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A Hopf differential for constant mean curvature surfaces in $\mathbf{S}^2\times\mathbf{R}$ and $\mathbf{H}^2\times\mathbf{R}$

by

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Dedicated to Hermann Karcher on the occasion of his 65th birthday

Introduction

In 1955 H. Hopf [13] discovered that the complexification of the traceless part of the second fundamental form h_Σ of an immersed surface Σ^2 with constant mean curvature H in Euclidean 3-space is a *holomorphic quadratic differential* Q on Σ^2. This observation has been the key to his well-known theorem that any immersed constant mean curvature (cmc) sphere $S^2\looparrowright\mathbf{R}^3$ is in fact a standard distance sphere with radius $1/H$.

Hopf's result has been extended to immersed cmc spheres S^2 in the sphere $\mathbf{S}^3$ or in hyperbolic space $\mathbf{H}^3$. In other words, the result has been extended to immersed cmc spheres $S^2\looparrowright M^3_\varkappa$ in space forms with arbitrary curvature $\varkappa$. Furthermore, W.-T. and W.-Y. Hsiang have conjectured [14, Remark (i) on p. 51] that immersed cmc spheres in the product space $\mathbf{H}^2\times\mathbf{R}$ should be embedded, rotationally invariant vertical bigraphs.

The holomorphic quadratic differential Q itself has other important applications: it guarantees the existence of conformal curvature line coordinates on cmc tori in space forms, and thus it has played a significant role in the discovery of the Wente tori [25] and in the subsequent development of a general theory of cmc tori in $\mathbf{R}^3$ (see [1], [21], [6], [12] and [7]). For cmc surfaces Σ^2_g of genus $g>1$ the holomorphic quadratic differential still provides a link between the genus of the surface and the number and index of its umbilics, a link that has been very useful in investigating the geometric properties of such cmc surfaces (see [8], [11] and [22]).

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1. Main results

Our main goal is to introduce a generalized *quadratic differential* Q for immersed surfaces Σ^2 in the product spaces $\mathbf{S}^2\times\mathbf{R}$ and $\mathbf{H}^2\times\mathbf{R}$, and to extend Hopf's result to cmc spheres in these target spaces; we shall prove that such immersed spheres are surfaces of revolution. In order to treat the two cases simultaneously, we write the target space as $M^2_\varkappa\times\mathbf{R}$, where $M^2_\varkappa$ stands for the complete simply-connected surface with constant curvature $\varkappa$.

A lot of information about the product structure of these target spaces is encoded in the properties of the height function $\xi\colon M^2_\varkappa\times\mathbf{R}\to\mathbf{R}$, that is induced by the standard coordinate function on the real axis. We note in particular that ξ has a parallel gradient field of norm 1, and that the fibers of ξ are the leaves $M^2_\varkappa\times\{\xi_0\}$. In addition to the function ξ we only need the second fundamental form $h_\Sigma(X,Y)=\langle X,AY\rangle$, the mean curvature $H=\frac{1}{2}\operatorname{tr}A$, and the induced (almost) complex structure J in order to define the *quadratic differential Q for immersed surfaces Σ^2 in product spaces $M^2_\varkappa\times\mathbf{R}$.* We set

$$q(X,Y):=2Hh_\Sigma(X,Y)-\varkappa\,d\xi(X)\,d\xi(Y) \tag{1}$$

and

$$Q(X,Y):=\tfrac{1}{2}(q(X,Y)-q(JX,JY))-\tfrac{1}{2}i(q(JX,Y)+q(X,JY)). \tag{2}$$

Clearly, $Q(JX,Y)=Q(X,JY)=iQ(X,Y)$. In dimension 2, traceless symmetric endomorphisms anticommute with the almost complex structure J, whereas multiples of the identity commute with J. Thus the quadratic differential Q defined in (2) depends only on the traceless part q_0 of the symmetric bilinear form q introduced in (1), and, conversely, it is the traceless part q_0 that can be recovered as the real part of Q.

THEOREM 1. *Let $\Sigma^2\looparrowright M^2_\varkappa\times\mathbf{R}$ be an immersed cmc surface in a product space. Then its quadratic differential Q as introduced in* (2) *is* holomorphic *with respect to the induced complex structure J on the surface Σ^2.*

The proof of this theorem is a straightforward computation based on the Codazzi equations for surfaces $\Sigma^2\looparrowright M^2_\varkappa\times\mathbf{R}$, which differ from the more familiar Codazzi equations for surfaces in space forms $M^3_\varkappa$ by an additional curvature term. The details will be given in §3.

Following H. Hopf, we want to determine the geometry of immersed cmc spheres in product spaces $M^2_\varkappa\times\mathbf{R}$ using Theorem 1. In these target spaces one has the embedded cmc spheres S^2_H studied by W.-Y. Hsiang [14] and R. Pedrosa and M. Ritoré [20] that are invariant under an isometric $\mathsf{SO}(2)$-action, rotation about a vertical geodesic. From now on we shall refer to them as the embedded rotationally invariant cmc spheres $S_H^2\subset M_\varkappa^2\times\mathbf{R}$.

Their geometry will be described in Proposition 2.5 (i) in the next section. In contrast to cmc spheres in space forms, they are *not* totally umbilical unless $\varkappa=0$ or $\varkappa>0$ and $H=0$. The traceless part q_0 of the symmetric bilinear form introduced in (1), however, vanishes on all of them. For us, this observation has been the major clue when guessing the proper expression for the quadratic differential Q for cmc surfaces in such product spaces.

THEOREM 2. *Any immersed cmc sphere $S^2\looparrowright M_\varkappa^2\times\mathbf{R}$ in a product space is actually one of the embedded rotationally invariant cmc spheres $S_H^2\subset M_\varkappa^2\times\mathbf{R}$.*

This theorem in particular establishes Hsiang's conjecture from [14, remark (i) on p. 51] about immersed cmc spheres in $\mathbf{H}^2\times\mathbf{R}$.

It is a well-known fact that any holomorphic quadratic differential on the 2-sphere vanishes identically. Thus Theorem 1 implies that the quadratic differential Q of an immersed cmc sphere $S^2\looparrowright M_\varkappa^2\times\mathbf{R}$ vanishes. Hence Theorem 2 follows directly from the following classification result:

THEOREM 3. *There are four distinct classes of complete, possibly immersed, cmc surfaces $\Sigma^2\looparrowright M_\varkappa^2\times\mathbf{R}$ with vanishing quadratic differential Q.*

Three of these classes are comprised of embedded rotationally invariant surfaces; they are the cmc spheres $S_H^2\subset M_\varkappa^2\times\mathbf{R}$ of Hsiang and Pedrosa, their non-compact cousins D_H^2, and the surfaces C_H^2 of catenoidal type. The fourth class is comprised of certain orbits P_H^2 of 2-dimensional solvable groups of isometries of $M_\varkappa^2\times\mathbf{R}$.

The *geometry* of the rotationally invariant embedded cmc surfaces of type S_H^2, D_H^2 and C_H^2 will be *described in detail* in Propositions 2.5 and 2.9. The fact that they all have $Q\equiv 0$ will be verified in §2, too. The remaining family P_H^2, where the parabolic symmetries are prevalent, will only be introduced in Proposition 4.9 in §4. Explicit formulas for the meridians of all these surfaces will be provided in the appendix; for some pictures thereof, see Figures 1–4.

Remark 4. If $\varkappa\geqslant 0$, then only the spheres $S_H^2\subset M_\varkappa^2\times\mathbf{R}$ occur, whereas for $\varkappa<0$ all four cases do actually occur.

The fact that there are *no other cmc surfaces with* $Q\equiv 0$ will be established in §4. A systematic approach to this classification problem is to study a suitable overdetermined system of ordinary differential equations for the unit normal field of the surfaces to be classified. In doing so, the surfaces P_H^2 in fact show up quite naturally.

In a sense the preceding theorems amount to saying that, when studying surfaces Σ^2 in product spaces $M^2_\varkappa \times \mathbf{R}$ rather than in space forms, the *role of umbilics* is taken over by the points where Σ^2 approximates one of the embedded cmc surfaces S^2_H, D^2_H, C^2_H and P^2_H up to second order.

Theorem 2 provides *substantial information* even when studying the geometry of embedded cmc spheres $S^2 \subset \mathbf{S}^2 \times \mathbf{R}$. Moving planes arguments along the lines of A. D. Alexandrov [3] only prove that a closed embedded cmc surface $\Sigma^2 \subset \mathbf{S}^2 \times \mathbf{R}$ is a vertical *bigraph*. In contrast to target spaces with non-positive sectional curvature, moving planes arguments do *not* imply that such a surface $\Sigma^2 \subset \mathbf{S}^2 \times \mathbf{R}$ is rotationally invariant. The latter claim would only follow if we assumed that Σ^2 were contained in the product of a *hemisphere* with the real axis. Such an assumption, however, is by far too strong. It does not even hold for the embedded rotationally invariant cmc spheres $S^2_H \subset M^2_\varkappa \times \mathbf{R}$ if $0 < 4H^2 < \varkappa$. This particular aspect of their geometry is explained in more detail in Remark 2.8 in §2.4.

Furthermore—unlike in Euclidean space $\mathbf{R}^3$—there even exists a family of *embedded cmc tori* in $\mathbf{S}^2_\varkappa \times \mathbf{R}$, most of them *not* being rotationally invariant. They rather look approximately like "undoloids" around some great circle in a leaf $\mathbf{S}^2_\varkappa \times \{\xi_0\} \subset \mathbf{S}^2_\varkappa \times \mathbf{R}$, and are thus not contained in the product of a hemisphere with the real axis either. They have been constructed by J. de Lira [17] by a bifurcation argument along the lines of the work done by R. Mazzeo and F. Pacard [18]. Presumably one could also obtain these surfaces using a singular perturbation argument similar to the one that N. Kapouleas' work is based on [16].

Finally, we observe that the holomorphic quadratic differential Q of Theorem 1 is likely to open the doors for investigating the geometry of cmc tori in product spaces $M^2_\varkappa \times \mathbf{R}$, and—as in the case of space forms—there might be interesting connections to the theory of integrable systems. A somewhat more subtle question is whether all the above can be generalized to cmc surfaces Σ^2 in homogeneous 3-manifolds.

2. The rotationally invariant cmc spheres $S^2_H \subset M^2_\varkappa \times \mathbf{R}$

In this section we shall describe the embedded rotationally invariant cmc spheres $S^2_H \subset M^2_\varkappa \times \mathbf{R}$ of Hsiang and Pedrosa. Our main goal is to compute the symmetric bilinear form q introduced in (1) for these cmc spheres, and thus verify that their quadratic differential Q vanishes. This goal will be accomplished in Proposition 2.7.

The work of both authors, Hsiang and Pedrosa, has been somewhat more general than what we need here. They have studied rotationally invariant cmc spheres S^n_H in product spaces including $\mathbf{H}^n \times \mathbf{R}$ [14] or in $\mathbf{S}^n \times \mathbf{R}$ [20], respectively. For our purposes we are only interested in properties of the surfaces S^2_H, and, specializing our summary accordingly, we thus work on product manifolds $M^3:=M^2_\varkappa\times\mathbf{R}$ that are equipped with an isometric $\mathsf{SO}(2)$-action.

2.1. Isometric $\mathsf{SO}(2)$-actions on the product spaces $M^2_\varkappa\times\mathbf{R}$

Such an action necessarily operates trivially on the second factor. In other words, the group $\mathsf{SO}(2)\subset\mathrm{Isom}_0(M^2_\varkappa\times\mathbf{R})$ can be considered as the isotropy group consisting of those *rotations* that preserve some fixed axis $l_0:=\{\hat{x}_0\}\times\mathbf{R}$. If $\varkappa\leqslant 0$, the fixed-point set of this $\mathsf{SO}(2)$-action is precisely the axis l_0 itself; however, if $\varkappa>0$, it encompasses the antipodal line $\hat{l}_0=\{-\hat{x}_0\}\times\mathbf{R}$ as well.

In order to understand this action better, it is useful to pick a normal geodesic $\gamma:\mathbf{R}\to M^2_\varkappa$ with $\gamma(0)=\hat{x}_0$. The product map $\gamma\times\mathrm{id}:\mathbf{R}^2\to M^2_\varkappa\times\mathbf{R}$ is a totally geodesic isometric immersion. Its image intersects all $\mathsf{SO}(2)$-orbits orthogonally. The existence of such a *slice* means that the group action under consideration is *polar*.

The subset $I_\varkappa\times\mathbf{R}\subset\mathbf{R}^2$, where $I_\varkappa$ denotes the half-axis $(0,\infty)$ if $\varkappa\leqslant 0$ and the interval $\left(0,\pi/\sqrt{\varkappa}\right)$ if $\varkappa>0$, is a *fundamental domain* whose image under the immersion $\gamma\times\mathrm{id}$ intersects each principal $\mathsf{SO}(2)$-orbit in $M^2_\varkappa\times\mathbf{R}$ precisely once. Its closure $\bar{I}_\varkappa\times\mathbf{R}$ represents the entire orbit space $\mathsf{SO}(2)\backslash(M^2_\varkappa\times\mathbf{R})$. The canonical projection p onto this orbit space lifts to a map

$$\hat{p}: M^2_\varkappa\times\mathbf{R}\to\bar{I}_\varkappa\times\mathbf{R}$$

such that the composition $\hat{p}\circ(\gamma\times\mathrm{id})$ is the identity on $\bar{I}_\varkappa\times\mathbf{R}$. This lift is given by $\hat{p}(x)=(r(x),\xi(x))$, where $r(x):=\mathrm{dist}(x,l_0)$ and where ξ is the height function introduced above. The restriction of $\hat{p}$ to the union of all principal orbits is clearly a Riemannian submersion.

