Curvilinear enumerative geometry

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Contents

A good part of Enumerative Geometry, in its modern version, may be viewed as seeking to compute and "understand" fundamental classes of loci of configurations of figures, say points on a variety, satisfying natural geometric conditions. The difficulty of the problem often has much to do with the degenerate configurations, i.e. those whose points may coalesce in complicated ways. The *curvilinear* configurations are those which can degenerate at most like points on a smooth curve. The purpose of this paper is to develop a point of view, going back to Severi [23] and Le Barz [14], which leads to a solution of a good number of enumerative problems involving curvilinear configurations. This point of view consists in realizing natural loci of interest as *intersections*, in the following manner:

We are given an embedding $X\subset Z$; X_k or Z_k are suitable spaces parametrizing k-tuples on X or Z (which need not be precisely defined here), and $B^k\subset Z_k$ is a certain subspace, which should be thought of as well-understood and well-behaved. Then the locus of interest is the intersection.

$$X_k\cap B^k\subset Z_k.$$

Provided all these spaces can be reasonably defined, this viewpoint clearly shifts, in a sense, the weight of the problem to performing the above intersection; for this, however, we have the well-oiled machine of intersection theory at our disposal.

Examples. (1) The case originally considered by Severi is where X is a curve in $Z=\mathbf{P}^3$, $k=3$, and B^3 consists of the triples of points which are *aligned*, i.e. lie on a line. Thus $X_3 \cap B^3$ consists of the aligned triples on X, i.e. essentially the trisecant lines of X. Severi's idea was recently resurrected and modernized by Le Barz [14], [15], who took for Z_k the curvilinear Hilbert scheme (see below), and was able by this method essentially to compute the cycles of multisecant lines to curves and surfaces in arbitrary projective spaces.

(2) Let $f: X \to Y$ be a map, take $Z=X\times Y$ and let $\Gamma\subset Z$ be the graph of f. Note the natural embedding $X_k \times Y \hookrightarrow Z_k$ and let B^k be its image. The $\tilde{U}_k=\Gamma_k \cap B^k$, the *united k-tuple locus* of f, parametrizes essentially the k-tuples on X mapped to a single point by f, and contains in particular information about the *multiple points* of f (cf. Kleiman [9] who, however, takes a different point of view on these).

In order to make good on this approach for enumerative geometry what one must do is to compute the intersection-cycle $[X_k].[B^k]$, where $[\cdot]$ denotes fundamental class; actually, in view of the well-understood nature of the embedding $B^k \hookrightarrow Z_k$, it turns out to be sufficient to compute $[X_k]$. More precisely, what one must do is to *define* X_k and Z_k suitably, so that $[X_k]$ can be computed. This causes some difficulty if one considers *arbitrary* k-tuples, and for this reason we restrict ourselves to curvilinear ones. For these, however, it turns out that an adequate parameter space has already been constructed by Kleiman [8]: it is a smooth compactification of the space of "ordered" curvilinear length-k schemes.

The main results obtained in this paper are as follows:

(a) A general formula, essentially in terms of Chern classes, for the fundamental class $[X_k] \in A^\cdot(Z_k)$, for an embedding of smooth varieties $X \hookrightarrow Z/S$ (Theorems 4.2 and 4.3).

(b) A representation of the loci of aligned k-tuples (k-secant lines) of a family of projective varieties, or k-secant spaces of arbitrary dimension to a family of smooth curves, as the zero-set of a section of a vector bundle, refining a construction of Schwarzenberger [22]. The Chern classes of these vector bundles are computed. In the case of curves this simplifies somewhat, and extends to the relative case, formulas of MacDonald [17] and Mattuck [18]. Applying this in the usual manner to special divisors, one obtains formulas for various cycles associated with these and in particular formulas for the classes of the loci, in the moduli space of curves, of curves carrying a given type of linear series; these all turn out to be polynomials in certain standard classes introduced by Mumford [19] (Propositions 5.3 and 5.4, Corollary 5.5, Theorem 5.6 and Example 5.7).

