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Holomorphic representation theory II

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Introduction

The starting point in the theory of holomorphic extensions of unitary representations was Ol'shanskiĭ's observation that, if W is a pointed generating invariant cone in a simple Lie algebra $\mathfrak{g}$, G a corresponding linear connected group, and $G_{\mathbf{C}}$ its universal complexification, then the set $S_W = G\exp(iW)$ is a closed subsemigroup of $G_{\mathbf{C}}$ ([O]). This theorem has been generalized by Hilgert and Ólafsson to solvable groups ([HO]) and the most general result of this type, due to Lawson ([La]), is that if $G_{\mathbf{C}}$ is a complex Lie group with an antiholomorphic involution inducing the complex conjugation on $\mathfrak{g}_{\mathbf{C}} = \mathbf{L}(G_{\mathbf{C}})$, then the set $S_W = G\exp(iW)$ is a closed subsemigroup of $G_{\mathbf{C}}$. The class of semigroups obtained by this construction is not sufficient for many applications in representation theory. For instance Howe's oscillator semigroup (cf. [How]) is a 2-fold covering of such a semigroup, but it does not fit into any group. In [Ne6] we have shown that given a Lie algebra $\mathfrak{g}$, a generating invariant convex cone $W \subseteq \mathfrak{g}$, and a discrete central subgroup of the simply connected group corresponding to the Lie algebra $\mathfrak{g} + i(W \cap (-W))$ which is invariant under complex conjugation, there exists a semigroup $S = \Gamma(\mathfrak{g}, W, D)$ called the *Ol'shanskiĭ semigroup* defined by this data. This semigroup is the quotient $\widetilde{S}/D$, where $\widetilde{S}$ is the universal covering semigroup of S (cf. [Ne3]) and $D \cong \pi_1(S)$ is a discrete central subgroup of $\widetilde{S}$. Moreover, the semigroup $\widetilde{S}$, also denoted $\Gamma(\mathfrak{g}, W)$ can be obtained as the universal covering semigroup of the subsemigroup $\langle \exp(\mathfrak{g} + iW)\rangle$ of the simply connected complex Lie group $G_{\mathbf{C}}$ with Lie algebra $\mathfrak{g}_{\mathbf{C}}$.

A *holomorphic representation* of a complex Ol'shanskiĭ semigroup S is a weakly continuous monoid morphism $\pi: S \to B(\mathcal{H})$ into the algebra of bounded operators on a Hilbert space $\mathcal{H}$ such that π is holomorphic on the interior $\mathrm{int}(S)$ of S and π is *involutive*, i.e., $\pi(s^*) = \pi(s)^*$ holds for all $s \in S$. This set is a dense semigroup ideal which is a complex manifold. One can think of representations of S as analytic continuations of unitary representations of the subgroup $U(S) = \{s \in S : s^* s = \mathbf{1}\}$ of unitary elements in S.

In this paper we consider the two principal problems of representation theory for this setting:

(P1) Describe the irreducible holomorphic representations of S.

(P2) Decompose a holomorphic representation of S into irreducible representations.

We will obtain a complete solution of (P2) under the assumption that $\mathfrak{g}$ is a *(CA) Lie algebra*, i.e., the group of inner automorphisms of $\mathfrak{g}$ is closed in the group $\mathrm{Aut}(\mathfrak{g})$ of all automorphisms of $\mathfrak{g}$. As we will see in Section IV, this condition is a rather natural one since it entails that every connected group G with $\mathbf{L}(G)=\mathfrak{g}$ is a type I group.

In Section I we will prove a criterion which makes it rather easy to check whether a given Lie algebra is (CA) or not. We recall in particular that a Lie algebra $\mathfrak{g}$ is (CA) if and only if its radical has this property (cf. [vE3]).

Let us say that a subalgebra $\mathfrak{a}\subseteq\mathfrak{g}$ is *compactly embedded* if the group generated by $e^{\mathrm{ad}\,\mathfrak{a}}$ has compact closure in $\mathrm{Aut}(\mathfrak{g})$. In Section II we investigate highest weight modules for Lie algebras containing a compactly embedded Cartan algebra $\mathfrak{t}$. This section is purely Lie algebraic. It contains some generalizations of results which are well known for semisimple Lie algebras.

In the third section we turn to unitary representations of a Lie group G with Lie algebra $\mathfrak{g}$, where $\mathfrak{g}$ is a (CA) Lie algebra. Let $\mathfrak{k}\subseteq\mathfrak{g}$ be a maximal compactly embedded subalgebra of $\mathfrak{g}$ and $K=\exp\mathfrak{k}$ the corresponding subgroup of G. Then we show that for every irreducible representation $(\pi,\mathcal{H})$ of G the space $\mathcal{H}^{K,\omega}$ of K-finite analytic vectors is dense in $\mathcal{H}$. Note that we do not assume that the group K is compact, we only have that it is compact modulo the center of G (cf. Section I). These results generalize well known facts from the representation theory of real reductive Lie groups (cf. [Wal1], [War]). The crucial observation is that by using K-invariant heat kernels on G it is possible to approximate elements in $\mathcal{H}$ in a K-equivariant way. So far these results are purely group theoretic and do not concern holomorphic extensions.

Next we combine these results with the fact that for a holomorphic representation $(\pi,\mathcal{H})$ of the Ol'shanskiĭ semigroup S all the self-adjoint operators $id\pi(X)$, $X\in W$ have a spectrum which is bounded from above (cf. [Ne6]). We use this observation to show that for every irreducible representation $(\pi,\mathcal{H})$ of the Ol'shanskiĭ semigroup S the space $\mathcal{H}^K$ is a highest weight module of the Lie algebra $\mathfrak{g}_{\mathbb{C}}$ and that $\pi(\mathrm{int}\,S)$ consists of trace class operators.

In Section IV we apply these results to obtain a rather satisfactory disintegration and character theory for Ol'shanskiĭ semigroups. It is based on the C^*-algebras defined in [Ne6] which, in view of the insights from Section III, turn out to be liminal.

The best known examples for representations which fit into this theory are the irreducible representations of compact Lie groups, the *holomorphic discrete series* representations of simple Hermitean Lie groups, the *metaplectic representation* of the 2-fold cover $H_n \rtimes \mathrm{Mp}(n, \mathbf{R})$ of $H_n \rtimes \mathrm{Sp}(n, \mathbf{R})$, where H_n denotes the $(2n+1)$-dimensional Heisenberg group, and the *oscillator representation* of the $(2n+2)$-dimensional oscillator group. Other examples are the *ladder representations* of the subgroups of $\mathrm{Mp}(n, \mathbf{R})$ obtained by restriction of the metaplectic representations.

In a subsequent paper we will obtain a classification of the irreducible representations and we will show that the holomorphic representations separate the points if and only if $H(W)$ is a compact Lie algebra and the Lie algebra $\mathfrak{g} \oplus \mathbf{R}$ contains a pointed generating invariant cone (cf. [Ne4]).

I. (CA) Lie algebras and groups

Definition I.1. A finite dimensional real Lie algebra $\mathfrak{g}$ is said to be a *(CA) Lie algebra* if the group $\mathrm{Inn}_{\mathfrak{g}} := \langle e^{\mathrm{ad}\, \mathfrak{g}} \rangle$ of inner automorphisms is closed. We say that a connected Lie group G is a (CA) Lie group if its Lie algebra has this property.

This notion has first been introduced by van Est [vE1] who proved for example that $\mathfrak{g}$ is (CA) if and only if its radical is (CA). A related fact is that a connected Lie group G has the property that every injective homomorphism into another Lie group is closed if and only if $\mathbf{L}(G)$ is a (CA) Lie algebra and the center of G is compact ([Go]). One can also construct (CA)-hulls of given Lie algebras with appropriate universal properties (cf. [Z]). In this section our approach will be via compactly embedded abelian subalgebras (cf. [HN1, Chapter 8], [Ste]).

Let $\mathfrak{g}$ be a finite dimensional real Lie algebra, $\mathfrak{r} := \mathrm{Rad}(\mathfrak{g})$ the radical of $\mathfrak{g}$, $\mathfrak{k} \subseteq \mathfrak{g}$ a maximal compactly embedded subalgebra, and $\mathfrak{t} \subseteq \mathfrak{k}$ a Cartan algebra, i.e., a maximal compactly embedded abelian subalgebra of $\mathfrak{g}$.

According to [HN1, III.7.15] we find a Levi subalgebra $\mathfrak{s} \subseteq \mathfrak{g}$ with the following properties:

(S1) $[\mathfrak{k}, \mathfrak{s}] \subseteq \mathfrak{s}$,

(S2) $[\mathfrak{k} \cap \mathfrak{r}, \mathfrak{s}] = \{0\}$,

(S3) $\mathfrak{k} = (\mathfrak{k} \cap \mathfrak{r}) \oplus (\mathfrak{k} \cap \mathfrak{s})$,

(S4) $\mathfrak{k}' \subseteq \mathfrak{s}$, and

(S5) $\mathfrak{k}_{\mathfrak{s}} := \mathfrak{k} \cap \mathfrak{s}$ is maximal compactly embedded in $\mathfrak{s}$.

From (S2) and (S3) we infer that $\mathfrak{k}_{\mathfrak{r}} := \mathfrak{k} \cap \mathfrak{r} \subseteq Z(\mathfrak{k})$, so that $\mathfrak{k}_{\mathfrak{r}} \subseteq \mathfrak{t}$. It follows that $\mathfrak{t} = \mathfrak{t}_{\mathfrak{r}} \oplus \mathfrak{t}_{\mathfrak{s}}$, where $\mathfrak{t}_{\mathfrak{r}} := \mathfrak{k}_{\mathfrak{r}}$ and $\mathfrak{t}_{\mathfrak{s}} := \mathfrak{t} \cap \mathfrak{s}$ is maximal compactly embedded abelian in $\mathfrak{s}$. For a connected Lie group G with $\mathbf{L}(G) = \mathfrak{g}$ we define $R := \langle \exp \mathfrak{r} \rangle$, $S := \langle \exp \mathfrak{s} \rangle$, $T := \exp \mathfrak{t}$, and $K := \exp \mathfrak{k}$.

PROPOSITION I.2. *The following conditions are equivalent:*

(1) *The Lie algebra* $\mathfrak{g}$ *is a (CA) Lie algebra.*

(2) $e^{\mathrm{ad}\,\mathfrak{t}}$ *is closed.*

(3) $e^{\mathrm{ad}\,\mathfrak{t}_\mathfrak{r}}$ *is closed.*

(4) *The radical* $\mathfrak{r}$ *of* $\mathfrak{g}$ *is a (CA) Lie algebra.*

Proof. Let G be a Lie group with $\mathbf{L}(G)=\mathfrak{g}$.

(1) $\Rightarrow$ (2): The condition that $\mathfrak{g}$ is (CA) means that $\mathrm{Ad}(G)$ is closed. The subalgebra $\mathfrak{t}' := \mathrm{ad}^{-1}(\mathbf{L}(\overline{e^{\mathrm{ad}\,\mathfrak{t}}}))$ is a compactly embedded subalgebra of $\mathfrak{g}$ containing $\mathfrak{t}$ as a central subalgebra. According to the maximality of $\mathfrak{t}$ as a compactly embedded abelian subalgebra, we see that $\mathfrak{t}'=\mathfrak{t}$, so that $e^{\mathrm{ad}\,\mathfrak{t}}$ is closed because $\mathrm{Ad}(G)$ is closed.

(2) $\Rightarrow$ (1): This follows from [HN1, III.8.14].

(1) $\Rightarrow$ (3): Since R is the radical of G, the group $\mathrm{Ad}(R)$ is the radical of $\mathrm{Ad}(G)$, and therefore closed. Hence (2) implies that the group $\mathrm{Ad}(R) \cap e^{\mathrm{ad}\,\mathfrak{t}}$ which has the Lie algebra $\mathrm{ad}(\mathfrak{r} \cap \mathfrak{t}) = \mathrm{ad}\,\mathfrak{k}_\mathfrak{r}$ is closed. It follows that $e^{\mathrm{ad}\,\mathfrak{k}_\mathfrak{r}}$ is closed.

(3) $\Rightarrow$ (2): Since $\mathfrak{t}_\mathfrak{s} \subseteq \mathfrak{s}$ is maximal compactly embedded abelian, $e^{\mathrm{ad}\,\mathfrak{t}_\mathfrak{s}}$ is a maximal torus in $\mathrm{Ad}(S)$, hence it is closed (cf. [HN1, III.6.16]). Now

$$e^{\mathrm{ad}\,\mathfrak{t}} = e^{\mathrm{ad}\,\mathfrak{k}_\mathfrak{r}} e^{\mathrm{ad}\,\mathfrak{t}_\mathfrak{s}}$$

is the product of two compact groups, so it is compact.

