Acta Math., 173 (1994), 61–101

# The topology of the space of rational maps into generalized flag manifolds

by

C. P. BOYER

*University of New Mexico*
*Albuquerque, NM, U.S.A.*

J. C. HURTUBISE

*McGill University*
*Montréal, QC, Canada*

B. M. MANN

*University of New Mexico*
*Albuquerque, NM, U.S.A.*

and

R. J. MILGRAM(1)

*University of New Mexico*
*Albuquerque, NM, U.S.A.*

## 1. Introduction

The energy functional for smooth based maps of a surface $\Sigma$ into a smooth manifold $M$ and the Yang–Mills functional for connections on a smooth four-manifold $X$ have many points in common. Both arise in mathematical physics, the first in the guise of the non-linear $\sigma$-model, the second in gauge theories. Both correspond to borderline cases for the analytic conditions for a good Morse functional, and in both cases one can still salvage a good deal of information about the critical points. Indeed such information has proven extremely useful in the recent work of Donaldson, Sachs–Uhlenbeck, Taubes, and Yau, among others.

There is also a more precise technical sense in which the two are related, in that if in the first case, one chooses $\Sigma$ to be the Riemann sphere and $M$ to be a flag manifold, then the space of maps from $\Sigma$ to $M$ is homotopy equivalent to a space of Yang–Mills–Higgs fields on $\mathbf{R}^3$ with appropriate boundary conditions [Ta1], and the space of minima (in one case, rational maps, in the other, monopoles) of the respective energies are the same ([D], [Hur2], [BM]).

One intriguing aspect of these two problems is a topological stability phenomenon. Indeed, they both have associated to them a natural degree, or charge, and one can consider the inclusion

$$\iota_k\colon \mathrm{Min}_k \to \mathrm{Maps}_k$$

(1) During the preparation of this work the first, third, and fourth authors were supported by NSF grants and the second author by NSERC and FCAR grants.

of the minima of the functional at charge $k$ into the whole based mapping space at charge $k$. In several cases, it has been shown that the inclusions $\iota_k$ induce isomorphisms in homology groups $H_q$ or in homotopy groups $\pi_q$ for $q$ in a range $0\leqslant q\leqslant q(k)$ where $q(k)$ increases with $k$. In such cases we say that the inclusion is a homology or homotopy equivalence through dimension $q(k)$. A reason for desiring such a result is that the topology of the full mapping space $\mathrm{Maps}_k$ is usually fairly easy to describe, so one obtains important information about the minima $\mathrm{Min}_k$.

For example, for the case of the Yang–Mills functional for connections on a principal SU(2) bundle over $S^4$, Atiyah and Jones [AJ] conjectured that the inclusion of the moduli space $\mathcal{M}_k$ of based instantons of charge $k$ into the space of connections modulo based gauge transformations exhibited just such a behavior. A proof of this conjecture was given recently by the authors in [BHMM], where it was shown that the range $q(k)$ was at least $\left[\frac{1}{2}k\right]-2$. Recently Y. Tian [Ti] and F. Kirwan [Ki2] have proved the Atiyah–Jones conjecture for $\mathrm{SU}(n)$ instantons when $n>2$.

In this paper we consider the case when the target manifold $M$ of the maps $\Sigma\to M$ is a generalized flag manifold $G/P$. Here $G$ is a complex semi-simple Lie group, and $P$ is a parabolic subgroup. In this case, the minima of the energy correspond to based holomorphic maps. We will denote the space of such maps by $\mathrm{Hol}(\Sigma, G/P)$, and when $\Sigma=\mathbf{P}^1$, by $\mathrm{Rat}(G/P)$. For general $G/P$ the components of $\mathrm{Rat}(G/P)$ are indexed by sequences $\mathbf{k}$ of non-negative numbers (each $\mathbf{k}$ is called a multi-degree and defined in equation (1.3) below). We denote the $\mathbf{k}$ component of $\mathrm{Rat}(G/P)$ by $\mathrm{Rat}_{\mathbf{k}}(G/P)$. Again for $\Sigma=\mathbf{P}^1$ the space of based continuous maps is the two-fold loop space $\Omega^2(G/P)$. Forgetting the holomorphic structure, one has an inclusion

$$i(\mathbf{k})\colon \mathrm{Rat}_{\mathbf{k}}(G/P)\to\Omega^2_{\mathbf{k}}(G/P). \tag{1.1}$$

Occasionally, we need to consider the unbased mapping spaces. These spaces will be denoted by writing an 'overtilde', e.g. $\widetilde{\mathrm{Rat}}(M)$ denotes the space of unbased rational maps $\mathbf{P}^1\to M$.

The space $\mathrm{Rat}(G/P)$ was studied extensively by Segal [S], in the case $G/P=\mathbf{P}^n$. The relevant charge here is just the degree $k$ of the map, and for $\Sigma=\mathbf{P}^1$, Segal proved,

THEOREM 1.2 (Segal). *The inclusion $\iota_k\colon \mathrm{Rat}_k(\mathbf{P}^n)\to\Omega^2_k(\mathbf{P}^n)$ is a homotopy equivalence through dimension $(2n-1)k$.*

Segal conjectured that a similar theorem should hold for any $G/P$. Indeed, since then, for $G=\mathrm{SL}(n,\mathbf{C})$ ("classical flags") much has been done:

(1) Kirwan [Ki1] proved the analogous result for Grassmannians $G(n,n+m)$ with a range $k/(l^2+3l)-1$, where $l=\min(n,m)$. This range was later increased in [MM1], [MM3] to $2k+1$.

(2) Guest [Gu1] proved the analogous result in homology for complete $\mathrm{SL}(n,\mathbf{C})$ flag manifolds of the form $V_1 \subset V_2 \subset \ldots \subset V_{m-1} \subset V_m \subset V_n$ with $\dim V_i = i$. In this case the charge is a multi-degree $\mathbf{k}$ and the range of dimensions $q(\mathbf{k})$ tends to infinity as all the components $k_i$ of $\mathbf{k}$ tend to infinity.

(3) Mann and Milgram [MM2] proved a homology stability result for any classical $\mathrm{SL}(n,\mathbf{C})$ flag manifold. The range here is an explicit function of the multi-degree; cf. [MM2, Corollary B].

In this paper we verify Segal's conjecture in full generality for both homology and homotopy groups, for any generalized flag manifold $G/P$.

To begin recall that the components of $\Omega^2(G/P)$ are labelled by a multi-degree

$$\mathbf{k} = (k_1, \ldots, k_{n(\mathfrak{p})}) \in \pi_2(G/P) \cong \bigoplus_{j=1}^{n(\mathfrak{p})} \mathbf{Z} \tag{1.3}$$

whose $j$th component $k_j$ is the intersection number of the image of $S^2$ with the closure of a codimension one Bruhat cell. Equivalently, $k_j$ is the degree obtained as the first Chern class of the pull-back of a line bundle $\mathcal{O}_\mathfrak{p}(\lambda_i)$ to $\mathbf{P}^1$ [BE]. Similarly, it can be shown that the components of $\mathrm{Rat}(G/P)$ are also indexed by these same multi-degrees in (1.3) but where all the $k_j \geqslant 0$. Forgetting the holomorphic structure induces the natural inclusion $\iota(\mathbf{k})$ given above in equation (1.1). Furthermore, while all the components of $\Omega^2(G/P)$ are naturally homotopy equivalent, the components of $\mathrm{Rat}(G/P)$ are finite dimensional complex manifolds whose dimension and homotopy type depends on $\mathbf{k}$.

In this paper we prove

THEOREM A. *Let $G$ be any complex, semi-simple Lie group and $P$ any parabolic subgroup of $G$. For all $\mathbf{k}$ the inclusion $\iota(\mathbf{k})$ induces an isomorphism in homology with $\mathbf{Z}$ coefficients through dimension $q(\mathbf{k})$; i.e.,*

$$(\iota(\mathbf{k}))_t\colon H_t(\mathrm{Rat}_\mathbf{k}(G/P);\mathbf{Z}) \cong H_t(\Omega^2_\mathbf{k}(G/P);\mathbf{Z})$$

*for $t \leqslant q(\mathbf{k}) = \left[\min\left(\frac{1}{2}, c(G/P)\right) l(\mathbf{k})\right] - 1$. Here $[x]$ is the greatest integer less than or equal to $x$, $c(G/P)$ is a positive constant, which is defined in Proposition 6.6 and depends only on the space $G/P$, and $l(\mathbf{k}) = \min(k_i)$.*

Since $\mathrm{Rat}_\mathbf{k}(G/P)$ and $\Omega^2(G/P)$ are not always simply connected, our second main result does not trivially follow from Theorem A but rather is proved in §9. Set

$$r(\mathbf{k}) = \begin{cases} q(\mathbf{k}), & \text{if } H_1(\Omega^2_\mathbf{k}(G/P)) = \pi_1(\Omega^2_\mathbf{k}(G/P)) \text{ is torsion free,} \\ q(\mathbf{k}) - 1 & \text{otherwise.} \end{cases}$$

THEOREM B. *Let $G$ be any complex, semi-simple Lie group and $P$ any parabolic subgroup of $G$. For all $\mathbf{k}$ the inclusion $\iota(\mathbf{k}(G/P))$ is a homotopy equivalence through dimension $r(\mathbf{k})$ as defined above.*

We note that the range of stability increases linearly with the charge. The constant $c(G/P)$ reflects, in essence, the singularity structure of the closure of the codimension one Bruhat cells ("the subspace at infinity") in $G/P$. If this subspace at infinity is smooth, then $c(G/P)=1$.

Gravesen [Gr] constructed a stabilization map $\mathrm{Rat}_{\mathbf{k}}(G/P)\to\mathrm{Rat}_{\mathbf{k}'}(G/P)$ which increases every coordinate of the multi-degree and proved a stable result that showed there is a homology equivalence between the direct limit of the $\mathrm{Rat}_{\mathbf{k}}(G/P)$ and a component of $\Omega^2(G/P)$. However, his techniques gave no information at any finite level.

Before explaining how this paper is organized we point out two related results.

(1) Both Segal and Kirwan's stability theorems cited above extend, in homology, to cover the case when the domain is a compact Riemann surface. As yet, our techniques do not allow us to prove the full stability theorem for maps from compact Riemann surfaces into general $G/P$. One can, however, obtain a "limit stability" theorem similar to Gravesen's theorem in this more general case.

(2) Guest [Gu2] proved a stability theorem for maps from $\mathbf{P}^1$ into toric varieties. These toric varieties seem to be a very natural class of target spaces and would probably be well worth further study.

In order to explain the general strategy used in this paper we give a brief outline of the proof of the stability theorem in the simplest case: based holomorphic self maps of $\mathbf{P}^1$ where the basing condition is given by $f(\infty)=0$. We then explain how this approach must be modified to analyze the case for general $G/P$. Any such map can be written as a sum:

$$f(z)=\sum_{i=1}^{n}\frac{p_i(z-z_i)}{(z-z_i)^{k_i}}$$

where each $p_i$ is a polynomial of degree less than $k_i$, $p_i(0)\neq 0$, and $\sum k_i=k$ is the degree of $f$. This description allows us to stratify $\mathrm{Rat}_k(\mathbf{P}^1)$ according to the pattern of the multiplicities of the poles of the holomorphic map as follows: Let $K=(k_1,\dots,k_n)$ be a partition of $k$ and set

$$S_K=\{f\in\mathrm{Rat}_k(\mathbf{P}^1)\mid f \text{ has poles of multiplicities } k_i \text{ for } 1\leqslant i\leqslant n\}.$$

$S_K$ is a subset of a labelled configuration space; more precisely, it is the space of points $(z_1,\dots,z_n)$ in the complex plane where each point is labelled by a polynomial $p_i$ (of degree $<k_i$ and $p_i(z_i)\neq 0$). The labelling polynomials are parametrized by $\mathbf{C}^*\times\mathbf{C}^{k_i-1}$ and this space of labels is called the $k_i$th *principal parts space.*

The complex codimension of $S_K$ in $\mathrm{Rat}_k(\mathbf{P}^1)$ is $\sum(k_i-1)$ so there is only one maximal stratum which consists of holomorphic maps with $k$ simple poles at $k$ distinct points. The codimension one stratum consists of maps with $k-2$ simple poles and one double pole and as the multiplicities which index the strata increase so do the codimensions of the associated strata. This stratification leads to a Leray spectral sequence converging to the homology of $\mathrm{Rat}_k(\mathbf{P}^1)$ where the filtered terms at the $E_1$ level are the homologies of the Thom spaces of the normal bundles of the individual strata. Thus, by the Thom isomorphism theorem, the homology of each $S_K$ appears in the spectral sequence shifted up in dimension by its *real* codimension.

There is a stabilization map, first considered by Segal [S],

$$i(k,k+1)\colon \mathrm{Rat}_k(\mathbf{P}^1)\to \mathrm{Rat}_{k+1}(\mathbf{P}^1)$$

defined by adding a simple pole near infinity with a fixed residue. This stabilization map preserves the stratification and is filtration preserving with respect to the Leray spectral sequences for the domain and range. Thus, to prove that $i(k,k+1)$ induces an isomorphism in homology through a range determined by $k$ we begin by analyzing $i(k,k+1)$ on each stratum.

Since each stratum is a labelled configuration space of points in the complex plane the homology can be computed by techniques from iterated loop space theory [BHMM]. In particular, one can show that if $j$ is the number of simple poles in the stratum $S_K$ (so that there are $j+1$ simple poles in the corresponding stratum of $\mathrm{Rat}_{k+1}(\mathbf{P}^1)$ containing $i(k,k+1)(S_K)$), then $i(k,k+1)$ restricted to $S_K$ induces an isomorphism in homology through dimension $\left[\frac{1}{2}j\right]$.

Thus, the simple poles contribute to the stability result by increasing the range of equivalence on each individual stratum while the poles of higher multiplicity contribute to the stability result by increasing the codimension of the strata and hence, as mentioned above, shifting up the dimension where the homology of the $S_K$ appears in the associated Leray spectral sequence by the real codimension of $S_K$. A simple calculation shows that the smallest possible range of equivalence at the $E_1$ level occurs for the generic stratum. Therefore, taking possible differentials into account one may conclude that $i(k,k+1)$ induces a homology isomorphism through dimension $\left[\frac{1}{2}k\right]-1$. Finally, to strengthen the stability result to a statement in homotopy, one must analyze the induced map on universal covers. Here, the key geometric fact is that $\pi_1(\mathrm{Rat}_k(\mathbf{P}^1))\cong\mathbf{Z}$ is generated by a loop which fixes the configuration of the poles and moves a single principal part $p\in\mathbf{C}^*$ along the generator of $\pi_1(\mathbf{C}^*)$.

This is not the best possible result for self maps of $\mathbf{P}^1$ as, using a different stratification, [C$^2$M$^2$] gives the entire homology of $\mathrm{Rat}_k(\mathbf{P}^1)$. However, the method described above does have one great virtue in that it requires very little knowledge of the principal parts spaces $(\mathbf{C}^* \times \mathbf{C}^{k_i-1})$. Most important, all one needs is some knowledge of the codimensions of the strata that are indexed by non-simple poles. Consequently, this method can be extended to arbitrary $G/P$. We do so in the following four steps:

(1) We define suitable principal parts for holomorphic maps $f: \mathbf{P}^1 \to G/P$. Here we follow Segal and Gravesen so that the based holomorphic mapping spaces $\mathrm{Rat}_{\mathbf{k}}(G/P)$ are again stratified by subsets of certain labelled configurations spaces. We do this in §§2 and 3.

(2) We obtain an estimate on the codimension of the strata in terms of their multiplicity pattern so that the "more generic" strata will have more simple poles. In the special case described above where $G/P = \mathbf{P}^1$ this is trivial. However, in general this step is non-trivial and, in fact, in the end our estimates depend on the resolution of singularities theorem. This step is carried out in §5.

(3) In §§6, 7, and 8, we prove Theorem A by applying the techniques of [BHMM].

(4) We conclude this paper by proving Theorem B. Here we analyze $\pi_1(\mathrm{Rat}_{\mathbf{k}}(G/P))$ and its effect on the associated universal cover. We do this in §9 using information about the multiplicity one principal parts space studied in §4.

The authors would like to thank R. Cohen and D. Freed for helpful discussions.

## 2. Review of $G/P$

In this section we give a presentation of some facts about $G/P$. We refer to standard references [Hum], [BE] for more details.

