

Torus embeddings and deformations of simple singularities of space curves

by

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Introduction

Let $(X_0, 0)$ be the germ of a complex analytic complete intersection with isolated singularity. A deformation of $(X_0, 0)$ is a flat holomorphic map germ

$$(X, 0) \xrightarrow{f} (S, 0)$$

together with an isomorphism of $(X_0, 0)$ with the special fibre $(f^{-1}(0), 0)$. The subgerm $(D, 0) \subset (S, 0)$ parametrizing the singular fibres of f is called the discriminant of f ; it is an analytic subgerm of $(S, 0)$.

As is well-known $(X_0, 0)$ admits a semi-universal deformation f : if X_0 is given by equations then f may be written down explicitly using the criterion of Kas and Schlessinger [20]. Nevertheless, even for quite simple X_0 the explicit form of f offers little help towards understanding the geometry of the deformation. A natural question

to ask would concern the exact position of $(X_0, 0)$ in the hierarchy of singularities: given a particular $(X_0, 0)$, can the singularities of the fibres of f be listed in terms of the classification of singularities (supposed to be known so far as to include all types that might conceivably occur in a fibre of f)? This is the so-called adjacency problem. A refined question of the same type would consider not individual singular points but the constellations of singularities that occur in one and the same fibre of f . There is not likely to be any feasible general method to extract such information from the equations describing f .

On the other hand it has turned out that the discriminant $(D, 0)$ is a very fine invariant of the semi-universal deformation f , and, a fortiori, of the singularity $(X_0, 0)$ itself. Indeed it was shown in Wirthmüller [53] that $(X_0, 0)$ is essentially determined by the analytic isomorphism class of the germ $(D, 0)$. This in particular allows to decide questions of adjacency once the geometry of $(D, 0)$ is understood sufficiently well. The discriminant of a semi-universal deformation is an apparently simple object conceptually—a reduced hypersurface in the smooth germ $(S, 0)$, and there are a few general results on such discriminants, see Vohmann [50], Greuel and Lê [14] and Teissier [44]. In view of the discussion above though, it is not surprising that a satisfactory understanding of the discriminant has been achieved only in special cases which, in fact, constitute the very beginning of the classification of complete intersections with isolated singularity.

The best-known of these cases is that of simple hypersurface singularities, which were first considered and classified by Arnol'd [1]. From the point of view of deformation theory the dimension of a hypersurface singularity may be arbitrarily increased by suspension, and all simple hypersurface singularities may be realized as surface singularities in three-space. They then turn out to coincide with the class of Kleinian singularities alias rational double points, see du Val [49] and Artin [3]. For these singularities Brieskorn achieved a good description of the discriminant, as follows (Brieskorn [6]). Let $f: X \rightarrow S$ be a suitable global representative of the semiuniversal deformation. The fundamental group of the complement $S \setminus D$ acts as a monodromy group on the (second) Milnor homology H of X_0 . The associated monodromy group is a finite subgroup of $GL(H)$ and there corresponds a finite Galois cover

$$(S \setminus D)^{\sim} \xrightarrow{\varrho} S \setminus D$$

that trivializes the monodromy action. ϱ extends as a branched cover $\tilde{S} \xrightarrow{\varrho} S$, and the extended cover turns out to be a well-known object in the theory of root systems. In fact, if $R \subset V$ is the abstract root system corresponding to the hypersurface singularity in

Arnol'd's classification then \bar{S} may be identified with the complex vector space $V_{\mathbb{C}} = \mathbb{C} \otimes_{\mathbb{R}} V$ in such a way that the monodromy group acts on $V_{\mathbb{C}}$ as the Weyl group W of R . Thus we have a diagram

$$\begin{array}{ccc} V_{\mathbb{C}} & \xrightarrow{\cong} & \bar{S} \\ \downarrow & & \downarrow \varrho \\ V_{\mathbb{C}}/W & \xrightarrow{\cong} & S \end{array}$$

which identifies ϱ with the quotient morphism of the natural W -action on $V_{\mathbb{C}}$. Under this identification the discriminant $D \subset S$ corresponds to the set of singular W -orbits, and this set is covered by the union of all reflection hyperplanes in $V_{\mathbb{C}}$. Thus at the cost of passing to a finite branched Galois cover the discriminant is expressed in terms of linear, in fact even combinatorial data. Likewise the configuration of singularities in a fibre $X_{\varrho(r)}$ ($r \in \bar{S}$) is given by a conjugacy class of isotropy groups of W , and isotropy groups correspond to the various intersections of reflection hyperplanes in $V_{\mathbb{C}}$. Up to W -conjugacy these intersections are classified by the full subdiagrams of the Dynkin diagram, and one arrives at the popular diagram rule that governs the occurrence of singular points in the fibres of f . The unbranched cover $(S \setminus D) \xrightarrow{\sim} S \setminus D$ also is a useful tool in investigating the topology of the complement of the discriminant. The fundamental group turns out to be an Artin group—see Brieskorn and Saito [8], and Deligne [9]. In the same paper Deligne has shown that the higher homotopy groups of $S \setminus D$ vanish.

Brieskorn's construction actually shows a deeper connection between simple singularities and simple Lie algebras; his proofs and results were greatly simplified and improved by Slodowy [42], [43]. For the more limited purpose discussed above—the description of D as the branch locus of a finite Galois cover—a different proof was given by Looijenga [26], [27]. In his proof the root system $R \subset V$ occurs naturally as the set of vanishing cycles in the second cohomology group of the Milnor fibre while the connection between $V_{\mathbb{C}}$ and \bar{S} is provided by a period mapping.

Going up in the classification of hypersurface singularities—see the lists of Arnol'd [2]—we note that all but the simple ones have infinite monodromy groups, and straightforward extension of the result above would lead to Galois covers with branching of infinite order. Nevertheless it was proposed by Looijenga to study the open part $S_f \subset S$ that parametrizes fibres with only simple singularities, hence finite (local) monodromy groups. If the global geometry of S_f could be understood in terms of a Galois cover $\mathcal{X}_f \rightarrow S_f$ with finite branching then it might be possible to recover S as a partial

compactification of S_f . Looijenga has successfully carried out this approach for all unimodular hypersurface singularities, certain (two-dimensional) cusp singularities, and triangle singularities which are not hypersurfaces or even complete intersections. For references see Looijenga [26], [29], [30], [31]. For simply-elliptic and cusp singularities the construction of the covering space \mathcal{X} is expressed in terms of the generalized root system formed by the vanishing cycles of the singularity; the partial compactification associated with such a root system is described in Looijenga [28]. Again the isomorphism between the quotient \mathcal{X}/W and S is induced by some period mapping.

Looijenga's results on simply-elliptic singularities were independently found by Pinkham [39], using a different method. Like Looijenga, he uses ubiquitous C^* -actions to compactify the fibres of the deformations. But then he observes that the resulting families of compact surfaces (which are classically known as del Pezzo surfaces) can be studied by essentially classical methods of projective geometry. This method also applies to the simple hypersurface singularities of E type—see also Tyurina [46]—and is closely related to work of du Val [48]. Using an approach similar to Pinkham's Knörrer [22], [23] was able to describe the discriminant for complete intersections of two quadrics in arbitrary dimension. Apart from the odd-dimensional hypersurface singularities (which have the same deformation theory as their suspensions) Knörrer's description is the first to include complete intersections of odd dimension. For those singularities the Galois group W of $\mathcal{X} \rightarrow \mathcal{X}/W$, which should be a reflection group in some sense, is of a nature completely different from that of the monodromy group. In fact, as the intersection form on the Milnor fibre is skew-symmetric the monodromy group is generated by transvections rather than reflections. To remedy the situation Knörrer relates the semi-universal deformation of a $(2n-1)$ -dimensional intersection of two quadrics with a (non-versal) deformation of a singularity of dimension $2n$ and thereby reduces the problem to the even case.

The present work is concerned with some of the space curve singularities which are complete intersections, and are simple in the sense of Arnol'd. Such singularities were classified by Giusti [12], [13]; his list contains one infinite series S_μ ($\mu=5, 6, 7, \dots$), as well as ten "exceptional" types $T_7, T_8, T_9, U_7, U_8, U_9, W_8, W_9, Z_9, Z_{10}$. Giusti's classification was extended by Wall [52] so as to include the unimodular cases.

It was noted in Wirthmüller [54] that the discriminants of certain fat (that is, non-reduced) points allow a description as above, with \mathcal{X} a torus embedding which is most conveniently described in terms of a root system of A type. This result will serve as a model for the cases at hand. Formalizing the construction of \mathcal{X} , we shall consider extensions of the classical Dynkin diagrams by some extra combinatorial data. These

extended diagrams \mathcal{D} and their associated torus embeddings $\mathcal{X}(\mathcal{D})$ are the subject of the first three sections of this paper. The construction proper of $\mathcal{X}(\mathcal{D})$ is explained in Section 1 while in the following section we study geometric properties of $\mathcal{X}=\mathcal{X}(\mathcal{D})$ and its natural stratification. We also show that the quotient of \mathcal{X} by the Weyl group $W=W(\mathcal{D})$ (as well as certain finite extensions W_m , $m=1, 2, \dots$) is an affine space with distinguished C^* -actions. In the third section we look at the discriminant (the set of singular W -orbits) $\Delta \subset \mathcal{X}/W$ from a topological point of view: we give a natural presentation of the fundamental group of the complement $(\mathcal{X}/W) \setminus \Delta$.

The next two sections are devoted to Giusti's series S_μ ($\mu \geq 5$). The main result (4.2) states that the discriminant $D \subset S$ is isomorphic to the branch locus of the Galois cover $\mathcal{X}(\mathcal{D}) \rightarrow \mathcal{X}(\mathcal{D})/W(\mathcal{D})_2$ where \mathcal{D} is a particular diagram labelled $D_k[\ast]$, $k=\mu-1$. We use this same label for the singularity itself, rather than Giusti's notation S_μ . The proof of Theorem 4.2 occupies Section 4. The main point is to recognize a sufficiently big part of the semi-universal deformation as a family of hyperelliptic curves; there is a natural common target line for the canonical mapping of each curve, and this provides an intrinsic parametrization of the family by the branch points of the canonical mapping. The description of the discriminant includes a relation between the isotropy groups of the W -action on \mathcal{X} and the types of singular fibres of the deformation; this is discussed in Section 5. As a by-result of the description of D we can show that the complement of D is an Eilenberg-MacLane space, thereby generalizing a result of Knörrer [24]. Finally we present a natural basis of the Milnor homology which is weakly distinguished (in a weak sense), and compute the intersection form.

The next three sections deal with the singularities T_7 , T_8 , T_9 , which we re-label as $E_6[\ast]$, $E_7[\ast]$, and $E_8[\ast]$, respectively. For these we obtain results analogous to those on $D_k[\ast]$, except that we do not know whether the higher homotopy groups of $S \setminus D$ vanish. The main result, describing the discriminant D , is Theorem 6.2. Though its statement is in perfect analogy with Theorem 4.2 its proof is rather different and occupies Sections 6 and 7. We first show that the general fibre of the deformation can be interpreted as the ramification curve of a double covering projection, the covering surface being a del Pezzo surface. This allows us to pass to a projective family of del Pezzo surfaces; these surfaces carry a distinguished anti-canonical divisor at infinity. The resulting situation is close to that studied by Pinkham in the simply-elliptic and simple hypersurface cases, the main difference being the type of the anti-canonical curves, compare Pinkham [39] and Merindol [34]. We go on to define a characteristic mapping between a cover of an open part of S and an algebraic torus; this map induces a morphism from that open part of S to the quotient \mathcal{X}/W . In Section 8 we extend the

latter morphism over all S . To this end we study certain degenerations of the del Pezzo surfaces in question. Only in the $E_6[\ast]$ case these are themselves del Pezzo surfaces (with an ordinary double point) while in the other two cases the special surface acquires a cyclic quotient singularity of multiplicity 4 or 5. The degeneration is not realized on the Artin component of this singularity, so that the latter cannot be resolved in the family. In order to extend the characteristic mapping to the special surfaces we look for configurations of exceptional curves on the regular part of these surfaces. We then use the fact that such curves are stable under deformations, and apply classical knowledge of exceptional curves on del Pezzo surfaces to identify the curves in question on the general surface of the family. This finally allows to extend the characteristic mapping and conclude the proof of Theorem 6.2.

In Section 8 we discuss some consequences of the results on $E_k[\ast]$. In particular we explain in some detail the intermediate position of $E_k[\ast]$ between the simple singularity E_k and the simply-elliptic \tilde{E}_k .

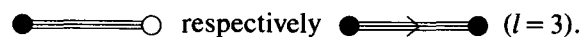
The final section takes up the construction of torus embeddings from Section 1. We use a recent construction of Looijenga [33] to assign spaces \mathcal{X} and \mathcal{X}/W to situations involving a generalized root system (with infinite Weyl group). These results are the basis for a description of the discriminant of the unimodular space curve singularities P_{kl} —see the list in Wall [52]. Details will appear in a subsequent paper.

1. Diagrams and cones

Let \mathcal{D} be a finite graph. A subgraph \mathcal{D}' of \mathcal{D} is called *full* if it contains all edges of \mathcal{D} with both end points in \mathcal{D}' . We consider graphs \mathcal{D} with the following additional structures.

- (1) A valuation which assigns to each vertex of \mathcal{D} its *colour*, black or white. In particular, this singles out full subgraphs $\mathcal{D}_{\text{black}}$ and $\mathcal{D}_{\text{white}}$.
- (2) A *weight* function on the set of edges of \mathcal{D} , with positive integral values.
- (3) An *orientation* of each edge in $\mathcal{D}_{\text{black}}$ with weight greater than 1.

Clearly, if $\mathcal{D}' \subset \mathcal{D}$ is a subgraph, structures of the same type are induced on \mathcal{D}' . The structured graph \mathcal{D} is conveniently represented visually by drawing the underlying graph, with vertices coloured according to (1), and each (oriented) edge of weight l drawn as an l -fold (directed) line:



Graphs with only black vertices include the classical Dynkin diagrams of reduced root systems as well as their completions (but \tilde{A}_1), compare Bourbaki [4], Chapter VI, §4, no. 2.

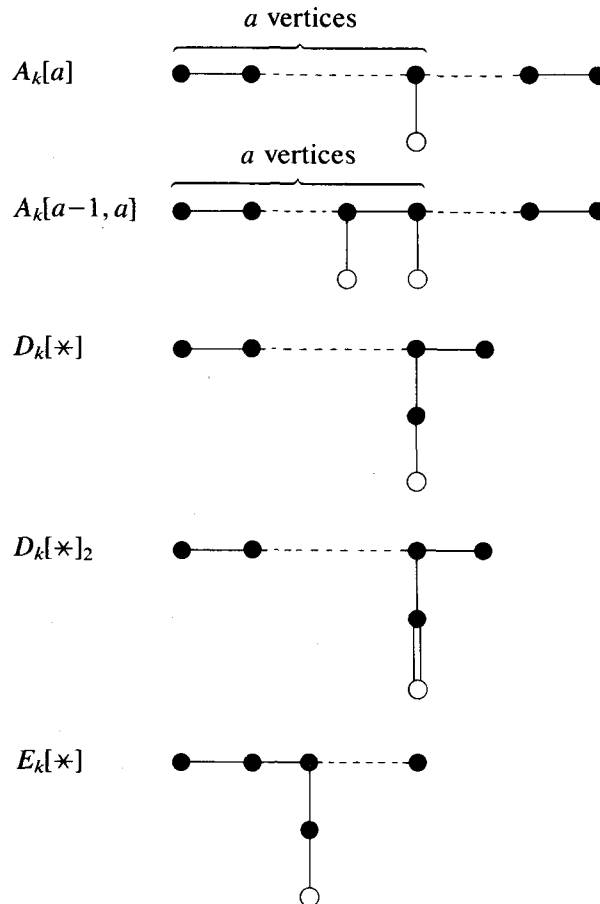
We call \mathcal{D} a *diagram* if the following axioms hold.

(D1) $\mathcal{D}_{\text{black}}$ is the Dynkin diagram of a reduced root system (which may be reducible).

(D2) $\mathcal{D}_{\text{white}}$ is discrete.

(D3) Each connected component of \mathcal{D} contains a white vertex.

(1.1) *Examples.* Diagrams of particular interest to us include the following types.



Every diagram \mathcal{D} gives rise to the following construction. We let A and B denote the sets of black and white vertices, respectively, and write $|\mathcal{D}|$ for the total number of vertices in \mathcal{D} . We fix a real vector space V of dimension $|\mathcal{D}|$, with dual space $V^\vee = \text{Hom}(V, \mathbf{R})$, and choose embeddings

$$A \subset V, B \subset V, \text{ and } A \rightarrow A^\vee \subset V^\vee$$

$$\alpha \mapsto \alpha^\vee$$

with the following properties.

(B1) $A \cup B \subset V$ is a basis of V .

(B2) The embeddings of A in V and V^\vee form a root basis with Dynkin diagram $\mathcal{D}_{\text{black}}$, see Looijenga [28].

(B3) $\langle \beta, \alpha^\vee \rangle = -l$ if $\alpha \in A$ and $\beta \in B$ span an edge of weight l in \mathcal{D} , and $\langle \beta, \alpha^\vee \rangle = 0$ if α and β are not connected by an edge.

Explicitly, the meaning of provision (B2) is this: for each $\alpha \in A$ we have $\langle \alpha, \alpha^\vee \rangle = 2$; for distinct $\alpha, \alpha' \in A$ we require $\langle \alpha, \alpha'^\vee \rangle = -l$ and $\langle \alpha', \alpha^\vee \rangle = -1$ if there is a directed edge of weight l from α to α' ; $\langle \alpha', \alpha^\vee \rangle = -1$ if α and α' span an edge of weight 1, and $\langle \alpha', \alpha^\vee \rangle = 0$ else.

We let $\Lambda \subset V$ be the lattice spanned by $A \cup B \subset V$, put $V_{\mathbf{C}} = \mathbf{C} \otimes_{\mathbf{R}} V$, and let $\mathcal{T} = V_{\mathbf{C}} / \Lambda$ be the corresponding algebraic torus. Each base root $\alpha \in A$ defines a reflection

$$w_\alpha: x \mapsto x - \langle x, \alpha^\vee \rangle \alpha,$$

and thereby a finite root system $R \subset V$ is generated, with Weyl group $W \subset GL(V)$. Note that R is a classical root system in the vector space $\mathbf{R}R \subset V$. The Weyl group serves to define a natural partial compactification \mathcal{X} of \mathcal{T} , as follows. Let $K \subset V$ be the convex cone spanned by the W -orbits of all $\beta \in B$. Then K is a convex polyhedral cone in the sense of Kempf et al. [21], Chapter I. Furthermore K is rational with respect to Λ and does not contain any line. We let \mathcal{X} be the equivariant affine embedding of \mathcal{T} corresponding to K ; compare [loc. cit.]. Clearly W acts on \mathcal{T} , and this action extends to one on \mathcal{X} permuting the \mathcal{T} -orbits.

As a matter of fact all these objects are assigned to the diagram \mathcal{D} , for up to canonical isomorphism in an obvious sense they do not depend on the choices made. The dependence on \mathcal{D} will be indicated by the notation $A = A(\mathcal{D})$, $V = V(\mathcal{D})$ etc. whenever this is necessary to avoid ambiguity.

The *closed facets* of K are, by definition, the sets of the form

$$\{x \in K \mid \langle x, \alpha^\vee \rangle = 0\}$$

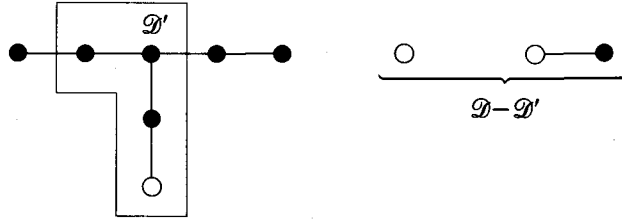


Figure 1.4

where $x^\vee \in V^\vee$ is a linear form which is non-negative on K . The closed facets may be described as follows.

A full subgraph of \mathcal{D} that itself is a diagram will be called a subdiagram of \mathcal{D} . If $\mathcal{D}' \subset \mathcal{D}$ is a subdiagram then the cone $K(\mathcal{D}')$ may be identified with a subset of $K(\mathcal{D})$.

THEOREM 1.2. (a) *For each subdiagram $\mathcal{D}' \subset \mathcal{D}$ the cone $K(\mathcal{D}')$ is a $|\mathcal{D}'|$ -dimensional closed facet of $K(\mathcal{D})$. In particular, $K(\mathcal{D}) \subset V(\mathcal{D})$ has non-empty interior.*

(b) *Assigning $K(\mathcal{D}')$ to \mathcal{D}' induces a bijection between the set of subdiagrams of \mathcal{D} and the set of $W(\mathcal{D})$ -orbits of closed facets in $K(\mathcal{D})$.*

Proof. This will be proved in Section 9 in a more general setting, see Theorem 9.5 and the discussion following it.

If \mathcal{D}' is a subdiagram of \mathcal{D} we define its *complement* $\mathcal{D} - \mathcal{D}'$ as follows. The sets of coloured vertices of $\mathcal{D} - \mathcal{D}'$ are

$$A(\mathcal{D} - \mathcal{D}') = \{\alpha \in A(\mathcal{D}) \setminus A(\mathcal{D}') \mid \text{there is no vertex } \gamma \text{ of } \mathcal{D}' \text{ such that } \alpha \text{ and } \gamma \text{ span an edge in } \mathcal{D}\},$$

$$B(\mathcal{D} - \mathcal{D}') = A(\mathcal{D}) \setminus (A(\mathcal{D}') \cup A(\mathcal{D} - \mathcal{D}')) \cup B(\mathcal{D}) \setminus B(\mathcal{D}').$$

Two vertices of $\mathcal{D} - \mathcal{D}'$ including a black one span an edge in $\mathcal{D} - \mathcal{D}'$ if they do so in \mathcal{D} .

(1.3) *Example.* Let \mathcal{D} be the diagram of type $E_6[*]$. Then the complement of the encircled subdiagram \mathcal{D}' is as shown in Figure 1.4.

It is clear from the definition that the actions of $W(\mathcal{D}')$ and $W(\mathcal{D} - \mathcal{D}')$ on $V(\mathcal{D})$ commute, so we have a canonical embedding

$$W(\mathcal{D}') \times W(\mathcal{D} - \mathcal{D}') \subset W(\mathcal{D}).$$

PROPOSITION 1.5. *The stabilizer of the closed facet $K(\mathcal{D}')$ is $W(\mathcal{D}') \times W(\mathcal{D} - \mathcal{D}')$.*

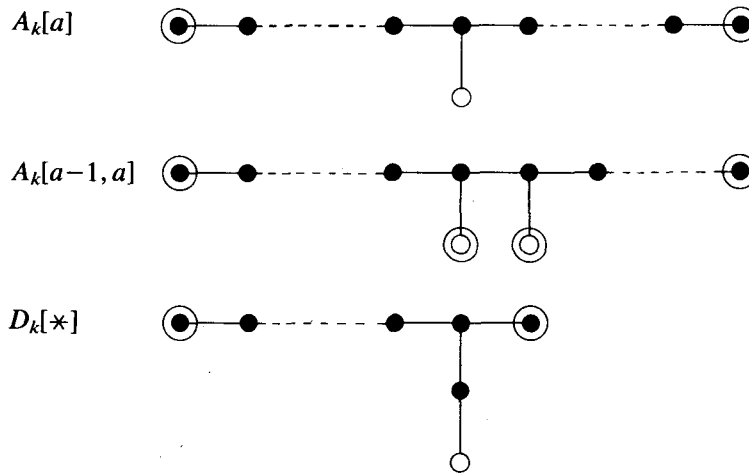
Proof. See Proposition 9.12.

A vertex γ of \mathcal{D} is called *extremal* if the full subgraph of \mathcal{D} obtained by deleting the open star of γ is a subdiagram. To each extremal vertex γ corresponds an *extremal form* $x_\gamma^\vee \in V^\vee(\mathcal{D})$ which takes value 1 on γ and vanishes on all other vertices of \mathcal{D} .

Applying Theorem 1.2 to the faces of K we obtain:

COROLLARY 1.6. *The union of the W -orbits of extremal forms is a basis of the dual cone $K^\vee = \{x^\vee \in V^\vee \mid x^\vee \geq 0 \text{ on } K\}$.* Q.E.D.

(1.7) *Examples (the encircled vertices are the extremal ones):*



(1.8) *Remark.* In general, the W -transforms of extremal forms fail to generate the semi-group $K^\vee \cap \Lambda^\vee$.

Let \mathcal{D} be a diagram and $\mathcal{D}' \subset \mathcal{D}$ a subdiagram. There is a canonical exact sequence

$$0 \rightarrow V(\mathcal{D}') \rightarrow V(\mathcal{D}) \xrightarrow{\pi} V(\mathcal{D} - \mathcal{D}') \rightarrow 0$$

of $W(\mathcal{D}') \times W(\mathcal{D} - \mathcal{D}')$ -modules. In the next section, the following proposition will serve to describe the local geometry of $\mathcal{X}(\mathcal{D})$.

PROPOSITION 1.9. *The cone $\pi K(\mathcal{D}) \subset V(\mathcal{D} - \mathcal{D}')$ is spanned by the set $W(\mathcal{D} - \mathcal{D}') \cdot B(\mathcal{D} - \mathcal{D}')$.*

Proof. An obvious induction reduces this question to the case $|\mathcal{D}'|=1$. Then \mathcal{D}' consists of a single white vertex β' , and we have

$$A' := A(\mathcal{D} - \mathcal{D}') = \{\alpha \in A(\mathcal{D}) \mid \langle \beta', \alpha^\vee \rangle = 0\}$$

$$B' := B(\mathcal{D} - \mathcal{D}') = \{\alpha \in A(\mathcal{D}) \mid \langle \beta', \alpha^\vee \rangle < 0\} \cup B(\mathcal{D}) \setminus \{\beta'\}.$$

We put $W' = W(\mathcal{D} - \mathcal{D}')$; let us show that $W'B' \subset \pi K(\mathcal{D})$. Clearly, we have $B(\mathcal{D}) \setminus \{\beta'\} \subset \pi K(\mathcal{D})$. If $\alpha \in A(\mathcal{D})$ is such that $\langle \beta', \alpha^\vee \rangle < 0$ then $w_\alpha(\beta') = \beta' - \langle \beta', \alpha^\vee \rangle \alpha$ implies that

$$-\langle \beta', \alpha \rangle \pi \alpha = \pi(\beta' - \langle \beta', \alpha^\vee \rangle \alpha) = \pi w_\alpha(\beta') \in \pi(W(\mathcal{D})\beta') \subset \pi K(\mathcal{D}).$$

Thus $B' \subset \pi K(\mathcal{D})$, and the inclusion $W'B' \subset \pi K(\mathcal{D})$ follows as π is W' -equivariant.

It remains to prove that $\pi(W(\mathcal{D}) \cdot B(\mathcal{D}))$ is contained in the cone spanned by $W'B'$. Thus let $\beta \in B(\mathcal{D})$ and $\gamma \in W(\mathcal{D})\beta$. Choose an element $w \in W(\mathcal{D})$ of minimal length $l(w)$ such that $\gamma = w\beta$, see Bourbaki [4], Chapter IV, § 1, no. 1 or Looijenga [28], (1.5). We prove that $\pi\gamma \in \mathbf{Z}_+(W'B')$, by induction on $l(w)$. The case $l(w) = 0$ is trivial. If $l(w) > 0$ we know from Looijenga [28], (1.11) that there is a base root $\alpha \in A(\mathcal{D})$ with $\langle \gamma, \alpha^\vee \rangle > 0$. Then $l(w_\alpha w) = l(w) - 1$, and by the inductive hypothesis $\pi w_\alpha(\gamma)$ belongs to $\mathbf{Z}_+(W'B')$. Now α belongs to either A' or B' . In the former case we note $w_\alpha \in W'$ while in the latter we write $\gamma = w_\alpha \gamma + \langle \gamma, \alpha^\vee \rangle \alpha$. In either case it follows that $\gamma \in \mathbf{Z}_+(W'B')$, and the proof is complete. Q.E.D.