(c) A formula enumerating the united k-tuples of a map, valid "away from the $\bar{S}_2$-locus" refining Kleiman's results [9, 10] and extending their range of validity (Theorems 6.5 and 6.8 and Corollary 6.9).

It is an essential feature of our (and Le Barz's, but not Kleiman's) approach that the formulas obtained are shown to be valid *whenever they make sense*, i.e. whenever the locus in question has its "expected dimension" (many of the concrete consequences of our results could also be obtained, but with further hypotheses, by existing methods). The importance of proving enumerative formulas in this generality was already stressed by Hilbert in his 15th problem. In fact in applications of enumerative geometry, such generality is often crucial. For instance in one common sort of such application (see for instance [12]), especially to situations over which one does not have good control, one wants to conclude that a certain locus is nonempty by showing that the formula for its fundamental class yields a nonzero answer (for this type of application, Kleiman's results [9, 10] are useless, because of his "genericity" hypotheses). As examples of such applications of the methods of this paper, see [20], [21].

This paper is mostly devoted to general principles and methods. Further applications, examples, special cases and computations will be given elsewhere.

In this paper everything will be done over a fixed, arbitrary, base scheme S (unless otherwise mentioned). We will generally deal with smooth spaces Z/S (I haven't seriously considered the singular case). We will use as intersection theory the Chow ring $A^{\cdot}$, but any coarser theory will do as well.

Acknowledgement. This paper, like any paper in this hoary subject, owes a good deal to previous works. I have been particularly influenced by ideas of Le Barz and of Fulton and Laksov. In addition, I would especially like to thank W. Fulton and F.-O. Schreyer for stimulating conversations and encouragement.

1. The spaces Z_k

In this section, we shall recount Kleiman's construction ([8] p. 390) of certain spaces Z_k parametrizing k-tuples of points on a variety Z. As Kleiman's description of these spaces is not quite sufficient for our purposes, it will be convenient to start from scratch.

The construction of Z_k can be motivated by the classical concept of "infinitely-near points": e.g. if Z is the plane and Z' is the blow-up of Z at the origin 0, with exceptional divisor E', then points $z' \in E'$ represent "pairs of infinitely near points" on Z containing 0; if Z'' is the blow-up of Z' at z' with exceptional divisor E'', then points $z'' \in E''$ represent triples of infinitely-near points on Z containing z', and so on. The space Z_k then is a suitable blow-up of the cartesian product of Z which parametrizes k-tuples of points, some (or all) of which may be infinitely near.

Let p: $Z \to S$ be a smooth morphism. The spaces Z_k, and associated objects and maps, are constructed by induction. $Z_0=S$, $Z_1=Z$,

$$p^1 = p_0^1 \colon Z_1 \to Z_0$$

is just p, and $\iota_j \colon Z_j \to Z_j$ is the identity, $j=0, 1$. By induction, suppose we have constructed the following:

(a) spaces $Z_0, \ldots, Z_k$;

(b) "projection" maps $p^j_{i_1,i_2} \colon Z_j \to Z_{i_2-i_1+1}$ for all i_1, i_2 with $0 \leq i_1 - 1 \leq i_2 \leq j$

(c) maps $\iota_j \colon Z_j \to Z_j$ with $p^j_{i_1,i_2} \circ \iota_j = p^j_{j-i_2,j-i_1}$ and $\iota_j \circ \iota_j = \text{identity}$.

(d) divisors $D^j_{i_1,i_2} \subset Z_j$ for all $1 \leq i_1 < i_2 \leq j$.