Let $G$ denote any complex semi-simple Lie group, and $\mathfrak{g}$ its Lie algebra. We fix a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ and let $\mathfrak{u}^+$ and $\mathfrak{u}^-$ denote the positive and negative root spaces, respectively, with respect to $\mathfrak{h}$. One then has Borel subalgebras

$$\begin{aligned} \mathfrak{b}^+ &= \mathfrak{h} + \mathfrak{u}^+, \\ \mathfrak{b}^- &= \mathfrak{h} + \mathfrak{u}^-. \end{aligned} \tag{2.1}$$

We consider *parabolic* subalgebras $\mathfrak{p}$ obtained by adjoining to $\mathfrak{b}^+$ a certain number of negative root spaces. Then, $\mathfrak{p}$ has a natural complement $\mathfrak{n}^-$ in $\mathfrak{g}$ with $\mathfrak{n}^- \subset \mathfrak{u}^-$ and $\mathfrak{b}^+ \subset \mathfrak{p}$, viz.

$$\mathfrak{g} = \mathfrak{p} + \mathfrak{n}^-. \tag{2.2}$$

We will denote by $B^\pm$, $P$, $U^\pm$, and $N^\pm$ the Borel, parabolic, and unipotent subgroups of $G$ corresponding to the Lie algebras $\mathfrak{b}^\pm$, $\mathfrak{p}$, $\mathfrak{u}^\pm$, and $\mathfrak{n}^\pm$, respectively.

Let $W$ denote the Weyl group of $\mathfrak{g}$. Each element $w \in W$ has a length $\ell(w)$, namely the minimal number of simple reflections in a wall of a Weyl chamber. $W$ has a structure of a directed graph which is compatible with the length $\ell(w)$, and this gives $W$ a partial ordering $\leqslant$ by saying that $w \leqslant w'$ if either $w = w'$ or there is a directed path in $W$ with $\ell(w) < \ell(w')$. If we fix a dominant integral weight $\lambda(\mathfrak{p})$ for the parabolic subalgebra $\mathfrak{p}$, then the orbit of $\lambda(\mathfrak{p})$ under the right action of $W$ can be identified with a directed subgraph $W^{\mathfrak{p}}$ of $W$, known as the Hasse diagram of $\mathfrak{p}$. The Hasse diagram $W^{\mathfrak{p}}$ gives rise to a double coset decomposition called *the Bruhat decomposition* of $G$:

$$G = \bigsqcup_{w \in W^{\mathfrak{p}}} U^- wP. \tag{2.3}$$

(This is not quite the usual convention, but it is more convenient for our purposes.) From this we have a cell decomposition of $G/P$ as $U^-$ orbits, viz.

$$G/P = \bigsqcup_{w \in W^{\mathfrak{p}}} U^- \cdot (wP). \tag{2.4}$$

Let $X_w$ denote the $U^-$ orbit through $(wP)$. The codimension of $X_w$ is the length $\ell(w)$ of $w$. Each Hasse diagram $W^{\mathfrak{p}}$ contains the identity $e \in W$ and a unique maximal length element $w_0$. The "big cell" $X_e$ is determined by the identity $e \in W^{\mathfrak{p}}$, and the maximal element $w_0$ determines the "0" cell $X_{w_0}$. The closure $Z_w = \overline{X}_w$ of the cell $X_w$ is the *Schubert variety*

$$Z_w = \bigsqcup_{w' \in W^{\mathfrak{p}} : w' \geqslant w} X_{w'}. \tag{2.5}$$

Schubert varieties for $G/P$ have been extensively studied, since they freely generate the integral homology of $G/P$.

Of particular interest are the codimension one cells $X_w$ with $\ell(w) = 1$; that is, those $w_\alpha \in W^{\mathfrak{p}}$ corresponding to a simple reflection in the wall $W_\alpha$. These are in one-to-one correspondence with the simple root spaces $V_\alpha$ which are not in $\mathfrak{p}$. We write these cells as $X_\alpha$ and their closure $Z_\alpha$ can be realized as the zero locus of a section $s_\alpha$ of a holomorphic line bundle $\mathcal{O}_{\mathfrak{p}}(\lambda_\alpha)$ on $G/P$ so that the first Chern class $c_1(\lambda_\alpha)$ of the line bundle $\mathcal{O}_{\mathfrak{p}}(\lambda_\alpha)$ is Poincaré dual to $Z_\alpha$. These Chern classes freely generate $H^2(G/P, \mathbf{Z})$ whose dimension equals the dimension of the central part of the Levi factor $\mathfrak{l}$ of the Levi decomposition of $\mathfrak{p}$. We shall denote this number by $n(\mathfrak{p})$. Thus, elements of $H^2(G/P, \mathbf{Z})$, which index the path components of $\Omega^2(G/P)$, correspond to multi-degrees $\mathbf{k} = (k_1, ..., k_{n(\mathfrak{p})})$. Given a map $f: \mathbf{P}^1 \to G/P$, the integer $c_1(f^*(\mathcal{O}(\lambda_\alpha)))$ is $k_\alpha$.

A fundamental theorem in representation theory is the Bott–Borel–Weil theorem which describes finite dimensional irreducible representations of $G$ in terms of certain sheaf cohomology groups. We give only part of this theorem in a version that is convenient for us.

THEOREM 2.6 (Bott–Borel–Weil). *$H^0(G/P, \mathcal{O}_{\mathfrak{p}}(\lambda_\alpha))$ is in a natural way an irreducible representation of $G$, and the section $s_\alpha$ is a highest weight vector stabilized by the parabolic subgroup $P$. Furthermore, all other cohomology groups vanish.*

This theorem allows one to map $G/P$ to $\mathbf{P}(V_\alpha)$ as a projective algebraic variety in such a way that the image of $Z_\alpha$ is cut out in $\mathbf{P}(V_\alpha)$ by the hyperplane at "infinity". This motivates the definition:

*Definition* 2.7. We denote by *infinity* the union

$$Z = \bigcup_\alpha Z_\alpha.$$

Note that $Z$ is the complement of the big cell $X_e \simeq N^-$ in the Bruhat decomposition of $G/P$.

In the case that $P$ is maximal parabolic, one has a smooth embedding. That is, if we denote the embedding map by $e_\alpha$, put $N = \dim \mathbf{P}(V_\alpha)$ and let $\mathbf{P}^{N-1}$ denote the hyperplane at infinity obtained by putting the first homogeneous coordinate on $\mathbf{P}^N = \mathbf{P}(V_\alpha)$ equal to zero, we have

$$Z_\alpha = e_\alpha(G/P) \cap \mathbf{P}^{N-1}.$$

$Z_\alpha$ is an algebraic variety of complex dimension $\dim \mathfrak{g} - \dim \mathfrak{p} - 1$, but in general it is not smooth. Its smooth locus $Z_\alpha^*$ contains the codimension one Bruhat cell $X_\alpha$, and the cells which make up its boundary must have $\ell(w) > 1$. In fact, more is true. It was shown by Ramanan and Ramanathan [RR] that Schubert varieties of $G/P$ are projectively normal. But for normal varieties singular sets have complex codimension $\geqslant 2$ ([I, p. 129]). Thus,

PROPOSITION 2.8. *The cells that make up the singular locus of $Z_\alpha$ have complex codimension at least 3 in $G/P$, i.e., $\ell(w_\alpha) \geqslant 3$.*

We shall be interested in the action of the "opposite unipotent" group $N^- \subset U^-$ on the various strata in the decomposition (2.4). In particular, one has:

PROPOSITION 2.9. *$N^-$ acts freely and transitively on the "big" cell $X_e$. This identifies $X_e$ with $N^-$ itself.*

On the other $U^-$ orbits, the action of $N^-$ is far from free. For example, for $G/P = \mathbf{P}^n$, the action $N^-$ is trivial on all cells except the "big" cell where it is transitive. It is important to realize that $N^-$ is compatible with the cell decomposition (2.5); that is, $N^-$ acts on each cell $X_w$ separately.

## 3. Poles and principal parts

In this section we describe the poles and principal parts of based holomorphic maps $f: \mathbf{P}^1 \to G/P$. This picture is due to Gravesen and Segal, and generalizes our description of maps from $\mathbf{P}^1$ to $\mathbf{P}^1$. The basic idea is that a holomorphic map $f: \mathbf{P}^1 \to G/P$ is determined by its poles (i.e., the points mapping to "infinity", the complement of the big cell in $G/P$) along with some extra data concentrated at these poles (the local principal parts).

For any complex analytic set $X$ we let $\mathcal{O}(X)$ denote the sheaf of germs of holomorphic maps from $\mathbf{P}^1$ into $X$, and $\mathcal{O}(U, X)$ denote the holomorphic sections of $\mathcal{O}(X)$ over the open set $U \subset \mathbf{P}^1$. If the space $X$ has a base point, $\mathcal{O}(X)$ will be a sheaf of pointed sets. Recall from Definition 2.7 that $Z$ denotes infinity of $G/P$. Then the presheaf of meromorphic maps to the opposite unipotent $N^-$ is

$$\mathcal{M}(U) = \mathcal{O}(U, G/P) \backslash \mathcal{O}(U, Z) \tag{3.1}$$

and we let $\mathcal{M}$ denote its associated sheaf. The natural action of $N^-$ on $G/P$ induces a free action of $\mathcal{O}(U, N^-)$ on $\mathcal{M}(U)$, which leaves the position of the poles in $\mathbf{P}^1$ fixed as well as their orders, and the quotient sheaf of principal parts $\mathcal{PP} = \mathcal{M}/\mathcal{O}(N^-)$ is well defined. As the action of $N^-$ on the big cell is free and transitive, there is the trivial principal part, corresponding to maps whose image lies entirely in the big cell; other, non-trivial principal parts will be defined by maps whose image intersects infinity. Thus, we have an exact sequence of sheaves of pointed sets,

$$0 \to \mathcal{O}(N^-) \to \mathcal{M} \to \mathcal{PP} \to 0. \tag{3.2}$$

A global section in $H^0(\mathbf{P}^1, \mathcal{PP})$ is called a *configuration of principal parts.* It is rather similar to a global section of the sheaf of divisors, in that it consists of a finite number of points $z_i \in \mathbf{P}^1$ (location of the poles), together with the *local* principal parts data, a non-trivial element in the stalk $\mathcal{PP}_{z_i} \simeq \mathcal{M}_{z_i}/\mathcal{O}(N^-)_{z_i}$ at each point $z_i$. As $\mathcal{PP}_z$ is independent of the point $z$ we denote this space of local principal parts by $\mathcal{LPP}$.

Let us now consider the basing condition. Recall that $\Omega^2(G/P)$ is the space of continuous based maps. We choose the base point in $\mathbf{P}^1 \simeq S^2 \simeq \mathbf{C} \cup \{\infty\}$ to be the north pole $\{\infty\}$ and the base point of $G/P$ to be a fixed point $b \in X_e \simeq N^-$. This precludes a map in $\mathrm{Rat}_{\mathbf{k}}$ having a pole at $\infty$. Let $H^0(\mathbf{C}, \mathcal{PP})$ be the subset of $H^0(\mathbf{P}^1, \mathcal{PP})$ whose poles are all located in $\mathbf{C} \simeq \mathbf{P}^1 - \{\infty\}$. Then, exactly as for $G/P = \mathbf{P}^1$, we find:

THEOREM 3.3. *Each element of $H^0(\mathbf{C}, \mathcal{PP})$ determines a unique based map, so that the space of based holomorphic maps can be identified with the space of configurations of principal parts*:

$$\mathrm{Rat}(G/P) \simeq H^0(\mathbf{C}, \mathcal{PP}).$$

The proof is given in Gravesen [Gr]. The key point is that while $N^-$ is non-Abelian in general, by using the fact that it is solvable, we can still show that $H^1(\mathbf{P}^1, \mathcal{O}(N^-))=0$ and that (3.2) induces an exact sequence of pointed sets

$$0 \to \mathbf{C}^d \to H^0(\mathbf{P}^1, \mathcal{M}) \to H^0(\mathbf{P}^1, \mathcal{PP}) \to 0,$$

where $\dim N^- = d$ and the first factor $\mathbf{C}^d$ includes into the $\mathbf{k}=(0,\dots,0)$ component of $H^0(\mathbf{P}^1, \mathcal{M})$ as the constant maps. (There is only the trivial principal part in degree 0.) The basing condition removes the $\mathbf{C}^d$ ambiguity.

According to our previous discussion a pole of $f \in \mathrm{Rat}(G/P)$ at $z_0 \in \mathbf{C} \subset \mathbf{P}^1$ corresponds to the condition $f(z_0) \in Z$. This is preserved by the action of $N^-$. Also, as remarked above, the action of $\mathcal{O}(N^-)$ on maps $f$ into $G/P$ preserves the order of vanishing of the sections $f^* s_\alpha$ which cut out infinity. Thus principal parts have a natural multiplicity:

*Definition* 3.4. The point $z_0$ is called an *$\alpha$-pole* of $f$ if $f(z_0)$ lies in the $\alpha$-component $Z_\alpha$ of $Z$. The *$\alpha$-multiplicity* $m_\alpha$ of this pole is the order of vanishing of $f^* s_\alpha$ at $z_0$, where $s_\alpha$ is the holomorphic section of the line bundle $\mathcal{O}_{\mathfrak{p}}(\lambda_\alpha)$ whose zero locus defines $Z_\alpha$. The total *multiplicity* of the pole is the multi-index $\mathbf{m}=(m_1,\dots,m_{n(\mathfrak{p})})$.

For a map $f$, the $\alpha$-degree $k_\alpha$ of $f$ (i.e., the $\alpha$ component of $\mathbf{k}$) is the intersection number of $f(\mathbf{P}^1)$ with $Z_\alpha$. This number is precisely the number of zeroes counting multiplicity of $f^* s_\alpha$. Let $m^i_\alpha$ denote the multiplicity of the $i$th $\alpha$-pole in $Z_\alpha$ where $1 \leqslant i \leqslant r_\alpha$, and $r_\alpha$ is the number of $\alpha$-poles (excluding multiplicity). Then we have

$$\sum_{i=1}^{r_\alpha} m^i_\alpha = k_\alpha, \tag{3.5}$$

for $1 \leqslant \alpha \leqslant n(\mathfrak{p})$. We can consider all multiplicities at once, so that if one has $r$ poles with multiplicity $\mathbf{m}^i$, for $i=1,\dots,r$; that is, the $i$th pole has multiplicity $\mathbf{m}^i=(m^i_1,\dots,m^i_{n(\mathfrak{p})})$ with $\alpha$-component $m^i_\alpha$, then equation (3.5) then becomes

$$\sum_{i=1}^{r} \mathbf{m}^i = \mathbf{k}. \tag{3.6}$$

We also define the *scalar multiplicity* of the $i$th pole by

$$|\mathbf{m}^i| = \sum_{\alpha=1}^{n(\mathfrak{p})} m^i_\alpha. \tag{3.7}$$

In particular, the scalar degree is $|\mathbf{k}| = \sum_\alpha k_\alpha$.

In order to understand the local principal parts space $\mathcal{LPP}$ as well as illustrate our point of view we give two examples. The first is very well understood from a classical point of view, but it is worth describing from our point of view to set the stage for the second more interesting example.

*Example* 3.8. $\mathrm{Rat}_{\mathbf{k}}(\mathbf{P}^1)$. We can represent $\mathbf{P}^1$ as the coset space $\mathrm{SL}(2,\mathbf{C})/B$ where $B$ is the Borel subgroup of upper triangular matrices. In this identification the affine neighborhood in $\mathbf{P}^1$ given in homogeneous coordinates by $[1,a]$ can be represented by matrices of the form

$$g_a = \begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix} \tag{3.9a}$$

while the "point at infinity" $[0,1]$ can be represented by

$$g_\infty = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{3.9b}$$

The cosets $gB$ with $g \in \mathrm{SL}(2,\mathbf{C})$ that are represented by matrices of the form (3.9a) describe the "big cell" $X_e$ in $\mathrm{SL}(2,\mathbf{C})/B$, and also viewed as a subgroup of $\mathrm{SL}(2,\mathbf{C})$ the matrices of the form (3.9a) describe the "opposite unipotent" subgroup $N^-$. Clearly, $N^-$ acts transitively on $X_e$ and there is a set identification of $X_e$ with $N^-$. Thus, there are precisely two $N^-$ orbits; the big cell $X_e$, and the "point at infinity" $g_\infty B$. The action of $N^-$ on the big cell is simply $g_a(g_bB) = g_{a+b}B$.

Now consider a based holomorphic map $f\colon \mathbf{P}^1 \to \mathbf{P}^1$ defined by $f(z) = [g(z), h(z)]$, where $g(z)$ and $h(z)$ are polynomials in $\mathbf{C}$. We take the base point in the domain $\mathbf{P}^1$ to be the north pole $[0,1]$ and in the target $\mathbf{P}^1$ to be $[1,1]$. This amounts to taking $g(z)$ and $h(z)$ to be monic of the same degree. The poles of $f$ are the zeroes of $g$. In the $G/P$ model the base point is represented by the matrix (3.9a) with $a=1$. Away from the poles the corresponding coset in $\mathrm{SL}(2,\mathbf{C})/B$ can be represented by

$$\begin{pmatrix} 1 & 0 \\ h(z)/g(z) & 1 \end{pmatrix}. \tag{3.10}$$

As a holomorphic map this is ill-defined at a pole; however, the matrix (3.10) is equivalent by an element of the group $\mathcal{O}(B)$ of holomorphic maps from $\mathbf{C}$ into $B$ to the matrix

$$\begin{pmatrix} g(z) & -1/h(z) \\ h(z) & 0 \end{pmatrix} \tag{3.11}$$

which is well-defined at the pole and equals

$$\begin{pmatrix} 0 & -1/h(z) \\ h(z) & 0 \end{pmatrix}$$

there. By another transformation in $\mathcal{O}(B)$ this is equivalent to "infinity" in $\mathrm{SL}(2,\mathbf{C})/B$, namely the matrix $g_\infty$ of (3.9b).

