. *Let $U, V \subsetneq \mathbf{C}$ be open hyperbolic subsets, i.e. carrying hyperbolic metrics λ_U and λ_V respectively, and let $d_U(\cdot,\cdot)$ denote the corresponding distance function on U. Suppose that U is simply-connected and let $f: U \to V$ be a univalent map. For $z, w \in U$ arbitrary let $T = \|D_z f\|_{\lambda_V, \lambda_U}$ and $R = d_U(z, w)$. Then*
$$\frac{\sinh R}{\exp R} \cdot \frac{T}{\lambda_V(f(z))} \leqslant |f(w) - f(z)| \leqslant \frac{T}{\lambda_V(f(z))} \cdot \frac{\sinh R}{\exp(-R)}.$$

Proof. Let $z \in U$ be arbitrary and let $\phi: \mathbf{D} \to U$ be biholomorphic with $\phi(0) = z$. Then ϕ is a hyperbolic isometry, so that we can suppose that $U = \mathbf{D}$. The lemma then follows from the estimates
$$\frac{|w|}{(1+|w|)^2} |g'(0)| \leqslant |g(w) - g(0)| \leqslant |g'(0)| \frac{|w|}{(1-|w|)^2}$$
for univalent maps $g: \mathbf{D} \to \mathbf{C}$. □

LEMMA 3.15. *Let $U \subsetneq \mathbf{C}$ be an open simply-connected subset. Let λ_U be the coefficient function of the hyperbolic metric and let $d_U(\cdot,\cdot)$ denote the hyperbolic distance function. For $z, w \in U$ arbitrary let $R = d_U(z, w)$. Then*
$$\exp(-2R) \leqslant \frac{\lambda_U(w)}{\lambda_U(z)} \leqslant \exp(2R).$$

Proof. Easy consequence of the distortion theorem for univalent mappings of the disc. □

THEOREM 3.16. *Let $\theta \in]0,1[- \mathbf{Q}$ be of constant type, let $R > 0$ and $0 < \alpha \leqslant 1$ be as in Proposition 3.13 and let $\alpha' = \min\{\frac{1}{2}, \alpha \cdot e^{(-12R)}\} > 0$. Then for all $z \in \widetilde{X}_0^*$,*
$$\liminf_{r \to 0} \frac{\mathrm{mes}(B_r(z) \cap \tilde{J}_\theta)}{\mathrm{mes}(B_r(z))} \leqslant 1 - \alpha' < 1.$$

Proof. Let $\theta \in]0,1[- \mathbf{Q}$ of constant type be given. Define Cdens: $\mathbf{H}_+ \times \mathbf{R}_+ \to [0,1]$ by
$$\mathrm{Cdens}(z, r) = \frac{\mathrm{mes}(B_r(z) - \tilde{J}_\theta)}{\mathrm{mes}(B_r(z))}.$$

The statement of the theorem is that $\limsup_{r\to 0} \mathrm{Cdens}(z,r) \geqslant \alpha'$ for any $z \in \widetilde{X}_0^*$. For any $z \in \exp^{-1}(J_\theta^{\mathrm{skeleton}}) \cap \widetilde{X}_0^*$ we have $\limsup_{r\to 0} \mathrm{Cdens}(z,r) \geqslant \frac{1}{2} \geqslant \alpha'$, because any inverse image of $\partial U_0 - \{1\}$ is an analytic arc with $\mathbf{C} - J_\theta$ on one side. Thus we need only consider points $z \in \widetilde{X}_0^*$ with $\exp(z)$ of infinite address.

Let $z \in \widetilde{X}_0^*$ be a point with $\exp(z)$ of infinite address (itinerary) $\underline{s} \in \mathbf{N}^{\mathbf{N}}$. Let $z_m \in \widetilde{X}_0^*$ be given by $\exp(z_m) = f_\theta^{(s_1+\ldots+s_m)}(\exp(z)) = F_\theta^m(\exp(z))$ so that $z = G_{s_1,\ldots,s_m}(z_m)$ for all $m \geqslant 1$. Let $\tilde{\delta}_m$ denote the hyperbolic metric on $\widetilde{W}_{s_1+\ldots+s_m}$ so that $G_{s_1,\ldots,s_m}$ is an isometry with respect to $\tilde{\lambda}$ and $\tilde{\delta}_m$. Define

$$\varpi_m(z) = G_{s_1,\ldots,s_m}(B_{R,\tilde{\lambda}}(z_m)) = B_{R,\delta_m}(z).$$

Let $0 < M < 1$ be as in the proof of Theorem 3.12. Then $\|D_{z_m} G_{s_1,\ldots,s_m}\|_{\tilde{\lambda}} \leqslant M^m \to 0$ when $m \to \infty$. Thus the Euclidean diameter of $\varpi_m(z)$ tends to 0 when $m \to \infty$ by Lemma 3.14. Moreover,

$$\frac{\mathrm{Area}(\varpi_m(z) - \tilde{J}_\theta, \delta_m)}{\mathrm{Area}(\varpi_m(z), \delta_m)} = \frac{\mathrm{Area}(B_{\tilde{\lambda},R}(z_m) - \tilde{J}_\theta, \tilde{\lambda})}{\mathrm{Area}(B_{\tilde{\lambda},R}(z_m), \tilde{\lambda})} \geqslant \alpha,$$

because $G_{s_1,\ldots,s_m}$ is an isometry. Next Lemma 3.15 implies that

$$\frac{\mathrm{mes}(\varpi_m(z) - \tilde{J}_\theta)}{\mathrm{mes}(\varpi_m(z))} \geqslant \alpha \exp(-8R).$$

Finally Lemma 3.14 implies that there exists $r_m > 0$ such that

$$\overline{\mathbf{D}}_{r_m}(z) \subset \varpi_m \subset \mathbf{D}_{r_m \exp(2R)}(z).$$

Hence $\mathrm{Cdens}(z, e^{2R} r_m) \geqslant \alpha \exp(-12R)$. This completes the proof. □

COROLLARY 3.17. *For $\theta \in]0,1[- \mathbf{Q}$ of constant type, the set J_θ has zero Lebesgue measure and so has J_{P_θ}, the Julia set of the quadratic polynomial P_θ.*

Proof. Let $K \subset \mathbf{R}^2$ be a compact set. A point $z \in K$ is called a density point for K if

$$\lim_{r\to 0} \frac{\mathrm{mes}(K \cap B_r(z))}{\mathrm{mes}(B_r(z))} = 1.$$

The Lebesgue density theorem states that: For any compact set $K \subset \mathbf{R}^2$ almost all points of K are density points for K.