A conceptually nice way of thinking about the slice $\mathrm{im}(\gamma\times\mathrm{id})$ is to *extend* the $\mathsf{SO}(2)$-action to some larger subgroup of $\mathrm{Isom}(M^2_\varkappa\times\mathbf{R})$. The natural candidate is the group $\mathsf{O}(2)$ consisting of *all isometries* of $M^2_\varkappa\times\mathbf{R}$ that fix the axis l_0. The cosets of $\mathsf{SO}(2)\subset\mathsf{O}(2)$ are represented by the subgroup consisting of the identity and of the reflection ϱ at the slice. The extended action has the *same orbits* as the $\mathsf{SO}(2)$-action that we began with. However, the principal isotropy group of the extended action is $\mathbf{Z}_2=\langle\varrho\rangle$ rather than the trivial group.

The slice $\mathrm{im}(\gamma\times\mathrm{id})$ can be recovered as the fixed-point set of the reflection ϱ. Moreover, ϱ preserves the height function ξ and thus also its gradient. In particular, $\mathrm{grad}\,\xi$ *is tangential* to the slice everywhere.

2.2. Connected SO(2)-invariant surfaces $\Sigma^2 \looparrowright M_\varkappa^2 \times \mathbf{R}$

As explained above, the given action extends to an isometric O(2)-action on $M_\varkappa^2 \times \mathbf{R}$ with precisely the same orbits. Hence any rotationally invariant cmc surface $\Sigma^2 \looparrowright M_\varkappa^2 \times \mathbf{R}$ is also invariant under the full O(2)-action. The additional symmetry tells a lot about the geometry of the surfaces in question:

PROPOSITION 2.1. *Any rotationally invariant surface* $\Sigma^2 \looparrowright M_\varkappa^2 \times \mathbf{R}$ *intersects the slice* $\operatorname{im}(\gamma \times \mathrm{id})$ *orthogonally in a set of regular curves c that are simultaneously geodesics and curvature lines on* Σ^2.

If the surface Σ^2 is connected, then so is its image under the projection p from $M_\varkappa^2 \times \mathbf{R}$ onto the orbit space. Therefore this image can be described by a single regular curve $\tilde{c}$ in $\bar{I}_\varkappa \times \mathbf{R}$ that may start and/or end at the boundary and that can meet the boundary only orthogonally. The surface $\Sigma^2 \looparrowright M_\varkappa^2 \times \mathbf{R}$ can be recovered from $\tilde{c}$ by letting SO(2) act on its image $c := (\gamma \times \mathrm{id}) \circ \tilde{c}$ in $M_\varkappa^2 \times \mathbf{R}$. Moreover, any regular curve $s \mapsto \tilde{c}(s) \in \bar{I}_\varkappa \times \mathbf{R}$ with the proper behavior at the boundary of $\bar{I}_\varkappa \times \mathbf{R}$ is the *generating curve* for some connected, smooth, rotationally invariant surface $\Sigma^2 \looparrowright M_\varkappa^2 \times \mathbf{R}$.

The unit normal field ν of the surface is best described in terms of its angle function $s \mapsto \theta(s)$ along the generating curve $s \mapsto \tilde{c}(s) = (r(s), \xi(s))$, a function that is defined by the equation

$$\nu|_{((\gamma \times \mathrm{id}) \circ \tilde{c})(s)} = d(\gamma \times \mathrm{id})|_{\tilde{c}(s)} \cdot (\cos\theta(s), \sin\theta(s)).$$

We assume that the generating curve $\tilde{c}$ is parametrized by arc length and that its tangent vector is $\tilde{c}'(s) = (-\sin\theta(s), \cos\theta(s))$. Our convention is to work with "exterior" unit normal vectors and write the Weingarten equation with a "+"-sign, i.e. to write $A = D\nu$. In particular, the principal curvature of Σ^2 in the direction of the meridian $c = (\gamma \times \mathrm{id}) \circ \tilde{c}$ is given by $(\partial\theta/\partial s)(s)$.

Using the formula of Meusnier to determine the other principal curvature, it is straightforward to compute the second fundamental form h_Σ and the tensor field $d\xi^2|_\Sigma$ along the meridian c on Σ^2 with respect to the orthonormal basis consisting of the vectors $c'(s)$ and $Jc'(s)$:

$$h_\Sigma = \begin{pmatrix} \partial\theta/\partial s & 0 \\ 0 & \cos(\theta)\,\mathrm{ct}_\varkappa(r) \end{pmatrix} \quad \text{and} \quad d\xi^2|_\Sigma = \begin{pmatrix} \cos^2\theta & 0 \\ 0 & 0 \end{pmatrix}. \tag{3}$$

Here $\mathrm{ct}_\varkappa$ stands for the generalized cotangent function;(1) the value $\mathrm{ct}_\varkappa(r)$ is the curvature of the circle of radius $r > 0$ in the surface $M_\varkappa^2$ of constant curvature $\varkappa$.

(1) Recall that the *generalized cotangent function* $\mathrm{ct}_\varkappa$ is by definition the logarithmic derivative of the *generalized sine function* $\mathrm{sn}_\varkappa$, which in turn is defined to be the solution of the differential equation $y'' + \varkappa y = 0$ with initial data $y(0) = 0$ and $y'(0) = 1$.

2.3. Differential equations for SO(2)-invariant cmc surfaces Σ^2

The mean curvature H of a rotationally invariant surface $\Sigma^2\hookrightarrow M^2_\varkappa\times\mathbf{R}$ is given by the identity $2H=\operatorname{tr} A=\operatorname{tr}_g(h_\Sigma)$. Thus the preceding considerations yield the following system of ordinary differential equations for the generating curve $s\mapsto\tilde c(s)=(r(s),\xi(s))$:

$$\begin{aligned}\frac{\partial r}{\partial s}&=-\sin\theta,\\ \frac{\partial \xi}{\partial s}&=\cos\theta,\\ \frac{\partial \theta}{\partial s}&=2H-\cos(\theta)\operatorname{ct}_\varkappa(r).\end{aligned}\tag{4}$$

The key to solving this system of ordinary differential equations is the observation that its right-hand side depends neither explicitly on s nor on ξ. The latter property is a direct consequence of the fact that the Euclidean factor in the product spaces $M^2_\varkappa\times\mathbf{R}$ is 1-dimensional. It follows that the system (4) of ordinary differential equations has a *first integral.* In fact, W.-Y. Hsiang already found that

$$L:=\cos(\theta)\operatorname{sn}_\varkappa(r)-2H\int_0^r \operatorname{sn}_\varkappa(t)\,dt=\cos(\theta)\operatorname{sn}_\varkappa(r)-4H\operatorname{sn}_\varkappa\big(\tfrac12 r\big)^2\tag{5}$$

is constant along any solution of (4).

Remark 2.2. The value of L can be used to characterize the *special solutions* of (4) that correspond to particularly *simple cmc surfaces in* $M^2_\varkappa\times\mathbf{R}$:

(i) If $H=0$, *there are the totally geodesic leaves* $M^2_\varkappa\times\{\xi_0\}$. These leaves correspond to solutions with $\xi=\xi_0$ and $\theta=\pm\frac12\pi$, and thus they can be characterized by the condition $L=0$.

(ii) If $H=\frac12\operatorname{ct}_\varkappa(r_0)$ for some $r_0\in I_\varkappa$, *there is the cylinder of radius* r_0 *around the axis* l_0. This surface corresponds to solutions with $r=r_0$ and $\theta=0$, and thus it can be characterized by the condition $L=\operatorname{ct}_\varkappa(\frac12 r_0)^{-1}=\big(2H+\sqrt{4H^2+\varkappa}\big)^{-1}$.

If $\varkappa$ is positive, the vertical cylinders with radius $\pi/2\sqrt{\varkappa}$ have mean curvature $H\equiv0$, too. Moreover, there exist vertical undoloids with $H\equiv0$ that oscillate around these cylinders. These undoloids are discussed from different points of view in [20] and in [23, §§ 1 and 3]. Together, all these examples give a good indication of how rich the class of rotationally invariant minimal surfaces in $M^2_\varkappa\times\mathbf{R}$ is.

Furthermore, there is a 2-parameter family of minimal annuli whose level curves are geodesic circles which are not rotationally invariant. We expect the quadratic differential Q to play a key role in studying these examples.

PROPOSITION 2.3. *A connected, rotationally invariant, immersed cmc surface* $\Sigma^2 \looparrowright M^2_\varkappa \times \mathbf{R}$ *is diffeomorphic to a torus or an annulus, unless the first integral L for its generating curve $\tilde{c}$ vanishes, or, unless $\varkappa > 0$ and $L = -4H/\varkappa$.*

Proof. The orbit structure described in §2.1 implies that any connected, SO(2)-invariant surface $\Sigma^2 \looparrowright M^2_\varkappa \times \mathbf{R}$ is topologically the product of its generating curve $\tilde{c}$ in $\bar{I}_\varkappa \times \mathbf{R}$ and a circle, provided that the generating curve stays in the interior of $I_\varkappa \times \mathbf{R}$. If $\tilde{c}$ has an end point on the boundary of $I_\varkappa \times \mathbf{R}$, a disk must be glued to the corresponding boundary component of the remaining part of Σ^2.

Inspecting the expression in (5), it is evident that L must vanish if $\tilde{c}$ touches the component of the boundary of $\bar{I}_\varkappa \times \mathbf{R}$ where $r \equiv 0$. If $\varkappa \leqslant 0$, this component is already the entire boundary of the fundamental domain $I_\varkappa \times \mathbf{R}$. However, if $\varkappa > 0$, the boundary of $I_\varkappa \times \mathbf{R}$ has one further component where $r \equiv \pi/\sqrt{\varkappa}$. Clearly, $\tilde{c}$ can only reach the latter component if $L = -4H/\varkappa$. □

Remark 2.4. The components of the boundary of $\bar{I}_\varkappa \times \mathbf{R}$ correspond to the components of the fixed-point set of the SO(2)-action. In particular, interchanging the axes l_0 and $\hat{l}_0$ induces a symmetry of (4) that interchanges the cases $L=0$ and $L=-4H/\varkappa$ as well.

2.4. SO(2)-invariant cmc surfaces $\Sigma^2 \looparrowright M^2_\varkappa \times \mathbf{R}$ with $L=0$

In this subsection we shall see that a rotationally invariant cmc surface with $L=0$ does indeed intersect the axis l_0. Thus, by the argument used in the proof of Proposition 2.3 the surface Σ^2 must be either a sphere S^2_H or a disk D^2_H.

Since $\mathrm{sn}_\varkappa(\frac{1}{2}r) > 0$ for any $r \in I_\varkappa$, it is possible to rewrite the condition that the first integral L introduced in (5) vanishes as follows:

$$\cos(\theta)\,\mathrm{ct}_\varkappa\bigl(\tfrac{1}{2}r\bigr) = 2H. \tag{6}$$

This equation has a number of important consequences.

It shows in particular that $\cos\theta$ converges to 0 as $r \to 0$. This property essentially reflects the regular singular nature of the system of ordinary differential equations in (4); it implies that the generating curve $\tilde{c}$ can only meet the boundary of $\bar{I}_\varkappa \times \mathbf{R}$ *orthogonally*. Hence the cmc surface Σ^2 is automatically *smooth* at all the points where it intersects the axis l_0.

PROPOSITION 2.5. *Let $\Sigma^2 \looparrowright M^2_\varkappa \times \mathbf{R}$ be a connected,* SO(2)*-invariant cmc surface with mean curvature $H \neq 0$ that intersects the axis l_0. Then either*

(i) *$4H^2 + \varkappa > 0$, and Σ^2 is an embedded sphere $S^2_H \subset M^2_\varkappa \times \mathbf{R}$, or*

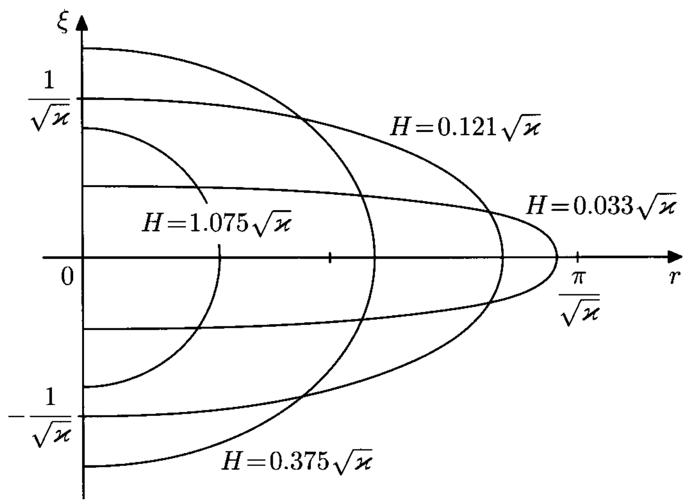

Fig. 1. The meridians of the spheres S_H^2 for $\varkappa>0$.