To compute the local principal parts space $\mathcal{LPP}$ we consider a pole of multiplicity $m$ at $z_0$ given by $g(z)=(z-z_0)^m$. According to the exact sequence (3.2) we compute the principal parts by normalizing by the right action of $\mathcal{O}(N^-)$ on the matrix

$$\begin{pmatrix} 1 & 0 \\ h(z)/(z-z_0)^m & 1 \end{pmatrix}.$$

This action allows us to add to $h(z)/(z-z_0)^m$ any function holomorphic near $z_0$, and so one can reduce $h(z)$ to a polynomial of degree less than $m$, with $h(z_0)$ non-zero. This gives the well known classical result that the local principal parts space $\mathcal{LPP}_m$ for a pole of multiplicity $m$ is just $\mathbf{C}^* \times \mathbf{C}^{m-1}$.

*Example* 3.12. Full $\mathrm{SL}(3,\mathbf{C})$ flags. Let $F$ be the space defined by

$$F=\{(V_1,V_2)\in \mathbf{P}^2\times(\mathbf{P}^2)^* \mid V_1\in V_2\}, \tag{3.13}$$

or, in words, $F$ is the set of pairs $(V_1,V_2)$ where $V_2$ is a projective line in $\mathbf{P}^2$ and $V_1$ is a point of $\mathbf{P}^2$ lying on the line $V_2$. Now $F$ can be realized as the homogeneous space $\mathrm{SL}(3,\mathbf{C})/B$ where $B$ is the Borel subgroup of upper triangular matrices. The opposite unipotent subgroup is

$$N^-=\left\{\begin{pmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ c & b & 1 \end{pmatrix} \middle| \; a,b,c\in\mathbf{C}\right\}. \tag{3.14}$$

We now describe the cell decomposition, (2.4), explicitly for this case. Since $P=B$ a Borel subgroup, the Hasse diagram $W^{\mathfrak{p}}$ coincides with the Weyl group $W$. The directed graph structure is given by Example 4.1.5 of [BE]. We shall adopt the notation of that example by using subscripts $1,2,12,\ldots$ to label the cells instead of elements of $W$. Then the number of subscripts equals the complex codimension of the cells. Thus, we call the big cell $X$ instead of $X_e$. Now let $R_1$ denote the point in $\mathbf{P}^2$ given in homogeneous coordinates by $[0,0,1]$, and let $R_2$ denote the projective line in $\mathbf{P}^2$ given in homogeneous coordinates by $[0,x,y]$. Then the big cell is

$$X=\{(V_1,V_2)\in F \mid V_1\notin R_2, R_1\notin V_2\}, \tag{3.15a}$$

while the codimension one Bruhat cells $X_\alpha$ are

$$X_1=\{(V_1,V_2)\in F \mid V_1\in R_2, R_\alpha\neq V_\alpha\}, \tag{3.15b}$$

$$X_2=\{(V_1,V_2)\in F \mid R_1\in V_2, R_\alpha\neq V_\alpha\}. \tag{3.15c}$$

The codimension two Bruhat cells are

$$X_{12}=\{(V_1,V_2)\in F \mid V_1=R_1, R_2\neq V_2\}, \tag{3.15d}$$

$$X_{21}=\{(V_1,V_2)\in F \mid V_2=R_2, R_1\neq V_1\}, \tag{3.15e}$$

and finally the codimension three (0-cell) Bruhat cell is

$$X_{121} = \{(V_1, V_2) \in F \mid V_1 = R_1, R_2 = V_2\}. \tag{3.15f}$$

Infinity is the singular variety $\overline{X}_1 \cup \overline{X}_2$ and we have the set identifications

$$\overline{X}_1 \cap \overline{X}_2 = \overline{X}_{12} \cup \overline{X}_{21}, \quad \overline{X}_{12} \cap \overline{X}_{21} = X_{121}.$$

Now let $f \in \mathrm{Rat}(F)$ have a pole at $z_0$ of multiplicity $\mathbf{m} = (m_1, m_2)$. In analogy with the previous example this can be represented by a matrix of the form (3.14) where $a, b, c$ are now viewed as functions of $z$ with possible poles at $z_0$. As before, this representation is not well-defined at the poles; however, it is equivalent under the right action of $\mathcal{O}(B)$ to a coset representative which is well-defined at the pole and lies in the complement of the big cell. Thus, we need matrix representatives for elements of $Z$. For example, the codimension one Bruhat cell $X_1$ is an $N^-$ orbit that can be represented by matrices of the form

$$\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ l(z) & m(z) & m(z)+1 \end{pmatrix}.$$

In the intersection of the big cell and a suitable neighborhood of $X_1$ (where $l(z)$ and $m(z)$ are two of the three coordinates in the neighborhood) we have $l(z) = b(z)$ and $m(z) = c(z) - a(z)b(z)$ where $a$, $b$, and $c$ are given in equation (3.14). This representation is well-defined at $z_0$ as long as $b$ and $c - ab$ are regular there. Thus a pole in $N$ will give an image in $X_1$, away from $X_2$, if $b, c - ab$ remain finite. Similarly, if $a, c$ remain finite, one obtains an image in $X_2$, away from $X_1$. Generally, poles of $a$ or $c$ correspond to image points in $\overline{X}_1$, and poles of $b$ or $c - ab$ correspond to image points in $\overline{X}_2$. Moreover, if

$$\lim_{z \to z_0} \frac{a}{c} = 0$$

then $z_0$ corresponds to an image point in $\overline{X}_{12}$ whereas if

$$\lim_{z \to z_0} \frac{b}{c - ab} = 0$$

then $z_0$ corresponds to an image point in $\overline{X}_{21}$. In this way one sees that the multiplicity $\mathbf{m}$ at a pole $z_0$ is given by

$$\mathbf{m} = (\max(-\mathrm{ord}_{z_0} a, -\mathrm{ord}_{z_0} c), \max(-\mathrm{ord}_{z_0} b, -\mathrm{ord}_{z_0} c - ab)). \tag{3.16}$$

It is easy to check that this is invariant under the action by left multiplication of elements of $\mathcal{O}(N^-)$.

Let us now compute the local principal part space $\mathcal{LPP}$ for three cases, namely, poles of multiplicity $(1,0)$, $(0,1)$, or $(1,1)$. A pole of multiplicity $(1,0)$ at $z_0$ can be normalized (under the action of $\mathcal{O}(N^-)$) to an element of the form

$$\begin{pmatrix} 1 & 0 & 0 \\ \alpha/(z-z_0) & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{3.17a}$$

whereas a pole of multiplicity $(0,1)$ can be normalized to

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \beta/(z-z_0) & 1 \end{pmatrix}. \tag{3.17b}$$

Here $\alpha, \beta \in \mathbf{C}^*$ so that the local principal parts spaces $\mathcal{LPP}_{(1,0)}$ and $\mathcal{LPP}_{(0,1)}$ equal $\mathbf{C}^*$ in both cases. Poles of multiplicity $(1,1)$ corresponding to image points of $f$ in $\overline{X}_{12}$ can be normalized to the form

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \gamma/(z-z_0) & \beta/(z-z_0) & 1 \end{pmatrix}, \tag{3.17c}$$

whereas poles of multiplicity $(1,1)$ corresponding to image points in $\overline{X}_{21}$ can be normalized to the form

$$\begin{pmatrix} 1 & 0 & 0 \\ \alpha/(z-z_0) & 1 & 0 \\ \gamma/(z-z_0) & 0 & 1 \end{pmatrix}, \tag{3.17d}$$

where $\alpha, \beta \in \mathbf{C}$ and $\gamma \in \mathbf{C}^*$. Then the local principal part space $\mathcal{LPP}_{(1,1)}$ is given by the singular variety

$$\mathbf{C}^* \times \{(\alpha, \beta) \in \mathbf{C}^2 \mid \alpha\beta = 0\}. \tag{3.18}$$

Let us now return to the general situation of maps $f \in \mathrm{Rat}_{\mathbf{k}}(G/P)$.

If elements $f$ and $g$ of $\mathrm{Rat}(G/P)$ have no poles in common, then viewing both $f$ and $g$ as configurations of principal parts, they can be "added" in a continuous way by considering their union $f \cup g$ as configurations of principal parts in the obvious way. This gives $\mathrm{Rat}(G/P)$ a homotopy-associative monoid structure by first defining homotopies which map the poles of $f$ and those of $g$ into disjoint discs in $\mathbf{C}$, and then taking their union as configurations of principal parts. This gives rise (up to homotopy) to a continuous map

$$\mathrm{Rat}_{\mathbf{k}}(G/P) \times \mathrm{Rat}_{\mathbf{l}}(G/P) \to \mathrm{Rat}_{\mathbf{k}+\mathbf{l}}(G/P), \tag{3.19}$$

where $\mathbf{k}+\mathbf{l}$ has components $k_\alpha + l_\alpha$. This "additivity" of principal parts will play a very important role in our development.

Actually more is true, namely the construction in (3.19) enriches to give

$$\coprod_{\mathbf{k}} \mathrm{Rat}_{\mathbf{k}}(G/P)$$

a $C_2$ operad space structure which is homotopy compatible with the standard two-fold loop space structure on $\Omega^2(G/P)$. This fact was used in [BM] and [C$^2$M$^2$] when $G/P=\mathbf{P}^n$, and, in a different guise, in [MM2] and [MM3] when $G=\mathrm{SL}(n,\mathbf{C})$.

Finally, recall $\mathrm{SP}^k(X)$ denotes the $k$-fold symmetric product of $X$. Our previous analysis implies:

PROPOSITION 3.20. *There is a natural holomorphic projection*

$$\Pi\colon \mathrm{Rat}_{\mathbf{k}}(G/P) \to \mathrm{SP}^{|\mathbf{k}|}(\mathbf{C}) \simeq \mathbf{C}^{|\mathbf{k}|}$$

*which associates to each element of* $\mathrm{Rat}_{\mathbf{k}}(G/P)$ *its polar divisor.*

We refer to $\Pi$ as the "pole location map". Its fibres are subsets of $\mathcal{LPP}$.

## 4. The Space Rat($G/P$) is a smooth complex manifold

In this section we prove that the space $\mathrm{Rat}_{\mathbf{k}}(G/P)$ is a smooth complex manifold of a certain dimension determined by multi-degree $\mathbf{k}$ and the dimension of the stabilizer of the action of the opposite unipotent group $N^-$ at a smooth point in the closure of the codimension one Bruhat cell $Z$. Explicitly,

THEOREM 4.1. *The space* $\mathrm{Rat}_{\mathbf{k}}(G/P)$ *is a smooth complex manifold of complex dimension* $\sum_{\alpha=1}^{n(\mathrm{p})} k_\alpha(\dim \mathfrak{s}_\alpha + 1)$, *where* $\mathfrak{s}_\alpha$ *is the Lie algebra of the stability subgroup of the point* $w_\alpha P$ *of* $Z_\alpha$.

*Proof.* To prove smoothness we apply Kodaira's deformation theory [Ko] to the map $F\colon \mathbf{P}^1 \to \mathbf{P}^1 \times G/P$ with $F=(\mathrm{id}, f)$. First consider the space $\widetilde{\mathrm{Rat}}(\mathbf{P}^1, G/P)$ of unbased holomorphic maps from $\mathbf{P}^1$ to $G/P$ and the pullback bundle $f^*T(G/P)$ over $\mathbf{P}^1$. Considering based holomorphic maps amounts to twisting $f^*T(G/P)$ by the tautological line bundle $\mathcal{O}(-1)$ on $\mathbf{P}^1$. Now the smoothness of $\mathrm{Rat}_{\mathbf{k}}$ will follow from [Ko] if we can show that

$$H^1(\mathbf{P}^1, f^*T(G/P)(-1)) = 0. \tag{4.2}$$

Furthermore, when (4.2) holds the tangent space $T_f\,\mathrm{Rat}(\mathbf{P}^1, G/P)$ is the space of holomorphic sections of $f^*T(G/P)(-1)$, that is

$$T_f\,\mathrm{Rat}(G/P) \simeq H^0(\mathbf{P}^1, f^*T(G/P)(-1)). \tag{4.3}$$

To prove that (4.2) holds, we note that the action of $N^-$ on $G/P$ induces a morphism of sheaves

$$\mathcal{O}^{\oplus d} \to T(G/P) \tag{4.4}$$

which is an isomorphism over the big cell. Pulling back via $f$, we have

$$\phi: \mathcal{O}^{\oplus d} \to f^*T(G/P) \tag{4.5}$$

which is again an isomorphism over some open set $U$. Over $\mathbf{P}^1$, $f^*T(G/P)$ splits as a direct sum $\bigoplus_i(\mathcal{O}(\lambda_i))$, by the Birkhoff–Grothendieck theorem. Since $\phi$ is a morphism of sheaves we have a commutative diagram,

$$\begin{tikzcd} \mathcal{O}^{\oplus d}(\mathbf{P}^1) \arrow[r, "\phi(\mathbf{P}^1)"] \arrow[d] & \bigoplus_i \mathcal{O}(\lambda_i)(\mathbf{P}^1) \arrow[d] \\ \mathcal{O}^{\oplus d}(U) \arrow[r, "\phi(U)"] & \bigoplus_i \mathcal{O}(\lambda_i)(U), \end{tikzcd} \tag{4.6}$$

where the vertical maps are the natural restrictions and $\phi(U)$ is an isomorphism. If one of the $\lambda_i$ were negative, the map $\phi(\mathbf{P}^1)$ on global sections would vanish on this factor. But then, by the commutativity of the diagram, the lower map $\phi(U)$ would not be injective which would be a contradiction. Hence, we must have $\lambda_i \geqslant 0$ for all $i$. Thus,

$$H^1(\mathbf{P}^1, f^*T(G/P)(-1)) = H^1\Big(\mathbf{P}^1, \bigoplus_i \mathcal{O}(\lambda_i)(-1)\Big) = 0. \tag{4.7}$$

Since generically $f \in \mathrm{Rat}_{\mathbf{k}}(G/P)$ is given by simple poles at $k_\alpha$ points in the $\alpha$th component $Z_\alpha$ of the closure of the codimension one Bruhat cell (see [Gr]) and the "local" principal parts data is additive, to compute the dimension of the space $\mathrm{Rat}_{\mathbf{k}}(G/P)$, it is enough for dimensional purposes to prove the result for degree one rational maps, that is elements of $\mathrm{Rat}(G/P)$ of $\alpha$-degree one having multi-index $\mathbf{k} = (0, ..., 1, ..., 0)$ where the $k_\alpha = 1$ and all other components of $\mathbf{k}$ are zero. This corresponds to a single pole of multiplicity one. Thus, we shall prove:

LEMMA 4.8. *The smooth manifold* $\mathrm{Rat}_{(0,\ldots,1,\ldots,0)}(G/P)$ *of rational maps of* $\alpha$*-degree one has complex dimension* $\dim \mathfrak{s}_\alpha + 1$.