Theorem 3.16 implies that $X_0^* = X_0 - \{1\}$ does not contain any density points for J_θ, because $\widetilde{X}_0^*$ does not contain any density points for $\tilde{J}_\theta$ and exp is locally biholomorphic. Thus X_0^* is a null set by the Lebesgue density theorem. Finally $J_\theta = f_\theta(X_0^*) \cup \{v\}$ is a null set because f_θ is holomorphic and injective on X_0^*. □

Controlling the Core of Nests for all irrational θ. This subsection is devoted to spreading local connectivity to all points z with infinite address. The special case θ of constant type was handled in Theorem 3.12. Here we give a slightly different proof independent of the combinatorics of θ. We shall use frequently all of the subsection "Lifting to the exponential" except Theorem 3.12.

For each irrational θ we single out the following set of exceptional addresses:

$$\mathcal{E}_\theta = \{j = kq_n + q_{n-1} \mid n \geqslant 1,\ 0 < k \leqslant a_n\}.$$

We divide $\mathbf{N}^{\mathbf{N}}$ into the following three classes:

$$\begin{aligned}
\mathrm{It}_{EB}(\theta) &= \{(s_1, \ldots, s_m, \ldots) \mid \liminf s_m < \infty\},\\
\mathrm{It}_{RB}(\theta) &= \{(s_1, \ldots, s_m, \ldots) \notin \mathrm{It}_{EB}(\theta) \mid \sup\{m \mid s_m \notin \mathcal{E}\} = \infty\},\\
\mathrm{It}_{SR}(\theta) &= \{(s_1, \ldots, s_m, \ldots) \notin (\mathrm{It}_{EB}(\theta) \cup \mathrm{It}_{RB}(\theta))\}\\
&= \{(s_1, \ldots, s_m, \ldots) \mid s_m \underset{m\to\infty}{\longrightarrow} \infty \text{ and } \exists m_0 : \forall m \geqslant m_0, s_m \in \mathcal{E}\}.
\end{aligned}$$

Addresses in the first two classes can be handled with essentially the information at hand, but in order to handle also the more difficult addresses in $\mathrm{It}_{SR}(\theta)$ we shall obtain a new family of curves by cutting and pasting iterated preimages of the Σ_n.

PROPOSITION AND DEFINITION 3.18. *There exists a family of Jordan curves* $\{\Upsilon_{n,0}\}_{n\geqslant 3}$, $\Upsilon_{n,0} = \Upsilon'_n \cup [1, x_{q_n}]$, *and a constant* $L'_{\Upsilon,\theta} > 0$ *such that*

$$l_{J_{n,0}}(\Upsilon'_n) \leqslant L'_{\Upsilon,\theta} \quad \forall n \geqslant 3, \tag{1}$$

$$Y_{\alpha q_n + q_{n-1}} \subset D(\Upsilon_{n,0}) \quad \forall n \geqslant 3 \textit{ and for each } 0 < \alpha \leqslant a_n. \tag{2}$$

Proof. For $n \geqslant 1$ let $g_n : D(\Sigma_n) \to D(\Gamma_n)$ be the Gain of Σ_n. Then g_n is a local branch of $f_\theta^{-q_{n+1}}$ mapping 1 to $y_{q_{n+1}}$ and $D(\Gamma_n) \subset D(\Sigma_n)$ by Proposition 1.10. If $a_{n+1} > 1$ then the Gain of Γ_n coincides with the restriction of g_n to $D(\Gamma_n)$, because it is a local inverse of $f_\theta^{q_{n+1}}$ mapping 1 to $y_{q_{n+1}}$. Let $H_n = g_n^{a_{n+1}}$,

$$\Sigma_n(x_{q_n}, y_{t_n}, x_{q_n, t_{n-1}}) \xrightarrow[a_{n+1}\text{ Gains}]{H_n} \Xi_{n+2}(x_{q_{n+2}}, y_{q_{n+1}}, x_{q_{n+2}, t_{n-1}}),$$

be the long composition of the a_{n+1} consecutive Gains starting from Σ_n. Essentially the curve Ξ_{n+2} is the curve we want, except that its $\lambda_{J_{n,0}}$-length grows linearly with a_{n+1}. Thus if $a_{n+1} \leqslant 2$ we define $\Upsilon_{n+2,0} = \Xi_{n+2}$ and $\Upsilon'_{n+2} = \Upsilon_{n+2,0} -]1, x_{q_{n+2}}[$. If $a_{n+1} \geqslant 3$ we shall replace the part of Ξ_{n+2} whose $\lambda_{J_{n,0}}$-length is proportional to a_{n+1} by a shortcut of bounded length.

