On Waring's problem

by

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Introduction

Landau [13] and Linnik [15], respectively, have shewn that all large numbers can be expressed as the sum of eight and of seven non-negative cubes. It is therefore a notable anomaly that no asymptotic formulae have yet been validated for the number of ways in which integers can be thus represented, the best that has been currently achieved being a formula for nine cubes. A formula for eight cubes is indeed narrowly missed by the circle method but radically new ideas would seem to be needed in order to bridge the present margin of failure.

As a contribution to the elimination of this and other lacunae in the theory of Waring's problem, we study in this memoir the effect of assuming the truth of the Riemann hypothesis for certain Hasse-Weil global L-functions defined over cubic three-folds. On this hypothesis, the precise form of which will be indicated in the text (Section 6, Chapter I), we shall indeed establish asymptotic formulae for seven and for eight cubes that are of a type previous theory would have led us to predict. But even the unconditional proof of these formulae would by no means exhaust this area of enquiry because it would still leave open the important question of the existence and number of representations of large integers as the sum of four non-negative cubes. Davenport [2], in fact, shewed that almost all numbers were representable in this manner but failed to obtain the stronger conclusion that the corresponding asymptotic formula was almost always true. We therefore partially repair this omission by deriving this formula almost always on the basis of our hypothesis, incidentally obtaining an improved estimate for the exceptional set of numbers not expressible as a sum of four non-negative cubes. As specialists in the field will recognize, this problem is in depth

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roughly equivalent to that of the asymptotic formula for eight cubes, which as a matter of fact could be inferred as soon as our almost all results were known (see comments in our memoir [9], II.8).

There are applications to mixed problems involving the expression of a large number as a sum of one square and a set of non-negative cubes. Here we obtain the asymptotic formula when five cubes are present, improving conditionally upon Watson's [22] existence theorem for representations of this type. Previously an asymptotic formula had only been known in this situation when six or more cubes were present (Sinnadurai [18]; for an alternative proof, vid. Hooley [9]).

In much the same vein we establish an asymptotic formula for the number of ways of writing large numbers as the sum of six non-negative cubes and two biquadrates, even the existence theorem implied by this being a new conditional result.

We also study the interesting question of how many integers less than a large number x are expressible as a sum of three non-negative cubes, obtaining the lower bound $x^{\frac{18}{19}-\varepsilon}$ that conditionally sharpens Davenport's bound $x^{\frac{47}{54}-\varepsilon}$ [3].

The heart of the memoir is concerned with the proof of the inequality

$$R(x)=\int_0^1\left|\sum_{0\leq m\leq x^{1/3}} e^{2\pi i m^3\theta}\right|^6 d\theta = O(x^{\frac{20}{19}+\varepsilon}) \tag{1}$$

that is tantamount to

$$\sum_{0<m\leq x} r_3^2(m) = O(x^{\frac{20}{19}+\varepsilon}),$$

where $r_3(m)$ is the number of representations of m as the sum of three non-negative cubes. This constitutes a considerable improvement on the previously best known upper bound

$$R(x)=O(x^{\frac{7}{6}+\varepsilon})$$

due to Hua(2) and does not fall all that short of the trivial bound $R(x)>x$ that follows from Hölder's inequality. Moreover, it is a conditional improvement in the direction of the inequality

(2) Use Hua's inequalities with Hölder's inequality. The bound may be improved to

$$O(x^{\frac{7}{6}}\log^{\frac{1}{2}(\sqrt{3}-1)+\varepsilon} x)$$

by using the author's results in [8], Chapter 4, Section 4.

$$R(x) = O(x^{1+\epsilon}) \tag{2}$$

that has long been conjectured to hold and that is a serviceable substitute for Hardy and Littlewood's false conjecture K (vid. [6] and Mahler [16]). Very possibly $R(x) \sim Ax$ as $x \to \infty$ but this is a matter to which we intend to return in a subsequent paper.

To derive (1) we use a refinement of the circle method that is more ambitious than the one usually associated with Kloosterman's name. A Kloosterman refinement has now come to mean the employment of a technique in the circle method whereby we assess non-trivially the collective contribution of the remainder terms arising from all Farey arcs centred by rationals h/k with given denominator k. But it was suggested in our address at the I.C.M., Warsaw [10] that to make further significant advances in additive number theory it might be necessary to go beyond the Kloosterman refinement and to consider cancellations between contributions due to different values of k. Never made before, this advance is practicable in our present work because special features associated with the left side of (1) make its introduction more manageable than for most problems (vid. comments in Section 5, Chapter I). This technique, which for want of a better term we call a *double Kloosterman refinement*, is perhaps the most noteworthy aspect of the paper. Also worthy of mention is the sub-division of the arcs into the *senior* and *junior* categories, corresponding, respectively, to those to which the double and ordinary Kloosterman refinements are applicable. Furthermore, it should be remarked that in this instance our implementation of the double refinement leads to the introduction of certain geometrically natural multiplicative functions that can be interpreted in terms of Hasse-Weil L-functions and that can be studied with some exactitude if the Riemann hypothesis be assumed.

The deduction of our main theorems from (1) is along fairly familiar lines. Indeed, our path does not altogether diverge from that first blazed by Hardy and Littlewood in P.N.VI [6], although more nicety in the reasoning is needed here because (1) is not as sharp as the hypothetical (2) that was the foundation of Hardy and Littlewood's researches.

The removal of the dependence of our work on the Riemann hypothesis is an obvious desideratum. Some weakening of the hypothesis is certainly possible either by substituting some form of zero density requirement or by insisting merely that the zeros of the Hasse-Weil L-functions be to the left of some vertical line lying to the right of the critical line $\sigma=2$. Yet is has not seemed worthwhile to explore such developments here because the principles of the method would be obscured and because we cannot predict the precise form of the first serviceable alternative to the Riemann hypothesis that might subsequently be established. Let it therefore suffice us to mention that the asymptotic formulae for eight cubes and for seven cubes (i.e. the main terms given in Theorems 2 and 3, respectively) would remain valid if the Hasse-Weil L-functions had no zeros to the right of $\sigma=\frac{1}{44}(139-\sqrt{1457})+\varepsilon$ and $\sigma=\frac{1}{92}(277-\sqrt{7361})+\varepsilon$.

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Notation

The meaning of the notation being usually clear from the context in which it occurs, it is unnecessary to define all the symbols used.

The letters d, k, n, N, λ, μ are usually positive integers; h, l, r are non-negative integers; m is an integer that in certain clearly defined situations is restricted to be non-zero; p is a (positive) prime number.

The letter x denotes a real variable that is to be regarded as tending to infinity; y, z are positive real numbers; u, v are real numbers; t is real as is s save when s is the complex variable $\sigma+it$.

Ordered sextuples are denoted by lower case letters in bold Roman font, the components being denoted by the corresponding lower case letters in italic font with subscript attached; thus $\mathbf{a}=(a_1,\dots,a_6)$. When $a_1,\dots,a_6$ are real we denote $\max|a_i|$ by $\|\mathbf{a}\|$; also the notation $\mathbf{a}<u$ signifies that $a_i<u$ for $1\leqslant i\leqslant 6$, where a corresponding meaning is to be given to the other three possible symbols of inequality; $\mathbf{ab}$ is the scalar product $\Sigma_{1\leqslant i\leqslant 6}\, a_i b_i$.

Positive absolute constants are denoted by $A, A_1, A_2, \dots$; ε is an arbitrarily small positive number that is not necessarily the same on all occasions; $A(\varepsilon)$ and $A(\eta)$ are positive numbers that depend at most on ε and η, respectively; according to the context, the constants implied by the O-notation are either absolute or depend at most on either ε or η.

The highest common factor of a, b is (a, b) but that of $a_1,\dots,a_6$ is h.c.f. $(a_1,\dots,a_6)$; $\sigma_{-\alpha}(n)=\Sigma_{d|n}\, d^{-\alpha}$; $d(n)$ is the number of divisors of n; $\omega(n)$ is the number of distinct prime factors of n; $r_s(m)$ is the number of representations of m as the sum of s non-negative cubes.

Chapter I. Estimation of the Integral $R(x)$

1. The method initiated

To describe the genesis of our method we introduce the exponential sum

$$f(\theta)=f(\theta,y)=\sum_{0\leqslant m\leqslant y} e^{2\pi i m^3\theta} \tag{3}$$

and the analogous sum

$$F(\theta)=F(\theta,y)=\sum_{-y\leqslant m\leqslant y} \gamma(m/y)\, e^{2\pi i m^3\theta}$$

whose terms are affected by weights defined in terms of the function

$$\gamma(t)=\begin{cases} e^{-1/(1-t^2)} & (|t|<1),\\ 0 & (|t|=1).\end{cases}$$

Then, since the coefficients in the exponential sum $f^3(\theta, y)$ are non-negative and do not exceed e^4 times the corresponding coefficients in $F^3(\theta, 2y)$, our objective of estimating

$$R(x)=\int_0^1 |f(\theta, x^{1/3}|^6\, d\theta$$

can be reached by finding an upper bound for

$$R^*(x)=\int_0^1 |F(\theta, x^{1/3}|^6\, d\theta,$$

between which and $R(x)$ there is the relation

$$R(x)\leqslant e^8 R^*(8x). \tag{4}$$

Save in certain hypothetical circumstances that the author has yet to encounter in practice, the obliquity of the procedure does not adversely affect the quality of our results but leads to important simplifications in the analysis. Similarly, no penalty is incurred by using the inequality

$$\sum_{0<m\leqslant x} r_3^2(m)=O\{R^*(8x)\} \tag{5}$$

that is associated with (4) and that will be needed in the final section of Chapter II.

To treat $R^*(x)$ we write $X=x^{1/3}$ when convenient and use the Farey's series of order

$$M=\left[x^{\frac{1}{2}+\delta}\right]$$

of fractions h/k, where

$$0 < \delta < \tfrac{1}{6} \tag{6}$$

and where $0 \leqslant h < k$ and $(h, k) = 1$. Since by Dirichlet's theorem or by the theory of the Farey dissection the interval $[-1/M, 1-1/M]$ is covered by the arcs

$$\left|\theta - \frac{h}{k}\right| \leqslant \frac{1}{Mk},$$

we deduce our second basic inequality

$$\begin{aligned} R^*(x) &\leqslant \sum_{k \leqslant M} \sum_{\substack{0 \leqslant h < k \\ (h,k)=1}} \int_{h/k-1/Mk}^{h/k+1/Mk} |F(\theta, X)|^6 \, d\theta \\ &= \sum_{k \leqslant M} \int_{-1/Mk}^{1/Mk} \sum_{\substack{0 \leqslant h < k \\ (h,k)=1}} \left| F\left(\frac{h}{k} + \varphi, X\right)\right|^6 d\varphi \\ &= \sum_{k \leqslant M} \int_{-1/Mk}^{1/Mk} G(\varphi, k) \, d\varphi, \quad \text{say,} \end{aligned} \tag{7}$$

in virtue of the non-negativity of the integrand. Here all arcs are to be regarded as *major* according to the usual understanding of the language of Hardy and Littlewood because at no point in this chapter do we use the technique of applying Weyl's inequality to exponential sums lifted from the integral. Nevertheless, not all arcs are treated in the same way and it is necessary for some purposes to divide them into the two classes of *senior arcs* and *junior arcs*. The precise nature of the sub-division and its sphere of applicability being as yet unimportant, it is enough here to indicate that senior arcs correspond to the larger values of k for which a double Kloosterman refinement is appropriate while the junior arcs form the complementary set of arcs to which the usual Kloosterman refinement is applied. In some instances, moreover, it will prove convenient to widen the arcs slightly in order to take advantage of the non-negativity of integrands that are derived from $G(\varphi, k)$.

This completes the foundation of the method, the next stage being the investigation of $F(h/k+\varphi, X)$.

2. Formula for $F(h/k+\varphi)$

We have

$$F\left(\frac{h}{k} + \varphi\right) = \sum_{0 \leqslant l < k} e^{2\pi i h l^3/k} \sum_{\substack{-X \leqslant m \leqslant X \\ m \equiv l, \bmod k}} \gamma(m/X) \, e^{2\pi i m^3 \varphi}, \tag{8}$$

where throughout for brevity we omit X in the notation for $F(h/k+\varphi, X)$. Here, by the Poisson summation formula, the inner sum is

$$\sum_{(-X-l)/k\leq r\leq (X-l)/k} \gamma\{(l+rk)/X\}\, e^{2\pi i(l+rk)^3\varphi} = \sum_{m=-\infty}^{\infty} \int_{(-X-l)/k}^{(X-l)/k} \gamma\{(l+wk)/X\}\, e^{2\pi i\{\varphi(l+wk)^3+mw\}}\, dw$$
$$= \frac{1}{k}\sum_{m=-\infty}^{\infty} e^{-2\pi iml/k}\int_{-X}^{X}\gamma(t/X)\, e^{2\pi i(\varphi t^3+mt/k)}\, dt,$$

since all derivatives of $\gamma(t/X)\, e^{2\pi i\varphi t^3}$ exist for $|t|\leq X$ and vanish at $t=\pm X$. Next, in order to express the effect of substituting this in (8), we introduce the important exponential sum

$$S(a, b; k) = \sum_{0\leq l<k} e^{2\pi i(al^3-bl)/k} \tag{9}$$

and its integral analogue

$$J(u, v; X) = \int_{-X}^{X} \gamma(t/X)\, e^{2\pi i(ut^3+vt)}\, dt,$$

where for brevity we write

$$S(a, k) = S(a, 0; k) \quad\text{and}\quad J(u; X) = J(u, 0; X). \tag{10}$$

Accordingly

$$\begin{aligned} F\left(\frac{h}{k}+\varphi\right) &= \frac{1}{k}S(h, k)\, J(\varphi; X)+\frac{1}{k}\sum_{m\neq 0} S(h, m; k)\, J(\varphi, m/k; X) \\ &= F_1(h, k; \varphi)+F_1^*(h, k; \varphi), \quad \text{say.} \end{aligned} \tag{11}$$

The sum $F_1^*(h, k; \varphi)$ revealed by the above transformation gives rise to the main difficulties in the problem, its investigation and application being initiated in the next section by a study of the integral $J(u, v; X)$.

3. The integral $J(u, v; X)$

The methods of partial integration (sometimes disguised in the form of Bonnet's mean-value theorem) and of stationary phase are used to obtain estimates for $J(u, v; X)$ that suffice for our design but that are not always necessarily best possible. The results are embodied in a series of lemmata, the first of which is

LEMMA 1. *If* $|v/u|\geqslant 6X^2$ *then*

$$J(u,v;X)=O(X\,e^{-A_0(X|v|)^{1/3}})$$

for some sufficiently small positive constant A_0.

It suffices to consider the special integral $J(u,v)=J(u,v;1)$ and then to assume that $|v|>A_1$, since $J(u,v;X)=XJ(uX^3,vX;1)$ and since the estimate for $J(u,v)$ supplied by the lemma is trivial for $|v|\leqslant A_1$. In these circumstances, as the derivative $\varphi(t)=3ut^2+v$ of ut^3+vt does not vanish for $|t|\leqslant 1$ while all derivatives of $\gamma(t)$ are zero at $t=\pm 1$, integration by parts gives

$$J(u,v)=\frac{1}{(2\pi i)^r}\int_{-1}^{1}G_r(t)\,e^{2\pi i(ut^3+vt)}\,dt, \tag{12}$$

where $G_r(t)$ is defined iteratively by

$$G_0(t)=\gamma(t),\quad G_{r+1}(t)=-\frac{d}{dt}\left(\frac{G_r(t)}{\varphi(t)}\right).$$

Next, setting

$$G_r(t)=\frac{\gamma(t)\,\psi_r(t)}{\varphi^{2r}(t)\,(1-t^2)^{2r}} \tag{13}$$

for $|t|<1$, we verify that $\psi_r(t)$ is a polynomial that satisfies the relations

$$\psi_0(t)=1,$$

$$\psi_{r+1}(t)=\{2t\varphi(t)+(2r+1)\,\varphi'(t)\,(1-t^2)^2-4rt\varphi(t)\,(1-t^2)\}\,\psi_r(t)-\varphi(t)\,(1-t^2)^2\,\psi_r'(t).$$

Clearly the degree of $\psi_{r+1}(t)$ does not exceed that of $\psi_r(t)$ by more than 5 so that $\psi_r(t)$ has degree at most $5r$. Hence the numerically greatest coefficient of $\psi_{r+1}(t)$ does not exceed $A_2(r+1)\,|v|$ times that of $\psi_r(t)$, the inference being that

$$|\psi_r(t)|\leqslant(r+1)\,r!\,(A_3|v|)^r\leqslant r!\,(A_4|v|)^r$$

for $|t|<1$.