The first step in the proof of Lemma 4.8 is to reduce to the case when $P$ is a maximal parabolic subgroup. This is done in the following

LEMMA 4.9. *There is a complex simple Lie group* $\widehat{G}$ *and a maximal parabolic subgroup* $\widehat{P} \subset \widehat{G}$ *such that*

$$\mathrm{Rat}_{(0,\ldots,1,\ldots,0)}(G/P) \cong \mathrm{Rat}_1(\widehat{G}/\widehat{P}).$$

*Furthermore, this equivalence is compatible with the cell decompositions, and the stabilizer $\mathfrak{s}_\alpha$ is isomorphic to the corresponding stabilizer algebra $\hat{\mathfrak{s}}$ for the action of corresponding opposite unipotent group $\widehat{N}^-$ on $\widehat{G}/\widehat{P}$.*

*Proof.* For notational convenience let $r=n(\mathfrak{p})$ and assume that $r>1$. There is a parabolic subgroup $P'$ with $P\subset P'\subset G$, $\pi_2(G/P')=\mathbf{Z}^{r-1}$ such that the fibration

$$P'/P\to G/P\xrightarrow{\varrho} G/P' \tag{4.10}$$

induces the projection

$$\varrho_*:\mathbf{Z}^r\to\mathbf{Z}^{r-1} \tag{4.11}$$

on the second homotopy groups given by forgetting the $\alpha$th coordinate of the domain. Thus, any map $f$ in $\mathrm{Rat}_{(0,\dots,1,\dots,0)}(G/P)$ projects to a map of degree zero, which must be constant and, since the map is based, it lies in the fibre $\mathrm{Rat}_0(P'/P)$. It follows that

$$\mathrm{Rat}_{(0,\dots,1,\dots,0)}(G/P)\cong\mathrm{Rat}_1(P'/P).$$

But the quotient $P'/P$ can be rewritten as $\widehat{G}/\widehat{P}$, with $\widehat{G}$ both simple and a summand of the Levi factor of $P'$, and $\widehat{P}$ maximal parabolic in $\widehat{G}$. This proves the first statement. The second statement follows from the fact that the cell decomposition (2.4) of any $G/P$ is determined by the Hasse diagram of $P$, and the Hasse diagram $W^{\hat{\mathfrak{p}}}$ of $\widehat{P}\subset\widehat{G}$ is the relative Hasse diagram $W^{\mathfrak{p}}_{\mathfrak{p}'}$ (see [BE]). □

*Proof of Lemma* 4.8. By the previous lemma we can assume that $P$ is a maximal parabolic subgroup. Recall from §2 that there is a projective embedding

$$I:G/P\to\mathbf{P}(V_\lambda)=\mathbf{P}^N \tag{4.12}$$

such that the intersection of $I(G/P)$ with the $\mathbf{P}^{N-1}$ at infinity is $Z$. Here $Z=X_1\cup X_{>1}$ is the complement of the generic cell $X_e$ in the Bruhat decomposition (2.4), where $X_1$ is the codimension one cell and $X_{>1}$ is the union of the codimension greater than one cells. Now, if $f\in\mathrm{Rat}_1(G/P)$ with $f(\infty)$ the base point in $X_e$ and $f(0)\in Z$ then the composition

$$I\circ f:\mathbf{P}^1\to G/P\to\mathbf{P}^N \tag{4.13}$$

must pull back the $\mathbf{P}^{N-1}$ at infinity to a single point in the Riemann sphere. This forces the composition $I\circ f$ to be linear. Thus, we have that elements $f\in\mathrm{Rat}_1(G/P)$ are determined (up to reparametrization of $\mathbf{P}^1$ by elements of $\mathrm{SL}(2,\mathbf{C})$ fixing $\infty$) by the point in $Z$ that they intersect, as the line is determined by $f(0)$ and $f(\infty)$. Therefore, the local degree one principal parts space, $\mathcal{LPP}_1$, which is equivalent to the set of $f\in\mathrm{Rat}_1(G/P)$

with $f(0)\in Z$, maps to points in $Z$ with fiber the automorphisms of $\mathbf{P}^1$ fixing 0 and $\infty$; that is, $\mathbf{C}^*$. This means we have a principal fibration

$$\mathbf{C}^* \to \mathcal{LPP}_1 \to L \tag{4.14}$$

where the base space $L$ is the "line space" of points in $Z$ that are hit by elements of $\mathrm{Rat}_1(G/P)$.

Define the quotient sheaf $Q$ over $\mathbf{P}^1$ corresponding to the action of $N^-$ by the exact sequence

$$0 \to \mathcal{O}^{\oplus d} \to f^*T(G/P) \to Q \to 0. \tag{4.15}$$

In some sense, the dimension of $Q$, which is supported over a point, counts the dimension of the stabilizer algebra of that point, with the proviso that if a vector field in $\mathfrak{s}$ vanishes to order $k$ along the image of the map, it gets counted $k$ times. By twisting with $\mathcal{O}(-1)$ and taking the long exact sequence in cohomology it follows that

$$H^0(\mathbf{P}^1, f^*T(G/P)(-1)) \simeq H^0(\mathbf{P}^1, Q(-1)), \tag{4.16}$$

and since the quotient sheaf $Q(-1)$ is supported at a single point we have

$$h^0(\mathbf{P}^1, f^*T(G/P)(-1)) = \dim Q. \tag{4.17}$$

Furthermore, as $f(\mathbf{P}^1)$ is a line in $P^N$, there is an exact sequence

$$0 \to \mathcal{O}(2) = T(\mathbf{P}^1) \to f^*T(\mathbf{P}^N) \to \mathcal{O}(1)^{\oplus N-1} \to 0 \tag{4.18}$$

with the stabilizer algebra $\mathfrak{s}$ mapping injectively to $f^*T(\mathbf{P}^N)$ over the open cell. There are then two possibilities:

(1) There is an element $Y$ of $\mathfrak{s}$ giving a section of $T(\mathbf{P}^1)$, in which case $Y$ vanishes to order 2 at $z=0$. All elements of $\mathfrak{s}$ in a subspace complementary to the subspace $\langle Y\rangle$ generated by $Y$ vanish to order 1 at $z=0$, and are nowhere else tangent to $\mathbf{P}^1$. In this case, $\dim Q = \dim \mathfrak{s} + 1$.

(2) All elements of $\mathfrak{s}$ vanish to order 1 at $z=0$, and are nowhere else tangent to $f(\mathbf{P}^1)$. In this case, $\dim Q = \dim \mathfrak{s}$.

Now, instead of viewing $H^0(\mathbf{P}^1, f^*T(G/P)(-1))$ as infinitesimal deformations of holomorphic maps $\mathbf{P}^1 \to G/P$ that send $\infty \in \mathbf{P}^1$ to the base point in $G/P$, we can fix $x\in Z$ and view $H^0(\mathbf{P}^1, f^*T(G/P)(-1))$ as infinitesimal deformations of holomorphic maps $\mathbf{P}^1 \to G/P$ that send $0\in\mathbf{P}^1$ to $x\in G/P$. One then has an inclusion of sheaves corresponding to the action of $\mathfrak{s}$

$$\mathcal{O}^{\oplus \dim(\mathfrak{s})} \to f^*T(G/P)(-1) \tag{4.19}$$

with a corresponding inclusion on the level of global sections, as a subspace of sections of $f^*T(G/P)$ vanishing only at $z=0$. However, there is a section of $f^*T(G/P)$ which vanishes at two points, corresponding to the action of $\mathbf{C}^*$ on $\mathbf{P}^1$. Thus, the two sheaves in (4.19) have spaces of sections of different dimension, and so case (1) above holds. This proves Lemma 4.8, and thus finishes the proof of Theorem 4.1. □

*Remarks* 4.20. (1) One can show that the line space $L$ is isomorphic to the quotient of the smooth locus of $Z$ by the action of $N^-$.

(2) There is a filtration of the Lie algebra $\mathfrak{n}^-$: $\langle Y\rangle\subset\mathfrak{s}\subset\mathfrak{n}^-$ corresponding to the order of vanishing at infinity.

Finally, returning to an arbitrary parabolic subgroup $P$ we have

$$\mathbf{C}^*\to\mathcal{LPP}_1^\alpha\to L \tag{4.21}$$

for every $\alpha$ between 1 and $n(\mathfrak{p})$. Furthermore, we have

LEMMA 4.22. *For all $\alpha$ the local principal part space $\mathcal{LPP}_1^\alpha$ is smooth and connected.*

*Proof.* Smoothness follows from the decomposition of the space of maps of degree one into a product $\mathbf{C}\times\mathcal{LPP}_1^\alpha$. From (4.21) a local principal part is determined by its point of intersection with $Z$ and its normal derivative along $Z$. Given any two local principal parts $(p(0),a(0))$ and $(p(1),a(1))$ corresponding to points $p(i)$ in the smooth locus $Z^*$ of $Z$ and normal data $a(i)\in\mathbf{C}^*$, we can find a family of germs $g(t)$ corresponding to $(p(t),a(t))$, $t\in[0,1]$ interpolating between the two, as $Z^*$ is connected.

## 5. Rats and jets

A key ingredient in the proof of our topological theorems will be the fact that the set of maps in $\mathrm{Rat}_{\mathbf{k}}(G/P)$ with a pole of multiplicity greater than $m$ has a codimension which is bounded below by some linear function of $m$ in $\mathrm{Rat}_{\mathbf{k}}(G/P)$. This section is devoted to proving this result. We proceed in two steps: the first reduces the problem from proving the theorem for the space of rational maps to proving a similar result for the space of jets of maps from $\mathbf{C}$ into $G/P$, and the second is to show that the result does indeed hold for these spaces of jets.

Let $J^r(\mathbf{C},M)$ denote the space of $r$-jets "centered at 0" of holomorphic maps from $\mathbf{C}$ to the variety $M$ that send $0\in\mathbf{C}$ to any point of $M$; alternatively, $J^r(\mathbf{C},M)$ can be thought of as the space of maps from the $r$th formal neighborhood of the origin in $\mathbf{C}$ into $M$. Let $M=G/P$, and consider a component $Z_\alpha$ of $Z$ which is represented as the zero set of a section $s_\alpha$ of the holomorphic line bundle $\mathcal{O}_{\mathfrak{p}}(\lambda_\alpha)$ on $G/P$. For $f\in\mathrm{Rat}_{\mathbf{k}}(G/P)$

with an $\alpha$-pole at $z=0$, the multiplicity $m_\alpha$ of this pole is the order of vanishing of $f^*s_\alpha$ at 0. This places obvious constraints $F^0=F^1=\ldots=F^{m-1}=0$ on the $m$-jet $j^m f$ at 0, where $F^i$ is simply the $i$th derivative. In general there are $m$ constraints, but depending on the explicit form of $s_\alpha$ these may or may not be independent. The question arises whether independent constraints on $J^m(\mathbf{C},G/P)$ pull back to independent constraints on $\mathrm{Rat}_{\mathbf{k}}(G/P)$ under the natural evaluation map.

For each $z\in\mathbf{C}$ and each $r$ we have the natural map

$$e_z^r\colon \mathrm{Rat}_{\mathbf{k}} \to J^r(\mathbf{C},G/P) \tag{5.1}$$

defined by sending $f\in\mathrm{Rat}_{\mathbf{k}}(G/P)$ to its $r$-jet at $z$; i.e., $e_z^r(f)=j^r f(z)$.

PROPOSITION 5.2. *Let $F^{i_1},\ldots,F^{i_n}$, for all $i_j\leqslant k$ be functionally independent near $p$ on $J^k(\mathbf{C},G/P)$, with $F^0(p)=F^1(p)=\ldots=F^k(p)=0$ then*

$$(e_z^k)^*F^{i_1},\ldots,(e_z^k)^*F^{i_n}$$

*are functionally independent near the inverse image of $p$ in $\mathrm{Rat}_{\mathbf{k}}(G/P)$ where $k=k_\alpha$ is the $\alpha$ component of $\mathbf{k}$.*

*Proof.* For notational simplicity we consider only one component $Z_\alpha$ of $Z$. Consider the rational map $f_0$ represented by a principal part $P_0$ consisting of a pole at $z_0$ of order $k$. Using the additivity of principal parts, we can add to $P_0$ fixed principal parts $P_1,\ldots,P_s$ of order $k_1,\ldots,k_s$, at points $z_1,\ldots,z_s$, respectively. This gives a commutative diagram

$$\begin{tikzcd} \mathrm{Rat}_K \arrow[dr, "e_{z_0}^k"] & \\ \mathrm{Rat}_k \arrow[u] \arrow[r] & J^k(\mathbf{C},G/P) \end{tikzcd} \tag{5.3}$$

where $K=\sum_{i=0}^s k_i$ and $k_0=k$. If we can show that the diagonal map $e_{z_0}^k$ is a submersion at the rational map $f$ determined by $P_0,P_1,\ldots,P_s$, then, by the local additivity property $\mathrm{Rat}_K\simeq\mathrm{Rat}_k\times\mathrm{Rat}_{K-k}$, the result will follow for these particular constraints when they take the value zero. Thus, it suffices to show:

LEMMA 5.4. *$e_{z_0}^k$ is a submersion at $f$.*

*Proof.* Identifying the tangent space to $\mathrm{Rat}_K$ with the sections $s$ of the twisted pullback bundle $H^0(\mathbf{P}^1,f^*T(G/P)(-1))$ as in (4.3), we see that the differential of $e_{z_0}^k$ at $f$ is

$$De_{z_0}^k(f)(s)=j^k s(z_0). \tag{5.5}$$

We will have a submersion at $f$ if given any element in the tangent space of $J^k(\mathbf{C}, G/P)$ it can be realized as the $k$-jet of a section $s \in H^0(\mathbf{P}^1, f^*T(G/P)(-1))$. Certainly, this will hold if we can show that $f^*T(G/P)(-1)$ splits as the sum of line bundles $\mathcal{O}(j_i)$ with $j_i \geqslant k$ for all $i=1, ..., n=\dim(G/P)$. To show that this is indeed the case we choose $f \in \mathrm{Rat}_K$ so that we add to $\mathrm{Rat}_k$ a single pole of order $k+1$ at $z_1$ with target point $p \in G/P$ that is stabilized by all of $N^-$. (Such points always exist, for example, the point corresponding to the 0-cell of the Bruhat decomposition.) Recall that the action of $N^-$ on $G/P$ gives a sheaf morphism (4.5) which is an isomorphism on an open set. But, by construction in our current case, the image of $\phi$ is generated by sections which vanish to order $k+1$ at $z_1$. This gives a generically bijective map

$$\mathcal{O}(k) \to f^*T(G/P)(-1) \tag{5.6}$$

corresponding to "removing the zero" at $z_1$, and, as with diagram (4.6), this forces the pull back bundle $f^*T(G/P)(-1)$ to split as

$$f^*T(G/P)(-1) \simeq \bigoplus_i \mathcal{O}(j_i) \tag{5.7}$$

with all $j_i \geqslant k$. This finishes the proof of Lemma 5.4 and thus of Proposition 5.2. □

We are interested in configurations of principal parts which have one of its poles at a fixed point which, for convenience, we take at $z=0$. By the above, to show that multiple poles occur with increasing codimension, it will suffice to prove the analogous result for jets. Let $M$ be a smooth variety, and $Z$ a (not necessarily smooth) reduced hypersurface in $M$, cut out by $s=0$, where $s$ is a section of some line bundle. Let $J^r(M)=J^r(\mathbf{C}, M)$, and let $J^r(l, M)$ be the subvariety of those jets $f$ such that the pull-back $f^*s$ vanishes to order at least $l$; that is, $s \circ f = (s \circ f)' = ... = (s \circ f)^{(l-1)} = 0$.

PROPOSITION 5.8. *If $Z$ is smooth, then the codimension of $J^r(l, M)$ in $J^r(M)$ is $l$.*

*Proof.* It suffices to compute in local coordinates $x_i$, with $Z$ cut out by $x_1=0$. □

We now turn to handling the singular set. Our method of controlling the codimension relies on Hironaka's resolution of singularities theorem [I]. There is a finite sequence of blowing up:

$$\begin{array}{ccccccc} M^m & \longrightarrow & M^{m-1} & \longrightarrow & \cdots & \longrightarrow & M^0 = M \\ \uparrow & & \uparrow & & & & \uparrow \\ Z^m & \longrightarrow & Z^{m-1} & \longrightarrow & \cdots & \longrightarrow & Z^0 = Z \end{array} \tag{5.9}$$

where each pair $(M^i, Z^i)$ is obtained by blowing up along a smooth $d^i$-dimensional submanifold $V^i$ of $Z^i$, $Z^i$ is the proper transform of $Z^{i-1}$, and $Z^m \subset M^m$ is smooth. Let $Z^i$ be cut out by $s^{(i)}=0$. We set $J^{k,i}=J^k(M^i)$ and $J^{k,i}(l)=J^k(l, M^i)$.