Consider the cartesian diagram

$$\begin{tikzcd} Z_k \underset{Z_{k-1}}{\times} Z_k \arrow[r, "\mathrm{pr}_2"] \arrow[d, "\mathrm{pr}_1"] & Z_k \arrow[d, "p^k_{1,k-1}"] \\ Z_k \arrow[r, "p^k_{2,k}"'] & Z_{k-1} \end{tikzcd}$$

Let Δ_k denote the image of the embedding

$$\text{identity} \times \iota_k \colon Z_k \to Z_k \underset{Z_{k-1}}{\times} Z_k$$

and let

$$b_{k+1} \colon Z_{k+1} \to Z_k \underset{Z_{k-1}}{\times} Z_k$$

be the blowing up of Δ_k, with exceptional divisor $D^{k+1}_{1,k+1}$. Define new projections $p^{k+1}_{i_1,i_2}$ by

$$\begin{aligned} p^{k+1}_{i_1,i_2} &= p^k_{i_1,i_2} \circ \mathrm{pr}_1 \circ b_{k+1} \quad \text{if } i_2 \leq k \\ &= p^k_{i_1-1,i_2-1} \circ \mathrm{pr}_2 \circ b_{k+1} \quad \text{if } i_1 \geq 2 \\ &= \text{identity} \quad \text{otherwise} \end{aligned}$$

(note that these formulas are consistent). Also let $p^{k+1}: Z_{k+1} \to S$ be the structure morphism, and $p_i^{k+1} = p_{i,i}^{k+1}$. When this causes no confusion, superscripts will be omitted. As

$$\Delta_k \hookrightarrow Z_k \underset{Z_{k-1}}{\times} Z_k$$

is a smooth embedding over either pr_1 or pr_2, it follows that $p_{1,k}^{k+1}$ and $p_{2,k+1}^{k+1}$ are smooth, whence by induction p_{i_1,i_2}^k are smooth for all k, i_1, i_2. Also note that $(\iota_k \circ \mathrm{pr}_2, \iota_k \circ \mathrm{pr}_1)$ defines an involution of $Z_k \times_{Z_{k-1}} Z_k$ preserving Δ_k, hence lifts to an involution ι_{k+1} of Z_{k+1}. Define further divisors $D_{i_1,i_2}^{k+1} \subset Z_{k+1}$ by

$$D_{i_1,i_2}^{k+1} = (p_{i_1,i_2}^{k+1})^* D_{1,i_2-i_1+1}^{i_2-i_1+1}, \quad 1 \leq i_1 < i_2 \leq k+1, i_2 - i_1 < k.$$

For any partition $\mathbf{k} = (k_1, \ldots, k_r)$ of k, define subvarieties $V_{\mathbf{k}} \subset Z_k$ by

$$V_{\mathbf{k}} = \bigcap_{l=0}^{r} \bigcap_{k_1+\ldots k_l+1 < j \leq k_1+\ldots k_{l+1}} D_{1+k_1+\ldots+k_l,j}^k.$$

These should be thought of as parametrizing k-tuples "of type $\mathbf{k}$".

Example. $k=3$. General elements of $V_{\mathbf{k}}$ can be described as follows:

$$V_{(1,1,1)} = Z_3 \leftrightarrow \text{general triples.}$$

$$V_{(2,1)} = D_{1,2}^3 \leftrightarrow \text{triples containing an infinitely near pair; pictorially: } \bullet\!\!\to, \bullet.$$

$$V_{(3)} = D_{1,2}^3 \cap D_{1,3}^3 \leftrightarrow \text{infinitely-near triples; pictorially } \bullet\!\!\xrightarrow[3]{}.$$

Note that $D_{1,2}^3 \cap D_{2,3}^3$, on the other hand, corresponds to pairs of pairs of infinitely-near points, based at the same point; pictorially $\overset{\uparrow}{\bullet}\!\!\to$. This shows that Z_3 is not "symmetric", i.e. permutations don't in general act on it (although the involution (13) does, corresponding to ι_3). Note, finally, the "self regenerating" nature of the construction of Z_k: namely if we rename $p_{1,\ldots k-1}^k$ as p^1 then the "new" p_1^2 will be the same as the "old" $p_{1,k}^{k+1}$, and so on.