(1) $\Leftrightarrow$ (4): [vE3, Theorems 2, 2a]. □

Note that the subalgebra $\mathfrak{t}_\mathfrak{r}$ is remarkably small in $\mathfrak{g}$, so that (3) in the preceding proposition is a condition which is fairly easy to check. Note also that $\mathfrak{t}_\mathfrak{r}$ need not be maximal compactly embedded in $\mathfrak{r}$. This is false for the example $\mathbf{R}^2 \rtimes \mathfrak{sl}(2,\mathbf{R})$, where the action of $\mathfrak{sl}(2,\mathbf{R})$ is the usual one. Here $\mathfrak{r}=\mathbf{R}^2$ and $\mathfrak{t}_\mathfrak{r}=\{0\}$.

The following corollary describes a property of (CA) Lie groups which will be crucial in the sequel.

COROLLARY I.3. *Let* $T=\exp\mathfrak{t}$ *and* $K=\exp\mathfrak{k}$ *be the analytic subgroups corresponding to* $\mathfrak{t}$ *and* $\mathfrak{k}$. *Then* $Z(G)\subseteq T$ *and the following are equivalent:*

(i) G *is a (CA) Lie group,*

(ii) $K/Z(G)$ *is compact, and*

(iii) $T/Z(G)$ *is a torus.*

Proof. It follows from [HN1, III.7.11] that $Z(G)\subseteq T$.

(i) $\Rightarrow$ (ii): The subalgebra $\tilde{\mathfrak{k}} := \mathrm{ad}^{-1}(\mathbf{L}(\overline{\mathrm{Ad}(K)}))$ is compactly embedded. Hence $\tilde{\mathfrak{k}}=\mathfrak{k}$ by maximality. Now the closedness of $\mathrm{Ad}(G)$ entails that

$$\overline{\mathrm{Ad}(K)} = \langle e^{\mathrm{ad}\,\mathfrak{k}} \rangle = \mathrm{Ad}(K) \cong K/Z(G)$$

is compact.

(ii) $\Rightarrow$ (iii): If $\mathrm{Ad}(K) \cong K/Z(G)$ is compact, then $\mathrm{Ad}(T) \cong T/Z(G)$ is a maximal torus in $K/Z(G)$ because $\mathrm{ad}\, \mathfrak{t}$ is a Cartan subalgebra of $\mathrm{ad}\, \mathfrak{k}$.

(iii) $\Rightarrow$ (i): In view of $T/Z(G) \cong e^{\mathrm{ad}\, \mathfrak{t}}$, this implication follows from Proposition I.2. □

Let G be a locally compact group and μ_G a left Haar measure on G. Then $C^*(G)$ is defined to be enveloping C^*-algebra of the Banach $*$-algebra $L^1(G, \mu_G)$. The following result shows that the (CA) property of a Lie group implies that it does not have a "wild" representation theory (cf. Remark IV.13).

THEOREM I.4. *Let G be a (CA) Lie group such that $\mathfrak{g}$ contains a compactly embedded Cartan algebra. Then $C^*(G)$ is a C^*-algebra of type* I.

Proof. In view of [D2, 2.1], we only have to prove that G is locally isomorphic to a real linear algebraic group, i.e., that $\mathfrak{g}$ is the Lie algebra of a real linear algebraic group. According to [Hoch2], it even suffices to show that $\mathrm{ad}\, \mathfrak{g} \subseteq \mathrm{End}(\mathfrak{g})$ is an algebraic Lie algebra. Since the group $\mathrm{Inn}_{\mathfrak{g}} = \langle e^{\mathrm{ad}\, \mathfrak{g}} \rangle$ is closed, it contains the torus $T := \overline{e^{\mathrm{ad}\, \mathfrak{t}}}$ and therefore it is almost algebraic by Corollary II.27 in [Ne7]. This means that it is the $\mathbf{1}$-component of a real algebraic group and in particular that $\mathrm{ad}\, \mathfrak{g}$ is algebraic. □

In Section III we will investigate irreducible unitary representations of (CA) Lie groups. Note that this class includes in particular all reductive Lie groups and the class of $(2n+2)$-dimensional oscillator groups.

II. Highest weight modules

In this section we collect some generalities on highest weight modules of complex Lie algebras which are complexifications of real Lie algebras containing compactly embedded Cartan algebras. Since we do not assume that the Lie algebra in question is semisimple, we will have to prove some of the classical results which are well known for semisimple complex Lie algebras in a more general setting.

In this section $\mathfrak{g}$ denotes a finite dimensional real Lie algebra containing a compactly embedded Cartan subalgebra $\mathfrak{t}$. Associated to the Cartan subalgebra $\mathfrak{t}_{\mathbb{C}}$ in the complexification $\mathfrak{g}_{\mathbb{C}}$ is a root decomposition as follows (cf. Theorem III.4 in [Ne6]). For a linear functional $\lambda \in \mathfrak{t}_{\mathbb{C}}^*$ we set

$$\mathfrak{g}_{\mathbb{C}}^{\lambda} := \{X \in \mathfrak{g}_{\mathbb{C}} : (\forall Y \in \mathfrak{t}_{\mathbb{C}})\ [Y, X] = \lambda(Y)X\}$$

and

$$\Delta := \Delta(\mathfrak{g}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}}) := \{\lambda \in \mathfrak{t}_{\mathbb{C}}^* \setminus \{0\} : \mathfrak{g}_{\mathbb{C}}^{\lambda} \neq \{0\}\}.$$

Then

$$\mathfrak{g}_{\mathbf{C}} = \mathfrak{t}_{\mathbf{C}} \oplus \bigoplus_{\lambda \in \Delta} \mathfrak{g}_{\mathbf{C}}^{\lambda},$$

$\lambda(\mathfrak{t}) \subseteq i\mathbf{R}$ for all $\lambda \in \Delta$ and $\sigma(\mathfrak{g}_{\mathbf{C}}^{\lambda}) = \mathfrak{g}_{\mathbf{C}}^{-\lambda}$, where σ denotes complex conjugation on $\mathfrak{g}_{\mathbf{C}}$ with respect to $\mathfrak{g}$. Let $\mathfrak{k} \supseteq \mathfrak{t}$ denote a maximal compactly embedded subalgebra. Then a root is said to be *compact* if $\mathfrak{g}_{\mathbf{C}}^{\lambda} \subseteq \mathfrak{k}_{\mathbf{C}}$. We write Δ_k for the set of compact roots and Δ_p for the set of noncompact roots.

Positive and parabolic systems of roots

Definition II.1. (a) A subset $\Delta^+ \subseteq \Delta$ is called a *positive system* if there exists $X_0 \in i\mathfrak{t}$ such that

$$\Delta^+ = \{\lambda \in \Delta : \lambda(X_0) > 0\}.$$

A positive system is said to be $\mathfrak{k}$-*adapted* if

$$\lambda(X_0) > \mu(X_0) \quad \forall \mu \in \Delta_k, \ \lambda \in \Delta_p^+.$$

Let $\Delta^+ \subseteq \Delta$ be a positive system of roots. For a subset M of a vector space V we write $\operatorname{cone}(M)$ for the smallest closed convex cone containing M and for a cone C in V the set $C^\star := \{\nu \in V^* : \nu(C) \subseteq \mathbf{R}^+\}$ is called the *dual cone.*

We define the *maximal cone* and the *minimal cone*

$$C_{\max} := C_{\max}(\Delta^+) := (i\Delta_p^+)^\star \subseteq \mathfrak{t},$$
$$C_{\min} := C_{\min}(\Delta^+) := \operatorname{cone}\{i[\overline{X}, X] : X \in \mathfrak{g}_{\mathbf{C}}^{\lambda}, \lambda \in \Delta_p^+\} \subseteq \mathfrak{t}.$$

(b) A subset $\Sigma \subseteq \Delta$ is called *parabolic* if there exists $E \in i\mathfrak{t}$ such that

$$\Sigma = \{\lambda \in \Delta : \lambda(E) \geqslant 0\}.$$

Note that this definition generalizes the notion of a parabolic set of roots in the root system of a complex semisimple Lie algebra (cf. [Bou2, Chapter VI, §1, No. 7, Proposition 20]). A subset $\Sigma \subseteq \Delta$ is said to be *closed* if

$$(\Sigma + \Sigma) \cap \Delta \subseteq \Sigma.$$

Note that a parabolic subset is automatically closed.

(c) For a closed subset $\Sigma \subseteq \Delta$ we set

$$\mathfrak{g}_{\mathbf{C}}(\Sigma) := \mathfrak{t}_{\mathbf{C}} \oplus \bigoplus_{\lambda \in \Sigma} \mathfrak{g}_{\mathbf{C}}^{\lambda}.$$

Note that the closedness condition implies that $\mathfrak{g}_{\mathbf{C}}(\Sigma)$ is a subalgebra of $\mathfrak{g}_{\mathbf{C}}$. □

LEMMA II.2. *Let* $\Sigma=\{\lambda\in\Delta:\lambda(E)\geqslant 0\}$ *be a parabolic subset. Then the following assertions hold*:

(i) *The set* Σ *contains a positive system.*

(ii) *Let*

$$\Sigma^0:=\Sigma\cap-\Sigma=\{\lambda\in\Delta:\lambda(E)=0\}$$

and

$$\Sigma^+:=\Sigma\setminus-\Sigma=\{\lambda\in\Delta:\lambda(E)>0\}.$$

Then $(\Sigma+\Sigma^+)\cap\Sigma\subseteq\Sigma^+$.

(iii) *We define*

$$\mathfrak{p}_\Sigma:=\mathfrak{g}_{\mathbf{C}}(\Sigma),\quad \mathfrak{s}_\Sigma:=\mathfrak{g}_{\mathbf{C}}(\Sigma^0),\quad \textit{and}\quad \mathfrak{n}_\Sigma:=\bigoplus_{\alpha\in\Sigma^+}\mathfrak{g}_{\mathbf{C}}^\alpha.$$

Then the following assertions hold:

(a) $\mathfrak{n}_\Sigma$ *is a nilpotent ideal in* $\mathfrak{p}_\Sigma$.

(b) $\mathfrak{s}_\Sigma=\mathfrak{p}_\Sigma\cap\overline{\mathfrak{p}_\Sigma}$ *is a subalgebra of* $\mathfrak{p}_\Sigma$.

(c) $\mathfrak{p}_\Sigma=\mathfrak{n}_\Sigma\rtimes\mathfrak{s}_\Sigma$.

Proof. (i) The set of all elements in $i\mathfrak{t}$ where no root vanishes is open and dense. Therefore we find an element E' in this set such that $\lambda(E')>0$ for all $\lambda\in\Sigma^+$. Then $\Delta^+:=\{\lambda\in\Delta:\lambda(E')>0\}$ is a positive system contained in Δ.

(ii) This follows immediately from the definitions.

(iii) This is a consequence of (ii) and the fact that $\overline{\mathfrak{g}_{\mathbf{C}}^\lambda}=\mathfrak{g}_{\mathbf{C}}^{-\lambda}$. □

Definition II.3. Let $\mathfrak{t}\subseteq\mathfrak{g}$ be a compactly embedded Cartan algebra and $\mathfrak{k}\supseteq\mathfrak{t}$ the unique maximal compactly embedded subalgebra ([HHL, A.2.40]). We write $\mathcal{W}_\mathfrak{k}$ for the *Weyl group* of the compact Lie algebra $\mathfrak{k}$. This is the group of linear transformations of $\mathfrak{t}$ generated by the orthogonal reflections in the hyperplanes $\ker\omega$, $\omega\in\Delta_k$, with respect to the restriction of a $\mathfrak{k}$-invariant scalar product to $\mathfrak{t}$ (cf. [Ne4, III.1]). The connected components of the set

$$\mathfrak{t}\setminus\bigcup_{\alpha\in\Delta_k}\ker\alpha$$

are called the Δ_k*-chambers*. The Weyl group $\mathcal{W}_\mathfrak{k}$ acts simply transitive on the set of Δ_k-chambers ([Bou2, Chapter V, §3, No. 2, Theorem 1]). □

LEMMA II.4. *Let* $\mathfrak{k}\subseteq\mathfrak{g}$ *be maximal compactly embedded with* $\mathfrak{t}\subseteq\mathfrak{k}$. *Then the following assertions hold*:

(i) $Z(\mathfrak{k})=\{X\in\mathfrak{t}:(\forall\gamma\in\mathcal{W}_\mathfrak{k})\ \gamma(X)=X\}$.