(ii) $4H^2+\varkappa\leqslant 0$, *and* Σ^2 *is a convex rotationally invariant graph* D_H^2 *over the horizontal leaves* $M_\varkappa^2\times\{\xi_0\}$, *which is asymptotically conical whenever the inequality for* $4H^2+\varkappa$ *is strict. There are two possibilities for the range of the normal angle* θ; *the image of* $\sin\theta$ *is one of the two half-open intervals* $[-1,-\sin\theta_0)$ *and* $(\sin\theta_0,1]$, *where* $\theta_0:=\arcsin\sqrt{1+4H^2/\varkappa}$.

The surfaces described in this proposition are the *embedded rotationally invariant cmc spheres* $S_H^2\subset M_\varkappa^2\times\mathbf{R}$ of Hsiang and Pedrosa and their *non-compact cousins* D_H^2. The meridians generating these cmc surfaces are shown in Figures 1 and 2 for positive and negative values of $\varkappa$, respectively. As indicated in these figures, the surfaces converge on compact subsets of $M_\varkappa^2\times\mathbf{R}$ towards the $\mathsf{SO}(2)$-invariant *minimal surfaces* described in Remark 2.2 (i), provided their mean curvature H approaches 0. Thus it is sometimes convenient to denote the leaves $M_\varkappa^2\times\{\xi_0\}$ as S_0^2 and D_0^2, respectively.

Proof. The idea is to evaluate the condition $L=0$ at all points of the generating curve $s\mapsto\tilde{c}(s)=(r(s),\xi(s))$ that lie in the interior of $\bar{I}_\varkappa\times\mathbf{R}$. Using the monotonicity properties of the functions $r\mapsto \mathrm{ct}_\varkappa\left(\frac{1}{2}r\right)$ defined on the open intervals $I_\varkappa$, it is easy to determine their range, too. Thus one finds that the pair of inequalities $H\cos\theta>0$ and $4H^2+\varkappa\cos^2\theta>0$ is a necessary and sufficient condition in order to solve equation (6) for r, and, moreover, that this solution is always unique.

In particular, identity (6) can be used to eliminate the factor $\mathrm{ct}_\varkappa(r)$ from the last equation in (4); one finds that

$$\frac{\partial\theta}{\partial s}=\frac{1}{4H}(4H^2+\varkappa\cos^2\theta). \tag{7}$$

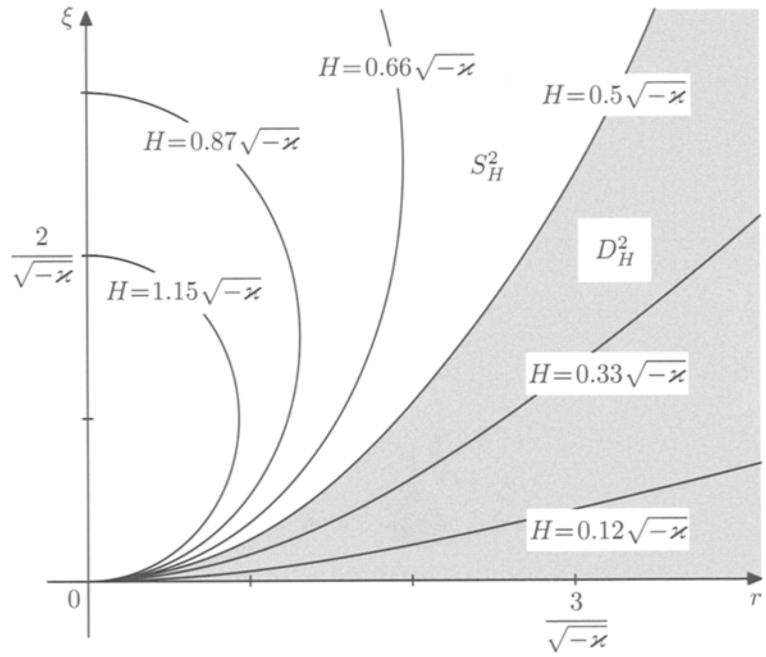

Fig. 2. The meridians of the spheres S_H^2 and the disk-like surfaces D_H^2 for $\varkappa<0$.

The right-hand side of this differential equation is uniformly bounded, and thus its solutions are defined on the entire real axis. As explained above, $H\cos\theta(s)>0$ for all points $\tilde{c}(s)$ on the generating curve that lie in the interior of $I_\varkappa\times\mathbf{R}$. By equation (6), the curve $\tilde{c}(s)$ approaches the boundary component of $\bar{I}_\varkappa\times\mathbf{R}$ corresponding to the axis l_0 whenever $\cos\theta(s)$ converges to 0, and the points on $\tilde{c}$ that lie in the interior of $I_\varkappa\times\mathbf{R}$ correspond to a maximal interval $(s_1,s_2)\subset\mathbf{R}$ such that $H\cos\theta(s)>0$ for all $s\in(s_1,s_2)$.

The discussion gets much more intuitive when observing that the angle function θ can be used as a regular parameter on $\tilde{c}$. In order to see this, we recall that by our analysis of equation (6) we have $4H^2+\varkappa\cos^2\theta>0$. Thus by equation (7) the principal curvature $\partial\theta/\partial s$ has always the same sign as the mean curvature H. In particular, θ is a strictly monotonic function of s. The same holds for the function $\sin\theta$. It remains to analyze whether the image of (s_1,s_2) under $\sin\theta$ is the entire interval $(-1,1)$ or just some subinterval thereof.

If $4H^2+\varkappa>0$, equation (7) implies that $\partial\theta/\partial s$ is bounded away from zero. Hence the range of $\sin\theta$ is the entire interval $[-1,1]$, and the generating curve $\tilde{c}$ is an embedded arc that begins and ends at the boundary component of $\bar{I}_\varkappa\times\mathbf{R}$ corresponding to the axis l_0. Thus by the argument in the proof of Proposition 2.3, the surface must be a sphere.

If $4H^2+\varkappa\leqslant 0$, the condition $4H^2+\varkappa\cos^2\theta>0$ implies that $\sin\theta$ does not vanish anywhere along $\tilde{c}$. In fact, $\sin\theta\notin[-\sin\theta_0,\sin\theta_0]$, where $\theta_0:=\arcsin\sqrt{1+4H^2/\varkappa}$, and the function $\theta(s)$ is asymptotic to one of the stationary solutions θ_0 and $-\theta_0$ of equation (7). This explains the claim about the range of the normal angle θ, and again, reasoning as in the proof of Proposition 2.3, the surface must be homeomorphic to a disk.

Moreover, if $4H^2+\varkappa<0$, the asymptotic normal angle θ_0 is different from 0. In other words, $\sin\theta(s)$ is bounded away from 0, and thus the radius function r is a regular parameter for the generating curve $\tilde{c}$ as well; the range of this parameter is the entire half-axis $I_\varkappa=[0,\infty)$. Finally, equation (7) implies that

$$\frac{\partial}{\partial s}(\theta(s)\pm\theta_0)=\frac{\varkappa}{4H}(\sin\theta_0-\sin\theta(s))(\sin\theta_0+\sin\theta(s)).$$

Hence $\theta(s)$ converges exponentially to θ_0 and $-\theta_0$, respectively. Thus the generating curve $\tilde{c}$ has real asymptotes rather than merely some asymptotic slope, and the generated surface is asymptotically conical as claimed.

On the other hand, if $4H^2+\varkappa=0$, we find that $\theta_0=0$, and $\theta(s)$ decays only like $O(1/s)$. In particular, the integral $\int_s^\infty \sin\theta(\sigma)\,d\sigma$ does not converge; hence in this case there do not exist any asymptotes. Yet, the radius function r is still a regular parameter along $\tilde{c}$, and its range is still the entire half-axis $I_\varkappa=[0,\infty)$. □

Remark 2.6. W.-Y. Hsiang has already computed both the area and the enclosed volume of the embedded cmc spheres $S^2_H\subset\mathbf{H}^2\times\mathbf{R}$. In [20] Pedrosa has extended these computations to embedded cmc spheres $S^2_H\subset\mathbf{S}^2\times\mathbf{R}$, and has then used this information to determine candidates for the isoperimetric profiles of the product spaces $M^2_\varkappa\times\mathbf{R}$.

PROPOSITION 2.7. *Let $\Sigma^2\looparrowright M^2_\varkappa\times\mathbf{R}$ be a connected, rotationally invariant cmc surface that intersects the axis l_0. Then the traceless part of the tensor field q introduced in* (1) *vanishes identically, and so does the quadratic differential Q of Σ^2. More precisely,*

$$q=2Hh_\Sigma-\varkappa\,d\xi^2|_\Sigma=\left(2H^2-\tfrac{1}{2}\varkappa\cos^2\theta\right)g, \tag{8}$$

where g denotes the induced Riemannian metric on the surface Σ^2.

As pointed out in the introduction, it is this simple proposition that has been the key to finding the proper expression for the quadratic differential of cmc surfaces in the product spaces $M^2_\varkappa\times\mathbf{R}$.

Proof. If $H=0$, we consider some point $\tilde{c}(s)$ on the generating curve where the surface intersects the axis l_0. At such a point $\mathrm{sn}_\varkappa(r(s))=0$, and thus $L=0$. It follows from (6) that $\cos\theta(s)\equiv 0$, and therefore Σ^2 must be one of the totally geodesic leaves

$M^2_\varkappa \times \{\xi_0\}$ described in Remark 2.2 (i). For these surfaces equation (8) holds, as both its sides evidently vanish identically.

If $H \neq 0$, we may simplify the expression for the second fundamental form h_Σ from (3) using equations (4) and (7):

$$h_\Sigma = \begin{pmatrix} H+(\varkappa/4H)\cos^2\theta & 0 \\ 0 & H-(\varkappa/4H)\cos^2\theta \end{pmatrix}. \tag{9}$$

Combining this formula with the expression for $d\xi^2|_\Sigma$ from $(\dot{3})$, we again arrive at equation (8). □

Remark 2.8. The rotationally invariant cmc spheres $S^2_H \subset M^2_\varkappa \times \mathbf{R}$ of Pedrosa are *not convex* if $0 < 4H^2 < \varkappa$. In fact, the principal curvatures computed in (9) have opposite signs at all points on $S^2_H \subset M^2_\varkappa \times \mathbf{R}$ that are closer to the antipodal axis $\hat{l}_0$ than to l_0 itself.

2.5. SO(2)-invariant cmc surfaces $\Sigma^2 \looparrowright M^2_\varkappa \times \mathbf{R}$ of catenoidal type

In this subsection we are going to describe another family of rotationally invariant cmc surfaces with *vanishing* quadratic differential Q. For this purpose we study the solutions of (4) with first integral $L = -4H/\varkappa$.

In case $\varkappa > 0$, the surfaces corresponding to solutions with $L = -4H/\varkappa$ are precisely the surfaces that intersect the antipodal axis $\hat{l}_0$ rather than l_0. As explained in Remark 2.4, they are congruent to the spheres that correspond to the solutions with $L=0$ and that have been studied by Pedrosa.

In case $\varkappa < 0$, however, the surfaces obtained from the solutions of the system (4) of ordinary differential equations with $L = -4H/\varkappa$ are obviously *not congruent* to the cmc spheres of Hsiang, anymore. Yet, it is conceivable that they still have $Q \equiv 0$. We shall establish this property in Proposition 2.10. Using the expression for the first integral from (5), the condition $L = -4H/\varkappa$ reads

$$0 = \varkappa \cos(\theta)\, \mathrm{sn}_\varkappa(r) + 4H\bigl[1 - \varkappa\, \mathrm{sn}_\varkappa\bigl(\tfrac{1}{2}r\bigr)^2\bigr].$$

In particular, we find that the radius r is bounded away from 0, and hence we may rewrite this equation in a form similar to (6):

$$-\varkappa \cos\theta = 2H\, \mathrm{ct}_\varkappa\bigl(\tfrac{1}{2}r\bigr). \tag{6_{cat}}$$

In fact, there is an extremely close relationship between the equations (6) and (6_{cat}).

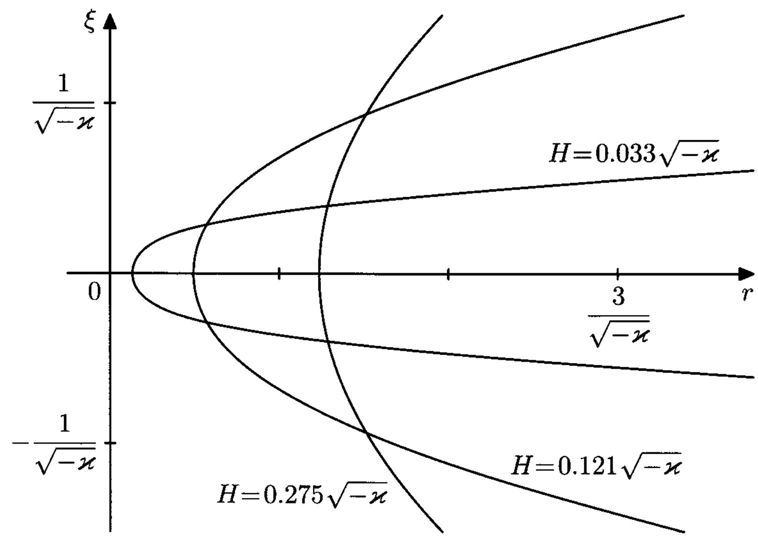

Fig. 3. The meridians of the surfaces C_H^2 of catenoidal type.