Substituting (13) and the inequality $|\varphi(t)|\geqslant\frac{1}{2}|v|$ in (12) when $r\geqslant 1$, we obtain

$$|J(u,v)|<\frac{r!\,A_5^r}{|v|^r}\int_0^1\frac{\gamma(t)\,dt}{(1-t^2)^{2r}}=\frac{r!\,A_5^r}{2|v|^r}\int_1^\infty\frac{e^{-s}s^{2r-3/2}\,ds}{(s-1)^{1/2}}$$

$$< \frac{A_6 r! A_5^r}{|v|^r} \Gamma(2r-\tfrac{1}{2})$$
$$< \frac{r!(2r)! A_7^r}{|v|^r} < \left(\frac{A_8 r}{|v|^{1/3}}\right)^{3r},$$

from which the lemma follows on choosing $r=[|v|^{1/3}/A_8 e]$ and $A_1=(2A_8 e)^3$.

LEMMA 2. *If* $|v|<|u|^{1/3}$, *then*

$$J(u, v; X) = O(|u|^{-1/3}),$$

but, if $|v| \geqslant |u|^{1/3}>0$, *then*

$$J(u, v; X) = O(|uv|^{-1/4}).$$

Thus

$$J(u, v; X) = O(|uv|^{-1/4})$$

whenever $u, v \neq 0$.

The final conclusion is all that is needed but, as we shall see, is most naturally derived by considering the two earlier cases separately.

It being sufficient to assume that u is positive when estimating

$$J(u, v; X) = 2\int_0^X \gamma(t/X) \cos 2\pi(ut^3+vt)\, dt,$$

we first estimate the associated integral

$$J(u, v; \alpha, \beta) = 2\int_\alpha^\beta \cos 2\pi(ut^3+vt)\, dt$$

for $0 \leqslant \alpha < \beta$ by using

$$J(u, v; \alpha', \beta') = O(\beta'-\alpha') \tag{14}$$

in combination with the inequality

$$J(u, v; \alpha', \beta') = O\left(\max_{\alpha' \leqslant t \leqslant \beta'} |3ut^2+v|^{-1}\right) \tag{15}$$

that is valid for $\alpha' \geqslant 0$ if $3ut^2+v$ do not vanish in $[\alpha', \beta']$. The proof of the latter involves the treatment of several cases which are fully exemplified by considering the situation where v is negative and $\alpha'>(-v/3u)^{1/2}$. Here, as $3ut^2+v$ is positive and increasing, Bonnet's form of the second mean-value theorem shews there is a number ξ strictly between α' and β' such that

$$\begin{aligned} J(u,v;\alpha',\beta') &= \frac{1}{\pi}\int_{\alpha'}^{\beta'} \frac{1}{3ut^2+v}\frac{d}{dt}\sin 2\pi(ut^3+vt)\,dt \\ &= \frac{1}{\pi(3u\alpha'^2+v)}\int_{\alpha'}^{\xi}\frac{d}{dt}\sin 2\pi(ut^3+vt)\,dt = O\left(\frac{1}{3u\alpha'^2+v}\right), \end{aligned}$$

as required.

Take first the easier case where v is non-negative. Then, if $v\geqslant u^{1/3}>0$, estimate (14) gives $J(u,v;\alpha,\beta)=O(v^{-1})=O\{(uv)^{-1/4}\}$. On the other hand, if $v<u^{1/3}$, then there are the three possibilities $\alpha<\beta\leqslant u^{-1/3}$, $u^{-1/3}<\alpha<\beta$, and $\alpha\leqslant u^{-1/3}<\beta$ that are treated by using, respectively, estimate (14) only, estimate (15) only, and estimates (14) and (15) for the integrals obtained by splitting the range of integration at $u^{-1/3}$; in each instance the estimate $J(u,v;\alpha,\beta)=O(u^{-1/3})$ is obtained.

In the case where v is negative, write $T=t-(|v|/3u)^{1/2}$ so that

$$3ut^2+v = 2(3u|v|)^{1/2}T+3uT^2. \tag{16}$$

First, suppose that $|v|\geqslant u^{1/3}>0$ and extract from $[\alpha,\beta]$ any part of the interval

$$0<(|v|/3u)^{1/2}-\tfrac{1}{2}(u|v|)^{-1/4}\leqslant t\leqslant(|v|/3u)^{1/2}+\tfrac{1}{2}(u|v|)^{-1/4}$$

that lies within it. The contribution of this to $J(u,v;\alpha,\beta)$ being at most $O\{(u|v|)^{-1/4}\}$ by (14), any complementary set remaining consists of one or two intervals that give rise to an effect $O\{(u|v|)^{-1/4}\}$ by (15) and (16). Secondly, if $|v|<u^{1/3}$, the part of $[\alpha,\beta]$ lying in $0\leqslant t\leqslant u^{-1/3}$ contributes $O(u^{-1/3})$ by (14), while there is a like contribution from any remaining part by (15). Thus the previous estimates for $J(u,v;\alpha,\beta)$ still obtain when v is negative.

Finally, the bounds obtained for $J(u,v;\alpha,\beta)$ are applicable to $J(u,v;X)$ because

$$J(u,v;X)=\gamma(0)\,J(u,v;0,\xi)$$

for some ξ between 0 and X.

Lastly, there is

LEMMA 3. *Let*

$$I(u,v;X)=\int_{-X}^{X} t\gamma(t/X)\,e^{2\pi i(ut^3+vt)}\,dt.$$

Then

$$I(u,v;X)=O(X|uv|^{-1/4})$$

for u, $v\neq 0$.

Since the bounds obtained for $J(u,v;\alpha,\beta)$ obviously apply equally well to

$$I(u,v;\alpha,\beta)=2i\int_\alpha^\beta \sin 2\pi(ut^3+vt)\,dt,$$

the lemma follows on observing that

$$\begin{aligned}I(u,v;X)&=2i\int_0^X t\gamma(t/X)\sin 2\pi(ut^3+vt)\,dt\\&=2iX\int_{\xi_1}^X \gamma(t/X)\sin 2\pi(ut^3+vt)\,dt\\&=X\gamma(\xi_1/X)\,I(u,v;\xi_1,\xi_2)\end{aligned}$$

for suitable numbers ξ_1,ξ_2 such that $0<\xi_1<\xi_2<X$.

4. Second transformation of $R^*(x)$

We return to $F_1^*(h,k;\varphi)$ and use Lemma 1 to continue our transformation of $R^*(x)$.

As a prelude, we introduce the numbers

$$Y_j=Y_j(x)=2^{-j}M \tag{17}$$

for $1\leqslant j\leqslant M_1$, where

$$M_1=[\log M/\log 2]+1 \tag{18}$$

and where therefore Y_{M_1} is the greatest number of form (17) that is less than 1. Next, if $k\leqslant M$, choose $Y=Y(x,k)$ to be that Y_j satisfying $Y_j<k\leqslant 2Y_j$ and then define $W=W(x,k,\varphi)$ by

$$W=\max(X^2Y|\varphi|\log^4 x,\ YX^{-1}\log^4 x). \tag{19}$$

Furthermore, note that W is independent of h and that

$$W\leqslant x. \tag{20}$$

Expressing $F_1^*(h,k;\varphi)$ in (11) as

$$\begin{aligned}\frac{1}{k}\sum_{0<|m|\leq W} S(h,m;k)\,J(\varphi,m/k;X)+\frac{1}{k}\sum_{|m|>W} S(h,m;k)\,J(\varphi,m/k;X)\\ = F_2(h,k;\varphi)+F_3(h,k;\varphi),\quad \text{say},\end{aligned}\tag{21}$$

where in some circumstances the first sum may be empty, we have

$$\begin{aligned}G(\varphi,k) &= O\Bigg(\sum_{\substack{0\leq h<k\\(h,k)=1}}|F_1(h,k;\varphi)|^6+\sum_{\substack{0\leq h<k\\(h,k)=1}}|F_2(h,k;\varphi)|^6+\sum_{\substack{0\leq h<k\\(h,k)=1}}|F_3(h,k;\varphi)|^6\Bigg)\\ &= O\{G_1(\varphi,k)+G_2(\varphi,k)+G_3(\varphi,k)\},\quad \text{say},\end{aligned}\tag{22}$$

by (7). Hence

$$\begin{aligned}R^*(x) &= O\Bigg(\sum_{k\leq M}\int_{-1/Mk}^{1/Mk} G_1(\varphi,k)\,d\varphi\Bigg)+O\Bigg(\sum_{k\leq M}\int_{-1/Mk}^{1/Mk} G_2(\varphi,k)\,d\varphi\Bigg)\\ &\quad +O\Bigg(\sum_{k\leq M}\int_{-1/Mk}^{1/Mk} G_3(\varphi,k)\,d\varphi\Bigg)\\ &= O\{R_1^*(x)\}+O\{R_2^*(x)\}+O\{R_3^*(x)\},\quad \text{say}.\end{aligned}\tag{23}$$

The first term $R_1^*(x)$ is akin to familiar expressions that occur in the customary theory. An upper bound being all that is required, we take the inequality $J(\varphi;X)=O(|\varphi|^{-1/3})$ from Lemma 2 and find that

$$\int_{-\infty}^{\infty}|J(\varphi;X)|^6\,d\varphi = O\Bigg(X^6\int_0^{X^{-3}}d\varphi\Bigg)+O\Bigg(\int_{X^{-3}}^{\infty}\frac{d\varphi}{\varphi^2}\Bigg) = O(X^3) = O(x).$$

Therefore, by (23), (22), and (11),

$$\begin{aligned}R_1^*(x) &\leq \sum_{k\leq M}\frac{1}{k^6}\sum_{\substack{0\leq h<k\\(h,k)=1}}|S(h,k)|^6\int_{-\infty}^{\infty}|J(\varphi;X)|^6\,d\varphi\\ &= O\Bigg(x\sum_{k\leq M}\frac{1}{k^6}\sum_{\substack{0\leq h<k\\(h,k)=1}}|S(h,k)|^6\Bigg) = O(x\mathfrak{S}_M),\quad \text{say},\end{aligned}\tag{24}$$

in which $\mathfrak{S}_M$ is a partial sum of the singular series originating in the indeterminate equation $x_1^3+x_2^3+x_3^3-x_4^3-x_5^3-x_6^3=0$. Since unfortunately there is no accessible reference to this particular series, we let $q(k)$ be the multiplicative function defined by

$$q(p^{\alpha})=\begin{cases}A_9 p^{\alpha/2}, & \text{if } \alpha=1 \text{ or } 2,\\ A_9 p^{2\alpha/3}, & \text{if } \alpha>2,\end{cases} \tag{25}$$

and avail ourselves of the estimate

$$|S(h,k)|\leqslant q(k) \tag{26}$$

that holds for $(h,k)=1$ in view of well known bounds ([14], [20], [21]) for $S(h,p^{\alpha})$ and the quasi-multiplicativity of $S(h,k)$. In consequence

$$\mathfrak{S}_M\leqslant\sum_{k\leqslant M}\frac{q^6(k)}{k^5}\leqslant\prod_p\left(1+\frac{A_{10}}{p^2}\right)=A_{11} \tag{27}$$

and we conclude from (24) that

$$R_1^*(x)=O(x). \tag{28}$$

The third term $R_3^*(x)$ can be even more summarily dismissed with the aid of the simple

LEMMA 4. *For any positive numbers, ξ, B, we have*

$$\sum_{r>\xi} e^{-A_{12}(Br)^{1/3}}<(1+A_{13}B^{-1})\,e^{-\frac{1}{2}A_{12}(B\xi)^{1/3}},$$

where $A_{13}=A_{13}(A_{12})$.

The result is obtained from the inequalities

$$\begin{aligned}\sum_{r>\xi} e^{-A_{12}(Br)^{1/3}}&<e^{-A_{12}(B\xi)^{1/3}}+\int_{\xi}^{\infty}e^{-A_{12}(Bt)^{1/3}}\,dt\\&=e^{-A_{12}(B\xi)^{1/3}}+\frac{3}{B}\int_{(B\xi)^{1/3}}^{\infty}s^2e^{-A_{12}s}\,ds<e^{-A_{12}(B\xi)^{1/3}}+\frac{48}{e^2A_{12}^2B}\int_{(B\xi)^{1/3}}^{\infty}e^{-\frac{1}{2}A_{12}s}\,ds\\&<(1+96\,e^{-2}A_{12}^{-3}B^{-1})\,e^{-\frac{1}{2}A_{12}(B\xi)^{1/3}}\end{aligned}$$

Since (19) implies that $|m/k|>6X^2|\varphi|$ for $|m|>W$, Lemma 1 may be used to estimate the integral $J(\varphi,m/k;X)$ appearing in the formula (21) for $F_3(h,k;\varphi)$, where it is to be assumed that $k\leqslant M$. This gives

$$J(\varphi,m/k;X)=O(X\,e^{-A_0(|m|X/k)^{1/3}}),$$

whence

$$F_3(h,k;\varphi)=O\left(X\sum_{|m|>6kX^{-1}\log^4 x} e^{-A_0(|m|X/k)^{1/3}}\right)$$
$$=O((X+k)\,e^{-A_{14}\log^{4/3}x})=O(1)$$

because of Lemma 4 and the inequality $|S(h,m;k)|\leq k$. Hence $G_3(\varphi,k)=O(k)$ by (22), and the required estimate

$$R_3^*(x)=O\left(\frac{1}{M}\sum_{k\leq M}1\right)=O(1) \tag{29}$$

follows from (23).

The easier constituents of (23) having been estimated by (28) and (29), we first conclude that

$$R^*(x)=O\{R_2^*(x)\}+O(x) \tag{30}$$

and then await the treatment of $R_2^*(x)$ in the following sections.

5. Transformation of $G_2(\varphi, k)$ and the properties of $Q(m; k)$

In anticipation of the transformation of $G_2(\varphi,k)$ we recall the previously introduced notation for ordered sets and augment it by insisting that $\mathbf{m}$ have *non-zero* integral components m_i. Consequently, since we do not distinguish notationally between integers and the members of a finite field to which they correspond, the notation for this and certain other ordered sets has a natural alternative meaning that will be adopted when we work in $\mathbf{F}_p$.

With these conventions understood, we set

$$H(\varphi,\mathbf{m}/k;X)=\prod_{1\leq i\leq 6} J(\varphi,m_i/k;X) \tag{31}$$

and

$$Q(\mathbf{m};k)=\sum_{\substack{0\leq h<k\\(h,k)=1}}\prod_{1\leq i\leq 6} S(h,m_i;k), \tag{32}$$

whereupon the formula for $G_2(\varphi,k)$ implicit in (21) and (22) may be expressed in the form

$$G_2(\varphi, k) = \frac{1}{k^6} \sum_{\substack{0 \leq h < k \\ (h,k)=1}} \sum_{0<|m_1|,\ldots,|m_6| \leq W} \prod_{1 \leq i \leq 6} J(\varphi, m_i/k; X) \prod_{1 \leq i \leq 6} S(h, m_i; k)$$

$$= \frac{1}{k^6} \sum_{\|\mathbf{m}\| \leq W} H(\varphi, \mathbf{m}/k; X)\, Q(\mathbf{m}; k) \tag{33}$$

since $J(\varphi, m_i/k; X)$ and $S(h, m_i; k)$ are real.