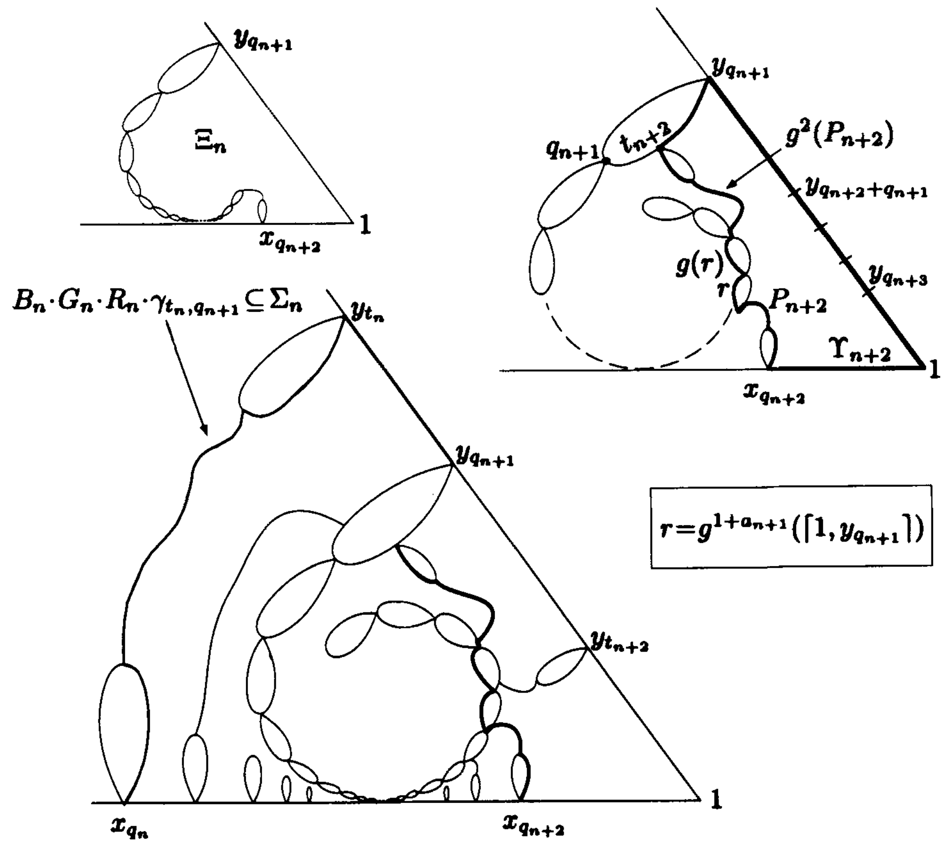

Fig. 23. Constructing the new Jordan curves $\Upsilon_{n,0}$ cutting and pasting bits and pieces

Define $P_{n+2} = H_n(B_n \cdot G_n \cdot R_n \cdot \gamma_{t_n, q_{n+1}})$ if $a_n \neq 1$ and $P_{n+2} = H_n(B_n \cdot G_n \cdot R_n)$ if $a_n = 1$. Thus

$$P_{n+2} = H_n(\Sigma_n - \lceil 1, x_{q_n} \lceil - \lceil 1, y_{q_{n+1}} \lceil)$$

and it starts at $x_{q_{n+2}}$, leaves $\partial U_{q_{n+2}}$ at $x_{q_{n+2}, t_{n-1}}$ and ends at

$$y_{\underbrace{q_{n+1}, \ldots, q_{n+1}}_{1+a_{n+1} \text{ times}}}$$

(see also Figure 23). Suppose that $a_{n+1} \geqslant 3$, and define

$$\begin{aligned} \Upsilon'_{n+2} = P_{n+2} \cdot g_n^{1+a_{n+1}}(\lceil 1, y_{q_{n+1}} \rceil) \cdot g_n^{2+a_{n+1}}(\lceil 1, y_{q_{n+1}} \rceil) \\ \cdot (g_n^2(-P_{n+2})) \cdot g_n(\lceil y_{t_{n+2}}, 1 \rceil) \cdot \lceil y_{q_{n+1}}, 1 \rceil \end{aligned}$$

and $\Upsilon_{n+2,0} = \Upsilon'_n \cup \lceil 1, x_{q_{n+2}} \rceil$. We have $\Upsilon_{n+2,0} \cap J_{f_\theta} \subset Y_{q_{n+1}} \cup \partial U_0 \cup \mathbf{S}^1 \cup X_{q_{n+2}}$ by construction. Thus $Y_{\alpha q_{n+2} + q_{n+1}} \subset D(\Upsilon_{n+2,0})$ for each $0 < \alpha \leqslant a_{n+2}$. This proves property (2).

To prove (1) and thus the existence of $L'_{\Upsilon,\theta}$ we prove that there exists a universal constant L'_{Υ}, i.e. independent of θ such that

$$\limsup l_{J_{n+2,0}}(\Upsilon'_{n+2,0}) \leqslant L'_{\Upsilon}. \tag{3}$$

We shall prove (3) by proving that the $\lambda_{J_{n+2,0}}$-lengths of the 6 constituent subarcs of $\Upsilon'_{n+2,0}$ are asymptotically universally bounded. Whenever a constituent is compactly contained in $\mathbf{C}-\overline{\mathbf{D}}$ it suffices to give a bound for its λ-length, because $\mathbf{C}-\overline{\mathbf{D}}\subset A_{J_{n,0}}$. Moreover, g_n is infinitesimally contracting with respect to λ, so it suffices to give bounds for the following four lengths:

$$l_{J_{n+2,0}}(P_{n+2}), \tag{4}$$
$$l_\lambda(g_n(\lceil 1, y_{q_{n+1}}\rceil)) \quad (\lceil 1, y_{t_{n+2}}\rceil \subset \lceil 1, y_{q_{n+1}}\rceil), \tag{5}$$
$$l_\lambda(g_n(P_{n+2})), \tag{6}$$
$$l_{J_{n+2,0}}(\lceil 1, y_{q_{n+1}}\rceil). \tag{7}$$

The careful reader may easily verify that this also suffices to cover the cases $a_{n+1}\leqslant 2$. We shall concentrate on (4) and (6), as the two others essentially are treated in Lemma 3.3 and the proof of Theorem 2.2 (4). Furthermore, $P_{n+2}=H_n(B_n)\cdot H_n(G_n\cdot R_n\cdot\gamma_{t_n,q_{n+1}})$ with $\gamma_{t_n,q_{n+1}}$ possibly being a point. As we have asymptotically universal bounds for $l_\lambda(G_n\cdot R_n\cdot\gamma_{t_n,q_{n+1}})$ (for $\gamma_{t_n,q_{n+1}}$ see Proposition and Definition 3.8) we are left with only $l_{J_{n+2,0}}(H_n(B_n))$ and $l_\lambda(g_n(H_n(B_n)))$.