2. Geometry of the torus embedding \mathcal{X}

Let \mathcal{D} be a diagram. The variety $\mathcal{X}(\mathcal{D})$ is stratified by orbits under the action of $\mathcal{T}(\mathcal{D})$, and these orbits correspond bijectivity to the (closed) facets of $K(\mathcal{D})$; see Kempf et al. [21], Chapter I. We wish to describe the geometry of $\mathcal{X}(\mathcal{D})$ along the orbits in terms of the subdiagrams of \mathcal{D} , taking into account the action of the Weyl group $W(\mathcal{D})$ on $\mathcal{X}(\mathcal{D})$. More generally, let $Q(\mathcal{D}) = \mathbf{Z}A(\mathcal{D})$ be the root lattice and consider the semi-direct product

$$\bar{W}_m(\mathcal{D}) = \left(\frac{1}{m} Q(\mathcal{D}) + \Lambda(\mathcal{D})\right) \cdot W(\mathcal{D})$$

where m is any positive integer. This group acts naturally on $V(\mathcal{D})$ and $\mathcal{X}(\mathcal{D})$; dividing by the kernel of the latter action we obtain an action of the finite group

$$W_m(\mathcal{D}) = \frac{\frac{1}{m} Q(\mathcal{D}) + \Lambda(\mathcal{D})}{\Lambda(\mathcal{D})} \cdot W(\mathcal{D})$$

on $\mathcal{X}(\mathcal{D})$. The Weyl group is recovered as the special case $m=1$.

For each subdiagram $\mathcal{D}' \subset \mathcal{D}$ we let $\mathcal{I}_{\mathcal{D}'} \subset \mathcal{X}(\mathcal{D})$ denote the $\mathcal{I}(\mathcal{D})$ -orbit corresponding to the facet $K(\mathcal{D}') \subset K(\mathcal{D})$. The open star of $\mathcal{I}_{\mathcal{D}'}$ is the union of all $\mathcal{I}(\mathcal{D})$ -orbits that contain $\mathcal{I}_{\mathcal{D}'}$ in its closure; we denote it by $\text{St } \mathcal{I}_{\mathcal{D}'}$. Recall from Kempf et al. [21], p. 15 that there is a canonical $\mathcal{I}(\mathcal{D})$ -equivariant retraction of $\mathcal{X}(\mathcal{D})$ to the closure of $\mathcal{I}_{\mathcal{D}'}$, with $\text{St } \mathcal{I}_{\mathcal{D}'}$ the inverse image of $\mathcal{I}_{\mathcal{D}'}$. We let

$$r: \text{St } \mathcal{I}_{\mathcal{D}'} \rightarrow \mathcal{I}_{\mathcal{D}'}$$

be the restriction. From the complex analytic point of view, r is a locally trivial fibre bundle with typical fibre $\mathcal{X}(\mathcal{D}')$. Finally, we let $\tilde{W}_m(\mathcal{D})_{\mathcal{D}'}$ be the stabilizer of $\mathcal{I}_{\mathcal{D}'}$; this subgroup of $\tilde{W}_m(\mathcal{D})$ consists of all elements that send $\mathcal{I}_{\mathcal{D}'}$ onto itself. The stabilizer is determined as follows.

PROPOSITION 2.1. *$\tilde{W}_m(\mathcal{D})_{\mathcal{D}'}$ is the semi-direct product*

$$G_{\mathcal{D}'} \cdot \tilde{W}_m(\mathcal{D}') \subset \tilde{W}_m(\mathcal{D}),$$

with

$$G_{\mathcal{D}'} = \frac{1}{m} (Q(\mathcal{D}) \cap \Lambda(\mathcal{D} - \mathcal{D}')) \cdot W(\mathcal{D} - \mathcal{D}') = \frac{1}{m} (Q(\mathcal{D})/Q(\mathcal{D}')) \cdot W(\mathcal{D} - \mathcal{D}').$$

In particular $\tilde{W}_m(\mathcal{D})_{\mathcal{D}'}$ contains the direct product $\tilde{W}_m(\mathcal{D} - \mathcal{D}') \times \tilde{W}_m(\mathcal{D}')$ as a normal subgroup.

Proof. By Proposition 1.5, the stabilizer of $K(\mathcal{D}')$ in $W(\mathcal{D})$ is $W(\mathcal{D} - \mathcal{D}') \times W(\mathcal{D}')$. The proposition follows because $\tilde{W}_m(\mathcal{D})_{\mathcal{D}'}$ includes all translations. Q.E.D.

The geometry of the $\mathcal{I}(\mathcal{D})$ -orbits is given by

PROPOSITION 2.2. *The canonical isomorphism*

$$\mathcal{I}_{\mathcal{D}'} = \mathcal{I}(\mathcal{D})/\mathcal{I}(\mathcal{D}') = \mathcal{I}(\mathcal{D} - \mathcal{D}')$$

extends to an isomorphism between the closure $\overline{\mathcal{I}_{\mathcal{D}'}}$ and $\mathcal{X}(\mathcal{D} - \mathcal{D}')$.

Proof. This follows at once from (1.9). Q.E.D.

The proposed description of $\mathcal{X}(\mathcal{D})$ along $\mathcal{I}_{\mathcal{D}'}$ is achieved by the following theorem.

THEOREM 2.3. (a) *There exists a finite étale Galois cover $e: \tilde{\mathcal{I}} \rightarrow \mathcal{I}_{\mathcal{D}'}$ such that base change by e trivializes the retraction r . More precisely: there is a morphism \tilde{e} such that the diagram*

$$\begin{array}{ccc}
 \tilde{\mathcal{F}} \times \mathcal{X}(\mathcal{D}') & \xrightarrow{\tilde{e}} & \text{St } \mathcal{T}_{\mathcal{D}'} \\
 \downarrow \text{pr} & & \downarrow r \\
 \tilde{\mathcal{F}} & \xrightarrow{e} & \mathcal{T}_{\mathcal{D}'}
 \end{array} \tag{2.4}$$

exhibits $\tilde{\mathcal{F}} \times \mathcal{X}(\mathcal{D})$ as a fibred product.

(b) The action of $\tilde{W}_m(\mathcal{D})_{\mathcal{D}'}$ lifts to $\tilde{\mathcal{F}}$ and $\tilde{\mathcal{F}} \times \mathcal{X}(\mathcal{D})$, making the diagram (2.4) equivariant. The lifted actions of $\tilde{W}_m(\mathcal{D})_{\mathcal{D}'}$ cover those of the Galois group.

(c) The subgroup $\tilde{W}_m(\mathcal{D}-\mathcal{D}') \times \tilde{W}_m(\mathcal{D}') \subset \tilde{W}_m(\mathcal{D})_{\mathcal{D}'}$ acts on $\tilde{\mathcal{F}} \times \mathcal{X}(\mathcal{D}')$ like the direct product of a $\tilde{W}_m(\mathcal{D}-\mathcal{D}')$ -action on $\tilde{\mathcal{F}}$ and the natural $\tilde{W}_m(\mathcal{D}')$ -action on $\mathcal{X}(\mathcal{D}')$. The induced effective action of $\tilde{W}_m(\mathcal{D})_{\mathcal{D}'} / (\tilde{W}_m(\mathcal{D}-\mathcal{D}') \times \tilde{W}_m(\mathcal{D}'))$ on $\tilde{\mathcal{F}} / \tilde{W}_m(\mathcal{D}-\mathcal{D}') \times \mathcal{X}(\mathcal{D}') / \tilde{W}_m(\mathcal{D}')$ is free, so there is an étale \tilde{e} that makes the diagram

$$\begin{array}{ccc}
 \tilde{\mathcal{F}} \times \mathcal{X}(\mathcal{D}') & \xrightarrow{\tilde{e}} & \text{St } \mathcal{T}_{\mathcal{D}'} \\
 \downarrow & & \downarrow \\
 \tilde{\mathcal{F}} / \tilde{W}_m(\mathcal{D}-\mathcal{D}') \times \mathcal{X}(\mathcal{D}') / \tilde{W}_m(\mathcal{D}') & \xrightarrow{\tilde{e}} & \text{St } \mathcal{T}_{\mathcal{D}'} / \tilde{W}_m(\mathcal{D})_{\mathcal{D}'}
 \end{array}$$

commutative.

(d) $\tilde{\mathcal{F}}$ may be chosen a direct product of algebraic tori

$$\mathbf{C}Q(\mathcal{D}-\mathcal{D}') / Q(\mathcal{D}-\mathcal{D}') \times \mathcal{T}'$$

such that the quotient morphism $\tilde{\mathcal{F}} \rightarrow \tilde{\mathcal{F}} / \tilde{W}_m(\mathcal{D}-\mathcal{D}')$ factors like

$$\begin{array}{ccc}
 \mathbf{C}Q(\mathcal{D}-\mathcal{D}') / Q(\mathcal{D}-\mathcal{D}') \times \mathcal{T}' & \xlongequal{\quad} & \tilde{\mathcal{F}} \\
 \downarrow & & \downarrow \\
 \mathbf{C}Q(\mathcal{D}-\mathcal{D}') / \frac{1}{m}Q(\mathcal{D}-\mathcal{D}') \times \mathcal{T}' & \xrightarrow{e'} & \tilde{\mathcal{F}} / \tilde{W}_m(\mathcal{D}-\mathcal{D}')
 \end{array}$$

for some étale Galois cover e' .

Proof. Let $P: V(\mathcal{D}) \rightarrow V(\mathcal{D}')$ be the projector onto the fixed space of $W(\mathcal{D}')$:

$$Pt = \sum_{w \in W(\mathcal{D}')} wt / |W(\mathcal{D}')|,$$

and put $U = PV(\mathcal{D}-\mathcal{D}')$. As the action of $W(\mathcal{D}')$ on $V(\mathcal{D})/V(\mathcal{D}')$ is trivial U is a $W(\mathcal{D}-\mathcal{D}')$ -stable complement of $V(\mathcal{D}')$ in $V(\mathcal{D})$, and P induces an equivariant isomorphism $P': V(\mathcal{D}-\mathcal{D}') \rightarrow U$. (See Figure 2.5.)

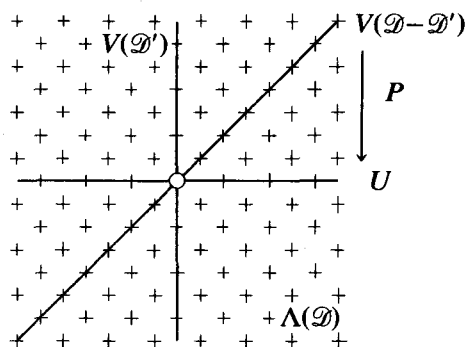


Figure 2.5

Provisionally, we put

$$\tilde{\mathcal{T}} = \mathbf{C}U / (\Lambda(\mathcal{D}) \cap U);$$

this is an algebraic torus because U is rational with respect to $\Lambda(\mathcal{D})$. As P induces the identity on $V(\mathcal{D})/V(\mathcal{D}')$ we have

$$\Lambda(\mathcal{D}) \cap U \subset P\Lambda(\mathcal{D}-\mathcal{D}'). \tag{2.6}$$

Thus the inverse of P' induces an étale covering morphism $e: \tilde{\mathcal{T}} \rightarrow \mathcal{T}_{\mathcal{D}'}$ with Galois group $P\Lambda(\mathcal{D}-\mathcal{D}') / (\Lambda(\mathcal{D}) \cap U)$. Likewise, the square diagram (2.4) is induced from the diagram

$$\begin{array}{ccc}
 U \times V(\mathcal{D}') & \xrightarrow{\quad} & V(\mathcal{D}) \\
 \downarrow \text{pr} & \searrow \supset & \downarrow \text{projection along } V(\mathcal{D}') \\
 (\Lambda(\mathcal{D}) \cap U) \times \Lambda(\mathcal{D}') & \xrightarrow{\quad} & \Lambda(\mathcal{D}) \\
 \downarrow & & \downarrow \\
 \Lambda(\mathcal{D}) \cap U & \xrightarrow{\quad} & \Lambda(\mathcal{D}-\mathcal{D}') \\
 \downarrow & \nearrow \supset & \downarrow \\
 U & \xrightarrow{P'^{-1}} & V(\mathcal{D}-\mathcal{D}')
 \end{array} \tag{2.7}$$

of vector spaces and lattices. It now is straightforward to verify (a).

We put on $U \times V(\mathcal{D}')$ the unique action of $\tilde{W}_m(\mathcal{D})_{\mathcal{D}'}$ that makes the isomorphism at the top of (2.7) equivariant. $\tilde{W}_m(\mathcal{D})_{\mathcal{D}'}$ then acts on the whole diagram (2.4), and (b) follows.

$\tilde{W}_m(\mathcal{D}-\mathcal{D}') \times \tilde{W}_m(\mathcal{D}')$ acts like a direct product as claimed, for $Q(\mathcal{D}-\mathcal{D}')$ is contained in U . Furthermore all elements of $\tilde{W}_m(\mathcal{D})_{\mathcal{D}'}/(\tilde{W}_m(\mathcal{D}-\mathcal{D}') \times \tilde{W}_m(\mathcal{D}'))$ are represented by translations in $(1/m)\mathbf{Z}B(\mathcal{D}-\mathcal{D}')$. If any of these, say γ , has a fixed point on

$\tilde{\mathcal{T}}/\tilde{W}_m(\mathcal{D}-\mathcal{D}')$ it must satisfy $P\gamma \in \Lambda(\mathcal{D}) \cap U$, hence $P\gamma \in P\Lambda(\mathcal{D}-\mathcal{D}')$ by (2.6). As $P|V(\mathcal{D}-\mathcal{D}')$ is injective this implies $\gamma \in \Lambda(\mathcal{D}-\mathcal{D}')$ and it follows that

$$\gamma = P\gamma + (\gamma - P\gamma) \in (\Lambda(\mathcal{D}) \cap U) \times \Lambda(\mathcal{D}')$$

acts trivially on $\tilde{\mathcal{T}} \times \mathcal{X}(\mathcal{D}')$. This proves (c).

Let $\Lambda' \subset \Lambda(\mathcal{D}) \cap U$ be the fixed lattice under the action of $W(\mathcal{D}-\mathcal{D}')$. Then $Q(\mathcal{D}-\mathcal{D}') + \Lambda'$ is a sublattice of finite index in $\Lambda(\mathcal{D}) \cap U$, and we may replace the former choice of $\tilde{\mathcal{T}}$ with its étale cover

$$\tilde{\mathcal{T}} = \mathbf{C}U / (Q(\mathcal{D}-\mathcal{D}') + \Lambda') = \mathbf{C}Q(\mathcal{D}-\mathcal{D}') / Q(\mathcal{D}-\mathcal{D}') \times \mathbf{C}\Lambda' / \Lambda'.$$

This substitution does not affect the already proven parts of the theorem. Part (d) now follows in view of the exact sequence

$$0 \rightarrow \left(\frac{1}{m} Q(\mathcal{D}-\mathcal{D}') + \Lambda' \right) W(\mathcal{D}-\mathcal{D}') \rightarrow \tilde{W}_m(\mathcal{D}-\mathcal{D}') \rightarrow \frac{\frac{1}{m} Q(\mathcal{D}-\mathcal{D}') + P\Lambda(\mathcal{D}-\mathcal{D}')}{\frac{1}{m} Q(\mathcal{D}-\mathcal{D}') + \Lambda'} \rightarrow 0.$$

Q.E.D.

Let $\tau \in \mathcal{F}_{\mathcal{D}'} \subset \mathcal{X}(\mathcal{D})$, and choose a $t \in V(\mathcal{D})_{\mathbf{C}}$ representing it via

$$V(\mathcal{D})_{\mathbf{C}} \rightarrow \mathcal{F}(\mathcal{D}) / \mathcal{F}(\mathcal{D}') = \mathcal{F}_{\mathcal{D}'}.$$

The set

$$R_{\tau} = \left\{ \alpha \in R(\mathcal{D}-\mathcal{D}') \mid \langle t, \alpha^{\vee} \rangle \in \frac{1}{m} \mathbf{Z} \right\}$$

is a root system in $V(\mathcal{D}-\mathcal{D}')$ and does not depend on the particular choice of t . Let $W(R_{\tau}) \subset GL(V(\mathcal{D}-\mathcal{D}'))$ be its Weyl group.

COROLLARY 2.8. *In the automorphism group of $\mathcal{X}(\mathcal{D})$, the isotropy group $W_m(\mathcal{D})_{\tau}$ is conjugate to the group*

$$W(R_{\tau}) \times W_m(\mathcal{D}') \subset W_m(\mathcal{D}).$$

Proof. Using the notation of the previous proof we may choose t in $\mathbf{C}U$. Note that $(1/m)Q(\mathcal{D}-\mathcal{D}')W(\mathcal{D}-\mathcal{D}')$ acts on $\mathbf{C}U$ like an affine Weyl group. Therefore the isotropy group

$$\left(\frac{1}{m} Q(\mathcal{D}-\mathcal{D}') W(\mathcal{D}-\mathcal{D}') \right)_t$$

is generated by reflections, see Bourbaki [4], Chapter IV, § 3, Proposition 1. Translation by t identifies this isotropy group with $W(R_\tau)$. The diagrams from Theorem 2.3, parts (c) and (d), combine to yield

$$\begin{array}{ccc}
 \mathbf{C}Q(\mathcal{D}-\mathcal{D}')/Q(\mathcal{D}-\mathcal{D}') \times \mathcal{F}' \times \mathcal{X}(\mathcal{D}') & \xrightarrow{\bar{e}} & \text{St } \mathcal{F}_{\mathcal{D}'} \\
 \downarrow & & \downarrow \\
 \mathbf{C}Q(\mathcal{D}-\mathcal{D}') / \frac{1}{m} Q(\mathcal{D}-\mathcal{D}') W(\mathcal{D}-\mathcal{D}') \times \mathcal{F}' \times \mathcal{X}(\mathcal{D}') / \bar{W}_m(\mathcal{D}') & \xrightarrow{e''} & \text{St } \mathcal{F}_{\mathcal{D}'} / \bar{W}_m(\mathcal{D}')_{\mathcal{D}'}
 \end{array} \tag{2.9}$$

with étale Galois covers \bar{e} and e'' . From this diagram it is clear that the subgroup

$$\left(\frac{1}{m} Q(\mathcal{D}-\mathcal{D}') W(\mathcal{D}-\mathcal{D}') \right)_t \times \bar{W}_m(\mathcal{D}') \subset \bar{W}_m(\mathcal{D})$$

projects into $W_m(\mathcal{D})_\tau$. In fact, its image is all $W_m(\mathcal{D})_\tau$. For, any $g \in W_m(\mathcal{D})_\tau$ lifts to an element $\bar{g} \in \bar{W}_m(\mathcal{D})_t$, and as e'' is étale \bar{g} acts on $\mathbf{C}Q(\mathcal{D}-\mathcal{D}')/Q(\mathcal{D}-\mathcal{D}') \times \mathcal{F}' \times \mathcal{X}(\mathcal{D}')$ like an element of $(1/m) Q(\mathcal{D}-\mathcal{D}') W(\mathcal{D}-\mathcal{D}') \times \bar{W}_m(\mathcal{D}')$. As $t \in U$ acts trivially on $\bar{W}_m(\mathcal{D}')$ we obtain

$$t^{-1} W_m(\mathcal{D})_\tau t = W(R_\tau) \times W_m(\mathcal{D}') \subset W_m(\mathcal{D}) \tag{2.10}$$

and the proof is complete. Q.E.D.

Among the objects associated with the diagram \mathcal{D} and the integer m the one of principal interest in view of its bearing on deformation theory is the *discriminant* $\Delta_m(\mathcal{D})$. This is, by definition, the reduced branch locus of the quotient morphism

$$\mathcal{X}(\mathcal{D}) \rightarrow \mathcal{X}(\mathcal{D})/W_m(\mathcal{D}).$$

Similarly, if $R \subset V$ is a root system with Weyl group $W(R)$ we use the symbol $\Delta(V, R)$ to denote the discriminant (that is, the reduced branch locus) of $V_{\mathbb{C}} \rightarrow V_{\mathbb{C}}/W(R)$.

In terms of local complex analytic geometry the substance of Theorem 2.3 reduces to the following.

COROLLARY 2.11. *The germ of the pair $(\mathcal{X}(\mathcal{D})/W_m(\mathcal{D}), \Delta_m(\mathcal{D}))$ at τ is equivalent to the cartesian product of the germ*

$$(V(\mathcal{D}-\mathcal{D}')_{\mathbb{C}}/W(R_\tau), \Delta(V(\mathcal{D}-\mathcal{D}'), R_\tau))$$

at the origin with the germ

$$(\mathcal{X}(\mathcal{D}')/W_m(\mathcal{D}'), \Delta_m(\mathcal{D}'))$$

at the $\mathcal{T}(\mathcal{D}')$ -fixed point of $\mathcal{X}(\mathcal{D}')$.

Proof. See (2.9).

Q.E.D.

(2.12) *Remark.* The groups $W(R_\tau)$ that occur for some $\tau \in \mathcal{T}_{\mathcal{D}'}$ are exactly the linear parts of the isotropy groups of the affine Weyl groups $Q(\mathcal{D}-\mathcal{D}') W(\mathcal{D}-\mathcal{D}')$, acting on $CQ(\mathcal{D}-\mathcal{D}')$. This is clear from the definition of R_τ .

The global geometry of \mathcal{X}/W_m is determined by the invariant theory of W_m , which is similar to the exponential invariant theory of root systems as described in Bourbaki [4], Chapter VI, § 3, and Looijenga [28], Section 4. Let us fix a diagram \mathcal{D} as well as an integer $m > 0$. If $\mathcal{R}(\mathcal{X})$ denotes the coordinate ring of \mathcal{X} then \mathcal{X}/W_m is the spectrum of the subring $\mathcal{R}(\mathcal{X})^{W_m}$ of invariant functions on \mathcal{X} . We also introduce the $\mathcal{R}(\mathcal{X})^{W_m}$ -module of anti-invariant functions, this is

$$\mathcal{R}(\mathcal{X})^{-W_m} = \{f \in \mathcal{R}(\mathcal{X}) \mid wf = \chi(w)f \text{ for all } w \in W_m\}$$

where $\chi: W_m \rightarrow \{\pm 1\}$ is the character that coincides with the determinant on W and is trivial on translations.

To describe the structure of $\mathcal{R}(\mathcal{X})^{\pm W_m}$ let $\Lambda^\vee = \text{Hom}(\Lambda, \mathbf{Z}) \subset V^\vee$ be the dual lattice; then

$$\mathcal{R}(\mathcal{X}) = \mathbf{C}[\Lambda^\vee \cap K^\vee]$$

is the \mathbf{C} -algebra of the semi-group $\Lambda^\vee \cap K^\vee$. Likewise we have

$$\mathcal{R}(\mathcal{X})^{(1/m)Q} = \mathbf{C}[\Lambda_m^\vee \cap K^\vee] \quad \text{with} \quad \Lambda_m^\vee = \text{Hom}\left(\frac{1}{m}Q + \Lambda, \mathbf{Z}\right) \subset V^\vee.$$

If we let $e^p \in \mathbf{C}[\Lambda_m^\vee]$ denote the character corresponding to $p \in \Lambda_m^\vee$ each function $f \in \mathbf{C}[\Lambda_m^\vee \cap K^\vee]$ can be written in a unique way as a finite sum

$$f = \sum_{p \in \Lambda_m^\vee \cap K^\vee} f_p e^p \quad (f_p \in \mathbf{C}). \tag{2.13}$$

We put

$$\text{Supp}(f) = \{p \in \Lambda_m^\vee \cap K^\vee \mid f_p \neq 0\}.$$

The lattice Λ_m^\vee carries a natural partial order given by

$$p \leq p' \Leftrightarrow \{(p' - p)/m \text{ is a sum of positive dual roots}\}.$$

The *maximal support* $\text{Max Supp}(f)$ is, by definition, the set of maximal elements in $\text{Supp}(f)$. For f as in (2.13) we call

$$\text{In}(f) = \sum_{p \in \text{Max Supp}(f)} f_p e^p$$

the *initial form* of f .

Let $\{\alpha^* | \alpha \in A\} \cup \{\beta^* | \beta \in B\}$ be the basis of Λ_m^\vee that is dual to the basis

$$\left\{ \frac{1}{m} \alpha | \alpha \in A \right\} \cup B \subset \frac{1}{m} Q + \Lambda,$$

and put

$$e^* = \sum_{\alpha \in A} \alpha^*.$$

THEOREM 2.14. *Let $S_\gamma \in \mathcal{R}(\mathcal{X})^{W_m}$ ($\gamma \in A \cup B$) and $J \in \mathcal{R}(\mathcal{X})^{-W_m}$ be functions with initial forms $\text{In}(S_\gamma) = e^{\gamma^*}$, $\text{In}(J) = e^{e^*}$. Then the homomorphism*

$$\mathbb{C}[X_\gamma]_{\gamma \in A \cup B} \rightarrow \mathcal{R}(\mathcal{X})^{W_m}$$

sending the indeterminate X_γ to S_γ is bijective. Multiplication by J restricts to a bijection

$$\mathcal{R}(\mathcal{X})^{W_m} \rightarrow \mathcal{R}(\mathcal{X})^{-W_m}.$$

Proof. The relevant part of the proof in Looijenga [28], (4.2) applies verbatim. Q.E.D.

(2.15) *Remark.* A possible choice for S_γ and J is

$$S_\gamma = \sum_{w \in W} e^{w\gamma^*} \quad (= |W| \cdot e^{w\gamma^*} \text{ if } \gamma \in B),$$

$$J = \sum_{w \in W} (\det w) e^{w e^*}.$$

COROLLARY 2.16. *\mathcal{R}/W_m is an affine $|\mathcal{D}|$ -space, and for each stratum $\mathcal{T}_{\mathcal{D}}$ in \mathcal{X} , the closure $\overline{\mathcal{T}_{\mathcal{D}}}$ maps to an affine subspace. The invariant polynomial J^2 generates the ideal of Δ_m in $\mathcal{R}(\mathcal{X})^{W_m}$.*

Proof. \mathcal{X}/W_m is the spectrum of $\mathcal{R}(\mathcal{X})^{W_m}$, and the ideal of the subvariety in question is generated by those $S_\gamma = \sum_{\omega \in W} e^{\omega\gamma^*}$ with γ a vertex of \mathcal{D}' . This proves the first statement. Let $W_m^+ \subset W_m$ be the kernel of the character χ , and let $\Delta' \subset \mathcal{X}/W_m$ be the discriminant of the branched double cover $\mathcal{X}/W_m^+ \rightarrow \mathcal{X}/W_m$. We claim that Δ_m and Δ' coincide. For let $\tau \in \mathcal{T}$ have a non-trivial isotropy group in W_m . By Corollary 2.8, $(W_m)_\tau$ contains a reflection, so τ maps to Δ' . If $\tau \in \mathcal{X}$ belongs to a \mathcal{T} -orbit of codimension one we may assume $\tau \in \mathcal{T}_{\mathcal{D}'} \subset \mathcal{X}$ for some subdiagram $\mathcal{D}' \subset \mathcal{D}$ consisting of a single white vertex. Then the same reasoning applies to show that τ maps to Δ' . Thus Δ_m and Δ' coincide in dimension $|\mathcal{D}|-1$. As \mathcal{X} and \mathcal{X}/W_m^+ are normal varieties the theorem on purity of branch loci, Nagata [36], (41.1) makes sure that both Δ_m and Δ' are hypersurfaces in \mathcal{X}/W_m , and the claim follows. This also proves the corollary as J^2 clearly generates the ideal of Δ' . Q.E.D.

Given a diagram \mathcal{D} and a positive integer m there is a diagram \mathcal{D}_m which differs from \mathcal{D} only by the weights of the edges joining black to white vertices: in \mathcal{D}_m these are m times those in \mathcal{D} . The notation $D_k[\ast]_2$ in (1.1) is consistent with this definition. The following is easily seen:

COROLLARY 2.17. *The discriminants $\Delta_m(\mathcal{D})$ and $\Delta_1(\mathcal{D}_m)$ are canonically isomorphic.* Q.E.D.

(2.18) *Remark.* Looijenga's proof of Theorem 2.14 consists in an algorithm that, in principle, allows to compute J^2 as a polynomial in the S_γ .

Let C^* denote the multiplicative group $C \setminus \{0\}$. As any diagram obeys axiom (D3) the fixed lattice Λ^W intersects the interior of K non-trivially. Each element $\omega \in \Lambda^W \cap K$ determines a C^* -action on \mathcal{X} which extends to a morphism $C \times \mathcal{X} \rightarrow \mathcal{X}$. As this action commutes with W_m a C^* -action on \mathcal{X}/W_m is induced.