For instance,[2] either one of them implies that

$$\operatorname{ct}_\varkappa(r)=\tfrac{1}{2}\big(\operatorname{ct}_\varkappa\big(\tfrac{1}{2}r\big)-\varkappa\operatorname{ct}_\varkappa\big(\tfrac{1}{2}r\big)^{-1}\big)=H\cos^{-1}\theta-\frac{\varkappa}{4H}\cos\theta.$$

Inserting this expression into the third equation in (4), we obtain of course the same differential equation for θ as in the proof of Proposition 2.5:

$$\frac{\partial\theta}{\partial s}=\frac{1}{4H}(4H^2+\varkappa\cos^2\theta). \tag{7_{cat}}$$

Thus it should not be surprising that most arguments in the proof of Proposition 2.5 can be carried over to this situation. As explained above, we are only interested in studying the case $\varkappa<0$.

PROPOSITION 2.9. *Let $\varkappa<0$, and let $\Sigma^2\looparrowright M_\varkappa^2\times\mathbf{R}$ be a connected,* $\mathsf{SO}(2)$*-invariant cmc surface whose generating curve can be described as a solution of* (4) *with first integral $L=-4H/\varkappa$. Then the surface is an embedded annulus C_H^2 with two asymptotically conical ends. It is generated by rotating a strictly concave curve with asymptotic slopes $dr/d\xi=\mp\tan\theta_0$, where $\theta_0:=\arccos(2H/\sqrt{-\varkappa})$. Moreover, the range of θ is the interval $(-\theta_0,\theta_0)$.*

---

(2) Another consequence of either one of equations (6) and (6_{cat}) is that the Pfaffian

$$(4H^2+\varkappa\cos^2\theta)\,dr+4H\sin(\theta)\,d\theta$$

vanishes along the corresponding solutions of (4). This in effect explains how to *recover* (6) and (6_{cat}) and thus eventually the system (4) of ordinary differential equations from the differential equations (23) when doing the classification in §4.

The name C^2_H has been picked for this family of surfaces, since their shapes resemble the shape of a catenoid or rather the shapes of the analogous surfaces in hyperbolic 3-space. We shall refer to them as the *rotationally invariant cmc surfaces of catenoidal type.* When H approaches zero, they converge to the double covering of some leaf $M^2_\varkappa \times \{\xi_0\}$ with a singularity at the origin. This collapse is indicated in Figure 3; it is pretty similar to the collapse of the minimal catenoids described in [23, §6] and [19].

Proof. Since $r \in I_\varkappa$ by construction and since $\varkappa < 0$ by hypothesis, equation (6_{cat}) implies that $H \cos\theta > 0$ and $4H^2 + \varkappa \cos^2\theta < 0$. Hence the maximal existence interval for the solutions of the differential equation (7_{cat}) is the entire real axis. The range of the function $s \mapsto \theta(s)$ is as claimed, and—because of the first two equations in (4)—all the other assertions are straightforward consequences hereof. □

PROPOSITION 2.10. *Let $\varkappa < 0$, and let $\Sigma^2 \looparrowright M^2_\varkappa \times \mathbf{R}$ be a connected, $\mathrm{SO}(2)$-invariant cmc surface whose generating curve can be described as a solution of* (4) *with first integral $L = -4H/\varkappa$. Then the traceless part of the tensor field q introduced in* (1) *vanishes identically, and so does the quadratic differential Q of Σ^2. More precisely,*

$$2H h_\Sigma - \varkappa\, d\xi^2|_\Sigma = \left(2H^2 - \tfrac{1}{2}\varkappa \cos^2\theta\right) g.$$

Proof. The proof of Proposition 2.7 only uses equations (3), (4) and (7). The first two of these sets of formulas are clearly valid in the present context as well. Moreover, the differential equation for θ obtained in (7_{cat}) happens to coincide with (7), as has been observed beforehand. Thus the proof of Proposition 2.7 carries over verbatim. □

3. $\bar\partial$-operators, quadratic differentials and Codazzi equations

The purpose of this section is to prove Theorem 1. In other words, we want to compute the $\bar\partial$-derivative of the quadratic differential Q defined in formula (2) in §1 and verify that $\bar\partial Q$ indeed vanishes identically. The *key ingredients* for this computation are, firstly, a formula that expresses the $\bar\partial$-operator in terms of covariant derivatives and, secondly, the Codazzi equations for surfaces Σ^2 in the product spaces $M^2_\varkappa \times \mathbf{R}$. These are somewhat more complicated than the Codazzi equations for surfaces in space forms. Yet, all this is still essentially standard material, and so we shall summarize the necessary details in Lemmas 3.1 and 3.4 as we go along.

3.1. Geometry and complex analysis on Riemann surfaces

In this subsection we recall the basic facts about the complex analytic structure on the surface (Σ^2, g). In particular, we explain how the $\bar\partial$-operator that comes with this complex structure is related to covariant differentiation. Passing to a 2-fold covering if necessary, we may assume that Σ^2 is oriented. Thus there exists a unique almost complex structure $J\in C^\infty(\operatorname{End} T\Sigma^2)$ that is compatible with the Riemannian metric g and the given orientation.

A celebrated theorem first stated by C. F. Gauss [10] guarantees the existence of *isothermal coordinates.* In such a coordinate chart $\psi: U\subset\Sigma^2\to\mathbf{C}=\mathbf{R}^2$ the almost complex structure J is given as multiplication by i, and the metric g is of the form $e^{2\lambda}g_0$ for some C^∞-function $\lambda: U\to\mathbf{R}$; here we have used g_0 to denote the standard Euclidean metric on $\mathbf{R}^2$. The transition functions between such charts are clearly orientation-preserving conformal maps, and are thus holomorphic. In other words, the existence of isothermal coordinates turns the surface (Σ^2, J) into a *complex* 1*-dimensional manifold* Σ. Its complex tangent bundle $T\Sigma$, its cotangent bundle $K:=T^*\Sigma$ and all its tensor powers $K^{\otimes m}$ are thus *holomorphic line bundles.* In other words, these are complex vector bundles that come with a natural $\bar\partial$-operator.

By construction, the quadratic differential Q of an immersed cmc surface $\Sigma^2\looparrowright M^2_\varkappa\times\mathbf{R}$ is a section in a subbundle of the bundle of complex-valued symmetric bilinear 2-forms on the real tangent bundle of Σ^2, a subbundle that can be canonically identified with $K^{\otimes 2}$. As mentioned above, we want to express the $\bar\partial$-operator in terms of the *Levi–Civita connection* ∇ associated to the Riemannian metric g. The basic link between the real and the complex picture is the fact that on Riemann surfaces ∇J vanishes(3) identically.

LEMMA 3.1. *Let $\eta\in C^\infty(K^{\otimes m})$, $m\in\mathbf{Z}$, be a section in the m-th power of the canonical bundle on a Riemann surface (Σ^2, g, J). Then $\bar\partial\eta$ can be computed in terms of Riemannian covariant differentiation as follows*:

$$(\bar\partial\eta)(X)=\tfrac12(\nabla_X\eta+i\nabla_{JX}\eta)\quad \textit{for all } X\in C^\infty(T\Sigma^2). \tag{10}$$

The lemma is an immediate consequence of a couple of basic facts. Firstly, whenever a *holomorphic vector bundle E* over some complex manifold is equipped with a Hermitian inner product g, there exists a unique metrical connection $\widehat\nabla$ such that $(\bar\partial\eta)(X)=\frac12(\widehat\nabla_X\eta+i\widehat\nabla_{JX}\eta)$. This connection is known as the *Hermitian connection of* (E,g). On the tangent bundle of a *Kähler manifold* the Hermitian connection $\widehat\nabla$ *coincides* with the Levi–Civita connection (see [24]). The extension of this identity to arbitrary tensor powers is then done using the standard product rules for ∇ and $\widehat\nabla$.

(3) In the higher-dimensional case this fact does not hold anymore for an arbitrary complex manifold with a Hermitian inner product on the tangent bundle. In fact, the condition $\nabla J=0$ is one way of saying that (M^n, g, J) is Kählerian.

By definition the $\bar{\partial}$-operator depends only on the *almost complex structure* J, i.e. on the conformal structure and the orientation, whereas ∇ depends also on the choice of the *metric* g representing the conformal class. Clearly, this dependence must *cancel* when taking the particular linear combination of covariant derivatives appearing on the right-hand side of (10). This observation can be used to verify the lemma directly.

Elementary proof. By the very definition of the $\bar{\partial}$-operator, formula (10) holds in isothermal coordinates provided that covariant differentiation is replaced by Euclidean differentiation d with respect to that particular isothermal chart. In such a coordinate system the metric is by definition conformal to the Euclidean metric g_0, i.e. $g=e^{2\lambda}g_0$, and the Christoffel formula specializes to

$$\nabla_X Y - d_X Y = d_X\lambda\cdot Y + d_Y\lambda\cdot X - g_0(X,Y)\,\mathrm{grad}_{g_0}\lambda.$$

Because of the product rules for $\bar{\partial}$ and ∇ it is sufficient to handle the case $m=1$. For this purpose we consider a $\mathbf{C}$-valued 1-form η of type $(1,0)$. This means that $\eta(JY)=i\eta(Y)$, i.e. that η represents a section of the complex cotangent bundle $K=T^*\Sigma$. Thus a straightforward computation based on the Christoffel formula shows that

$$\begin{aligned}
2(\bar{\partial}\eta)(X;Y)-(\nabla_X\eta\cdot Y+i\nabla_{JX}\eta\cdot Y) &= (d_X\eta-\nabla_X\eta)\cdot Y+i(d_{JX}\eta-\nabla_{JX}\eta)\cdot Y\\
&= \eta(\nabla_X Y-d_X Y)+i\eta(\nabla_{JX}Y-d_{JX}Y)\\
&= (d_X\lambda+id_{JX}\lambda)\cdot\eta(Y)+d_Y\lambda\cdot(\eta(X)+i\eta(JX))\\
&\quad -(g_0(X,Y)+ig_0(JX,Y))\,\eta(\mathrm{grad}_{g_0}\lambda)\\
&= g_0(X,\mathrm{grad}_{g_0}\lambda)\,\eta(Y)+g_0(JX,\mathrm{grad}_{g_0}\lambda)\,\eta(JY)\\
&\quad -g_0(X,Y)\,\eta(\mathrm{grad}_{g_0}\lambda)-g_0(JX,Y)\,\eta(J\,\mathrm{grad}_{g_0}\lambda).
\end{aligned}$$

In order to explain the third equality sign, we simply observe that the coefficient of the factor $d_Y\lambda$ vanishes, as η is of type $(1,0)$. Picking an orthonormal basis $e_1, e_2=Je_1$ such that Y is a multiple of e_1, we find that

$$\begin{aligned}
g_0(X,Y)\,\eta(\mathrm{grad}_{g_0}\lambda) &= g_0(X,e_1)\,g_0(e_1,\mathrm{grad}_{g_0}\lambda)\,\eta(Y)\\
&\quad +g_0(X,e_1)\,g_0(e_2,\mathrm{grad}_{g_0}\lambda)\,\eta(JY),\\
g_0(JX,Y)\,\eta(J\,\mathrm{grad}_{g_0}\lambda) &= g_0(JX,e_1)\,g_0(e_1,\mathrm{grad}_{g_0}\lambda)\,\eta(JY)\\
&\quad -g_0(JX,e_1)\,g_0(e_2,\mathrm{grad}_{g_0}\lambda)\,\eta(Y).
\end{aligned}$$

These identities reveal that the last line in the preceding display does indeed vanish. $\square$

3.2. Surface theory in the product spaces $M^2_\varkappa\times\mathbf{R}$

There are two aspects in which the surface theory in the product spaces $M^2_\varkappa\times\mathbf{R}$ differs significantly from the surface theory in space forms. Firstly, there is the global parallel vector field $\operatorname{grad}\xi$, and secondly the Codazzi equations are more involved. Except for some inevitable changes caused by these differences, the computation of the $\bar\partial$-derivative $\bar\partial Q$ of the quadratic differential introduced in formulas (1) and (2) follows the same basic pattern as the corresponding computation in the case of space forms. Clearly,

$$\begin{aligned} Q(Y_1,Y_2) = {} & H(\langle Y_1,AY_2\rangle-\langle JY_1,AJY_2\rangle) \\ & -iH(\langle JY_1,AY_2\rangle+\langle Y_1,AJY_2\rangle) \\ & -\tfrac{1}{2}\varkappa(d\xi(Y_1)\,d\xi(Y_2)-d\xi(JY_1)\,d\xi(JY_2)) \\ & +\tfrac{1}{2}i\varkappa(d\xi(JY_1)\,d\xi(Y_2)+d\xi(Y_1)\,d\xi(JY_2)). \end{aligned} \tag{11}$$

Remark 3.2. Since the quadratic differential is a field of type $(2,0)$, its $\bar\partial$-derivative $\bar\partial Q$ is a field of type $(2,1)$, i.e. $\bar\partial Q$ is a section of the bundle $K^{\otimes 2}\otimes\bar K$. This means in particular that

$$\begin{aligned} &\bar\partial Q(X;JY_1,Y_2)=\bar\partial Q(X;Y_1,JY_2)=i\bar\partial Q(X;Y_1,Y_2), \\ &\bar\partial Q(JX;Y_1,Y_2)=-i\bar\partial Q(X;Y_1,Y_2). \end{aligned}$$

In other words, $\bar\partial Q$ is invariant under some T^3-action, and we are going to arrange the terms in the subsequent computations accordingly.