Note that $f_\theta^{q_{n+2}}$ maps $H_n(B_n)$ diffeomorphically onto $\lceil 1, y_{t_{n-1}}\rceil\subset Q_{n-2}$, because $B_n=\lceil x_{q_n}, x_{q_n,t_{n-1}}\rceil\subset\partial U_{q_n}$ by construction of Σ_n (recall Definition 2.7). Moreover, $f_\theta^{q_{n+2}}$ maps the arc $K'_{n+2}=\lceil 1, x_{-q_{n+3}+q_{n+2}}\rceil\subset J_{n+2,0}$ diffeomorphically onto K_{n+2}. Let ϱ_{n+2} denote the hyperbolic metric on $W_{J_{n+2,q_{n+1}}}$ and note that $f_\theta^{q_{n+1}}\circ g_n=\mathrm{Id}$. Combining Lemma 3.2 with Proposition 2.8 and Lemma 2.9 we obtain

$$\begin{aligned} l_{\varrho_{n+2}}(g_n(H_n(B_n))) &\leqslant l_{J_{n+2,0}}(H_n(B_n)) \\ &< l_{K'_{n+2}}(H_n(B_n)) \leqslant l_{K_{n+2}}(Q_{n-2}) \leqslant L_{4,\theta}. \end{aligned} \tag{8}$$

This takes care of $l_{J_{n+2,0}}(H_n(B_n))$. To obtain a bound for $l_\lambda(g_n(H_n(B_n)))$ from (8) we combine Lemma 2.4 with Lemma 2.5 as in the proof of Lemma 3.6. This completes the proof. □

Definition 3.19. For $1\leqslant n$ and $0<j<q_{n+1}$ let $\Upsilon_{n,j}$ be the unique lift of $\Upsilon_{n,0}$ to f_θ^j intersecting ∂U_0. Note that the previously defined arc $I_{n,0}$ equals both the I of $\Sigma_{n,0}$ and the intersection $\Upsilon_{n,0}\cap\mathbf{S}^1$ for $n\geqslant 3$. Let $I'_{n,j}=\Upsilon_{n,j}\cap f_\theta^{-j}(I_{n,0})$ for each $n\geqslant 3$ and $0<j<q_{n+1}$.

LEMMA 3.20. *For each* $\theta\in]0,1[-\mathbf{Q}$ *there exist constants* $L'_{\Upsilon,\theta}, L_{\Upsilon,\theta}>0$ ($L'_{\Upsilon,\theta}$ *equals the constant of Proposition and Definition* 3.18) *such that* $\forall n\geqslant 3$ *and* $0<j<q_{n+1}$,

$$l_{J_{n,j-1}}(f_\theta(\Upsilon_{n,j}-I'_{n,j}))\leqslant L'_{\Upsilon,\theta}, \tag{1}$$

$$l_\lambda(\Upsilon_{n,j})\leqslant L_{\Upsilon,\theta}. \tag{2}$$

Proof. The proof is a simple copy of the proof of Lemma 3.6 and is left to the reader. □

For $n\geqslant 1$ and $0<j<q_{n+1}$ let $\tilde{\Delta}_{n,j}$ be the connected component of $\exp^{-1}(\Delta_{n,j})$ intersecting $\partial\widetilde{U}_0$. Moreover, for $n\geqslant 3$ and $0<j<q_{n+1}$ let $\widetilde{\Upsilon}_{n,j}$ be the connected component of $\exp^{-1}(\Upsilon_{n,j})$ intersecting $\partial\widetilde{U}_0$. Recall the definition of $\Omega(T)$, $T>0$, from the subsection "Lifting to the exponential".

LEMMA 3.21. *For each* $\theta\in]0,1[-\mathbf{Q}$ *there exist constants*

$$T_{\Delta,\theta},\ T_{\Upsilon,\theta}\geqslant T_C=\log\bigl(1+\sqrt{2}\bigr)$$

such that

$$\mathring{D}(\tilde{\Delta}_{n,j})\subset\Omega(T_{\Delta,\theta})\quad \forall n\geqslant 1,\ 0<j<q_{n+1}, \tag{1}$$

$$\mathring{D}(\widetilde{\Upsilon}_{n,j})\subset\Omega(T_{\Upsilon,\theta})\quad \forall n\geqslant 3,\ 0<j<q_{n+1}. \tag{2}$$

Proof. We prove (1) and leave the similar proof of (2) to the reader. Let $\tilde{J}_{n,j-1}$ be the connected component of $\exp^{-1}(J_{n,j-1})$ contained in J_0 and let $\tilde{I}'_{n,j}=\exp^{-1}(I'_{n,j})\cap\tilde{\Delta}_{n,j}$. We shall prove the following slightly better statement, from which (1) follows. Let $\delta_{n,j-1}$ denote the hyperbolic metric on the doubly slit plane $\mathbf{C}_{\tilde{J}_{n,j-1}}$. There exists $T_{\Delta,\theta}>0$ such that

$$l_{\delta_{n,j-1}}(\tilde{f}_\theta(\tilde{\Delta}_{n,j}-\tilde{I}'_{n,j}))\leqslant 2T_{\Delta,\theta},\quad 1\leqslant n,\ 0<j<q_{n+1}. \tag{3}$$

To prove (3) it suffices to prove that

$$\limsup_{n\to\infty} l_{\delta_{n,j-1}}(\tilde{f}_\theta(\tilde{\Delta}_{n,j}-\tilde{I}'_{n,j}))\leqslant\limsup_{n\to\infty} l_{J_{n,j-1}}(\Sigma_{n,j-1}-I_{n,j-1})\leqslant L_{\Sigma,\theta}, \tag{4}$$

because the length $l_{\delta_{n,j-1}}(\tilde{f}_\theta(\tilde{\Delta}_{n,j}))$ is always finite. Let $\tilde{A}_{n,j-1}=\exp^{-1}(A_{J_{n,j-1}})$ and let $\tilde{\lambda}_{n,j-1}$ denote the hyperbolic metric on $\tilde{A}_{n,j-1}$. Then

$$l_{\tilde{\lambda}_{n,j-1}}(\tilde{f}_\theta(\tilde{\Delta}_{n,j}-I'_{n,j}))=l_{J_{n,j-1}}(\Sigma_{n,j-1}-I_{n,j-1})\leqslant L_{\Sigma,\theta}, \tag{5}$$

by Lemma 3.6 and because the restriction $\exp\colon\tilde{A}_{n,j-1}\to A_{J_{n,j-1}}$ is a local hyperbolic isometry. Moreover, by the same argument

$$d_{\tilde{\lambda}_{n,j-1}}(\tilde{f}_\theta(\tilde{\Delta}_{n,j}),\partial\mathbf{C}_{\tilde{J}_{n,j-1}})\geqslant E(l_e(J_{n,j-1}))-L_{\Sigma,\theta}. \tag{6}$$