The nature of our investigation endows the sums $Q(\mathbf{m}; k)$ with some important properties that have not been previously identified in treatments of Waring's problem, the underlying causes being the homogeneity of the problem and the equality in length of the arcs corresponding to a given denominator. These and the other required properties of $Q(\mathbf{m}; k)$ are evolved in a series of lemmata that entail the appearance of the linear form $\mathbf{mx}$, the cubic form

$$g(\mathbf{x}) = x_1^3 + \ldots + x_6^3,$$

and the discriminant

$$\Delta(\mathbf{m}) = 3 \prod (m_1^{3/2} \pm m_2^{3/2} \pm \ldots \pm m_6^{3/2}), \tag{34}$$

the vanishing of which expresses a necessary and sufficient condition that the array

$$\begin{pmatrix} \partial g/\partial x_j \\ m_j \end{pmatrix} = \begin{pmatrix} 3x_j^2 \\ m_j \end{pmatrix} \quad (j = 1, \ldots, 6)$$

have rank not exceeding 1 for some non-zero solution of $g(\mathbf{x}) = \mathbf{mx} = 0$; the factor 3 in (34) does not occur naturally in the process of elimination but has been included in order that a subsequent interpretation, modulo 3, should be valid. In particular, Lemmata 6 and 7 will involve the congruence $g(\mathbf{x}) \equiv 0$, $\bmod k$, and the simultaneous congruences $g(\mathbf{x}) \equiv 0$, $\bmod k$, and $\mathbf{mx} \equiv 0$, $\bmod k$, the numbers of whose incongruent solutions are denoted, respectively, by $\nu(k)$ and $\nu(\mathbf{m}; k)$. Since all we currently need to know about $\Delta(\mathbf{m})$ is that it is not identically zero, we can postpone its further study and can enunciate at once

LEMMA 5. *For any given* $\mathbf{m}$, *the sum* $Q(\mathbf{m}; k)$ *is a (properly) multiplicative function of* k, *viz., if* $k = k'k''$ *where* $(k', k'') = 1$, *then* $Q(\mathbf{m}; k) = Q(\mathbf{m}; k')\, Q(\mathbf{m}; k'')$.

Let us expand the product in (32) by means of (9) to obtain

$$Q(\mathbf{m};k)=\sum_{\substack{0\leqslant h<k\\(h,k)=1}}\sum_{0\leqslant \mathbf{l}<k} e^{2\pi i\{hg(\mathbf{l})-\mathbf{ml}\}/k}, \tag{35}$$

denoting by h', $\mathbf{l}'$ and h'', $\mathbf{l}''$ the variables of summation in the corresponding formulae for $Q(\mathbf{m};k')$ and $Q(\mathbf{m};k'')$, respectively. Next, if $\bar{k}'$, $\bar{k}''$ be defined, modulis k'', k', respectively, by $\bar{k}'k'\equiv 1$, $\bmod k''$ and $\bar{k}''k''\equiv 1$, $\bmod k'$, then all variables of summation in (35) are obtained once and once only through the formulae

$$h\equiv k''\bar{k}''^3h'+k'\bar{k}'^3h'', \bmod k'k'', \quad \mathbf{l}\equiv k''\mathbf{l}'+k'\mathbf{l}'', \bmod k'k''.$$

Hence

$$hg(\mathbf{l})-\mathbf{ml}\equiv k''h'g(\mathbf{l}')+k'h''g(\mathbf{l}'')-k''\mathbf{ml}'-k'\mathbf{ml}'', \bmod k'k'',$$

and then

$$\{hg(\mathbf{l})-\mathbf{ml}\}/k\equiv\{h'g(\mathbf{l}')-\mathbf{ml}'\}/k'+\{h''g(\mathbf{l}'')-\mathbf{ml}''\}/k'', \bmod 1,$$

from which the lemma flows.

It being therefore enough to restrict attention to the case where $k=p^\alpha$ with $\alpha>0$, we prepare for the next two lemmata by the transformation

$$\begin{aligned}Q(\mathbf{m};p^\alpha)&=\sum_{0\leqslant\mathbf{l}<p^\alpha}\left(\sum_{0\leqslant h<p^\alpha} e^{2\pi i\{hg(\mathbf{l})-\mathbf{ml}\}/p^\alpha}-\sum_{0\leqslant h'<p^{\alpha-1}} e^{2\pi i\{ph'g(\mathbf{l})-\mathbf{ml}\}/p^\alpha)}\right)\\&=p^\alpha\sum_{\substack{g(\mathbf{l})\equiv 0,\bmod p^\alpha\\0\leqslant\mathbf{l}<p^\alpha}} e^{2\pi i\mathbf{ml}/p^\alpha}-p^{\alpha-1}\sum_{\substack{g(\mathbf{l})\equiv 0,\bmod p^{\alpha-1}\\0\leqslant\mathbf{l}<p^\alpha}} e^{2\pi i\mathbf{ml}/p^\alpha}\\&=p^\alpha Q_1(\mathbf{m};p^\alpha)-p^{\alpha-1}Q_2(\mathbf{m};p^\alpha),\quad\text{say.}\end{aligned}$$

But, since in the sum defining $Q_2(\mathbf{m};p^\alpha)$ we may write $\mathbf{l}=\mathbf{l}'+\mathbf{r}p^{\alpha-1}$ where $g(\mathbf{l}')\equiv 0$, $\bmod p^{\alpha-1}$, and $0\leqslant\mathbf{l}'<p^{\alpha-1}$, we have

$$Q_2(\mathbf{m};p^\alpha)=\sum_{\substack{g(\mathbf{l}')\equiv 0,\bmod p^{\alpha-1}\\0\leqslant\mathbf{l}'<p^{\alpha-1}}} e^{2\pi i\mathbf{ml}'/p^\alpha}\sum_{0\leqslant\mathbf{r}<p} e^{2\pi i\mathbf{mr}/p},$$

in which the inner sum is zero unless $\mathbf{m}\equiv 0$, $\bmod p$. Hence certainly

$$Q(\mathbf{m};p^\alpha)=p^\alpha Q_1(\mathbf{m};p^\alpha) \tag{36}$$

when $\Delta(\mathbf{m})\not\equiv 0$, $\bmod p$.

At this point the treatments for the two cases $\alpha=1$ and $\alpha>1$ diverge. Suppose first that $\alpha=1$. Then, $g(\mathbf{x})$ being homogeneous, the substitution $\mathbf{l}\equiv h\mathbf{l}'$, $\bmod p$, transforms $Q_1(\mathbf{m};p)$ into $Q_1(h\mathbf{m};p)$ if $p\nmid h$. Hence

$$\begin{aligned} Q_1(\mathbf{m};p) &= \frac{1}{p-1}\Bigg(\sum_{0\le h<p} Q_1(h\mathbf{m};p)-Q_1(0;p)\Bigg) \\ &= \frac{1}{p-1}\Bigg(\sum_{\substack{g(\mathbf{l})\equiv 0,\bmod p\\ 0\le \mathbf{l}<p}}\ \sum_{0\le h<p} e^{2\pi i h\mathbf{m}\mathbf{l}/p}-\nu(p)\Bigg) \\ &= \frac{1}{p-1}\Bigg(p\sum_{\substack{g(\mathbf{l})\equiv\mathbf{m}\mathbf{l}\equiv 0,\bmod p\\ 0\le \mathbf{l}<p}} 1-\nu(p)\Bigg) \\ &= \frac{1}{p-1}(p\nu(\mathbf{m};p)-\nu(p)), \end{aligned}$$

which together with (36) yields

LEMMA 6. *If* $\Delta(\mathbf{m})\not\equiv 0$, $\bmod p$, *then*

$$Q(\mathbf{m};p)=\frac{p}{p-1}(p\nu(\mathbf{m};p)-\nu(p)).$$

When $\alpha>1$ we first determine the effect on $Q_1(\mathbf{m};p^\alpha)$ of those $\mathbf{l}$ in the summation that are divisible by p, replacing $\mathbf{l}$ by $p\mathbf{l}'$ so that[1] $pg(\mathbf{l}')\equiv 0$, $\bmod p^{\alpha-2}$, and $0\le\mathbf{l}'<p^{\alpha-1}$. Then since these conditions amount to $\mathbf{l}'=\mathbf{l}''+\mathbf{r}p^{\alpha-2}$ where $pg(\mathbf{l}'')\equiv 0$, $\bmod p^{\alpha-2}$, $0\le\mathbf{l}''<p^{\alpha-2}$, and $0\le\mathbf{r}<p$, we infer that this part of the sum is

$$\sum_{\substack{pg(\mathbf{l}'')\equiv 0,\bmod p^{\alpha-2}\\ 0\le\mathbf{l}''<p^{\alpha-2}}} e^{2\pi i\mathbf{m}\mathbf{l}''/p^{\alpha-1}}\sum_{0\le\mathbf{r}<p} e^{2\pi i\mathbf{m}\mathbf{r}/p}=0$$

when $\mathbf{m}\not\equiv 0$, $\bmod p$ and hence when $\Delta(\mathbf{m})\not\equiv 0$, $\bmod p$.

The remaining part $Q_1'(\mathbf{m};p^\alpha)$ of $Q_1(\mathbf{m};p^\alpha)$ is unchanged if $\mathbf{m}$ be replaced by $h\mathbf{m}$ and if $h\not\equiv 0$, $\bmod p$. Therefore, in emulation of the argument used to derive (36), we deduce that

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[1] The conditions are framed in this way in order to take care of the case $\alpha=2$.

$$Q_1'(\mathbf{m};p^\alpha)=\frac{1}{\varphi(p^\alpha)}\left\{\sum_{\substack{g(\mathbf{l})\equiv 0,\ \mathrm{mod}\, p^\alpha\\ \mathbf{l}\not\equiv 0,\ \mathrm{mod}\, p;\, 0\leqslant \mathbf{l}<p^\alpha}}\left(\sum_{0\leqslant h<p^\alpha} e^{2\pi ih\mathbf{ml}/p^\alpha}-\sum_{0\leqslant h<p^{\alpha-1}} e^{2\pi ih\mathbf{ml}/p^{\alpha-1}}\right)\right\}$$
$$=\frac{1}{\varphi(p^\alpha)}\{p^\alpha\nu_1(\mathbf{m};p^\alpha)-p^{\alpha-1}\nu_2(\mathbf{m};p^\alpha)\},\tag{37}$$

where $\nu_1(\mathbf{m};p^\alpha)$ and $\nu_2(\mathbf{m};p^\alpha)$ are, respectively, the number of incongruent solutions, $\mathrm{mod}\, p^\alpha$, of the simultaneous conditions

$$g(\mathbf{l})\equiv 0,\ \mathrm{mod}\, p^\alpha;\quad \mathbf{ml}\equiv 0,\ \mathrm{mod}\, p^\alpha;\quad \mathbf{l}\not\equiv 0,\ \mathrm{mod}\, p, \tag*{$(38)_\alpha$}$$

and

$$g(\mathbf{l})\equiv 0,\ \mathrm{mod}\, p^\alpha;\quad \mathbf{ml}\equiv 0,\ \mathrm{mod}\, p^{\alpha-1};\quad \mathbf{l}\not\equiv 0,\ \mathrm{mod}\, p. \tag{39}$$

We proceed by comparing $\nu_1(\mathbf{m};p^\alpha)$ and $\nu_2(\mathbf{m};p^\alpha)$ with $\nu_1(\mathbf{m};p^{\alpha-1})$. Since every solution $\mathbf{l}$ of either $(38)_\alpha$ or (39) is a solution of $(38)_{\alpha-1}$, we may suppose in either case $\mathbf{l}=\mathbf{l}'+\mathbf{r}p^{\alpha-1}$, where $\mathbf{l}'$ satisfies $(38)_{\alpha-1}$, $0\leqslant\mathbf{l}'<p^{\alpha-1}$, and where $0\leqslant\mathbf{r}<p$. In the second instance, we obtain the condition

$$3\sum_{1\leqslant i\leqslant 6} l_i'^2 r_i\equiv -g(\mathbf{l}')/p^{\alpha-1},\ \mathrm{mod}\, p,$$

which, having p^5 incongruent solutions $\mathbf{r}$ for each apposite $\mathbf{l}'$ when $p\neq 3$, certainly shews that

$$\nu_2(\mathbf{m};p^\alpha)=p^5\nu_2(\mathbf{m};p^{\alpha-1}) \tag{40}$$

if $\Delta(\mathbf{m})\not\equiv 0$, $\mathrm{mod}\, p$. In the first instance, we obtain the simultaneous linear congruences

$$3\sum_{1\leqslant i\leqslant 6} l_i'^2 r_i\equiv -g(\mathbf{l}')/p^{\alpha-1},\ \mathrm{mod}\, p,\quad \sum_{1\leqslant i\leqslant 6} m_i r_i\equiv -\mathbf{ml}'/p^{\alpha-1},\ \mathrm{mod}\, p,$$

in which the left-hand sides are linearly independent, $\mathrm{mod}\, p$, when $\Delta(\mathbf{m})\not\equiv 0$, $\mathrm{mod}\, p$, $g(\mathbf{l}')\equiv 0$, $\mathrm{mod}\, p$, and $\mathbf{l}'\not\equiv 0$, $\mathrm{mod}\, p$. Thus, if $\Delta(\mathbf{m})\not\equiv 0$, $\mathrm{mod}\, p$, then

$$\nu_1(\mathbf{m},p^\alpha)=p^4\nu_2(\mathbf{m},p^{\alpha-1})$$

and therefore $Q_1'(\mathbf{m};p^\alpha)=0$ by (40) and (37). In summation, this and (36) then give

LEMMA 7. *If* $\Delta(\mathbf{m})\not\equiv 0$, $\mathrm{mod}\, p$ *and* $\alpha>1$, *then*

$$Q(\mathbf{m},p^\alpha)=0.$$

When $p|\Delta(\mathbf{m})$ the above procedures partially break down and we make do in this situation with a universal bound we derive from a refinement of the Hua-Weil inequality ([12], [20])

$$S(h, a; p^\alpha) = O\{p^{\alpha/2}(a, p^\alpha)\} \quad (p \nmid h). \tag{41}$$

In obtaining this improvement, we already assume for brevity the truth of (41) and the other Hua inequality ([11], [20])

$$S(h, a; p^\alpha) = O(p^{2\alpha/3}) \quad (p \nmid h), \tag{42}$$

even though a direct verification would in principle result in an intrinsically more straightforward proof.

Let(2) $p^\beta \| a$. Then our first aim being to shew that

$$S(h, a; p^\alpha) = O\{p^{\alpha/2}(a, p^\alpha)^{1/4}\} \quad (p \nmid h), \tag{43}$$

we pass over the obvious case $\beta=0$ already covered by (41) and then ignore until later the atypical case $p=3$.

First suppose that $\beta=1$. Then, by the usual estimate for the cubic Gaussian sum,

$$S(h, a; p) = S(h, p) = O(p^{1/2}),$$

which is a stronger form of (43) for $\alpha=1$. But, if $\alpha>1$, then

$$\begin{aligned} S(h, a; p^\alpha) &= \sum_{0 \leq l' < p^{\alpha-1}} \sum_{0 \leq r < p} e^{2\pi i\{h(l'+rp^{\alpha-1})^3 + a(l'+rp^{\alpha-1})\}/p^\alpha} \\ &= \sum_{0 \leq l' < p^{\alpha-1}} e^{2\pi i(hl'^3+al')/p^\alpha} \sum_{0 \leq r < p} e^{6\pi i h l'^2 r/p}, \end{aligned}$$

where the inner sum is zero unless $p|l'$. Hence

$$S(h, a; p^\alpha) = p \sum_{0 \leq l'' < p^{\alpha-2}} e^{2\pi i(hpl''^3 + ap^{-1}l'')/p^{\alpha-2}}, \tag{44}$$

and we deduce that

$$S(h, a; p^2) = O(p)$$

but that

(2) The case $\beta=\infty$ is covered by the treatment.

$$S(h, a; p^\alpha) = p \sum_{0 \leq l''' < p^{\alpha-3}} e^{2\pi i(hpl'''^3 + ap^{-1}l''')/p^{\alpha-2}} \sum_{0 \leq r < p} e^{2\pi i ap^{-1}r/p} = 0$$

for $\alpha > 2$; both these estimates are improved forms of (43).