PROPOSITION 2.19. *The induced C^* -action on \mathcal{X}/W_m is equivalent to the linear action with weights $\langle \omega, \gamma^* \rangle$ ($\gamma \in A \cup B$). These weights are non-negative; they are all positive exactly when ω lies in the interior of K . The discriminant Δ_m is a quasi-homogeneous hypersurface with respect to these weights and its degree is $2\sum_{\alpha \in A} \langle \omega, \alpha^* \rangle$.*

Proof. We know that the forms γ^* are non-negative on WB . Therefore, if ω is in the interior of K the integers $\langle \omega, \gamma^* \rangle$ are all positive. Conversely, if this is the case then in

particular $\langle \omega, l^\vee \rangle > 0$ holds for each extremal form l^\vee . As ω is W -invariant this is still true if l^\vee is in the W -orbit of an extremal form. Then, by Corollary 1.6, ω is in the interior of K .

The rest of the proposition follows at once from Theorem 2.14 when the special choice (2.15) for S_γ and J is used. Q.E.D.

3. The fundamental group of the complement of the discriminant

Let \mathcal{D} be a diagram and fix an integer $m > 0$. The purpose of this section is to describe a natural presentation of the fundamental group $\pi_1((\mathcal{R}/W_m) \setminus \Delta_m)$. For the result see Theorem 3.10 below. A special case was already done in Wirthmüller [54], Section 4 by the same method.

In view of Corollary 2.17 it suffices to consider the case $m=1$. We first study the action of W on \mathcal{T} . By definition it is induced from an action of $\tilde{W} = \Lambda W$ on the universal cover V_C . The group \tilde{W} as well as the affine Weyl group $W' := QW$ act on V_C by affine transformations, and together with the group of translations ZB they fit into the exact sequence

$$0 \rightarrow W' \rightarrow \tilde{W} \rightarrow ZB \rightarrow 0. \quad (3.1)$$

At this point it is convenient to introduce a bit of general notation. Let X be a topological space with a properly discontinuous group action (assumed to be implied by the context). For such X we let X^{reg} denote the union of all regular orbits, that is, the open subset of points with trivial isotropy group. In this notation we may write $(\mathcal{R}/W) \setminus \Delta = \mathcal{R}^{\text{reg}}/W$.

The fundamental group of V_C^{reg}/W' may be computed by the method of Brieskorn [7] and will turn out to be an Artin group, see Brieskorn and Saito [8]. Let \tilde{A} be the set of vertices of the completed Dynkin diagram $\tilde{\mathcal{D}}_{\text{black}}$; thus $\tilde{A} \setminus A$ is in one-to-one correspondence with the irreducible factors of the root system R . The embedding of A in V is extended naturally by assigning to each irreducible factor of R minus its greatest root. Similarly we have $\tilde{A} \subset V^\vee$, and this embedding defines the fundamental alcove

$$C = \{ \alpha^\vee > 0 \text{ if } \alpha \in A; \alpha^\vee > -1 \text{ if } \alpha \in \tilde{A} \setminus A \} \subset V.$$

C is a fundamental domain for the action of W' on V^{reg} , and W' is generated by the reflections in the walls of C ,

$$w_\alpha: x \mapsto \begin{cases} x - \langle x, \alpha^\vee \rangle \alpha & (\alpha \in A) \\ x - (1 + \langle x, \alpha^\vee \rangle) \alpha & (\alpha \in \tilde{A} \setminus A), \end{cases}$$

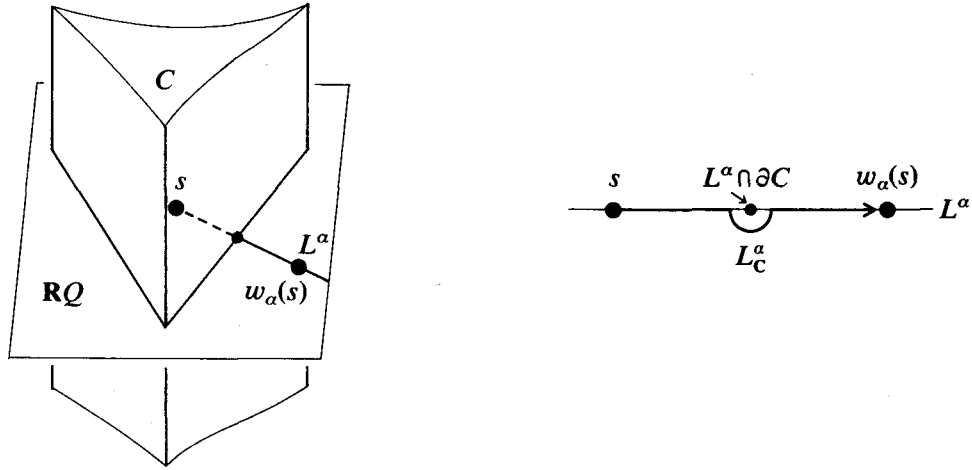


Figure 3.2

see Bourbaki [4], Chapter VI, §2, no. 1. Let $s \in V_C$ be the barycentre of the bounded set $C \cap RQ$. The projections of s in V_C^{reg}/W' and V_C^{reg}/\tilde{W} will serve as base points for the fundamental groups. For each $\alpha \in \tilde{A}$ we let L^α (and L_C^α) be the real (or complex) affine line through s and $w_\alpha(s)$. The element

$$a_\alpha \in \pi_1(V_C^{reg}/W')$$

is, by definition, represented by the path in L_C^α that follows the real segment joining s to $w_\alpha(s)$ but avoids the point of intersection with the reflection hyperplane of w_α on a small positively oriented semi-circle (see Figure 3.2).

LEMMA 3.3. $\pi_1(V_C^{reg}/W')$ is the Artin group with generators $a_\alpha (\alpha \in \tilde{A})$ and relations according to the completed Dynkin diagram \tilde{D}_{black} .

Proof. This is the extension of the result of Brieskorn [7] to affine Weyl groups; the proof is virtually the same as Brieskorn's for finite reflection groups. Q.E.D.

Next we study the covering projection

$$V_C^{reg}/W' \rightarrow V_C^{reg}/\tilde{W} = \mathcal{J}^{reg}/W$$

with Galois group ZB (as the latter still acts freely on V_C/W' the apparent ambiguity in the notation does not matter). For each $\beta \in B$ let w_β be the unique element of W' that

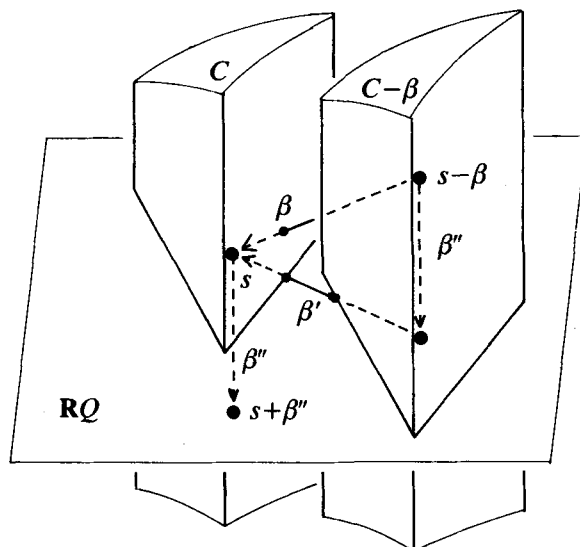


Figure 3.4

sends C to the alcove $C-\beta$. As $\beta \cdot w_\beta \in \bar{W}$ leaves C invariant it induces a permutation on \bar{A} which we denote $\alpha \mapsto \beta(\alpha)$. We split $\beta = \beta' + \beta''$ into $\beta' \in RQ$ and a W -fixed part β'' . (See Figure 3.4.) Then the segment connecting s to $s + \beta''$ projects to a loop in V_C^{reg}/\bar{W} and defines a homotopy class

$$h_\beta \in \pi_1(V_C^{\text{reg}}/\bar{W}).$$

Clearly the h_β ($\beta \in B$) represent a basis for the Galois group ZB . The action of h_β on $\pi_1(V_C^{\text{reg}}/W')$ is read off from Figure 3.5. The unlabelled path is the image of the path defining a_α under the transformation $\beta \cdot w_\beta \in \bar{W}$. As it is clearly homotopic to $h_\beta^{-1} a_{\beta(\alpha)} h_\beta$ we see that

$$h_\beta^{-1} a_{\beta(\alpha)} h_\beta = a_\alpha \tag{3.6}$$

holds in $\pi_1(V_C^{\text{reg}}/\bar{W})$. This describes the action of ZB on $\pi_1(V_C^{\text{reg}}/W')$ completely, and we have shown:

LEMMA 3.7. $\pi_1(V_C^{\text{reg}}/\bar{W})$ is the extension of the Artin group $\pi_1(V_C^{\text{reg}}/W')$ by the free abelian group ZB , acting via (3.6). Q.E.D.

We finally determine $\pi_1(\mathcal{X}^{\text{reg}}/W)$. Let \mathcal{X}'' be the union of \mathcal{T} and the one-codimen-

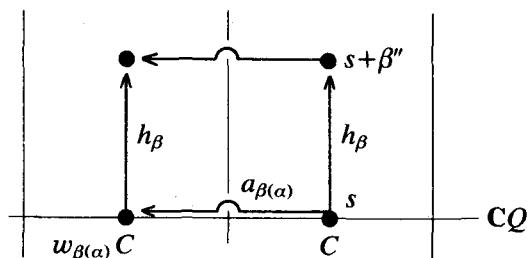


Figure 3.5

sional \mathcal{T} -orbits in \mathcal{X} . As \mathcal{X}/W_m is smooth the inclusion $\mathcal{X}'' \subset \mathcal{X}$ induces an isomorphism of fundamental groups $\pi_1(\mathcal{X}''^{\text{reg}}/W) \rightarrow \pi_1(\mathcal{X}^{\text{reg}}/W)$. In order to obtain $\mathcal{T}^{\text{reg}}/W$ from $\mathcal{X}''^{\text{reg}}/W$ one has to remove a finite number of closed connected complex hypersurfaces, one for each $\beta \in B$. Therefore the canonical homomorphism

$$\pi_1(\mathcal{T}^{\text{reg}}/W) \rightarrow \pi_1(\mathcal{X}^{\text{reg}}/W) \tag{3.8}$$

is surjective and the kernel is generated, as a normal subgroup, by loops of the following type. For each $\beta \in B$ let $\mathcal{T}_\beta \subset \mathcal{X}$ be the corresponding \mathcal{T} -orbit, and choose a disc in $\mathcal{X}''^{\text{reg}}/W$ which meets the image of $\mathcal{T}_\beta \cap \mathcal{X}^{\text{reg}}$ transversely in a single point. Then consider a loop that first connects the base point s to the boundary of the disc, then goes once around the boundary, and follows the initial part back to s .

We make a particular choice as follows. Let L_C^β be the complex affine line through s and $s + \beta$ (see Figure 3.9). The shaded region plus the point $s + i\infty\beta$ project to a disc in $\mathcal{X}''^{\text{reg}}/W$, and the arrows indicate a path of the desired type. Using the decomposition $\beta = \beta' + \beta''$ we may deform this path into a product $a_\beta h_\beta^{-1}$ where h_β is as above and a_β is

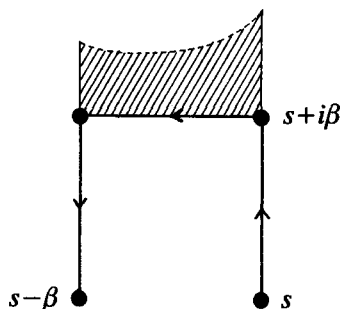


Figure 3.9

represented as follows. Connect s linearly to $s-\beta'$ but avoid the reflection hyperplanes on small positively oriented semi-circles in the complex line through s and $s-\beta'$.

Finally, we sum up the results of this section.

THEOREM 3.10. *The fundamental group $\pi_1(\mathcal{X}^{\text{reg}}/W)$ is the quotient of the Artin group*

$$\langle a_\alpha | \alpha \in \tilde{A} \rangle$$

by the normal subgroup generated by the relations

$$a_\beta^{-1} a_{\beta(\alpha)} a_\beta = a_\alpha \quad (\alpha \in \tilde{A}, \beta \in B)$$

$$a_\beta a_{\beta'} = a_{\beta'} a_\beta \quad (\beta, \beta' \in B).$$

The element $a_\beta \in \langle a_\alpha | \alpha \in \tilde{A} \rangle$ may be determined as follows. If $w_\beta \in W'$ is written as a word of minimal length in the generating reflections

$$w_\beta = w_{\alpha_1} \cdots w_{\alpha_l} \quad (\alpha_1, \dots, \alpha_l \in \tilde{A})$$

then

$$a_\beta = a_{\alpha_1} \cdots a_{\alpha_l}.$$

Proof. All but the characterization of a_β follows from the discussion preceding the theorem. In our geometric setting the canonical homomorphism from the Artin group $\langle a_\alpha | \alpha \in \tilde{A} \rangle$ to its associated Coxeter group—see Brieskorn and Saito [8]—is just the homomorphism $\pi_1(V_{\mathbb{C}}^{\text{reg}}/W') \rightarrow W'$ that describes the Galois cover $V_{\mathbb{C}}^{\text{reg}} \rightarrow V_{\mathbb{C}}^{\text{reg}}/W'$. Thus a representative path in $V_{\mathbb{C}}^{\text{reg}}$, starting at s and ending at $t \in W's$, is sent to the unique $w \in W'$ with $t=ws$. Now recall the definition of a_β and consider the real segment joining s to $s-\beta'$. Of course, this segment can meet each reflection hyperplane at most once, and after a C^1 -small deformation in V it will also meet only one at a time.

Such a deformation determines a new path representing a_β , and in view of the orientation convention a_β is thereby expressed as a word in the $a_\alpha (\alpha \in \tilde{A})$. In W' this word projects to a representation of w_β as a word in the w_α . The length of this word is the number of reflection hyperplanes that separate s and $s-\beta'$, and therefore is minimal among all such representations of w_β . On the other hand all words in the a_α with this minimality property represent the same element in the Artin group $\langle a_\alpha | \alpha \in \tilde{A} \rangle$. This follows from Tits [45], Théorème 3 (alternatively, a geometric proof is that of Deligne [9], (1.12)). This completes the proof of the theorem. Q.E.D.

4. $D_k[\ast]$: The discriminant

We fix an integer $k \geq 4$ and consider the affine algebraic space curve

$$X_0 = \{x^2 = y^2 + z^{k-2}, yz = 0\} \subset \mathbb{C}^3. \tag{4.1}$$

X_0 is always reducible; depending on the parity of k there are three or four components which are all defined over the reals.

The singularity of type $D_k[\ast]$ is the analytic equivalence class of the germ $(X_0, 0)$ at the origin. We let \mathbb{C}^* act linearly on \mathbb{C}^3 with

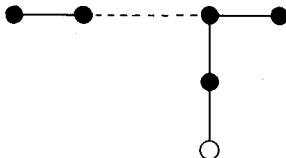
$$\text{weight}(x) = \text{weight}(y) = 2k - 4, \text{weight}(z) = 4.$$

The semi-universal deformation of $(X_0, 0)$ has a global \mathbb{C}^* -equivariant representative

$$X \xrightarrow{\pi} S$$

with affine spaces X and S . We let $D \subset S$ be the discriminant of π .

Recall the diagram $D_k[\ast]$,



introduced in (1.1).

THEOREM 4.2. *There exists a \mathbb{C}^* -equivariant isomorphism*

$$\mathcal{A}(D_k[\ast])/W_2(D_k[\ast]) \xrightarrow{\Phi} S$$

which respects the discriminants, that is, $\Phi(\Delta_2(D_k[\ast])) = D$.

The proof of this result is the purpose of the present section while supplements and consequences of the theorem will be discussed in the next.

Using the criterion of Kas and Schlessinger [20] a semi-universal deformation of $(X_0, 0)$ may be constructed explicitly. We let P^{k-2} be the affine $(k-2)$ -space of unitary polynomials of degree $k-2$,

$$p(Z) = Z^{k-2} + \sum_{j=0}^{k-3} p_j Z^j,$$

and put

$$S = \{(b, u, v, p)\} = \mathbb{C}^3 \times P^{k-2}.$$

We let $X \subset S \times \mathbb{C}^3 = \mathbb{C}^3$ be the variety

$$X = \left\{ \begin{array}{l} x^2 = y^2 + 2by + p(z) \\ yz = ux + v \end{array} \right\};$$

then the projection $X \xrightarrow{\pi} S$ is a semi-universal deformation of $(X_0, 0)$, the special fibre X_0 sitting over $(0, Z^{k-2}) \in \mathbb{C}^3 \times P^{k-2} = S$. This deformation carries a \mathbb{C}^* -action given by the following table of weights.

x	y	z	b	u	v	p_j
$2k-4$	$2k-4$	4	$2k-4$	4	$2k$	$4(k-j-2)$

Table 4.3

The fibres of π may be compactified by embedding \mathbb{C}^3 in a suitable weighted projective space ('espace projectif anisotrope' in Delorme [10]). For any sequence $(\alpha_0, \alpha_1, \dots, \alpha_n)$ of positive integers, any n of which are relatively prime, we let $\mathbb{P}_{\alpha_0, \alpha_1, \dots, \alpha_n}^n$ be the projective n -space with weights $(\alpha_0, \alpha_1, \dots, \alpha_n)$. If $(x_0, x_1, \dots, x_n) \in \mathbb{C}^{n+1}$ is a non-zero vector $[x_0: x_1: \dots: x_n]$ will denote the point of $\mathbb{P}_{\alpha_0, \alpha_1, \dots, \alpha_n}^n$ represented by it. Any integer l defines a sheaf $\mathcal{O}(l)$ on $\mathbb{P}_{\alpha_0, \alpha_1, \dots, \alpha_n}^n$; its local sections are the homogeneous regular functions of degree l (with respect to the weights). $\mathcal{O}(l)$ is a reflexive sheaf of rank one; it is invertible exactly if $\alpha_0, \alpha_1, \dots, \alpha_n$ all divide l , see [loc. cit.], 1.5.

Returning to the situation at hand, we define the integer m by

$$k = 2m+1 \quad \text{or} \quad k = 2m+2$$

and embed \mathbb{C}^3 in \mathbb{P}_{1mm1}^3 , sending (x, y, z) to $[1:x:y:z]$. We let Y be the closure of X in $\mathbb{P}_{1mm1, S}^3$; thus π extends to a projective morphism $Y \xrightarrow{\pi} S$. The polynomial p has the homogenized form of degree $2m$

$$p(W, Z) = W^{2m}p(Z/W),$$

and Y is defined in $\mathbb{P}_{1mm1, S}^3$ by the equations

$$\begin{aligned} x^2 &= y^2 + 2bw^m y + p(w, z), \\ yz &= uwx + vw^{m+1}. \end{aligned} \tag{4.4}$$

Let $Z \subset \mathbf{P}^3_{1mm1, S}$ be the relative surface defined by the second equation (4.4), and let $Z_{S'}$ be its restriction over $S' = S \setminus \{u=0\}$.

PROPOSITION 4.5. *There exists an S' -morphism $L: Z_{S'} \rightarrow \mathbf{P}^1_{S'}$ which makes $Z_{S'}$ a smooth family of rational ruled surfaces with invariant $m-1$.*

Proof. The morphism L is defined by the formula

$$L([w: x: y: z]) = [w: z] = [y: ux + vw^m].$$

A straightforward computation proves that $Z_{S'}$ is smooth over S' , and that the fibre of L over $[\xi: \eta] \in \mathbf{P}^1$ is the smooth rational curve

$$L_{[\xi: \eta]} := \left\{ \begin{aligned} \eta w &= \xi z \\ \eta y &= \xi(ux + vw^m) \end{aligned} \right\} \subset \mathbf{P}^3_{1mm1, S'}. \tag{4.6}$$

It remains to determine the invariant of the ruled surfaces $Z_s (s \in S')$. To this end we consider two particular sections of L ,

$$\begin{aligned} \Sigma_0: [\xi: \eta] &\mapsto \left[\xi: \frac{-v}{u} \xi^m: 0: \eta \right] \\ \Sigma_\infty: [\xi: \eta] &\mapsto [0: \eta: u\xi: 0]. \end{aligned}$$

Ambiguously, we use the same symbols Σ_0 and Σ_∞ to denote the images of these sections in $Z_{S'}$. Note that Σ_0 and Σ_∞ do not intersect.

The homogeneous forms w^m and y are sections of the invertible sheaf $\mathcal{O}(m)$ on \mathbf{P}^3_{1mm1} . On $Z_{S'}$ they cut out the divisors

$$(w^m) = \Sigma_\infty + mL_{[0: 1]}, \tag{4.7}$$

respectively

$$(y) = \Sigma_0 + L_{[0: 1]}. \tag{4.8}$$

Fix $s \in S'$. Letting L denote the divisor class of the lines on the ruled surface Z_s we compute intersection numbers in Z_s :

$$\Sigma_\infty^2 + 2m = (w^m)^2 = (w^m)(y) = m + 1,$$

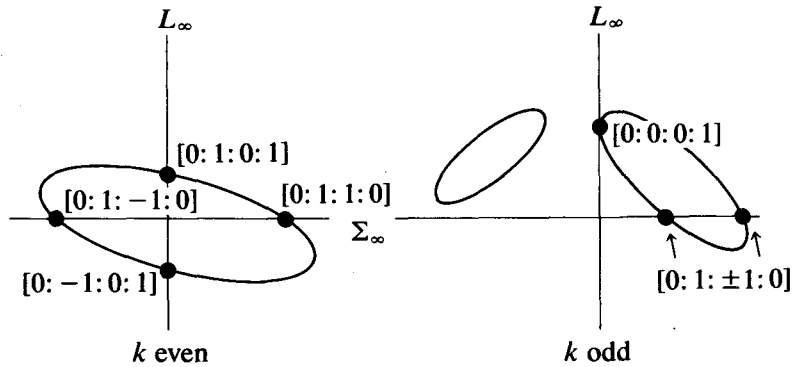


Figure 4.10

hence $\Sigma_\infty^2 = -(m-1)$. On a ruled surface the only section with negative self-intersection number is the section at infinity—see Hartshorne [17], Chapter V, Proposition 2.20. This completes the proof. Q.E.D.

It will be convenient to use an affine coordinate on the base \mathbf{P}^1 . When doing so we shall identify $[1:\eta] \in \mathbf{P}^1$ with $\eta \in \mathbf{C}$, and $[0:1]$ with infinity.

We now turn to the fibres of the completed deformation $Y \xrightarrow{\pi} S$.

PROPOSITION 4.9. *Y is a flat curve of arithmetic genus m over S . Its points at infinity, that is, on $Z \cap \{w=0\} = \Sigma_\infty \cup L_\infty$, are as shown in Figure 4.10.*

Proof. The stated behaviour at infinity follows at once from the equations (4.4). In particular $Y \xrightarrow{\pi} S$ is smooth at infinity, so it is flat everywhere. It follows that the arithmetic genus of the fibre Y_s does not depend on the choice of $s \in S$: Hartshorne [17], Chapter III, Corollary 9.10. We choose $s \in S'$ and apply the adjunction formula in Z_s (p the arithmetic genus, κ the canonical divisor class):

$$\begin{aligned} 2p(Y_s) - 2 &= \deg \kappa(Y_s) \\ &= (\kappa(Z_s) + (Y_s)) \cdot (Y_s) \\ &= (-2(\Sigma_\infty) - (m+1)L + 2(w^m)) \cdot 2(w^m) \\ &= 2m - 2, \end{aligned}$$

using (4.7).

Q.E.D.

Note the intersection number

$$L \cdot (Y_s) = 2 \quad \text{in } Z_s \quad (s \in S').$$

For general $s \in S'$ the curve Y_s together with the projection $Y_s \xrightarrow{L} \mathbf{P}_s^1$ is hyperelliptic (in order to include the case $m=1$ we use the term ‘hyperelliptic curve’ as a synonym for ‘branched double cover of \mathbf{P}^1 ’). The S' -morphism $Y_{S'} \xrightarrow{L} \mathbf{P}_{S'}^1$ fails to define a family of hyperelliptic curves because it is not finite: for special values of s the curve Y_s may contain one or both of the lines $L_{\pm u} \subset Z_s$. Nevertheless it is possible to relate $Y_{S'}$ to a true family of hyperelliptic curves by a suitable modification of the relative surface $Z_{S'}$.

We construct \tilde{Z} from $Z_{S'}$ by blowing up the S' -valued points $[0: \pm 1: 1: 0] \in \Sigma_\infty$, and let \tilde{Y} , $\tilde{\Sigma}_0$, $\tilde{\Sigma}_\infty$, \tilde{L}_η denote the strict transforms of $Y_{S'}$, Σ_0 , Σ_∞ , L_η , respectively. The natural map $\tilde{Y} \rightarrow Y_{S'}$ is an isomorphism because Y is smooth along the centres of the blowing-ups. \tilde{L}_u and \tilde{L}_{-u} are smooth S' -families of exceptional curves of the first kind on \tilde{Z} . By Castelnuovo’s criterion they can be blown down to S' -valued points. Let $\tilde{Z} \xrightarrow{\pi} S'$ be the resulting surface over S' , and let \tilde{Y} , $\tilde{\Sigma}_0$, and $\tilde{\Sigma}_\infty$ be the images of \tilde{Y} , $\tilde{\Sigma}_0$, and $\tilde{\Sigma}_\infty$ in \tilde{Z} . It is clear by construction that \tilde{L} induces an S' -morphism $\tilde{L}: \tilde{Z} \rightarrow \mathbf{P}_{S'}^1$. The situation is summarized in the diagram of S' -morphisms:

$$\begin{array}{ccccc}
 & & \tilde{Z} & & \\
 & \swarrow & \cup & \searrow & \\
 Z_{S'} \supset Y_{S'} & \xleftarrow{\cong} & \tilde{Y} & \longrightarrow & \tilde{Y} \subset \tilde{Z} \\
 & \searrow L & \downarrow & \swarrow \tilde{L} & \\
 & & \mathbf{P}_{S'}^1 & &
 \end{array} \tag{4.11}$$

\tilde{L} gives \tilde{Z} the structure of a smooth family of ruled surfaces over S' , all with invariant $m+1$, for each individual surface \tilde{Z}_s ($s \in S'$) is obtained from Z_s by two elementary transforms with centres on the section at infinity—Nagata [35], Section 2, (3).

Note that the restriction $\tilde{Y} \xrightarrow{L} \mathbf{P}_{S'}^1$ is a finite morphism of degree 2 and therefore defines a flat family of hyperelliptic curves over S' . This fact will now be used to trivialize the family $\tilde{Z} \xrightarrow{\pi} S'$. We first set up models for \tilde{Z} and \tilde{Y} .

A standard model for the ruled surface F_{m+1} (with invariant $m+1$) is obtained from the weighted projective plane $\mathbf{P}_{1,1,m+1}^2$ by blowing up the singular point at $[0:0:1]$. The structure of a ruled surface is given by the projection

$$l: F_{m+1} \rightarrow \mathbf{P}^1,$$

induced from $\mathbf{P}_{1,1,m+1}^2 \ni [\xi: \eta: \zeta] \mapsto [\xi: \eta] \in \mathbf{P}^1$. The exceptional divisor of the resolution is the section at infinity $\sigma_\infty \subset F_{m+1}$.

With $S' = \text{Spec } \mathcal{R}$ we have $\mathbf{P}_{S'}^1 = \text{Proj } \mathcal{R}[\Xi, H]$. As the branch locus of $\bar{L}: \bar{Y} \rightarrow \mathbf{P}_{S'}^1$ is a hypersurface in $\mathbf{P}_{S'}^1$, its ideal is generated by a homogeneous polynomial $g(\Xi, H) \in \mathcal{R}[\Xi, H]$ of degree $2m+2$. To make g unique we require that the inhomogeneous polynomial $g(H) := g(1, H)$ be unitary. For all $s \in S'$ the point $\infty \in \mathbf{P}^1$ is either no branch point ($k=2m+2$) or a simple branch point ($k=2m+1$) of $\bar{L}_s: \bar{Y}_s \rightarrow \mathbf{P}_s^1$, see Figure 4.10. Therefore the degree of $g(H)$ is exactly k . Our model for \bar{Y} is the relative hyperelliptic curve

$$G \subset F_{m+1, S'}$$

with equation $\zeta^2 = g(\xi, \eta)$.

THEOREM 4.12. *There exist exactly two $\mathbf{P}_{S'}^1$ -isomorphisms*

$$\bar{Z} \xrightarrow{\varphi} F_{m+1, S'}$$

with $\varphi(\Sigma_\infty) = \sigma_\infty$ and $\varphi(\bar{Y}) = G$.