The right hand side of (6) diverges to ∞ as n diverges to ∞, by Lemma 3.4. We obtain (4) by combining this fact with Lemma 2.4 and (5). Recall Lemma 3.10. □

LEMMA 3.22. *For each* $\theta\in]0,1[-\mathbf{Q}$ *there exist constants* $L_{N,\theta}>0$ *and* $T_{N,\theta}\geqslant T_C$, $1\leqslant N$, *such that for each* $0<s\leqslant N$,

$$\operatorname{diam}_{\tilde\lambda}(\widetilde{Y}_s)\leqslant L_{N,\theta}, \tag{1}$$

and moreover, for all compact subsets $K\subset\widetilde{X}_0^*$,

$$\operatorname{diam}_{\tilde\lambda}(G_s(K))\leqslant M(T_{N,\theta})\cdot\operatorname{diam}_{\tilde\lambda}(K). \tag{2}$$

Proof. The sets Y_s, $0<s\leqslant N$, are compact subsets of $\mathbf{H}_+$. This proves the existence of L_N as in (1). We have $\bigcup_{s=1}^N \widetilde{X}_{s-1}\subset\omega(T_{N,\theta})$ for $T_{n,\theta}$ sufficiently big, because the former is a compact subset of $\mathbf{C}_0$. Moreover, $\widetilde{Y}_s=G_s(\widetilde{X}_0)=G_\theta(\widetilde{X}_{s-1})$ for each $s=1,...,N$, and in particular $Y_s^*\subset\Omega(T_{N,\theta})$. Increasing $T_{N,\theta}$ if necessary we can suppose that $T_C\leqslant T_{N,\theta}$. Since G_s is Lipschitz with constant 1 for the hyperbolic metrics $\tilde\lambda$ and $\tilde\varrho$ we have

$$\operatorname{diam}_{\tilde\varrho}(G_s(K))\leqslant\operatorname{diam}_{\tilde\lambda}(K)$$

for any compact subset $K\subset X_0^*$. Furthermore, Lemma 3.10 implies

$$\operatorname{diam}_{\tilde\lambda}(G_s(K))\leqslant M(T_{N,\theta})\cdot\operatorname{diam}_{\tilde\varrho}(G_s(K)).$$ □

Recall that $\mathcal{E}=\{j=kq_n+q_{n-1}\,|\,n\geqslant1,\,0<k\leqslant a_n\}$.

LEMMA 3.23. *Suppose that* $s\notin\mathcal{E}$ *and* $s>q_2+1$. *Then*

$$\operatorname{diam}_{\tilde\lambda}(\widetilde{Y}_s)\leqslant L_{\Delta,\theta} \tag{1}$$

and for all compact subsets $K\subset\widetilde{X}_0^*$,

$$\operatorname{diam}_{\tilde\lambda}(G_s(K))\leqslant M(T_{\Delta,\theta})\cdot\operatorname{diam}_{\tilde\lambda}(K). \tag{2}$$

Proof. Let $q_2+1<s\notin\mathcal{E}$ be arbitrary, so that s is of the form $s=\alpha q_{n+1}+q_n+j$ for some $n\geqslant1$, $0<\alpha\leqslant a_{n+1}$ and $0<j<q_{n+1}$. As $Y_{\alpha q_{n+1}+q_n+j}\subset D(\Delta_{n,j})$ and the covering map $\exp\colon\mathbf{H}_+\to\mathbf{C}-\overline{\mathbf{D}}$ is a hyperbolic isometry, we obtain (1) from Lemma 3.6 and we obtain

$$\widetilde{Y}^*_{\alpha q_{n+1}+q_n+j}=G_{\alpha q_{n+1}+q_n+j}(\widetilde{X}_0^*)\subset\Omega(T_{\Delta,\theta})$$

by Lemma 3.21 (1). Applying Lemma 3.10 as in the above proof of Lemma 3.22 we obtain (2). □

LEMMA 3.24. *For $s_2=\alpha q_n+q_{n-1}\in\mathcal{E}$, $n\geqslant 3$, $0<\alpha\leqslant a_n$ and $s_1<s_2$, we have*

$$\operatorname{diam}_{\tilde\lambda}(\widetilde{Y}_{s_1,s_2})\leqslant L_{\Upsilon,\theta} \tag{1}$$

and for all compact subsets $K\subset\widetilde{Y}_{s_2}$,

$$\operatorname{diam}_{\tilde\lambda}(G_{s_1}(K))\leqslant M(T_{\Upsilon,\theta})\cdot\operatorname{diam}_{\tilde\lambda}(K). \tag{2}$$

Proof. We have $s_1<s_2=\alpha q_n+q_{n-1}\leqslant q_{n+1}$, $0<\alpha\leqslant a_n$ and $Y_{s_2}\subset D(\Upsilon_{n,0})$, which implies that $Y_{s_1,s_2}\subset D(\Upsilon_{n,s_1})$. From here and onwards the proof goes as in the previous lemma, except that we use Lemma 3.20 to obtain (1) and Lemma 3.21 (2) to obtain (2). □

THEOREM 3.25. *For each $\theta\in\,]0,1[-\mathbf{Q}$, $\operatorname{Core}(\mathcal{Y}_{\theta,\underline{s}})$ is trivial for any $\underline{s}\in\mathbf{N}^{\mathbf{N}}$. In particular, J_θ is locally connected for each irrational θ.*

Proof. Let $\theta\in\,]0,1[-\mathbf{Q}$ be given. We shall prove the following equivalent statement of the theorem: Let $\underline{s}=(s_1,...,s_m,...)$ be arbitrary. There exists $\tilde z_{\underline{s}}\in\mathbf{H}_+$ such that

$$\bigcap_{m\geqslant 1}\widetilde{Y}_{s_1,...,s_m}=\{\tilde z_{\underline{s}}\}.$$