Proceeding to the case $\beta \geq 2$, we observe that the last equation still holds and then gives

$$S(h, a; p^\alpha) = p^2 S(h, ap^{-2}; p^{\alpha-3}) \tag{45}$$

whenever $\alpha > 3$. Next suppose that $\beta < \frac{2}{3}\alpha$ so that the previous condition holds. Then, setting $\beta_1 = \left[\frac{1}{2}\beta\right]$ and noting that $ap^{-2\left[\frac{1}{2}\beta\right]} \not\equiv 0$, mod p^2, we deduce that

$$\begin{aligned} S(h, a; p^\alpha) &= p^{2\left[\frac{1}{2}\beta\right]} S(h, ap^{-2\left[\frac{1}{2}\beta\right]}; p^{\alpha - 3\left[\frac{1}{2}\beta\right]}) = O(p^{2\left[\frac{1}{2}\beta\right] + \frac{1}{2}\alpha - \frac{3}{2}\left[\frac{1}{2}\beta\right]}) \\ &= O(p^{\frac{1}{2}\alpha + \frac{1}{4}\beta}) \\ &= O(p^{\frac{1}{2}\alpha}(a, p^\alpha)^{\frac{1}{4}}) \end{aligned} \tag{46}$$

in view of our results for $\beta = 0$ and 1. On the other hand, if $\beta \geq \frac{2}{3}\alpha$, then (42) implies that

$$S(h, a; p^\alpha) = O(p^{\frac{2}{3}\alpha}) = O(p^{\frac{1}{2}\alpha + \frac{1}{6}\alpha}) = O(p^{\frac{1}{2}\alpha}(a, p^\alpha)^{\frac{1}{4}}),$$

which with (46) completes the proof of (43) if $p \neq 3$.

When $p = 3$ the above procedure requires some modifications, which we do not have time to describe in full. First, since p is now bounded, the case $\beta = 1$ is also covered by (41) and the highest common factor can be omitted from the estimate. Secondly, we can still shew that (45) is true provided that $\beta \geq 2$ and $\alpha > 3$, although it now stems directly from the transformation $l = l' + rp^{\alpha-2}$, where $0 \leq l' < p^{\alpha-2}$ and $0 \leq r < p^2$. The remainder of the proof of (43) being the same as before, we obtain the following lemma on referring to (32) and Lemma 5.(3)

LEMMA 8. *We have*

$$Q(\mathbf{m}; k) = O\left(A_{15}^{\omega(k)} k^4 \prod_{1 \leq i \leq 6} (m_i, k)^{1/4} \right).$$

The earlier remarks about the case $\beta = 1$ enable one to frame a useful alternative version of this result, which we enunciate in

(3) Or, alternatively, by using the pseudo-multiplicativity of the sum $S(h, a; k)$.

LEMMA 9. *Let $\bar{\omega}(l)$ be the multiplicative function of l that is defined by $\bar{\omega}(p)=1$ and by $\bar{\omega}(p^\alpha)=p^{\alpha/4}$ if $\alpha>1$. Then*

$$Q(\mathbf{m};k)=O\left(A_{15}^{\omega(k)}k^4\prod_{1\leqslant i\leqslant 6}\bar{\omega}(m_i)\right).$$

The position has been prepared for the entrance of algebraic geometry, by means of which the study of $Q(\mathbf{m};k)$ will be resumed in the next section.

6. Applications to $Q(\mathbf{m};k)$ of local and global L-functions

When considered geometrically the numbers $\nu(p)$ and $\nu(\mathbf{m};p)$ can be interpreted in terms of certain cones in affine spaces. But, to apply the theory of local L-functions to their study when $\Delta(\mathbf{m})\neq 0$, it is desirable to work with the corresponding underlying projective varieties $\mathscr{V}$ and $\mathscr{V}(\mathbf{m})$ over the field $\mathbf{Q}$ that are given, respectively, by $g(\Xi)=0$ and by the simultaneous equations $g(\Xi)=\mathbf{m}\Xi=0$, where $\Xi=(\Xi_1,\ldots,\Xi_6)$ denotes the coordinates of a point in five-dimensional projective space over $\mathbf{Q}$. If $p\nmid\Delta(\mathbf{m})$ and therefore $p\neq 3$, the interpretation of these equations as congruences, $\bmod p$, or as equations in the field $\mathbf{F}_p$ leads to the parallel reduced non-singular varieties $\mathscr{V}(p)$ and $\mathscr{V}(\mathbf{m};p)$ that are defined over $\mathbf{F}_p$. The former variety is a hypersurface, while the latter is an embedding in five-space of a hypersurface lying in a four-space. Next, let $\varrho(p^r)$ and $\varrho(\mathbf{m};p^r)$ be the number of points on $\mathscr{V}(p)$ and $\mathscr{V}(\mathbf{m};p)$ having coordinates in $\mathbf{F}_{p^r}$. Then $\nu(p)=(p-1)\varrho(p)+1$ and $\nu(\mathbf{m};p)=(p-1)\varrho(\mathbf{m};p)+1$ with the consequence that $Q(\mathbf{m};p)=p\{p\varrho(\mathbf{m};p)-\varrho(p)+1\}$ by Lemma 6. Therefore, since

$$Q(\mathbf{m};p)=p(pE(\mathbf{m};p)-E(p)) \tag{47}$$

if $E(p^r)=\varrho(p^r)-(p^{5r}-1)/(p^r-1)$ and $E(\mathbf{m},p^r)=\varrho(\mathbf{m};p^r)-(p^{4r}-1)/(p^r-1)$, we are led to consider the L-functions

$$\begin{aligned} L(p;T)&=\exp\left(-\sum_{r=1}^{\infty}\frac{E(p^r)T^r}{r}\right)\\ L(\mathbf{m};p;T)&=\exp\left(-\sum_{r=1}^{\infty}\frac{E(\mathbf{m};p^r)T^r}{r}\right)\end{aligned} \tag{48}$$

that are the quotients of the zeta functions of five-space or four-space and those of $\mathscr{V}(p)$ or $\mathscr{V}(\mathbf{m};p)$, respectively. Taking the latter function first, we know that Bombieri and Swinnerton-Dyer [1] anticipated Deligne's more general work [4] by shewing that

$$L(\mathbf{m};p;T)=\prod_{1\leqslant j\leqslant 10}(1-\lambda_{j,p}T)^{-1}, \tag{49}$$

where

$$|\lambda_{j,p}|=p^{3/2}. \tag{50}$$

Hence, if we equate the coefficients of T in the identical expressions in (48) and (49), we obtain

$$E(\mathbf{m};p)=-\sum_{1\leqslant j\leqslant 10}\lambda_{j,p}=O(p^{3/2}), \tag{51}$$

while the estimate

$$E(p)=O(p^2) \tag{52}$$

follows similarly from Deligne's theory or, indeed, more elementarily from the theory of cyclotomy. Thus $Q(\mathbf{m};p)=O(p^{7/2})$ by (47), (51), and (52), and we infer from Lemmata 6 and 7 that

$$Q(\mathbf{m};k)=O(A_{16}^{\omega(k)}k^{7/2}) \tag{53}$$

when $(k,\Delta(\mathbf{m}))=1$.

Taken with Lemma 8, this estimate is the basis for the Kloosterman refinement used on the junior arcs. For the senior arcs, however, the previous analysis in this section must be developed further in order that a double Kloosterman refinement can be brought into play.

To extend the study of $Q(\mathbf{m};k)$ let

$$Q_3(\mathbf{m};k)=\frac{1}{k^2}Q(\mathbf{m};k) \tag{54}$$

and let $L_p(\mathbf{m};s)$ denote the value of $L(\mathbf{m};p;T)$ obtained from the specialization $T=p^{-s}$ in (49). Then, for $\sigma>2$ and $p\nmid\Delta(\mathbf{m})$, we have

$$1+\frac{Q_3(\mathbf{m};p)}{p^s}=1-\frac{1}{p^s}\sum_{1\leqslant j\leqslant 10}\lambda_{j,p}+O\left(\frac{1}{p^{\sigma-1}}\right)$$

and

$$\frac{1}{L_p(\mathbf{m};s)}=1-\frac{1}{p^s}\sum_{1\leqslant j\leqslant 10}\lambda_{j,p}+O\left(\frac{1}{p^{2\sigma-3}}\right)$$

by (54), (47), (51), (49), and (52), whence

$$1+\frac{Q_3(\mathbf{m};p)}{p^s}=\frac{1}{L_p(\mathbf{m};s)}+O\left(\frac{1}{p^{\sigma-1}}\right)=\frac{1}{L_p(\mathbf{m};s)}\left\{1+O\left(\frac{1}{p^{\sigma-1}}\right)\right\} \tag{55}$$

since $|L_p(\mathbf{m};s)|\leqslant(1-2^{-1/2})^{-10}$. Also, by (50) and then by Lemmata 6 and 7, the function

$$\Psi(\mathbf{m};s)=\sum_{\substack{k=1\\(k,\Delta(\mathbf{m}))=1}}^{\infty}\frac{Q_3(\mathbf{m};k)}{k^s} \tag{56}$$

is regular and equals the Euler product

$$\prod_{p\nmid\Delta(\mathbf{m})}\left(1+\frac{Q_3(\mathbf{m};p)}{p^s}\right)$$

if $\sigma>\frac{5}{2}$. Hence, bringing in the Hasse-Weil L-function

$$L^*(\mathbf{m};s)=L^*(\mathscr{V}(\mathbf{m});s)=\prod_{p\nmid\Delta(\mathbf{m})}L_p(\mathbf{m};s)$$

which is also regular for $\sigma>\frac{5}{2}$, we deduce from (55) that

$$\Psi(\mathbf{m};s)=\frac{\Theta(\mathbf{m};s)}{L^*(\mathbf{m};s)}, \tag{57}$$

where

$$\Theta(\mathbf{m};s)=\prod_{p\nmid\Delta(\mathbf{m})}\left\{1+O\left(\frac{1}{p^{\sigma-1}}\right)\right\}$$

is regular and bounded for $\sigma\geqslant\sigma_0>2$. Thus when $\Delta(\mathbf{m})\neq0$ the problem of estimating coefficient sums of the form

$$T^*(\mathbf{m};y)=\sum_{\substack{k\leqslant y\\(k,\Delta(\mathbf{m}))=1}}\frac{Q_3(\mathbf{m};k)}{k^{3/2}} \tag{58}$$

is related to the properties of the Hasse-Weil L-functions defined over $\mathscr{V}(\mathbf{m})$.

The Hasse-Weil L-functions are generalizations of the Riemann zeta function and our further progress is partially contingent on our assuming the truth of the widely held belief—first expressed by Hasse—that their main properties are very similar to those of the latter function. The properties involved fall into two categories of unequal profundity. On the one hand, there are the analytic continuation and the functional equation, which have been shewn to hold for L-functions over several types of varieties but not yet unfortunately for those over $\mathscr{V}(\mathbf{m})$. On the other, there is the generalized Riemann hypothesis but there are as yet no instances in which this has been proved.

With the intention of expressing some of our conjectures in the form given by Serre [17], we take for each p dividing $\Delta(\mathbf{m})$ a certain factor

$$L_p(\mathbf{m}; s) = \prod_{1 \leq j \leq 10} (1 - \lambda_{j,p} p^{-s})^{-1}$$

that is defined by Serre in such a way that(4) $|\lambda_{j,p}| = 0$, p, or $p^{3/2}$. These determine a modified Hasse-Weil L-function.

$$L(\mathbf{m}; s) = \prod_p L_p(\mathbf{m}; s) = \prod_{p \mid \Delta(\mathbf{m})} L_p(\mathbf{m}; s) \cdot L^*(\mathbf{m}; s) = \Lambda(\mathbf{m}; s) L^*(\mathbf{m}; s), \quad \text{say}, \tag{59}$$

whose conductor $B(\mathbf{m})$ is given by

$$B(\mathbf{m}) = \prod_{p \mid \Delta(\mathbf{m})} p^{a_p}$$

where $0 \leq a_p \leq 200$ and where therefore

$$B(\mathbf{m}) \leq |\Delta(\mathbf{m})|^{200} = O(\|\mathbf{m}\|^{A_{17}}). \tag{60}$$

Then, setting

$$\xi(\mathbf{m}; s) = (2\pi)^{-5s} \Gamma^5(s) B^{s/2}(\mathbf{m}) L(\mathbf{m}; s) \tag{61}$$

for $\sigma > \frac{5}{2}$ in the first place, we state

Hypothesis HW. *If* $\Delta(\mathbf{m}) \neq 0$, *then*

(i) $\xi(\mathbf{m}; s)$ *is a meromorphic function of finite order*(5) *that is regular everywhere save possibly for poles at* $s = \frac{5}{2}$ *and* $\frac{3}{2}$;

---

(4) 1 and $p^{1/2}$ are not listed as possible values for $\lambda_{j,p}$ because Bombieri and Swinnerton-Dyer shew that any non-singular cubic three-fold is related to a curve. But, according to Serre's conjectures, they would have to be included if a general three-fold were involved.

(5) In the sense usually adopted in the general theory of integral functions, i.e. there is some number $c(\mathbf{m})$ such that $e^{-|z|^c} \xi(\mathbf{m}; z) \to 0$ as $|z| \to \infty$. As we shall see, suppositions (i) and (ii) imply that $L(\mathbf{m}; s)$ is of finite order in the normal language of the theory of Dirichlet's series.

(ii) $\xi(\mathbf{m};s)$ *satisfies a functional equation*

$$\xi(\mathbf{m};s)=w(\mathbf{m})\,\xi(\mathbf{m};4-s),$$

where $w(\mathbf{m})=\pm 1$;

(iii) $\xi(\mathbf{m};s)\neq 0$ *if* $\sigma\neq 2$ *(Riemann hypothesis).*

It is almost inconceivable that $\xi(\mathbf{m};s)$ have a pole at $s=\frac{5}{2}$ and hence at $s=\frac{3}{2}$ but it seems an unwarranted indulgence to debar this possibility in circumstances that do not require us to do so. Note, also, that any such pole cannot have multiplicity greater than 10 in view of the first of the (unconditional) inequalities

$$|L(\mathbf{m};s)|\leqslant\prod_p\left(1-\frac{1}{p^{\sigma-3/2}}\right)^{-1}=\zeta^{10}(\sigma-3/2), \tag{62}$$

$$|\log L(\mathbf{m};s)|\leqslant 10\sum_{p,\alpha}\frac{1}{\alpha p^{\alpha(\sigma-3/2)}}=10\log\zeta(\sigma-3/2), \tag{63}$$

which are valid for $\sigma>\frac{5}{2}$. Nevertheless, having made these remarks, we shall illustrate the way our conclusions are drawn from Hypothesis HW by mainly referring to the case where $\xi(\mathbf{m};s)$ is entire, since the procedure can easily accommodate the extra complication caused by a pole of absolutely bounded multiplicity.

When $L(\mathbf{m};s)$ has no poles the arguments are similar to some that have been previously applied to the Riemann zeta function and the Dirichlet's L-functions (vid. Titchmarsh [19], Chapter XIV). It therefore being unnecessary to supply full details, we first express the functional equation as

$$L(\mathbf{m};s)=w(\mathbf{m})\,(2\pi)^{10(s-2)}B^{2-s}(\mathbf{m})\,\Gamma^{-5}(s)\,\Gamma^{5}(4-s)\,L(\mathbf{m};4-s)$$

and deduce from (62) that

$$L(\mathbf{m};3+it)=O(1) \tag{64}$$

and

$$L(\mathbf{m};1+it)=O\{B(\mathbf{m})\,(|t|+2)^{10}\}, \tag{65}$$

since $|\Gamma(3-it)/\Gamma(1+it)|=|(1+it)\,(2+it)|=O\{(|t|+2)^2\}$. Secondly, applying the Phragmén-Lindelöf principle by dividing $L(\mathbf{m};s)$ by a function of the form $B(\mathbf{m})\,s^{10}e^{\varepsilon\cos\gamma s}$ where $0<\gamma<\pi/6$, we infer from (64), (65), supposition (i), and (61) that $L(\mathbf{m};s)=O\{B(\mathbf{m})\,(|t|+2)^{10}\}$ for $1\leqslant\sigma\leqslant 3$ and hence for $\sigma\geqslant 1$. Therefore, since $\log L(\mathbf{m};s)$ is regular for $\sigma>2$ and $\mathscr{R}\log L(\mathbf{m};s)<A_{18}\log(\|\mathbf{m}\|\,(|t|+2))$, we can use (63) and the

Borel-Carathéodory theorem with circles with centre $3+it_0$ and radii $1-\frac{1}{2}\eta$ and $1-\eta$ to obtain

$$|\log L(\mathbf{m};s)| \leqslant 4A_{18}\eta^{-1}\log\{\|\mathbf{m}\|(|t_0|+2)\}+41\eta^{-1}|\log\zeta(3/2)| \quad (\eta<1)$$

within and on the smaller circle, whence in particular

$$|\log L(\mathbf{m};s)| \leqslant A_{19}\eta^{-1}\log\{\|\mathbf{m}\|(|t|+2)\} \tag{66}$$

for $2+\eta\leqslant\sigma\leqslant 3$ and thus for $\sigma\geqslant 2+\eta$.