Proof. We first show that there are exactly two $\mathbf{P}_{S'}^1$ -isomorphisms $\bar{Y} \xrightarrow{\psi} G$. Such a ψ is the same as a $\mathbf{P}_{S'}^1$ -map between the branched covers $\bar{Y} \rightarrow S' \times \mathbf{P}^1$ and $G \rightarrow S' \times \mathbf{P}^1$ which is a mere homeomorphism (with respect to the classical topology). For \bar{Y} is a hypersurface in the smooth variety \bar{Z} , and is regular in codimension one, hence is normal. Likewise G is normal, and in view of Riemann's extension theorem every $\mathbf{P}_{S'}^1$ -homeomorphism ψ is biholomorphic. In fact, ψ is even algebraic as its graph is an irreducible component of the algebraic variety $\bar{Y} \times_{S' \times \mathbf{P}^1} G$, by Grothendieck et al. [15], Exposé XII, Proposition 2.4.

To solve the topological problem we look at the restriction of the branched double cover $\bar{Y} \rightarrow S' \times \mathbf{P}^1$ over $S' \times \{\infty\}$. There, Proposition 4.9 either trivializes the cover (k even) or provides a common coordinate on the tangent spaces to \bar{Y} (k odd). As the degree of g is exactly k the equation $\zeta^2 = g(\xi, \eta)$ does the same for the restricted cover $G_{S' \times \{\infty\}} \rightarrow S' \times \{\infty\}$. It follows that the two covers are isomorphic over some neighborhood of $S' \times \{\infty\}$, and there are just two isomorphisms. As \bar{Y} and G have by definition identical branching everywhere on $S' \times \mathbf{P}^1$ these isomorphisms extend global-

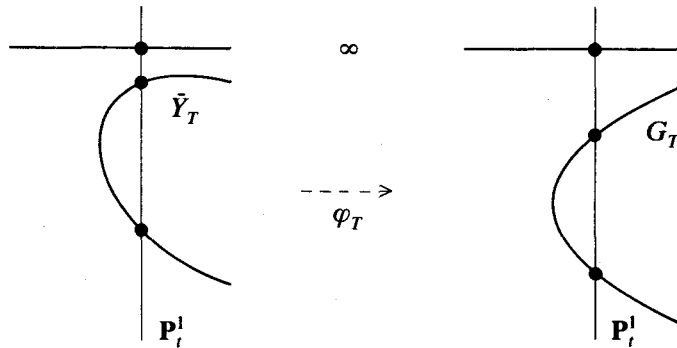


Figure 4.13

ly. This completes the first part of the proof; we must still see that each $\mathbf{P}_{S'}^1$ -isomorphism $\tilde{Y} \xrightarrow{\psi} G$ has a unique extension $\tilde{Z} \xrightarrow{\varphi} F_{m+1, S'}$ with $\varphi(\Sigma_\infty) = \sigma_\infty$.

\tilde{Z} and $F_{m+1, S'}$ are smooth families of rational curves over $\mathbf{P}_{S'}^1$, with a distinguished section at infinity. It follows that both are locally trivial \mathbf{P}^1 -bundles over $\mathbf{P}_{S'}^1$, compare Hartshorne [17], Chapter V, Proposition 2.2. The extension problem is local over $\mathbf{P}_{S'}^1$; over suitable open sets $T \subset \mathbf{P}_{S'}^1$, it reads

$$\begin{array}{ccc}
 \tilde{Y}_T & \xrightarrow{\psi_T} & G_T \\
 \cap & & \cap \\
 \mathbf{P}_T^1 & \xrightarrow{\varphi_T} & \mathbf{P}_T^1 \\
 \cup & & \cup \\
 \{\infty\}_T & \xlongequal{\quad} & \{\infty\}_T.
 \end{array}$$

Note that neither \tilde{Y} nor G meets the section at infinity (see Figure 4.13). \tilde{Y}_T is a relative divisor in \mathbf{P}_T^1 of degree 2 while the morphism ψ_T defines a section in $H^0(\tilde{Y}_T, \mathcal{O}(1))$. Solutions φ_T correspond to extensions of this section to $H^0(\mathbf{P}_T^1, \mathcal{O}(1))$. But for each $t \in T$ the restriction homomorphism $H^0(\mathbf{P}_t^1, \mathcal{O}(1)) \rightarrow H^0(\tilde{Y}_t, \mathcal{O}(1))$ is bijective, and as the higher cohomology vanishes the theorem on cohomology and base change implies that $H^0(\mathbf{P}_T^1, \mathcal{O}(1)) \rightarrow H^0(\tilde{Y}_T, \mathcal{O}(1))$ is also bijective. Thus there are unique local extensions φ_T , which yield the global extension φ by patching. Q.E.D.

The polynomial $g \in \mathcal{R}[H]$ can be computed explicitly. By (4.4) and (4.6) the points of intersection of $Y_{S'}$ with the line L_η are the solutions of

$$\begin{aligned}x^2 &= y^2 + 2bw^m y + p(w, z), \\ \eta w &= z, \\ \eta y &= ux + vw^m.\end{aligned}$$

This system reduces to a single quadratic equation

$$(u^2 - \eta^2)y^2 + 2(bu^2 + \eta v)w^m y + (p(\eta)u^2 - v^2)w^{2m} = 0$$

in $[w: y] \in \mathbf{P}_{1m}^1$. Its discriminant is

$$(bu^2 + \eta v)^2 - (u^2 - \eta^2)(p(\eta)u^2 - v^2),$$

and dividing by the leading coefficient we obtain $g(\eta) = \eta^k + \sum_{j=0}^{k-1} g_j \eta^j$ with:

$$\begin{aligned}g_{k-1} &= p_{k-3} \\ g_{k-2} &= p_{k-4} - u^2 \\ g_{k-3} &= p_{k-5} - u^2 p_{k-3} \\ &\vdots \\ g_2 &= p_0 - u^2 p_2 \\ g_1 &= 2bv - u^2 p_1 \\ g_0 &= u^2 b^2 + v^2 - u^2 p_0\end{aligned}$$

Table 4.14

PROPOSITION 4.15. *The morphism $Y_S \xrightarrow{L} \mathbf{P}_S^1$ extends to $Y \xrightarrow{L} \mathbf{P}_S^1$, and G extends to a hyperelliptic curve $G_S \subset F_{m+1, S} \xrightarrow{l} \mathbf{P}_S^1$. The composition*

$$Y_S \simeq \tilde{Y} \rightarrow \tilde{Y} \xrightarrow{\varphi} G \tag{4.16}$$

extends to a \mathbf{P}_S^1 -morphism $\psi: Y \rightarrow G_S$.

Proof. The first statement is clear from the definition (4.6), the second from Table 4.14. The morphism ψ exists because Y is normal and G_S is finite over \mathbf{P}_S^1 . Q.E.D.

By Proposition 4.9 the original family of curves $Y \xrightarrow{\pi} S$ is equipped with distinguished sections at infinity. Those on L_∞ have already served to construct the morphism $Y \xrightarrow{\psi} G_S$ and lose their special significance when mapped to G_S . On the other hand, ψ sends the sections $[0: 1: \pm 1: 0] \in Y \cap \Sigma_\infty$ to sections σ_\pm of $G_S \xrightarrow{\pi} S$ which put an extra structure on this family.

In affine coordinates ($\xi \equiv 1$ on F_{m+1}) we have $\sigma_{\pm} = (\pm u, \zeta_{\pm})$ with $\zeta_{\pm}^2 = g(\pm u) = (ub \pm v)^2$, by Table 4.14. To pin down the signs of ζ_{\pm} we test a particular value of $s \in S$: put $b = u = 0$, $v = 1$, and $p(Z) = Z^{k-2}$. Then $g(H) = H^k + 1$, both Y_s and G_s are regular, and ψ maps Y_s isomorphically to G_s . In particular ζ_+ and ζ_- are distinct at s , hence either $\zeta_{\pm} = ub \pm v$ or $\zeta_{\pm} = -(ub \pm v)$ holds throughout. These cases correspond to the two choices of φ allowed by Theorem 4.12, and we fix φ so that

$$\zeta_{\pm} = ub \pm v. \tag{4.17}$$

Recall that P^k is the space of unitary complex polynomials of degree k , and put

$$T = \{(u, g, \zeta_+, \zeta_-) \in \mathbb{C} \times P^k \times \mathbb{C}^2 \mid \zeta_{\pm}^2 = g(\pm u)\}.$$

The family G_s together with its sections σ_{\pm} is completely described by the morphism

$$\begin{aligned} \bar{g}: S &\rightarrow T \\ s &\mapsto (u, g_s, \zeta_+, \zeta_-). \end{aligned}$$

\bar{g} is not finite but it restricts to an isomorphism between large open subsets of S and T . We put

$$\begin{aligned} S'' &= S \setminus \{u = v = 0\} \\ T'' &= T \setminus \{u = 0, \zeta_+ = \zeta_-\}. \end{aligned}$$

PROPOSITION 4.18. *\bar{g} restricts to an isomorphism*

$$\bar{g}'': S'' \rightarrow T''.$$

It sends the discriminant $D \cap S''$ to the hypersurface

$$\Delta := \{(u, g, \zeta_+, \zeta_-) \in T'' \mid g \text{ has a multiple root}\}.$$

Proof. Table 4.14 and (4.17) show that \bar{g}'' has an inverse. For any $s \in S'$ the curve \bar{Y}_s is, by construction, singular exactly if Y_s is singular. As \bar{Y}_s is isomorphic to the hyperelliptic curve G_s it follows that \bar{g} maps $D \cap S'$ to $\Delta \setminus \{u=0\}$. This proves $\bar{g}(D \cap S'') = \Delta$, for both $D \cap S'' \subset S''$ and $\Delta \subset T''$ are hypersurfaces and neither contains $\{u=0\}$. Q.E.D.

Let $T' = T \setminus \{u=0\}$. We shall identify T' with the quotient $\mathcal{T}(D_k[*])/W_2(D_k[*])$, as follows. Let $R \subset \mathbb{R}^k$ be the standard model of the root system of type D_k , see Bourbaki

[4], Planche IV. The base roots are

$$\begin{aligned}\alpha_1 &= (1, -1, 0, \dots, 0) \\ \alpha_2 &= (0, 1, -1, 0, \dots, 0) \\ &\vdots \\ \alpha_{k-1} &= (0, \dots, 0, 1, -1) \\ \alpha_k &= (0, \dots, 0, 1, 1),\end{aligned}$$

and each root α is identified with its dual α^\vee via the standard Euclidean metric on \mathbf{R}^k . We put $V = \mathbf{R} \times \mathbf{R}^k$ and embed \mathbf{R}^k as $\{0\} \times \mathbf{R}^k$; then the white vertex of the diagram $D_k[\ast]$ may be realized as

$$\beta = \frac{1}{2}(1, -1, \dots, -1) \in V.$$

The root lattice is

$$Q = \{t = (t_0; t_1, \dots, t_k) \in \mathbf{Z} \times \mathbf{Z}^k \mid t_0 = 0, \sum_{j=1}^k t_j \equiv 0(2)\}.$$

Recall that $\mathcal{T} = (\mathbf{C} \times \mathbf{C}^k) / \Lambda$ with

$$\Lambda = Q + \mathbf{Z}\beta = \{t \in \frac{1}{2}\mathbf{Z} \times \mathbf{Z}^k \mid kt_0 + \sum_{j=1}^k t_j \equiv 0(2)\}.$$

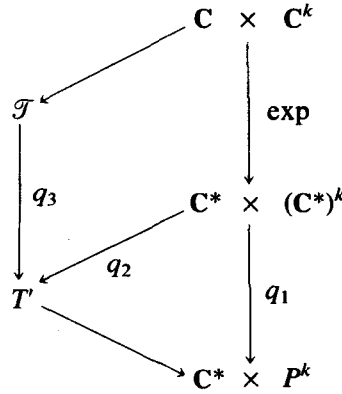
An element of the Weyl group W acts on t by permuting the last k components and changing the signs of an even number of them.

Let $q: \mathbf{C}^* \rightarrow \mathbf{C}$ send $\tau \in \mathbf{C}^*$ to $\frac{1}{2}(\tau^2 + \tau^{-2}) \in \mathbf{C}$; this is a fourfold Galois cover and its Galois group Γ is generated by the involutions

$$\begin{aligned}\gamma: \tau &\mapsto -\tau, \\ \delta: \tau &\mapsto \tau^{-1}.\end{aligned}\tag{4.19}$$

The branch locus of q is $\{\pm 1\} \subset \mathbf{C}$.

We set up the following diagram of branched Galois covers, of which q is an ingredient. The unlabelled arrow $\mathbf{C} \times \mathbf{C}^k \rightarrow \mathcal{T}$ indicates the quotient by Λ while $T' \rightarrow \mathbf{C}^* \times P^k$ projects $(u, g, \zeta_+, \zeta_-) \in \mathbf{C}^* \times P^k \times \mathbf{C}^2$ to (u, g) . The covering projection q_1 consists, essentially, of k copies of q followed by the quotient map of the symmetric group $\text{Sym}(k)$, acting on



\mathbb{C}^k by coordinate permutations. q_1 sends $(\tau_0; \tau_1, \dots, \tau_k) \in \mathbb{C}^* \times (\mathbb{C}^*)^k$ to

$$\left(\tau_0^2; \prod_{j=1}^k (H - \tau_0^2 q(\tau_j)) \right) \in \mathbb{C}^* \times P^k. \tag{4.20}$$

To factorize q_1 through T' we define q_2 by

$$\xi_{\pm} = \pm (\tau_0 / i\sqrt{2})^k \prod_{j=1}^k (\tau_j \mp \tau_j^{-1}). \tag{4.21}$$

The Galois group of q_1 is the direct product of the group $\{\pm 1\}$ (acting on τ_0) with the wreath product

$$(\Gamma \times \Gamma) \wr \text{Sym}(k),$$

acting on (τ_1, \dots, τ_k) via (4.19). The subgroup corresponding to T' is the kernel of the homomorphism

$$\begin{aligned} & \{\pm 1\} \times (\Gamma \times \Gamma) \wr \text{Sym}(k) \rightarrow \Gamma \\ & (\varepsilon; \gamma_1, \delta_1, \dots, \gamma_k, \delta_k; \sigma) \mapsto \left(\varepsilon^k \prod_{j=1}^k \gamma_j, \prod_{j=1}^k \delta_j \right). \end{aligned}$$

It follows that the composition $q_2 \circ \exp$ factors through \mathcal{T} as indicated. Furthermore the Galois group of q_3 is just the group W_2 defined in Section 2, so that q_3 identifies T' with the quotient \mathcal{T}/W_2 .

As before, let \mathcal{X}''' denote the torus embedding of \mathcal{T} that corresponds to the set of rays

$$\{\mathbf{R}_+ \cdot (w\beta) | w \in W\}$$

in V .

PROPOSITION 4.22. q_3 extends to a morphism $q_3'': \mathcal{X}'' \rightarrow T''$.

Proof. As q_3 is W -invariant it suffices to extend it over the \mathcal{T} -orbit \mathcal{T}_β corresponding to the ray $\mathbf{R}_+\beta$. For each j ($1 \leq j \leq k$) the characters $\tau_0 \tau_j$ and τ_0 / τ_j extend over \mathcal{T}_β , hence so do the functions defined by (4.20) and (4.21), with values in $\mathbf{C} \times \mathbf{P}^k$ and \mathbf{C}^2 , respectively. Likewise it is clear from the latter formula that ζ_+ and ζ_- take distinct values everywhere on \mathcal{T}_β . Thus q_3 extends as a morphism into T'' . Q.E.D.

We are now in a position to prove Theorem 4.2. We shall make use of the following simple fact.

LEMMA 4.23. *Let V be a complex vector space on which \mathbf{C}^* acts linearly with positive weights. If $\Phi: V \rightarrow V$ is a \mathbf{C}^* -equivariant dominant morphism then Φ is an isomorphism (of algebraic varieties).*

Proof. The Jacobian determinant of Φ has zero weight and does not vanish identically, hence is a non-zero constant. In particular Φ has a local analytic inverse at the origin. As the latter is in the boundary of each \mathbf{C}^* -orbit this local inverse is given by polynomials and extends globally. Q.E.D.

Proof of Theorem 4.2. The composition $(\tilde{g}'')^{-1} \circ q_3''$ induces a morphism

$$\Phi'': \mathcal{X}''/W_2 \rightarrow S''$$

which, by Proposition 4.18, restricts to an isomorphism $\Phi': \mathcal{T}/W_2 \rightarrow S'$. As \mathcal{X} is normal Φ'' extends to

$$\Phi: \mathcal{X}/W_2 \rightarrow S.$$

We put on \mathcal{X} the \mathbf{C}^* -action corresponding to the vector

$$\omega = (2; 0, \dots, 0) \in \Lambda^W \cap K.$$

According to Proposition 2.19 the weights of the induced action on \mathcal{X}/W_2 are the numbers $\langle \omega, \gamma^* \rangle$ where γ^* runs through the basis dual to $(\alpha_1, \dots, \alpha_k, \beta)$. As $\frac{1}{2}\omega$ is the sum of β and the fundamental weight $\bar{\omega}_k$ these numbers are conveniently read off from Table IV in Bourbaki [4]; the result coincides with Table 4.3. Checking through the

definitions in Table 4.14, (4.17), (4.20), and (4.21) the morphism Φ is readily seen to be equivariant. Therefore Lemma 4.23 applies, and Φ is an isomorphism.

Finally, Φ respects the discriminants by Proposition 4.18. Q.E.D.

5. $D_k[*]$: Applications

We draw some consequences from the results of the previous section. First of all we relate the singularities in the fibres of $X \xrightarrow{\pi} S$ to the isotropy groups of the W_2 -action on \mathcal{X} . Let $\Phi: \mathcal{X}/W_2 \rightarrow S$ be an isomorphism as in Theorem 4.2.

THEOREM 5.1. *Let $\tau \in \mathcal{X}$ and put $s = \Phi(\tau \bmod W_2) \in S$. Assume that $x_1, \dots, x_r \in X_s$ are the singular points of the fibre over s . Then each (complex analytic) singularity (X_s, x_ρ) is either a plane curve singularity of type A_{l-1} or D_l with $l \leq k$, or a singularity of type $D_l[*]$ with $l \leq k$. In the first case let M_ρ be the Weyl group corresponding to (X_s, x_ρ) , and put $M_\rho = W_2(D_l[*])$ in the second. Then the isotropy group of W_2 at $\tau \in \mathcal{X}$ is isomorphic to the direct product*

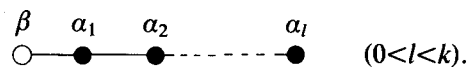
$$M_1 \times \dots \times M_r.$$

Proof. In view of Theorem 1.2 we may assume that τ belongs to the \mathcal{F} -orbit $\mathcal{F}_{\mathcal{D}'} \subset \mathcal{X}$ for some subdiagram \mathcal{D}' of $D_k[*]$. Let $\mathcal{D}'' = D_k[*] - \mathcal{D}'$ be the complement. By Corollary 2.11 the germ of the pair $(\mathcal{X}/W_2, \Delta_2)$ at the image of τ is analytically equivalent to a direct product. Its first factor is

$$(V(\mathcal{D}'')_{\mathbb{C}}/W(R_\tau), \Delta(V(\mathcal{D}''), R_\tau))$$

for some root system R_τ contained in R . Each irreducible factor of $\Delta(V(\mathcal{D}''), R_\tau)$ is known to be the discriminant of a versal deformation of a simple plane curve singularity, and $W(R_\tau)$ splits accordingly into direct factors which are the Weyl groups of these singularities—see Brieskorn [6], Slodowy [42], and Looijenga [27].

The second factor in the decomposition of $(\mathcal{X}/W_2, \Delta_2)$, is the germ of $(\mathcal{X}(\mathcal{D}')/W_2(\mathcal{D}'), \Delta_2(\mathcal{D}'))$ at its $\mathcal{F}(\mathcal{D}')$ -fixed point. Of course $\Delta_2(\mathcal{D}')$ is the discriminant of the semi-universal deformation of the singularity $D_l[*]$ if \mathcal{D}' is of this type. The only other subdiagrams \mathcal{D}' of $D_k[*]$ with non-trivial group $W_2(\mathcal{D}')$ are isomorphic to



We claim that then $\Delta_2(\mathcal{D}')$ is the discriminant of the plane curve singularity of type D_{l+1} (as usual, for small values of l we interpret D_2 as A_1+A_1 , and D_3 as A_3). To prove the claim we realize \mathcal{D}' in \mathbf{R}^{l+1} ; we put

$$\begin{aligned}\alpha_1 &= (-1, 1, 0, \dots, 0) \\ \alpha_2 &= (0, -1, 1, 0, \dots, 0) \\ &\vdots \\ \alpha_l &= (0, \dots, -1, 1) \\ \beta &= (1, 0, \dots, 0)\end{aligned}$$

and let α_j^\vee be the dual of α_j with respect to the standard Euclidean metric. The exponential map

$$\begin{aligned}\mathbf{C}^{l+1} &\rightarrow (\mathbf{C}^*)^{l+1} \\ (t_0, \dots, t_l) &\mapsto (e^{-2\pi i t_0}, \dots, e^{-2\pi i t_l})\end{aligned}$$

induces an isomorphism $\mathcal{X}(\mathcal{D}') \simeq \mathbf{C}^{l+1}$, and this isomorphism takes the action of $W_2(\mathcal{D}')$ on $\mathcal{X}(\mathcal{D}')$ to the standard representation of the Weyl group of D_{l+1} , compare Bourbaki [4], Planche IV. This proves the claim.

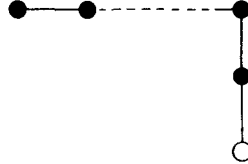
In view of Corollary 2.8 we have shown that the decomposition of Δ_2 into irreducible analytic components at $\tau \pmod{W_2}$ corresponds to a decomposition of $(W_2)_\tau$ into direct factors. Furthermore, we have identified each component of Δ_2 as the discriminant of a versal deformation of one of the singularities listed in the theorem. In fact, this singularity is realized in the fibre X_s , for by Wirthmüller [53] an isolated complete intersection singularity is determined, up to analytic equivalence, by the discriminant of a versal deformation. This completes the proof. Q.E.D.

(5.2) *Remark.* The statement of the theorem becomes false when $\mathcal{X}(D_k[\ast])$ and $W_2(D_k[\ast])$ are replaced by $\mathcal{X}(D_k[\ast]_2)$ and $W(D_k[\ast]_2)$.

The discriminant D of the semiuniversal deformation $X \xrightarrow{\pi} S$ has a natural decomposition into strata: each is characterized by the constellation of singularities in the fibres over it. In connection with Corollary 2.8 the theorem enables one to list all constellations that occur. Likewise some more precise information on the geometry of the strata may be obtained.

(5.3) *Example.* Let $S(D_k) \subset D$ be the stratum which corresponds to one D_k -singularity in the fibre. As this must be one-dimensional it is the union of finitely many \mathbf{C}^* -

orbits, and a fibre over $S(D_k)$ cannot contain any other singularity. The inverse image of $S(D_k)$ in \mathcal{X} consists of the W_2 -fixed point set in \mathcal{T} and of the \mathcal{T} -orbits corresponding to the subdiagram of $D_k[\ast]$,



and, if $k=4$, also



Each of these subdiagrams contributes one \mathbb{C}^* -orbit in $S(D_k)$ while the orbits covered by \mathcal{T}^{W_2} correspond bijectively to the connected components of the abelian group $\frac{1}{2}P/(\frac{1}{2}Q+\Lambda)$ where

$$P = \{x \in V \mid \langle x, \alpha^\vee \rangle \in \mathbb{Z} \text{ for all roots } \alpha\}$$

is the group of weights. Thus the total number of \mathbb{C}^* -orbits in $S(D_k)$ is 6 ($k=4$), 5 if $k>4$ is even, and 3 if k is odd.

For the curve singularities of type $D_k[\ast]$ we have a complete description of the homotopy type of $S \setminus D$, the complement of the discriminant. As the fundamental group was already determined in Section 3 this goal is achieved by the following theorem which includes the corresponding result of Knörrer [24] for $k=4$.

THEOREM 5.4. $\pi_n(S \setminus D) = 0$ if $n > 1$.

Proof. The discriminant contains $\{u=v=0\} \subset S$. Therefore, in the notation of Proposition 4.18, the complement $S \setminus D = S'' \setminus D$ is isomorphic to $T'' \setminus \Delta$. We put

$$K = \{(\eta_+, \eta_-, g, \zeta_+, \zeta_-) \in \mathbb{C}^2 \times P^k \times \mathbb{C}^2 \mid \zeta_\pm^2 = g(\eta_\pm), (\eta_+, \zeta_+) \neq (\eta_-, \zeta_-)\}$$

and observe the isomorphism

$$\begin{aligned} \mathbb{C} \times T'' &\longrightarrow K \\ (\eta, u, g, \zeta_+, \zeta_-) &\longmapsto (\eta+u, \eta-u, g(H-\eta), \zeta_+, \zeta_-) \end{aligned}$$

which sends $\mathbb{C} \times \Delta$ onto

$$\Delta_K := \{g \text{ has a multiple root}\} \subset K.$$

The cartesian projections induce C^∞ fibrations

$$\begin{array}{c} K \setminus \Delta_K \\ \downarrow \\ \{(\eta, g, \zeta) \in \mathbb{C} \times P^k \times \mathbb{C} \mid \zeta^2 = g(\eta), g \text{ has only simple roots}\} \\ \downarrow \\ \{g \in P^k \mid g \text{ has only simple roots}\} \end{array}$$

with smooth affine curves as fibres. The assertion now follows from the exact homotopy sequences of these fibrations. Q.E.D.

Recall from Theorem 3.10 that we have a natural presentation of $\pi_1(S \setminus D, s)$ with generators a_α , indexed by the vertices of the affine Dynkin diagram \tilde{D}_k , and relations which include the Artin relations with respect to this diagram. By its special nature each generator a_α determines an unoriented vanishing cycle $\pm v_\alpha$ in the Milnor homology $H := H_1(X_s, \mathbb{Z})$. The action of a_α on H is governed by the Picard-Lefschetz formula

$$a_\alpha(x) = x - \langle x, v_\alpha \rangle v_\alpha \tag{5.5}$$

where $\langle \cdot, \cdot \rangle$ is the intersection form on X_s , see Looijenga [32], (7.4).

A basis of H consisting of vanishing cycles is called *weakly distinguished* if the corresponding transformations (5.5) generate the monodromy group (this definition is slightly weaker than the one usually used in this context).

THEOREM 5.6. *The v_α form a weakly distinguished basis of H , and their intersection diagram is \tilde{D}_k (as this is a tree the intersection numbers need be specified up to sign only).*

Proof. It is known that the set of all vanishing cycles in H generates H , and is an orbit under the action of the monodromy, see Looijenga [32], (7.5), (7.8). Therefore the v_α generate H . As $k+1$ is the Milnor number they form a basis which is weakly distinguished by definition. The intersection numbers follow from the Artin relations: if α and α' are orthogonal base roots then a_α and $a_{\alpha'}$ commute, in particular we have

$$\begin{aligned} a_{\alpha'} a_\alpha(v_\alpha) &= a_\alpha a_{\alpha'}(v_\alpha) \\ v_\alpha - \langle v_\alpha, v_{\alpha'} \rangle v_{\alpha'} &= (1 - \langle v_\alpha, v_{\alpha'} \rangle^2) v_\alpha - \langle v_\alpha, v_{\alpha'} \rangle v_{\alpha'}, \end{aligned}$$

hence $\langle v_\alpha, v_{\alpha'} \rangle = 0$. Similarly, if α and α' span an edge in \tilde{D}_k the relation $a_\alpha a_{\alpha'} a_\alpha(v_\alpha) = a_{\alpha'} a_\alpha a_{\alpha'}(v_\alpha)$ implies $\langle v_\alpha, v_{\alpha'} \rangle \in \{-1, 0, 1\}$. If this intersection number were zero then the actions of a_α and $a_{\alpha'}$ on H would commute and, in view of the Artin relation, would coincide. But then the orbit of a vanishing cycle could not have generated H . We conclude that $\langle v_\alpha, v_{\alpha'} \rangle = \pm 1$, and the theorem is proved. Q.E.D.