We shall treat the three types of addresses $\operatorname{It}_{EB}(\theta)$, $\operatorname{It}_{RR}(\theta)$ and $\operatorname{It}_{SR}(\theta)$ separately. Suppose first that $\underline{s}\in\operatorname{It}_{EB}(\theta)$ and let $N=\liminf s_m$. Let $L_{N,\theta}$ and $T_{N,\theta}$ be constants as in Lemma 3.22. Define $M_N=M(T_{N,\theta})$, where $M(\cdot)$ is the function of Lemma 3.10. Moreover, define $\chi(m)=\#\{l<m\,|\,s_l\leqslant N\}$, where $\#(\cdot)$ denotes the cardinality. Then for any m with $s_m\leqslant N$,

$$\operatorname{diam}_{\tilde\lambda}(Y_{s_1,...,s_m})=\operatorname{diam}_{\tilde\lambda}(G_{s_1}\circ...\circ G_{s_{m-1}}(Y_{s_m}))\leqslant L_{N,\theta}\cdot(M_N)^{\chi(m)}\underset{m\to\infty}{\longrightarrow}0.$$

This proves the case $\underline{s}\in\operatorname{It}_{EB}(\theta)$.

Suppose next that $\underline{s}\in\operatorname{It}_{RR}(\theta)$. Shifting $\underline{s}$ some number of times if necessary, we can suppose that $s_m>q_2+1$ for all m, so that Lemma 3.23 applies whenever $s_m\notin\mathcal{E}$. Define $M_\Delta=M(T_{\Delta,\theta})$ and $\chi(m)=\#\{l<m\,|\,s_l\notin\mathcal{E}\}$. Then for any m with $s_m\notin\mathcal{E}$,

$$\operatorname{diam}_{\tilde\lambda}(Y_{s_1,...,s_m})=\operatorname{diam}_{\tilde\lambda}(G_{s_1}\circ...\circ G_{s_{m-1}}(Y_{s_m}))\leqslant L_{\Delta,\theta}\cdot(M_\Delta)^{\chi(m)}\underset{m\to\infty}{\longrightarrow}0.$$

This proves the case $\underline{s}\in\operatorname{It}_{RR}(\theta)$.

Finally suppose that $\underline{s}\in\operatorname{It}_{SR}(\theta)$. Shifting $\underline{s}$ some number of times if necessary, we can suppose that $s_m\in\mathcal{E}$ and $s_m>q_3$ for all m.

Let $L=L_{\Upsilon,\theta}$ and $M=M(T_{\Upsilon,\theta})$. Define $\chi(m)=\#\{l<m-1\,|\,s_l>s_{l-1}\}$. Then for any m with $s_m>s_{m-1}$,

$$\operatorname{diam}_{\tilde\lambda}(Y_{s_1,...,s_m})=\operatorname{diam}_{\tilde\lambda}(G_{s_1}\circ...\circ G_{s_{m-2}}(Y_{s_{m-1},s_m}))\leqslant L\cdot M^{\chi(m)}\underset{m\to\infty}{\longrightarrow}0.$$

This proves the case $\underline{s}\in\operatorname{It}_{SR}(\theta)$ and completes the proof. □

THEOREM 3.26 (1). *For all $\theta\in]0,1[-\mathbf{Q}$ and for all $z\in Y_\theta$ we have*

$$\|D_zF_\theta\|_\lambda=\frac{\lambda(F_\theta(z))}{\lambda(z)}|F'_\theta(z)|>1 \quad \textit{and} \quad \|D_zF_\theta^m\|_\lambda \underset{m\to\infty}{\longrightarrow}\infty.$$

Proof. Let $\theta\in]0,1[-\mathbf{Q}$ be given. The first inequality follows from

$$\|D_zf_\theta\|_\lambda=\|D_zf_\theta\|_{\lambda,\varrho}\cdot\frac{\varrho(z)}{\lambda(z)}=\frac{\varrho(z)}{\lambda(z)}>1 \quad \forall z\in W_1,$$

and the definition of F_θ. For each $T>0$ we have $\varrho(z)/\lambda(z)\geqslant 1/M(T)$ for all $z\in\exp(\Omega(T))$ (recall Lemma 3.10), because exp is a local isometry for both the pair of metrics $\tilde\lambda,\lambda$ and $\tilde\varrho,\varrho$.

We proceed to prove the second part of the theorem. Let $z\in Y_\theta$ with itinerary $(s_1,...,s_m,...)=\underline{s}\in\mathbf{N}^\mathbf{N}$ be arbitrary. We shall consider separately the three cases of $\underline{s}$ belonging to the three different classes of addresses $\mathrm{It}_{EB},\mathrm{It}_{RR},\mathrm{It}_{SR}$.

Suppose first that $\underline{s}\in\mathrm{It}_{EB}$. Let $N=\liminf s_m$ and let $T_{N,\theta}$ be as in the proof of Lemma 3.22, so that $Y_s\in\exp(\Omega(T_N))$ for all $0<s\leqslant N$. Let $M=M(T_{N,\theta})$, where $M(\cdot)$ is the function of Lemma 3.10. Moreover, define $\chi(m)=\#\{l\leqslant m\,|\,s_l\leqslant N\}$, where $\#(\cdot)$ denotes the cardinality. Then

$$\|D_zF_\theta^m\|_\lambda\geqslant M^{-\chi(m)}\underset{m\to\infty}{\longrightarrow}\infty. \tag{1}$$

This proves the case $\underline{s}\in\mathrm{It}_{EB}(\theta)$.

Secondly suppose that $\underline{s}\in\mathrm{It}_{RR}(\theta)$. Shifting $\underline{s}$ some number of times if necessary, we can suppose that $s_m>q_2+1$ for all m. Recall from the proof of Lemma 3.23 that $Y_s\subset\exp(\Omega(T_{\Delta,\theta}))$ for all $s\notin\mathcal{E}$. Define $M=M(T_{\Delta,\theta})$ and $\chi(m)=\#\{l\leqslant m\,|\,s_l\notin\mathcal{E}\}$. Then (1) holds again. This proves the case $\underline{s}\in\mathrm{It}_{RR}(\theta)$.