(ii) *Let $C\subseteq\mathfrak{t}$ be a generating $\mathcal{W}_{\mathfrak{k}}$-invariant cone and $p_Z\colon\mathfrak{t}\to Z(\mathfrak{k})$ the orthogonal projection along $\mathfrak{t}\cap[\mathfrak{k},\mathfrak{k}]$. Then*

$$C\cap Z(\mathfrak{k})=p_Z(C) \quad \textit{and} \quad p_Z(\operatorname{int} C)=\operatorname{int} p_Z(C)\neq\varnothing.$$

(iii) *If W is an invariant wedge in $\mathfrak{g}$ and $p\colon\mathfrak{g}\to\mathfrak{t}$ the orthogonal projection along $[\mathfrak{t},\mathfrak{g}]=\mathfrak{g}_{\mathrm{eff}}$, then*

$$p(W)=W\cap\mathfrak{t}$$

is a $\mathcal{W}_{\mathfrak{k}}$-invariant cone in $\mathfrak{t}$.

(iv) *$(\Delta_p+\Delta_k)\cap\Delta\subseteq\Delta_p$ and Δ_p is invariant under the Weyl group $\mathcal{W}_{\mathfrak{k}}$.*

(v) *Let $X\in Z(\mathfrak{k})$ and $\Sigma_k\subseteq\Delta_k$ be a parabolic system. Then every neighborhood of X contains an element Y with $\Sigma_k=\{\alpha\in\Delta_k:\alpha(Y)\geqslant 0\}$.*

Proof. (i) [Ne4, III.8].

(ii) In view of (i), this is a consequence of Theorem I.10 in [Ne4] because the group $\mathcal{W}_{\mathfrak{k}}$ is finite and therefore compact.

(iii) [Ne4, III.7].

(iv) Since $\mathfrak{k}$ acts semisimply on $\mathfrak{g}$, there exists a $\mathfrak{k}$-invariant subspace $\mathfrak{p}\subseteq\mathfrak{g}$ with $\mathfrak{g}=\mathfrak{k}\oplus\mathfrak{p}$. Then $\mathfrak{t}\subseteq\mathfrak{k}$ entails that

$$\mathfrak{k}_{\mathbf{C}}=\mathfrak{t}_{\mathbf{C}}\oplus\bigoplus_{\lambda\in\Delta_k}\mathfrak{g}_{\mathbf{C}}^{\lambda} \quad\text{and}\quad \mathfrak{p}_{\mathbf{C}}=\bigoplus_{\lambda\in\Delta_p}\mathfrak{g}_{\mathbf{C}}^{\lambda}.$$

Moreover, the $\mathfrak{k}$-invariance of $\mathfrak{p}$ implies that for $\lambda\in\Delta_p$ and $\mu\in\Delta_k$ the sum $\lambda+\mu$ is noncompact whenever

$$[\mathfrak{g}_{\mathbf{C}}^{\lambda},\mathfrak{g}_{\mathbf{C}}^{\mu}]\neq\{0\}.$$

For $\gamma\in\mathcal{W}_{\mathfrak{k}}$ there exists $k\in N_K(\mathfrak{t}_{\mathbf{C}})=\{k\in K:\operatorname{Ad}(k)\mathfrak{t}=\mathfrak{t}\}$ such that $\gamma=\operatorname{Ad}(k)|_{\mathfrak{t}}$ ([Ne4, III.1]). Hence

$$\operatorname{Ad}(k)\mathfrak{g}_{\mathbf{C}}^{\alpha}=\mathfrak{g}_{\mathbf{C}}^{\gamma.\alpha}=\mathfrak{g}_{\mathbf{C}}^{\alpha\circ\gamma^{-1}}$$

and therefore $\alpha\circ\gamma^{-1}\in\Delta_p$ whenever $\alpha\in\Delta_p$ and similarly for $\alpha\in\Delta_k$.

(v) Since we may without loss of generality assume that $Z(\mathfrak{k})=\{0\}$ and therefore that $X=0$, this follows from ([Bou2, Chapter V, §3, No. 3, Theorem 2]). ☐

PROPOSITION II.5. *For a positive system $\Delta^+\subseteq\Delta$ the following are equivalent:*

(1) *Δ^+ is $\mathfrak{k}$-adapted.*

(2) *Δ_p^+ is $\mathcal{W}_{\mathfrak{k}}$-invariant.*

(3) *$C_{\max}(\Delta^+)$ is invariant under $\mathcal{W}_{\mathfrak{k}}$.*

(4) *$\Delta_k\cup\Delta_p^+$ is a parabolic set of roots.*

Proof. (1) ⇒ (2): Pick $X_0 \in it$ such that

$$\lambda(X_0) > \mu(X_0) > 0$$

holds for all $\lambda \in \Delta_p^+$ and $\mu \in \Delta_k^+$. Set $\mathfrak{p}^+ := \bigoplus_{\lambda \in \Delta_p^+} \mathfrak{g}_{\mathbf{C}}^\alpha$. For $\lambda \in \Delta_p^+$ and $\mu \in \Delta_k$ we have that $(\lambda \pm \mu)(X_0) > 0$ so that $[\mathfrak{k}_{\mathbf{C}}, \mathfrak{p}^+] \subseteq \mathfrak{p}^+$. Hence $\mathrm{Ad}(K)\mathfrak{p}^+ = \mathfrak{p}^+$ and consequently $\mathcal{W}_{\mathfrak{k}}.\Delta_p^+ = \Delta_p^+$.

(2) ⇒ (3): If Δ_p^+ is invariant under $\mathcal{W}_{\mathfrak{k}}$, then the same holds for $i\Delta_p^+$ and for the cone dual to this set which is $C_{\max}$.

(3) ⇒ (4): According to Lemma II.4 (ii), there exists $X \in \mathrm{int}\, C_{\max} \cap Z(\mathfrak{k})$. Then

$$\Delta_k \cup \Delta_p^+ = \{\alpha \in \Delta : \alpha(X) \geqslant 0\}$$

is a parabolic system of roots.

(4) ⇒ (1): Pick $X \in it$ such that

$$\Delta_k \cup \Delta_p^+ = \{\alpha \in \Delta : \alpha(X) \geqslant 0\}.$$

Then every compact root vanishes on X and every noncompact positive root is positive on X. Hence there exists X_1 near to X such that

$$\lambda(X_1) > \mu(X_1) > 0 \quad \forall \mu \in \Delta_k^+,\ \lambda \in \Delta_p^+.$$

This means that Δ^+ is $\mathfrak{k}$-adapted. □

LEMMA II.6. *Let $\Delta^+ \subseteq \Delta$ be a $\mathfrak{k}$-adapted positive system. Then a subset $\Sigma \subseteq \Delta$ with $\Sigma \cap \Delta_p = \Delta_p^+$ is parabolic if and only if $\Sigma \cap \Delta_k$ is parabolic.*

Proof. "⇒": Pick $X_1 \in \mathrm{int}\, C_{\max} \cap Z(\mathfrak{k})$ (Lemma II.4 (ii), (iv), Proposition II.5). Then every compact root vanishes on X_1 and hence we find with Lemma II.4 (v) an element $X_0 \in \mathrm{int}\, C_{\max}$ arbitrary near to X_1 such that $\Sigma \cap \Delta_k = \{\alpha \in \Delta_k : \alpha(X_0) \geqslant 0\}$. Hence

$$\Sigma = \{\alpha \in \Delta : \alpha(X_0) \geqslant 0\}$$

is parabolic.

"⇐": This is trivial because $\mathfrak{t} \subseteq \mathfrak{k}$. □

PROPOSITION II.7. *The following are equivalent:*

(1) *There exists a $\mathfrak{k}$-adapted positive system.*

(2) $Z(Z(\mathfrak{k})) = \mathfrak{k}$.

Proof. (1) $\Rightarrow$ (2): If Δ^+ is a $\mathfrak{k}$-adapted positive system, then $\Sigma := \Delta_k \cup \Delta_p^+$ is a parabolic system (Proposition II.5) and therefore we find an element $X \in i\mathfrak{t}$ such that

$$\alpha(X) = 0 \quad \forall \alpha \in \Delta_k,$$

and

$$\alpha(X) > 0 \quad \forall \alpha \in \Delta_p^+.$$

Then $X \in Z(\mathfrak{k})$ and $Z(Z(\mathfrak{k})) \subseteq \ker \operatorname{ad} X \subseteq \mathfrak{k}$ follows from the fact that $(\ker \operatorname{ad} X)_{\mathbf{C}}$ contains no root space for a noncompact root.

(2) $\Rightarrow$ (1): If $Z(Z(\mathfrak{k})) = \mathfrak{k}$, then there exists $X \in Z(\mathfrak{k})$ such that $\ker \operatorname{ad} X = \mathfrak{k}$. Then

$$\Sigma := \{\alpha \in \Delta : \alpha(iX) \geqslant 0\}$$

is a parabolic system of roots which contains Δ_k. Let $\Delta^+ \subseteq \Sigma$ be a positive system. Then $\Sigma = \Delta_k \cup \Delta_p^+$ is parabolic so that Δ^+ is $\mathfrak{k}$-adapted by Proposition II.5 (4). □

We recall that Lie algebras with a compactly embedded Cartan algebra satisfying the two equivalent conditions of Proposition II.7 are called *quasi-Hermitean.*

Highest weight modules

Definition II.8. Let $\Delta^+ \subseteq \Delta$ denote a positive system.

(a) We set $\mathfrak{b} := \mathfrak{b}(\Delta^+) := \mathfrak{p}_{\Delta^+}$, $\mathfrak{g}_{\mathbf{C}}^+ := [\mathfrak{b}, \mathfrak{b}]$, and $\mathfrak{g}_{\mathbf{C}}^- := \overline{\mathfrak{g}_{\mathbf{C}}^+}$.

(b) Let V be a $\mathfrak{g}_{\mathbf{C}}$-module and $v \in V$. We say that v is a *primitive element of* V (with respect to Δ^+) if $v \neq 0$ and $\mathfrak{b}.v \subseteq \mathbf{C}.v$.

(c) For a $\mathfrak{g}$-module V and $\lambda \in \mathfrak{t}_{\mathbf{C}}^*$ we set

$$V^\lambda := \{v \in V : (\forall X \in \mathfrak{t}_{\mathbf{C}})\ X.v = \lambda(X)v\}.$$

This space is called the *weight space of weight* λ and λ is said to be a *weight* of V is $V^\lambda \neq \{0\}$. We write $\mathcal{P}_V$ for the set of weights of V.

(d) A $\mathfrak{g}_{\mathbf{C}}$-module V is called a *highest weight module* with highest weight λ (with respect to Δ^+) if it is generated by a primitive element of weight λ.

(e) A $\mathfrak{g}$-module V is called $\mathfrak{k}$*-finite* if it consists of $\mathfrak{k}$-finite vectors, i.e., if $\dim \mathcal{U}(\mathfrak{k}_{\mathbf{C}}).v < \infty$ for all $v \in V$.

(f) Let V be a complex $\mathfrak{g}_{\mathbf{C}}$-module. A pseudo-Hermitean form, $\langle \cdot, \cdot \rangle$ on V is said to be *contravariant* if

$$\langle X.v, w\rangle = -\langle v, \overline{X}.w\rangle$$

for all $X \in \mathfrak{g}_{\mathbf{C}}$, $v, w \in V$.

(g) A highest weight module V is said to be *unitarizable* if there exists a unitary representation $(\pi, \mathcal{H})$ of the simply connected Lie group G with $\mathbf{L}(G)=\mathfrak{g}$ such that V is isomorphic to the $\mathfrak{g}_{\mathbf{C}}$-module $\mathcal{H}^{K,\infty}$ of K-finite smooth vectors in $\mathcal{H}$ (see Section III). Note that this implies in particular that V is endowed with a positive definite contravariant Hermitean form. □

THEOREM II.9. *Let V be a $\mathfrak{g}$-module of highest weight λ and v a corresponding primitive element in V of weight λ. Then the following assertions hold*:

(i) $V=\mathcal{U}(\mathfrak{g}_{\mathbf{C}}^-).v$.

(ii) $V=\bigoplus_{\beta\in\mathcal{P}_V} V^\beta$, *every V^α is finite dimensional*, $\dim V^\lambda=1$, *and every weight of V may be written as $\lambda-\sum_{\alpha\in\Delta^+} n_\alpha\alpha$, where the n_α are nonnegative integers.*

(iii) *Let $X\in it$ such that $\alpha(X)>0$ holds for all $\alpha\in\Delta^+$ and set $\pi(X)(v):=X.v$ for $v\in V$. We set $m(\beta):=\dim V^\beta$. Then*

$$\operatorname{tr} e^{\pi(X)} := \sum_{\beta\in\mathcal{P}_V} m(\beta)e^{\beta(X)}$$

is finite.