The triple $(H, \langle, \rangle \{\text{vanishing cycles}\})$ is a skew-symmetric vanishing lattice in the sense of Janssen [18]. Most of this structure is preserved upon passing to coefficients in the field F_2 . Vanishing lattices over F_2 are classified in [loc. cit.], and the case at hand is easily identified by the dimensions of H and the kernel of the intersection form, and the number of vanishing cycles in $H/2H$. The latter is just the number of different roots mod $2H$ in the affine root system \tilde{D}_k , which is $2k(k-1)$. Therefore the vanishing lattice is of type $A^{\text{odd}}(k-2, 3, F_2)$ if k is even, and of type $A^{\text{ev}}(k-1, 2, F_2)$ if k is odd.

In a recent paper, Janssen [19], it is shown how the classification over F_2 lifts to a complete classification of integral (skew-symmetric) vanishing lattices. The lattices in question are easily seen to be

$$\begin{aligned} A^{\text{odd}}(1; 3; 0) & \quad (k = 4) \\ A^{\text{odd}}(1, \dots, 1; 3; \infty) & \quad (k > 4 \text{ even}) \\ A^{\text{ev}}(1, \dots, 1; 2) & \quad (k \text{ odd}). \end{aligned}$$

6. $E_k[*]$ ($k=6, 7, 8$): The discriminant

In the hierarchy of simple singularities the series $D_k[*]$ ($k \geq 4$) is followed by ten exceptional ones, including those with defining ideal

$$(x^2 - y^3 - z^{k-3}, yz)$$

($k=6, 7, 8$). The main result of this and the following section is the analogue of Theorem 4.2 for these singularities.

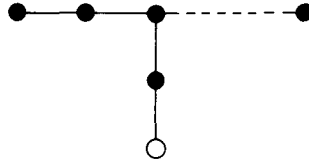
The curve $X_0 = \{x^2 = y^3 + z^{k-3}, yz = 0\} \subset \mathbb{C}^3$ admits a \mathbb{C}^* -action with weights

k	x	y	z	(6.1)
6	3	2	2	
7	12	8	6	
8	15	10	6	

As in the previous chapter we let

$$X \xrightarrow{\pi} S$$

denote a \mathbf{C}^* -equivariant semi-universal deformation of $(X_0, 0)$ with discriminant $D \subset S$. The diagram $E_k[\ast]$,



has been introduced in (1.1).

THEOREM 6.2. *There exists a \mathbf{C}^* -equivariant isomorphism*

$$\mathcal{A}(E_k[\ast])/W_2(E_k[\ast]) \xrightarrow{\Phi} S$$

which respects the discriminants.

For the proof we use the explicit form of the semi-universal deformation provided by the criterion of Kas and Schlessinger [20]. Let P^l denote the affine space of unitary polynomials of degree l , and let $P'_0 \subset P^l$ contain those with vanishing constant term. We put

$$S = \{(u, v, p, q)\} = \mathbf{C}^2 \times P_0^3 \times P^{k-3} \cong \mathbf{C}^{k+1}$$

and define $X \subset \mathbf{C}_S^3$ by

$$X = \left\{ \begin{array}{l} x^2 = p(y) + q(z) \\ yz = ux + v \end{array} \right\}.$$

The projection $X \xrightarrow{\pi} S$ is a \mathbf{C}^* -equivariant deformation of $(X_0, 0)$ with respect to the weights (6.1). Note that the curve X lies on the relative surface

$$Y := \{2xyz - ux^2 - 2vx = up(y) + uq(z)\} \subset \mathbf{C}_S^3.$$

Over the hyperplane $\{u=0\} \subset S$ this surface decomposes into two components but the substitution $x \mapsto ux$ gives a new surface

$$Z' := \{2xyz - u^2x^2 - 2vx = p(y) + q(z)\} \subset \mathbf{C}_S^3 \tag{6.3}$$

which will turn out to be less degenerate over $\{u=0\}$. In order to compactify Z' we embed \mathbb{C}^3 in $\mathbb{P}^3_{1\alpha\beta 1}$ with weights

k	α	β
6	1	1
7	2	1
8	3	2

sending (x, y, z) to $[1: x: y: z]$. (Note that these weights differ from those of (6.1).) We let Z be the closure of Z' in $\mathbb{P}^3_{1\alpha\beta 1, S}$. As in each case the weights divide $\alpha+\beta+1$ the sheaf $\mathcal{O}(\alpha+\beta+1)$ on $\mathbb{P}^3_{1\alpha\beta 1}$ is invertible, and the hypersurface Z is the Cartier divisor defined by the section

$$f(w, x, y, z) := p(w, y) + q(w, z) + u^2 w^{\beta-\alpha+1} x^2 + 2vw^{\beta+1} x - 2xyz, \tag{6.4}$$

with $p(w, y) = w^{\alpha+\beta+1} p(y/w)$ and $q(w, z) = w^{\alpha+\beta+1} q(z/w)$. Therefore Z is a flat family of surfaces over S . It comes naturally equipped with the effective Weil divisor at infinity $Z_\infty := \{w=0\}$. In $\mathbb{P}^2_{\alpha\beta 1, S}$ this is the Cartier divisor with equation

$$\begin{aligned} 2xyz &= y^3 + z^3 & (k=6), \\ 2xyz &= u^2 x^2 + z^4 & (k=7), \\ 2xyz &= u^2 x^2 + y^3 & (k=8). \end{aligned} \tag{6.5}$$

In particular Z_∞ is also flat over S .

Our interest in the surface Z is caused by the fact that at least the restriction $X_{S'} \rightarrow S' := S \setminus \{u=0\}$ may be recovered from it in a natural way, as follows. First note that the surface $Z_{S'}$ is smooth along $Z_{\infty, S'}$. If $k=6$ then $Z_{S'}$ contains the S' -point $[0: 1: 0: 0]$ and we let \tilde{Z} denote the blow-up of $Z_{S'}$ along this point. We put $\tilde{Z} = Z_{S'}$ else. Then the linear projection from $[0: 1: 0: 0]$ in $\mathbb{P}^3_{1\alpha\beta 1}$ restricts to an S' -morphism

$$c: \tilde{Z} \rightarrow \{x=0\} = \mathbb{P}^2_{1\beta 1}.$$

PROPOSITION 6.6. *c is a double cover, ramified along an S' -curve X' and, if $k=8$, also at an isolated point $[0: 1: -u^{2/3}: 0]$. If d denotes the scaling automorphism*

$$\begin{aligned} \mathbb{P}^3_{1\alpha\beta 1, S'} &\rightarrow \mathbb{P}^3_{1\alpha\beta 1, S'} \\ [w: x: y: z] &\mapsto [w: ux: y: z] \end{aligned}$$

then X' is just the closure of $d(X_{S'})$ in $\mathbb{P}^3_{1\alpha\beta 1, S'}$.

Proof. In affine coordinates ($w=1$) we have

$$f = p(y) + q(z) + u^2x^2 + 2(v - yz)x$$

which shows that c is a double cover. Its ramification points are the common zeros of f and

$$\frac{\partial f}{\partial x} = 2(u^2x + v - yz) = 2(u(ux) + v - yz);$$

these functions generate the same ideal as $\partial f/\partial x$ and

$$f - x \frac{\partial f}{\partial x} = p(y) + q(z) - (ux)^2$$

A trivial verification at infinity completes the proof.

Q.E.D.

COROLLARY 6.7. *Let $s \in S'$. Then X_s is singular at $a \in \mathbb{C}^3$ exactly if Z_s is singular at $d(a)$. In this case the singularity $(Z_s, d(a))$ is isomorphic to the suspension of the (plane) curve singularity (X_s, a) . In particular all singularities of Z_s are isolated, and $D \cap S'$ is the discriminant of $Z_{S'} \rightarrow S'$.*

Q.E.D.

PROPOSITION 6.8. *The restriction $Z_{S'} \xrightarrow{\pi} S'$ is a flat family of (possibly singular) del Pezzo surfaces of degree $9-k$, and $Z_{\infty, S'}$ is an anti-canonical divisor relative S' .*

Proof. Let $s \in S'$. Then Z_s does not meet the singular points of $\mathbb{P}_{1\alpha\beta 1}^3$, and the sheaf $\mathcal{O}(1)$ restricts to an invertible sheaf on Z_s . This sheaf is ample; indeed, on the regular part of $\mathbb{P}_{1\alpha\beta 1}^3$, the sheaf $\mathcal{O}(\beta) = \mathcal{O}(1)^\beta$ is very ample. Using the fact that the singularities of Z_s are isolated it is easily seen that $\mathcal{O}(1)$ admits a smooth divisor on Z_s . The main result of Pinkham [39] (see Merindol [34], Théorème 6.1 for a more detailed account) then implies that Z_s is a del Pezzo surface of degree $9-k$, and that $\mathcal{O}(1)$ is the anti-canonical sheaf. It was noted earlier that Z and Z_∞ are flat over S , and the proposition follows.

Q.E.D.

Flat families of del Pezzo surfaces are well-understood. In the sequel we apply the results of Pinkham [39], [40], Merindol [34], Looijenga [26], [27] and others to the situation at hand.

For each $s \in S' \setminus D$ the surface Z_s may be obtained from the projective plane by blowing up k points in general position. Therefore the canonical homomorphism $\text{Pic}(Z_s) \rightarrow H_2(Z_s, \mathbb{Z})$ is bijective. We fix a point $s_0 \in S' \setminus D$. The fundamental group $\pi_1(S' \setminus D, s_0)$ acts as a monodromy group on $H := H_2(Z_{s_0}, \mathbb{Z}) = \text{Pic}(Z_{s_0})$. This action

defines an étale Galois cover

$$(S' \setminus D)^{\sim} \xrightarrow{e'} S' \setminus D$$

such that the monodromy is trivial on the induced family $\varrho'^*Z \rightarrow (S' \setminus D)^{\sim}$. It is well-known that ϱ' then extends as a branched analytic cover

$$\tilde{S}' \xrightarrow{e} S'$$

and that the induced family of del Pezzo surfaces ϱ^*Z admits a simultaneous resolution $\tilde{Z} \xrightarrow{\sigma} \varrho^*Z$:

$$\begin{array}{ccccc} \tilde{Z} & \xrightarrow{\sigma} & \varrho^*Z & \longrightarrow & Z_{S'} \\ & \searrow & \downarrow & \lrcorner & \downarrow \\ & & \tilde{S}' & \xrightarrow{e} & S' \end{array}$$

The smooth family $\tilde{Z} \rightarrow \tilde{S}'$ is obtained from a suitable bundle of projective planes by blowing up k sections consecutively, see Merindol [34] for details.

The S' -divisor $Z_{\infty, S'}$ on $Z_{S'}$ is a family of rational curves with a node at the point

$$\begin{aligned} [1: 0: 0] \in \mathbf{P}^2 & \quad \text{if } k=6, \\ [0: 1: 0] \in \mathbf{P}_{211}^2 & \quad \text{if } k=7, \\ [0: 0: 1] \in \mathbf{P}_{321}^2 & \quad \text{if } k=8. \end{aligned}$$

We parametrize the regular part of $Z_{\infty, S'}$ by the S' -morphism

$$\begin{aligned} \mathbf{C}_{S'}^* & \xrightarrow{\varepsilon} Z_{\infty, S'} \subset \mathbf{P}_{\alpha\beta 1}^2 \\ \lambda & \rightarrow \left[\frac{1}{2}(u^6 + \lambda^3), u^4\lambda, u^2\lambda^2 \right] \quad (k=6), \\ \lambda & \rightarrow \left[\lambda^3, \frac{1}{2}(u^6 + \lambda^2), u^2\lambda \right] \quad (k=7), \\ \lambda & \rightarrow \left[u^8\lambda, u^4\lambda, \frac{1}{2}(u^6 + \lambda) \right] \quad (k=8). \end{aligned} \tag{6.9}$$

The inverse of ε_s extends to a unique homomorphism

$$\varphi_s: \text{Pic}(Z_{\infty, S'}) \rightarrow \mathbf{C}^*;$$

it sends the divisor $\Sigma_{j=1}^d \{\varepsilon_s(\lambda_j)\}$ to $\prod_{j=1}^d \lambda_j \in \mathbf{C}^*$. As $Z_{S'}$ is smooth along $Z_{\infty, S'}$ the pull-back ϱ^*Z_{∞} lifts isomorphically to a relative anti-canonical divisor \tilde{Z}_{∞} on \tilde{Z} . Thus

for each $r \in \tilde{S}'$ we have a homomorphism

$$\varphi_r: \text{Pic}(\tilde{Z}_{\infty, r}) \rightarrow \mathbf{C}^*$$

which is, essentially, $\varphi_{\varrho(r)}$.

We use this homomorphism to set up a *characteristic mapping*

$$\tilde{S}' \xrightarrow{\psi} \text{Hom}(H, \mathbf{C}^*)$$

for the family $\tilde{Z} \rightarrow \tilde{S}'$. Fix $r_0 \in \varrho^{-1}(s_0)$. Given $r \in \tilde{S}'$, each homology class $h \in H$ corresponds to a divisor class $\mathcal{L}(h) \in \text{Pic}(\tilde{Z}_r)$ via $H \simeq H_2(\tilde{Z}_{r_0}, \mathbf{Z}) \simeq H_2(\tilde{Z}_r, \mathbf{Z}) \simeq \text{Pic}(\tilde{Z}_r)$, and we define

$$\psi(r)(h) = \varphi_r(\mathcal{L}(h)|_{\tilde{Z}_{\infty, r}}) \in \mathbf{C}^*. \quad (6.10)$$

For the following it will be convenient to have an explicit basis of H at our disposal. Following Demazure [11], II.2 we choose a birational morphism $Z_{s_0} \rightarrow \mathbf{P}^2$ which contracts k disjoint exceptional curves $E_1, s_0, \dots, E_k, s_0$. We let h_j ($1 \leq j \leq k$) be minus the homology class of E_j, s_0 , and denote the class of the total transform of a line in \mathbf{P}^2 by h_0 . Then (h_0, h_1, \dots, h_k) is a basis for H . The canonical class is

$$\kappa = -3h_0 - \sum_{j=1}^k h_j$$

and the intersection form is determined by the self-intersection numbers

$$h_0^2 = 1, \quad h_j^2 = -1 \quad (j > 0).$$

Likewise, for any $r \in \tilde{S}'$ we may consider (h_0, \dots, h_k) a basis of $H_2(\tilde{Z}_r, \mathbf{Z})$ via the canonical isomorphism $H \simeq H_2(\tilde{Z}_{r_0}, \mathbf{Z}) \simeq H_2(\tilde{Z}_r, \mathbf{Z})$.

LEMMA 6.11. *The characteristic mapping ψ is analytic.*

Proof. Let $r \in \tilde{S}'$. Over some neighbourhood U of r the element $h_0 \in H$ is represented by a divisor $L \subset \tilde{Z}_U$ which is smooth over U and avoids the singular point of $\tilde{Z}_{\infty, U}$. Similarly, h_1, \dots, h_k are represented by the exceptional divisors E_1, \dots, E_k in \tilde{Z} which are flat over \tilde{S}' . From this the assertion follows easily. Q.E.D.

Because H is a free abelian group the exponential sequence

$$0 \rightarrow \mathbf{Z} \rightarrow \mathbf{C} \xrightarrow{\exp} \mathbf{C}^* \rightarrow 1 \quad (\exp(t) := e^{2\pi it})$$

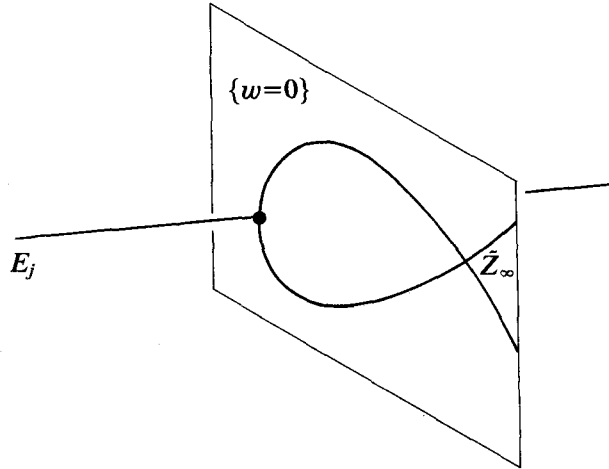


Figure 6.12

induces an exact sequence

$$0 \rightarrow \text{Hom}(H, \mathbf{Z}) \rightarrow \text{Hom}(H, \mathbf{C}) \rightarrow \text{Hom}(H, \mathbf{C}^*) \rightarrow 1.$$

We let $\Lambda \subset \text{Hom}(H, \mathbf{Z})$ be the sublattice

$$\Lambda = \{t \in \text{Hom}(H, \mathbf{Z}) \mid \langle \kappa, t \rangle \in 6\mathbf{Z}\}$$

and put $\mathcal{T} = \text{Hom}(H, \mathbf{C})/\Lambda$. As the canonical class κ is indivisible in H the algebraic torus \mathcal{T} is a sixfold cyclic cover of $\text{Hom}(H, \mathbf{C})/\text{Hom}(H, \mathbf{Z}) = \text{Hom}(H, \mathbf{C}^*)$.

PROPOSITION 6.13. *The characteristic map ψ lifts to an analytic map $\psi_1: \tilde{S}' \rightarrow \mathcal{T}$.*

Proof. A glance at the definition of ε verifies the relation

$$(u \circ \varrho)^6 = (-1)^{k-1} \psi(r)(-\kappa) \tag{6.14}$$

for each $r \in \tilde{S}'$. If $\tilde{\tilde{S}}'$ denotes the universal cover of \tilde{S}' then ψ may be lifted to an analytic map

$$\tilde{\psi}: \tilde{\tilde{S}}' \rightarrow \text{Hom}(H, \mathbf{C}),$$

and there exists a logarithm $\log(u \circ \varrho): \tilde{\tilde{S}}' \rightarrow \mathbf{C}$. If $\tilde{r}, \tilde{r}' \in \tilde{\tilde{S}}'$ represent the same point in \tilde{S}' then in view of (6.14) we have

$$\tilde{\psi}(\tilde{r})(-\kappa) - \tilde{\psi}(\tilde{r}')(-\kappa) = 6 \log(u \circ \varrho)(\tilde{r}) - 6 \log(u \circ \varrho)(\tilde{r}') \in 6\mathbf{Z}.$$

Therefore $\tilde{\psi}$ drops to a well-defined map $\psi_1: \tilde{S}' \rightarrow \mathcal{T}$.

Q.E.D.

Following Pinkham [39] and Merindol [34] we describe the action of the monodromy on H . Of course the canonical class κ is invariant, as is the intersection form. The orthogonal complement of κ in H is the lattice Q generated by the set

$$R = \{h \in H \mid (\kappa, h) = 0, h^2 = -2\}.$$

Minus the intersection form restricts to a Euclidean inner product on the vector space $\mathbf{R}Q = \{h \in \mathbf{R}H \mid (\kappa, h) = 0\}$, and with respect to this structure R is a root system of type E_k . The group $\pi_1(S' \setminus D, s_0)$ acts on H by transformations of the Weyl group W of that root system.

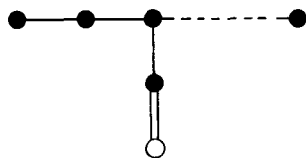
As the intersection form is unimodular on H we may use it to identify H and $\text{Hom}(H, \mathbf{Z})$. Thus we consider R as a subset of $\text{Hom}(H, \mathbf{Z})$; with respect to the standard basis (h_0, \dots, h_k) a root basis of R is given by the components

$$\begin{aligned} \alpha_1 &= (0; -1, 1, 0, \dots, 0) \\ \alpha_2 &= (0; 0, -1, 1, 0, 0, \dots, 0) \\ &\vdots \\ \alpha_{k-1} &= (0; 0, \dots, 0, -1, 1) \\ \alpha_k &= (1; 1, 1, 1, 0, \dots, 0) \end{aligned}$$

We define $\beta \in \text{Hom}(H, \mathbf{Z})$ by its components

$$\beta = (2; 0, \dots, 0).$$

Then $(\alpha_1, \dots, \alpha_k, \beta)$ is a basis of the lattice Λ , and this basis together with the duals of $\alpha_1, \dots, \alpha_k$ constitute a realization of the diagram $E_k[\times]_2$:



It is clear from the very definitions of the characteristic mappings ψ and ψ_1 that $\psi_1: \tilde{S}' \rightarrow \mathcal{T}$ induces an analytic morphism

$$S' \xrightarrow{\psi_2} \mathcal{T}/W.$$

The main step in the proof of Theorem 6.2 is the extension of ψ_2 over the general point

of the hyperplane $\{u=0\} \subset S$. To this end we shall study the fibres of the family $Z \xrightarrow{\pi} S$ over such points in the next section.

Here we note for later use:

PROPOSITION 6.15. ψ_2 is \mathbb{C}^* -equivariant with respect to the action on S' determined by the weights (6.1), and that on

$$\mathcal{T}/W = \mathcal{T}(E_k[\ast])/W_2(E_k[\ast])$$

considered in Proposition 2.19.

Proof. The weights (6.1) define a \mathbb{C}^* -action on the family $Z \xrightarrow{\pi} S$ with weights

k	w	x	y	z	u
6	0	2	2	2	1
7	0	10	8	6	2
8	0	14	10	6	1.

For $\mu \in \mathbb{C}^*$ close to 1 and $r \in \tilde{S}'$ the point $\mu \cdot r \in \tilde{S}'$ is well-defined, and the definition of the parametrization $\varepsilon: \mathbb{C}_S^* \rightarrow Z_{\infty, S'}$ shows that

$$\psi(\mu \cdot r)(h) = \mu^{c \cdot \deg(\mathcal{L}(h)|Z_{\infty, S'})} \cdot \psi(r)(h) = \mu^{-c \cdot (\alpha, h)} \cdot \psi(r)(h)$$

holds for all $h \in H$, with $c=2$ ($k=6$) or $c=6$ ($k \neq 6$). On the other hand, the \mathbb{C}^* -action on \mathcal{T}/W is induced by the W -fixed vector $\omega \in \text{Hom}(H, \mathbb{C})$ as defined in Proposition 2.19. This vector is just $-\alpha$, and the assertion follows. Q.E.D.

7. Extension of the characteristic mapping

Let $s \in \{u=0\} \subset S$. We study the surface $Y := Z_s$ and its distinguished (Weil) divisor $W := Z_{\infty, s} = \{w=0\} \subset Z_s$. The latter is given in $\mathbb{P}_{\alpha\beta 1}^2 = \{w=0\} \subset \mathbb{P}_{1\alpha\beta 1}^3$ by the equation

$$\begin{aligned} 2xyz &= y^3 + z^3 & (k=6), \\ 2xyz &= z^4 & (k=7), \\ 2xyz &= y^3 & (k=8). \end{aligned}$$

We have a nodal cubic if $k=6$, while for $k \neq 6$ the divisor W decomposes:

$$\begin{aligned} W &= \{2xy = z^3\} \cup \{z=0\} & (k=7) \\ W &= \{2xz = y^2\} \cup \{y=0\} & (k=8). \end{aligned}$$

In either case W is reduced, and the components

$$W_0 = \{2xy = z^3\}, \quad W_1 = \{z = 0\},$$

respectively

$$W' = \{2xz = y^2\}, \quad W'' = \{y = 0\},$$

are easily seen to be smooth rational curves. We let $n \in W$ denote the point $[1:0:0]$.

PROPOSITION 7.1. *If $s \in \{u=0\} \subset S$ is chosen such that $v \neq 0$ then Y has a quotient singularity at n , of the following type.*




k	resolution graph	type—see Brieskorn [5]
6	-2 	$C_{2,1} = A_1$
7	-4 	$C_{4,1}$
8	$-2 \quad -5$ 	$C_{9,2}$

Table 7.2

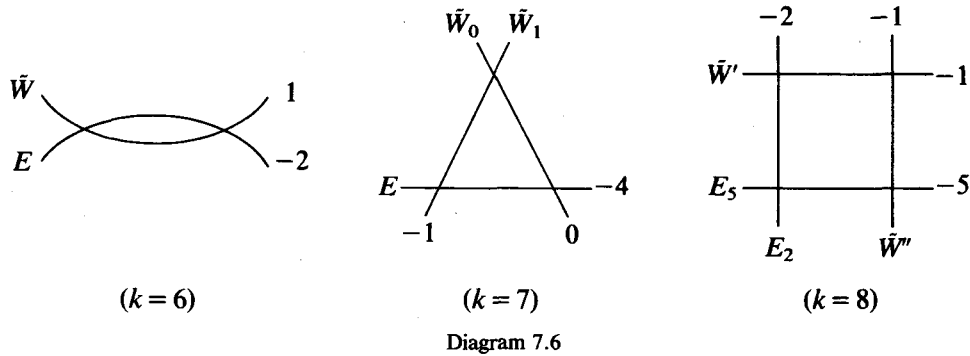
Proof. This follows by analyzing the affine equation for Y ,

$$2yz - 2vw^{\beta+1} = p(w, y) + q(w, z). \quad (7.3)$$

Q.E.D

(7.4) *Remarks.* If v is the only non-zero component of $s \in S$ then Y is smooth off the singularity n . Hence the same holds for generic $s \in \{u=0\} \subset S$, and in the sequel we assume that s is so chosen. Variation of the coordinate $u: (S, s) \rightarrow \mathbb{C}$ defines a one-parameter deformation of the quotient singularity (Y, n) . For $k \neq 6$ this deformation is easily seen to be one of the ω^* -constant deformations considered by Wahl [51]—compare his Theorem (2.7), with $(n, q) = (2, 1)$ if $k=7$, and $(n, q) = (3, 1)$ if $k=8$. These deformations are not on the Artin component of the singularity, so we do not have simultaneous resolution at our disposal.

Let $\tilde{Y} \xrightarrow{\sigma} Y$ be the minimal resolution of the singular point $n \in Y$, and let $E \subset \tilde{Y}$ be the (reduced) exceptional fibre. In case $k=8$ the latter consists of two curves E_2 and E_5 with self-intersection $E_i^2 = -i$. We let $\tilde{W} \subset \tilde{Y}$ denote the strict inverse image of $W \subset Y$; similarly, \tilde{W}_j , \tilde{W}' , and \tilde{W}'' .



PROPOSITION 7.5. \tilde{Y} is a rational surface, and $E + \tilde{W}$ is an anticanonical divisor. Its geometry is shown in Diagram 7.6 which includes the self-intersection numbers.

Proof. These data may be worked out by explicitly resolving the singularity at n . We omit the details. Q.E.D.

COROLLARY 7.7. The rank of $\text{Pic}(\tilde{Y})$ is $2k - 5$.

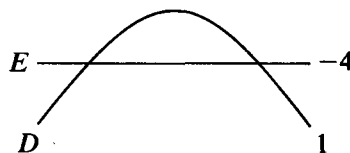
Proof. This follows from the relation

$$\text{rk Pic}(\tilde{Y}) = 10 - \kappa^2$$

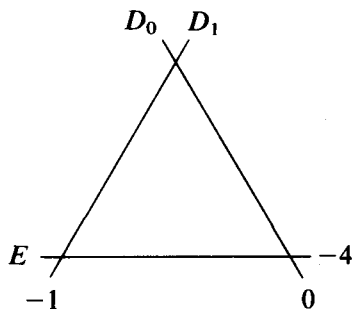
for a rational surface. Q.E.D.

In order to be able to extend the characteristic mapping we shall make sure that the surface $Y \setminus \{n\} = \tilde{Y} \setminus E$ contains sufficiently many exceptional curves (of the first kind, that is, embedded copies of \mathbb{P}^1 with self-intersection -1). As a preliminary we prove the following statement.

PROPOSITION 7.8. Let F be a smooth rational surface with $\text{rk Pic}(F) = 9$. Let $E \subset F$ be a smooth rational curve with $E^2 = -4$, and assume that the anti-canonical sheaf is ample on $F \setminus E$. Suppose further that E is part of an anti-canonical cycle $E + D$ of the form



or



Then there exist seven disjoint exceptional curves C_1, \dots, C_7 on F that do not meet E . In the second case each C_j can be chosen so as to intersect $E+D$ in a single point of D_1 .