The differential $d\xi$ of the height function is clearly a parallel 1-form with respect to the Levi–Civita connection D of the target manifold $M^2_\varkappa\times\mathbf{R}$. However, its restriction is not parallel with respect to the induced Levi–Civita connection ∇ on the surface; one rather has the following result:

LEMMA 3.3. *The covariant derivative of the restriction of $d\xi$ to any surface $\Sigma^2\hookrightarrow M^2_\varkappa\times\mathbf{R}$ can be expressed in terms of the gradient of the height function ξ, the unit normal field ν of the surface, and its Weingarten map $A=D\nu$ as follows:*(4)

$$\nabla_X(d\xi)\cdot Y=-\langle\nu,\operatorname{grad}\xi\rangle\langle X,AY\rangle.$$

For intuition, we think of the term $d\xi(\nu)=\langle\nu,\operatorname{grad}\xi\rangle$ as the sine of some angle θ.

(4) The symmetric endomorphism $A=D\nu$ is sometimes also called the *shape operator*, and the associated bilinear form is the *second fundamental form* h_Σ.

Proof. Extending Y to a tangential vector field, a straightforward computation shows that

$$\begin{aligned}\nabla_X(d\xi)\cdot Y &= d_X(d\xi\cdot Y)-d\xi(\nabla_X Y)\\ &= D_X(d\xi)\cdot Y+d\xi(D_X Y-\nabla_X Y)\\ &= D_X(d\xi)\cdot Y+d\xi(\nu)\langle\nu, D_X Y\rangle\\ &= D_X(d\xi)\cdot Y-\langle\nu,\operatorname{grad}\xi\rangle\langle X, AY\rangle.\end{aligned}$$

The lemma follows when taking into account that the vector field $\operatorname{grad}\xi$ is globally parallel, i.e. that $D_X(\operatorname{grad}\xi)$ and $D_X(d\xi)$ vanish. □

Differentiating formula (11) using Lemmas 3.1 and 3.3, we find that

$$\bar\partial Q(X;Y_1,Y_2)=HT_1(X;Y_1,Y_2)+\varkappa\langle\nu,\operatorname{grad}\xi\rangle T_2(X;Y_1,Y_2), \tag{12}$$

where

$$\begin{aligned}T_1(X;Y_1,Y_2) := \tfrac{1}{2}[&\langle Y_1,\nabla_X A\cdot Y_2\rangle-\langle JY_1,\nabla_X A\cdot JY_2\rangle\\ &+\langle JY_1,\nabla_{JX}A\cdot Y_2\rangle+\langle Y_1,\nabla_{JX}A\cdot JY_2\rangle]\\ -\tfrac{1}{2}i[&\langle JY_1,\nabla_X A\cdot Y_2\rangle+\langle Y_1,\nabla_X A\cdot JY_2\rangle\\ &-\langle Y_1,\nabla_{JX}A\cdot Y_2\rangle+\langle JY_1,\nabla_{JX}A\cdot JY_2\rangle],\end{aligned}$$

$$\begin{aligned}T_2(X;Y_1,Y_2) := \tfrac{1}{4}[&\langle Y_1,AX\rangle\, d\xi(Y_2)+d\xi(Y_1)\langle AX,Y_2\rangle\\ &-\langle JY_1,AX\rangle\, d\xi(JY_2)-d\xi(JY_1)\langle AX,JY_2\rangle\\ &+\langle JY_1,AJX\rangle\, d\xi(Y_2)+d\xi(JY_1)\langle AJX,Y_2\rangle\\ &+\langle Y_1,AJX\rangle\, d\xi(JY_2)+d\xi(Y_1)\langle AJX,JY_2\rangle]\\ -\tfrac{1}{4}i[&\langle JY_1,AX\rangle\, d\xi(Y_2)+d\xi(JY_1)\langle AX,Y_2\rangle\\ &+\langle Y_1,AX\rangle\, d\xi(JY_2)+d\xi(Y_1)\langle AX,JY_2\rangle\\ &-\langle Y_1,AJX\rangle\, d\xi(Y_2)-d\xi(Y_1)\langle AJX,Y_2\rangle\\ &+\langle JY_1,AJX\rangle\, d\xi(JY_2)+d\xi(JY_1)\langle AJX,JY_2\rangle].\end{aligned}$$

Clearly, T_1 is just four times the $(2,1)$-part of ∇A. It is the same expression that appears when proving that the standard Hopf differential for cmc surfaces in space forms $M^3_\varkappa$ is holomorphic. It will be evaluated in Lemma 3.5 using the Codazzi equations.

The other term, however, is new. Since in dimension 2 the traceless part A_0 of the Weingarten map anticommutes with the almost complex structure J, it follows that the tensor T_2 depends only on the mean curvature H but not on the traceless part $A_0 := A - H\cdot\mathrm{id}$ of the Weingarten map A. Thus

$$T_2(X;Y_1,Y_2) = HT_2^{\mathrm{red}}(X;Y_1,Y_2), \tag{13}$$

where

$$\begin{aligned} T_2^{\mathrm{red}}(X;Y_1,Y_2) = \tfrac{1}{2}[&\langle Y_1,X\rangle\, d\xi(Y_2) + d\xi(Y_1)\langle X,Y_2\rangle \\ &- \langle JY_1,X\rangle\, d\xi(JY_2) - d\xi(JY_1)\langle X,JY_2\rangle] \\ - \tfrac{1}{2}i[&\langle JY_1,X\rangle\, d\xi(Y_2) + d\xi(JY_1)\langle X,Y_2\rangle \\ &+ \langle Y_1,X\rangle\, d\xi(JY_2) + d\xi(Y_1)\langle X,JY_2\rangle]. \end{aligned} \tag{14}$$

The expression on the right-hand side in this formula is actually the *simplest tensor* of type $(2,1)$ that can be constructed from $\langle\cdot,\cdot\rangle\, d\xi$ and that is symmetric with respect to interchanging Y_1 and Y_2.

LEMMA 3.4. *The Codazzi equations for surfaces* $\Sigma^2 \looparrowright M^2_\varkappa \times \mathbf{R}$ *are*

$$\begin{aligned} \langle \nabla_X A\cdot Y - \nabla_Y A\cdot X, Z\rangle &= \langle R^D(X,Y)\nu, Z\rangle \\ &= \varkappa\langle \nu, \operatorname{grad}\xi\rangle(\langle Y,Z\rangle\, d\xi(X) - \langle X,Z\rangle\, d\xi(Y)). \end{aligned}$$

Here R^D denotes the Riemannian curvature tensor of the product space $M^2_\varkappa\times\mathbf{R}$, and, with our sign conventions, the Weingarten equation is $A = D\nu$, and the sectional curvature K of a 2-dimensional subspace in $T(M^2_\varkappa\times\mathbf{R})$ is given by $K(\mathrm{span}\{X,Y\}) = \|X\wedge Y\|^{-2}\langle R^D(X,Y)Y,X\rangle$.

Proof. In principle the first half of the claimed formula is well known. However, in order to make sure that the sign of the curvature term is correct, it is easiest to include the short computation. Working locally, we may assume that Σ^2 is embedded, and thus we may extend X, Y and ν as smooth vector fields onto a neighborhood of the surface:

$$\begin{aligned} \langle \nabla_X A\cdot Y - \nabla_Y A\cdot X, Z\rangle &= \langle \nabla_X(AY) - \nabla_Y(AX) - A[X,Y], Z\rangle \\ &= \langle D_X(AY) - D_Y(AX) - A[X,Y], Z\rangle \\ &= \langle D_X(D_Y\nu) - D_Y(D_X\nu) - D_{[X,Y]}\nu, Z\rangle \\ &= \langle R^D(X,Y)\nu, Z\rangle. \end{aligned}$$

The second equality sign in the claimed formula is obtained upon computing the curvature tensor R^D for the product manifolds $M^2_\varkappa\times\mathbf{R}$. Indeed, the bilinear form $g - d\xi^2$ represents the metric on the leaves $M^2_\varkappa\times\{\xi_0\}$; hence

$$R^D = \varkappa(g - d\xi^2)\otimes(g - d\xi^2) = \varkappa(g\otimes g - 2g\otimes d\xi^2).$$

Like in the case of space forms, the tensor $g \otimes g$ vanishes on the relevant combinations of arguments, and therefore

$$\begin{aligned}\langle R^D(X,Y)\nu, Z\rangle &= -2\varkappa g \otimes d\xi^2(X,Y;\nu,Z)\\ &= -\varkappa[\langle X,Z\rangle d\xi(Y)\,d\xi(\nu) - \langle Y,Z\rangle d\xi(X)\,d\xi(\nu)\\ &\qquad -\langle X,\nu\rangle d\xi(Y)\,d\xi(Z) + \langle Y,\nu\rangle d\xi(X)\,d\xi(Z)]\\ &= \varkappa\langle \nu, \operatorname{grad}\xi\rangle(\langle Y,Z\rangle d\xi(X) - \langle X,Z\rangle d\xi(Y)).\end{aligned}$$ □

LEMMA 3.5. *For cmc surfaces* $\Sigma^2 \looparrowright M^2_\varkappa \times \mathbf{R}$ *the tensor field* T_1 *introduced in the context of formula* (12) *can be evaluated as*

$$T_1(X;Y_1,Y_2) = -\varkappa\langle \nu, \operatorname{grad}\xi\rangle\, T_2^{\mathrm{red}}(X;Y_1,Y_2),$$

where T_2^{red} *is the field introduced in* (14).

Proof. On cmc surfaces it is evident that the tensor $\nabla_w A$ is traceless and therefore anticommutes with the almost complex structure J for any vector $w \in T\Sigma^2$. We conclude that the real part of T_1 can be rewritten as

$$\operatorname{Re} T_1(X;Y_1,Y_2) = \langle Y_1, \nabla_X A\cdot Y_2\rangle + \langle JY_1, \nabla_{JX} A\cdot Y_2\rangle,$$

and that the sum $\langle Y_1, \nabla_{Y_2} A\cdot X\rangle + \langle JY_1, \nabla_{Y_2} A\cdot JX\rangle$ vanishes for all vectors X, Y_1 and Y_2. Subtracting this sum from the expression for $\operatorname{Re} T_1$, we may use the Codazzi equations from Lemma 3.4 in order to evaluate the differences $\langle Y_1, \nabla_X A\cdot Y_2 - \nabla_{Y_2} A\cdot X\rangle$ and $\langle JY_1, \nabla_{JX} A\cdot Y_2 - \nabla_{Y_2} A\cdot JX\rangle$. Hence we find that

$$\operatorname{Re} T_1(X;Y_1,Y_2) = \varkappa\langle \nu, \operatorname{grad}\xi\rangle[\langle Y_1,Y_2\rangle d\xi(X) + \langle JY_1,Y_2\rangle d\xi(JX) - 2\langle X,Y_1\rangle d\xi(Y_2)].$$

By construction, T_1 is a tensor field of type $(2,1)$. Firstly, this implies that $T_1(X;Y_1,Y_2) = -T_1(X;JY_1,JY_2)$, and thus

$$\begin{aligned}\operatorname{Re} T_1(X;Y_1,Y_2) &= \tfrac{1}{2}[\operatorname{Re} T_1(X;Y_1,Y_2) - \operatorname{Re} T_1(X;JY_1,JY_2)]\\ &= \varkappa\langle \nu, \operatorname{grad}\xi\rangle[\langle X,JY_1\rangle d\xi(JY_2) - \langle X,Y_1\rangle d\xi(Y_2)].\end{aligned}$$

Moreover, the field T_1 is clearly symmetrical with respect to the permutation of Y_1 and Y_2, and thus we may symmetrize the right-hand side accordingly. We conclude that $\operatorname{Re} T_1 = -\varkappa\langle \nu, \operatorname{grad}\xi\rangle \operatorname{Re} T_2^{\mathrm{red}}$, as claimed. Referring to the invariance properties of type $(2,1)$-tensors one more time, we can deduce that the imaginary parts are equal, too. □

Proof of Theorem 1. Combining equations (12) and (13), one obtains

$$\bar\partial Q = H\,[T_1 + \varkappa\langle\nu, \operatorname{grad}\xi\rangle\, T_2^{\mathrm{red}}],$$

and by Lemma 3.5 the expression on the right-hand side of this equation vanishes. □

Remark 3.6. The proof of Theorem 1 is *much more robust* than it might appear at first: The tensors T_1, T_2 and T_2^{red} are the $(2,1)$-parts of basic geometric objects like ∇A or the trilinear form $\langle\,\cdot\,,\cdot\,\rangle\, d\xi$. When applying the Codazzi equations to T_1, we evidently obtain a tensor of type $(2,1)$ that is a sum of terms of the form $\varkappa\langle\nu, \operatorname{grad}\xi\rangle\langle\,\cdot\,,\cdot\,\rangle\, d\xi$. The upshot of such structural considerations is that $\bar\partial Q$ must be a multiple of $\varkappa H\langle\nu, \operatorname{grad}\xi\rangle\, T_2^{\mathrm{red}}$ with some factor that is a universal constant.

There is also an *independent argument* that this constant factor must indeed be zero. One simply considers the rotationally invariant cmc spheres $S^2_H\subset M^2_\varkappa\times\mathbf{R}$ of Hsiang and Pedrosa, and observes that their quadratic differential Q vanishes identically. Therefore $\bar\partial Q\equiv 0$, too. On the other hand, however, neither the function $\varkappa H\langle\nu, \operatorname{grad}\xi\rangle$ nor the field T_2^{red} vanish on any set of positive measure in $S^2_H\subset M^2_\varkappa\times\mathbf{R}$.