PROPOSITION 5.10. *There are constants $c_i>0$ and $c'_i\geqslant 0$ both independent of $k$ and $l$ such that the codimension of $J^{k,i}(l)$ in $J^{k,i}$ is at least $c_i\cdot l-c'_i$ for all $i$.*

*Proof.* We will proceed by descending induction on $i$. By Proposition 5.8, the result is true for $i=m$, with $c_m=1$ and $c'_m=0$. Now suppose that the result holds for $J^{k,i}(l)$. It is sufficient to work locally. Choose coordinates $x^1,\dots,x^n$ on $M^{i-1}$ such that locally the subvariety $V=V^{i-1}$ is given by $x^1=\dots=x^{n-d}=0$ with $d=d^{i-1}$. In these coordinates, blowing-up along $V$ is given in one coordinate patch by

$$\begin{aligned} y^1 &= x^1,\\ y^2 &= \frac{x^2}{x^1},\\ &\vdots\\ y^{n-d} &= \frac{x^{n-d}}{x^1},\\ y^j &= x^j, \end{aligned} \tag{5.11}$$

for $j=n-d+1,\dots,n$; where the $y^j$ are then coordinates on the blow-up $M^i$. We stratify the jets into $M^{i-1}$ according to their degree of osculation of the variety $V$. Let $\sigma\geqslant 1$ be the degree of this osculation (if $\sigma=0$ then the jet is not on $V$, and there is nothing to prove). We then consider a jet $f\in J^{k,i-1}$, with $f(0)\in V$. In coordinates we have:

$$x^j\circ f(z)=a_{\sigma,j}z^\sigma+a_{\sigma+1,j}z^{\sigma+1}+\dots \tag{5.12}$$

for $j=1,\dots,n-d$ and $a_{\sigma,1}\neq 0$. By reparametrization in $\mathbf{C}$ we can take $x^1(z)=z^\sigma$. Lifting the jet $f$ to a jet $\tilde f$ on $M^i$ gives in coordinates on $M^i$,

$$\begin{aligned} y^1\circ\tilde f(z) &= z^\sigma,\\ y^2\circ\tilde f(z) &= a_{\sigma,2}+a_{\sigma+1,2}z+\dots,\\ &\vdots\\ y^{n-d}\circ\tilde f(z) &= a_{\sigma,n-d}+a_{\sigma+1,n-d}z+\dots, \end{aligned} \tag{5.13}$$

and $y^j\circ\tilde f(z)=x^j\circ\tilde f(z)$ for $j=n-d+1,\dots,n$. Now suppose that the section $s^{(i-1)}$ vanishes to order $r$ on $V$, so the proper transform $Z^i$ of the divisor $Z^{i-1}$ is given by $(s^{(i)})=((x^1)^{-r}s^{(i-1)})$. So we have

$$\tilde f^* s^{(i)}(z)=\frac{f^* s^{(i-1)}}{z^{\sigma r}}, \tag{5.14}$$

and this implies

LEMMA 5.15. *The first $m \geqslant \sigma r$ derivatives of $f^* s^{(i-1)}$ vanish if and only if the first $m-\sigma r$ derivatives of $\tilde{f}^* s^{(i)}$ vanish.*

Let us now examine what this implies for our codimensions. The jets on $M^{i-1}$ in our stratum correspond to the jets on $M^i$ whose first coordinate vanishes to order $\sigma$: this is a subspace $W_\sigma$ of codimension $\sigma$ in the space of all jets. By hypothesis, the space $J^{k,i}(l')$ has codimension in $J^{k,i}$ bounded below by $c_i l' - c'_i$. Its intersection with $W_\sigma$ then has codimension at least

$$\max(0, c_i l' - c'_i - \sigma). \tag{5.16}$$

In $M^{i-1}$, these correspond to jets in $J^{k,i-1}(l)$, where by Lemma 5.15,

$$l = r\sigma + l'. \tag{5.17}$$

The stratum of jets osculating $V$ to order $\sigma$ is of codimension $(n-d)\sigma$ in $J^{k,i-1}$ (note that $(n-d) \geqslant 2$), and inside this stratum, those that belong to $J^{k,i-1}(l)$ have codimension bounded by (5.16). Thus, we have a lower bound on the codimension inside the whole jet space $J^{k,i-1}$ of

$$\max((n-d)\sigma, c_i l - c'_i + (n-d-c_i r-1)\sigma). \tag{5.18}$$

It is then an easy exercise to check that this can be bounded below by

$$c_{i-1} l - c'_{i-1} \tag{5.19}$$

for some $c_{i-1} > 0$ and $c'_{i-1} \geqslant 0$ both independent of $s$. This completes the proof of Proposition 5.10. □

We close this section by summarizing how the results of Proposition 5.10 apply to our maps. Let us fix the Schubert variety $Z_\alpha$ with corresponding $\alpha$-degree $k_\alpha$. We let $\mathrm{Rat}_{\mathbf{k},k_\alpha}(G/P)$ denote the subspace of $\mathrm{Rat}_{\mathbf{k}}(G/P)$ of those maps with only one $\alpha$-pole of multiplicity $k_\alpha$. More generally, for any subvariety $V \subset Z_\alpha$, we define $\mathrm{Rat}_{\mathbf{k},k_\alpha}(G/P, V)$ to be the subset of $\mathrm{Rat}_{\mathbf{k},k_\alpha}(G/P)$ consisting of those maps $f$ whose $\alpha$-pole lies in $V$. In particular, we are interested in the cases when $V = Z^*_\alpha$, the smooth locus at infinity, or $V = Z^s_\alpha = Z_\alpha - Z^*_\alpha$, the singular locus at infinity. Also, if $\mathbf{k}$ is a multi-index, we let $\mathrm{Rat}_{\mathbf{k},\mathbf{k}}(G/P)$ be the subvariety of $\mathrm{Rat}_{\mathbf{k}}(G/P)$ with only one pole of multiplicity $\mathbf{k}$, i.e., one pole whose $\alpha$-multiplicity for each $\alpha = 1, ..., n(\mathfrak{p})$ is $k_\alpha$. Let $\mathcal{LPP}_{\mathbf{k}}$ be the subspace of $\mathrm{Rat}_{\mathbf{k},\mathbf{k}}(G/P)$ with the pole at 0, so that $\mathrm{Rat}_{\mathbf{k},\mathbf{k}}(G/P) = \mathcal{LPP}_{\mathbf{k}} \times \mathbf{C}$.

THEOREM 5.20. *Taking "codimension" to mean codimension in* $\mathrm{Rat}_{\mathbf{k}}(G/P)$,

(i) *there are constants $0 < C_\alpha \leqslant 1$ and $C'_\alpha \geqslant 0$ depending only on $\alpha$ and $G/P$ such that the codimension of* $\mathrm{Rat}_{\mathbf{k},k_\alpha}(G/P)$ *is bounded below by $C_\alpha k_\alpha - C'_\alpha$;*

(ii) *the codimension of* $\mathrm{Rat}_{\mathbf{k},k_\alpha}(G/P, Z_\alpha^*)$ *is* $k_\alpha - 1$;

(iii) *the codimension of* $\mathrm{Rat}_{\mathbf{k},k_\alpha}(G/P, Z_\alpha^s)$ *is bounded below by* $\max\{C_\alpha k_\alpha - C'_\alpha, 2\}$;

(iv) *there are constants* $0<\widehat{C}\leqslant 1$ *and* $\widehat{C}'\geqslant 0$ *depending only on* $G/P$ *such that the codimension of* $\mathrm{Rat}_{\mathbf{k},\mathbf{k}}(G/P)$ *is bounded below by* $\widehat{C}|\mathbf{k}|-\widehat{C}'$;

(v) *if* $|\mathbf{k}|>2$, *the codimension of* $\mathrm{Rat}_{\mathbf{k},\mathbf{k}}(G/P)$ *is* $\geqslant 2$;

(vi) *if* $|\mathbf{k}|=2$, *the codimension of* $\mathrm{Rat}_{\mathbf{k},\mathbf{k}}(G/P)$ *is* 1.

*Proof.* Proposition 5.10 gives a linear lower bound on the codimension of jets vanishing to a given order. This lower bound is realized in terms of constraints on the jet space $J^r(G/P)$, and Proposition 5.2 says that these constraints remain independent when pulled back to $\mathrm{Rat}_{\mathbf{k}}(G/P)$. Thus, since $k_\alpha$ is precisely the order of vanishing of $f^*s_\alpha$, the first result holds. Statement (ii) follows immediately from Propositions 5.2 and 5.8. To prove (iii) we exhibit for each map $f$ with $f(0)\in Z_\alpha^s$, a two parameter family $f_{t,x}$ of maps with $f_{0,0}=f$ such that the only map in the family meeting $Z_\alpha^s$ is $f$. By Proposition 2.8 $Z_\alpha^s$ has codimension 3 in $G/P$. As $G$ acts transitively on $G/P$, the map $\varrho\colon \mathbf{P}^1\times G\to\mathbf{P}^1\times G/P$ sending $(z,g)$ to $(z, g\cdot f(z))$ is a submersion, and so the inverse image $\varrho^{-1}(\mathbf{P}^1\times Z_\alpha^s)$ is of codimension 3, and its projection onto $G$ is of codimension 2. Hence, by general position arguments there exists a two parameter holomorphic family $g(x,t)$ of elements of $G$, $g(0,0)=\mathrm{Id}$, such that the graph $\{(z, g(x,t)\cdot f(z))\,|\,z\in\mathbf{P}^1\}\subset\mathbf{P}^1\times G/P$ only intersects $\mathbf{P}^1\times Y$ if $(x,t)=(0,0)$. Of course, the map $g(x,t)\cdot f(z)$ does not respect the base point, but one can use the action of $N^-$ to correct this. There is a unique holomorphic family $\{n(x,t)\}$ with $n(x,t)\in N^-$ for each $(x,t)$ such that $n(x,t)f(\infty)=g(x,t)f(\infty)$. One then sets,

$$f_{x,t}(z)=n(x,t)^{-1}g(x,t)f(z). \tag{5.21}$$

We remark that, as the action of $N^-$ preserves the Bruhat cells, it cannot move us back into $Z_\alpha^s$ for $(x,t)\neq(0,0)$. This proves (iii).

Part (iv) is obtained in exactly the same way as part (i), except that one considers intersections of the image of the map with $\bigcup_\alpha Z_\alpha$ instead of just one $Z_\alpha$. For (v), one has for a map in $\mathrm{Rat}_{\mathbf{k},\mathbf{k}}(G/P)$ that either:

(i) The map meets the singular set of one of the $Z_\alpha$.

(ii) The map meets a triple intersection of the $Z_\alpha$.

(iii) The map meets just two of the $Z_\alpha$, but in a codimension $>2$ cell of the Bruhat decomposition.

(iv) The image of the map meets just two of the $Z_\alpha$, at a transverse point of their intersection, on a codimension two cell of the Bruhat decomposition.

In cases (i) to (iii), the argument given to prove (iii) applies, to give a lower bound of 2 for the codimension. Case (iv) can be studied explicitly, by taking a coordinate system in which the two $Z_\alpha$'s correspond to coordinate planes.

The proof of (vi) is similar to that of (v). $\square$

## 6. The stratification of $\mathrm{Rat}(G/P)$

In order to stratify $\mathrm{Rat}_{\mathbf{k}}(G/P)$ we consider a collection of multi-indices

$$\mathcal{M}=\{\mathbf{m}^1,\dots,\mathbf{m}^r\} \tag{6.1}$$

satisfying the constraint (3.6). For any given collection $\mathcal{M}$ we denote the number of poles $r$ by $|\mathcal{M}|$.

*Definition* 6.2. Let $S_{\mathcal{M}}$ be the subset of all elements of $\mathrm{Rat}_{\mathbf{k}}(G/P)$ that have $r$ poles at the distinct points $z_1,\dots,z_r$ of $\mathbf{C}$ such that the multiplicity of the pole at $z_i$ is $\mathbf{m}^i$.

The strata $S_{\mathcal{M}}$ are thus the sets of maps with a fixed pattern of multiplicities. We let $\mathbf{DP}^r(\mathbf{C})$ denote the deleted $r$-fold product; that is, the space of $r$ distinct unordered points in $\mathbf{C}$. This is a smooth complex manifold of complex dimension $r$. The pole location map of Proposition 3.8 restricts to a locally trivial fibration

$$\Pi_{\mathcal{M}}: S_{\mathcal{M}}\to \mathbf{DP}^r(\mathbf{C}) \tag{6.3}$$

with fiber

$$\prod_{i=1}^{r}\mathcal{LPP}_{\mathbf{m}^i}. \tag{6.4}$$

Note that $S_{\mathcal{M}}$ is not necessarily smooth, as the $\mathcal{LPP}^i_{\mathbf{m}}$ are not necessarily smooth. $S_{\mathcal{M}}$ is, however, a variety, as it is cut out of $\mathrm{Rat}_{\mathbf{k}}(G/P)$ by the vanishing of derivatives of locally defined analytic functions, and so

PROPOSITION 6.5. *There is a finite well-ordered set* $(J=J_{\mathcal{M}},\leqslant)$ *and a decomposition*

$$S_{\mathcal{M}}=\bigsqcup_{j\in J}S_{\mathcal{M},j}$$

*where* $S_{\mathcal{M},j}$ *is a smooth complex variety for each* $\mathcal{M},j$ *which satisfies the following conditions*:

(i) $\dim S_{\mathcal{M},j}<\dim S_{\mathcal{M},i}$ *only if* $i<j$,

(ii) $S_{\mathcal{M},i}\subset\overline{S_{\mathcal{M},j}}$ *only if* $j\leqslant i$.

This decomposition can be done in a way which is compatible with similar decompositions of the $\mathcal{LPP}_{\mathbf{m}^i}$. While we have not computed the complex codimension of $S_{\mathcal{M},j}$ explicitly, Proposition 6.5 and Theorem 5.20 together with the additivity of principal parts imply

PROPOSITION 6.6. *For each $G/P$ there is a positive constant $c(G/P)$, which is independent of the stratum indices $\mathcal{M}, j$ and the multi-degree $\mathbf{k}$, so that the complex codimension of $S_{\mathcal{M},j}$ in $\mathrm{Rat}_{\mathbf{k}}(G/P)$,*

$$\operatorname{codim}(\mathcal{M}, j; \mathbf{k}) = \operatorname{codim}(S_{\mathcal{M},j} \subset \mathrm{Rat}_{\mathbf{k}}(G/P))$$

*is bounded below by*

$$c(G/P) \sum_i (|\mathbf{m}^i| - 1) = c(G/P)(|\mathbf{k}| - r)$$

*where $r$ is the number of poles.*

The smooth locus of $S_{\mathcal{M}}$ is denoted by $S_{\mathcal{M},0}$ and corresponds to the index $j=0\in J_{\mathcal{M}}$. With this in mind we write $\mathrm{cd}(\mathcal{M},\mathbf{k})=\operatorname{codim}(\mathcal{M},0;\mathbf{k})$ and generally $\mathrm{cd}(\mathcal{M},j,\mathbf{k})=\operatorname{codim}(\mathcal{M},j;\mathbf{k})$ as the complex codimension of $S_{\mathcal{M},j}$ in $\mathrm{Rat}_{\mathbf{k}}(G/P)$.

Next we show that the stratification of $\mathrm{Rat}_{\mathbf{k}}(G/P)$ by the $S_{\mathcal{M},j}$ gives an *$L$-stratification* as used in [BHMM] and [MM1], [MM2], [MM3]. For completeness we recall

*Definition* 6.7. A smooth manifold $M$ is $L$-stratified if there is a decomposition of $M$ into disjoint smooth submanifolds $M(K)$ such that:

(L.1) The index set $\mathcal{K}=\{K\}$ is finite with a given fixed well ordering $\leqslant$.

(L.2) If $K_0$ is the smallest element in $(\mathcal{K},\leqslant)$ then $M(K_0)$ is an open dense subset of $M$.

(L.3) For all $K\in\mathcal{K}$ the union of the submanifolds of same or smaller order

$$Z(K) = \bigcup_{K'\leqslant K} M(K')$$

is an open dense submanifold of $M$.

(L.4) For all $K\in\mathcal{K}$ the normal bundle, $\nu(K)$, of $M(K)$ in $M$, is orientable.

We denote a particular $L$-stratification of $M$ by $\{M(K),\mathcal{K}\}$.

Given any smooth $L$-stratified manifold $M$ the submanifolds $Z(K)$, as defined in (L.3), give an increasing filtration of $M$ by open dense submanifolds

*Definition* 6.8.
$$\mathcal{F}_K[M] = Z(K) = \bigcup_{K'\leqslant K} M(K').$$

Notice that condition (L.3) implies $\nu(K)$ is contained in $Z(K)$ and that the successive quotients of this filtration of $M$ are Thom spaces

$$Z_K/Z_{K-1} \cong T(\nu(Z_K - Z_{K-1})) \cong T(\nu(K)). \tag{6.9}$$

Hence, given any such filtration and any coefficient ring $A$ there is an associated homology Leray spectral sequence with $E^1$ term isomorphic to

$$\bigoplus_{K\in\mathcal{K}} H_*(T(\nu(K));A)\cong H_*(\Sigma^{\tau(K)}(M(K))_+;A). \tag{6.10}$$

Here $\tau(K)$ is the real codimension of $M(K)$ in $M$ and the last isomorphism follows from condition (L.4) of Definition 6.7 and the Thom isomorphism theorem. Furthermore, since the index set $\mathcal{K}$ is assumed to be finite, the spectral sequence must converge to a filtration of $H_*(M;A)$.

Let $\mathcal{I}_{\mathbf{k}}$ denote the index set consisting of all sets of multiplicities given by (6.1) satisfying $\sum_{i=1}^{r}\mathbf{m}^i=\mathbf{k}$ together with the sets $J_{\mathcal{M}}$ defined in Proposition 6.5. Then the set $\{S_{\mathcal{M},j}\}$ of all strata of $\mathrm{Rat}_{\mathbf{k}}(G/P)$ of Proposition 6.5 is indexed by $\mathcal{I}_{\mathbf{k}}$. We begin by dividing $\mathcal{I}_{\mathbf{k}}$ into two subsets. Let $\mathcal{I}^1_{\mathbf{k}}$ be the subset consisting of all $(\mathcal{M},j)$ with $j\in J_{\mathcal{M}}$ where for every $\alpha$ there is some $i$ such that $\mathbf{m}^i_\alpha=1$, i.e., a simple pole of type $\alpha$ (recall $1\leqslant\alpha\leqslant n(\mathfrak{p})$) and let $\mathcal{I}^2_{\mathbf{k}}$ be the complement of $\mathcal{I}^1_{\mathbf{k}}$ in $\mathcal{I}_{\mathbf{k}}$. We shall now give $\mathcal{I}_{\mathbf{k}}$ a well ordering $\leqslant$:

*Definition* 6.11. First if $(\mathcal{M},j)\in\mathcal{I}^1_{\mathbf{k}}$ and $(\mathcal{M}',j')\in\mathcal{I}^2_{\mathbf{k}}$ then $(\mathcal{M},j)<(\mathcal{M}',j')$. The ordering within each $\mathcal{I}^i_{\mathbf{k}}$ is identical. So within a given $\mathcal{I}^i_{\mathbf{k}}$ we set:

(1) $(\mathcal{M},j)<(\mathcal{M}',j')$ if the cardinalities satisfy $\mathrm{card}(\mathcal{M})>\mathrm{card}(\mathcal{M}')$.