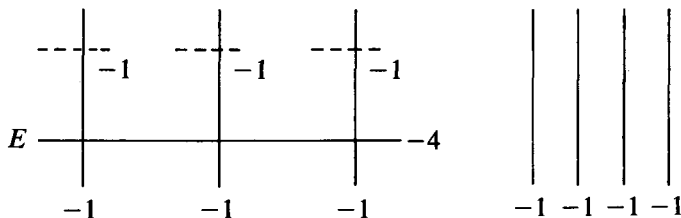
Proof. Besides E there cannot be any other irreducible curve C on F with $C^2 < 1$, for the adjunction formula for C reads

$$2p(C) - 2 = C \cdot (C + \kappa) = C^2 - C \cdot (-\kappa),$$

and the last term is positive because $-\kappa$ is ample on $F \setminus E$. Likewise, if C is any effective divisor on F with $E \not\subset C$ and $C^2 = -1$ then C is an exceptional curve. For, by the adjunction formula $p(C) = 0$, and if C were reducible then one of its components would have self-intersection smaller than -1 , which we have just seen to be impossible.

Let $\sigma: F \rightarrow \bar{F}$ be a birational morphism to a minimal model \bar{F} . By the classification of rational surfaces we may assume $\bar{F} = \mathbf{P}^2$, $\bar{F} = \mathbf{P}^1 \times \mathbf{P}^1$, or that \bar{F} is a ruled surface F_n ($n > 1$). Thus σ is a composition of eight or seven σ -processes, respectively. In fact we need not consider the case $\bar{F} = \mathbf{P}^1 \times \mathbf{P}^1$, as $\mathbf{P}^1 \times \mathbf{P}^1$ with one point blown up maps birationally to \mathbf{P}^2 . Thus either $\bar{F} = \mathbf{P}^2$ (case α), or $\bar{F} = F_n$ (case β). We further distinguish two cases according to whether σ contracts E (case 1) or not (case 2). Let us discuss these in turn.

Case (1 α). F is obtained from \mathbf{P}^2 by blowing up five points and three directions through one of them, say P . The configuration of non-trivial fibres of σ is that shown by the solid lines:

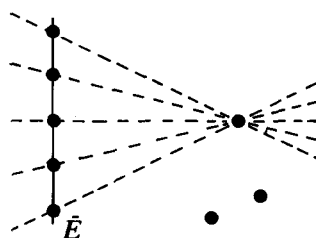


The broken lines indicate the strict transforms of the three lines of distinguished direction through P . We thus see seven disjoint exceptional curves not meeting E , as claimed.

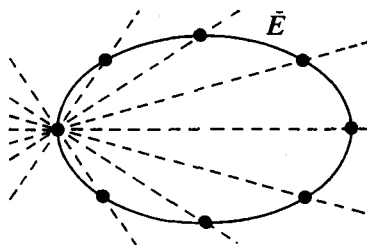
Case (1 β) cannot occur because when E is contracted all curves of F will have self-intersection at least -1 .

We turn to case (2) and its subcases. Here all non-trivial fibres of σ are irreducible. D_1 may be one of them while any other must meet the anti-canonical divisor $E+D$ transversely in a single point. In particular, any non-trivial fibre of σ that meets E does so transversely in one point. It follows that $\bar{E}:=\sigma(E)$ is a smooth rational curve.

Case (2 α). $\bar{E}\subset\mathbf{P}^2$ is either a line or an irreducible conic. Thus σ represents F as a projective plane with eight points blown up, as indicated:



respectively



In either case there is an obvious choice of the seven exceptional curves including the strict transforms of the broken lines.

Case (2 β). $\bar{E}\subset F_n$ must be the unique curve of negative self-intersection, that is, the section at infinity of class $(1,0)\in\text{Pic}(F_n)$. Thus F is obtained from F_n by blowing up seven points, necessarily on different fibres of the ruling morphism $F_n\rightarrow\mathbf{P}^1$. Of these exactly $4-n$ are on \bar{E} . Performing elementary transforms—see Nagata [35],

Section 2, (3)—we obtain a birational morphism $F \rightarrow F_4$ with the seven non-trivial fibres disjoint from E .

Note that this last situation is achieved in any case, simply by contracting the seven exceptional curves C_1, \dots, C_7 . Thus we have

$$F \xrightarrow{\sigma} F_4$$

with $\bar{E} = \sigma(E)$ the section at infinity. Let $\bar{D}_j = \sigma(D_j)$, $\bar{D} = \sigma(D)$. Then $\bar{E} + \bar{D}$ is an anti-canonical divisor on F_4 , hence of type $(2, 6) \in \text{Pic}(F_4)$. In case $D = D_0 + D_1$ the decomposition $\bar{D}_0 + \bar{D}_1$ must be of type

$$(0, 1) + (1, 5)$$

or vice versa. In view of the self-intersection numbers, either all curves C_1, \dots, C_7 meet D_1 (which is what we want), or one, say C_7 , meets D_1 , and the other six meet D_0 . In this latter case we put $P_j = \sigma(C_j)$, and consider the linear system of curves of type $(1, 4)$ through all but one of the points P_1, \dots, P_6 ; say P_1 . By the Riemann-Roch formula this system is non-empty. Let C'_1 be a member, considered as a curve on F . If E were a component of C'_1 then C'_1 would have to decompose into E plus four fibres which is impossible. We further have $\rho(C'_1) = 0$ and $C_1'^2 = -1$. If C'_1 were reducible it would contain some component of self-intersection smaller than -1 which is likewise impossible. It follows that C'_1 is an exceptional curve (in particular, the only curve in the linear system). Similarly we find curves C'_2, \dots, C'_6 . Together with C_7 they form a set of seven curves which satisfy the last clause of the proposition. Q.E.D.

Returning to the situation that was the starting-point of this section we are now able to prove that the surface Y contains certain configurations of exceptional curves.

THEOREM 7.9. *If $k=6$ then in $Y \setminus \{n\}$ there exist exceptional curves C_{ij} ($1 \leq i < j \leq 6$) with*

$$C_{ij} \cdot C_{lm} = \begin{cases} 1 & \text{if } \{i, j\} \cap \{l, m\} = \emptyset \\ 0 & \text{else.} \end{cases}$$

If $k=7$ then $Y \setminus \{n\}$ contains seven disjoint exceptional curves C_1, \dots, C_7 . For each j the point of intersection of C_j with W lies in $W_1 \setminus W_0$.

If $k=8$ then there are seven disjoint exceptional curves $C_1, \dots, C_7 \subset Y \setminus \{n\}$. For each pair (j, l) with $1 \leq j < l \leq 7$ there exist exceptional curves C_{jl} and C'_{jl} in $Y \setminus \{n\}$ with

$$C_{ij} \cdot C_{lm} = C'_{ij} \cdot C'_{lm} = \begin{cases} 1 & \text{if } \{i, j\} \cap \{l, m\} = \emptyset \\ 0 & \text{else,} \end{cases}$$

$$C_i \cdot C_{jl} = C_i \cdot C'_{jl} = \begin{cases} 0 & \text{if } i \in \{j, l\} \\ 1 & \text{else,} \end{cases}$$

$$C_{ij} \cdot C'_{lm} = 2 - c \quad \text{if } \{i, j\} \cap \{l, m\} \text{ contains } c \text{ elements.}$$

The curves C_i and C_{jl} intersect W in a point of $W'' \setminus W'$ each, while $C'_{jl} \cap W$ is a point of $W' \setminus W''$.

Proof. For $k=6$ the surface Y is a del Pezzo surface with an ordinary double point at n . Its minimal resolution \tilde{Y} may be identified with a projective plane on which six points P_1, \dots, P_6 of an irreducible conic have been blown up: Demazure [11], V, Proposition 1. The exceptional fibre of the resolution is the strict transform of this conic. We let C_{ij} be the strict transform of the line through P_i and P_j . Clearly the C_{ij} are contained in $\tilde{Y} \setminus E = Y \setminus \{n\}$, and their incidence relations are as required.

The case $k=7$ is an immediate consequence of Proposition 7.8, applied to $F = \tilde{Y}$, $D_j = \tilde{W}_j$.

Thus assume $k=8$. Using the notation of Proposition 7.5 we let F be the surface obtained from \tilde{Y} by contracting \tilde{W}' and E_2 . To this F Proposition 7.8 applies, with D the image of \tilde{W}'' . Therefore F is isomorphic to F_4 with points P_1, \dots, P_7 blown up, all in different fibres of the ruling morphism and none on the section at infinity \tilde{E}_5 . The image of \tilde{W}'' is a curve \tilde{W}'' of type (1, 6) while \tilde{W}' and E_2 map to one point P of $\tilde{W}'' \cap \tilde{E}_5$ and the infinitesimally near point Q determined by the tangent of \tilde{W}'' at P . We let $C_i \subset \tilde{Y} \setminus E = Y \setminus \{n\}$ be the fibre over P_i . To construct C_{jl} we consider the linear system of curves on F_4 that have type (1,4) and pass through all P_i with $i \notin \{j, l\}$. The Riemann-Roch formula shows that the corresponding complete system on \tilde{Y} is non-empty. Let $C_{jl} \subset \tilde{Y}$ be a member, with image \tilde{C} in F_4 . The curve C_{jl} cannot contain E_5 , for that would force \tilde{C} to split into \tilde{E}_5 plus four lines, which cannot pass through all the base points. It follows that \tilde{C} is an irreducible curve of type (1,4); hence C_{jl} is irreducible and meets neither E_5 nor E_2 . By the adjunction formula $p(C_{jl})=0$, and as $C_{jl}^2 = -1$ we have an exceptional curve.

Similarly, $C'_{jl} \subset \tilde{Y} \setminus E$ may be obtained from the unique curve in the linear system of type (1, 5) with assigned base points P, Q , and P_i ($i \in \{1, \dots, 7\} \setminus \{j, l\}$).

One easily verifies the incidence relations stated in the theorem. Q.E.D.

Our next aim is to identify configurations as described above on smooth del Pezzo surfaces.

PROPOSITION 7.10. *Let F be a smooth del Pezzo surface of degree $9-k$, and suppose on F a configuration of exceptional curves is given, of the kind described in Theorem 7.9. Then there exist a birational morphism $F \xrightarrow{\sigma} \mathbf{P}^2$, and points $P_1, \dots, P_k \in \mathbf{P}^2$ such that the following holds:*

(1) $\sigma^{-1}(P_1), \dots, \sigma^{-1}(P_k)$ are the non-trivial fibres of σ .

(2) If $k=6$ then σ sends C_{ij} to the line through P_i and P_j . If $k=7$ then $C_j = \sigma^{-1}(P_j)$ for $i=1, \dots, 7$. For $k=8$, $\sigma(C_i)$ is the line through P_i and P_8 ($i=1, \dots, 7$). The remaining part of the configuration may be mapped in two ways. Either $\sigma(C_{jl})$ is the line joining P_j and P_l , for each pair (j, l) , and $\sigma(C'_{ij})$ is the quartic through P_1, \dots, P_8 with double points at P_j, P_l , and P_8 , or vice versa.

Proof. The case $k=7$ is trivial, for contracting C_1, \dots, C_7 must yield a projective plane. Thus let $k \neq 7$. Recall that the Weyl group $W = W(E_k)$ acts naturally on $\text{Pic}(F)$, thereby permuting the k -tuples of disjoint exceptional curves. A $(k-1)$ -tuple of disjoint exceptional curves in F is either maximal or part of a k -tuple of such curves; accordingly there are two orbits under W —see Demazure [11], II, Proposition 4. Thus there exists a birational morphism $\sigma: F \rightarrow \mathbf{P}^2$ such that either

$$\sigma(C_{i6}) = P_i \quad (k=6; i=1, \dots, 5), \quad \sigma(C_i) = P_i \quad (k=8; i=1, \dots, 7)$$

(the non-maximal case), or $\sigma(C_{i6})$, respectively $\sigma(C_i)$, is the line joining P_i to P_k ($i=1, \dots, k-1$). We claim that the first case is impossible. To prove this we note that for $k=6$ the exceptional curve C_{12} meets exactly three among the curves C_{i6} ($i=1, \dots, 5$). Likewise, for $k=8$, both C_{12} and C'_{12} meet exactly five of the C_i ($i=1, \dots, 7$). Inspection of Demazure [11], II, Table 3 shows that this could not happen in the non-maximal case.

Thus $\sigma(C_{i6})$, respectively $\sigma(C_i)$, is the line through P_i and P_k ($i=1, \dots, k-1$). Comparing the known incidences on F to the table [loc. cit.] it is readily verified that σ maps the remaining exceptional curves as stated. Q.E.D.

THEOREM 7.11. *The characteristic mapping $\psi_2: S' \rightarrow \mathcal{T}/W$ extends as an analytic morphism*

$$S \xrightarrow{\psi_3} \mathcal{X}/W.$$

Proof. S is smooth and \mathcal{X}/W is affine, it therefore suffices to show that for general $s \in \{u=0\} \subset S$ there exists a neighbourhood U of s in S such that ψ_2 is bounded on $S' \cap U$.

Thus let $s \in \{u=0\} \subset S$ be generic in the sense of (7.4). Still using the notation introduced at the beginning of the section we consider an exceptional curve C_s on the surface $Y \setminus \{n\} = Z_s \setminus \{n\}$. The family $Z \xrightarrow{\pi} S$ is smooth near C_s and by Kodaira [25], Theorem 1, it follows that $C_s \rightarrow \{s\}$ extends to a smooth family of exceptional curves over some neighbourhood U of s in S ,

$$\begin{array}{ccccc} C_s & \subset & C & \subset & Z_U \\ & \searrow & \downarrow & \swarrow & \downarrow \pi \\ & & s \in U & & \end{array}$$

We choose U small enough so that it serves for all curves $C_s \subset Y$ of the configuration described in Theorem 7.9. Shrinking U further, we may achieve that U does not meet the discriminant $D \subset S$, and intersects S' in a connected set. Passing to the Galois cover $\tilde{S}' \xrightarrow{\varrho} S'$ we fix a connected component V of $\varrho^{-1}(S' \cap U)$.

Pick some $r \in V$. Thus $(\varrho^*Z)_r \cong Z_{\varrho(r)}$ is a smooth del Pezzo surface, and Proposition 7.10 provides a birational morphism $(\varrho^*Z)_r \rightarrow \mathbf{P}^2$, which singles out a basis (h_0, h_1, \dots, h_k) of

$$H = H_2((\varrho^*Z)_r, \mathbf{Z}) = \text{Pic}((\varrho^*Z)_r),$$

as discussed in Section 6. By parallel transport we now have a distinguished basis of $\text{Pic}((\varrho^*Z)_r)$, for each $r \in V$.

Each exceptional curve C_t ($t \in U$) intersects $Z_{\infty, t}$ in a single point which we simply denote by $C \cap Z_{\infty, t}$. By definition of the characteristic map we have

$$\psi(r)(C) = \varepsilon_{\varrho(r)}^{-1}(C \cap Z_{\infty, \varrho(r)}) \in \mathbf{C}^*$$

for each $r \in V$. We wish to study the behaviour of $\psi(r)(C)$ as r varies in V . Let us discuss the three cases $k=6, 7$, and 8 in turn.

$k=6$: If r varies such that $\varrho(r)$ converges to s then $C \cap Z_{\infty, \varrho(r)}$ converges to a smooth point of $W = Z_{\infty, s}$. In view of the definition of ε , see (6.9), this means that $\psi(r)(C)/u^2$ has a limit in \mathbf{C}^* . This holds for $C = C_{jl}$ ($1 \leq j < l \leq 6$), and by Theorem 7.9 the class of C_{jl} is

$$C_{jl} = h_0 + h_j + h_l.$$

Re-writing this via (6.14) we obtain that

$$(\psi(r)(h_0+h_j+h_l))^3/\psi(r)(-\kappa) = \psi(r)\left(3h_j+3h_l-\sum_{i=1}^6 h_i\right)$$

converges in \mathbf{C}^* .

The subspace of $\mathbf{Q} \otimes H$ spanned by the classes

$$3h_j+3h_l-\sum_{i=1}^6 h_i \quad (1 \leq j < l \leq 6)$$

is just the kernel of the linear form

$$\beta = (2; 0, \dots, 0) \in \text{Hom}(H, \mathbf{Z}).$$

As, by (6.14),

$$\psi(r)(-\kappa) \rightarrow 0$$

it follows that there is a neighbourhood U' of s in U such that for each $h \in \mathbf{Q} \otimes H$ with $\langle h, \beta \rangle \geq 0$ the function $r \mapsto \psi(r)(h)$ is bounded on $V \cap \varrho^{-1}(U')$. Those h which are also integral on the lattice $\Lambda \subset \text{Hom}(H, \mathbf{Z})$ are, by definition, those characters of $\mathcal{T} = \text{Hom}(H, \mathbf{C})/\Lambda$ which extend to a complex-valued function on \mathcal{X} . Therefore the characteristic mapping

$$S' \xrightarrow{\psi_1} \mathcal{T} \subset \mathcal{X}$$

is bounded on $V \cap \varrho^{-1}(U')$. Dividing by the action of the monodromy, we conclude that

$$S' \xrightarrow{\psi_2} \mathcal{T}/W \subset \mathcal{X}/W$$

is bounded on $S' \cap U'$, and therefore extends over U' . This completes the proof for $k=6$.

The case $k=7$ is even simpler. We put $C=C_i$ ($i=1, \dots, 7$). Then as $\varrho(r) \rightarrow s$ ($r \in V$) the point $C \cap Z_{\infty, \varrho(r)}$ converges to a regular point of $W_1 \subset Z_{\infty, s}$, see Theorem 7.9. Then

$$\psi(r)(C) = \psi(r)(-h_i)$$

converges in \mathbf{C}^* and we conclude as above.

$k=8$: This is slightly more involved. We first observe that for a smooth family of exceptional curves C over U we have

$$C_s \cap W \in W' \Leftrightarrow \psi(r)(C)/u^4 \text{ converges in } \mathbf{C}^*$$

$$C_s \cap W \in W'' \Leftrightarrow \psi(r)(C)/u^{10} \text{ converges in } \mathbf{C}^* \quad (\varrho(r) \rightarrow s).$$

Again this follows from the formula (6.9) defining the parametrization ε . Putting $C=C_j$ ($1 \leq j \leq 7$) we obtain that

$$\psi(r) \left(-12h_0 + 3h_j + 3h_8 - 5 \sum_{i=1}^8 h_i \right)$$

converges in \mathbf{C}^* . On the other hand for $C=C_{lm}$ and $C=C'_{lm}$ application of Proposition 7.10 only gives the ambiguous result that either

$$(\psi(r)(h_0 + h_l + h_m))^3 / (\psi(r)(-\varkappa))^5$$

and

$$\left(\psi(r) \left(4h_0 + h_l + h_m + h_8 + \sum_{i=1}^8 h_i \right) \right)^3 / (\psi(r)(-\varkappa))^2,$$

or

$$(\psi(r)((h_0 + h_l + h_m)))^3 / (\psi(r)(-\varkappa))^2$$

and

$$\left(\psi(r) \left(4h_0 + h_l + h_m + h_8 + \sum_{i=1}^8 h_i \right) \right)^3 / (\psi(r)(-\varkappa))^5,$$

converge in \mathbf{C}^* as $\varrho(r) \rightarrow s$. But the first case may be ruled out: The classes

$$-12h_0 + 3h_j + 3h_8 - 5 \sum_{i=1}^8 h_i \quad (1 \leq j \leq 7),$$

$$3(h_0 + h_l + h_m) + 5\varkappa = -12h_0 + 3h_l + 3h_m - 5 \sum_{i=1}^8 h_i$$

and

$$3 \left(4h_0 + h_l + h_m + h_8 + \sum_{i=1}^8 h_i \right) + 2\varkappa = 6h_0 + 3h_l + 3h_m + 3h_8 - \sum_{i=1}^8 h_i$$

($1 \leq l < m \leq 7$) generate the vector space $\mathbf{Q} \otimes H$, and if $\psi(r)$ were to have a limit (in \mathbf{C}^*) on all then in particular on \varkappa ; but this is absurd by (6.14). This leaves only the second

possibility. Here the classes

$$3(h_0 + h_l + h_m) + 2\kappa = -3h_0 + 3h_l + 3h_m - 2 \sum_{i=1}^8 h_i$$

$$3 \left(4h_0 + h_l + h_m + h_8 + \sum_{i=1}^8 h_i \right) + 5\kappa = -3h_0 + 3h_l + 3h_m + 3h_8 - 2 \sum_{i=1}^8 h_i$$

($1 \leq l < m \leq 7$) span the kernel of the form

$$\beta_1 := (8; 3, \dots, 3, 0) \in \text{Hom}(H, \mathbf{Z}).$$

As $\langle \beta_1, -\kappa \rangle = 3$ is positive the proof will be complete if we can show that some positive multiple of β_1 is in the W -orbit of β . To this end note that $\langle \beta_1, -\kappa \rangle = 3 = \langle \frac{1}{2}\beta, -\kappa \rangle$, and that the orthogonal projections of β_1 and $\frac{1}{2}\beta$ to Q are

$$\beta'_1 = -(1; 0, \dots, 0, 3)$$

and

$$\beta' = -(8; 3, \dots, 3).$$

We have $(\beta'_1, \beta'_1) = -8 = (\beta', \beta')$, and neither $\frac{1}{2}\beta'_1$ nor $\frac{1}{2}\beta'$ is a root. The result now is a consequence of the following lemma. Q.E.D.

LEMMA 7.12. *Let Q be the root lattice of a root system of type E_8 , equipped with the invariant quadratic form q that takes value 2 on each root. Then the Weyl group acts transitively on*

$$\{x \in Q \mid q(x) = 8 \text{ but } \frac{1}{2}x \text{ is not a root}\}.$$

Proof. Using coordinates as above, q is minus the intersection form, and β' belongs to the set in question. The isotropy group of β' (which must be a reflection group) is the symmetric group $\text{Sym}(8)$ generated by $\alpha_1, \dots, \alpha_7$. Thus $W\beta'$ has

$$|W|/|\text{Sym}(8)| = 2^2 \cdot 3^2 \cdot 4^2 \cdot 5 \cdot 6 = 240 \cdot 72$$

elements. Adding the 240 double roots we obtain $240 \cdot 73$ vectors x with $q(x) = 8$. By the theory of modular forms (see Gunning [16], § 12) there are no others. Q.E.D.

The proof of Theorem 6.2 is now quickly completed. By Proposition 6.15 the map $\psi_3: S \rightarrow \mathcal{H}/W$ is \mathbf{C}^* -equivariant; in particular it is an algebraic morphism. By Lemma 4.23 it is an isomorphism, and we put

$$\Phi = \psi_3^{-1}: \mathcal{X}/W \rightarrow S.$$

Let $r \in \tilde{S}'$ be such that $s = \varrho(r) \in D$. Then the curve X_s is singular, hence so is $(\varrho^*Z)_r \cong Z_s$. By Demazure [11], V, Proposition 1 there exists a root $h \in H$ which is represented by an effective divisor on the resolution \tilde{Z}_r (which is contracted in Z_s). Therefore $\psi(r)(h) = 1$; hence $\langle h, \psi_1(r) \rangle \in \mathbf{Z}$. This implies

$$\psi_2(s) \in \Delta = \Delta(E_k[\ast]_2).$$

As both $D \subset S$ and $\Delta \subset \mathcal{X}/W$ are irreducible hypersurfaces it follows that $\psi_3(D) = \Delta$, that is, $\Phi(\Delta) = D$. Q.E.D.

8. $E_k[\ast]$: Supplements

Theorem 6.2 has consequences for the deformations of the singularity $E_k[\ast]$ similar to the $D_k[\ast]$ -case. Still using the notation of the previous section, we have the following analogue of Theorem 5.1.

THEOREM 8.1. *Let $\tau \in \mathcal{X}$ and put $s = \Phi(\tau \bmod W_2) \in S$. Assume that $x_1, \dots, x_r \in X_s$ are the singular points of the fibre of the deformation $X \xrightarrow{\pi} S$ over s . Then the plane curve singularities among the (X_s, x_ϱ) are of type A_{l-1} , D_{l-1} , or E_l , with $l \leq k$. There is at most one more singularity, and this has type $D_{l-1}[\ast]$ or $E_l[\ast]$ ($l \leq k$). The isotropy group of W_2 at $\tau \in \mathcal{X}$ is isomorphic to the direct product $M_1 \times \dots \times M_r$, where the factors are defined as in Theorem 5.1.*

Proof. See Theorem 5.1.

Q.E.D

The deformation theory of the singularity $E_k[\ast]$ puts it in an intermediate position between the simple hypersurface singularity E_k and the simply-elliptic singularity \tilde{E}_k . Each of the corresponding semi-universal deformations can be described by a family of del Pezzo surfaces of degree $9-k$ with a distinguished anticanonical divisor. The latter has arithmetic genus one and is a rational curve with a cusp (E_k , see Tyurina [46], or Pinkham [40], 5); or a rational curve with a node, allowed to decompose ($E_k[\ast]$); or an elliptic curve whose j -invariant is allowed to vary (\tilde{E}_k , see Looijenga [26], Pinkham [39], Merindol [34]). Accordingly a description of the discriminant is obtained via an action of the Weyl group $W(E_k)$ on a vector space, an algebraic torus, or a family of abelian varieties.

These distinctions also imply different adjacency relations. As is well-known the

singularity E_k deforms into configurations of simple singularities that correspond to root systems contained in the system E_k as rationally closed subsets; these are classified by the full subgraphs of the Dynkin diagram E_k .

In the other two cases adjacencies correspond to closed and symmetric subsets of the root system E_k , but not all of those need occur (du Val [48], Merindol [34], Urabe [47]). Let us illustrate this point by one example.

(8.2) *Example.* The maximal number of ordinary double points that can occur in a fibre of the semi-universal deformation is 4 for E_7 , 5 for $E_7[*]$, and 6 for \tilde{E}_7 .

Proof. This is clear for E_7 while the case of \tilde{E}_7 is treated in detail in the references just quoted. Turning to $E_7[*]$ we first look at points in $\mathcal{T} \subset \mathcal{X}$.

Let R be the root system of type E_7 . Following Urabe [47] the closed symmetric subsets of type $5A_1$, $6A_1$, and $7A_1$ form one W -orbit each. The corresponding manipulations of the Dynkin diagram are:

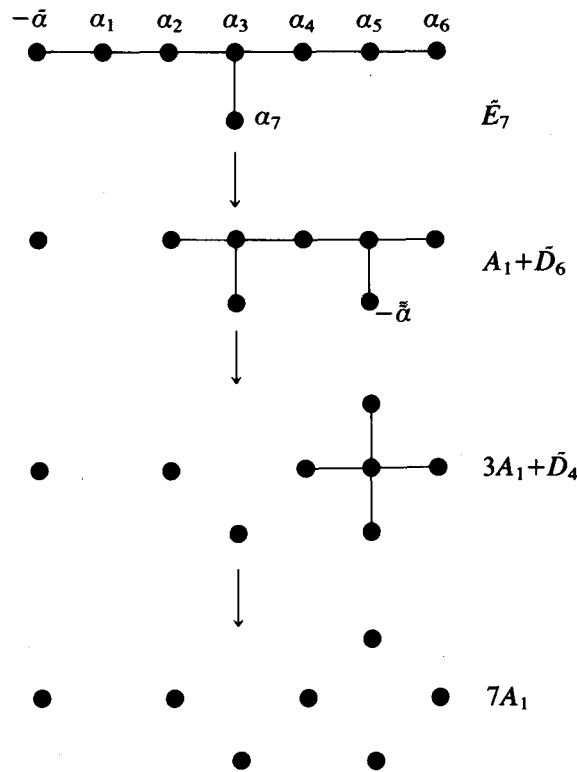


Table 8.3

By (2.12), in order to realize the configuration $7A_1$, we must try and find $t \in \mathbb{R}^7$ with the property that the roots in the last diagram together with their negatives are just those roots of R with integral values on t . Let $\bar{\alpha} \in R$ denote the greatest root; it then suffices to consider t in the closed fundamental alcove

$$\{\alpha_j^\vee \geq 0 \ (j=1, \dots, 7), \bar{\alpha}^\vee \leq 1\}.$$

Using standard coordinates as in the previous section the conditions on $t=(t_0; t_1, \dots, t_7)$ include

$$\begin{aligned} \langle t, \alpha_2^\vee \rangle &= t_2 - t_3 = 0 \\ \langle t, \alpha_4^\vee \rangle &= t_4 - t_5 = 0 \\ \langle t, \alpha_6^\vee \rangle &= t_6 - t_7 = 0 \\ \langle t, \alpha_7^\vee \rangle &= t_0 - t_1 - t_2 - t_3 = 0 \\ \langle t, \bar{\alpha}^\vee \rangle &= 2t_0 - t_2 - \dots - t_7 = 1 \\ \langle t, \bar{\alpha}^\vee \rangle &= t_0 - t_1 - t_6 - t_7 \in \{0, 1\} \\ \langle t, \alpha_1^\vee \rangle &= t_1 - t_2 \in (0, 1) \\ \langle t, \alpha_3^\vee \rangle &= t_3 - t_4 \in (0, 1) \\ \langle t, \alpha_5^\vee \rangle &= t_5 - t_6 \in (0, 1) \end{aligned} \tag{*}$$

Thus t must have the form

$$t = (t_0; t_1, t_2, t_2, t_4, t_4, t_6, t_6)$$

with

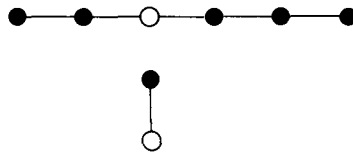
$$\begin{aligned} t_0 &= t_1 + 2t_2 \\ 2t_0 &= 2(t_2 + t_4 + t_6) + 1 \\ t_0 - t_1 - 2t_6 &\in \{0, 1\} \\ t_1 &> t_2 > t_4 > t_6. \end{aligned} \tag{*}$$

These conditions are incompatible; therefore not even the type $6A_1$ can be realized. On the other hand, if we are looking for $5A_1$ the conditions marked (*) may be dropped, and we have a three-parameter family of solutions

$$t = (t_2 + t_4 + t_6 + \frac{1}{2}; -t_2 + t_4 + t_6 + \frac{1}{2}, t_2, t_2, t_4, t_4, t_6, t_6)$$

with $t_2 > t_4 > t_6$, $t_4 + t_6 > 2t_2 + \frac{1}{2}$. Therefore $5A_1$ does occur as a configuration of singular points.