To refine this initial bound let η and $\sigma_0=\sigma_0(\eta)$ be, respectively, any small positive constant and a sufficiently large constant, supposing then that $2+\eta\leqslant\sigma\leqslant\frac{5}{2}+\frac{1}{2}\eta$. Apply Hadamard's three circles theorem to the function $\log L(\mathbf{m};s)$, using centre σ_0+it and radii $r_1=\sigma_0-\frac{5}{2}-\frac{1}{2}\eta$, $r_2=\sigma_0-\sigma$, and $r_3=\sigma_0-2-\frac{1}{2}\eta$. Then, if

$$\lambda=\log(r_2/r_1)/\log(r_3/r_1),$$

we have

$$|\log L(\mathbf{m};s)| \leqslant (A_{20}\eta^{-1}\log\{\|\mathbf{m}\|(|t|+2)\})^{\lambda}(A_1(\eta))^{1-\lambda}$$

by (63) and by (66) for ordinates differing from the given value of t by less than σ_0. But

$$\frac{r_2}{r_1}=1+\frac{\frac{5}{2}+\frac{1}{2}\eta-\sigma}{\sigma_0-\frac{5}{2}-\frac{1}{2}\eta} \quad\text{and}\quad \frac{r_3}{r_1}=1+\frac{\frac{1}{2}}{\sigma_0-\frac{5}{2}-\frac{1}{2}\eta}$$

so that

$$\lambda\leqslant(2+2\eta)(\tfrac{5}{2}+\tfrac{1}{2}\eta-\sigma)\leqslant 1-\eta^2$$

so long as σ_0 be sufficiently large in terms of η. All this gives

$$1/L(\mathbf{m};s)=O\{\|\mathbf{m}\|^{\eta}(|t|+2)^{\eta}\}$$

for $2+\eta\leqslant\sigma\leqslant\frac{5}{2}+\frac{1}{2}\eta$ and hence for $\sigma\geqslant 2+\eta$ by (63).

If $L(\mathbf{m};s)$ have poles of multiplicity l at $s=\frac{3}{2}$ and $\frac{5}{2}$, then direct the Phragmén-Lindelöf principle to the entire function $Z(\mathbf{m};s)=(s-\frac{3}{2})^l(s-\frac{5}{2})^l L(\mathbf{m};s)$ and deduce that $L(\mathbf{m};s)=O\{B(\mathbf{m})(|t|+2)^{10}\}$ for $|t|>A_2(\eta)$ and $Z(\mathbf{m};s)=O\{B(\mathbf{m})\}$ for $|t|<3A_2(\eta)$, where $A_2(\eta)>\sigma_0$. For $|t|\geqslant 2A_2(\eta)$, the previous method is still efficacious and yields the same bound for $1/L(\mathbf{m};s)$ as before. In like manner, a parallel treatment of $Z(\mathbf{m};s)$ for $|t|<2A_2(\eta)$ produces the inequality $1/Z(\mathbf{m};s)=O\{\|\mathbf{m}\|^{\eta}\}$ and hence once again the previous bound for $L(\mathbf{m};s)$.

This bound is the foundation of our key

LEMMA 10. *Let $T^*(\mathbf{m};y)$ be defined as in equation* (58). *Then, on Hypothesis* HW, *we have*

$$T^*(\mathbf{m};y)=O(\|\mathbf{m}\|^{\varepsilon}y^{\frac{1}{2}+\varepsilon}).$$

It suffices to consider the case where $y-\frac{1}{2}$ is a positive integer. Then, as

$$\frac{1}{2\pi i}\int_{c\pm iT}^{c\pm i\infty}\frac{u^s}{s}\,ds=O\left(\frac{u^c}{T|\log u|}\right)$$

when the ambiguous signs are the same, we have

$$\begin{aligned}T^*(\mathbf{m};y)&=\frac{1}{2\pi i}\int_{2-iT}^{2+iT}\frac{\Theta(\mathbf{m};s+\frac{3}{2})\Lambda(\mathbf{m};s+\frac{3}{2})}{L(\mathbf{m};s+\frac{3}{2})}\frac{y^s\,ds}{s}+O\left(\frac{y^3}{T}\sum_{\substack{k=1\\(k,\Delta(\mathbf{m}))=1}}^{\infty}\frac{|Q_3(\mathbf{m};k)|}{k^{7/2}}\right)\\&=\frac{1}{2\pi i}\int_{2-iT}^{2+iT}\frac{\Theta(\mathbf{m};s+\frac{3}{2})\Lambda(\mathbf{m};s+\frac{3}{2})}{L(\mathbf{m};s+\frac{3}{2})}\frac{y^s\,ds}{s}+O\left(\frac{y^3}{T}\right)\end{aligned}$$

by (56), (57), (59), (53), and (54). Now change the contour of integration so that it consists of the other three sides of the rectangle with vertices $2-iT$, $\frac{1}{2}+\varepsilon-iT$, $\frac{1}{2}+\varepsilon+iT$, $2+iT$, choosing $T=y^3$. The lemma is then deduced from Cauchy's theorem, the above estimate for $1/L(\mathbf{m};s)$, and the bound

$$\Lambda(\mathbf{m};s+\tfrac{3}{2})=O\left\{\prod_{p|\Delta(\mathbf{m})}\left(1+\frac{1}{p^{1/2}}\right)^{10}\right\}=O\{\|\mathbf{m}\|^{\varepsilon}\}$$

that holds for $\sigma\geq\frac{1}{2}$.

7. Decomposition of $R_2^*(x)$

We are now equipped to follow up the analysis of Section 4 and the first part of Section 5 provided that we recall the definition of the numbers Y_j introduced in (17). By the positivity of $G_2(\varphi,k)$ and by (23), we have

$$R_2^*(x)=\sum_{1\leq j\leq M_1}\sum_{Y_j<k\leq 2Y_j}\int_{-1/MY_j}^{1/MY_j}G_2(\varphi,k)\,d\varphi=\sum_{1\leq j\leq M_1}P(Y_j),\quad\text{say,}$$

where (33) shews that

$$\begin{aligned}
P(Y) &\leqslant \frac{1}{Y^{5/2}} \sum_{Y<k\leqslant 2Y} \frac{1}{k^{7/2}} \int_{-1/MY}^{1/MY} \sum_{\|\mathbf{m}\|\leqslant W} H(\varphi, \mathbf{m}/k; X)\, Q(\mathbf{m}; k)\cdot d\varphi \\
&\leqslant \frac{1}{Y^{5/2}} \sum_{Y<k\leqslant 2Y} \frac{1}{k^{7/2}} \int_{-1/MY}^{1/MY} \sum_{\substack{\|\mathbf{m}\|\leqslant W \\ \Delta(\mathbf{m})\neq 0}} H(\varphi, \mathbf{m}/k; X)\, Q(\mathbf{m}; k)\cdot d\varphi \\
&\quad + \frac{2}{Y^2} \sum_{Y<k\leqslant 2Y} \frac{1}{k^4} \int_{-1/MY}^{1/MY} \sum_{\substack{\|\mathbf{m}\|\leqslant W \\ \Delta(\mathbf{m})=0}} |H(\varphi, \mathbf{m}/k; X)|\, |Q(\mathbf{m}; k)|\cdot d\varphi \\
&= P_1(Y)+P_2(Y), \quad \text{say,}
\end{aligned} \tag{67}$$

the subscript j being implicit in the notation. Next let δ_1 be a positive absolute constant less than $\frac{1}{12}$ that is to be chosen later and let M_2 be that value of j for which

$$x^{\frac{1}{3}-\delta_1} < Y_j \leqslant 2x^{\frac{1}{3}-\delta_1}.$$

Then

$$\begin{aligned}
R_2^*(x) &\leqslant \sum_{1\leqslant j\leqslant M_2} P_1(Y_j) + \sum_{M_2<j\leqslant M_1} P_1(Y_j) + \sum_{1\leqslant j\leqslant M_1} P_2(Y_j) \\
&= R_4^*(x)+R_5^*(x)+R_6^*(x), \quad \text{say,}
\end{aligned} \tag{68}$$

in which the three sums are estimated separately by different methods in the forthcoming sections.

8. The senior arcs—estimation of $R_4^*(x)$

The estimation of $R_4^*(x)$ takes place on the senior arcs and will entail the assumption of Hypothesis HW at a point to be duly indicated. Rearranging the orders of summation and integration in the formula for $P_1(Y)$ given by (67), we get

$$\begin{aligned}
P_1(Y) &= \frac{1}{Y^{5/2}} \int_{-1/MY}^{1/MY} \sum_{\substack{\|\mathbf{m}\|\leqslant W \\ \Delta(\mathbf{m})\neq 0}} \sum_{Y<k\leqslant 2Y} \frac{H(\varphi, \mathbf{m}/k; X)\, Q(\mathbf{m}; k)}{k^{7/2}}\cdot d\varphi \\
&= \frac{1}{Y^{5/2}} \int_{-1/MY}^{1/MY} \sum_{\substack{\|\mathbf{m}\|\leqslant W \\ \Delta(\mathbf{m})\neq 0}} U(\varphi, \mathbf{m}, Y)\cdot d\varphi \\
&= \frac{1}{Y^{5/2}} \int_{-1/MY}^{1/MY} \mathscr{Y}(\varphi, Y)\, d\varphi, \quad \text{say,}
\end{aligned} \tag{69}$$

and then need a bound for $U(\varphi, \mathbf{m}, Y)$ for

$$Y > x^{\frac{1}{3}-\delta_1}. \tag{70}$$

To meet this requirement we set

$$T(\mathbf{m}; y) = \sum_{k \leq y} \frac{Q(\mathbf{m}; k)}{k^{7/2}} \tag{71}$$

in analogy with (58) and then use the formula

$$U(\varphi, \mathbf{m}, Y) = \int_Y^{2Y} H(\varphi, \mathbf{m}/y; X)\, dT(\mathbf{m}; y). \tag{72}$$

But before proceeding further we need a notational convention in order to express as simply as possible the effect of the integrals $J(u, v; X)$ on our work. For each relevant set of m, Y, φ we shall estimate $J(\varphi, m/y; X)$ in the range $Y \leq y \leq 2Y$ either trivially or by Lemma 2 according to a well defined procedure and shall see that we can denote the effect of so doing by writing

$$J(\varphi, m/y; X) = O(a_m), \tag{73}$$

where $a_m = a_m(\varphi, Y, X)$. Now, if

$$J'(u, v; X) = \frac{\partial}{\partial v} J(u, v; X),$$

then

$$J'(\varphi, m/y; X) = 2\pi i I(\varphi, m/y; X) = O(Xa_m)$$

either trivially or on account of Lemmata 2 and 3. Consequently, by this, (31), and (73), we also have

$$\begin{aligned} \frac{\partial}{\partial y} H(\varphi, \mathbf{m}/y; X) &= -\frac{1}{y^2} \sum_{1 \leq j \leq 6} m_j J'(\varphi, m_j/y; X) \prod_{i \neq j} J(\varphi, m_i/y; X) \\ &= O\left(\frac{X\|\mathbf{m}\|}{Y^2} \prod_{1 \leq j \leq 6} a_{m_j} \right). \end{aligned} \tag{74}$$

It will also be helpful to have the simple

LEMMA 11. *Let $k^\dagger$ denote, generally, a positive integer (possibly 1) all of whose prime factors divide a given non-zero integer Δ. Then, if $|\Delta| \leq z^{A_{21}}$, we have*

$$\sum_{k^\dagger \leqslant z} 1 = O(z^\varepsilon).$$

The left side of the proposed inequality does not exceed

$$z^\varepsilon \sum_{k^\dagger \leqslant z} \frac{1}{k^{\dagger\varepsilon}} \leqslant z^\varepsilon \prod_{p|\Delta}\left(1-\frac{1}{p^\varepsilon}\right)^{-1} \leqslant z^\varepsilon \prod_{p^\varepsilon<2}\left(1-\frac{1}{p^\varepsilon}\right)^{-1}\prod_{p|\Delta} 2$$
$$\leqslant A(\varepsilon)\, z^\varepsilon d(\Delta) = O(z^\varepsilon),$$

as stated.

In treating $T(\mathbf{m};y)$ we suppose that $\mathbf{m}$ and y satisfy the conditions implicit in (69) and (72) so that $\|\mathbf{m}\| = O(y^{A_{22}})$ by (20) and (70). Furthermore, for a given $\mathbf{m}$ subject to this proviso, let k_1 denote, generally, a positive integer relatively prime to $\Delta(\mathbf{m})$ and k_2 a positive integer whose prime factors all divide $\Delta(\mathbf{m})$. Then, now assuming Hypothesis HW and appealing to (71), (58), and Lemmata 5 and 10, we have

$$T(\mathbf{m};y) = \sum_{k_1 k_2 \leqslant y} \frac{Q(\mathbf{m};k_1)\,Q(\mathbf{m};k_2)}{k_1^{7/2}k_2^{7/2}} = \sum_{k_2\leqslant y}\frac{Q(\mathbf{m};k_2)}{k_2^{7/2}}\sum_{k_1\leqslant y/k_2}\frac{Q_3(\mathbf{m};k_1)}{k_1^{3/2}}$$
$$= O\left(y^{\frac{1}{2}+\varepsilon}\sum_{k_2\leqslant y}\frac{|Q(\mathbf{m};k_2)|}{k_2^4}\right),$$

whence

$$T(\mathbf{m};y) = O\left(y^{\frac{1}{2}+\varepsilon}\prod_{1\leqslant j\leqslant 6}\tilde{\omega}(m_j)\sum_{k_2\leqslant y} 1\right) = O\left(y^{\frac{1}{2}+\varepsilon}\prod_{1\leqslant j\leqslant 6}\tilde{\omega}(m_j)\right)$$

by Lemmata 9 and 11. Therefore, since

$$U(\varphi,\mathbf{m},Y) = {}_{Y}^{2Y}[H(\varphi,\mathbf{m}/y;X)\,T(\mathbf{m};y)] - \int_Y^{2Y}\frac{\partial H(\varphi,\mathbf{m}/y;X)}{\partial y}\,T(\mathbf{m};y)\,dy$$

by (72) and since we may assume that $\|\mathbf{m}\|\leqslant W$, we infer that

$$U(\varphi,\mathbf{m},Y) = O\left(Y^{\frac{1}{2}+\varepsilon}\prod_{1\leqslant j\leqslant 6}a_{m_j}\tilde{\omega}(m_j)\right) + O\left(\frac{X\|\mathbf{m}\|}{Y^2}\prod_{1\leqslant j\leqslant 6}a_{m_j}\tilde{\omega}(m_j)\int_Y^{2Y} y^{\frac{1}{2}+\varepsilon}\,dy\right)$$
$$= O\left\{\left(Y^{\frac{1}{2}+\varepsilon}+\frac{XY^\varepsilon\|\mathbf{m}\|}{Y^{1/2}}\right)\prod_{1\leqslant j\leqslant 6}a_{m_j}\tilde{\omega}(m_j)\right\} \tag{75}$$
$$= O\left(\frac{XY^\varepsilon W}{Y^{1/2}}\prod_{1\leqslant j\leqslant 6}a_{m_j}\tilde{\omega}(m_j)\right)$$

with the aid of (73) and (74) and then (19).