(iv) *Every linear operator on V commuting with $\mathfrak{g}$ is a scalar multiple of the identity.*

(v) *Suppose that λ is real on it. Pick a highest weight vector v_λ. Then there exists a unique contravariant pseudo-Hermitean form $\langle\cdot\,,\cdot\rangle$ on V such that $\langle v_\lambda, v_\lambda\rangle=1$.*

(vi) *The radical $V^\perp$ of the contravariant pseudo-Hermitean form is a unique maximal proper submodule.*

Proof. (Cf. [Bou3, Chapter VIII, §6, No. 1, Proposition 1] for the semisimple case.)

(i) It follows from the Poincaré–Birkhoff–Witt theorem that

$$\mathcal{U}(\mathfrak{g}_{\mathbf{C}})=\mathcal{U}(\mathfrak{g}_{\mathbf{C}}^-)\mathcal{U}(\mathfrak{t}_{\mathbf{C}})\mathcal{U}(\mathfrak{g}_{\mathbf{C}}^+)=\mathcal{U}(\mathfrak{g}_{\mathbf{C}}^-)\mathcal{U}(\mathfrak{b})$$

([Bou1, Chapter I, §2, No. 7, Corollary 6]). Hence

$$V=\mathcal{U}(\mathfrak{g}_{\mathbf{C}}).V=\mathcal{U}(\mathfrak{g}_{\mathbf{C}}^-)\mathcal{U}(\mathfrak{b}).v=\mathcal{U}(\mathfrak{g}_{\mathbf{C}}^-).v. \tag{2.1}$$

(ii), (iii) We have seen above that $V=\mathcal{U}(\mathfrak{g}_{\mathbf{C}}^-).v$. This means that the mapping

$$\mathcal{U}(\mathfrak{g}_{\mathbf{C}}^-)\otimes\mathbf{C}\to V, \quad X\otimes z\mapsto zX.v,$$

is surjective. Moreover, one checks easily that this mapping is a morphism of $\mathfrak{t}_{\mathbf{C}}$-modules, where $\mathbf{C}=\mathbf{C}_\lambda$ is considered as a $\mathfrak{t}_{\mathbf{C}}$ module via $X.z:=\lambda(X)z$. Hence it suffices to prove the assertions of (ii) and (iii) for the $\mathfrak{t}_{\mathbf{C}}$-module $V':=\mathcal{U}(\mathfrak{g}_{\mathbf{C}}^-)\otimes\mathbf{C}_\lambda$ because the multiplicities of the quotient module V are smaller than in V'.

Now we use the Poincaré–Birkhoff–Witt theorem once again to see that $\mathcal{U}(\mathfrak{g}_{\mathbf{C}}^{-})\cong\mathcal{S}(\mathfrak{g}_{\mathbf{C}}^{-})$ as $\mathfrak{t}_{\mathbf{C}}$-module, where the latter denotes the symmetric algebra of $\mathfrak{g}_{\mathbf{C}}^{-}$. Moreover

$$\mathcal{S}(\mathfrak{g}_{\mathbf{C}}^{-})\cong\bigotimes_{\alpha\in\Delta^{+}}\mathcal{S}(\mathfrak{g}_{\mathbf{C}}^{-\alpha}).$$

Let $m_{\alpha}:=\dim_{\mathbf{C}}\mathfrak{g}_{\mathbf{C}}^{\alpha}$. Then $\mathcal{S}(\mathfrak{g}_{\mathbf{C}}^{-\alpha})\cong\mathcal{S}(\mathbf{C}_{-\alpha})^{\otimes m_{\alpha}}$.

For the module $\mathcal{S}(\mathbf{C}_{\beta})$ with $\beta(X)<0$ and the morphism $\pi_{\beta}:\mathfrak{t}_{\mathbf{C}}\to\mathrm{End}(\mathcal{S}(\mathbf{C}_{\beta}))$, it is immediately clear that each weight space has multiplicity 1 and that

$$\mathrm{tr}\, e^{\pi_{\beta}(X)}=\sum_{n=0}^{\infty}e^{n\beta(X)}=\frac{1}{1-e^{\beta(X)}}.$$

Now let $\pi':\mathfrak{t}_{\mathbf{C}}\to\mathrm{End}(V')$ denote the representations of $\mathfrak{t}_{\mathbf{C}}$ on V'. Then the tensor product decomposition yields

$$\mathrm{tr}\, e^{\pi'(X)}=e^{\lambda(X)}\prod_{\alpha\in\Delta^{+}}\frac{1}{(1-e^{-\alpha(X)})^{m_{\alpha}}}<\infty.$$

We conclude in particular that all the multiplicities are finite because they arise as the coefficient of $e^{\beta(X)}$ for $\beta\in\mathcal{P}_{V'}$.

(iv) Let A be a linear operator on V commuting with $\mathfrak{g}$. Then $AV^{\beta}\subseteq V^{\beta}$ holds for each weight β. So $Av\in V^{\lambda}=\mathbf{C}v$. Let $Av=cv$. Then (2.1) entails that $A=c\,\mathrm{id}_{V}$.

(v) We write $X\mapsto X^{*}$ for the complex antilinear antiautomorphism of the universal enveloping algebra $\mathcal{U}(\mathfrak{g}_{\mathbf{C}})$ which extends the antilinear antiautomorphism $X\mapsto-\overline{X}$ of $\mathfrak{g}_{\mathbf{C}}$. For a vector u in V we define the *expectation value* $\langle u\rangle$ as the coefficient of v_{λ} in the additive decomposition of u into weight vectors. For $A\in\mathcal{U}(\mathfrak{g}_{\mathbf{C}})$ we claim that

$$\langle A^{*}.v_{\lambda}\rangle=\overline{\langle A.v_{\lambda}\rangle}.$$

To see this, according to the Poincaré–Birkhoff–Witt theorem, we may assume that $A=BCD$, where $B\in\mathcal{U}(\mathfrak{g}_{\mathbf{C}}^{+})$, $C\in\mathcal{U}(\mathfrak{t}_{\mathbf{C}})$, and $D\in\mathcal{U}(\mathfrak{g}_{\mathbf{C}}^{+})$ are monomials in a certain set of generators. Then $A^{*}=D^{*}C^{*}B^{*}$, and we see that

$$\langle A.v_{\lambda}\rangle=\langle BCD.v_{\lambda}\rangle=\{0\}=\langle D^{*}C^{*}B^{*}.v_{\lambda}\rangle=\langle A^{*}.v_{\lambda}\rangle$$

if $B,D\neq\mathbf{1}$. Therefore we may assume that $B=D=\mathbf{1}$, i.e., that $A\in\mathcal{U}(\mathfrak{t}_{\mathbf{C}})$. We write $\lambda:\mathcal{U}(\mathfrak{t}_{\mathbf{C}})\cong\mathcal{S}(\mathfrak{t}_{\mathbf{C}})\to\mathbf{C}$ for the unique algebra homomorphism extending λ. Then the fact that $\lambda(\mathfrak{t})\subseteq i\mathbf{R}$ entails that

$$\langle A^{*}.v_{\lambda}\rangle=\lambda(A^{*})=\overline{\lambda(A)}=\langle A.v_{\lambda}\rangle$$

because both homomorphisms $A \mapsto \lambda(A)$ and $A \mapsto \overline{\lambda(A^*)}$ are complex linear and extend the linear functional λ on $\mathfrak{t}_{\mathbf{C}}$.

With this information, it is immediate that the prescription

$$\langle A.v_\lambda, B.v_\lambda \rangle := \langle B^* A.v_\lambda \rangle$$

is well defined and defines a pseudo-Hermitean form on V. For $C \in \mathcal{U}(\mathfrak{g}_{\mathbf{C}})$ we have that

$$\langle CA.v_\lambda, B.v_\lambda \rangle = \langle B^*(CA).v_\lambda \rangle = \langle ((C^*B)^*A).v_\lambda \rangle = \langle A.v_\lambda, C^*B.v_\lambda \rangle.$$

Hence the form is contravariant.

(vi) Let $V^\perp = \{v \in V : \langle v, V \rangle = \{0\}\}$ denote the radical of the pseudo-Hermitean form. Then $V^\perp$ is a submodule of V and it does not contain v_λ.

Suppose, conversely, that $E \subseteq V$ is a proper submodule. Then $v_\lambda \in E^\perp$ since the weight space decomposition is orthogonal and therefore $\mathcal{U}_{\mathfrak{g}_{\mathbf{C}}}.v_\lambda = V \subseteq E^\perp$. Hence $E \subseteq V^\perp$. □

The proof of the following result carries over word for word from the semisimple case. For further results on highest weight modules see [Bou3, Chapter VIII, §6, No. 3] and [J].

Proposition II.10. *For every $\lambda \in \mathfrak{t}_{\mathbf{C}}^*$ there exists an irreducible highest weight module V_λ which is unique up to isomorphy. If $\lambda(\mathfrak{t}) \subseteq i\mathbf{R}$, then V_λ carries a nondegenerate contravariant pseudo-Hermitean form and if, conversely, V_λ' is a module with highest weight λ which carries a nondegenerate contravariant pseudo-Hermitean form, then V_λ' is irreducible and therefore isomorphic to V_λ.*

Proof. Let $\mathfrak{b} := \mathfrak{g}_{\mathbf{C}}^+ \rtimes \mathfrak{t}_{\mathbf{C}}$ be as above. Then the functional $\lambda \in \mathfrak{t}_{\mathbf{C}}^*$ defines a one-dimensional representation of $\mathfrak{b}$ on $\mathbf{C}$. Write L_λ for the corresponding $\mathcal{U}(\mathfrak{b})$-module. Since $\mathcal{U}(\mathfrak{g}_{\mathbf{C}})$ is a free right $\mathcal{U}(\mathfrak{b})$-module (Poincaré–Birkhoff–Witt), we set

$$Z(\lambda) := \mathcal{U}(\mathfrak{g}_{\mathbf{C}}) \otimes_{\mathcal{U}(\mathfrak{b})} L_\lambda.$$

Let $e := 1 \otimes 1 \in Z(\lambda)$. Then it is clear that e is a primitive element of weight λ because

$$X.e = X.(1 \otimes 1) = X \otimes 1 = 1 \otimes X = \lambda(X)(1 \otimes 1) = \lambda(X)e$$

holds for all $X \in \mathfrak{b}$, where λ is viewed as a functional on $\mathfrak{b}$ which vanishes on $\mathfrak{g}_{\mathbf{C}}^+$. According to Theorem II.9, the $\mathfrak{t}_{\mathbf{C}}$-module $Z(\lambda)$ is semisimple.

If $W \subseteq Z(\lambda)$ is a submodule, then the semisimplicity with respect to $\mathfrak{t}_{\mathbf{C}}$ yields that

$$W = \bigoplus_{\mu \in \mathfrak{t}_{\mathbf{C}}^*} W^\mu = \bigoplus_{\mu \in \mathfrak{t}_{\mathbf{C}}^*} (W \cap Z(\lambda)^\mu).$$

The hypothesis $W^\lambda \neq \{0\}$ implies that $W = Z(\lambda)$ because $Z(\lambda)$ is generated by e. If $W^\lambda = \{0\}$, then

$$W \subseteq Z(\lambda)_+ := \bigoplus_{\mu \neq \lambda \in \mathfrak{t}_{\mathbf{C}}^*} Z(\lambda)^\mu.$$

Let F_λ denote the sum of all submodules different from $Z(\lambda)$. Then $F_\lambda \subseteq Z(\lambda)_+$. Hence F_λ is the largest proper submodule of $Z(\lambda)$ different from $Z(\lambda)$, so $E(\lambda) := Z(\lambda)/F_\lambda$ is irreducible and the image of e is a primitive element of weight λ.