4. Cmc surfaces with vanishing quadratic differential Q

Our goal is to classify the cmc surfaces $\Sigma^2\looparrowright M^2_\varkappa\times\mathbf{R}$ with vanishing quadratic differential Q, unless H and $\varkappa$ vanish simultaneously. By the very definition of the quadratic differential Q in equations (1) and (2) we shall in effect classify complete surfaces in $M^2_\varkappa\times\mathbf{R}$ such that

$$2Hh_\Sigma - \varkappa\, d\xi\otimes d\xi|_\Sigma = (2H^2 - \tfrac{1}{2}\varkappa\|d\xi|_\Sigma\|^2)\, g. \tag{15}$$

At this point it gets very clear why we have to exclude the case when H and $\varkappa$ vanish simultaneously; the preceding condition holds trivially despite the fact that there is an ample supply of interesting minimal surfaces in Euclidean 3-space.

The remainder of the *minimal surface case* on the other hand is easy: if $H=0$ and $\varkappa\neq 0$, it follows directly from equation (15) that $d\xi\otimes d\xi|_\Sigma = 0$, and thus $d\xi|_\Sigma$ vanishes identically. Therefore Σ^2 must be a totally geodesic leaf $M^2_\varkappa\times\{\xi_0\}$. These considerations tie in nicely with the fact that by general theory the height function ξ has to be harmonic, provided that Σ^2 is compact.

From now on we shall *concentrate on the case* $H\neq 0$. In this case equation (15) can be solved for the second fundamental form h_Σ. Clearly, for any surface $\Sigma^2\looparrowright M^2_\varkappa\times\mathbf{R}$, the vector fields $(\operatorname{grad}\xi)^{\tan} := \operatorname{grad}\xi - \langle\nu, \operatorname{grad}\xi\rangle\nu$ and $J\cdot(\operatorname{grad}\xi)^{\tan}$ are principal directions. This implies that $\langle\nu, \operatorname{grad}\xi\rangle$ and $\|(\operatorname{grad}\xi)^{\tan}\|^2 = 1 - \langle\nu, \operatorname{grad}\xi\rangle^2$ are constant along horizontal sections, and that, moreover, these level curves have constant curvature.

The proper way to formalize the treatment of equation (15) is to prolong the system once and interpret (15) as a problem about integral surfaces of a suitable distribution E_H in the unit tangent bundle of $M^2_\varkappa\times\mathbf{R}$. This prolongation will be defined in §4.1. In §4.2 we reap the easy consequences that are implied by the still fairly large isometry groups of the product spaces $M^2_\varkappa\times\mathbf{R}$ in the presence of the rotationally invariant examples described in §2. The properties of the distribution E_H will then be analyzed in full detail in §4.3 based on the simple observations described in the preceding paragraph. The argument culminates in the proof of Theorem 3 at the end of this section.

4.1. The Gauss section of cmc surfaces with $Q\equiv 0$

In order to handle equation (15) in the non-minimal case in a conceptually nice way, we find it better not to work with the immersion $F:\Sigma^2\looparrowright M^2_\varkappa\times\mathbf{R}$ itself, but to think of the unit normal field ν as the *primary* unknown. It will be easiest to consider the Gauss section ν simply as an immersion of Σ^2 into the total space $N^5:=T_1M^3$ of the unit tangent bundle of the manifold $M^3:=M^2_\varkappa\times\mathbf{R}$. Clearly, $F=\pi\circ\nu$, where $\pi: N^5=T_1M^3\to M^3$ denotes the standard projection map. Thus the immersion F can be recovered, once its lift $\nu:\Sigma^2\looparrowright N^5$ is known.

It is well known that any affine connection on a manifold M can be understood as a vector bundle isomorphism $TTM\to\pi^*(TM\oplus TM)$ which identifies the bitangent bundle TTM with a more familiar vector bundle over TM. In particular, the Levi–Civita connection D on $M^3=M^2_\varkappa\times\mathbf{R}$ induces an injective bundle map

$$\Phi_D: TN^5 \longrightarrow \pi^*(TM^3\oplus TM^3) \tag{16}$$

such that

$$\Phi_D\left(\frac{\partial}{\partial s}\nu(s)\right)=\left(\frac{\partial}{\partial s}(\pi\circ\nu(s)),\frac{D}{\partial s}\nu(s)\right)$$

for any smooth curve $s\mapsto\nu(s)\in N^5$. The image of this map is the 5-dimensional subbundle given by

$$\Phi_D(T_vN^5)=\{(w_1,w_2)\in T_{\pi(v)}M^3\oplus T_{\pi(v)}M^3\mid\langle v,w_2\rangle=0\}.$$

With these preparations it is now possible to *translate* the classification problem for cmc surfaces in $M^3:=M^2_\varkappa\times\mathbf{R}$ with mean curvature $H\neq 0$ and vanishing quadratic differential Q into the classification problem for integral surfaces of some 2-dimensional distribution $E_H\subset TN^5$.

PROPOSITION 4.1. *Let $\Sigma^2\looparrowright M^3=M^2_\varkappa\times\mathbf{R}$ be a cmc surface with $H\neq0$ and vanishing quadratic differential Q. Then its Gauss section $\nu:\Sigma^2\to N^5:=T_1M^3$ is necessarily* *an integral surface of the 2-dimensional distribution $E_H\subset TN^5$ given by*

$$\Phi_D((E_H)_v)=\{(w,A_v\cdot w)\mid w\in v^\perp\},$$

where

$$A_v\cdot w:=\frac{\varkappa}{2H}\langle w,(\operatorname{grad}\xi)^{\tan}\rangle(\operatorname{grad}\xi)^{\tan}+\Big[H-\frac{\varkappa}{4H}(1-\langle v,\operatorname{grad}\xi\rangle^2)\Big](w-\langle v,w\rangle v)$$

for all $w\in T_{\pi(v)}M^3$, and where $(\operatorname{grad}\xi)^{\tan}:=\operatorname{grad}\xi-\langle v,\operatorname{grad}\xi\rangle v$.

Conversely, for any number $H\neq0$ an integral surface of the distribution E_H necessarily projects onto a surface $\Sigma^2\looparrowright M^2_\varkappa\times\mathbf{R}$ with constant mean curvature equal to H and with vanishing quadratic differential Q.

In this proposition we do *not* claim yet that any of the distributions E_H with $H\neq0$ are integrable.

Proof. Each unit vector v clearly lies in the kernel of the corresponding symmetric endomorphism A_v introduced in the proposition. Moreover, the restriction of A_v to $v^\perp$ is precisely the tensor that one obtains when solving equation (15) for the Weingarten map of the surface $\Sigma^2\looparrowright M^2_\varkappa\times\mathbf{R}$, provided that v is chosen to be the unit normal vector ν at the point under consideration.

Thus the proposition immediately follows when reading this expression with the Weingarten equation $A=D\nu$ and with the definition of the isomorphism Φ_D in mind. □

4.2. Symmetries of the distributions E_H

In many cases the integral surfaces of the distribution $E_H\subset TN^5$ can be determined quite easily using the invariance properties of E_H, and the symmetry properties themselves can be established without much effort, too.

In fact, there is an induced action $\mathsf{G}\times N^5\to N^5$ of the 4-dimensional Lie group $G:=\operatorname{Isom}_0(M^2_\varkappa\times\mathbf{R})$ on the unit tangent bundle $N^5:=T_1(M^2_\varkappa\times\mathbf{R})$. Since the vector field $\operatorname{grad}\xi$ is even invariant under all isometries of $M^2_\varkappa\times\mathbf{R}$ that preserve the orientation of the second factor, it is clear that the function

$$\begin{aligned}\Theta\colon N^5&\longrightarrow\left[-\tfrac12\pi,\tfrac12\pi\right],\\ v&\longmapsto\arcsin\langle v,\operatorname{grad}\xi\rangle\end{aligned}\tag{17}$$

is *invariant* under the action of G on N^5. Since the isotropy group G_x at any point $x\in M^2_\varkappa\times\mathbf{R}$ acts on $T_x(M^2_\varkappa\times\mathbf{R})$ as the group of rotations preserving $(\operatorname{grad}\xi)|_x$, it follows that the function Θ *separates* the G-orbits in N^5. The fibers over $-\frac12\pi$ and $\frac12\pi$ correspond to 3-dimensional singular orbits with isotropy group isomorphic to $\mathsf{SO}(2)$, whereas all other fibers of Θ correspond to regular orbits; the principal isotropy group of the G-action on N^5 is trivial.

LEMMA 4.2. *For any $H\neq 0$ the distribution $E_H\subset TN^5$ introduced in Proposition* 4.1 *is invariant under the action of* $\mathsf{G}=\mathrm{Isom}_0(M^2_\varkappa\times\mathbf{R})$.

Proof. We may think of v as a G-invariant section of π^*TM^3, where π denotes the canonical projection $N^5=T_1M^3\to M^3=M^2_\varkappa\times\mathbf{R}$. From this point of view, the fields $(\operatorname{grad}\xi)^{\tan}$ and A_v are G-invariant sections of π^*TM^3 and $\pi^*\mathrm{End}(TM^3)$, respectively. □

PROPOSITION 4.3. *Suppose that $4H^2+\varkappa>0$ and $H\neq 0$. Then the distribution $E_H\subset TN^5$ introduced in Proposition* 4.1 *is integrable, and all its integral surfaces are congruent to Gauss sections of the embedded rotationally invariant cmc spheres $S^2_H\subset M^2_\varkappa\times\mathbf{R}$.*

This proposition actually proves the part of Theorem 3 about the classification in the case that $4H^2+\varkappa>0$.

Proof. Let $v_0\in N^5$ be arbitrary. The monotonicity properties established in the proof of Proposition 2.5 (i) show that there exists some point $\tilde c(s_0)$ on the generating curve of the sphere $S^2_H\subset M^2_\varkappa\times\mathbf{R}$ such that $\Theta(v_0)=\theta(s_0)=\Theta(\nu(s_0))$. Thus there exists an isometry $\psi\in\mathsf{G}=\mathrm{Isom}_0(M^2_\varkappa\times\mathbf{R})$ that maps the point $\tilde c(s_0)$ to the foot point $\pi(v_0)$, and $\nu(s_0)$ to v_0 itself.

By Proposition 2.7 the quadratic differential Q of the sphere S^2_H vanishes. Thus Proposition 4.1 implies that the Gauss sections of S^2_H and of all its isometric images are integral surfaces of the distribution E_H, and the Gauss section of $\psi(S^2_H)$ is an integral surface through the given point $v_0\in N^5$. □

PROPOSITION 4.4. *Suppose that $4H^2+\varkappa\leqslant 0$ and $H\neq 0$. Then the distribution $E_H\subset TN^5$ introduced in Proposition* 4.1 *has complete non-compact integral surfaces through any point $v_0\in N^5$ such that $4H^2+\varkappa\cos^2\Theta(v_0)\neq 0$. These integral surfaces are congruent to the Gauss sections of*

(i) *one of the non-compact cousins $D^2_H\subset M^2_\varkappa\times\mathbf{R}$ of the rotationally invariant cmc spheres $S^2_H\subset M^2_\varkappa\times\mathbf{R}$, or*

(ii) *one of the embedded, rotationally invariant cmc surfaces $C^2_H\subset M^2_\varkappa\times\mathbf{R}$ of catenoidal type.*

The first case occurs if and only if $4H^2+\varkappa\cos^2\Theta(v_0)>0$, whereas the second case occurs if and only if $4H^2+\varkappa\cos^2\Theta(v_0)<0$.

Proof. The argument is essentially the same as in the proof of the preceding proposition. Note that by hypothesis, $4H^2+\varkappa\cos^2\Theta(v_0)$ is non-zero.

Looking at the ranges of the angle function $s\mapsto\theta(s)$ that have been determined in Propositions 2.5 (ii) and 2.9, we see that there exists a point $\tilde c(s_0)$ on either the generating curve of D^2_H, or the generating curve of C^2_H such that $\Theta(v_0)=\theta(s_0)=\Theta(\nu(s_0))$. Thus there exists an isometry $\psi\in\mathsf{G}=\mathrm{Isom}_0(M^2_\varkappa\times\mathbf{R})$ that maps the point $\tilde{c}(s_0)$ to the foot point $\pi(v_0)$, and $\nu(s_0)$ to v_0 itself.

By Propositions 2.7 and 2.10 the quadratic differentials of both kinds of surfaces vanish identically, and so we have identified the integral surface through v_0 as the Gauss section of D^2_H and C^2_H, respectively. $\square$

The preceding proposition, however, does *not* finish the proof of Theorem 3. This is so despite the fact that we have identified the integral surfaces of the distribution E_H through an open dense set of points $v\in N^5$.

It remains to study the distribution E_H on the hypersurface in N^5 given by the equation $4H^2+\varkappa\cos^2\Theta=0$. As we shall see in the next subsection, the integral surfaces of E_H that are contained in this hypersurface are *not congruent* to the Gauss section of any rotationally invariant cmc surface in $M^2_\varkappa\times\mathbf{R}$.

4.3. Integral surfaces of E_H

In this subsection we shall determine *all integral surfaces* of the 2-dimensional distribution E_H in the unit tangent bundles N^5 of the product spaces $M^3=M^2_\varkappa\times\mathbf{R}$ in a *systematic* way.