Next, inserting (75) into the formula for $\mathcal{Y}(\varphi, Y)$ implied by (69), we obtain the relation

$$\mathcal{Y}(\varphi, Y) = O\left(\frac{XY^{\varepsilon}W}{Y^{1/2}}\left\{\sum_{0<|m|\leq W} a_m \tilde{\omega}(m)\right\}^6\right) = O(XY^{-\frac{1}{2}+\varepsilon}W\Psi^6(\varphi, Y)), \quad \text{say,} \tag{76}$$

whose consequences will be examined by means of

LEMMA 12. *If $b=0$ or $\frac{1}{4}$, then*

$$\sum_{0<m\leq z} \frac{\tilde{\omega}(m)}{m^b} = O(z^{1-b})$$

for $z>0$.

For each m let $m_1 = \Pi_{p\|m}\, p$ and $m = m_1 m_2$ so that $\tilde{\omega}(m) = m_2^{1/4}$. Then, because any number of the form m_2 can be represented (not necessarily uniquely) as $\lambda^2\mu^3$, we get

$$\begin{aligned}\sum_{0<m\leq z} \tilde{\omega}(m) &\leq \sum_{l\lambda^2\mu^3\leq z} \lambda^{1/2}\mu^{3/4} = \sum_{\lambda^2\mu^3\leq z} \lambda^{1/2}\mu^{3/4} \sum_{l\leq z/\lambda^2\mu^3} 1 \\ &\leq z \sum_{\lambda,\mu} \frac{1}{\lambda^{3/2}\mu^{9/4}} = O(z),\end{aligned}$$

and then deduce the result for the other exponent by partial summation.

Since Lemma 2 permits us to define a_m in (73) by

$$a_m = \begin{cases} X, & \text{if } |\varphi| \leq 1/X^3, \\ Y^{1/4}|\varphi|^{-1/4}|m|^{-1/4}, & \text{if } |\varphi| > 1/X^3, \end{cases} \tag{77}$$

we deduce from (76) and Lemma 12 that

$$\Psi(\varphi, Y) = O\left(X \sum_{0<m\leq W} \tilde{\omega}(m)\right) = O(XW)$$

for $|\varphi| \leq 1/X^3$ but that

$$\Psi(\varphi, Y) = O\left(\frac{Y^{1/4}}{|\varphi|^{1/4}} \sum_{0<m\leq W} \frac{\tilde{\omega}(m)}{m^{1/4}}\right) = O\left(\frac{Y^{1/4}W^{3/4}}{|\varphi|^{1/4}}\right)$$

for $|\varphi| > 1/X^3$, in which the appropriate values of W are, respectively,

$$YX^{-1}\log^4 x \quad \text{and} \quad X^2Y|\varphi|\log^4 x \tag{78}$$

by (19). Hence

$$\Psi(\varphi, Y) = O(Y \log^4 x) \tag{79}$$

if $|\varphi| \leq 1/X^3$, while

$$\Psi(\varphi, Y) = O(X^{3/2} Y |\varphi|^{1/2} \log^3 x) \tag{80}$$

if $|\varphi| > 1/X^3$.

We bring the estimation over the senior arcs to a close by analyzing separately the two cases

$$\begin{aligned} &\text{(a) } X^3 M^{-1} \leq Y \leq M, \\ &\text{(b) } x^{\frac{1}{3}-\delta_1} < Y < X^3 M^{-1}, \end{aligned} \tag{81}$$

neither of which is nugatory because of (6). Taking case (a) in which $|\varphi| \leq 1/MY \leq 1/X^3$, we have

$$\mathscr{Y}(\varphi, Y) = O(Y^{\frac{13}{2}+\varepsilon})$$

by (76), (78), and (79), whence (69) yields

$$P_1(Y) = O\left(Y^{4+\varepsilon} \int_0^{1/MY} d\varphi \right) = O\left(\frac{Y^{3+\varepsilon}}{M} \right) \tag{82}$$

in this instance. In case (b) the above estimate for $\mathscr{Y}(\varphi, Y)$ still obtains when $|\varphi| \leq 1/X^3$ but most otherwise be supplanted by the estimate

$$\mathscr{Y}(\varphi, Y) = O(X^{12} Y^{\frac{13}{2}+\varepsilon} |\varphi|^4)$$

that is supplied by (76), (78), and (80); here, therefore, we get([6])

$$\begin{aligned} P_1(Y) &= O\left(Y^{4+\varepsilon} \int_0^{1/X^3} d\varphi \right) + O\left(X^{12} Y^{4+\varepsilon} \int_{1/X^3}^{1/MY} \varphi^4 \, d\varphi \right) \\ &= O\left(\frac{Y^{4+\varepsilon}}{X^3} \right) + O\left(\frac{X^{12+\varepsilon}}{M^5 Y} \right) = O\left(\frac{X^{12+\varepsilon}}{M^5 Y} \right) \end{aligned} \tag{83}$$

since $Y < X^3 M^{-1}$. Finally, (68), (81), (82), and (83) produce the estimate

$$R_4^*(x) = O(M^{2+\varepsilon}) + O\left(\frac{X^{12+\varepsilon}}{M^5 x^{\frac{1}{3}-\delta_1}} \right) = O(x^{1+2\delta+\varepsilon}) + O(x^{\frac{7}{6}-5\delta+\delta_1+\varepsilon}) \tag{84}$$

by means of a simple summation over j.

([6]) If $Y < X \log^{-4} x$ in (b), then there are values of φ for which $W < 1$ and for which therefore the sums $\mathscr{Y}(\varphi, Y)$ and $\Psi(\varphi, Y)$ are empty. But replacing our estimates in these instances by zero would not confer any benefit and would only complicate the argument.

9. The junior arcs—estimation of $R_5^*(x)$

The estimation of $R_5^*(x)$ is effected on the junior arcs and does not involve the supposition of any unproved hypothesis. Equation (69) being still applicable, we want a bound for $U(\varphi, \mathbf{m}, Y)$ when $Y \leqslant x^{\frac{1}{3}-\delta_1}$ and derive it from the inequality

$$|U(\varphi, \mathbf{m}, Y)| \leqslant \sum_{Y<k\leqslant 2Y} \frac{|H(\varphi, \mathbf{m}/k; X)|\,|Q(\mathbf{m}; k)|}{k^{7/2}} = O\left(\prod_{1\leqslant j\leqslant 6} a_{m_j} \sum_{k\leqslant 2Y} \frac{|Q(\mathbf{m}; k)|}{k^{7/2}}\right) \tag{85}$$

that flows from (31) and (73). Here, by Lemma 5, (53), and Lemma 9 and then by (20) and the implied condition $\Delta(\mathbf{m}) \neq 0$, we have

$$\begin{aligned}
\sum_{k\leqslant 2Y} \frac{|Q(\mathbf{m}; k)|}{k^{7/2}} &= O\left(Y^{\varepsilon} \prod_{1\leqslant j\leqslant 6} \bar{\omega}(m_j) \sum_{k_1 k_2\leqslant 2Y} k_2^{1/2}\right)\\
&= O\left(Y^{\varepsilon} \prod_{1\leqslant j\leqslant 6} \bar{\omega}(m_j) \sum_{k_2\leqslant 2Y} k_2^{1/2} \sum_{k_1\leqslant 2Y/k_2} 1\right)\\
&= O\left(Y^{1+\varepsilon} \prod_{1\leqslant j\leqslant 6} \bar{\omega}(m_j) \sum_{k_2\leqslant 2Y} \frac{1}{k_2^{1/2}}\right)\\
&= O\left\{Y^{1+\varepsilon} \prod_{1\leqslant j\leqslant 6} \bar{\omega}(m_j) \prod_{p|\Delta(m)} \left(1-\frac{1}{p^{1/2}}\right)^{-1}\right\}\\
&= O\left(x^{\varepsilon} Y \prod_{1\leqslant j\leqslant 6} \bar{\omega}(m_j)\right)
\end{aligned} \tag{86}$$

in the notation used to estimate $T(\mathbf{m}; y)$ in the previous section. Therefore, as the counterpart of (76), we obtain

$$\mathcal{Y}(\varphi, Y) = O(x^{\varepsilon} Y \Psi^6(\varphi, Y))$$

from (69), (85), and (86).

In the current situation $W<1$ if $|\varphi| \leqslant 1/X^3$ so that (80) may be used whenever $\Psi(\varphi, Y)$ is not an empty sum. Hence $\mathcal{Y}(\varphi, Y) = O(X^{9+\varepsilon} Y^7 |\varphi|^3)$ and thus (69) implies that

$$P_1(Y) = O\left(X^{9+\varepsilon} Y^{9/2} \int_0^{1/MY} \varphi^3\, d\varphi\right) = O\left(\frac{X^{9+\varepsilon} Y^{1/2}}{M^4}\right).$$

Consequently, summing over j, we conclude from (68) that

$$R_5^*(x) = O(x^{\frac{7}{6}-4\delta-\frac{1}{2}\delta_1+\varepsilon}). \tag{87}$$

10. Estimation of $R_6^*(x)$

No form of Kloosterman refinement appears in the treatment of $R_6^*(x)$ because the estimation of $Q(\mathbf{m}; k)$ now stems from Lemma 8 alone; in particular, therefore, the distinction between junior arcs and senior arcs becomes irrelevant.

The particular reordering of integration and summations used in the metamorphosis of $P_1(Y)$ being inappropriate for $P_2(Y)$, we deduce from (67) that

$$\begin{aligned} P_2(Y) &= \frac{2}{Y^2}\int_{-1/MY}^{1/MY} \sum_{Y<k\leqslant 2Y} \frac{1}{k^4} \sum_{\substack{\|\mathbf{m}\|\leqslant W \\ \Delta(\mathbf{m})=0}} |H(\varphi, \mathbf{m}/k; X)|\,|Q(\mathbf{m}; k)|\cdot d\varphi \\ &= \int_{-1/MY}^{1/MY} \frac{2}{Y^2} \sum_{Y<k\leqslant 2Y} V(\varphi, k, Y)\cdot d\varphi \qquad (88) \\ &= \int_{-1/MY}^{1/MY} \mathscr{K}(\varphi, Y)\, d\varphi, \quad \text{say,} \end{aligned}$$

in which

$$V(\varphi, k, Y) = O\left(x^{\varepsilon} \sum_{\substack{\|\mathbf{m}\|\leqslant W \\ \Delta(\mathbf{m})=0}} \prod_{1\leqslant j\leqslant 6} (k, m_j)^{1/4} a_{m_j} \right) = O(x^{\varepsilon} V_1(\varphi, k, Y)), \quad \text{say,} \tag{89}$$

by Lemma 8 and (73). Further progress is then dependent on our eliciting the nature of the integral solutions of the equation

$$\Delta(\mathbf{m}) = 0. \tag{90}$$

To attend to this question, let $a_j = m_j^3$ for $1\leqslant j\leqslant 6$ and then express each a_j uniquely as $b_j c_j^2$ where b_j is square-free and $c_j>0$, remembering that all the m_j are non-zero. Then (34) implies that any solution of (90) corresponds to a solution of

$$c_1\sqrt{b_1} \pm \ldots \pm c_6\sqrt{b_6} = 0 \tag{91}$$

for some choice of the ambiguous signs. Next, if for any such solution we denote the distinct values of $b_1, \ldots, b_6$ by $d_1, \ldots, d_l$, we deduce a relation of the type

$$e_1\sqrt{d_1} + \ldots + e_l\sqrt{d_l} = 0,$$

where $e_1, \ldots, e_l$ are integers. Hence, since it is well known (and easily proved) that $\sqrt{d_1}, \ldots, \sqrt{d_l}$ are linearly independent over the rationals even when one of them is 1, we see that $e_1 = \ldots = e_l = 0$ and that therefore (90) holds because of trivial cancellation between the coefficients c_j in (91). The circumstances in which this can potentially happen are exhausted by the four (not necessarily mutually exclusive) typical cases

(i) $b_1=\dots=b_6=b$, say;
(ii) $m_1=m_2$, $b_3=\dots=b_6$;
(iii) $b_1=b_2=b_3$, $b_4=b_5=b_6$;
(iv) $m_1=m_2$, $m_3=m_4$, $m_5=m_6$.

In case (i), for each j, $m_j^3=bc_j^2=bc^2c_j'^2$, where $c=\text{h.c.f}\,(c_1,\dots,c_6)$ and $1=\text{h.c.f.}\,(c_1',\dots,c_6')$. Therefore, being equal to $\{\text{h.c.f. } m_1,\dots,m_6)\}^3$, bc^2 is a perfect cube λ^3 and thus c_j' is a perfect cube $m_j'^3$, whence $\mathbf{m}=\lambda(m_1'^2,\dots,m_6'^2)$. In like manner $(m_3,\dots,m_6)=\lambda(m_3'^2,\dots,m_6'^2)$ in case (ii). Yet case (iii) can be rejected because the similar implication $(m_1,\dots,m_3)=\lambda(m_1'^2,\dots,m_3'^2)$ leads to the impossible Fermat equation $m_1'^3\pm m_2'^3\pm m_3'^3=0$ when the need for cancellation is also taken into account.

Summing up, we have

$$V_1(\varphi,k,Y)\leq V_2(\varphi,k,Y)+V_3(\varphi,k,Y)+V_4(\varphi,k,Y), \tag{92}$$

where $V_2(\varphi,k,Y)$, $V_3(\varphi,k,Y)$, and $V_4(\varphi,k,Y)$ are the (not necessarily mutually exclu sive) contributions to $V_1(\varphi,k,Y)$ due, respectively, to the cases that are typified by (i), (ii), and (iv). These sums are all very similar in character and share the feature of being estimated with the assistance of the following well known lemma.

LEMMA 13. *For a given positive integer d, we have*

$$\sum_{0<l\leq z}\frac{(d,l)^{1/2}}{l^b}=O(z^{1-b}\sigma_{-1/2}(d))$$

if $b=0$ or $1/2$.

The ambience of the calculations being still described by (89), we first suppose that $a_m=X$ and see that (92) gives

$$\begin{aligned}V_2(\varphi,k,Y)&=O\left(X^6\sum_{0<lm_1'^2,\dots,lm_6'^2\leq W}\ \prod_{1\leq j\leq 6}(k,lm_j'^2)^{1/4}\right)\\&=O\left\{X^6\sum_{l\leq W}l^{3/2}\left(\sum_{0<m\leq (W/l)^{1/2}}(k,m)^{1/2}\right)^6\right\}\\&=O\left(X^{6+\varepsilon}W^3\sum_{l\leq W}\frac{1}{l^{3/2}}\right)=O(X^{6+\varepsilon}W^3)\end{aligned} \tag{93}$$

by Lemma 13. But, if $a_m = Y^{1/4}|\varphi|^{-1/4}|m|^{-1/4}$, then

$$\begin{aligned} V_2(\varphi, k, Y) &= O\left(Y^{3/2}|\varphi|^{-3/2} \sum_{0<lm_1'^2, \ldots, lm_6'^2 \leq W} \prod_{1 \leq j \leq 6} \frac{(k, lm_j'^2)^{1/4}}{(lm_j'^2)^{1/4}} \right) \\ &= O\left\{ Y^{3/2}|\varphi|^{-3/2} \sum_{l \leq W} \left(\sum_{0<m\leq (W/l)^{1/2}} \frac{(k,m)^{1/2}}{m^{1/2}} \right)^6 \right\} \\ &= O\left(x^{\varepsilon} Y^{3/2}|\varphi|^{-3/2} W^{3/2} \sum_{l \leq W} \frac{1}{l^{3/2}} \right) = O(x^{\varepsilon} Y^{3/2} W^{3/2} |\varphi|^{-3/2}). \end{aligned} \tag{94}$$

Secondly,

$$\begin{aligned} V_4(\varphi, k, Y) &= O\left(X^6 \sum_{0<m_1, m_3, m_5 \leq W} \prod_{j=1,3,5} (k, m_j)^{1/2} \right) \\ &= O\left\{ X^6 \left(\sum_{0<m\leq W} (k,m)^{1/2} \right)^3 \right\} = O(X^{6+\varepsilon} W^3) \end{aligned} \tag{95}$$

when $a_m = X$, whereas

$$\begin{aligned} V_4(\varphi, k, Y) &= O\left(Y^{3/2}|\varphi|^{-3/2} \sum_{0<m_1, m_2, m_3 \leq W} \prod_{j=1,2,3} \frac{(k, m_j)^{1/2}}{m_j^{1/2}} \right) \\ &= O\left\{ Y^{3/2}|\varphi|^{-3/2} \left(\sum_{0<m\leq W} \frac{(k,m)^{1/2}}{m^{1/2}} \right)^3 \right\} = O(x^{\varepsilon} Y^{3/2} W^{3/2} |\varphi|^{-3/2}) \end{aligned} \tag{96}$$

when $a_m = Y^{1/4} m^{-1/4} |\varphi|^{-1/4}$.