2. Schubert calculus

As we shall see below, the fundamental classes of naturally occuring cycles on Z_k can all be shown to be polynomials in the divisors D_{i_1,i_2}^k, and classes pulled back from Z. In order to compute those the essential step, since $D^k_{i_1,i_2}$ are distinct divisors, is to compute powers $(D^k_{i_1,i_2})^r$. As our projection maps are flat, we are reduced by induction to computing $(D^k_{1,k})^r$. This can be done inductively as follows:

Let T_k denote the "vertical tangent bundle" of $p^k_{2,k}: Z_k \to Z_{k-1}$; thus we have an exact sequence

$$0 \to T_k \to TZ_k \xrightarrow{dp^k_{2,k}} (p^k_{2,k})^* TZ_{k-1} \to 0. \tag{2.1}$$

Note that the normal bundle to Δ_k in $Z_k \times_{Z_{k-1}} Z_k$ is $N_{\Delta_k} = \mathrm{pr}_1^* T_k|_{\Delta_k}$. By definition, we have $D^k_{1,k} = \mathbf{P}(N_{\Delta_k})$, hence

$$\mathcal{O}_{D^k_{1,k}}(D^k_{1,k}) = \mathcal{O}_{\mathbf{P}(N_{\Delta_k})}(-1). \tag{2.2}$$

In view of the isomorphism $Z_k \to \Delta_k$, standard results from intersection theory allow us to compute powers $(D^k_{1,k})^r$ provided we know the Chern classes of T_k. To compute these, inductively, let Q_k be the tautological quotient bundle on $D^k_{1,k} = \mathbf{P}(N_{\Delta_k})$, i.e. we have an exact sequence

$$0 \to \mathcal{O}_{\mathbf{P}(N_{\Delta_k})}(-1) \to \pi_k^* N_{\Delta_k} \to Q_k \to 0 \tag{2.3}$$

where $\pi_k: \mathbf{P}(N_{\Delta_k}) \to \Delta_k$ is the projection. Then standard facts about blowing up yield the following sequence

$$0 \to T_{k+1} \to (p^{k+1}_{1,k+1})^* T_k \to Q_k \to 0 \tag{2.4}$$

from which we can compute what we want.

3. Z_k and the Hilbert scheme

In order to make sense of using Z_k as a parameter space for k-tuples, we have to consider its relation to the canonical such parameter space, the Hilbert scheme. As it turns out, there is a close connection between a suitable open subset of Z_k and the "curvilinear" open subset of the Hilbert scheme. But this is, essentially, as far as it goes.

Let $\mathrm{Hilb}_k(Z/S)$ denote the Hilbert scheme of length-k subschemes of Z and let $\varphi_k: Z_k \to \mathrm{Hilb}_k(Z/S)$ be the natural rational map. Let $\mathrm{Hilb}^c_k(Z/S) \subset \mathrm{Hilb}_k(Z/S)$ denote the open subset consisting of *curvilinear* schemes, i.e. those lying on smooth curves or equivalently those whose ideal is, locally at each point of their support, of the form $(z_1^r, z_2, \dots, z_n)$ for a suitable system of parameters $z_1, \dots, z_n$. Let $\mathrm{Hilb}_k^c(Z/S) \to \mathrm{Sym}^k(Z/S)$ be the "cycle map", put

$$Z^{[k]} = \mathrm{Hilb}_k^c(Z/S) \underset{\mathrm{sym}^k(Z/S)}{\times} (Z/S)^k, \quad (Z/S)^k = \text{cartesian product}$$

and let $\tilde{\varphi}_k: Z_k \to Z^{[k]}$ be the natural rational map.