Finally suppose that $\underline{s}\in\mathrm{It}_{SR}(\theta)$. Shifting $\underline{s}$ some number of times if necessary, we can suppose that $s_m\in\mathcal{E}$ and $s_m>q_3$ for all m. Recall from the proof of Lemma 3.24 that $Y_{s_1,s_2}\subset\exp(\Omega(T_{\Upsilon,\theta}))$ whenever $s_1<s_2\in\mathcal{E}$ and $s_2>q_3$.

Let $M=M(T_{\Upsilon,\theta})$. Define $\chi(m)=\#\{l\leqslant m\,|\,s_l<s_{l+1}\}$. Then (1) holds again. This proves the case $\underline{s}\in\mathrm{It}_{SR}(\theta)$ and completes the proof. □

4. Local connectivity of J_f

Let Z_θ be the subset of J_{f_θ} consisting of those points which pass infinitely often through $U_+=U_0$ and $U_-=\tau(U_+)$. In this section we shall prove the following theorem.

THEOREM 4.1. *For all $\theta\in]0,1[-\mathbf{Q}$ any point of Z_θ has a fundamental system of connected neighbourhoods in J_{f_θ}.*

For $s\geqslant 1$ we define $Z_{+s}=f_\theta^{-1}(X_{-(s-1)})\cap\overline{U}_0$ and $Z_{-s}=\tau(Z_{+s})$. The sets Z_{-s} and Z_{+s} are connected and $y_s\in Z_{+s}$. We shall rename y_s to y_{+s} and define $y_{-s}=\tau(y_{+s})$. We shall say that Z_{+s} is the internal limb of U_+ with root y_{+s} and that Z_{-s} is the internal limb of U_- with root y_{-s}.

Rename the hyperbolic metric λ on $\mathbf{C}-\mathbf{D}$ to λ_+ and let λ_- denote the hyperbolic metric on $\mathbf{D}^*=\mathbf{D}-\{0\}$. Moreover, let δ_+ denote the hyperbolic metric on $U_+^*=U_+-f_\theta^{-1}(0)$ and let δ_- denote the hyperbolic metric on $U_-^*=U_--f_\theta^{-1}(\infty)$. We note immediately that τ is an isometry with respect to both of the pairs of metrics $\lambda_\pm$ and $\delta_\pm$.

LEMMA 4.2. *The local inverse branches $f_\theta^{-1}\colon\mathbf{D}^*\to U_{+0}^*$ and $f_\theta^{-1}\colon\mathbf{C}-\overline{\mathbf{D}}\to U_{-0}^*$ are strong contractions with respect to the pair of hyperbolic metrics λ_+ and λ_-. More precisely, there exists a constant $0<C<1$ such that*

$$\|D_z f_\theta^{-1}\|_{\lambda_+,\lambda_-}\leqslant C\cdot\|D_z f_\theta^{-1}\|_{\delta_+,\lambda_-}=C\quad\forall z\in\mathbf{D}^*$$

and

$$\|D_z f_\theta^{-1}\|_{\lambda_-,\lambda_+}\leqslant C\|D_z f_\theta^{-1}\|_{\delta_-,\lambda_+}=C\quad\forall z\in\mathbf{C}-\overline{\mathbf{D}}.$$

Proof. Let us prove that there exists a constant $0<C<1$ such that $\lambda_+(z)/\delta_+(z)\leqslant C$ for all $z\in U_+$ and $\lambda_-(z)/\delta_-(z)\leqslant C$ for all $z\in U_-$. These two inequalities are equivalent because τ is a hyperbolic isometry.

The first inequality, say, follows by observing that $U_+^*\subset\mathbf{C}-\overline{\mathbf{D}}$ and observing that the boundary of U_+ makes an angle of $\frac{1}{3}\pi$ with $\mathbf{S}^1$ at 1, their unique point of intersection. □

LEMMA 4.3. *For each $\theta\in]0,1[-\mathbf{Q}$ there exists a constant $L_\theta>0$ such that for all $s\geqslant 1$,*

$$\operatorname{diam}_{\lambda_+}(Z_{+s})=\operatorname{diam}_{\lambda_-}(Z_{-s})\leqslant L_\theta.$$

Proof. The equality sign follows from τ being an isometry with respect to the pair of metrics λ_+ and λ_- and $\tau(Z_{+s})=Z_{-s}$.

We shall thus concentrate on giving an absolute upper bound for $\operatorname{diam}_{\lambda_-}(Z_{-s})$. For $n\geqslant 2$ write $\Sigma_n=\Sigma_n(I_n,B_n,G_n,R_n,O_n)$ where $\Sigma_n=\Sigma_{n,0}$ is the arc defined in §3. Define an arc $\Xi_n\subset\overline{U}_-$ by $f_\theta(\Xi_n)=f_\theta(B_n\cup G_n\cup R_n)$, so that Ξ_n connects y_{-q_n} and y_{-t_n} in U_-. Essentially repeating the arguments of Lemma 3.3 we find that

$$\limsup_{n\to\infty} l_{\lambda_-}(\Xi_n)\leqslant L_1+L_2+C(4L_{R,\theta}+L_{G,\theta}).\tag{1}$$

(Let $g:\Gamma(x_{q_{n-1}}, y_l)\to\Sigma_n(x_{q_n}, y_{t_n})$ be the last move in obtaining Σ_n. Let h be the long composition of inverse branches of f_θ in the construction of g. Consider instead of g, the composition of h with the previously unused inverse branch of f_θ, which maps $\overline{\mathbf{C}}-\mathbf{D}$ into $\overline{U}_-$. The details are left to the reader.)

We extend the notion $\lceil\cdot,\cdot\rceil$ to include also subarcs of U_- as well (as $\mathbf{S}^1$ and U_+ where it has previously been defined).