(2) If $\mathrm{card}(\mathcal{M})>\mathrm{card}(\mathcal{M}')$, and if the multi-indices $\mathbf{m}^i,\mathbf{m}'^i$ comprising $\mathcal{M}$ and $\mathcal{M}'$ are listed in declining lexicographical order (so that $\mathbf{m}^i>\mathbf{m}'^i$ means $m^i_1>m'^i_1$ or if $m^i_1=m'^i_1$, then $m^i_2>m'^i_2$, and so on), we set $\mathcal{M}<\mathcal{M}'$ if $\mathbf{m}^1<\mathbf{m}'^1$ or ($\mathbf{m}^1=\mathbf{m}'^1$ and $\mathbf{m}^2<\mathbf{m}'^2$), and so on.

(3) If $\mathcal{M}=\mathcal{M}'$ (all multi-indices are equal), then the well-ordering is that of $J_{\mathcal{M}}$.

We denote $\mathrm{Rat}_{\mathbf{k}}(G/P)$ with this stratification by the triple $(\mathrm{Rat}_{\mathbf{k}}(G/P),S_{\mathcal{M},j},\mathcal{I}_{\mathbf{k}})$. The main result of this section is:

THEOREM 6.12. *For every $G/P$ and each $\mathbf{k}$, the well-ordering on $\mathcal{I}_{\mathbf{k}}$ defined in Definition* 6.11 *gives an L-stratification for* $(\mathrm{Rat}_{\mathbf{k}}(G/P),S_{\mathcal{M},j},\mathcal{I}_{\mathbf{k}})$.

*Proof.* First we notice that condition (L.1) of Definition 6.7 is satisfied. Perturbation of an element of $S_{\mathcal{M},j}$ inside $\mathrm{Rat}_{\mathbf{k}}(G/P)$ either (1) keeps one inside the stratum, or (2) keeps the number of poles fixed, and hence leaves the multiplicities unchanged, but moves one from $S_{\mathcal{M},j}$ to $S_{\mathcal{M},j'}$, with $j<j'$, or (3) increases the number of poles, which moves us onto a lower stratum, and so condition (L.3) is verified. For (L.2), the lowest stratum consists of maps with only simple poles ($|\mathbf{m}^i|=1$) and $j=0$. Then (ii) of Theorem 5.20 together with additivity of principal parts tells us that this is open dense. (L.4) follows by the fact that one has a stratification by complex varieties. □

Thus, we have

COROLLARY 6.13. *For each* $\mathbf{k}$ *and all coefficient rings* $A$, *there are homology Leray spectral sequences* $E^r(\mathrm{Rat}_{\mathbf{k}}(G/P); A)$ *converging to filtrations of* $H_*(\mathrm{Rat}_{\mathbf{k}}(G/P); A)$ *with*

$$E^1(\mathrm{Rat}_{\mathbf{k}}(G/P); A) \cong \bigoplus_{\mathcal{I}_{\mathbf{k}}} H_*(\Sigma^{2\,\mathrm{cd}(\mathcal{M},j,\mathbf{k})}(S_{\mathcal{M},j})_+; A)$$

*where* $2\,\mathrm{cd}(\mathcal{M}, j, \mathbf{k})$ *is the real codimension of* $S_{\mathcal{M},j}$ *in* $\mathrm{Rat}_{\mathbf{k}}(G/P)$. *Furthermore, the inclusions*

$$\iota(\mathbf{k}, \mathbf{k}'): \mathrm{Rat}_{\mathbf{k}}(G/P) \to \mathrm{Rat}_{\mathbf{k}'}(G/P)$$

*induced by adding poles and principal parts all of scalar multiplicity one induce maps of spectral sequences*

$$\iota(\mathbf{k}, \mathbf{k}')(r): E^r(\mathrm{Rat}_{\mathbf{k}}(G/P); A) \to E^r(\mathrm{Rat}_{\mathbf{k}'}(G/P); A).$$

*Proof.* The fact that the Leray sequence exists and is of the stated form is immediate from Definitions 6.8, 6.11, and Theorem 6.12. Also, it is clear that the stabilization map $\iota(\mathbf{k}, \mathbf{k}')$ is filtration preserving with respect to the filtration defined by Corollary 6.13 and this implies that the induced map $\iota(\mathbf{k}, \mathbf{k}')(r)$ must be a map of associated spectral sequences. □

## 7. The stability of the principal parts space

We now prove our first main stability result by analyzing Corollary 6.13 in some detail. We begin by examining each term in the $E^1$ term of the Leray spectral sequence constructed there. Recall, from Theorem 6.12, that the individual strata $S_{\mathcal{M},j}$ of $\mathrm{Rat}_{\mathbf{k}}(G/P)$ are indexed by the multiplicity multi-sequences $\mathcal{M}$ given in Definition 6.2 and $j \in J_{\mathcal{M}}$. In addition, recall that we have inclusions of any component of $\mathrm{Rat}(G/P)$ into another component of higher total multiplicity induced by adding (possibly repeated) poles at new configuration points. We shall be especially interested in one specific such stabilization which is obtained as follows: First, fix one local principal part of multiplicity one for each of the $n(\mathfrak{p})$ components in the Bruhat decomposition and then add each of these principal parts at a new pole configured far away from the previously given data. In all that follows we will denote this particular stabilization by

$$\iota(\mathbf{k}, \mathbf{k}'): \mathrm{Rat}_{\mathbf{k}}(G/P) \to \mathrm{Rat}_{\mathbf{k}'}(G/P). \tag{7.1}$$

Using both Theorem 6.12 and the Thom isomorphism theorem we may rewrite $E^1(\mathrm{Rat}_{\mathbf{k}}(G/P); A)$ in Corollary 6.13 as

$$\bigoplus_{\mathcal{I}_{\mathbf{k}}} H_{*-2\,\mathrm{cd}(\mathcal{M},j,\mathbf{k})}(S_{\mathcal{M},j}; A) \tag{7.2}$$

in $\mathrm{Rat}_{\mathbf{k}}(G/P)$. Similarly, we may rewrite $E^1(\mathrm{Rat}_{\mathbf{k}'}(G/P);A)$ in Corollary 6.13 as

$$\bigoplus_{\mathcal{I}^1_{k'}} H_{*-2\,\mathrm{cd}(\mathcal{M},j,\mathbf{k}')}(S_{\mathcal{M},j};A)\oplus\bigoplus_{\mathcal{I}^2_{k'}} H_{*-2\,\mathrm{cd}(\mathcal{M},j,\mathbf{k}')}(S_{\mathcal{M},j};A). \tag{7.3}$$

Note that a class in $H_{*-2\,\mathrm{cd}(\mathcal{M},j,\mathbf{k})}(S_{\mathcal{M},j};A)$ surviving in (7.2) or (7.3) to $E^\infty$ represents a class in $H_*(\mathrm{Rat}(G/P);A)$. It is fundamental to keep track of this dimension shift in what follows. Next, recall that the strata indexed by elements in $\mathcal{I}^1_{\mathbf{k}}$ contain the image of the stabilization map (1.3) which increases the multiplicity in *every* Chern component. Furthermore, Theorem 6.12 implies that the stabilization map (7.1) restricted to each stratum is covered by a bundle map

$$\begin{tikzcd} \nu(S_{\mathcal{M},j}) \arrow[r, "\phi(\mathcal{M},\mathcal{M}')"] \arrow[d] & \nu(S_{\mathcal{M}',j}) \arrow[d] \\ S_{\mathcal{M},j} \arrow[r, "\iota(\mathcal{M},\mathcal{M}')"] & S_{\mathcal{M}',j}, \end{tikzcd} \tag{7.4}$$

where $\iota(\mathcal{M},\mathcal{M}')$ denotes the restriction of the stabilization map of (7.1) to the stratum $S_{\mathcal{M},j}$. Notice that the index $j$ does not change under stabilization. This is because the index set $J_{\mathcal{M}}$ is determined by the vanishing of derivatives of locally defined analytic functions, and stabilization involves adding simple poles where derivatives do not vanish. Although we have computed only a lower bound on the codimension of each stratum in $\mathrm{Rat}_{\mathbf{k}}(G/P)$, it follows again from the fact that (7.1) adds only generic multiplicity one type principal parts at new configuration points and the local analysis given in sections three and four that

$$\mathrm{cd}(\mathcal{M},j,\mathbf{k})=\mathrm{cd}(\mathcal{M}',j,\mathbf{k}'). \tag{7.5}$$

Hence, $\phi(\mathcal{M},\mathcal{M}')$ is not just a fibrewise injection but actually a fibrewise isomorphism; that is, $\phi(\mathcal{M},\mathcal{M}')$ is the inclusion of the pull-back

$$\iota(\mathcal{M},\mathcal{M}')^*(\nu(S_{\mathcal{M}',j}))=\nu(S_{\mathcal{M},j}). \tag{7.6}$$

Therefore, at the $E^1$ level, the induced map $\iota(\mathcal{M},\mathcal{M}')(1)$ maps (7.2) into the first summand in (7.3), taken over $\mathcal{I}^1_{\mathbf{k}'}$, precisely via the map $\iota(\mathcal{M},\mathcal{M}')_*$, and maps to zero in the second summand in (7.3), taken over $\mathcal{I}^2_{\mathbf{k}'}$.

Before proceeding we need one more bit of notation. Recall from Definition 6.2 that the multiplicity $\mathbf{m}^i$ of the $i$th pole consists of $n(\mathfrak{p})$ components $m^i_\alpha$ where the index $i$ runs from 1 to $r$. For each $\alpha$ let $s_\alpha$ be the number of $\alpha$-poles with $|\mathbf{m}|=1$ and let $s(\mathcal{M})=\min(s_1,\dots,s_{n(\mathfrak{p})})$.

The following technical result, which was proved in [BHMM] for $\mathbf{Z}_p$ coefficients, is fundamental in our analysis.

LEMMA 7.7. *Let $p$ be a prime and $\mathbf{F}_p$ be the finite field of $p$ elements. For all* $\mathbf{k}$ *and* $\mathcal{M}$ *the natural inclusion* $\iota(\mathbf{k},\mathbf{k}')$ *of* (7.1) *restricted to the* $S_{\mathcal{M},j}$ *stratum induces isomorphisms in* $\mathrm{mod}(p)$ *homology*

$$(\iota(\mathcal{M},\mathcal{M}'))_t\colon H_t(S_{\mathcal{M},j};\mathbf{F}_p)\cong H_t(S_{\mathcal{M}',j};\mathbf{F}_p)$$

*for* $t\leqslant q(\mathcal{M},\mathcal{M}')=\left[\frac{1}{2}s(\mathcal{M})\right]$.

*Proof.* The proof given in [BHMM] goes through word for word here as the fact that the multiplicity one label space in [BHMM] SO(3) is replaced by the multiplicity one principal parts spaces in each component in the Bruhat decomposition here is irrelevant to the argument. □

We are now able to prove our first main stability result.

THEOREM 7.8. *For all* $\mathbf{k}$ *and* $\mathbf{F}_p$ *coefficients, the inclusion*

$$\iota(\mathbf{k},\mathbf{k}')\colon \mathrm{Rat}_{\mathbf{k}}(G/P)\to\mathrm{Rat}_{\mathbf{k}'}(G/P)$$

*induces an isomorphism in homology with* $\mathbf{F}_p$ *coefficients*

$$(\iota(\mathbf{k},\mathbf{k}'))_t\colon H_t(\mathrm{Rat}_{\mathbf{k}}(G/P);\mathbf{F}_p)\cong H_t(\mathrm{Rat}_{\mathbf{k}'}(G/P);\mathbf{F}_p) \tag{7.9}$$

*for*

$$t\leqslant q=q(\mathbf{k},\mathbf{k}')=\left[\min\left(\tfrac{1}{2},c(G/P)\right)l(\mathbf{k})\right]-1.$$

*Here* $[x]$ *is the greatest integer less than or equal to* $x$, $c(G/P)$ *is the positive constant given in Proposition* 6.6 *and* $l(\mathbf{k})=\min(k_1,\dots,k_{n(\mathfrak{p})})$.

*Proof.* We begin by noting that for a given configuration of multiplicities $\mathcal{M}$, the degree $\mathbf{j}$ of the subconfiguration of multiple poles satisfies the inequality $|\mathbf{j}|\geqslant l(\mathbf{k})-s(\mathcal{M})$. As each multiple pole has scalar multiplicity at least two, we have from Proposition 6.6 that the real codimension of $S_{\mathcal{M},j}$ is at least $2c(G/P)\left[\frac{1}{2}(l(\mathbf{k})-s(\mathcal{M})+1)\right]$.

Returning to the $E^1$ terms associated to Corollary 6.13 for both $\mathrm{Rat}_{\mathbf{k}}(G/P)$ and $\mathrm{Rat}_{\mathbf{k}'}(G/P)$ we first note that as $s(\mathcal{M}')=0$ for the strata in $\mathcal{I}^2_{\mathbf{k}'}$, all non-trivial homology classes in the second summand in (7.3) occur in dimension greater than $q(\mathbf{k},\mathbf{k}')+1$ and thus can be ignored in the proof. Next, by Definition 6.8, Theorem 6.12, and the Thom isomorphism theorem, the homology of the stratum $S_{\mathcal{M},j}$ and its image appears in (7.2) and (7.3) respectively suspended $2\,\mathrm{cd}(\mathcal{M},j,\mathbf{k})\geqslant 2c(G/P)\left[\frac{1}{2}(l(\mathbf{k})-s(\mathcal{M})+1)\right]$ times (recall Proposition 6.6). Thus, Corollary 6.13 and (7.6) imply that the size of the real codimension of each $S_{\mathcal{M},j}$ in $\mathrm{Rat}_{\mathbf{k}}(G/P)$ and the number of stabilizing degree one principal parts, $l_j$, together ensure that

$$\iota(\mathbf{k},\mathbf{k}')(1)\colon E^1(\mathrm{Rat}_{\mathbf{k}}(G/P);\mathbf{F}_p)\to E^1(\mathrm{Rat}_{\mathbf{k}'}(G/P);\mathbf{F}_p)$$

is an isomorphism through dimension $q(\mathbf{k},\mathbf{k}')+1$. That is, the statement of the theorem holds at the $E^1$ level for $t\leqslant q(\mathbf{k},\mathbf{k}')+1$. Since differentials in Corollary 6.13 are natural, the only non-trivial differentials in $E^r(\mathrm{Rat}_{\mathbf{k}'}(G/P))$ that do not appear in $E^r(\mathrm{Rat}_{\mathbf{k}}(G/P))$ must leave classes of dimension at least $q(\mathbf{k},\mathbf{k}')+2$ and from this fact the result follows. □

COROLLARY 7.10. *Theorem 7.8 holds with $\mathbf{Z}$ as well as $\mathbf{F}_p$ coefficients.*

*Proof.* This follows from the general fact that if $f\colon X\to Y$ is any map that induces isomorphisms in homology with $\mathbf{Z}_p$ coefficients for all primes $p$ where $X$ and $Y$ both have finite type then $f$ induces isomorphisms on the $p$-primary parts of the integral homology groups as well. Clearly $\iota(\mathbf{k},\mathbf{k}')$ has this property through the stated range. □

Finally, one can take the direct limit induced by repeated application of (7.1) to obtain the inclusion

$$\iota(\mathbf{k})\colon \mathrm{Rat}_{\mathbf{k}}(G/P)\to \varinjlim \mathrm{Rat}(G/P)\cong \mathbf{RAT}(G/P). \tag{7.11}$$

Theorem 7.8 immediately implies

COROLLARY 7.12. *The inclusion given in* (7.11) *induces an isomorphism of homology groups*

$$\iota(\mathbf{k})_t\colon H_t(\mathrm{Rat}_{\mathbf{k}}(G/P))\to H_t(\mathbf{RAT}(G/P)) \tag{7.13}$$

*for $t$ less than the function $q(\mathbf{k},\mathbf{k}')+1$ in Theorem* 7.8. *Recall that, as the minimum of the $k_j$'s tends to infinity so does $q(\mathbf{k},\mathbf{k}')+1$.*

In the next section we will combine Theorem 7.8 and Corollary 7.12 with the stabilization theorem of Gravesen [Gr] to prove Theorem A of the introduction.

## 8. The Gravesen stabilization and Theorem A

When $G=\mathrm{SL}(n,\mathbf{C})$, so that $G/P$ is then the flag manifold of complex subspaces inside $\mathbf{C}^n$, Gravesen [Gr] studied the space of poles and principal parts considered in §3. Furthermore, Gravesen proved that the components of the mapping telescope formed obtained from repeated application of maps similar to (7.1) to his spaces has the same homology as the components of the double loop space $\Omega^2(G/P)$. We now show how to modify his analysis to finish the proof of Theorem A.