On the other hand put $Z_k^2 = \bigcup_{i_1<i_2<i_3} D_{i_1,i_2}^k \cap D_{i_2,i_3}^k$ and $Z_k^0 = Z_k \setminus Z_k^2$.

PROPOSITION 3.1. *$\tilde{\varphi}_k$ induces an isomorphism $Z_k^0 \to Z^{[k]}$.*

Proof. In order to save notation we will write out the proof in case $Z=\mathbf{A}^2$. By induction it clearly suffices to verify the proposition in a neighborhood of $D_{1,2}^k \cap D_{1,3}^k \cap \dots \cap D_{1,k}^k \cap Z_k^0$. Let x, y be coordinates on $\mathbf{A}^2 = Z$. Then on a typical open subset $U \subset Z_2 = Z_2^0$, local coordinates are given by x_1, y_1, x_2, u_2^2, with

$$y_2 = y_1 + u_2^2(x_2 - x_1),$$

where $x_i = x \circ p_i^2$, $y_i = y \circ p_i^2$. By induction, local coordinates in a neighborhood of $(p_{1,2}^k)^{-1}(U) \cap D_{1,2}^k \cap \dots \cap D_{1,k}^k \cap Z_k^0$ are given by

$$y_1, x_1, x_2, \dots, x_k, u_2^k, \dots, u_k^k$$

with

$$u_i^k = u_i^{k-1} \circ p_{1,k-1}^k, \quad i = 1, \dots, k-1$$

$$u_k^k \cdot (x_1 - x_k) = u_{k-1}^{k-1} \circ p_{1,k}^k - u_{k-1}^{k-1} \circ p_{2,k}^k.$$

To get a local chart for $Z^{[k]}$, note that for any curvilinear length-k scheme σ on which projection to the x-axis induces an embedding, there exist uniquely determined polynomials $f(x)$, $g(x)$ of degree $\leq k-1$, such that the ideal of σ is $(x^k - f(x), y - g(x))$. Here $x^k - f(x)$ defines the projection of σ to the x-axis, so that if $x^k - f(x) = \Pi_{i=1}^k (x - x_i)$ then the cycle corresponding to σ is just the formal sum $\Sigma_{i=1}^k (x_i, g(x_i))$. Thus ordering this cycle amounts to ordering the x's, i.e. replacing f by the ordered set $(x_1 \dots, x_k)$.

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(1) As the referee points out, this case was already known (see e.g. [4]), and in fact lies at the heart of the modern approach to curvilinear enumerative geometry.

Thus we see that $(x_1, \dots, x_k, g(x))$ give a local chart on $Z^{[k]}$.

In terms of this latter chart and the former one for Z_k^0, the map $\tilde{\varphi}_k$ is given by

$$x_i \mapsto x_i, \quad i=1, \dots, k,$$
$$g(x) \mapsto y_1 + u_2^k(x-x_1) + u_3^k(x-x_1)(x-x_2) + \dots + u_k^k(x-x_1)\dots(x-x_{k-1}).$$

This is obviously an isomorphism.

What happens to $\tilde{\varphi}_k$ or φ_k over the rest of Z_k? For $k \leqslant 3$ we're still OK because of

PROPOSITION 3.2. *For $k \leqslant 3$, φ_k gives a morphism $Z_k \to \mathrm{Hilb}_k(Z/S)$.*

Proof. For $k \leqslant 2$ this is easy and well-known. For $k=3$, the only problem is near $D_{1,3}^3 \cap D_{1,2}^3$. We will content ourselves to give a recipe for $\varphi_3(\mathbf{z})$ for $\mathbf{z} \in D_{1,3}^3 \cap D_{1,2}^3$, leaving to the reader the straightforward task of verifying that this makes φ_3 a morphism. Put $\mathbf{z}' = p_{1,2}^3(\mathbf{z}) = p_{1,3}^3(\mathbf{z})$. The main point is that the map