The idea is to compute the integral curves of some distinguished vertical and horizontal vector fields $\hat{e}_1$ and $\hat{e}_2$. The properties of these integral curves will then be used in Proposition 4.8 to verify that E_H is integrable and that each of its integral surfaces is invariant under some 1-parameter group of isometries. This will give us a *unified approach* to the classification result stated in Theorem 3.

The union of the *regular orbits* under the action of $\mathsf{G}=\mathrm{Isom}_0(M^2_\varkappa\times\mathbf{R})$ is the open dense subset $N^5_0:=\pi^{-1}\big((-\frac{1}{2}\pi,\frac{1}{2}\pi)\big)$. The vector fields v and $\operatorname{grad}\xi$ are linearly independent at all points in N^5_0. Applying the Gram–Schmidt process, we thus obtain *two adapted orthonormal bases of* $\pi^*TM^3|_{N^5_0}$ that are compatible with the given orientation of $M^3=M^2_\varkappa\times\mathbf{R}$ and that depend smoothly on the foot point v. The first one is of the form $\operatorname{grad}\xi, e^0_1, e^0_2=J_0e^0_1$, where J_0 denotes the (almost) complex structure of the leaf $M^2_\varkappa\subset M^2_\varkappa\times\mathbf{R}$ and where

$$e^0_1:=\frac{1}{\cos\Theta}\,(v-\langle v,\operatorname{grad}\xi\rangle\operatorname{grad}\xi) \tag{18}$$

denotes the horizontal unit vector in the plane spanned by v and $\operatorname{grad}\xi$ that has positive inner product with v. The other adapted frame is of the form v,e_1,e_2; it is given by

$$\begin{aligned} e_1&:=\frac{1}{\cos\Theta}\,(\operatorname{grad}\xi)^{\tan}\equiv\frac{1}{\cos\Theta}\,(\operatorname{grad}\xi-\langle v,\operatorname{grad}\xi\rangle v),\\ e_2&:=v\times e_1. \end{aligned} \tag{19}$$

It follows directly from the definition of the distribution E_H in Proposition 4.1 that the vector fields $\hat{e}_1$ and $\hat{e}_2$ defined by

$$\begin{aligned}\Phi_D(\hat{e}_1)&=(e_1,A_v\cdot e_1)=\Big(e_1,\Big(H+\frac{\varkappa}{4H}\cos^2\Theta\Big)e_1\Big),\\ \Phi_D(\hat{e}_2)&=(e_2,A_v\cdot e_2)=\Big(e_2,\Big(H-\frac{\varkappa}{4H}\cos^2\Theta\Big)e_2\Big)\end{aligned}\tag{20}$$

are a smooth basis for $E_H|_{N_0^5}$.

Our plan is to compute the *integral curves* of these two vector fields and then use this information to recover the integral surfaces of E_H. For these computations it will be useful to note that

$$\begin{aligned}v&=\sin(\Theta)\operatorname{grad}\xi+\cos(\Theta)e_1^0,\\ e_1&=\cos(\Theta)\operatorname{grad}\xi-\sin(\Theta)e_1^0,\\ e_2&=-e_2^0.\end{aligned}\tag{21}$$

LEMMA 4.5. (The meridians.) *Let $H\neq0$, and let v_0 be a point in the regular set N_0^5. Moreover, let $r\mapsto\gamma_0(r)$ denote the unit speed geodesic in the horizontal leaf $M_\varkappa^2$ such that $\pi(v_0)=(\gamma_0(0),\xi_0)$ for some $\xi_0\in\mathbf{R}$ and such that $\gamma_0'(0)=e_1^0|_{v_0}$. Then the integral curve $s\mapsto\nu_1(s)$ of the field $\hat{e}_1$ through v_0 and its projection $\pi\circ\nu_1$ onto $M^3=M_\varkappa^2\times\mathbf{R}$ satisfy*

$$\begin{aligned}\pi\circ\nu_1(s)&=(\gamma_0(r(s)),\xi(s)),\\ \nu_1(s)&=(\cos(\theta(s))\gamma_0'(r(s)),\sin\theta(s)),\end{aligned}\tag{22}$$

where the triple of functions $(r(s),\xi(s),\theta(s))$ is obtained as the solution of the system of ordinary differential equations

$$\begin{aligned}\frac{\partial r}{\partial s}&=-\sin\theta,\\ \frac{\partial\xi}{\partial s}&=\cos\theta,\\ \frac{\partial\theta}{\partial s}&=H+\frac{\varkappa}{4H}\cos^2\theta,\end{aligned}\tag{23}$$

with initial data $(r(0),\xi(0),\theta(0))=(0,\xi_0,\Theta(v_0))$.

Here the domain for the independent variable s is *not* the full existence interval for the differential equation. It is restricted by the condition that $\cos\theta(s)$ must remain strictly positive, as the integral curve must not leave the regular set N_0^5. In short, what matters are the solutions of the third equation in (23).

Moreover, we observe that the lemma is *consistent* with Propositions 4.3 and 4.4. The third equation in (23) is actually identical to the differential equations for θ obtained in (7) and (7_{cat}), respectively.

Remark 4.6. If $4H^2+\varkappa\leqslant 0$, the system of ordinary differential equations in (23) has one class of particularly *simple solutions* that do *not* correspond to any of the surfaces S_H^2, D_H^2 and C_H^2 discussed in §2. They are given by

$$r(s):=r_0-s\sin\theta_0,\quad \xi(s):=\xi_0+s\cos\theta_0 \quad\text{and}\quad \theta(s):=\theta_0,$$

where $\theta_0:=\pm\arcsin\sqrt{1+4H^2/\varkappa}$, and where r_0 and ξ_0 are arbitrary constants of integration. In fact, all the other solutions of (23) can be determined explicitly, too. The corresponding formulas will be given in the appendix.

Proof of Lemma 4.5. Using equations (21)–(23), it is easy to verify that

$$\frac{\partial}{\partial s}(\pi\circ\nu_1(s))=(-\sin(\theta(s))\gamma_0'(r(s)),\cos\theta(s))=e_1|_{\nu_1(s)},$$
$$\frac{D}{\partial s}\nu_1(s)=\frac{\partial}{\partial s}\theta(s)\cdot(-\sin(\theta(s))\gamma_0'(r(s)),\cos\theta(s))=\Big(H+\frac{\varkappa}{4H}\cos^2\theta(s)\Big)e_1|_{\nu_1(s)}.$$

By the first line in (20) the preceding equations can be identified as the two components of $(\partial/\partial s)\nu_1(s)=\hat{e}_1|_{\nu_1(s)}$. Thus we have verified that equations (22) and (23) in the lemma indeed define an integral curve of $\hat{e}_1$. □

The next step is to understand the horizontal curves on the immersed surface $\iota:\Sigma^2\looparrowright M^3=M_\varkappa^2\times\mathbf{R}$. Observe that their lifts to $N^5\subset TM^3$ are precisely the integral curves of the vector field $\hat{e}_2$. We want to use this information in order to recognize the horizontal curves themselves as circles of latitude on some surface of revolution.

LEMMA 4.7. (The horizontal curves.) *Let v_0 be a point in the regular set N_0^5. Then the integral curve $t\mapsto\nu_2(t)$ of the vector field $\hat{e}_2$ through v_0 projects to a curve of constant curvature*

$$k(v_0):=H\cos^{-1}\Theta(v_0)-\frac{\varkappa}{4H}\cos\Theta(v_0)$$

in the leaf $M_\varkappa^2\times\{\xi_0\}$ through the foot point $\pi(v_0)$.

Proof. Since by construction the gradient of the height function ξ is always perpendicular to the field e_2, it is clear that the projection $\pi\circ\nu_2$ remains inside the totally geodesic leaf $M_\varkappa^2\times\{\xi_0\}\subset M^3$ through $\pi(v_0)$. A straightforward computation based on the second equation in (20) shows that

$$\frac{\partial}{\partial t}\Theta(\nu_2(t))=\frac{\partial}{\partial t}\langle\nu_2(t),\operatorname{grad}\xi\rangle=\Big\langle\frac{D}{\partial t}\nu_2(t),\operatorname{grad}\xi\Big\rangle$$
$$=\Big[H-\frac{\varkappa}{4H}\cos^2\Theta(\nu_2(t))\Big]\langle e_2|_{\nu_2(t)},\operatorname{grad}\xi\rangle=0.$$

In other words, Θ is *constant* along any integral curve of $\hat{e}_2$. By construction e_1^0 is the (exterior) normal along the curve $t \mapsto \pi \circ \nu_2(t) \in M_\varkappa^2 \times \{\xi_0\}$. With the help of the first equation in (21) and the second equation in (20), we thus find that

$$\frac{D}{\partial t} e_1^0 = \frac{1}{\cos\Theta} \frac{D}{\partial t}(\nu_2(t) - \sin(\Theta)\operatorname{grad}\xi) = \frac{1}{\cos\Theta}\Big(H - \frac{\varkappa}{4H}\cos^2\Theta\Big) e_2|_{\nu_2(t)}. \qquad \square$$

PROPOSITION 4.8. *Let $H \neq 0$ be some constant. Then, for any regular point $v_0 \in N_0^5$, the 2-dimensional distribution E_H is integrable in some neighborhood of v_0. Moreover, all these local integral surfaces are invariant under the action of some 1-parameter subgroup $t \mapsto \phi_t \times \mathrm{id} \in \mathrm{Isom}_0(M_\varkappa^2 \times \mathbf{R})$.*

Proof. An abstract verification that E_H is integrable is not of much use when we want to determine the invariance properties of the local integral surfaces. Thus it seems better to construct some *explicit candidates* for these integral surfaces from the very beginning. We determine the integral curve $s \mapsto \nu_1(s)$ of the vector field $\hat{e}_1$ through the given point $v_0 \in N_0^5$ as explained in Lemma 4.5. Consider the orbit of this curve under the flow of $\hat{e}_2$. In other words, we consider the map $(s,t) \mapsto \nu(s,t) \in N_0^5$ such that

$$\frac{\partial}{\partial t}\nu(s,t) = \hat{e}_2|_{\nu(s,t)} \quad \text{and} \quad \nu(s,0) = \nu_1(s).$$

Firstly, we *claim* that the maps $t \mapsto \pi \circ \nu(s,t)$ yield a family of *parallel curves of constant curvature* in the surface $M_\varkappa^2$. In fact, by construction each of these maps describes a curve in the first factor of $M^3 = M_\varkappa^2 \times \mathbf{R}$ that emanates perpendicularly from the unit speed geodesic γ_0 at the point $\gamma_0(r(s))$, and by Lemma 4.5 this curve has constant geodesic curvature

$$k_g(s) := H\cos^{-1}\theta(s) - \frac{\varkappa}{4H}\cos\theta(s). \tag{24}$$

It is well known that the geodesic curvature $\hat{k}$ of a family of *parallel* curves of constant curvature emanating perpendicularly from γ_0 satisfies the Riccati equation

$$\frac{\partial \hat{k}}{\partial r} + \hat{k}^2 + \varkappa = 0. \tag{25}$$

The *converse* is not hard to prove either: if the geodesic curvature $\hat{k}$ of a family of curves of constant curvature that emanate perpendicularly from γ_0 satisfies this differential equation, then the family is in fact a family of parallel curves.

Thus we only need to show that the geodesic curvatures $k_g(s)$ of the curves $t \mapsto \nu(s,t)$ can be written as $k_g(s) = \hat{k}(r(s))$ for some function $\hat{k}$ satisfying (25). Using equations (23) and (24), we find that

$$\frac{\partial}{\partial s}k_g(s) = \sin(\theta(s))\Big(H\cos^{-2}\theta(s) + \frac{\varkappa}{4H}\Big)\frac{\partial}{\partial s}\theta(s) = \sin(\theta(s))(k_g(s)^2 + \varkappa). \tag{26}$$

Since $(\partial/\partial s)r(s)=-\sin\theta(s)$, the preceding differential equation indeed asserts that the function $s\mapsto k_g(s)$ factors over some function $r\mapsto\hat{k}(r)$ solving (25), thereby establishing the claim from the beginning of the preceding paragraph.

Parallel curves of constant curvature in the 2-dimensional space forms $M^2_\varkappa$ are known to be the *orbits of a suitable* 1*-parameter group* $t\mapsto\phi_t$ of isometries. This in turn means that

$$\pi\circ\nu(s,a(s)t)=(\phi_t\times\mathrm{id})(\pi\circ\nu_1(s)),$$

where

$$a(s):=\left\|\frac{\partial}{\partial t}\bigg|_{t=0}\phi_t\circ\gamma_0(r(s))\right\|.$$

By construction, the tangent vectors to the s- and t-parameter lines are linearly independent as long as $\nu(s,t)$ is contained in the regular set N^5_0. Passing to the 1-jet, we therefore obtain

$$\nu(s,a(s)t)=(\phi_t\times\mathrm{id})_*(\nu_1(s)). \tag{27}$$

Since the vector field $\hat{e}_1$ on N^5_0 is by construction invariant under isometries of $M^3=M^2_\varkappa\times\mathbf{R}$, we conclude that the image of ν_1 under each of the maps $(\phi_t\times\mathrm{id})_*$ is again an integral curve of $\hat{e}_1$. Hence the map ν is indeed an integral surface of E_H. By construction it passes through the given point v_0, and by formula (26) it is invariant under the induced action of the 1-parameter group $t\mapsto\phi_t\times\mathrm{id}\in\mathrm{Isom}_0(M^2_\varkappa\times\mathbf{R})$ of isometries. □

PROPOSITION 4.9. *Suppose that* $4H^2+\varkappa\leqslant 0$, *and let* $\Sigma^2\looparrowright M^2_\varkappa\times\mathbf{R}$ *be one of the cmc surfaces with* $Q\equiv 0$ *that corresponds to the special solutions of* (23) *from Remark* 4.6. *Then* Σ^2 *is embedded*; *it is an orbit under some* 2*-dimensional solvable subgroup* $\mathsf{A}\ltimes\mathsf{N}\subset\mathrm{Isom}_0(M^2_\varkappa\times\mathbf{R})$.