We show that $E(\lambda)$ is unique up to isomorphy. Suppose that V is an irreducible highest weight module with highest weight λ and v_λ a highest weight vector. Let K denote the kernel of the representation of $\mathcal{U}(\mathfrak{b})$ on L_λ. Then $\operatorname{codim} K = 1$ and $L_\lambda \cong \mathcal{U}(\mathfrak{b})/K$ as left $\mathcal{U}(\mathfrak{b})$-modules. Let $J := \mathcal{U}(\mathfrak{g}_{\mathbf{C}})K$. Then $Z(\lambda) \cong \mathcal{U}(\mathfrak{g}_{\mathbf{C}})/J$ as left $\mathcal{U}(\mathfrak{g}_{\mathbf{C}})$-modules. Since $K.v_\lambda = \{0\}$, we conclude that $J.v = \{0\}$. Hence we have a morphism

$$\psi \colon Z(\lambda) \to V$$

with $\psi(e) = v$. Since V is irreducible, $\ker \psi$ is a maximal submodule of $Z(\lambda)$ not containing $Z(\lambda)^\lambda$. Thus $\ker \psi = F_\lambda$ and therefore ψ induces an isomorphism of $E(\lambda)$ onto V.

Assume, in addition, that $\lambda(\mathfrak{t}) \subseteq i\mathbf{R}$. Then Theorem II.9 (v) shows that a highest weight module V_λ' with highest weight λ is irreducible if and only if the contravariant form is nondegenerate. This completes the proof. □

PROPOSITION II.11. *Let V be an irreducible highest weight module and $\Sigma \subseteq \Delta$ a parabolic system containing the positive system Δ^+. Then*

$$V_\Sigma^+ := \{v \in V : \mathfrak{n}_\Sigma.v = \{0\}\}$$

is an irreducible $\mathfrak{s}_\Sigma$-submodule of highest weight λ.

Proof. First we note that V_Σ^+ is an $\mathfrak{s}_\Sigma$-submodule because $\mathfrak{n}_\Sigma$ is an ideal in $\mathfrak{p}_\Sigma$. Note also that $v_\lambda \in V_\Sigma^+$.

Using the Poincaré–Birkhoff–Witt theorem, we get

$$\mathcal{U}(\mathfrak{g}_{\mathbf{C}}) = \mathcal{U}(\overline{\mathfrak{n}_\Sigma})\mathcal{U}(\mathfrak{s}_\Sigma)\mathcal{U}(\mathfrak{n}_\Sigma).$$

So

$$V = \mathcal{U}(\mathfrak{g}_{\mathbf{C}}).v_\lambda = \mathcal{U}(\overline{\mathfrak{n}_\Sigma})\mathcal{U}(\mathfrak{s}_\Sigma).v_\lambda.$$

Pick $X_0 \in i\mathfrak{t}$ such that

$$\Sigma = \{\alpha \in \Delta : \alpha(X_0) \geqslant 0\}.$$

Then $X_0 \in Z(\mathfrak{s}_\Sigma)$ and for $\beta \in -\Sigma \backslash \Sigma$, and $X \in \mathfrak{g}_{\mathbf{C}}^\beta \subseteq \overline{\mathfrak{n}_\Sigma}$ we have that

$$X.V^\mu \subseteq V^{\mu+\beta}$$

with $(\mu+\beta)(X_0)=\mu(X_0)+\beta(X_0)<\mu(X_0)$.

We conclude that

$$V^{\lambda(X_0)}(X_0):=\{v\in V: X_0.v=\lambda(X_0)v\}=\mathcal{U}(\mathfrak{s}_\Sigma).v_\lambda\subseteq V_\Sigma^+.$$

Suppose that

$$V^{\lambda(X_0)}(X_0)\neq V_\Sigma^+.$$

Then the fact that V_Σ^+ is a semisimple $\mathfrak{t}_{\mathbf{C}}$-module entails the existence of a weight μ and $v_\mu\in V^\mu\cap V_\Sigma^+$ such that $\mu(X_0)<\lambda(X_0)$.

Now

$$\mathcal{U}(\mathfrak{g}_{\mathbf{C}}).v_\mu=\mathcal{U}(\overline{\mathfrak{n}_\Sigma})\mathcal{U}(\mathfrak{s}_\Sigma).v_\mu\subseteq\bigoplus_{\beta(X_0)\leqslant\mu(X_0)}V^\beta$$

is a submodule which is strictly smaller than V, contradicting the irreducibility. Hence $V_\Sigma^+=\mathcal{U}(\mathfrak{s}_\Sigma).v_\lambda$ proves that V_Σ^+ is a highest weight module of weight λ for the Lie algebra $\mathfrak{s}_\Sigma$.

Let $W\subseteq V_\Sigma^+$ be an $\mathfrak{s}_\Sigma$-submodule. According to the irreducibility of V we now have that

$$V\subseteq\mathcal{U}(\mathfrak{g}_{\mathbf{C}}).W=\mathcal{U}(\overline{\mathfrak{n}_\Sigma})\mathcal{U}(\mathfrak{s}_\Sigma).W\subseteq\mathcal{U}(\overline{\mathfrak{n}_\Sigma}).W.$$

Hence

$$V^{\lambda(X_0)}(X_0)\subseteq W\subseteq V^{\lambda(X_0)}(X_0),$$

shows that $W=V_\Sigma^+$. □

COROLLARY II.12. *Let V be an irreducible highest weight module, Δ^+ a $\mathfrak{k}$-adapted positive system and $\mathfrak{p}^+:=\bigoplus_{\alpha\in\Delta_p^+}\mathfrak{g}_{\mathbf{C}}^\alpha$. Then $V_+:=\{v\in V:\mathfrak{p}^+.v=\{0\}\}$ is an irreducible $\mathfrak{k}_{\mathbf{C}}$-submodule. For each $\beta\in\mathcal{P}_V$ there exists a weight $\alpha\in\mathcal{P}_{V_+}$ such that*

$$\beta\in\alpha-\sum_{\gamma\in\Delta_p^+}\mathbf{N}_0\gamma.$$

Proof. In view of the preceding lemma, the first assertion follows from the fact that $\Sigma:=\Delta_k\cup\Delta_p^+$ is a parabolic system of roots (Proposition II.5).

For the second assertion we recall that

$$\mathcal{U}(\mathfrak{g}_{\mathbf{C}}).v_\lambda=\mathcal{U}(\mathfrak{g}_{\mathbf{C}}^-).v_\lambda=\mathcal{U}(\mathfrak{p}^-)\mathcal{U}(\mathfrak{k}_{\mathbf{C}}).v_\lambda=\mathcal{U}(\mathfrak{p}^-).\mathcal{H}_+$$

holds for every highest weight vector v_λ. □

III. Irreducible representations

In this section G denotes a simply connected Lie group, $\mathfrak{g}=\mathbf{L}(G)$ its Lie algebra. First we recall the notation of Section I. We pick a maximal compactly embedded subalgebra $\mathfrak{k}$ and choose a Cartan algebra $\mathfrak{t}\subseteq\mathfrak{k}$. We write $T:=\exp\mathfrak{t}$ and $K:=\exp\mathfrak{k}$.

Definition III.1. Let $(\pi,\mathcal{H})$ be a unitary representation of the group G, i.e., $\pi: G\to U(\mathcal{H})$ is a continuous homomorphism into the unitary group.

(a) We write $\mathcal{H}^\infty$ ($\mathcal{H}^\omega$) for the corresponding space of smooth (analytic) vectors. We write $d\pi$ for the derived representation of $\mathfrak{g}$ on $\mathcal{H}^\infty$. We extend this representation to a representation of the complexified Lie algebra $\mathfrak{g}_{\mathbf{C}}$ on the complex vector space $\mathcal{H}^\infty$.

(b) A vector $v\in\mathcal{H}$ is said to be K-finite if it is contained in a K-invariant finite dimensional subspace of $\mathcal{H}$. We write $\mathcal{H}^K$ for the set of K-finite vectors in $\mathcal{H}$.

(c) For a Lie group G we write $\widehat{G}$ for the set of equivalence classes of irreducible unitary representations of G.

(d) For an irreducible representation χ of K we write $[\chi]$ for the class of χ in $\widehat{K}$ and $\mathcal{H}_{[\chi]}=\mathcal{H}_{[\chi]}(K)$ for the corresponding isotypic subspace of $\mathcal{H}^K$. □

PROPOSITION III.2. *For an irreducible unitary representation $(\pi,\mathcal{H})$ of the (CA) Lie group G the following assertions hold*:

(i) *The space $\mathcal{H}^K$ of K-finite vectors is dense in $\mathcal{H}$.*

(ii) $\mathcal{H}=\widehat{\bigoplus}_{\chi\in\widehat{K}}\mathcal{H}_{[\chi]}(K)$.

Proof. (i) Since G is a (CA) Lie group, $K/Z(G)$ is a compact group (Corollary I.3). According to the assumption that π is irreducible, it follows from Schur's lemma that $\pi(Z(G))\subseteq\mathbf{C}\mathbf{1}$. Hence there exists a unitary character χ of $Z(G)$ such that

$$\pi(z)=\chi(z)\mathbf{1}\quad \forall z\in Z(G).$$

Let us consider $\pi: G\to U(\mathcal{H})$ as a morphism of topological groups and write $q: U(\mathcal{H})\to PU(\mathcal{H})$ for the quotient morphism onto the *projective unitary group*. We also set $G^\sharp:=G\times\mathbb{S}^1$ and extend π to a continuous unitary representation

$$\pi^\sharp: G^\sharp\to U(\mathcal{H}),\quad (g,z)\mapsto z\pi(g).$$

Then $q\circ\pi^\sharp$ is constant on $Z(G)\times\mathbb{S}^1$, hence factors to a morphism $\tilde\pi: G/Z(G)\to PU(\mathcal{H})$. Thus $q\circ p(K)=\tilde\pi(K/Z(G))$ is a compact group and

$$U:=q^{-1}(\tilde\pi(K/Z(G)))=\pi^\sharp(K^\sharp),$$

where $K^\sharp=K\times\mathbb{S}^1$ is also compact because the kernel of q is isomorphic to $\mathbb{S}^1\cong\pi^\sharp(\mathbb{S}^1)$.

The essential consequence of this observation is that the set of U-invariant subspaces of $\mathcal{H}$ equals the set of K-invariant subspaces of $\mathcal{H}$. This makes it possible to apply the representation theory of compact groups to decompose $\mathcal{H}$. Therefore (i) follows from the Peter–Weyl Theorem ([Wal1, p. 25]).

(ii) Since $\pi(K^\sharp)$ is compact, we have a Hilbert space direct sum

$$\mathcal{H} = \widehat{\bigoplus_{\alpha\in\widehat{K}^\sharp}} \mathcal{H}_{[\alpha]}(K^\sharp).$$

On the other hand

$$\mathcal{H}_{[\chi^\sharp]}(K^\sharp) = \mathcal{H}_{[\chi]}(K)$$

which proves the assertion. □

The next step is to prove that even the analytic K-finite vectors, i.e., the space $\mathcal{H}^{K,\omega} := \mathcal{H}^K \cap \mathcal{H}^\omega$ is dense in $\mathcal{H}$ whenever π is an irreducible representation of a (CA) Lie group. We will obtain this by a tool which also yields some information for groups which are not necessarily (CA).

We start by choosing a K-invariant positive definite scalar product on $\mathfrak{g}$. Let $X_1, \ldots, X_n$ denote an orthonormal basis of $\mathfrak{g}$ with respect to this scalar product. Further let $\Omega := \sum_{i=1}^n X_i^2 \in \mathcal{U}(\mathfrak{g})$. We claim that $\mathfrak{k}$ commutes with Ω. To see this, let $Z \in \mathfrak{k}$ and $[Z, X_j] = \sum_{i=1}^n a_{i,j} X_i$. Then $a_{i,j} = -a_{j,i}$ and therefore

$$\begin{aligned}\Omega Z - Z\Omega &= \sum_{i=1}^n X_i(X_i Z) - (Z X_i) X_i = \sum_{i=1}^n X_i(X_i Z - Z X_i) - (Z X_i - X_i Z) X_i \\ &= \sum_{i,j=1}^n -X_i a_{j,i} X_j - a_{j,i} X_j X_i = \sum_{i,j=1}^n (a_{i,j} - a_{i,j}) X_i X_j = 0.\end{aligned}$$

We also set $X_0 := \sum_{i=1}^n \operatorname{tr}(\operatorname{ad} X_i) X_i$. Then $\langle X_0, X\rangle = \operatorname{tr}\operatorname{ad} X$ for all $X \in \mathfrak{g}$. Since $\omega\colon X \mapsto \operatorname{tr}\operatorname{ad} X$ is a homomorphism of Lie algebras, we conclude that $\omega \in \mathfrak{g}^*$ is invariant under the coadjoint action. Hence the element X_0 is invariant under K because the scalar product is invariant under K.