First of all note that Gravesen's restriction to the case where $G=\mathrm{SL}(n,\mathbf{C})$ is unnecessary. That is, replacing the beginning of [Gr, §3] with §3 here one sees that his entire analysis holds word for word for the more general case when $G$ is any complex semi-simple Lie group and $P$ is any parabolic subgroup.

THEOREM 8.1 (Gravesen). *Let $G/P$ be a complex flag manifold. There are stabilization maps*

$$\iota(\mathbf{k},\mathbf{k}')\colon \mathrm{Rat}_{\mathbf{k}}(G/P)\to \mathrm{Rat}_{\mathbf{k}'}(G/P)$$

*increasing the multi-degrees in each component so that passing to the direct limit*

$$\varinjlim \mathrm{Rat}_{\mathbf{k}}(G/P)=\mathbf{RAT}(G/P)$$

*yields an induced map into the direct limit of the components of $\Omega^2(G/P)$*

$$\varrho\colon \mathbf{RAT}(G/P)\to \Omega^2_\infty(G/P)$$

*so that $\varrho_*$ induces an isomorphism in homology*

$$H_*(\mathbf{RAT}(G/P))\cong H_*(\Omega^2_\infty(G/P)).$$

*Thus, there is an isomorphism*

$$H_*(\Omega^2(G/P))\cong H_*(\widehat{\mathcal{P}}(G/P)) \tag{8.2}$$

*where $\widehat{\mathcal{P}}(G/P)$ is the mapping telescope of the Gravesen stabilized principal part space.*

We note that Gravesen's analysis extends to mapping spaces where the domain Riemann sphere is replaced by a compact Riemann surface $\Sigma_g$ and in this more general analysis one must also consider the principal parts space where the complex structures on $\Sigma_g$ are allowed to vary. However, we need not concern ourselves with this fact here. We now examine the Gravesen mapping telescope $\widehat{\mathcal{P}}(G/P)$ more carefully. $\mathcal{P}(G/P)$ is defined to be the union of all pole and principal parts spaces of all finite multiplicities with topology induced by the same topology as determined by local holomorphic data. Thus, the stabilization maps (7.1) extend to give a self-map of $\mathcal{P}(G/P)$ and we have the following commutative diagram:

$$\begin{array}{ccccccccc}
\mathrm{Rat}_{\mathbf{k}}(G/P) & \longrightarrow & \mathrm{Rat}_{\mathbf{k}'}(G/P) & \longrightarrow & \cdots & \longrightarrow & \mathrm{Rat}_{\mathbf{k}''}(G/P) & \longrightarrow & \cdots \\
\downarrow & & \downarrow & & & & \downarrow & & \\
\mathcal{P}(G/P) & \longrightarrow & \mathcal{P}(G/P) & \longrightarrow & \cdots & \longrightarrow & \mathcal{P}(G/P) & \longrightarrow & \cdots
\end{array} \tag{8.3}$$

where all the horizontal maps are the stabilization maps (7.1) and the vertical maps are inclusions of labelled configuration spaces. This induces a natural map on the mapping telescope level

$$\varrho(G/P)\colon \mathcal{RAT}(G/P)\to\widehat{\mathcal{P}}(G/P) \tag{8.4}$$

where $\mathcal{RAT}(G/P)$ is the mapping telescope of the top row in (8.3).

But for $t\leqslant q(\mathbf{k},\mathbf{k}')$ we have the isomorphisms:

$$\widetilde{H}_t(\mathrm{Rat}_{\mathbf{k}}(G/P))\cong\varinjlim \widetilde{H}_t(\mathrm{Rat}_{\mathbf{k}}(G/P))\cong\widetilde{H}_t(\mathcal{RAT}(G/P)). \tag{8.5}$$

Hence, Theorem A of the introduction follows immediately from the fact that the pole and principal part description of $\mathrm{Rat}_{\mathbf{k}}(G/P)$ form a cofinal subset in $\mathcal{P}(G/P)$ under the stabilization maps (7.1). That is, $\mathcal{RAT}(G/P))$ is a component of $\widehat{\mathcal{P}}(G/P)$.

*Remark* 8.6. In [BHMM, §1] we gave a proof that the Taubes stabilization for instantons on $S^4$ is homotopy equivalent to the stabilization map given by the addition of a fixed jumping line and used this fact in our proof of the Atiyah–Jones conjecture. The argument given here, suitably modified, can be used to combine [BHMM, 6.17] with [Gr, 7.8] to give an alternate proof of [BHMM, 6.20].

## 9. Homotopy theorems

The proof of Theorem B of the introduction will require two preliminary steps. First, we show, for $\mathbf{k}$ sufficiently large in all entries, that $\pi_1(\mathrm{Rat}_{\mathbf{k}}(G/P))$ is Abelian. Combined with Theorem A this implies, again for $\mathbf{k}$ sufficiently large in all entries, that $\pi_1(\mathrm{Rat}_{\mathbf{k}}(G/P))=\pi_1(\Omega^2(G/P))$. Second, we show that the action of $\pi_1(\mathrm{Rat}_{\mathbf{k}}(G/P))$ on $\pi_t(\mathrm{Rat}_{\mathbf{k}}(G/P))$ is simple for $t\leqslant q(\mathbf{k})$ where, once again, $\mathbf{k}$ is sufficiently large in all entries. The geometric basis for these facts is that one can "shove" representatives for generators of the fundamental group into the multiplicity one principal parts spaces.

Let $\mathrm{Rat}'_{\mathbf{k}}(G/P)$ be the open subset of $\mathrm{Rat}_{\mathbf{k}}(G/P)$ consisting of those maps whose image does not meet the singular subset of any of the $Z_\alpha$, and whose poles are all either simple, or of multiplicity $(0,\dots,0,1,0,\dots,0,1,0,\dots,0)$ or $(0,\dots,0,2,0,\dots,0)$.

PROPOSITION 9.1. $\pi_1(\mathrm{Rat}'_{\mathbf{k}}(G/P))=\pi_1(\mathrm{Rat}_{\mathbf{k}}(G/P))$.

*Proof.* We show that the difference $\mathrm{Rat}_{\mathbf{k}}(G/P)\backslash\mathrm{Rat}'_{\mathbf{k}}(G/P)$ is of complex codimension two. By construction $\mathrm{Rat}_{\mathbf{k}}(G/P)\backslash\mathrm{Rat}'_{\mathbf{k}}(G/P)$ consists of poles that either meet the singular set of one of the $Z_\alpha$'s or has a pole with scalar multiplicity $|\mathbf{m}|>2$. So the result follows from Theorem 5.20 and the local additivity of principal parts. □

PROPOSITION 9.2. *Let*

$$\Pi\colon \mathrm{Rat}_{\mathbf{k}}(G/P)\to \mathbf{C}^{|\mathbf{k}|}$$

*be the pole location map of Proposition* 3.20. *Then* $\Pi$ *has a "generic homotopy lifting property" for loops*; *that is, any loop in* $\mathrm{Rat}_{\mathbf{k}}(G/P)$ *can be homotoped into the generic fiber of* $\Pi$.

*Proof.* First, by Proposition 9.1, it suffices to consider loops in $\mathrm{Rat}'_{\mathbf{k}}(G/P)$. As the locus of maps with multiple poles is of complex codimension one, we can suppose that our loop to be deformed consists of maps with only simple poles. Since the range of $\Pi$ has trivial fundamental group and $\Pi$ is clearly a fibration when restricted to the locus with distinct poles (the generic stratum) we are reduced to showing that we can "shove" a path in $\mathrm{Rat}_{\mathbf{k}}(G/P)$ through $\Pi^{-1}(\mathrm{DL})$, where DL is the discriminental locus, for poles of the same type or through $\Pi^{-1}(\mathrm{FD})$, where FD is the fat diagonal, for poles of different type. To do this, we construct contractible loops in $\mathrm{Rat}'_{\mathbf{k}}(G/P)$ which project to small circles surrounding DL or FD. For DL, let $f(z)$ be a map with a pole of multiplicity $(0,\ldots,0,1,0,\ldots,0)$ at $z=0$. The composition $f(z^2-a)$ gives a map with a double pole when $a=0$, and two simple poles otherwise. Setting

$$g_\theta(z)=f(z^2-re^{i\theta}),\quad \theta\in[0,2\pi],\tag{9.3}$$

gives us the desired loop, after adding in fixed principal parts to bring the charge up to $\mathbf{k}$. To construct the loop $g'_\theta$ "around FD", let $p$ be a point where the smooth loci of $Z_\sigma$ and $Z_\varrho$ intersect transversely (one checks directly that such points always exist). If we intersect with a suitable transverse two dimensional surface $D$, we find that in suitable coordinates $(x,y)$ on the surface that $Z_\sigma\cap D$ and $Z_\varrho\cap D$ are the graphs of functions $y=f_\sigma(x)$, $y=f_\varrho(x)$, with $f_\sigma(x)=f_\varrho(x)$ only when $x=0$. Now consider the germs of maps $\mathbf{C}\to D$ given by $\phi_\theta(z)=(re^{i\theta},z)$. This defines a contractible loop in the space of principal parts, and so after adding suitable fixed principal parts, the desired loop $g'_\theta$ in $\mathrm{Rat}'_{\mathbf{k}}(G/P)$.

We now write $A$ for either DL or FD and $G_t$ for either $g_t$ or $g'_t$. One can deform the loop $p(t)\in\pi_1(\mathrm{Rat}_{\mathbf{k}}(G/P))$ so that $\Pi(p(t_0))=\Pi(G_0)$, for some $t_0$, as the loop $p$ lies in the generic locus. Recall that Lemma 4.22 showed that the local principal part spaces $\mathcal{LPP}_1^\alpha$ are path connected. Hence, there is a path

$$h\colon[0,1]\to\mathrm{Rat}_{\mathbf{k}}(G/P),\tag{9.4}$$

with $\Pi\circ h$ constant, connecting $p(t_0)$ and $G_0$. Hence, we can homotope $p$ in a small neighborhood of $t_0$ into the path $h\cdot G\cdot h^{-1}$, effectively "shoving" $p$ through $A$. In this manner we can continuously homotopy any path $p$ to $p_1$ so that $\Pi(p_1)$ does not wind around $A$ and is thus contractible in the image of the generic set under $\Pi$. This construction allows one to get around $A$ and thus homotope the "pole path", i.e., the image of $p$ under $\Pi$, into a constant path of distinct poles of multiplicity 1. $\square$

Thus, $\pi_1(\mathrm{Rat}_{\mathbf{k}}(G/P))$ is a quotient of

$$\pi_1((\mathcal{LPP}_1^1)^{k_1}\times\ldots\times(\mathcal{LPP}_1^{n(\mathfrak{p})})^{k_{n(\mathfrak{p})}})=(\pi_1(\mathcal{LPP}_1^1))^{k_1}\times\ldots\times(\pi_1(\mathcal{LPP}_1^{n(\mathfrak{p})}))^{k_{n(\mathfrak{p})}}$$

where $\mathcal{LPP}_1^\alpha$ is the space of local principal parts of multiplicity one in the $\alpha$th component and multiplicity zero in the $j$th component for all $j\neq\alpha$. An element of $\pi_1(\mathrm{Rat}_{\mathbf{k}}(G/P))$ can then be written as

$$((a_1^1,\ldots,a_{k_1}^1),(a_1^2,\ldots,a_{k_2}^2),\ldots,(a_1^{n(\mathfrak{p})},\ldots,a_{k_{n(\mathfrak{p})}}^{n(\mathfrak{p})})). \tag{9.5}$$

Geometrically, this means that loops can be thought of as moving in the principal parts space over fixed poles. The next proposition shows that these loops can be placed above the pole of one's choice amongst the poles of the same type.

PROPOSITION 9.6. *With suitable identifications due to the choice of base points, in the notation of* (9.5), *the loop*

$$a_j^i=\begin{cases} a, & \text{if } i=i_0 \text{ and } j=j_0,\\ 1 & \text{otherwise,}\end{cases} \tag{9.7}$$

*is homotopic in* $\pi_1(\mathrm{Rat}_{\mathbf{k}}(G/P))$ *to the loop*

$$b_j^i=\begin{cases} a, & \text{if } i=i_0 \text{ and } j=j_1,\\ 1 & \text{otherwise.}\end{cases} \tag{9.8}$$

*Proof.* It suffices to prove the proposition in the case when $n(\mathfrak{p})=1$, and there are only two poles, located, say, at $\pm r$. Once again recall that $\mathcal{LPP}_1^\alpha$ is path connected, Lemma 4.22, and that the loop $g_\theta$ of (9.3) defines principal parts $P^\pm(\theta)$ located at $\pm re^{i\theta/2}$, with $P^\pm(2\pi)=P^\mp(0)$. Notice that varying $\theta$ from 0 to $2\pi$ interchanges the poles. We choose as base point $P^\pm(\theta)$ for the principal parts over the poles $re^{i\theta/2}$, and set $V_\theta(t)$, $t\in[0,\theta]$ to be the path in $\mathrm{Rat}_2(G/P)$ obtained by fixing the poles at $\pm re^{i\theta/2}$ and letting the principal parts move along the path $P^\pm(\theta-t)$. Let $a(t)$ be a loop in $\mathcal{PP}_1$, with base point $P^+(0)$, and let $h_\theta(t)$ be the loop in $\mathrm{Rat}_2(G/P)$ with fixed poles at $\pm re^{i\theta/2}$ and principal parts $a(t)$ over $re^{i\theta}$, and staying at $P^-(0)$ over $-re^{i\theta/2}$. In the notation of the statement of the theorem, $h_0(t)$ is the path $(a,1)$, whereas (and this is what is meant by identification due to change of base point) $V_{2\pi}\cdot h_{2\pi}\cdot V_{2\pi}^{-1}$ represents $(1,a)$. Now we remark that $h_0(t)$ is homotopic to the composition $j_s(t)$ of:

(1) the path $g_t$, $t \in [0, 2\pi s]$;
(2) the path $V_{2\pi s}(t)$, $t \in [0, 2\pi s]$;
(3) the loop $h_{2\pi s}$;
(4) the inverse of the path in 2;
(5) the inverse of the path in 1.

When $s=1$, this is the composition $g \cdot V_{2\pi} \cdot h_{2\pi} \cdot V_{2\pi}^{-1} \cdot g^{-1}$. But, as $g$ is null homotopic, we are done. □

COROLLARY 9.9. *$\pi_1(\mathrm{Rat}_{\mathbf{k}}(G/P))$ is Abelian, for $\mathbf{k}=(k_1, ..., k_{n(\mathbf{p})})$ whenever all the $k_\alpha \geqslant 2$.*

*Proof.* In the notation given above, this simply reflects the fact that $(ab, 1)=(a, b)=(ba, 1)$. □

Now assume that $G$ is simple. Using the long exact sequence in homotopy for the fibration

$$P \to G \to G/P$$

and the fact that $P$ retracts onto its Levi factor (which up to a finite cover is a product of $\mathbf{C}^*$'s and simple groups) we have the short exact sequence

$$\mathbf{Z}^{n(\mathbf{p})} \xrightarrow{i_*} \mathbf{Z} \to \pi_3(G/P) \to 0. \tag{9.10}$$

Furthermore, the image of $i_*$ is computable in terms of ratios of longest roots (cf. [AHS, p. 455]) and it follows that $\pi_1(\Omega^2_{\mathbf{k}}(G/P))$ must be one of the following four groups: $\mathbf{Z}$ (when $P$ is a Borel subgroup), or 0, $\mathbf{Z}_2$, or $\mathbf{Z}_3$ (when $P$ is not a Borel subgroup).

Combining Theorem A and Corollary 9.9 with the observation that $q(\mathbf{k})>0$ implies that all $k_\alpha \geqslant 2$ (Theorem 7.8), we have

PROPOSITION 9.11. *For all $G$, $P$, and all $\mathbf{k}$ such that $q(\mathbf{k})>0$,*

$$\pi_1(\mathrm{Rat}_{\mathbf{k}}(G/P)) \cong \pi_1(\Omega^2(G/P)) \cong \pi_3(G/P). \tag{9.12}$$

Recall that §4 gives a description of the multiplicity one principal part spaces as principal fibrations $\mathbf{C}^* \to \mathcal{LPP}_1^\alpha \to L$. Our next step is to show that the fundamental group of $\mathrm{Rat}_{\mathbf{k}}(G/P)$ is "carried" by the images of $\pi_1(\mathbf{C}^*)$.