It is readily verified that $6A_1$ is neither realized on any lower dimensional \mathcal{T} -orbit in \mathcal{X} . This completes the proof. We remark, though, that $5A_1$ also occurs on the two-codimensional \mathcal{T} -orbits as is seen from the diagram



derived from $E_7[\ast]$.

Quite a different, if less systematic argument can be given in terms of the geometry of del Pezzo surfaces of degree 2. Any such surface is a double cover of the projective plane, branched along a quartic curve—see Demazure [11] V, 4. The singularities of the surface and the branch curve correspond as in Corollary 6.7. Thus the semi-universal deformations of E_7 , $E_7[\ast]$, and \tilde{E}_7 may also be described by families of plane quartics with a distinguished section “at infinity”. For E_7 , the latter consists of a simple point plus a triple point. Likewise, by (6.5) we have two simple plus one double point for $E_7[\ast]$ while the \tilde{E}_7 -case is characterized by four simple points at infinity.

By elementary geometry of the plane no quartic curve can have seven ordinary double points while one with six double points is a complete quadrilateral. This clearly checks with the numbers given in (8.2).

Recall from Theorem 3.10 that we have a natural presentation of the fundamental group $\pi_1(S \setminus D, s)$, with a geometrically distinguished generator a_α for each vertex α of the affine Dynkin diagram \tilde{E}_k . The following is proved in complete analogy with Theorem 5.6.

THEOREM 8.4. *The generators a_α determine a weakly distinguished basis of vanishing cycles in the Milnor homology of the curve $E_k[\ast]$, with intersection diagram \tilde{E}_k . Q.E.D.*

Again the corresponding vanishing lattice over the field \mathbf{F}_2 is determined by the relevant dimensions and the number of mod 2 vanishing cycles. These numbers are 72, 126, and 240 ($k=6, 7, 8$) and in the classification of integral vanishing lattices by Janssen

[19] we arrive at the types $O_1^\#(1, 1, 1; 1)$, $O^\#(1, 1, 1; 2; \infty)$, and $O_0^\#(1, 1, 1, 1; 1)$, respectively.

The group $\pi_1(S \setminus D, s)$ also acts, though in a different way, as a monodromy group on the Milnor homology of the family of affine surfaces

$$Z'_{S \setminus D} \xrightarrow{\pi} S \setminus D$$

where $Z' = Z \setminus Z_\infty$, see (6.3). This action may be identified as follows. Note that in fact $s \in S' \setminus D$, and look at the exact homology sequence (with integral coefficients) of the pair (Z_s, Z'_s) :

$$\begin{array}{ccccccc} H_3(Z_s) & \rightarrow & H_3(Z_s, Z'_s) & \xrightarrow{\partial} & H_2(Z'_s) & \rightarrow & H_2(Z_s) \rightarrow H_2(Z_s, Z'_s) \\ & & \parallel & & \parallel & & \parallel \\ & & H_3(T, \partial T) & & \text{Pic}(Z_s) & & H_2(T, \partial T) \end{array} \quad (8.5)$$

T is a compact ‘‘tubular’’ neighbourhood of the singular curve $Z_{\infty, s}$ in Z_s , that is, a compact four-manifold with boundary in Z_s of which $Z_{\infty, s}$ is a deformation retract. A simple way to construct such retractions is shown in Pickl [37], II, § 4.

By Lefschetz duality we have

$$\begin{aligned} H_3(T, \partial T) &\simeq H^1(T) = H^1(Z_{\infty, s}) \\ H_2(T, \partial T) &\simeq H^2(T) = H^2(Z_{\infty, s}). \end{aligned}$$

$Z_{\infty, s}$ is obtained from the 2-sphere by identifying two points p and q . Poincaré duality in the sphere yields (with $Z_{\infty, s}^{\text{reg}} \simeq \mathbb{C}^*$ the regular locus of $Z_{\infty, s}$):

$$H^1(Z_{\infty, s}) = H^1(S^2, \{p, q\}) \simeq H_1(S^2 \setminus \{p, q\}) = H_1(Z_{\infty, s}^{\text{reg}})$$

and $H^2(Z_{\infty, s}) \simeq H_0(Z_{\infty, s}^{\text{reg}})$. Thus the sequence (8.5) reads

$$\begin{array}{ccccccc} 0 & \rightarrow & H_1(Z_{\infty, s}^{\text{reg}}) & \rightarrow & H_2(Z'_s) & \rightarrow & \text{Pic}(Z_s) \rightarrow H_0(Z_{\infty, s}^{\text{reg}}) \\ & & \parallel & & \parallel & & \parallel \\ & & \mathbb{Z} & & \mathbb{Z}^{k+1} & & \mathbb{Z} \end{array} \quad (8.6)$$

where all homomorphisms have a simple geometric meaning: the retraction

$$r: T \rightarrow Z_{\infty, s}$$

may be chosen so as to restrict to a disk bundle projection over $Z_{\infty, s}^{\text{reg}}$, and the first non-

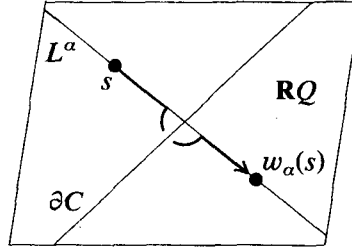


Figure 8.7

trivial arrow of the sequence sends a cycle to its inverse image in ∂T . The next arrow is induced by the inclusion, and the last assigns to each class its intersection number with the anticanonical curve $Z_{\infty, s}$.

For each of the generators $a_\alpha \in \pi_1(S \setminus D, s)$ ($\alpha = \alpha_1, \dots, \alpha_k, -\bar{\alpha}$) let us construct a vanishing cycle v_α in Z'_s . Recall the definition of a_α (see Figure 8.7). The base point s is the barycentre of $C \cap \mathbf{RQ}$, the intersection of the fundamental alcove with the hyperplane spanned by the roots. The defining representative $a_\alpha: [0, 1] \rightarrow L_C^\alpha$ is the path which follows the real segment from s to $w_\alpha(s)$, avoiding the fixed hyperplane of w_α on a small positively oriented semi-circle in L_C^α .

We identify S with $\mathcal{X}W$ via the isomorphism $\psi_3 = \Phi^{-1}$ constructed in the previous section. Note that the function u restricts to a non-zero constant on \mathbf{RQ} ; therefore the curve $Z_{\infty, t}$ does not vary with $t \in L_C^\alpha$ and we may put $W := Z_{\infty, t}$.

The dual root α^\vee may be conceived of as an element of

$$H = H_2(Z_s) = \text{Pic}(Z_s).$$

By Demazure [11] II, 2, the root $\alpha^\vee \in H$ is the difference of a pair of exceptional curves. We pick one such pair, say $\alpha^\vee = e_s - f_s$. As $\tau \in [0, 1]$ and hence $a_\alpha(\tau) \in S$ vary these curves vary in smooth families e and f ; we have $e_{w_\alpha(s)} = f_s$ and vice versa. The point of intersection $e \cap W$ thus forms a smooth path

$$\bar{a}_\alpha: [0, 1] \rightarrow W^{\text{reg}},$$

joining $e_s \cap W$ to $f_s \cap W$. This path lifts to a path \tilde{a}_α in the universal cover

$$C \rightarrow C/\mathbf{Z} \simeq W^{\text{reg}}.$$

By definition of the characteristic mapping ψ —see (6.10)—this path is linear, and the difference $\tilde{a}_\alpha(0) - \tilde{a}_\alpha(1)$ is just $\langle s, \alpha^\vee \rangle$. In particular \tilde{a}_α is actually an embedding.

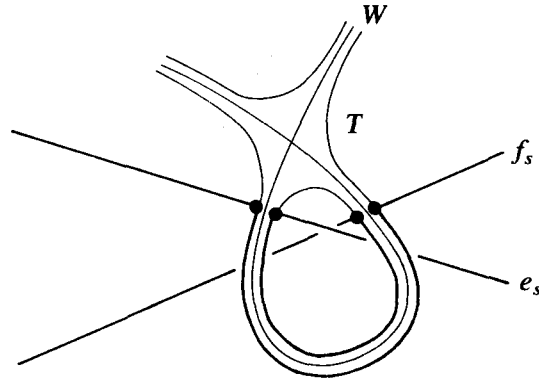


Figure 8.8

Let $v_\alpha \subset Z'_s$ be the sphere

$$(e_s \setminus \dot{T}) \cup (r^{-1}(\bar{a}_\alpha[0, 1]) \cap \partial T) \cup (f_s \setminus \dot{T})$$

as shown in the (real) Figure 8.8.

We give v_α the orientation which restricts to the natural orientation of $e_s \setminus T$.

THEOREM 8.9. (1) v_α is a vanishing cycle with respect to the path that linearly joins s to $\frac{1}{2}(s+w_\alpha(s))$.

(2) $(v_{\alpha_1}, \dots, v_{\alpha_k}, v_{-\bar{\alpha}})$ represents a basis of the homology group $H_2(Z'_s)$.

(3) This basis is a root basis of affine type \tilde{E}_k (with respect to the intersection form on $H_2(Z'_s)$), and $\pi_1(S \setminus D, s)$ acts as the affine Weyl group.

Proof. All this is clear by construction but the claim that the v_α ($\alpha = \alpha_1, \dots, \alpha_k, -\bar{\alpha}$) generate $H_2(Z'_s)$. They clearly generate the kernel of

$$\text{Pic}(Z_s) \rightarrow H_0(Z_{\infty, s}^{\text{reg}}),$$

and in view of the exact sequence (8.6) it suffices to exhibit a generator of the cyclic group $H_1(W)$ as a linear combination of the v_α . To this end let

$$\bar{\alpha} = \sum_{j=1}^k c_j \alpha_j$$

be the representation of the greatest root in the finite root system E_k , with basis $\alpha_1, \dots, \alpha_k$. Then the linear combination

$$v_{-\tilde{a}} + \sum_{j=1}^k c_j v_{\alpha_j} \in H_2(Z'_s)$$

maps to zero in $H_2(Z_s)$, and, therefore, is represented by a sum of arcs in W^{reg} . Lifting these to the universal cover $C \rightarrow C/Z \simeq W^{\text{reg}}$ we obtain segments of total length

$$-\langle s, \tilde{a}^\vee \rangle + \sum_{j=1}^k c_j \langle s, \tilde{\alpha}_j^\vee \rangle = -1. \quad \text{Q.E.D.}$$

Recall that the family of projective surfaces $Z \xrightarrow{\pi} S$ is given by the equation

$$0 = f(w, x, y, z) = p(w, y) + q(w, z) + u^2 w^{\beta-a+1} x^2 + 2vw^{\beta+1} x - 2xyz,$$

compare (6.4). It is clearly induced from the family

$$X \xrightarrow{\chi} T = \{(p, q, t, v)\} \simeq \mathbf{C}^{k+1}$$

given by

$$0 = g(w, x, y, z) = p(w, y) + q(w, z) + tw^{\beta-a+1} x^2 + 2vw^{\beta+1} x - 2xyz,$$

via the substitution $t = u^2$. For $k=6$ this latter family carries the fixed divisor

$$W = \{2xyz = y^3 + z^3\}$$

at infinity. The fibre of χ over the point $(Y^3, Z^3, 0, 0) \in T$ is the cone over W , and the whole family is naturally interpreted as the semi-universal projective deformation of that cone with fixed divisor W , compare Pinkham [38] Chapter I. The singularities of the fibres of χ are either rational double points or isolated line singularities in the sense of Siersma [41]. Our description of S as a quotient $\mathcal{A}(E_6[\ast])/W(E_6[\ast])$ also provides a description of the locus of singular fibres of χ . In fact the method of Section 6 still applies if $k \neq 6$, and serves to prove the following result which will only be stated.

Let W be a rational curve with node, and let X_0 be the projective cone over W with respect to a degree $9-k$ line bundle \mathcal{L} —that is,

$$X_0 = \text{Proj} \bigoplus_{j=0}^{\infty} H^0(W, \mathcal{L}^j)[w], \quad \text{weight}(w) = 1.$$

Let $X \xrightarrow{\chi} T$ be the semi-universal projective deformation of X_0 with fixed divisor $W = \{w=0\}$, and let $D \subset T$ be the locus of singular fibres.

THEOREM 8.10. *There is a C^* -equivariant isomorphism*

$$T \rightarrow \mathcal{X}(\tilde{E}_k) / \frac{1}{2} \Lambda(\tilde{E}_k) \cdot W(\tilde{E}_k)$$

which takes D to the discriminant of the quotient map

$$\mathcal{X} \rightarrow \mathcal{X} / \frac{1}{2} \Lambda \cdot W.$$

Here the extended Dynkin diagram \tilde{E}_k is read as a diagram in the sense of Chapter I, with the added root the unique white vertex.

Note that D is reducible as all fibres of χ over $\{t=0\}$ are singular at infinity.

9. Mixed root bases

In this section we generalize the notion of diagram from Section 1; we will allow a generalized root system in the sense of Looijenga [28] to play the role of the classical root system associated with the Dynkin diagram $\mathcal{D}_{\text{black}} \subset \mathcal{D}$. Using a recent construction of Looijenga [33] we shall see that in this situation there still is a naturally defined quotient \mathcal{X}/W . Rather than an affine space, \mathcal{X}/W will be a Stein manifold, equipped with a stratification that may be described combinatorially in terms of the root data.

We also include proofs of Theorem 1.2 and Proposition 1.5.

Let V be a real vector space of finite dimension. We study triples (A, δ, B) where A and B are disjoint subsets of V , and

$$\begin{aligned} \delta: A &= A^\vee \subset V^\vee \\ \alpha &\rightarrow \alpha^\vee \end{aligned}$$

is an embedding of A in the dual space V^\vee . This data is subject to the following axioms.

(R1) $A \cup B$ is a basis of V .

(R2) The pair (A, δ) is a (generalized) root basis, see Looijenga [28].

(R3) $\langle \beta, \alpha^\vee \rangle \leq 0$ for all $\alpha \in A, \beta \in B$.

Thus $\langle \alpha, \alpha^\vee \rangle = 2$ for $\alpha \in A$, while $\langle \gamma, \alpha^\vee \rangle$ is a non-positive integer for all $\alpha \in A, \gamma \in A \cup B, \alpha \neq \gamma$. In case this number vanishes for some $\gamma \in A$ then so does $\langle \alpha, \gamma^\vee \rangle$.

The matrix

$$(\langle \gamma, \alpha^\vee \rangle)_{\alpha \in A, \gamma \in A \cup B}$$

is called the *Cartan matrix* of (A, δ, B) . The corresponding *Dynkin diagram* $\mathcal{D} = \mathcal{D}(A, \delta, B)$ is, by definition, the Dynkin diagram of the root basis (A, δ) , extended by

one white vertex for each $\beta \in B$; this vertex and a vertex $\alpha \in A$ span an edge in \mathcal{D} if $-\langle \beta, \alpha^\vee \rangle > 0$, this number being the weight of the edge. Recall that two distinct black vertices α and α' span an edge of weight $\langle \alpha, \alpha'^\vee \rangle \cdot \langle \alpha', \alpha^\vee \rangle$ unless this is zero.

Thus if the root basis (A, δ) is known to satisfy

$$\langle \alpha, \alpha'^\vee \rangle \geq -1 \quad \text{or} \quad \langle \alpha', \alpha^\vee \rangle \geq -1, \quad \text{for all } \alpha, \alpha' \in A, \quad (9.1)$$

then the Cartan matrix of (A, δ, B) may be recovered from the Dynkin diagram. In particular the construction described in Section 1 assigns to a diagram \mathcal{D} the (essentially unique) triple (A, δ, B) that has \mathcal{D} as its Dynkin diagram, and satisfies (9.1).

In analogy with Section 1 we define: (A, δ, B) is a *mixed root basis* if:

(R4) Each connected component of the Dynkin diagram $\mathcal{D}(A, \delta, B)$ contains a white vertex.

If (A, δ, B) is a mixed root basis we let $Q = \mathbb{Z}A \subset V$ be the root lattice, and put $\Lambda = Q + \mathbb{Z}B$ as usual. The Weyl group $W \subset GL(V)$ generated by (A, δ) may now be infinite, and K , the convex cone spanned by the set $WB \subset V$, need not be closed in V .

The fundamental chamber

$$C = \{x \in V \mid \langle x, \alpha^\vee \rangle > 0 \text{ for all } \alpha \in A\}$$

defines the *Tits cone* $I = WC \subset V$, see Looijenga [28], (1.1). We have $B \subset -\bar{C}$, therefore $K \subset -I$, as well as

$$WB \subset B + \mathbb{Z}_+ A$$

([loc. cit.], (1.11)).

LEMMA 9.2. *K has non-empty interior, so $\dim K = \dim V$.*

Proof. By induction on the cardinality $|A|$. For $A = \emptyset$ the set B spans V , and the assertion is trivial. If $A \neq \emptyset$ then each black vertex α in \mathcal{D} may be joined to the subdiagram $\mathcal{D}_{\text{white}}$ by a path in \mathcal{D} . We let $d(\alpha, \mathcal{D}_{\text{white}})$ be the distance between α and $\mathcal{D}_{\text{white}}$, that is, the number of edges needed to set up such a path. Choose $\alpha \in A$ so that $d(\alpha, \mathcal{D}_{\text{white}})$ is maximal, and let $A' = A \setminus \{\alpha\}$. Then the triple $(A', \delta|_{A'}, B)$ is a mixed root basis in the hyperplane $V' \subset V$ which is spanned by $A' \cup B$. By induction hypothesis the corresponding cone $K' = K_{(A', \delta|_{A'}, B)}$ has non-empty interior in V' . As the dual root α^\vee does not vanish identically on V' it cannot vanish identically on K' . Thus $w_\alpha K'$ is not contained in V' , and the assertion follows. Q.E.D.

As the notion of root basis is self-dual we also have the dual Tits cone $I^\vee \subset V^\vee$. Our aim is to describe the dual of K , the cone

$$K^\vee = \{x^\vee \in V^\vee \mid x^\vee \geq 0 \text{ on } K\} \subset V^\vee,$$

which is closed by definition. We first prove:

PROPOSITION 9.3. $K^\vee \subset I^\vee$.

Proof. Assume there exists an $x^\vee \in K^\vee \setminus I^\vee$. We let $\Gamma \subset \mathbf{R}$ be the additive subgroup generated by the finite set $\{\langle \alpha, x^\vee \rangle \mid \alpha \in A\}$. Inductively, we construct a sequence of linear forms x_j^\vee ($j=0, 1, \dots$) with the following properties.

- (i) $x_j^\vee \in Wx^\vee \cap (x^\vee + \Gamma A^\vee)$
- (ii) $x_{j+1}^\vee - x_j^\vee = \gamma_j \alpha_j^\vee$ for some $\alpha_j \in A$, and some $\gamma_j \in \Gamma$, $\gamma_j > 0$.

We may start with $x_0^\vee = x^\vee$. If x_j^\vee has been defined then (i) implies that $x_j^\vee \notin I^\vee$. In particular $x_j^\vee \notin \bar{C}^\vee$, and we find an $\alpha_j \in A$ with $\langle \alpha_j, x_j^\vee \rangle < 0$. We put

$$x_{j+1}^\vee = w_{\alpha_j}(x_j^\vee) = x_j^\vee - \langle \alpha_j, x_j^\vee \rangle \alpha_j^\vee.$$

Then $\gamma_j = -\langle \alpha_j, x_j^\vee \rangle$ is positive and in Γ , for by (i) we have

$$\gamma_j = -\langle \alpha_j, x_j^\vee \rangle \equiv -\langle \alpha_j, x^\vee \rangle \pmod{\Gamma}.$$

This completes the induction.

As K has non-empty interior by Lemma 9.2, and is a W -invariant subset of $-I$ it meets the chamber $-C$. Thus we may pick some $x \in K \cap (-C)$. By (ii) the values $\langle x, x_j^\vee \rangle$ tend to $-\infty$ as $j \rightarrow \infty$. In view of (i) this contradicts the assumption $x^\vee \in K^\vee$. Q.E.D.

Let $X \subset A \cup B$ be any subset. We shall have to distinguish various types of such, recognizable by the corresponding full subgraphs \mathcal{D}_X of the Dynkin diagram $\mathcal{D} = \mathcal{D}_{A \cup B}$.

X is a *mixed subset* of $A \cup B$ if each connected component of \mathcal{D}_X contains a white vertex. In this case $(X \cap A, \delta \mid (X \cap A), X \cap B)$ is a mixed root basis in the vector space $\mathbf{R}X$.

X is a *special subset* of $A \cup B$ if it is contained in A , and is special in the sense of Looijenga [28], that is, if each connected component of \mathcal{D}_X generates an infinite Weyl group.

The subset $X \subset A \cup B$ is *admissible* if each connected component of \mathcal{D}_X corresponds to a subset of X which is either mixed or special.

Clearly, each subset $X \subset A \cup B$ contains greatest mixed, special, and admissible subsets; we denote these X^m , X^s , and X^a respectively.

For $X \subset A \cup B$ we put

$$X^* = \{\alpha \in A \mid \langle \gamma, \alpha^\vee \rangle = 0 \text{ for all } \gamma \in X\}.$$

Let us study the decomposition of the dual cone K^\vee into *facets*, defined as follows:

$$x^\vee, y^\vee \in K^\vee \text{ belong to the same facet if and only if } \bar{K} \cap \{x^\vee = 0\} = \bar{K} \cap \{y^\vee = 0\}.$$

LEMMA 9.4. *Let $\Phi \subset K^\vee$ be a facet. Then Φ is open in its supporting vector space $\mathbf{R}\Phi$.*

Proof. Let $x^\vee \in \Phi$ and let $V' \subset V$ be the subspace spanned by the set $\bar{K} \cap \{x^\vee = 0\}$. We choose a Euclidean norm on the quotient V/V' ; this norm lifts to V as a semi-norm σ that vanishes exactly on V' . As x^\vee is zero on V' it takes a positive minimum c on the set $\bar{K} \cap \{\sigma = 1\}$. If $y^\vee \in \mathbf{R}\Phi$ is sufficiently close to the origin then $|y^\vee| < c$ on $\bar{K} \cap \{\sigma = 1\}$. It follows that

$$x^\vee + y^\vee = 0 \quad \text{on } V',$$

and

$$x^\vee + y^\vee > 0 \quad \text{on } \bar{K} \setminus V'.$$

Therefore $x^\vee + y^\vee \in \Phi$, and the lemma follows. Q.E.D.

Lemma 9.2 implies that $\{0\}$ is a facet of K^\vee . For any subset $X \subset A \cup B$ we have a unique facet $\Phi(X)$ which contains the linear form x^\vee with

$$\begin{aligned} \langle \gamma, x^\vee \rangle &= 0 & \text{if } \gamma \in X \\ \langle \gamma, x^\vee \rangle &= 1 & \text{if } \gamma \in (A \cup B) \setminus X. \end{aligned}$$

Thus $\{0\} = \Phi(A \cup B)$.

We classify the facets of K^\vee as follows.

THEOREM 9.5. *There is a bijection $X \mapsto \Phi(X)$, between the set of admissible subsets of $A \cup B$, and the set of W -orbits of facets of K^\vee .*

As a first step we prove:

LEMMA 9.6. *Let $x^\vee \in K^\vee$. Then the orbit Wx^\vee meets $\Phi(X)$ for some subset $X \subset A \cup B$.*

Proof. By Proposition 9.3 we may assume that x^\vee belongs to the closed dual fundamental chamber \tilde{C}^\vee , so $\langle \alpha, x^\vee \rangle \geq 0$ for all $\alpha \in A$. As $x^\vee \in K^\vee$ we also have $\langle \beta, x^\vee \rangle \geq 0$ for all $\beta \in B$. We put

$$X = \{y \in A \cup B \mid \langle \gamma, x^\vee \rangle = 0\}$$

and let y^\vee be the linear form that defines $\Phi(X)$; $\langle \gamma, y^\vee \rangle = 0$ if $\gamma \in X$ and $\langle \gamma, y^\vee \rangle = 1$ if $\gamma \in (A \cup B) \setminus X$. As \tilde{K} is contained in $\mathbf{R}_+ \cdot (A \cup B)$ it follows that

$$\tilde{K} \cap \{x^\vee = 0\} = \left\{ \sum_{\gamma \in A \cup B} c_\gamma \cdot \gamma \in \tilde{K} \mid c_\gamma = 0 \text{ if } \langle \gamma, x^\vee \rangle > 0 \right\} = \tilde{K} \cap \{y^\vee = 0\}.$$

Thus $x^\vee \in \Phi(X)$.

Q.E.D.

PROPOSITION 9.7. $\Phi(X) = \Phi(X^a)$ for any subset $X \subset A \cup B$.

Proof. The root basis $(Y := X \setminus X^a, \delta \mid Y)$ generates a finite Weyl group $W_Y \subset GL(V)$. Let $\pi: V \rightarrow \mathbf{R}Y$ be the projection along the W_Y -fixed space

$$V^{W_Y} = \{x \in V \mid \langle x, \alpha^\vee \rangle = 0 \text{ for all } \alpha \in Y\}.$$

Composing π with a W_Y -invariant Euclidean norm on $\mathbf{R}Y$ we obtain a semi-norm $\sigma: V \rightarrow \mathbf{R}$ which is W_Y -invariant and vanishes exactly on V^{W_Y} .

Let $\beta \in B$, and consider any y in the W -orbit of β ,

$$y = \beta + \sum_{\alpha \in A} y_\alpha \alpha.$$

We put $c := \max_{\alpha \in A \setminus X} \sigma(\alpha)$ and claim that

$$\sigma(y) \leq \sigma(\beta) + c \cdot \sum_{\alpha \in A \setminus X} y_\alpha. \tag{9.8}$$

To prove this we choose an element $w \in W$ of minimal length $l(w)$, such that $w\beta = y$. If $l(w)$ is zero then $y = \beta$ and (9.8) holds trivially. If $l(w)$ is positive we argue by induction: for some $\alpha_0 \in A$ we have $\langle y, \alpha_0^\vee \rangle > 0$, and the inductive hypothesis applies to

$$y' = w_{\alpha_0}(y) = y - \langle y, \alpha_0^\vee \rangle \alpha_0.$$

Writing $y' = \sum_{\alpha \in A} y'_\alpha \alpha$ we thus have

$$y_\alpha = y'_\alpha \quad \text{if } \alpha \neq \alpha_0$$

$$y_{\alpha_0} = y'_{\alpha_0} + \langle y, \alpha_0^\vee \rangle.$$

We now distinguish three cases:

(i) If $\alpha_0 \in X^a$ then $\langle \alpha_0, \alpha^\vee \rangle = 0$ for all $\alpha \in Y$, hence $\alpha_0 \in V^{W_Y}$, and $\sigma(y) = \sigma(y')$; this implies (9.8).

(ii) If $\alpha_0 \in Y$ then $\sigma(y) = \sigma(y')$ because σ is W_Y -invariant, and the assertion follows again.