Now we write $\mathcal{X}_i$ for the left invariant differential operator on G defined by

$$\mathcal{X}_i f(g) := \frac{d}{dt}\bigg|_{t=0} f(g \exp t X_i).$$

Then the mapping $X_i \mapsto \mathcal{X}_i$ defines a homomorphism of Lie algebras, and the differential operator

$$N := \sum_{i=1}^n \mathcal{X}_i^2 + \operatorname{tr}(\operatorname{ad} X_i)\mathcal{X}_i$$

is biinvariant under K on $C^\infty(G)$.

Then $N_0 := N|_{C_c^\infty(G)}$ is a densely defined symmetric operator on the Hilbert space $L^2(G)$ with respect to a left Haar measure μ_G and $\Lambda := N_0^*$ is a negative self adjoint operator on $L^2(G)$ ([Gå, p. 81]). The corresponding one-parameter semigroup of contractions $(U_t)_{t \in \mathbf{R}^+}$ ([P, p. 15]) is given by

$$U_t(f) = f * p_t \quad \forall t > 0$$

([Ne1, IV.4]), where $p_t \in C^\omega(G)$ is a function with the following properties:

(1) $p_t(g) > 0$ for all $g \in G$.

(2) $\int_G p_t(x)\, dm_G(x) = 1$ for all $t > 0$.

(3) $p_t * p_s = p_{t+s}$ for $t, s > 0$.

Now the fact that N_0 and therefore $N_0^* = \Lambda$ commutes with K, entails that U_t commutes with the right action of K on $L^2(G)$. Hence p_t is invariant under conjugation with K, i.e., constant on the K-conjugacy classes.

LEMMA III.3. *Let G be a connected Lie group, $\mathfrak{k}$ a compactly embedded subalgebra of $\mathfrak{g} = \mathbf{L}(G)$, $K = \exp \mathfrak{k}$, and $(U_n)_{n \in \mathbf{N}}$ a basis of the filter of $\mathbf{1}$-neighborhoods. Then there exists a sequence of functions $f_n \in C^\infty(G)$ such that:*

(1) *$f_n \geq 0$,*

(2) *$\int_G f_n(g)\, d\mu_G(g) = 1$, where μ_G is a left Haar measure on G,*

(3) *$\operatorname{supp}(f_n) \subseteq U_n$,*

(4) *f_n commutes with K, i.e., $\delta_k * f_n = f_n * \delta_k$ for all $k \in K$.*

Proof. Let $V_n \subseteq U_n$ be a K-invariant compact neighborhood of $\mathbf{1}$. Let h_n satisfy (1)–(3) with V_n instead of U_n. Then we set

$$f_n := \int_U (h_n \circ \gamma)\, d\mu_U(\gamma),$$

where U denotes the closure of $\{I_k : k \in K\}$ in the Lie group

$$\operatorname{Aut}(G) = \{\gamma \in \operatorname{Aut}(\widetilde{G}) : \gamma(\pi_1(G)) = \pi_1(G)\} = \{\gamma \in \operatorname{Aut}(\mathfrak{g}) : \gamma(\pi_1(G)) = \pi_1(G)\}.$$

This subgroup is compact because it is a closed subgroup of the compact group $U \cong \overline{\operatorname{Ad}(K)} \subseteq \operatorname{Aut}(\mathfrak{g})$ which in turn is compact since $\mathfrak{k}$ is compactly embedded. □

We apply this to unitary representation. Note that the following proposition says nothing about the existence of K-finite vectors which is guaranteed by Proposition III.2 if G is a (CA) group.

PROPOSITION III.4. *Let $(\pi, \mathcal{H})$ be a unitary representation of the Lie group G. The space $\mathcal{H}^{K,\omega}$ of K-finite analytic vectors is dense in $\mathcal{H}^K$. If $[\lambda]\in\widehat{K}$ and $\mathcal{H}^K_{[\lambda]}$ is the corresponding isotypic component in $\mathcal{H}^K$, then the analytic vectors in $\mathcal{H}^K_{[\lambda]}$ are also dense.*

Proof. Since $\mathcal{H}^K$ is the sum of finite dimensional K-modules, it follows that

$$\mathcal{H}^K = \bigoplus_{[\lambda]\in\widehat{K}} \mathcal{H}^K_{[\lambda]}.$$

So it remains to show that the set of analytic vectors in $\mathcal{H}^K_{[\lambda]}$ is dense in this space. Let $v\in\mathcal{H}^K_{[\lambda]}$. We set $v_n:=\pi(f_n).v$ with f_n as in Lemma III.3 and

$$\pi(f_n).v = \int_G f_n(g)\pi(g).v\, d\mu_G(g),$$

where μ_G denotes Haar measure on G. Then $\pi(K)$ commutes with $\pi(f_n)$, so that $v_n\in\mathcal{H}^K_{[\lambda]}$. On the other hand $v_n\in\mathcal{H}^\infty$ and $v_n\to v$.

Next let $q_{n,t}:=f_n * p_t$ with p_t as above. Then

$$\|q_{n,t}\|_1 \leqslant \|f_n\|_1\|p_t\|_1 = 1$$

yields $q_{n,t}\in L^1(G)$.

The following argument follows the proof of Theorem 4.4.5.7 in [War]. We set $v_{n,t}:=\pi(q_{n,t}).v$. Then [War, 4.4.5.14] implies that $v_{n,t}\in\mathcal{H}^\omega$ with $\lim_{t\to 0} v_{n,t}=v_n$. On the other hand $q_{n,t}$ commutes with K so that the same argument as above shows that $v_{n,t}\in\mathcal{H}^{K,\omega}_{[\lambda]}$. We conclude that $\mathcal{H}^{K,\omega}_{[\lambda]}$ is dense in $\mathcal{H}^K_{[\lambda]}$. □

COROLLARY III.5. *If the spaces $\mathcal{H}^K_{[\lambda]}$ are finite dimensional for every $[\lambda]\in\widehat{K}$, then $\mathcal{H}^K\subseteq\mathcal{H}^\omega$.*

This is the point where representation theory of Lie algebras comes in. For $X\in\mathfrak{g}$ and $v\in\mathcal{H}^{K,\infty}$ we have that

$$\pi(k)d\pi(X)v = d\pi(kXk^{-1})\pi(k)v.$$

Thus

$$\operatorname{span}\pi(K)d\pi(X)v \subseteq \operatorname{span} d\pi(\mathfrak{g})\pi(K)v,$$

and the latter space is at most of dimension $(\dim\mathfrak{g})(\dim\pi(K)v)$. It follows that the representation of the Lie algebra $\mathfrak{g}_{\mathbb{C}}$ lives on the rather well behaved subspace $\mathcal{H}^{K,\infty}$ of $\mathcal{H}$. In the following we always consider this $\mathfrak{g}$-module as a module of the universal enveloping algebra $\mathcal{U}(\mathfrak{g}_{\mathbb{C}})$. From now on we only consider Lie algebras $\mathfrak{g}$ containing a compactly embedded Cartan algebra $\mathfrak{t}$.

THEOREM III.6. *Let $(\pi,\mathcal{H})$ be a unitary representation of connected Lie group G, $\mathfrak{t}\subseteq\mathfrak{g}$ a compactly embedded Cartan algebra, Δ^+ a positive system, and $X\in\operatorname{int}(i\Delta^+)^\star$. Then (1) $\Rightarrow$ (2) holds for the following assertions:*

(1) *The operator $id\pi(X)$ is bounded from above and $\mathcal{H}^K\neq\{0\}$.*

(2) *$\mathcal{H}^{K,\omega}$ contains a primitive element with respect to Δ^+.*

If (2) is satisfied, then the following are equivalent:

(3) *π is an irreducible representation.*

(4) *$\mathcal{H}^{K,\infty}$ is a highest weight module with respect to Δ^+ and $\mathcal{H}^K$ is dense.*

(5) *$\mathcal{H}^{K,\infty}$ is an irreducible highest weight module with respect to Δ^+ and $\mathcal{H}^K$ is dense.*

If (5) is satisfied, then $\mathcal{H}^K=\mathcal{H}^{K,\infty}$ consists of analytic vectors and (1) is satisfied.

Proof. (1) $\Rightarrow$ (2): Since $\mathcal{H}^K\neq\{0\}$ by assumption, Proposition III.4 shows that there exists a K-finite unitvector v_0 in $\mathcal{H}^\omega$. We may assume that $v_0\in\mathcal{H}^\mu$ for a functional $\mu\in\mathfrak{t}_{\mathbf{C}}^*$. Then

$$\mathcal{U}(\mathfrak{g}_{\mathbf{C}}^+).v_0\subseteq\bigoplus_{\beta\in\mu+\sum_{\alpha\in\Delta^+}\mathbf{N}_0\alpha}\mathcal{H}^{K^\beta},$$

and since $i\alpha(X)>0$ for all $\alpha\in\Delta^+$, and $id\pi(X)$ is bounded from above, there exists a $\lambda\in\mathfrak{t}_{\mathbf{C}}^*$ such that $i\lambda(X)$ is maximal among all those with $\mathcal{H}^\beta\cap\mathcal{U}(\mathfrak{g}_{\mathbf{C}}^+).v_0\neq\{0\}$. Pick a unit vector $v\in\mathcal{H}^\lambda\cap\mathcal{H}^K$ which is contained in $\mathcal{U}(\mathfrak{g}_{\mathbf{C}}^+).v_0$. Then $v\in\mathcal{H}^\omega$ and $\mathfrak{g}_{\mathbf{C}}^+.v=\{0\}$. Therefore v is a primitive element of weight λ with respect to Δ^+.

(3) $\Rightarrow$ (4): Suppose that π is irreducible and that (2) is satisfied. Set $\mathcal{H}_v:=\mathcal{U}(\mathfrak{g}_{\mathbf{C}}).v$, where v is a primitive element in $\mathcal{H}^{K,\omega}$. Then $\mathcal{H}_v$ is a highest weight module of highest weight λ. Thus all the subspaces $\mathcal{H}_v^\beta$, $\beta\in\mathcal{P}_{\mathcal{H}_v^K}$ are finite dimensional by Theorem II.9 (ii). Pick $\beta\in\mathcal{P}_{\mathcal{H}^K}$. Suppose that $(\mathcal{H}^K)^\beta\neq\mathcal{H}_v^\beta$. Then there exists a vector $v'\in(\mathcal{H}^K)^\beta$ which is orthogonal to the finite dimensional subspace $\mathcal{H}_v^\beta$. But this means that v' is orthogonal to the whole space $\mathcal{H}_v$.

Since $\mathcal{H}_v$ consists of analytic vectors, it is dense in $\mathcal{H}$ because it is invariant under $\mathfrak{g}_{\mathbf{C}}$, so that its closure is a G-invariant subspace of $\mathcal{H}$ ([War, 4.4.5.6]). On the other hand this subspace is orthogonal to v'. Hence $v'=0$ and therefore $\mathcal{H}^K=\mathcal{H}_v$. We conclude in particular that $\mathcal{H}^K\subseteq\mathcal{H}^\omega$ and that $\mathcal{H}^K$ is a highest weight module with highest weight λ with respect to Δ^+.

(4) $\Rightarrow$ (5): If $\mathcal{H}^{K,\infty}$ is a highest weight module with highest weight λ, then the scalar product on $\mathcal{H}$ induces a nondegenerate contravariant Hermitean form and therefore $\mathcal{H}^{K,\infty}$ is irreducible (Proposition II.10).

(5) $\Rightarrow$ (3): Suppose that $\mathcal{H}^{K,\infty}$ is an irreducible highest weight module with highest weight λ. All the $\mathfrak{t}_{\mathbf{C}}$-weight spaces in $\mathcal{H}^{K,\infty}$ are finite dimensional by Theorem II.9 (ii).

Hence all the weight spaces in $\mathcal{H}^K$ are finite dimensional and therefore $\mathcal{H}^K \subseteq \mathcal{H}^\omega$ (Corollary III.5).

Suppose that the representation π is not irreducible and that $\mathcal{H}=\mathcal{H}_1 \oplus \mathcal{H}_2$ is a nontrivial G-invariant orthogonal decomposition. Let P_j denote the orthogonal projection on $\mathcal{H}_j$. Then the P_j commute with K and therefore map $\mathcal{H}^K$ into $\mathcal{H}_j^K$. Since P_j also commutes with the action of $\mathfrak{g}_{\mathbf{C}}$ on $\mathcal{H}^\infty$, it follows that $\mathcal{H}^K \cong \mathcal{H}_1^K \oplus \mathcal{H}_2^K$ is a direct sum decomposition of $\mathfrak{g}_{\mathbf{C}}$-modules. Therefore the irreducibility of $\mathcal{H}^K$ yields a contradiction because no factor can be trivial since $\mathcal{H}^K$ is dense. This proves that the representation is irreducible.