Let $D_n\subset\overline{U}_-$, $n\geqslant 1$, be topological discs defined as follows. For $n=1$ we take D_1 to be bounded by Ξ_4 and the subarc $\lceil y_{-q_4}, y_{-t_4}\rceil\subset\partial U_-$ not containing 1. For $n\geqslant 2$ take D_n to be bounded by Ξ_n, $\tau(\gamma_{t_n,q_{n+3}})$, Ξ_{n+3} and $(-\tau(\gamma_{q_n,t_{n+3}}))$, where $\gamma_{l,l'}\subseteq\partial U_-$ are the curves of Proposition and Definition 3.8. Then it follows from (1) that

$$\limsup_{n\to\infty}\operatorname{diam}_{\lambda_-}(D_n)\leqslant L_2'+L_1+L_2+C(4L_{R,\theta}+L_{G,\theta}).$$

In particular, there exists a constant L_θ such that $\operatorname{diam}_{\lambda_-}(D_n)\leqslant L_\theta$ for all n, as each D_n is a compact subset of $\mathbf{D}^*$ and so has finite λ_--diameter. Next we easily check that any limb Z_{-s}, $s\geqslant 1$, is contained in at least one D_n, so that its λ_--diameter is at most L_θ. □

Proof of Theorem 4.1. Let $z\in Z_\theta$ be arbitrary. Let $n_1\geqslant 0$ be minimal with the property $z_1:=f_\theta^{n_1}(z)\in U_+\cup U_-$. Replacing z by $\tau(z)$ we can suppose that $z_1\in\overline{U}_+$. Hence $z_1\in Z_{+s_1}$ for some $s_1\geqslant 1$. Let $n_2\geqslant 1$ be minimal with $z_2:=f_\theta^{n_2}(z_1)\in U_-$ and let s_2 be given by $z_2\in Z_{-s_2}$. Define inductively $n_j\geqslant 1$, $z_j\in U_+\cup U_-$ and $s_j\geqslant 1$ by n_j being minimal with

$$z_j:=f_\theta^{n_j}(z_{j-1})\in\begin{cases} Z_{+s_j}\subset\overline{U}_+ & \text{if and only if } j \text{ is odd,}\\ Z_{-s_j}\subset\overline{U}_- & \text{if and only if } j \text{ is even.}\end{cases}$$

Next let ϖ_j denote the connected component, containing z, of $f_\theta^{-(n_1+n_2+\ldots+n_j)}(Z_{+s_j})$ if j is odd and of $f_\theta^{-(n_1+n_2+\ldots+n_j)}(Z_{-s_j})$ if j is even. Then each ϖ_j is a connected neighbourhood of z in J_{f_θ}. Moreover, applying first Lemma 4.3 and then Lemma 4.2 we obtain

$$\operatorname{diam}_{\lambda_+}(\varpi_j)\leqslant L_\theta\cdot C^j\underset{j\to\infty}{\longrightarrow}0,$$

which implies that the sequence $\{\varpi_j\}_{j\geqslant 1}$ forms a fundamental system of connected neighbourhoods of z in J_{f_θ}. As $z\in Z_\theta$ was arbitrary we have proved the theorem. □

References

[Be] BEARDON, A. F., *Iteration of Rational Functions.* Graduate Texts in Math., 132. Springer-Verlag, New York–Berlin, 1991.

[BH] BRANNER, B. & HUBBARD, J. H., The iteration of cubic polynomials, Part II: Patterns and parapatterns. *Acta Math.*, 169 (1992), 229–325.

[CG] Carleson, L. & Gamelin, T. W., *Complex Dynamics*. Universitext: Tracts in Mathematics. Springer-Verlag, New York, 1993.

[Do] Douady, A., Disques de Siegel et anneaux de Herman. *Sém. Bourbaki*, 39ème année, 1986/87, n° 677.

[He] Herman, M. R., Conjugaison quasi symmetrique des homéomorphismes analytiques du cercle a des rotations. Preliminary manuscript.

[Hu] Hubbard, J. H., Local connectivity of Julia sets and bifurcation loci: three theorems by Yoccoz, in *Topological Methods in Modern Mathematics* (*Stony Brook, NY,* 1991), pp. 467–511. Publish or Perish, Houston, TX, 1993.

[Ke] Keller, K., Symbolic dynamics for angle-doubling on the circle III. Sturmian sequences and the the quadratic map. *Ergodic Theory Dynamical Systems*, 14 (1994), 787–805.

[LV] Lehto, O. & Virtanen, K. I., *Quasiconformal Mappings in the Plane*, 2nd edition. Grundlehren Math. Wiss., 126. Springer-Verlag, New York–Berlin, 1973.

[Mc] McMullen, C. T., Self-similarity of Siegel disks and Hausdorff dimension of Julia sets. Manuscript, Univ. of California, Berkeley, CA, October 1995.

[Si] Siegel, L., Iteration of analytic functions. *Ann. of Math.* (*2*), 43 (1942), 607–612.

[St] Steinmetz, N., *Rational Iteration. Complex Analytic Dynamical Systems.* de Gruyter Stud. Math., 16. de Gruyter, Berlin, 1993.

[Su] Sullivan, D., Bounds, quadratic differentials, and renormalization conjectures, in *American Mathematical Society Centennial Publications, Vol.* II (*Providence, RI,* 1988), pp. 417–466. Amer. Math. Soc., Providence, RI, 1992.

[Sw] Świątec, G., Rational rotation numbers for maps of the circle. *Comm. Math. Phys.*, 119 (1988), 109–128.

[TY] Tan, L. & Yin, Y., Local connectivity of the Julia set for geometrically finite rational maps. Preprint, École Normale Supérieure de Lyon, UMPA–94–n° 121, 1994. To appear in *Acta Math. Sinica.*

[Ya] Yampolsky, M., Complex bounds for critical circle maps. Preprint, SUNY, StonyBrook, Institute for Mathematical Sciences, #1995/12.

[Yo1] Yoccoz, J.-C., Il n'y a pas de contre-exemple de Denjoy analytique. *C. R. Acad. Sci. Paris Sér. I Math.*, 298 (1984), 141–144.

[Yo2] — Structure des orbites des homéomorphismes analytiques possedant un point critique. Manuscript.

Carsten Lunde Petersen
IMFUFA
Roskilde University
Postbox 260
DK-4000 Roskilde
Denmark
lunde@mmf.ruc.dk

Received June 27, 1994
Received in revised form May 22, 1996