(iii) If $\alpha_0 \in A \setminus X$ then

$$\begin{aligned} \sigma(y) &\leq \sigma(y') + \langle y, \alpha_0^\vee \rangle \sigma(\alpha_0) \\ &\leq \sigma(\beta) + c \cdot \sum_{\alpha \in A \setminus X} y'_\alpha + \langle y, \alpha_0^\vee \rangle \sigma(\alpha_0) \\ &\leq \sigma(\beta) + c \cdot \left(\sum_{\alpha \in A \setminus X} y'_\alpha + \langle y, \alpha_0^\vee \rangle \right) \\ &\leq \sigma(\beta) + c \cdot \sum_{\alpha \in A \setminus X} y_\alpha. \end{aligned}$$

This establishes the estimate (9.8). As to the proof of the proposition, let $x \in K$ converge to $\bar{x} \in \bar{K}$, with $\bar{x} = \sum_{\gamma \in X} \bar{x}_\gamma \gamma$. We must show that $\bar{x}_\gamma = 0$ for $\gamma \in Y$. Now x may be written

$$x = \sum_{j \in J} \lambda_j y^{(j)}$$

with $y^{(j)} \in W\beta_j$ ($\beta_j \in B$), and $\lambda_j \geq 0$. Thus

$$x = \sum_{j \in J} \lambda_j \beta_j + \sum_{\substack{j \in J \\ \alpha \in A}} \lambda_j y_\alpha^{(j)} \alpha,$$

and the assumption of convergence implies

$$\sum_{\substack{j \in J \\ \beta_j = \beta}} \lambda_j \rightarrow 0 \quad \text{for each } \beta \in B \setminus X,$$

while

$$\sum_{j \in J} \lambda_j y_\alpha^{(j)} \rightarrow 0 \quad \text{for each } \alpha \in A \setminus X.$$

Note that $\beta \in B \cap X$ implies $\langle \beta, \alpha^\vee \rangle = 0$ for each $\alpha \in Y$, so $\beta \in V^{W_Y}$ and $\sigma(\beta) = 0$. The estimate (9.8) therefore implies

$$\begin{aligned} \sigma(x) &\leq \sum_{j \in J} \lambda_j \sigma(y^{(j)}) \\ &\leq \sum_{\substack{j \in J \\ \beta_j \in B \setminus X}} \lambda_j \sigma(\beta_j) + c \cdot \sum_{\substack{j \in J \\ \alpha \in A \setminus X}} \lambda_j y_\alpha^{(j)}. \end{aligned}$$

Taking limits we conclude $\sigma(\bar{x}) = 0$ and $\bar{x} \in V^{W_Y} \cap \mathbf{R}X = \mathbf{R}X^a$. Q.E.D.

Let $X \subset A$ be special. Recall that a *positive X-root* is an element

$$\sum_{\alpha \in X} x_\alpha \alpha \in -\bar{C}$$

with (strictly) positive integral coefficients. The existence of positive X -roots characterizes the special subsets X of A , see Looijenga [28] (1.18).

LEMMA 9.9. *Let $x^\vee \in \Phi(X)$. Then $\bar{K} \cap \{x^\vee = 0\}$ contains the positive X^s -roots.*

Proof. As $K \subset -I$ has non-empty interior we find a point $y \in (-K^\circ) \cap C$. Let x be a positive X^s -root. By (2.4) of [loc. cit.] the convex hull of the orbit Wy intersects \bar{C} in the set $(y - \mathbf{R}_+ A) \cap \bar{C}$. The latter set contains $y - tx$ for all $t > 0$, hence so does $-K$. Thus K contains $x - t^{-1}y$ for all $t > 0$, and taking the limit as $t \rightarrow \infty$ we obtain $x \in \bar{K}$. Clearly $\langle x, x^\vee \rangle = 0$ and the lemma follows. Q.E.D.

(9.10) *Example.* Consider the mixed root basis given by

$$\begin{aligned} A &= \{\alpha_1, \alpha_2\}, \quad B = \{\beta\}, \\ \langle \alpha_1, \alpha_2^\vee \rangle &= \langle \alpha_2, \alpha_1^\vee \rangle = -2, \\ \langle \beta, \alpha_1^\vee \rangle &= -1, \quad \langle \beta, \alpha_2^\vee \rangle = 0. \end{aligned}$$

Its Dynkin diagram is:



The root basis A is of affine type ([loc. cit.] Section 5), and the positive A -roots are the multiples of $\alpha_1 + \alpha_2$. The roots are $m\alpha_1 + n\alpha_2$, $m, n \in \mathbf{Z}$, $|m - n| = 1$ while the orbit of β consists of all points $\beta + x_1\alpha_1 + x_2\alpha_2$ with $x_1, x_2 \in \mathbf{Z}_+$ and $(x_1 - x_2)^2 = x_1$. Thus K is as

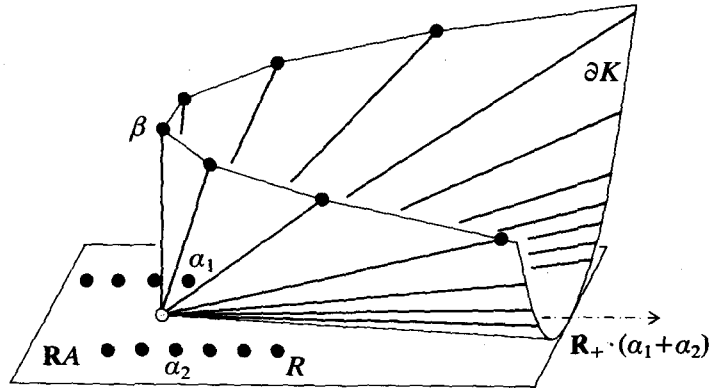


Figure 9.11

shown in Figure 9.11. The Tits cone I is the union of the open lower half space bounded by RA , and the line $R(\alpha_1 + \alpha_2)$. Note that in accordance with the lemma, the closure of K contains the ray spanned by the positive A -root $\alpha_1 + \alpha_2$.

PROPOSITION 9.12. *Let X and Y be admissible subsets of $A \cup B$. If $\Phi(X)$ and $\Phi(Y)$ are in one W -orbit then $X=Y$. The stabilizer of $\Phi(X)$ is the direct product $W_{X \cup X^*} = W_X \times W_{X^*}$.*

Proof. Let $x^\vee \in \Phi(X)$ be the defining form;

$$\langle \gamma, x^\vee \rangle = \begin{cases} 0 & \text{if } \gamma \in X, \\ 1 & \text{if } \gamma \in (A \cup B) \setminus X. \end{cases}$$

Then $x^\vee \in \bar{C}^\vee$, and $wx^\vee = x^\vee$ for $w \in W_X$. On the other hand, as \bar{K} is contained in $R_+(A \cup B)$ each $w \in W_{X^*}$ must leave $\bar{K} \cap \{x^\vee = 0\}$ pointwise fixed; therefore $wx^\vee \in \Phi(X)$. This shows that $W_{X \cup X^*}$ is contained in the stabilizer of $\Phi(X)$.

To prove the opposite inclusion, let $w \in W$ send $\Phi(X)$ to $\Phi(Y)$. By what is already proven we may assume $wx^\vee \in \bar{C}_{(A \cap Y) \cup Y^*}^\vee$; that is,

$$\langle \alpha, wx^\vee \rangle \geq 0 \quad \text{for all } \alpha \in (A \cap Y) \cup Y^*.$$

We show that this inequality holds, in fact, for all $\alpha \in A$. Thus let $\alpha \in A \setminus (Y \cup Y^*)$. We distinguish two cases.

If $Y^m \cup \{\alpha\}$ is a mixed subset of $A \cup B$ then the cone $K_{Y^m \cup \{\alpha\}}$ strictly contains K_{Y^m} —see Lemma 9.2. Thus we find some $y \in K_{Y^m}$ with $\langle y, \alpha^\vee \rangle < 0$. As wx^\vee is in $\Phi(Y)$

we have $\langle y, wx^\vee \rangle = 0$ but $\langle w_\alpha y, wx^\vee \rangle > 0$. From the relation

$$\langle w_\alpha y, wx^\vee \rangle = \langle y, wx^\vee \rangle - \langle y, \alpha^\vee \rangle \langle \alpha, wx^\vee \rangle$$

we conclude $\langle \alpha, wx^\vee \rangle > 0$.

In the second case $\alpha \in A \setminus (Y \cup Y^*)$ is such that $Y^m \cup \{\alpha\}$ is not mixed. Then $Y^s \cup \{\alpha\}$ is a special subset of A . By Lemma 9.9 the form $wx^\vee \in \Phi(Y)$ vanishes on the positive Y^s -roots, and as $\langle \alpha', wx^\vee \rangle \geq 0$ for all $\alpha' \in A \cap Y$ this implies $\langle \alpha', wx^\vee \rangle = 0$ if $\alpha' \in Y^s$. Now let

$$y = \sum_{\alpha' \in Y^s} y_{\alpha'} \alpha' + y_\alpha \alpha$$

be a positive $(Y^s \cup \{\alpha\})$ -root. On $y \in \bar{K}$ forms in $\Phi(Y)$ must be positive, and we obtain

$$0 < \langle y, wx^\vee \rangle = y_\alpha \langle \alpha, wx^\vee \rangle.$$

As $y_\alpha > 0$ we have $\langle \alpha, wx^\vee \rangle > 0$ too.

We now know that both x^\vee and wx^\vee belong to the dual fundamental chamber \bar{C}^\vee , and conclude that $x^\vee = wx^\vee$ and that $w \in W_X$. Thus $\Phi(X) = \Phi(Y)$, and the stabilizer of $\Phi(X)$ is exactly $W_{X \cup X^*}$ as claimed. Finally, we have $X = Y$, for Lemma 9.9 and the following lemma allow to recover X from the set $\bar{K} \cap \{x^\vee = 0\}$. Q.E.D.

LEMMA 9.13. *Let $X \subset A \cup B$ be a mixed subset, and let $x^\vee \in \Phi(X)$. Then*

$$K \cap \{x^\vee = 0\} = K_X.$$

In particular $\bar{K} \cap \{x^\vee = 0\}$ spans the vector space $\mathbf{R}X$.

Proof. We may assume that x^\vee is the standard linear form given by $\langle \gamma, x^\vee \rangle = 0$ ($\gamma \in X$), $\langle \gamma, x^\vee \rangle = 1$ ($\gamma \in (A \cup B) \setminus X$). Let $\beta \in B$, $x \in W\beta$ such that $\langle x, x^\vee \rangle = 0$. Then $\beta \in X$ and $x \in W_X\beta$. In fact, choosing among the $w \in W$ with $x = w\beta$ one of minimal length $l(w)$ we may argue by induction: the case $l(w) = 0$ is trivial, and if $l(w) > 0$ then

$$x = w_\alpha x' = x' + \langle x, \alpha^\vee \rangle \alpha$$

where $l(w_\alpha w) = l(w) - 1$ and $\langle x, \alpha^\vee \rangle > 0$. The inductive hypothesis applies to x' and it follows that $\alpha \in X$.

Now let $x \in K \cap \{x^\vee = 0\}$. Then $x = \sum_j \lambda_j \cdot w_j \beta_j$ with $\lambda_j > 0$, $w_j \in W$, $\beta_j \in B$, and necessarily $\langle w_j \beta_j, x^\vee \rangle = 0$. Therefore $w_j \in W_X$ and $\beta_j \in X$ whence $x \in K_X$. This proves

$$K \cap \{x^\vee = 0\} = K_X,$$

the other inclusion being obvious.

The last clause of the lemma holds because K_X has non-empty interior in $\mathbf{R}X$, by Lemma 9.2. Q.E.D.

Note that Theorem 9.5 now follows from Lemma 9.6 and Propositions 9.7 and 9.12. Likewise we have established the unproven statements made in Section 1. We need only remark that the cone K is closed if the root basis (A, δ) has a finite Weyl group. Thus Theorem 1.2 follows from Lemma 9.13, Lemma 9.2, and Theorem 9.5 while Proposition 1.5 follows from Proposition 9.12.

In Section 1 we have assigned to each diagram \mathcal{D} a torus embedding $\mathcal{T} \subset \mathcal{X}$. In the present more general context \mathcal{X} is obtained by a recent construction of Looijenga [33]. The object \mathcal{X} will be a mere topological space with W -action, but the quotient \mathcal{X}/W will carry a natural analytic structure and will turn out to be a Stein manifold.

We briefly recall Looijenga's construction as far as it is relevant to the problem.

The place of the torus $\mathcal{T} = V_{\mathbb{C}}/\Lambda$ is now taken by

$$\mathcal{T} = (V - iP^{\circ})/\Lambda;$$

P is the topological interior of the Tits cone I so that \mathcal{T} is an open subset of an algebraic torus. By Looijenga [28], (1.14) and (2.17), the Weyl group W acts properly discontinuously on \mathcal{T} , with finite reflection groups as isotropy groups. By Chevalley's Theorem the quotient \mathcal{T}/W is an analytic manifold. In order to describe the partial compactification \mathcal{X} of \mathcal{T} we need some preparatory notation.

Let $\Lambda^{\vee} \subset V^{\vee}$ be the dual of the lattice $\Lambda \subset V$.

LEMMA 9.14. *The convex hull of $K^{\vee} \cap \Lambda^{\vee}$ is K^{\vee} .*

Proof. As K^{\vee} is closed in V^{\vee} it is the convex hull of its extremal rays. The non-zero points of such a ray must form a one-dimensional facet of K^{\vee} , for by Lemma 9.4 each facet is open in its support. By Theorem 9.5 each ray is spanned by a lattice point, and the lemma follows. Q.E.D.

If Φ is a facet of K^{\vee} we put

$$V^{\Phi} = \{x \in V \mid \langle x, x^{\vee} \rangle = 0 \text{ for each } x^{\vee} \in \Phi\}.$$

We let π_{Φ} denote the projection

$$\pi_{\Phi}: V \rightarrow V/V^{\Phi},$$

or, ambiguously, the induced epimorphism

$$\pi_\Phi: V_C/\Lambda \rightarrow V_C/(V_C^\Phi + \Lambda).$$

Finally, we let $\mathcal{T}_\Phi \subset V_C/(V_C^\Phi + \Lambda)$ be the image of \mathcal{T} under π_Φ .

Then \mathcal{X} is, by definition, the disjoint union of the \mathcal{T}_Φ where Φ runs through the set of facets of K^\vee . The set \mathcal{X} is topologized as in Looijenga [33]. The group W acts naturally on \mathcal{X} . In fact, by Proposition 9.12 the stabilizer of a typical facet $\Phi(X)$ ($X \subset A \cup B$ admissible) is

$$W_{\Phi(X)} = W_X \times W_{X^*},$$

and as $\pi_{\Phi(X)}(I^p)$ is contained in $I_{X^*}^p \subset V/V^{\Phi(X)}$ —see Looijenga [28], (2.7)—the W -action permutes the strata $\mathcal{T}_\Phi \subset \mathcal{X}$. Thus the quotient \mathcal{X}/W is the disjoint union of the analytic manifolds

$$\mathcal{T}_{\Phi(X)}/W_{\Phi(X)} \quad (X \subset A \cup B \text{ admissible}).$$

Let $x^\vee \in K^\vee$. Then for any facet $\Phi \subset K^\vee$ the Fourier series

$$\sum_{y^\vee \in Wx^\vee \cap \Phi} e^{2\pi i y^\vee}$$

converges on compact subsets of $V - iI^p$, and these series define a W -invariant function S_{x^\vee} on \mathcal{X} as follows: if $\xi \in \mathcal{T}_\Phi$ is represented by $x \in V - iI^p$ then

$$S_{x^\vee}(\xi) = \sum_{y^\vee \in Wx^\vee \cap \Phi} e^{2\pi i \langle x, y^\vee \rangle}.$$

Looijenga shows that the topological quotient \mathcal{X}/W admits a unique normal analytic structure such that the functions S_{x^\vee} induce holomorphic functions on \mathcal{X}/W . In fact \mathcal{X}/W is a Stein space—see Looijenga [33].

The formal invariant theory of the situation still is virtually the same as that described in Looijenga [28], Section 4. One has to study the ring \mathcal{R} consisting of complex-valued functions f on $\Lambda^\vee \cap K^\vee$ which have finitely dominated support in the sense of [loc. cit.]. Thus each $f \in \mathcal{R}$ is a formal sum

$$f = \sum_{p \in \Lambda^\vee \cap K^\vee} f_p e^p \quad (f_p \in \mathbb{C}),$$

and there exists a finite subset $D \subset \bar{C}^\vee$ such that $f_p = 0$ unless $d - p \in \mathbb{Z}_+ A^\vee$ for some $d \in D$.

The lattice Λ^\vee is spanned by the basis

$$\{\gamma^* | \gamma \in A \cup B\} \subset V^\vee$$

which is dual to $A \cup B$. The ring \mathcal{R} contains the W -invariant functions

$$S_\gamma = \sum_{p \in W\gamma^*} e^p \quad (\gamma \in A \cup B).$$

As the γ^* are linearly independent the functions S_γ are algebraically independent in \mathcal{R} , and the embedding

$$\begin{aligned} \mathbb{C}[X_\gamma]_{\gamma \in A \cup B} &\rightarrow \mathcal{R}^W \\ X_\gamma &\mapsto S_\gamma \end{aligned}$$

makes \mathcal{R}^W an algebra over the polynomial ring $\mathbb{C}[X_\gamma]_{\gamma \in A \cup B}$. In general, though, this fails to be an isomorphism. We let

$$A \cup B = A^s \cup Y$$

be the decomposition of $A \cup B$ into its greatest special subset and its complement. Note that the function S_γ has finite support in $\Lambda^\vee \cap K^\vee$ if $\gamma \in Y$ while it is an infinite formal series for $\gamma \in A^s$.

LEMMA 9.15. *The set*

$$\alpha = \{f \in \mathcal{R} \mid \text{each } p \in \text{Supp}(f) \text{ is positive on some } a \in A^s\}$$

is an ideal in \mathcal{R} .

Proof. Let x be a positive A^s -root. By Proposition 9.3, K^\vee is contained in I^\vee , and by Looijenga [28], (2.2) each $p \in I^\vee$ is either positive on x or vanishes on each $a \in A^s$. Thus

$$\text{if } p, q \in K^\vee, \text{ and } \langle \alpha, p \rangle > 0 \text{ for some } \alpha \in A^s \text{ then } \langle x, p+q \rangle \geq \langle x, p \rangle > 0,$$

so $\langle \alpha, p+q \rangle > 0$ for some $\alpha \in A^s$. This implies the assertion. Q.E.D.

The following is the proper generalization of Theorem 2.14.

THEOREM 9.16. (a) *The embedding*

$$\begin{aligned} \mathbb{C}[X_\gamma]_{\gamma \in A \cup B} &\rightarrow \mathcal{R}^W \\ X_\gamma &\mapsto S_\gamma \end{aligned}$$

induces an isomorphism of $\mathbb{C}[X_\gamma]_{\gamma \in Y}$ -algebras between the formal power series ring

$$\mathbb{C}[X_\gamma][[X_\alpha]]_{\gamma \in Y, \alpha \in A},$$

and the completion of \mathcal{R}^W with respect to the ideal α^W .

(b) The \mathcal{R}^W -module \mathcal{R}^{-W} of anti-invariant functions in \mathcal{R} is freely generated by

$$J = \sum_{w \in W} (\det w) e^{w\varrho^*}$$

where $\varrho^* = \sum_{\alpha \in A} \alpha^*$.

Proof. The proof of Looijenga [28], (4.2) applies with only minor changes. Q.E.D.

The formal functions S_γ and J^2 actually correspond to holomorphic functions on \mathcal{X}/W , and the theorem has consequences for the geometry of \mathcal{X}/W similar to those of Theorem 2.14.

COROLLARY 9.17. (a) \mathcal{X}/W is smooth, and so is the closure of each stratum $\mathcal{T}_{\Phi(X)}/W_{\Phi(X)}$.

(b) The discriminant of $\mathcal{X} \rightarrow \mathcal{X}/W$ —the set of orbits with nontrivial isotropy group—is the analytic hypersurface defined by J^2 .

Proof. Similar to Looijenga [28], (5.5), (5.6). Q.E.D.

As in the case of finite W , the quotient \mathcal{X}/W carries natural \mathbb{C}^* -actions induced by lattice points $\omega \in \Lambda^W$. The weights $\langle \omega, \gamma^* \rangle$ ($\gamma \in A \cup B$) need no longer be positive, though. The discriminant is still quasi-homogeneous of degree $2 \sum_{\alpha \in A} \langle \omega, \alpha^* \rangle$.

References

- [1] ARNOL'D, V. I., Normal forms of functions near degenerate critical points, Weyl groups A_k , D_k , E_k and Lagrange singularities. *Functional Anal. Appl.*, 6 (1972), 254–272.
- [2] — Local normal forms of functions. *Invent. Math.*, 35 (1976), 87–109.
- [3] ARTIN, M., On isolated rational singularities of surfaces. *Amer. J. Math.*, 88 (1966), 129–136.
- [4] BOURBAKI, N., *Groupes et algèbres de Lie*. Chapitres 4, 5 et 6. Hermann, Paris 1968.
- [5] BRIESKORN, E., Rationale Singularitäten komplexer Flächen. *Invent. Math.*, 4 (1968), 336–358.
- [6] — Singular elements of semi-simple algebraic groups. In *Actes Congrès intern. Math.* 2, 1970, 279–284.
- [7] — Die Fundamentalgruppe des Raumes der regulären Orbits einer endlichen komplexen Spiegelungsgruppe. *Invent. Math.*, 12 (1971), 57–61.

- [8] BRIESKORN, E. & SAITO, K., Artin-Gruppen und Coxeter-Gruppen. *Invent. Math.*, 17 (1972), 245–271.
- [9] DELIGNE, P., Les immeubles des groupes de tresses généralisés. *Invent. Math.*, 17 (1972), 273–302.
- [10] DELORME, CH., Espaces projectifs anisotropes. *Bull. Soc. Math. Fr.*, 103 (1975), 203–223.
- [11] DEMAZURE, M., Surfaces de Del Pezzo. In *Seminaire sur les singularités des surfaces*. Lecture Notes in Math., 777 (1980), 21–69.
- [12] GIUSTI, M., Classification des singularités isolées d'intersections complètes simples. *C.R. Acad. Sci. Paris Sér. A*, 284 (1977), 167–170.
- [13] — Classification des singularités isolées simples d'intersections complètes. In *Singularities, Proc. Symp. Pure Math.*, 40:1 (1983), 457–494.
- [14] GREUEL, G.-M. & LÊ DŨNG TRÁNG, Spitzen, Doppelpunkte und vertikale Tangenten in der Diskriminante verseller Deformationen von vollständigen Durchschnitten. *Math. Ann.*, 222 (1976), 71–88.
- [15] GROTHENDIECK, A. ET AL., *Seminaire de géométrie algébrique*. 1. Revêtements étales et groupe fondamental. Lecture Notes in Math., 224 (1971).
- [16] GUNNING, R. C., *Lectures on modular forms*. Ann. of Math. Stud., 48 (1962).
- [17] HARTSHORNE, R., *Algebraic geometry*. Springer-Verlag, Heidelberg 1977.
- [18] JANSSEN, W., Skew-symmetric vanishing lattices and their monodromy groups. *Math. Ann.*, 266 (1983), 115–133.
- [19] — Skew-symmetric vanishing lattices and their monodromy groups II. *Math. Ann.*, 272 (1985), 17–22.
- [20] KAS, A. & SCHLESSINGER, M., On the versal deformation of a complex space with an isolated singularity. *Math. Ann.*, 196 (1972), 23–29.
- [21] KEMPF, G. ET AL., *Toroidal embeddings I*. Lecture Notes in Math., 339 (1973).
- [22] KNÖRRER, H., *Isolierte Singularitäten von Durchschnitten zweier Quadriken*. Bonn. Math. Schr., 117 (1980).
- [23] — Die Singularitäten vom Typ \tilde{D} . *Math. Ann.*, 251 (1980), 135–150.
- [24] — Zum $K(\pi, 1)$ -Problem für isolierte Singularitäten von vollständigen Durchschnitten. *Compositio Math.*, 45 (1982), 333–340.
- [25] KODAIRA, K., On stability of compact submanifolds of complex manifolds. *Amer. J. Math.*, 85 (1963), 79–94.
- [26] LOOIJENGA, E., On the semi-universal deformation of a simple-elliptic singularity II. *Topology*, 17 (1978), 23–40.
- [27] — Homogeneous spaces associated to certain semi-universal deformations. In *Proc. Intern. Congr. Math. 1978, vol. 2*. Acad. Sci. Fenn., Helsinki, 529–536 (1980).
- [28] — Invariant theory for generalized root systems. *Invent. Math.*, 61 (1980), 1–32.
- [29] — Rational surfaces with an anti-canonical cycle. *Ann. of Math. (2)*, 114 (1981), 267–322.
- [30] — The smoothing components of a triangle singularity I. In *Singularities, Proc. Symp. Pure Math.*, 40:2 (1983), 173–183.
- [31] — The smoothing components of a triangle singularity II. *Math. Ann.*, 269 (1984), 357–387.
- [32] — *Isolated singular points on complete intersections*. London Math. Soc. Lect. Note Ser., 77 (1984).
- [33] — *Invariant theory on convex cones*. Lecture at the Max-Planck-Institut, Bonn (24-1-84).
- [34] MÉRINDOL, J.-Y., *Deformations des surfaces de Del Pezzo, de points doubles rationnels et des cônes sur une courbe elliptique*. Thèse 3-ème cycle, Université Paris VII (1980).
- [35] NAGATA, M., On rational surfaces I. *Mem. Coll. Sci. Kyoto Ser. A*, 32 (1960), 351–370.
- [36] — *Local rings*. J. Wiley, New York 1962.
- [37] PICKL, A., *Die Homologie der Einhangung eines vollstandigen Durchschnitts mit isolierter Singularitat*. Diplomarbeit, Regensburg (1983).

- [38] PINKHAM, H., *Deformations of algebraic varieties with G_m action*. *Asterisque*, 20 (1974).
- [39] — Simple elliptic singularities, Del Pezzo surfaces and Cremona transformations. In *Several complex variables, Proc. Symp. Pure Math.*, 30:1 (1977), 69–71.
- [40] — Résolution simultanée de points doubles rationnels. In *Seminaire sur les singularités des surfaces*. *Lecture Notes in Math.*, 777 (1980), 179–203.
- [41] SIERSMA, D., Isolated line singularities. In *Singularities, Proc. Symp. Pure Math.*, 40:2 (1983), 485–496.
- [42] SLODOWY, P., *Simple singularities and simple algebraic groups*. *Lecture Notes in Math.*, 815 (1980).
- [43] — Chevalley groups over $C((t))$ and deformations of simply elliptic singularities. In *Algebraic geometry*. *Lecture Notes in Math.*, 961 (1982), 285–301.
- [44] TEISSIER, B., The hunting of invariants in the geometry of discriminants. In *Real and complex singularities*. Sijthoff & Noordhoff, Alphen aan den Rijn, (1977), 565–678.
- [45] TITS, J., Le problème des mots dans les groupes de Coxeter. *Symp. Math.*, 1 (1968), 175–185.
- [46] TYURINA, G. N., Resolution of singularities of plane deformations of double rational points. *Functional Anal. Appl.*, 4 (1970), 68–73.
- [47] URABE, T., On singularities on degenerate Del Pezzo surfaces of degree 1, 2. In *Singularities, Proc. Symp. Pure Math.*, 40:2 (1983), 587–591.
- [48] DU VAL, P., On isolated singularities of surfaces which do not affect the conditions of adjunction, III. *Math. Proc. Cambridge Philos. Soc.*, 30 (1934), 483–491.
- [49] — *Homographies, quaternions and rotations*. Clarendon Press, Oxford 1964.
- [50] VOHMANN, H. D., *Einige Eigenschaften der Diskriminante verseller Deformationen vollständiger Durchschnitte mit isolierter Singularität*. *Bonn. Math. Schr.*, 70 (1974).
- [51] WAHL, J., Elliptic deformations of minimally elliptic singularities. *Math. Ann.*, 253 (1980), 241–262.
- [52] WALL, C. T. C., Classification of unimodal isolated singularities of complete intersections. In *Singularities, Proc. Symp. Pure Math.*, 40:2 (1983), 625–640.
- [53] WIRTHMÜLLER, K., Singularities determined by their discriminant. *Math. Ann.*, 252 (1980), 237–245.
- [54] — Deformations of fat points of Boardman type 2,0. *Math. Ann.*, 259 (1982), 519–539.

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