(5) $\Rightarrow$ (1): If (5) holds, then it is clear that $i\lambda(X)$ is the maximal eigenvalue of $id\pi(X)$ on $\mathcal{H}^K$ and hence that the operator $id\pi(X)$ is bounded from above. □

In [Ne8] we will show that whenever there exist unitarizable highest weight modules with respect to a positive system Δ^+, this positive system must be $\mathfrak{k}$-adapted.

Note that there are two crucial properties of a unitary representation which are dealt with in the two parts of Theorem III.6, namely the existence of K-finite vectors and the density of the space of K-finite vectors. If one does not want to worry about these problems one has to impose the assumption that the representation under consideration is irreducible and that G is a (CA) Lie group (cf. Corollary III.7). As the subrepresentations of the regular representation of $\mathbf{R}$ show, one cannot expect to have any K-finite vector without imposing any restrictions on the type of the representation.

Corollary III.7. *Let* $(\pi, \mathcal{H})$ *be an irreducible unitary representation of the connected (CA) Lie group* G, $\mathfrak{t} \subseteq \mathfrak{g}$ *a compactly embedded Cartan algebra,* Δ^+ *a positive system, and* $X \in \operatorname{int}(i\Delta^+)^\star$. *Then the following are equivalent:*

(1) *The operator* $id\pi(X)$ *is bounded from above.*

(2) $\mathcal{H}^{K,\omega}$ *contains a primitive element with respect to* Δ^+.

(3) $\mathcal{H}^K$ *is an irreducible highest weight module with respect to* Δ^+.

If (1)–(3) *are satisfied, then* $\mathcal{H}^K$ *consists of analytic vectors.*

Proof. Since the space $\mathcal{H}^K$ is dense by Proposition III.2, the assertions follow immediately from Theorem III.6. □

In the following we write $B_1(\mathcal{H})$ for the space of trace class operators on the Hilbert space $\mathcal{H}$ (cf. [We, p. 167]). If $(\pi, \mathcal{H})$ is a holomorphic representation of an Ol'shanskiĭ semigroup S, then we recall that the *kernel* of π is defined by $\ker \pi := \pi^{-1}(\mathbf{1})$.

Theorem III.8. *Let* $S=\Gamma(\mathfrak{g}, W, D)$ *be an Ol'shanskiĭ semigroup,* $\mathfrak{g}$ *a (CA) Lie algebra containing a compactly embedded Cartan algebra, and* $(\pi, \mathcal{H})$ *an irreducible holomorphic representation. Then the following assertions hold:*

(i) $\mathcal{H}^K$ *is an irreducible highest weight module of the Lie algebra* $\mathfrak{g}_{\mathbf{C}}$.

(ii) *For every* $s\in\operatorname{int}(S)$ *the operator* $\pi(s)$ *is a trace class operator, i.e.,* $\pi(\operatorname{int} S)\subseteq B_1(\mathcal{H})$.

Proof. (i) First we choose a regular element $X\in W\cap\mathfrak{t}$. Then the fact that $(\pi,\mathcal{H})$ is a holomorphic representation of S entails that the operator $id\pi(X)$ is bounded from above ([Ne6, III.1]). Now Corollary III.7 yields that $\mathcal{H}^K$ is an irreducible highest weight module with respect to the positive system

$$\Delta^+ := \{\alpha\in\Delta : i\alpha(X)>0\}.$$

(ii) First let X be as above. Then Proposition II.9 implies that $\pi(\operatorname{Exp}(iX))= e^{id\pi(X)}\in B_1(\mathcal{H})$. Now pick $s\in\operatorname{int} S$. Then there exists $s'\in\operatorname{int} S$ and $\varepsilon>0$ such that $s=s'\operatorname{Exp}(i\varepsilon X)$ ([HN2, 3.19 (v)]). Hence

$$\pi(s)=\pi(s')\pi(\operatorname{Exp}(i\varepsilon X))\in B(\mathcal{H})B_1(\mathcal{H})\subseteq B_1(\mathcal{H})$$

([We, p. 165]). □

In [Ne8] we will see that the assumption that G is a (CA) group is not necessary in Theorem III.8.

IV. Disintegration and character theory

In the preceding section we have seen that for every irreducible holomorphic representation of an Ol'shanskiĭ semigroup $S=\Gamma(\mathfrak{g},W,D)$ the elements in the interior of S are mapped onto trace class operators whenever $\mathfrak{g}$ is a (CA) Lie algebra. Since the (CA) assumption is not really necessary (cf. [Ne8, Theorem IV.3]), we anticipate this result from [Ne8] and do not make this assumption in this section.

We will show how this fact can be used to derive a rather satisfactory disintegration theory for holomorphic representations of S. We also apply the theory of liminal C^*-algebras to show that two irreducible representations are equivalent if and only if their characters coincide.

Definition IV.1. (a) Let A be a C^*-algebra. Then A is called *liminal* or *CCR* (completely continuous representations), if for every irreducible representation $(\pi,\mathcal{H})$ of A the image $\pi(A)$ is contained in the algebra $K(\mathcal{H})$ of compact operators on $\mathcal{H}$. A C^*-algebra A is said to be *postliminal* if all nontrivial quotients of A contain a nonzero closed two-sided liminal ideal. Note that for separable A this means that A is a C^*-algebra of type I (cf. [D1, §9]).

(b) Let S be an Ol'shanskiĭ semigroup. A nonzero function $\alpha: S \to \mathbf{R}^+$ is called an *absolute value* if $\alpha(st) \leqslant \alpha(s)\alpha(t)$ for all $s, t \in S$ and $\alpha(s^*) = \alpha(s)$ for all $s \in S$. We write $\mathcal{A}(S)$ for the set of all locally bounded absolute values on S.

Let $\alpha \in \mathcal{A}(S)$. A representation $(\pi, \mathcal{H})$ of S is called *α-bounded* if $\|\pi(s)\| \leqslant \alpha(s)$ for all $s \in S$. □

In [Ne6] we have constructed for each $\alpha \in \mathcal{A}(S)$ a C^*-algebra $C^*(S, \alpha)$ whose representations are precisely the α-bounded holomorphic representations of S. More precisely we have the following theorem.

THEOREM IV.2. *The C^*-algebra $C^*(S, \alpha)$ has the following properties:*

(i) *There exists a homomorphism $j: S \to \mathcal{M}(C^*(S, \alpha))$ mapping* $\operatorname{int}(S)$ *into $C^*(S, \alpha)$ such that $j|_{\operatorname{int}(S)}$ is holomorphic and* $\operatorname{span} j(\operatorname{int} S)$ *is dense in $C^*(S, \alpha)$.*

(ii) *For every nondegenerate representation $(\pi, \mathcal{H})$ of $C^*(S, \alpha)$ we have an extension π' to a representation of the multiplier algebra and $(\pi' \circ j, \mathcal{H})$ defines a holomorphic α-bounded representation of S on $\mathcal{H}$.*

(iii) *For every α-bounded holomorphic representation $(\pi, \mathcal{H})$ of S there exists a unique representation $(\tilde{\pi}, \mathcal{H})$ of $C^*(S, \alpha)$ such that $\tilde{\pi}' \circ j = \pi$, where $\tilde{\pi}'$ denotes the extension of $\tilde{\pi}$ to the multiplier algebra.*

Proof. [Ne6, Theorem IV.2]. □

THEOREM IV.3. *Let S be an Ol'shanskiĭ semigroup and $\alpha \in \mathcal{A}(S)$. Then the C^*-algebra $C^*(S, \alpha)$ is liminal.*

Proof. Let $(\pi, \mathcal{H})$ be an irreducible representation of $C^*(S, \alpha)$. Then we use Theorem IV.2 to see that we have a corresponding α-bounded holomorphic representation $\tilde{\pi}$ of S on $\mathcal{H}$. Now Theorem III.8 (cf. [Ne8, Theorem IV.3]) entails that

$$\tilde{\pi}(\operatorname{int} S) \subseteq B_1(\mathcal{H}) \subseteq K(\mathcal{H}).$$

Since $\tilde{\pi}(\operatorname{int} S)$ spans a dense subspace of $\pi(C^*(S, \alpha))$ by Theorem IV.2, it follows that $\tilde{\pi}$ maps $C^*(S, \alpha)$ into $K(\mathcal{H})$. □

Definition IV.4. (i) Let S be an Ol'shanskiĭ semigroup. We write $\widehat{S}$ for the set of equivalence classes of irreducible holomorphic representations. This set is called the *dual of* S.

(ii) Let $\pi \in \widehat{S}$. For $s \in \operatorname{int} S$ we set

$$\Theta_\pi(s) := \operatorname{tr} \pi(s).$$

Note that this function is well defined (Theorem III.8). It is called the *character of* π. □

Our next objective is the result that two irreducible representations are equivalent if and only if their characters agree. For the proof we need the following lemma which is a generalisation of Proposition 4.2.5 in [D1].

LEMMA IV.5. *Let C be a liminal C^*-algebra and $\pi_1, \ldots, \pi_n$ a set of pairwise non-equivalent irreducible representations. Set $\pi := \bigoplus_{i=1}^n \pi_i$. Then*

$$\pi(C) = \bigoplus_{i=1}^{n} K(\mathcal{H}_i).$$

Proof. We prove the assertion by induction over n. For $n=1$ the image $\pi_1(C)$ is a closed $*$-invariant subalgebra of $K(\mathcal{H}_1)$ which acts irreducibly on $\mathcal{H}_1$ (cf. [D1, 1.8.3]), hence $\pi_1(C) = K(\mathcal{H}_1)$ follows from [Wal2, p. 293].

Assume that $n \geqslant 2$ and that the assertion holds for collections of $n-1$ representations. Let $J_n := \ker \pi_n$. Using [Wal2, p. 304], we see that the restrictions $\pi_i|_{J_n}$ are irreducible for $i=1, \ldots, n-1$. Since J_n is a liminal C^*-algebra ([Wal2, p. 303]), and the restrictions $\pi_i|_{J_n}$ are pairwise nonequivalent by [D1, 2.10.4], the induction hypothesis implies that

$$\pi(J_n) = \bigoplus_{i=1}^{n-1} K(\mathcal{H}_i).$$

Since on the other hand $\pi_n(C) = K(\mathcal{H}_n)$, we find that

$$\pi(C) = \pi(J_n) + \pi_n(C) = \bigoplus_{i=1}^{n} K(\mathcal{H}_i). \qquad \square$$

For the following we recall some facts from functional analysis concerning the space $B_2(\mathcal{H})$ of *Hilbert–Schmidt operators* on a Hilbert space $\mathcal{H}$.

PROPOSITION IV.6. *Let $\mathcal{H}$ be a Hilbert space and $\mathcal{H}^* \cong \overline{\mathcal{H}}$ its dual space endowed with the scalar product $\langle v, w\rangle' := \langle w, v\rangle$. For $v, w \in \mathcal{H}$ we write $p_{v,w}$ for the rank-one operator $x \mapsto \langle x, w\rangle v$. Then the following assertions hold:*

(i) *The mapping*

$$\mathcal{H} \otimes \overline{\mathcal{H}} \to B_2(\mathcal{H}), \quad v \otimes w \mapsto p_{v,w},$$

induces an isomorphism $\mathcal{H} \widehat{\otimes} \overline{\mathcal{H}} \cong B_2(\mathcal{H})$, where the scalar product on $B_2(\mathcal{H})$ is given by

$$\langle A, B\rangle = \operatorname{tr}(AB^*) = \operatorname{tr}(B^*A).$$

(ii) *The assignment $\pi(X).A := XA$ defines a symmetric representation of $B(\mathcal{H})$ on the Hilbert space $B_2(\mathcal{H})$.*

(iii) *$B_1(\mathcal{H}) \subseteq B_2(\mathcal{H})$ is a dense subspace.*

Proof. (i) [We, p. 170].

(ii) In view of [We, p. 165], it only remains to check that the representation is symmetric:

$$\begin{aligned}\langle \pi(X)^*.A, B\rangle &= \langle A, \pi(X).B\rangle = \operatorname{tr}(A(XB)^*)\\ &= \operatorname{tr}(AB^*X^*) = \operatorname{tr}(X^*AB^*) = \langle \pi(X^*).A, B\rangle.\end{aligned}$$

(iii) In view of [We, p. 162], this follows from the fact that

$$l^1(\mathbf{N})\cap l^\infty(\mathbf{N}) \subseteq l^2(\mathbf{N})$$

is a dense subspace. □

COROLLARY IV.7. *For each* $\pi\in\widehat{S}$ *the character* Θ_π *is a holomorphic function on* $\operatorname{int} S$.