The method for $V_3(\varphi, k, Y)$ is an amalgam of those used for the previous sums. If $a_m = X$, then

$$\begin{aligned} V_3(\varphi, k, Y) &= O\left(X^6 \sum_{0<lm_3'^2, \ldots, lm_6'^2 \leq W} \prod_{3 \leq j \leq 6} (k, lm_j'^2)^{1/4} \sum_{m_1 \leq W} (k, m_1)^{1/2} \right) \\ &= O\left\{ X^{6+\varepsilon} W \sum_{l \leq W} l \left(\sum_{0<m\leq (W/l)^{1/2}} (k,m)^{1/2} \right)^4 \right\} \\ &= O\left(X^{6+\varepsilon} W^3 \sum_{l \leq W} \frac{1}{l} \right) = O(X^{6+\varepsilon} W^3), \end{aligned} \tag{97}$$

similar reasoning shewing that

$$V_3(\varphi, k, Y) = O(x^{\varepsilon} Y^{3/2} W^{3/2} |\varphi|^{-3/2}) \tag{98}$$

for $a_m = Y^{1/4} m^{-1/4} |\varphi|^{-1/4}$.

Equations (93)–(98) with (92) and (88) shew that $\mathscr{X}(\varphi, Y)$ is $O(X^{6+\varepsilon}Y^{-1}W^3)$ or $O(x^{\varepsilon}Y^{1/2}W^{3/2}|\varphi|^{-3/2})$ according as $|\varphi|\leqslant 1/X^3$ or $|\varphi|>1/X^3$, whence

$$\mathscr{X}(\varphi, Y)=O(X^{3+\varepsilon}Y^2)$$

by (19). Consequently

$$P_2(Y)=O\left(\frac{X^{3+\varepsilon}Y}{M}\right),$$

and we end up with the bound

$$R_6^*(x)=O(x^{1+\varepsilon}) \tag{99}$$

after using (68).

11. The theorem on $R(x)$

Our theorem on $R(x)$ is now available. By (68), (84), (87), and (99),

$$R_2^*(x)=O(x^{1+2\delta+\varepsilon})+O(x^{\frac{7}{6}-5\delta+\delta_1+\varepsilon})+O(x^{\frac{7}{6}-4\delta-\frac{1}{2}\delta_1+\varepsilon})$$

through which by choosing $\delta=\frac{1}{38}$, $\delta_1=\frac{1}{57}$ we get

$$R_2^*(x)=O(x^{\frac{20}{19}+\varepsilon}). \tag{100}$$

Then, from (30) and then (4), we conclude that

$$R(x)=O(x^{\frac{20}{19}+\varepsilon}).$$

We thus have

THEOREM 1. *Let $f(\theta, y)$ be the cubic exponential sum defined by* (3) *and let*

$$R(x)=\int_0^1 |f(\theta, x^{1/3})|^6\, d\theta.$$

Then, as $x\to\infty$,

$$R(x)=O(x^{\frac{20}{19}+\varepsilon})$$

if Hypothesis HW *be true for the Hasse-Weil L-functions defined over the varieties* $\mathscr{V}(\mathbf{m})$.

Chapter II. Applications of Theorem 1

1. Introduction and notation

Theorem 1 is applied to Waring's problem through familiar procedures that do not in themselves involve the assumption of the Hypothesis HW. Our account can therefore be relatively brief even though we must enter into some details not fully covered by the existing literature. The remainder terms in the asymptotic formulae we derive are usually the best that can be achieved by a fairly careful use of such reasoning but there are some instances where improvements can be effected through more intensive methods.

Until the penultimate section we shall be wholly preoccupied with problems appertaining to the number of representations of a large number N as a sum of powers. We therefore first let

$$f_j(\theta) = \sum_{0 \leq m \leq N^{1/j}} e^{2\pi i m^j \theta}$$

and write

$$g(\theta) = f_2(\theta), \quad f(\theta) = f_3(\theta), \quad h(\theta) = f_4(\theta) \tag{101}$$

so that $f(\theta) = f(\theta, N^{1/3})$ according to (3). Then, by analogy with (10), we write

$$J_j(u) = \int_0^{N^{1/j}} e^{2\pi i u t^j}\, dt, \quad S_j(a, k) = \sum_{0 \leq l < k} e^{2\pi i a l^j/k},$$

suppressing without ambiguity the subscript when $j=3$ but noting that $J(u)$ is not the exact counterpart of $J(u; X)$.

For each problem we shall employ a Farey dissection of appropriate order $M = M(N)$—different from that used in Chapter I—to divide the unit interval of integration into arcs $\theta = h/k + \varphi$, where, for each pair h, k satisfying $0 \leq h < k$, $(h, k) = 1$, $k \leq M$, we have the inequalities

$$-a'_{h,k} \leq \varphi \leq a_{h,k} \tag{102}$$

and $1/2kM < a_{h,k}$, $a'_{h,k} < 1/kM$ (see [7]). The customary sub-division of the arcs is then brought about by introducing a number $M_1 = M_1(N)$ and by defining the major and minor arcs to be those arising from values of k satisfying $k \leq M_1$ and $k > M_1$, respectively;(¹) as usual, the aggregates of major and minor arcs are denoted by $\mathfrak{M}$ and $\mathfrak{m}$.

(¹) Provided that $M_1 \leq \frac{1}{2}M$ we can extend the major arcs at the expense of the minor arcs by taking $a_{h,k} = a'_{h,k} = 1/Mk$ for $k \leq M_1$. This permits a more accurate treatment of terms such as E_4 in the following equation (104), although in the present context the effect of such improvements is usually vitiated by the influence of the other terms under consideration.

To treat the minor arcs we shall require Weyl's inequality ([14], [20], [21]), as expressed by

LEMMA 14. *If* $|\varphi|\leq 1/k^2$, *then*

$$f_j\left(\frac{h}{k}+\varphi\right)=O\{N^{1/j+\varepsilon}(N^{-1/j}+k^{-1}+kN^{-1})^{2^{1-j}}\}.$$

But for some of the minor arcs we do better to exploit the following lemma, which is primarily needed for the analysis of the major arcs.

LEMMA 15. *We have*

$$f_j\left(\frac{h}{k}+\varphi\right)=\frac{1}{k}S_j(h,k)J_j(\varphi)+O(k^{\frac{1}{2}+\varepsilon})+O\{k^{\frac{1}{2}+\varepsilon}(N\varphi)^{1/2}\},$$

where the final remainder term can be dropped when $|\varphi|\leq 1/2jkN^{1-1/j}$.

There are various approaches to this result that involve either the Euler-Maclaurin sum formula, or the Poisson summation formula, or a hybrid of these two formulae. In all cases the Hua-Weil estimate for $S_j(h,k)$ is needed ([12], [20]), while an analogue of Lemma 2 is relevant to the situation where φ is unrestricted. A proof for the restricted case is given by Vaughan [20] but an alternative proof can be derived by developing the way Davenport handled a Lemma in [2]. The formula for the unrestricted case is quoted by Vaughan on p. 54 of [20] but can be proved by analyzing $f(h/k+\varphi)$ in much the same way as $F(h/k+\varphi)$ was examined in our Section 2, Chapter I.

2. The asymptotic formulae for seven cubes and eight cubes

As usual, we have

$$\begin{aligned} r_7(N)&=\int_0^1 f^7(\theta)\,e^{-2\pi iN\theta}\,d\theta=\int_{\mathfrak{M}} f^7(\theta)\,e^{-2\pi iN\theta}\,d\theta+\int_{\mathfrak{m}} f^7(\theta)\,e^{-2\pi iN\theta}\,d\theta \\ &=E_1+E_2,\quad\text{say,}\end{aligned}\tag{103}$$

where in this instance it is appropriate to take $M=[6N^{2/3}]+1$ and $M_1=N^{1/3}$ in order to define $\mathfrak{M}$ and $\mathfrak{m}$.

On a major arc centred by h/k

$$f(\theta)=f\left(\frac{h}{k}+\theta\right)=\frac{1}{k}S(h,k)\,J(\varphi)+O(k^{\frac{1}{2}+\varepsilon})$$

by Lemma 15, whence

$$f^7(\theta)=\frac{1}{k^7}S^7(h,k)J^7(\varphi)+O\left(\frac{N}{k^{11/2}}|S(h,k)|^6|J(\varphi)|^6\right)+O(k^{\frac{7}{2}+\varepsilon}).$$

Therefore

$$\begin{aligned}E_1=&\sum_{k\leq M_1}\frac{1}{k^7}\sum_{\substack{0\leq h<k\\(h,k)=1}}S^7(h,k)\,e^{-2\pi iNh/k}\int_{-\infty}^{\infty}J^7(\varphi)\,e^{-2\pi iN\varphi}\,d\varphi\\&+O\left(\sum_{k\leq M_1}\frac{1}{k^7}\sum_{\substack{0\leq h<k\\(h,k)=1}}|S(h,k)|^7\int_{1/2Mk}^{\infty}|J(\varphi)|^7\,d\varphi\right)\\&+O\left(M_1^{\frac{1}{2}+\varepsilon}\sum_{k\leq M_1}\frac{1}{k^6}\sum_{\substack{0\leq h<k\\(h,k)=1}}|S(h,k)|^6\int_0^{\infty}|J(\varphi)|^6\,d\varphi\right)+O\left(\sum_{k\leq M_1}k^{\frac{9}{2}+\varepsilon}\int_0^{1/Mk}d\varphi\right)\\&=E_3+O(E_4)+O(E_5)+O(E_6),\quad\text{say.}\end{aligned}\tag{104}$$

The remainder terms are easily assessed by (26), (25), and the bound $J(\varphi)=O(|\varphi|^{-1/3})$. In fact

$$\begin{aligned}E_4&=O\left(M^{4/3}\sum_{k\leq M_1}\frac{q^7(k)}{k^{14/3}}\right)=O\left(M^{4/3}\sum_{k\leq M_1}A_{23}^{\omega(k)}\right)\\&=O(M^{4/3}M_1^{1+\varepsilon})=O(N^{\frac{11}{9}+\varepsilon})\end{aligned}\tag{105}$$

by a crude argument that satisfies our present need. Also, repeating the reasoning used for (24) and (26), we have

$$E_5=O(M_1^{\frac{1}{2}+\varepsilon}N)=O(N^{\frac{7}{6}+\varepsilon}).\tag{106}$$

Since obviously

$$E_6=O(M^{-1}M_1^{\frac{9}{2}+\varepsilon})=O(N^{\frac{5}{6}+\varepsilon}),$$

we obtain

$$E_1=E_3+O(N^{\frac{11}{9}+\varepsilon})$$

from (104), (105), and (106).

Altogether, therefore, bearing in mind that

$$\sum_{k>M_1}\frac{q^7(k)}{k^6}=O\left(\sum_{k>M_1}\frac{A_{24}^{\omega(k)}}{k^{4/3}}\right)=O(M_1^{-\frac{1}{3}+\varepsilon}),$$

we conclude from Fourier's integral theorem that

$$E_1 = \tfrac{3}{4}\Gamma^6(\tfrac{4}{3})\,\mathfrak{S}(N)\,N^{4/3} + O(N^{\frac{11}{9}+\varepsilon}), \tag{107}$$

where $\mathfrak{S}(N)$ is the singular series.

On the other hand,

$$|E_2| \leqslant \overline{\mathrm{bd}}_{\theta\in\mathfrak{m}}\ |f(\theta)| \int_0^1 |f(\theta)|^6\,d\theta$$

so that Lemma 14 and Theorem 1 yield

$$E_2 = O(N^{\frac{1}{4}+\varepsilon}N^{\frac{20}{19}+\varepsilon}) = O(N^{\frac{99}{76}+\varepsilon}) \tag{108}$$

if Hypothesis HW be assumed.

Equations (103), (107), and (108) now give

THEOREM 2. *Let $r_7(N)$ be the number of representations of N as the sum of seven non-negative cubes. Then, if Hypothesis* HW *be true, we have*

$$r_7(N) = \tfrac{3}{4}\Gamma^6(\tfrac{4}{3})\,\mathfrak{S}(N)\,N^{\frac{4}{3}} + O(N^{\frac{99}{76}+\varepsilon}),$$

where $\mathfrak{S}(N)$ is the singular series.

The formula is a genuine asymptotic relation because it is known that $\mathfrak{S}(N) > A_{25}$. Similarly, we can prove

THEOREM 3. *On Hypothesis* HW, *we have*

$$r_8(N) = \frac{\Gamma^8(4/3)}{\Gamma(8/3)}\,\mathfrak{S}(N)\,N^{\frac{5}{3}} + O(N^{\frac{59}{38}+\varepsilon}),$$

where $\mathfrak{S}(N)$ is the singular series.

3. The asymptotic formula for a square and five cubes

The analysis becomes harder than in the last section because there is now a narrower margin between success and failure and because the presence of both $g(\theta)$ and $f(\theta)$ in the integrand creates technical complications. For example, it is desirable to introduce the notation

$$T_j(\varphi) = \min(N^{1/j}, |\varphi|^{-1/j}), \quad T(\varphi) = T_3(\varphi) \tag{109}$$

whose use would have previously seemed superfluous. Moreover, to determine the major and minor arcs we shall have to select δ carefully in

$$M = N^{\delta} \quad \text{and} \quad M_1 = N^{1-\delta} = NM^{-1} \tag{110}$$

so that

$$\tfrac{1}{2} < \delta < \tfrac{3}{5}. \tag{111}$$

With this implicit choice of $\mathfrak{M}$ and $\mathfrak{m}$ we then use the representation

$$\begin{aligned} \nu(N) &= \int_0^1 g(\theta) f^5(\theta) e^{-2\pi i N\theta} d\theta = \int_{\mathfrak{M}} g(\theta) f^5(\theta) e^{-2\pi i N\theta} d\theta + \int_{\mathfrak{m}} g(\theta) f^5(\theta) e^{-2\pi i N\theta} d\theta \\ &= E_7 + E_8, \quad \text{say,} \end{aligned} \tag{112}$$

where $\nu(N)$ is the number of ways of expressing N as the sum of a square and five non-negative cubes.