PROPOSITION 9.13. *There is a positive $\mathbf{k}$ for which $\pi_1(\mathrm{Rat}_{\mathbf{k}}(G/P))$ is generated by the loop $f_\theta(z)=f(e^{i\theta}z)$, $\theta \in [0, 2\pi]$. Here positive means that each $k_\alpha$ is $\geqslant 0$ and not all are equal to 0.*

*Proof.* There is a surjection $\pi_3(G) \to \pi_3(G/P)$. In turn, $\pi_3(G)$ is generated by a subgroup $\mathrm{SU}(2) \to G$, (see, e.g. [AHS, p. 455]), and this projects to a holomorphic map

$i\colon \mathbf{P}^1 = \mathrm{SU}(2)/S^1 \to G/P$, of positive multi-charge $\mathbf{k}$, again inducing a surjection on $\pi_3$. This reduces our problem to the case of maps of degree 1 from $\mathbf{P}^1$ to $\mathbf{P}^1$, with $f$ the identity map, where it is evident. □

This proposition in fact gives us just what we need to show that the images of $\pi_1(\mathbf{C}^*)$ "carry" $\pi_1(\mathrm{Rat}_\mathbf{k}(G/P))$. In the case when $\pi_1 = \mathbf{Z}$ (when $P$ is a Borel subgroup) the local principal part space $\mathcal{LPP}_1^\alpha$ is just $\mathbf{C}^*$, as the group $\widehat{G}$ of §4 is $\mathrm{SL}(2,\mathbf{C})$ [HM2], and so there is nothing to prove. When $\pi_1$ is $\mathbf{Z}_2$ or $\mathbf{Z}_3$, one can choose the $\mathbf{k}$ to be of the form $(0, ..., 0, 1, 0, ..., 0)$. Referring to our geometric description in §4 of the maps, the rotation action leaves fixed the element in the line space $L$, and describes the standard generator $\gamma$ of $\pi_1(\mathbf{C}^*)$.

PROPOSITION 9.14. *The element $\gamma$ acts trivially on $\pi_r(\mathcal{LPP}_1^\alpha)$.*

*Proof.* This follows easily from the fact that the fibration of $\mathcal{LPP}_1^\alpha$ is principal, which gives us a natural way of moving elements of $\pi_r(\mathcal{LPP}_1^\alpha)$ along $\gamma$. □

To prove Theorem B for the cases when the fundamental group is non-trivial it suffices to show that

$$\iota(\mathbf{k})_*\colon H_t(\widetilde{\mathrm{Rat}}_\mathbf{k}(G/P); \mathbf{F}_p) \to H_t(\widetilde{\Omega}^2_\mathbf{k}(G/P); \mathbf{F}_p)$$

is an isomorphism through the dimension $r(\mathbf{k})$ for all primes $p$ (this reduction to $\mathbf{F}_p$ coefficients follows from the fact that the integral homology groups of $\widetilde{\mathrm{Rat}}_\mathbf{k}(G/P)$ and $\widetilde{\Omega}^2_\mathbf{k}(G/P)$ are finitely generated). Here $\widetilde{X}$ is the universal cover of $X$. When $\pi_1$ is $\mathbf{Z}$ or $\mathbf{Z}_2$ this follows essentially as in [S] and [BHMM], respectively; in the $\mathbf{Z}_2$ case one loses a dimension in going from homology to homotopy. When $\pi_1 \cong \mathbf{Z}_3$ (and this happens only when $G = G_2$ and $P$ is a nilpotent extension of $\mathrm{SL}_2(\mathbf{C}) \times \mathbf{C}^*$) the argument is somewhat different so we present it here.

We want to verify that the homology map $\iota(\mathbf{k})_*$ is an isomorphism through a range when restricted to the universal cover over strata in $\mathrm{Rat}_\mathbf{k}(G_2/P)$ indexed by $\mathcal{I}^1_\mathbf{k}$. Using Corollary 9.9 and Proposition 9.14 we see that we are considering the commutative diagram of $\mathbf{C}^*$ fibrations

$$\begin{tikzcd} \mathbf{C}^* \arrow[r] \arrow[d, "z^3"] & \widetilde{S}_{\mathcal{M},j} \arrow[r] \arrow[d, "\pi"] & W \arrow[d, "\mathrm{id}"] \\ \mathbf{C}^* \arrow[r] & S_{\mathcal{M},j} \arrow[r] & W \end{tikzcd}$$

where $\mathbf{C}^*$ carries $\pi_1(\mathrm{Rat}_\mathbf{k}(G_2/P))$. Thus, by the argument of [BHMM, §6], away from the prime 3 the homology of this $\mathbf{Z}_3$-cover is acted on simply by $\mathbf{Z}_3$, and a direct comparison theorem gives the result.

7-945203 *Acta Mathematica* 173. Imprimé le 5 octobre 1994

At the prime 3 we consider the mod(3) homology structure of the 3-fold cover. To understand this consider the relative pair $(X,\widetilde{X})$ where $\widetilde{X}$ is any $\mathbf{Z}_3$-cover over $X$. This can be replaced by the mapping cylinder of $\pi: \widetilde{X} \to X$ and we have a fibration $\mathrm{Mcyl}(\pi) \to X$ with fibre over each point the "propeller" space given by the mapping cone of $\{1,2,3\} \to *$. Here the action of $\tau \in \mathbf{Z}_3$ just rotates the propeller. The natural inclusion $\widetilde{X} \to \mathrm{Mcyl}(\pi)$ has associated mapping cone $\mathrm{MC}(\pi)$ homotopy equivalent to the quotient space $\mathrm{Mcyl}(\pi)/\widetilde{X}$. There is a (relative) Serre spectral sequence converging to $\widetilde{H}_*(\mathrm{MC}(\pi);\mathbf{F}_3) \cong H_*(\mathrm{Mcyl}(\pi),\widetilde{X};\mathbf{F}_3)$ with $E^2$ term

$$H_*(X, \mathcal{H}_*(F_{\mathrm{Mcyl}(\pi)}, F_{\widetilde{X}}; \mathbf{F}_3)).$$

Here $\mathcal{H}_*$ represents homology with twisted coefficients. $\mathcal{H}_*(F_{\mathrm{Mcyl}(\pi)}, F_{\widetilde{X}}; \mathbf{F}_3)$ is obtained from the exact sequence of $\mathbf{F}_3(\mathbf{Z}_3)$ modules

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{H}_1(F_{\mathrm{Mcyl}(\pi)}, F_{\widetilde{X}}; \mathbf{F}_3) \arrow[r] \arrow[d, "="] & \mathcal{H}_0(F_{\widetilde{X}}; \mathbf{F}_3) \arrow[r] \arrow[d, "="] & \mathcal{H}_0(F_{\mathrm{Mcyl}(\pi)}; \mathbf{F}_3) \arrow[r] \arrow[d, "="] & 0 \\
0 \arrow[r] & \mathcal{H}_1(F_{\mathrm{Mcyl}(\pi)}, F_{\widetilde{X}}; \mathbf{F}_3) \arrow[r] & \mathbf{F}_3(\mathbf{Z}_3) \arrow[r, "\varepsilon"] & \mathbf{F}_3 \arrow[r] & 0
\end{tikzcd}$$

and hence

$$\mathcal{H}_*(F_{\mathrm{Mcyl}(\pi)}, F_{\widetilde{X}}; \mathbf{F}_3) = \begin{cases} 0 & \text{when } * \neq 1, \\ \mathcal{I} & \text{when } * = 1, \end{cases}$$

where $\mathcal{I}$ is the augmentation ideal

$$\mathcal{I} \to \mathbf{F}_3(\mathbf{Z}_3) \xrightarrow{\varepsilon} \mathbf{F}_3$$

with $\varepsilon(\sum n_i \tau^i) = \sum n_i$.

In other words the Serre spectral sequence converging to the homology of the mapping cone has exactly one non-zero row, $E^2_{*,1}$, so

$$E^\infty = E^2 = H_*(X;\mathcal{I}) = H_{*+1}((X,\widetilde{X});\mathbf{F}_3).$$

We analyze these homology groups as follows. First, $\mathbf{F}_3(\mathbf{Z}_3) \cong \mathbf{F}_3[x]/(x^3)$ where $x = \tau - 1$. Second, $\mathcal{I} \subset \mathbf{F}_3(\mathbf{Z}_3)$ corresponds to the ideal $(x)$ under this isomorphism. Third, there is a short exact sequence of $\mathbf{F}_3(\mathbf{Z}_3)$ modules

$$0 \to \mathbf{F}_3 \xrightarrow{j} \mathcal{I} \to \mathbf{F}_3 \to 0 \tag{9.15}$$

where $j$ includes $\mathbf{F}_3$ into $\mathcal{I}$ as $x^2\mathbf{F}_3[x]/(x^3)$. Thus, while (9.15) does not split and $\mathcal{I}$ is not a trivial $\mathbf{F}_3(\mathbf{Z}_3)$ module, (9.15) does give a long exact sequence

$$\ldots \to H_*(X;\mathbf{F}_3) \to H_*(X;\mathcal{I}) \to H_*(X;\mathbf{F}_3) \to H_{*-1}(X;\mathbf{F}_3) \to \ldots \tag{9.16}$$

that is clearly natural with respect to $\mathbf{Z}_3$-covers. In particular, if we have a map $f: X \to Y$ which induces isomorphisms in $\pi_1$ and in homology through degree $m$, then naturality of the Serre spectral sequence and (9.16) imply that the relative spaces $(\mathrm{Mcyl}(X), \widetilde{X})$ and $(\mathrm{Mcyl}(Y), \widetilde{Y})$ representing the mapping cones also map isomorphically under $f_*$ through dimension $m$ as well because the relative complex increases the dimension by 1.

Next, $f: X \to Y$ also induces the following commutative diagram

$$\begin{tikzcd}
{} \arrow[r] & H_*(X;\mathbf{F}_3) \arrow[r] \arrow[d, "f_*"] & H_*(X,\widetilde{X};\mathbf{F}_3) \arrow[r] \arrow[d, "f_*"] & H_{*-1}(\widetilde{X};\mathbf{F}_3) \arrow[r] \arrow[d, "f_*"] & H_{*-1}(X;\mathbf{F}_3) \arrow[r] \arrow[d, "f_*"] & {} \\
{} \arrow[r] & H_*(Y;\mathbf{F}_3) \arrow[r] & H_*(Y,\widetilde{Y};\mathbf{F}_3) \arrow[r] & H_{*-1}(\widetilde{Y};\mathbf{F}_3) \arrow[r] & H_{*-1}(Y;\mathbf{F}_3) \arrow[r] & {} .
\end{tikzcd}$$

Thus, we can now apply the five lemma with $X = \mathrm{Rat}_{\mathbf{k}}(G_2/P)$ and $Y = \Omega^2(G_2/P)$ to the spaces, their two $\mathbf{Z}_3$-covers, and the associated relative spaces to obtain isomorphisms in homology for the $\mathbf{Z}_3$-covers through dimension $m-1 = q(\mathbf{k})-1 = r(\mathbf{k})$. This finishes the proof of Theorem B when $G$ is simple in this last case.

Returning to the general case when $G$ is semi-simple, we use the fact that $G/P$ splits into a product of "simple" summands [Bo], giving a corresponding decomposition for both the Rat spaces and the mapping spaces. One then applies the above in each of the simple summands. This concludes the proof of Theorem B for the general case.

Finally, as noted in §4, $\mathrm{Rat}(G/P)$, thought of as the disjoint union of its components, has a natural $\mathcal{C}_2$ operad structure induced by the obvious $\mathcal{C}_2$ operad structure on the position of the poles. Thus, Theorem B implies that the natural inclusion

$$\iota: \mathrm{Rat}(G/P) \to \Omega^2(G/P)$$

is a group completion; more precisely, that

$$\Omega_0 B\big(\bigsqcup_{\mathbf{k}} \mathrm{Rat}_{\mathbf{k}}(G/P)\big) \to \Omega_0^2(G/P) \tag{9.17}$$

is a $\mathcal{C}_2$ operad equivalence of two-fold loop spaces.

## References

[AHS] Atiyah, M. F., Hitchin, N. J. & Singer, I., Self-duality in four dimensional Riemannian geometry. *Proc. Roy. Soc. London Ser. A*, 362 (1978), 425–461.

[AJ] Atiyah, M. F. & Jones, J. D., Topological aspects of Yang–Mills theory. *Comm. Math. Phys.*, 61 (1978), 97–118.

[BE] Baston, R. J. & Eastwood, M. G., *The Penrose Transform*. Oxford University Press, 1989.

[BHMM] BOYER, C. P., HURTUBISE, J. C., MANN, B. M. & MILGRAM, R. J., The topology of instanton moduli spaces. I: The Atiyah–Jones conjecture. *Ann. of Math.*, 137 (1993), 561–609.

[BM] BOYER, C. P. & MANN, B. M., Monopoles, non-linear $\sigma$ models, and two-fold loop spaces. *Comm. Math. Phys.*, 115 (1988), 571–594.

[Bo] BOREL, A., Kählerian coset spaces of semisimple Lie groups. *Proc. Nat. Acad. Sci. U.S.A.*, 40 (1954), 1147–1151.

[$C^2M^2$] COHEN, F. R., COHEN, R. L., MANN, B. M. & MILGRAM, R. J., The topology of rational functions and divisors of surfaces. *Acta Math.*, 166 (1991), 163–221.

[D] DONALDSON, S. K., Nahm's equations and the classification of monopoles. *Comm. Math. Phys.*, 96 (1988), 387–407.

[Gr] GRAVESEN, J., On the topology of spaces of holomorphic maps. *Acta Math.*, 162 (1989), 247–286.

[Gu1] GUEST, M. A., Topology of the space of absolute minima of the energy functional. *Amer. J. Math.*, 106 (1984), 21–42.

[Gu2] — The topology of the space of rational curves on a toric variety. Preprint, Rochester University, 1993.

[HM1] HURTUBISE, J. C. & MURRAY, M. K., On the construction of monopoles for the classical groups. *Comm. Math. Phys.*, 122 (1989), 35–89.

[HM2] — Monopoles and their spectral data. *Comm. Math. Phys.*, 133 (1990), 487–508.

[Hum] HUMPHREYS, J. E., *Linear Algebraic Groups*. Springer-Verlag, 1975.

[Hur1] HURTUBISE, J. C., Instantons and jumping lines. *Comm. Math. Phys.*, 105 (1986), 107–122.

[Hur2] — The classification of monopoles for the classical groups. *Comm. Math. Phys.*, 120 (1989), 613–641.

[I] IITAKA, S., *Algebraic Geometry*. Springer-Verlag, New York, 1982.

[Ki1] KIRWAN, F. C., On spaces of maps from Riemann surfaces to Grassmannians and applications to the cohomology of moduli of vector bundles. *Ark. Mat.*, 24 (1986), 221–275.

[Ki2] — Geometric invariant theory and the Atiyah–Jones conjecture. Preprint, Oxford University, 1992.

[Ko] KODAIRA, K., A theorem of completeness of characteristic systems for analytic families of compact submanifolds of complex manifolds. *Ann. of Math.*, 75 (1962), 146–162.

[Mc] MCDUFF, D., Configuration spaces of positive and negative particles. *Topology*, 14 (1975), 91–107.

[MF] MUMFORD, D. & FOGARTY, J., *Geometric Invariant Theory*, 2nd edition. Springer-Verlag, New York, 1982.

[Mi] MILGRAM, R. J., Iterated loop spaces. *Ann. of Math.*, 84 (1966), 386–403.

[MM1] MANN, B. M. & MILGRAM, R. J., Some spaces of holomorphic maps to complex Grassmann manifolds. *J. Differential Geom.*, 33 (1991), 301–324.

[MM2] — On the moduli space of $\mathrm{SU}(n)$ monopoles and holomorphic maps to flag manifolds. *J. Differential Geom.*, 38 (1993), 39–103.

[MM3] — The topology of rational maps to Grassmannians and a homotopy theoretic proof of the Kirwan stability theorem. *Contemp. Math. AMS*, 146 (1993), 251–275.

[Mu] MURRAY, M. K., Stratifying monopoles and rational maps. *Comm. Math. Phys.*, 125 (1989), 661–674.

[RR] RAMANAN, S. & RAMANATHAN, A., Projective normality of flag varieties and Schubert varieties. *Invent. Math.*, 79 (1985), 217–224.

[S] SEGAL, G., The topology of rational functions. *Acta Math.*, 143 (1979), 39–72.

[Ta1] TAUBES, C. H., Monopoles and maps from $S^2$ of $S^2$; the topology of configuration space. *Comm. Math. Phys.*, 95 (1984), 345–391.

[Ta2] — The stable topology of self-dual moduli spaces. *J. Differential Geom.*, 29 (1989), 163–230.

[Ti] TIAN, Y., The based SU($n$)-instanton moduli spaces. *Math. Ann.*, 298 (1994), 117–139.

C. P. BOYER
Department of Mathematics and Statistics
University of New Mexico
Albuquerque, NM 87131
U.S.A.
cboyer@ramanujan.unm.edu

J. C. HURTUBISE
Department of Mathematics
McGill University
805 Sherbrooke ouest
Montreal, QC
Canada H3A 2K6
hurtubis@gauss.math.mcgill.ca

B. M. MANN
Department of Mathematics and Statistics
University of New Mexico
Albuquerque, NM 87131
U.S.A.
mann@math.unm.edu

R. J. MILGRAM
Department of Mathematics
Stanford University
Stanford, CA 94305
U.S.A.
milgram@gauss.stanford.edu
and
Department of Mathematics and Statistics
University of New Mexico
Albuquerque, NM 87131
U.S.A.
milgram@math.unm.edu

*Received November 17, 1992*
*Received in revised form September 15, 1993*