Proof. It follows from Proposition IV.6 (ii) that the mapping $B(\mathcal{H})\to B(B_2(\mathcal{H}))$ defined by the left multiplication representation is holomorphic. Hence the mapping

$$\operatorname{int} S \to B(B_2(\mathcal{H})), \quad s\mapsto (A\mapsto \pi(s)\cdot A),$$

is holomorphic. It follows in particular that for each $t\in\operatorname{int} S$ the function

$$s\mapsto \langle \pi(s), \pi(t)\rangle = \operatorname{tr}(\pi(s)\pi(t)^*) = \operatorname{tr}(\pi(st^*))$$

is holomorphic. Now the assertion is a consequence of [HN2, Proposition 3.19] because we may without loss of generality assume that S is simply connected. □

LEMMA IV.8. *Let S be an Ol'shanskiĭ semigroup and $\pi_1, ..., \pi_n$ a set of pairwise nonequivalent irreducible representations. Set $\pi := \bigoplus_{i=1}^n \pi_i$. Then $\pi(S)$ spans a dense subspace in*

$$\bigoplus_{i=1}^n B_2(\mathcal{H}_i).$$

Proof. Let $X_i\in B_2(\mathcal{H}_i)$, $i=1,...,n$, and $X := \bigoplus_{i=1}^n X_i \in B_2(\mathcal{H})$. Suppose that

$$\operatorname{tr}(X\pi(S)) = \{0\}.$$

We have to show that $X=0$.

Let $\alpha(s) := \max\{\|\pi_i(s)\| : i=1,...,n\}$. Then α is a locally bounded absolute value on S and we obtain corresponding representations $\widehat{\pi_1}, ..., \widehat{\pi_n}$ of the liminal C^*-algebra $C^*(S,\alpha)$.

For $A\in B(\mathcal{H})$ and $s\in\operatorname{int} S$ we set $F_X(A):=\operatorname{tr}(X\pi(s)A)$. Using that

$$|\operatorname{tr}(X\pi(s)A)|\leqslant\|X\pi(s)A\|_1\leqslant\|X\pi(s)\|_1\|A\|,$$

([We, p. 165]), we conclude that F_X is continuous. Thus $F_X(\pi(S))=\{0\}$ entails that $F_X(\pi(C^*(S,\alpha)))=\{0\}$. Using Lemma IV.5, we deduce that $\operatorname{tr}(X_i\pi(s)K(\mathcal{H}_i))=\{0\}$ holds for $i=1,\dots,n$. It follows in particular that

$$\operatorname{tr}(X_i\pi_i(s)B_2(\mathcal{H}_i))=\{0\},$$

and therefore $X_i\pi_i(s)=0$. Finally $X_i=0$ follows from the fact that the irreducible representation $(\pi_i,\mathcal{H}_i)$ of S is cyclic. □

THEOREM IV.9. *If $\pi_1,\dots,\pi_n$ represent disctinct elements of $\widehat{S}$, then the characters $\Theta_{\pi_1},\dots,\Theta_{\pi_n}$ are linearly independent.*

Proof. Suppose that $\sum_{i=1}^n\lambda_i\Theta_{\pi_i}=0$ on $\operatorname{int} S$ and choose $X_i\in B_1(\mathcal{H}_i)$. We define a function F on $\bigoplus_{i=1}^n B_2(\mathcal{H}_i)$ by

$$F(A):=\sum_{i=1}^n\lambda_i\operatorname{tr}(X_iA_i).$$

Then this function is continuous on $\bigoplus_{i=1}^n B_2(\mathcal{H}_i)$ because $X_i\in B_2(\mathcal{H}_i)$ (Proposition IV.6 (iii)).

Now we make the special choice $X_i:=\pi_i(s)$ for a fixed $s\in\operatorname{int} S$. Then

$$F(\pi_1(t),\dots,\pi_n(t))=\sum_{i=1}^n\lambda_i\operatorname{tr}(\pi_i(st))=\sum_{i=1}^n\lambda_i\Theta_{\pi_i}(st)=0,$$

so that Lemma IV.8 implies that F vanishes on $\bigoplus_{i=1}^n B_2(\mathcal{H}_i)$. We conclude that

$$\lambda_i\operatorname{tr}(\pi_i(s)\pi_i(s)^*)=\lambda_i\|\pi_i(s)\|_2^2=0\quad\forall s\in\operatorname{int} S.$$

Picking s such that $\pi_i(s)\neq0$, it follows that $\lambda_i=0$. □

COROLLARY IV.10. *Two irreducible holomorphic representations of an Ol'shanskiĭ semigroup S are equivalent if and only if their characters agree.* □

For the following theorem we recall from [Ne6] that a holomorphic function ϕ on $\operatorname{int} S$ is said to be *positive definite* if for $s_1,\dots,s_n\in\operatorname{int} S$ the matrix $(\phi(s_i^*s_j))_{i,j=1,\dots,n}$ is positive semidefinite. For such a function ϕ one can construct a Hilbert space $\mathcal{H}_\phi$ of holomorphic functions on $\operatorname{int} S$ such that the function ϕ is a reproducing kernel, i.e., $f(s)=\langle f,\phi_{s^*}\rangle$, where $\phi_{s^*}(t)=\phi(ts^*)$ for all $s,t\in\operatorname{int} S$ (cf. [Ne6, Proposition II.9]). For a function f on $\operatorname{int} S$ we define the function $\pi_r(s)(f)\colon t\mapsto f(ts)$ on $\operatorname{int} S$ for each $s\in S$.

THEOREM IV.11. *Let $(\pi, \mathcal{H})$ be an irreducible holomorphic representation of the Ol'shanskiĭ semigroup S. Then Θ_π is a holomorphic positive definite function on* int S *and the mapping*

$$\mathcal{H}_{\Theta_\pi} \to B_2(\mathcal{H}), \quad \pi_r(s).\Theta_\pi \mapsto \pi(s),$$

induces a unitary isomorphism of the reproducing kernel Hilbert space $\mathcal{H}_{\Theta_\pi}$ onto the space $B_2(\mathcal{H})$ of Hilbert–Schmidt operators on $\mathcal{H}$. The inverse of this mapping is given by

$$A \mapsto s \mapsto \operatorname{tr}(A\pi(s)).$$

Proof. First we note that

$$\langle \pi(s), \pi(t) \rangle = \operatorname{tr}(\pi(t)^* \pi(s)) = \Theta_\pi(t^* s)$$

for $s, t \in \operatorname{int} S$. From these relations it is immediate that the span of $\pi_r(S).\Theta_\pi$ in $\mathcal{H}_{\Theta_\pi}$ is mapped isometrically onto $\pi(S) \subseteq B_2(\mathcal{H})$ ([Ne6, II.9]). Hence it extends to an isometry of the completion $\mathcal{H}_{\Theta_\pi}$ onto $B_2(\mathcal{H})$ which is onto since $\pi(S)$ spans a dense subset by Lemma IV.8.

For the same reason it suffices to check the formula for the inverse on $\pi(S)$ where it is trivial. □

The topology on the dual

Definition IV.12. We write $\widehat{S}_\alpha$ for the set of unitary equivalence classes of α-bounded irreducible holomorphic representations of S. Then

$$\widehat{S} = \bigcup_{\alpha \in \mathcal{A}(S)} \widehat{S}_\alpha$$

is a directed union of subspaces. We endow the sets $\widehat{S}_\alpha$ with the topology inherited by the bijection $\widehat{S}_\alpha \cong C^*(S, \alpha)\widehat{}$ which in turn is inherited from the bijection

$$C^*(S, \alpha)\widehat{} \to \operatorname{Prim}(C^*(S, \alpha)), \quad \pi \mapsto \ker \pi$$

([D1, 4.4]), where the space of prime ideals which in this case coincides with the set of maximal ideals is endowed with the *Jacobson topology*. In this topology the closure of a set A of ideals is given by the set of all ideals containing $\bigcap A$.

If $\alpha \leqslant \beta$ in $\mathcal{A}(S)$, then we have a canonical morphism $C^*(S, \beta) \to C^*(S, \alpha)$ of C^*-algebras since the identity representation of $C^*(S, \alpha)$ is β-bounded. This morphism is surjective because the image of int S generates both. Thus $C^*(S, \alpha)\widehat{}$ can be identified with a subset of $C^*(S, \beta)\widehat{}$ ([D1, 2.11.2, 3.2.1]).

We define a topology on $\widehat{S}$ by saying that a subset $A \subseteq \widehat{S}$ is closed if and only if the intersections $A \cap \widehat{S}_\alpha$ are closed for all $\alpha \in \mathcal{A}(S)$. □

PROPOSITION IV.13. *The topology on $\widehat{S}$ has the following properties:*

(i) *The subspaces $\widehat{S}_\alpha$ are closed subspaces of $\widehat{S}$.*

(ii) *Each $\widehat{S}_\alpha$ is a Baire space which is locally quasi-compact.*

(iii) *The points in $\widehat{S}$ are closed.*

(iv) *Let $s=s^* \in \operatorname{int} S$. Then the function $\pi \mapsto \Theta_\pi(s)$ is lower semicontinuous on $\widehat{S}$.*

Proof. (i) This is immediate from the definition.

(ii) [D1, 3.4.13] and [D1, 3.3.8].

(iii) This follows from the fact that all closed subspaces $\widehat{S}_\alpha$ have this property ([D1, 4.4.1]).

(iv) In view of the definition of the topology on $\widehat{S}$, it suffices to check this on the subspaces $\widehat{S}_\alpha$. Now the assertion follows from [D1, 3.5.9] since $\widehat{S}_\alpha = C^*(S,\alpha)\widehat{\ }$. □

Disintegration of representations

THEOREM IV.14. *Let $(\pi,\mathcal{H})$ be a holomorphic representation of the Ol'shanskiĭ semigroup S and $\alpha(s):=\|\pi(s)\|$. Then there exists a Borel measure μ on $\widehat{S}_\alpha \subseteq \widehat{S}$ and a direct integral of representations*

$$\left(\int_{\widehat{S}}^{\oplus} \pi_\omega \, d\mu(\omega), \int_{\widehat{S}}^{\oplus} \mathcal{H}_\omega \, d\mu(\omega)\right)$$

such that:

(i) *$(\pi,\mathcal{H})$ is unitarily equivalent to $\left(\int_{\widehat{S}}^{\oplus} \pi_\omega \, d\mu(\omega), \int_{\widehat{S}}^{\oplus} \mathcal{H}_\omega \, d\mu(\omega)\right)$.*

(ii) *There exists a subset N of $\widehat{S}$ such that $\mu(N)=\{0\}$ and if $\omega \in \widehat{S}\backslash N$, then $(\pi_\omega,\mathcal{H}_\omega)$ is equivalent to $(\tilde{\pi}_\omega \otimes I, \widetilde{\mathcal{H}}_\omega \widehat{\otimes} V_\omega)$ with $(\tilde{\pi}_\omega, \widetilde{\mathcal{H}}_\omega) \in \omega$ and V_ω a Hilbert space.*

(iii) *If $\omega \in \widehat{S}$, then set $n(\omega):=\dim V_\omega$. Then n is a μ-measurable function from $\widehat{S}$ to the extended positive axis $[0,\infty]$ which is called the multiplicity function.*

Proof. This follows from [Wal2, p. 334] if we extend π to a representation of $C^*(S,\alpha)$ and recall that $\widehat{S}_\alpha = C^*(S,\alpha)\widehat{\ }$. □

Remark IV.15. If G is a connected Lie group and S an Ol'shanskiĭ semigroup such that $G \cong U(S)_0$, then we can consider the C^*-algebra $C^*(G)$ and in this algebra the ideal consisting of all those elements which are annihilated by those representations which do not extend to holomorphic representations of S. Then $A:=C^*(G)/I$ is a C^*-algebra which describes the representation theory of S and if G is a (CA) group, then Theorem I.4 shows that A is postliminal. It would be interesting to know whether this C^*-algebra is also liminal or not.

Remark IV.16. Let us say that a holomorphic representation $(\pi,\mathcal{H})$ of S is *tracable* if $\pi(\operatorname{int} S) \subseteq B_1(\mathcal{H})$. Such a representation decomposes into a discrete direct sum because the operators of int S are represented by compact operators. How is it possible to reconstruct π from its character? We conjecture that two such representations are equivalent if and only if they have the same character. Note that if we associate to the character Θ_π its reproducing kernel Hilbert space on S, we loose the information on the multiplicities but not on the support in $\widehat{S}$.

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