On a major arc containing h/k, we have

$$g(\theta) = g\left(\frac{h}{k} + \varphi\right) = \frac{1}{k} S_2(h, k) J_2(\varphi) + O(k^{\frac{1}{2}+\varepsilon}),$$

$$\begin{aligned} f(\theta) = f\left(\frac{h}{k} + \varphi\right) &= \frac{1}{k} S(h, k) J(\varphi) + O(k^{\frac{1}{2}+\varepsilon}) + O\{(Nk^{1+\varepsilon}\varphi)^{\frac{1}{2}}\} \\ &= \frac{1}{k} S(h, k) J(\varphi) + O(N^{\frac{1}{2}+\varepsilon} M^{-\frac{1}{2}}) \end{aligned}$$

by Lemma 15 and then by (102), (110), and (111). In the former relation, using (109) and the estimate for the Gauss sum, we have $S_2(h, k) = O(k^{1/2})$ and $J_2(\varphi) = O\{T_2(\varphi)\}$ so that (102) and (110) imply that $g(\theta) = O\{k^{-1/2} T_2(\varphi)\}$ on $\mathfrak{M}$. Furthermore, by the latter relation and (26),

$$f^5(\theta) = \frac{1}{k^5} S^5(h, k) J^5(\varphi) + O\left(\frac{N^{\frac{1}{2}+\varepsilon} q^4(k) T^4(\varphi)}{M^{1/2} k^4}\right) + O\left(\frac{N^{\frac{5}{2}+\varepsilon}}{M^{5/2}}\right),$$

whence

$$\begin{aligned} g(\theta) f^5(\theta) = \frac{1}{k^6} S_2(h, k) S^5(h, k) J_2(\varphi) J^5(\varphi) + O\left(\frac{N^{\varepsilon} q^5(k) T^5(\varphi)}{k^{9/2}}\right) \\ + O\left(\frac{N^{\frac{1}{2}+\varepsilon} q^4(k) T_2(\varphi) T^4(\varphi)}{M^{1/2} k^{9/2}}\right) + O\left(\frac{N^{\frac{5}{2}+\varepsilon} T_2(\varphi)}{M^{5/2} k^{1/2}}\right). \end{aligned}$$

Therefore, since it may be verified from (25) that the second remainder term above can absorb the first, we conclude that

$$E_7 = \sum_{k\le M_1} \frac{1}{k^6} \sum_{\substack{0\le h<k \\ (h,k)=1}} S_2(h,k)\, S^5(h,k)\, e^{-2\pi i N h/k} \int_{-\infty}^{\infty} J_2(\varphi) J^5(\varphi)\, e^{-2\pi i N\varphi}\, d\varphi$$

$$+O\left(\sum_{k\le M_1} \frac{q^5(k)}{k^{9/2}} \int_{1/2Mk}^{\infty} T_2(\varphi)\, T^5(\varphi)\, d\varphi\right) + O\left(\frac{N^{\frac{1}{2}+\varepsilon}}{M^{1/2}} \sum_{k\le M_1} \frac{q^4(k)}{k^{7/2}} \int_0^{\infty} T_2(\varphi)\, T^4(\varphi)\, d\varphi\right)$$

$$+O\left(\frac{N^{\frac{5}{2}+\varepsilon}}{M^{5/2}} \sum_{k\le M_1} k^{1/2} \int_0^{1/Mk} T_2(\varphi)\, d\varphi\right) \tag{113}$$

$$= E_9 + O(E_{10}) + O(E_{11}) + O(E_{12}), \quad \text{say.}$$

We easily dispose of the remainder terms in (113) by means of (25), obtaining

$$E_{10} = O\left(M^{\frac{7}{6}+\varepsilon} \sum_{k\le M_1} 1\right) = O(M^{\frac{7}{6}+\varepsilon} M_1) = O(N^{1+\varepsilon} M^{1/6}),$$

$$E_{11} = O\left(\frac{N^{\frac{4}{3}+\varepsilon}}{M^{1/2}} \sum_{k\le M_1} \frac{q^4(k)}{k^{7/2}}\right) = O\left\{\frac{N^{\frac{4}{3}+\varepsilon}}{M^{1/2}} \prod_p \left(1 + \frac{A_{26}}{p^{5/2}} + \frac{A_{26}}{p^{5/2}}\right)\right\} = O\left(\frac{N^{\frac{4}{3}+\varepsilon}}{M^{1/2}}\right),$$

and

$$E_{12} = O\left(\frac{N^{\frac{5}{2}+\varepsilon}}{M^3} \sum_{k\le M_1} 1\right) = O\left(\frac{N^{\frac{5}{2}+\varepsilon} M_1}{M^3}\right) = O\left(\frac{N^{\frac{7}{2}+\varepsilon}}{M^4}\right).$$

Therefore, since the error involved in converting the series in E_9 into an infinite series is

$$O(N^{\frac{7}{6}+\varepsilon} M_1^{-1/6}) = O(N^{1+\varepsilon} M^{1/6}),$$

we deduce that

$$E_7 = \frac{\Gamma^5(1/3)\,\Gamma(1/2)}{\Gamma(13/6)}\, \mathfrak{S}(N)\, N^{7/6} + O(N^{\frac{7}{2}+\varepsilon} M^{-4}) \tag{114}$$

after taking detailed account of (111).

On the minor arcs,

$$g(\theta) = O(N^{\frac{1}{2}+\varepsilon} M_1^{-1/2}) + O(M^{\frac{1}{2}+\varepsilon}) = O(M^{\frac{1}{2}+\varepsilon})$$

by Lemma 14 or Lemma 1. Hence, using Theorem 1 and the well known estimate

$$\int_0^1 |f(\theta)|^4\, d\theta \leqslant \sum_{m\leqslant 2N} r_2^2(m) = O(N^{\frac{2}{3}+\varepsilon}),$$

we have

$$\begin{aligned}|E_8| &\leqslant \overline{\mathrm{bd}}_{\theta\in\mathfrak{m}}\ |g(\theta)| \int_0^1 |f(\theta)|^5\, d\theta = O\left\{M^{\frac{1}{2}+\varepsilon}\left(\int_0^1 |f(\theta)|^4\, d\theta\right)^{1/2}\left(\int_0^1 |f(\theta)|^6\, d\theta\right)^{1/2}\right\} \\ &= O(M^{1/2} N^{\frac{49}{57}+\varepsilon})\end{aligned} \tag{115}$$

on Hypothesis HW.

If M be chosen so that the O-terms in (114) and (115) contain functions of a common order, then $\delta = 301/513$ in conformity with (111). We therefore obtain

THEOREM 4. *Let $\nu(N)$ be the number of ways of representing N as the sum of a square and five non-negative cubes. Then, if Hypothesis* HW *be true, we have*

$$\nu(N) = \frac{\Gamma^5(4/3)\,\Gamma(3/2)}{\Gamma(13/6)}\,\mathfrak{S}(N)\,N^{\frac{7}{6}} + O(N^{\frac{1183}{1026}+\varepsilon}),$$

where $\mathfrak{S}(N)$ is the singular series.

Again we have a genuine asymptotic relation because it is easily shewn by standard methods that $\mathfrak{S}(N) > A_{27}$.

4. The asymptotic formula for six cubes and two biquadrates

Neither the asymptotic formula nor its derivation is as interesting for six cubes and two biquadrates as it was in the previous situations. We are therefore satisfied to state that we choose $M = [8N^{3/4}]+1$, $M_1 = N^{1/4}$ and then easily treat the major arcs by directing Lemma 15 to $f(\theta)$ and $h(\theta)$. But then the relevant integral over the minor arcs is

$$O\left\{\overline{\mathrm{bd}}_{\theta\in\mathfrak{m}}\ |f_4(\theta)|^2 \int_0^1 |f(\theta)|^6\, d\theta\right\} = O(N^{\frac{7}{16}+\frac{20}{19}+\varepsilon}) = O(N^{\frac{453}{304}+\varepsilon})$$

by Theorem 1 and Lemma 14. We can thus infer

THEOREM 5. *Let $\nu^*(N)$ be the number of ways of representing N as the sum of six non-negative cubes and two biquadrates. Then, on Hypothesis* HW,

$$\nu^*(N) = \frac{\Gamma^6(4/3)\,\Gamma^2(5/4)}{\Gamma(5/2)}\,\mathfrak{S}(N)\,N^{\frac{3}{2}} + O(N^{\frac{453}{304}+\varepsilon}),$$

where $\mathfrak{S}(N)$ is the singular series. All sufficiently large numbers are expressible in the proposed form.

In contrast with its predecessors, this theorem furnishes us with a hitherto unknown result—albeit conditional—concerning the existence of representations of all large numbers in a specified way.

5. The asymptotic formula for $r_4(n)$ for almost all numbers n

Let N be a large number (it is in fact immaterial that it be an integer) and then use the representation

$$\begin{aligned} r_4(n) &= \int_0^1 f^4(\theta)\, e^{-2\pi i n\theta}\, d\theta = \int_{\mathfrak{M}} f^4(\theta)\, e^{-2\pi i n\theta}\, d\theta + \int_{\mathfrak{m}} f^4(\theta)\, e^{-2\pi i n\theta}\, d\theta \\ &= r_4'(n, N) + r_4''(n, N), \quad \text{say,} \end{aligned} \tag{116}$$

for all positive integers n not exceeding N, where $f(\theta)$ is still $f(\theta, N^{1/3})$ and where N alone determines the major and minor arcs by means of

$$M = [6N^{2/3}+1], \quad M_1 = N^{\delta}, \quad \tfrac{1}{6} < \delta < \tfrac{1}{4}. \tag{117}$$

The first constituent $r_4'(n, N)$ in (116) is administered in the usual way, although more delicacy than before is needed in treating the singular series

$$\mathfrak{S}(n) = \sum_{k=1}^{\infty} \Omega(k, n)$$

whose terms spring from the sums $S(h, k)$ in the routine manner. Since

$$f\left(\frac{h}{k}+\varphi\right) = \frac{1}{k} S(h, k)\, J(\varphi) + O(k^{\frac{1}{2}+\varepsilon})$$

on the major arcs as before, we have

$$f^4\left(\frac{h}{k}+\varphi\right) = \frac{1}{k^4} S^4(h, k)\, J^4(\varphi) + O\left(\frac{N^{\varepsilon}}{k^{5/2}}\, q^3(k)\, T^3(\varphi)\right)$$

by (25) and (117), the latent parameter in $J(\varphi)$ and $T(\varphi)$ being still $N^{1/3}$. Therefore

$$r'(n, N) = \sum_{k \le M_1} \frac{1}{k^4} \sum_{\substack{0 \le h < k \\ (h,k)=1}} S^4(h, k)\, e^{-2\pi i n h/k} \int_{-\infty}^{\infty} J^4(\varphi)\, e^{-2\pi i n\varphi}\, d\varphi$$

$$+O\left(\sum_{k\le M_1}\frac{q^4(k)}{k^3}\int_{1/2Mk}^{\infty}T^4(\varphi)\,d\varphi\right)+O\left(N^\varepsilon\sum_{k\le M_1}\frac{q^3(k)}{k^{3/2}}\int_0^{1/Mk}T^3(\varphi)\,d\varphi\right) \tag{118}$$

$$=E_{13}+O(E_{14})+O(E_{15}),\quad \text{say.}$$

Here, by (25),

$$\begin{aligned} E_{14}&=O\left(M^{1/3}\sum_{k\le M_1}\frac{q^4(k)}{k^{8/3}}\right)=O\left(M^{1/3}M_1^{1/3}\sum_{k\le M_1}\frac{q^4(k)}{k^3}\right)\\ &=O\left\{M^{1/3}M_1^{1/3}\prod_{p\le M_1}\left(1+\frac{A_{27}}{p}\right)\right\}=O(M^{\frac13+\varepsilon}M_1^{\frac13}) \end{aligned} \tag{119}$$

and

$$\begin{aligned} E_{15}&=O\left(N^\varepsilon\sum_{k\le M_1}\frac{q^3(k)}{k^{3/2}}\right)=O\left(M_1^{1+\varepsilon}\sum_{k\le M_1}\frac{q^3(k)}{k^{5/2}}\right)\\ &=O\left\{M_1^{1+\varepsilon}\prod_{p\le M_1}\left(1+\frac{A_{28}}{p}\right)\right\}=O(M_1^{1+\varepsilon}). \end{aligned} \tag{120}$$

We calculate the effect of truncating the series in E_{13} at $k=M_1$ by means of the inequalities

$$\Omega(p^\alpha,n)=\begin{cases} O(p^{-3/2}), & \text{if } p\nmid 6n \text{ and } \alpha=1,\\ 0, & \text{if } p\nmid 6n \text{ and } \alpha>1,\\ O(p^{-3\alpha}q^4(p^\alpha)), & \text{if } p\mid 6n,\end{cases}$$

the first two of which are due to Hardy and Littlewood ([5] and [6]). As all these give

$$\begin{aligned} \sum_{k>M_1}|\Omega(k,n)|&\le M_1^{-\frac13+\varepsilon}\sum_{k=1}^{\infty}|\Omega(k,n)|\,k^{\frac13-\varepsilon}\le M_1^{-\frac13+\varepsilon}\prod_{p\nmid 6n}\left(1+\frac{A_{29}}{p^{7/6}}\right)\prod_{p\mid 6n}\left(1+\frac{A(\varepsilon)}{p^\varepsilon}\right)\\ &=O(M_1^{-\frac13+\varepsilon}), \end{aligned}$$

we infer that

$$E_{13}=\Gamma^3(4/3)\,\mathfrak{S}(n)\,n^{\frac13}+O(n^{\frac13}M_1^{-\frac13+\varepsilon})$$

from Fourier's integral theorem and the inequality $n\le N$. Together with (118), (119), and (120), this then yields

$$r_4'(n,N)=\Gamma^3(4/3)\,\mathfrak{S}(n)\,n^{\frac13}+O(M^{\frac13+\varepsilon}M_1^{\frac13}) \tag{121}$$

in virtue of (117).

Turning to the other expression $r''(n, N)$ defined in (116), we have

$$\sum_{n\leq N} |r_4''(n, N)|^2 \leq \int_{\mathfrak{m}} |f(\theta)|^8 \, d\theta$$

by Bessel's inequality, whence

$$\sum_{n\leq N} |r_4''(n, N)|^2 \leq \left(\overline{\text{bd}}_{\theta \in \mathfrak{m}} |f(\theta)| \right)^2 \int_0^1 |f(\theta)|^6 \, d\theta = O(N^{\frac{20}{19}+\frac{2}{3}+\varepsilon} M_1^{-\frac{2}{3}})$$

by Theorem 1, Lemmata 14 and 15, and (117). Hence, if $\mu = \mu_N$ be the number of integers n not exceeding N for which $r_4''(n, N) > M^{1/3} M_1^{1/3}$, then

$$\mu M^{\frac{2}{3}} M_1^{\frac{2}{3}} = O(N^{\frac{20}{19}+\frac{2}{3}+\varepsilon} M_1^{-\frac{2}{3}})$$

so that

$$\mu = O(N^{\frac{20}{19}+\frac{2}{9}+\varepsilon} M_1^{-\frac{4}{3}}).$$

Thus certainly $\mu = o(N)$ whenever we choose M_1 to be N^{α} with exponent a exceeding 47/228. Taken with (116) and (121), this fact means that

$$r_4(n) = \Gamma^3(4/3)\, \mathfrak{S}(n)\, n^{\frac{1}{3}} + O(N^{\frac{199}{684}+\varepsilon})$$

for all integers n not exceeding N save possibly for $o(N)$ exceptions. Since we may obviously replace N by n in the remainder term without jeopardising the conclusion, we obtain

THEOREM 6. *Let $r_4(n)$ be the number of representations of n as the sum of four non-negative cubes. Then, on Hypothesis* HW,

$$r_4(n) = \Gamma^3(4/3)\, \mathfrak{S}(n)\, n^{\frac{1}{3}} + O(n^{\frac{199}{684}+\varepsilon})$$

for almost all integers n.

Somewhat similarly, by choosing $M_1 = N^{\frac{1}{3}-\varepsilon}$ and using Lemma 14, we arrive at the parallel

THEOREM 7. *On Hypothesis* HW, *we have*

$$r_4(n) \sim \Gamma^3(4/3)\, \mathfrak{S}(n)\, n^{1/3}$$

as $n \to \infty$ through some sequence of integers that up to any limit N omits at most

$$O(N^{\frac{101}{114}+\varepsilon})$$

values. Thus all positive integers not exceeding N are sums of four non-negative cubes save possibly for

$$O(N^{\frac{101}{114}+\varepsilon})$$

exceptional values.

Each result contained in these theorems represents a conditional improvement in some direction of Davenport's work on four cubes [2].

6. The integers that are sums of three non-negative cubes

If $\varrho(x)$ denote the number of positive integers not exceeding x that are equal to a sum of three non-negative cubes, then the Cauchy-Schwarz inequality implies that

$$\left(\sum_{n\leq x} r_3(n)\right)^2 \leq \varrho(x) \sum_{n\leq x} r_3^2(n),$$

in which

$$\sum_{n\leq x} r_3(n) \sim \Gamma^3(4/3)\, x.$$

Hence, since

$$\sum_{n\leq x} r_3^2(n) = O(x^{\frac{20}{19}+\varepsilon})$$

by[2] (5), (30), and (100), we deduce the final

THEOREM 8. *On Hypothesis* HW, *we have*

$$\varrho(x) > x^{\frac{18}{19}-\varepsilon}$$

for $x > x_0(\varepsilon)$.

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[2] Or, equivalently, by Theorem 1 and the analogue of (5) with $R(x)$ replacing $R^*(8x)$.

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