If $4H^2+\varkappa<0$, the solvable group $\mathsf{A}\ltimes\mathsf{N}$ in the proposition is closely related to the Iwasawa decomposition $\mathrm{Isom}_0(M^2_\varkappa)=\mathsf{SO}^+(2,1)=\mathsf{KAN}$. However, if $4H^2+\varkappa=0$, the group N is still the nilpotent factor from the Iwasawa decomposition, whereas A degenerates into the group of vertical translations, and the semi-direct product turns into a direct product.

We consider the *parabolic nature* of the elements in the subgroup N as the characteristic property, and so we call this family of embedded cmc surfaces P^2_H. The surfaces P^2_H can be viewed as pointed limits of sequences of disk-like cmc surfaces D^2_H, or of sequences of cmc surfaces C^2_H of catenoidal type. In either case, the axis of the rotational symmetry disappears to infinity in the limit as indicated in Figure 4.

Proof. The formulas in Remark 4.6 directly imply that the expression

$$k_g(s)^2+\varkappa\equiv\left[H\cos^{-1}\theta(s)+\frac{\varkappa}{4H}\cos\theta(s)\right]^2$$

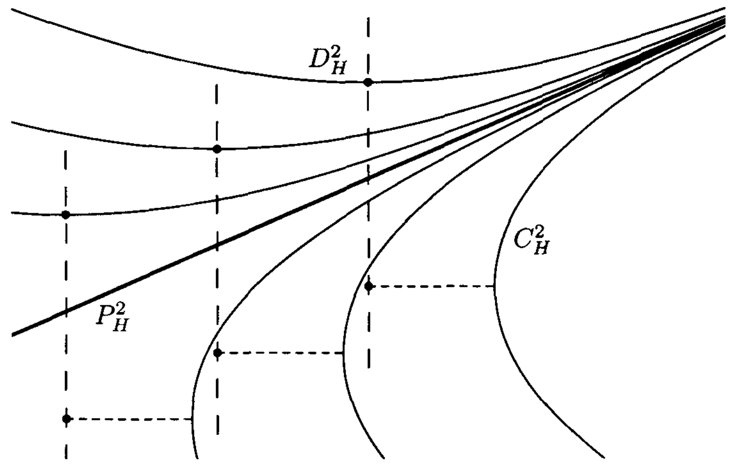

Fig. 4. The meridian of a parabolic surface P_H^2 comes as a limit of the meridians of the corresponding disk-like surfaces D_H^2 and surfaces C_H^2 of catenoidal type, as the axis is moved further and further out.

vanishes identically. Thus the horizontal curves $t \mapsto \nu(s,t)$ yield a family of *parallel horocycles in* $M_\varkappa^2$. This means that the isometries ϕ_t constructed in the proof of the preceding proposition must be *parabolic elements* in $\mathrm{Isom}_0(M_\varkappa^2)$. In other words, the image of the homomorphism $\mathbf{R} \to \mathrm{Isom}_0(M_\varkappa^2)$, $t \mapsto \phi_t$, constructed in the proof of the preceding proposition is a nilpotent subgroup $\mathsf{N} \subset \mathrm{Isom}_0(M_\varkappa^2)$.

The meridians $s \mapsto \nu(s,t)$ project to geodesics in $M_\varkappa^2 \times \mathbf{R}$. Each of them is the orbit under some 1-parameter subgroup $\mathsf{A}_t \subset \mathrm{Isom}_0(M_\varkappa^2 \times \mathbf{R})$ consisting of transvections. The action of A_t clearly maps the horizontal horocycles that intersect the given geodesic $s \mapsto \pi \circ \nu(s,t)$ perpendicularly to horocycles of the same kind. Hence the group A_t maps the surface $\Sigma^2 = \mathrm{im}(\pi \circ \nu)$ into itself. The various groups A_t associated to distinct meridians are mutually conjugate, and thus the semi-direct product $\mathsf{A}_t \ltimes \mathsf{N}$ does not depend on t. It acts isometrically and simply transitively on $\Sigma^2 \subset M_\varkappa^2 \times \mathbf{R}$. □

Proof of Theorem 3. It is a property of the Riccati equation (26) for the geodesic curvatures $k_g(s)$ of the horizontal curves $t \mapsto \pi \circ \nu(s,t)$ that the sign of the expression $k_g(s)^2 + \varkappa$ is independent of s. In fact, it follows directly from formula (24) that

$$k_g(s)^2 + \varkappa = \Big[H \cos^{-1}\theta(s) + \frac{\varkappa}{4H}\cos\theta(s)\Big]^2 \geqslant 0.$$

Hence there are just *two possibilities*: the expression $k_g(s)^2 + \varkappa$ may either be *strictly positive* for all s, or it may *vanish identically.* In the latter case the term in the square brackets itself must vanish, and so we are in the case analyzed in the preceding proposition.

In the other case, $k_g^2+\varkappa>0$ everywhere, and so the horizontal curves are *geodesic circles*, and the image of the homomorphism $\mathbf{R}\to\mathrm{Isom}_0(M_\varkappa^2\times\mathbf{R})$, $t\mapsto\phi_t\times\mathrm{id}$, is isomorphic to $\mathsf{SO}(2)$. Hence the integral surfaces must be congruent to the Gauss sections of the rotationally invariant cmc surfaces $\Sigma^2\looparrowright M_\varkappa^2\times\mathbf{R}$ described in §2. By Propositions 4.3 and 4.4 we know that we are only seeing the surfaces S_H^2, D_H^2 and C_H^2 described in Propositions 2.5 and 2.9. $\square$

Appendix. Explicit formulas for the meridians

It is not hard to see that the system (23) of differential equations in Lemma 4.5 is actually *integrable*. Its flow vector field lies in the kernel of the following set of Pfaffian 1-forms:

$$\begin{aligned}\eta_0 &:= \cos(\theta)\,dr+\sin(\theta)\,d\xi,\\ \eta_1 &:= (4H^2+\varkappa\cos^2\theta)\,dr+4H\sin(\theta)\,d\theta,\\ \eta_2 &:= (4H^2+\varkappa\cos^2\theta)\,d\xi-4H\cos(\theta)\,d\theta.\end{aligned}\tag{28}$$

In order to compute *explicit first integrals* from η_1 and η_2, it is necessary to integrate Riccati equations for $u_1:=\cos\theta$ and $u_2:=\sin\theta$, respectively. Substituting $u:=\tan\theta$ into the third equation in (23), we find that the relation between the angle θ and the arc length parameter s is given by a differential equation of Riccati type, too.

PROPOSITION A.1. *The meridian curves of the cmc surfaces $\Sigma^2\looparrowright M_\varkappa^2\times\mathbf{R}$ with vanishing quadratic differential Q and non-zero mean curvature H can be described as zero sets of suitable elementary functions*:

(i) *If $4H^2+\varkappa>0$, then Σ^2 is necessarily one of the* rotationally invariant spheres *S_H^2 of Hsiang and Pedrosa, and up to vertical translations the meridian is the smooth variety*

$$1=4H^2\,\mathrm{sn}_{-\varkappa}^2\left(\frac{1}{2}\xi\sqrt{1+\frac{\varkappa}{4H^2}}\right)+(4H^2+\varkappa)\,\mathrm{sn}_{\varkappa}^2\left(\frac{1}{2}r\right).\tag{29S}$$

(ii) *If $4H^2+\varkappa=0$, then Σ^2 must be either one of the* disk-like surfaces *D_H^2, or Σ^2 is a* cylinder over a horocycle, *which is a borderline case of a surface of type P_H^2. Up to vertical translations, their meridians are the smooth varieties*

$$H\xi=\pm\cosh Hr,\tag{30D}$$

$$r=r_0,\tag{30P}$$

respectively.

(iii) *If* $4H^2+\varkappa<0$, *then—depending on the slope of the meridian—the surface* Σ^2 *must be either a* disk-like surface D^2_H, *or a* parabolic surface P^2_H, *or a surface* C^2_H of catenoidal type. *Up to vertical translations, their meridians are the smooth varieties*

$$\sinh^2\frac{1}{2}\xi\sqrt{\varkappa\Big(1+\frac{\varkappa}{4H^2}\Big)}=-\Big(1+\frac{\varkappa}{4H^2}\Big)\cosh^2\frac{1}{2}r\sqrt{-\varkappa}, \tag{31D}$$

$$\sqrt{-(4H^2+\varkappa)}\,\xi=\pm 2Hr, \tag{31P}$$

$$\cosh^2\frac{1}{2}\xi\sqrt{\varkappa\Big(1+\frac{\varkappa}{4H^2}\Big)}=-\Big(1+\frac{\varkappa}{4H^2}\Big)\sinh^2\frac{1}{2}r\sqrt{-\varkappa}, \tag{31C}$$

respectively. The asymptotes to the meridians of D^2_H *and* C^2_H *intersect the axis at a point with* $\xi\sqrt{\varkappa(1+\varkappa/4H^2)}=\pm\log(-1-\varkappa/4H^2)$.

Some pictures of these meridians have been provided in Figures 1–3 in §2, and in Figure 4 in §4. In fact, the graphs shown there have been plotted using the explicit formulas from this proposition.

Remark A.2. If $\varkappa=0$, equation (29S) is just a somewhat involved way to write the standard equation $1=H^2\xi^2+H^2r^2$ for a circle with radius $1/H$ in Euclidean 3-space. If $\varkappa$ is non-zero, (29S) turns out to be a shorthand for one of the following two formulas:

$$\varkappa=4H^2\sinh^2\frac{1}{2}\xi\sqrt{\varkappa\Big(1+\frac{\varkappa}{4H^2}\Big)}+(4H^2+\varkappa)\sin^2\frac{1}{2}r\sqrt{\varkappa},$$

$$-\varkappa=4H^2\sin^2\frac{1}{2}\xi\sqrt{-\varkappa\Big(1+\frac{\varkappa}{4H^2}\Big)}+(4H^2+\varkappa)\sinh^2\frac{1}{2}r\sqrt{-\varkappa}.$$

Remark A.3. In order to make it very clear that the surfaces C^2_H *really resemble catenoids*, it is best to consider the following scaling limits: We pick some constant $\lambda>0$ and a sequence $(\varkappa_j)_{j=0}^{\infty}$ of negative numbers that converges to 0. Moreover, we set $H_j:=\varkappa_j/4\lambda$. Thus $4H_j^2+\varkappa_j<0$ for j sufficiently large, and so we are indeed dealing with a sequence of surfaces of catenoidal type. Passing to the limit in equation (31C) we find that the *meridians of these surfaces* converge to the zero set of the equation

$$\lambda r=\pm\cosh\lambda\xi,$$

which is obviously the equation for the meridian of a standard catenoid in Euclidean 3-space.

Proof of Proposition A.1. As suggested above the proof starts out integrating the Riccati equations corresponding to the Pfaffians η_1 and η_2. It follows directly from

Remark 4.6 and Proposition 4.9 that neither $\cos\theta$ can be independent of r, nor can $\sin\theta$ be independent of ξ, unless Σ^2 is a horizontal leaf or a surface of the parabolic type P^2_H. An explicit equation describing the meridians of the latter surfaces has already been given in Proposition 4.9.

It remains to compute the equations for the meridians of the surfaces S^2_H, D^2_H and C^2_H. From the discussion of the qualitative properties of these surfaces we already know that $\cos\theta$ must be an odd function of r. For the surfaces of type S^2_H and D^2_H, this function must vanish at $r{=}0$, whereas the radius function r must be bounded away from 0 for the surfaces of catenoidal type. In short, when integrating η_1, we recover equations (6) and (6_{cat}).

Clearly, when integrating the Pfaffian η_2 the solution only matters up to translation in the ξ-variable. We can normalize the solutions so that $\sin\theta$ is an odd function of ξ. For the surfaces S^2_H and C^2_H, we thus want $\sin\theta$ to vanish at $\xi{=}0$, whereas we want $\sin\theta$ to be bounded away from 0 for the disk-like surfaces D^2_H. Hence the first integrals obtained from η_2 are

$$4H^2+\varkappa=2H\sin(\theta)\,\mathrm{ct}_{-\varkappa(1+\varkappa/4H^2)}\big(\tfrac{1}{2}\xi\big), \tag{32}$$

$$\varkappa\sin\theta=2H\,\mathrm{ct}_{-\varkappa(1+\varkappa/4H^2)}\big(\tfrac{1}{2}\xi\big). \tag{32_{disk}}$$

We merely need to insert these expressions for $\cos\theta$ and $\sin\theta$ into the standard identity $\cos^2\theta+\sin^2\theta{=}1$ in order to obtain the claimed equations for the meridians. □

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