$$\varphi_2 \times p_1^2 \colon Z_2 \to \mathrm{Hilb}_2(Z/S) \underset{S}{\times} Z = \mathrm{Hilb}_2(Z/S) \underset{S}{\times} \mathrm{Hilb}_1(Z/S)$$

is an isomorphism onto the locus of pairs of ideals (I_2, I_1) of colengths 2, 1, respectively with $I_2 \subset I_1$. In view of the isomorphism, for any ideal J,

$$T_J(\mathrm{Hilb}) = \mathrm{Hom}(J, \mathcal{O}/J)$$

of Zariski tangent space, the tangent space to the latter locus consists of commutative diagrams

$$\begin{array}{ccc} I_1 & \longrightarrow & \mathcal{O}/I_1 \\ {\scriptstyle r}\uparrow & & \uparrow{\scriptstyle s} \\ I_2 & \longrightarrow & \mathcal{O}/I_2 \end{array}$$

where r, s are the natural maps. As

$$D_{1,3}^3 = \mathbf{P}(N_{\Delta_2}), \quad \Delta_2 \hookrightarrow Z_2 \underset{Z_1}{\times} Z_2,$$

we see that $\mathbf{z}$ yields a (not quite commutative) diagram

$$\begin{array}{ccc} I_1 & \xrightarrow{\quad B \quad} & \mathcal{O}/I_1 \\ {\scriptstyle r}\uparrow & \alpha_1 & \uparrow{\scriptstyle s} \\ I_2 & \underset{\alpha_2}{\rightrightarrows} & \mathcal{O}/I_2 \end{array}$$

with $s\circ\alpha_1=s\circ\alpha_2=\beta\circ r$ and $\alpha_1\neq\alpha_2$; here α_1 and α_2 are not well-defined, as they depend on the lifting of an element of the normal bundle to the tangent bundle. However, the difference $\alpha_1-\alpha_2$ is well-defined up to scalars and yields a *nonzero* map

$$I_2\to I_1/I_2.$$

Then define $\varphi_3(\mathbf{z})$ to be the kernel of this map, an ideal of colength 3.

Warning. Contrary to an assertion of Kleiman ([8] p. 390), it is by no means the case that φ_k extends to a morphism to Hilb if $k\geq 4$ and $\dim Z\geq 2$: e.g. if $\dim Z=2$ any element of $D^4_{1,2}\cap D^4_{2,3}\cap D^4_{3,4}$ corresponds to infinitely many length-4 schemes containing the total first-order neighborhood of some point.

4. The fundamental-class formula

As noted above, a pivotal role in our approach to enumerative geometry is played by a formula for the fundamental class of X_k on Z_k, where $X\subset Z$ is a subvariety. This formula will be given in this section, as an application of the "residual-intersection formulas" of Fulton and Laksov and Fulton and MacPherson [4], [5].

Fix a smooth embedding

$$j\colon X\to Z$$

over S. It induces smooth embeddings

$$j_k\colon X_k\to Z_k$$

Let $p^k_i, D^k_{i_1,i_2}$ etc. denote objects attached to Z, and $\mathscr{I}_{X_k}$ etc. denote ideal sheaves. Also put

$$D^{j,k}=\sum_{i<j} D^k_{i,j}.$$

In order to be able to apply the residual-intersection formula, what we need is the following

LEMMA 4.1. $(p^k_k)^*(\mathscr{I}_X)\equiv\mathscr{I}_{X_k}\cdot\mathscr{I}_{D^{k,k}} \mod (p^k_{1,k-1})^*(\mathscr{I}_{X_{k-1}})$.

The proof will be postponed to the end of this section. Assuming this, the formula of Fulton and Laksov applies and computes the fundamental class $[X_k]\in A^{\cdot}(Z_k)$. There is one case in which the result becomes particularly simple and explicit, and in fact this case already suffices for many applications, and in particular for the united-set formula of § 6, so we will give this case first. Let N denote the normal bundle of X in 2 and

$$c = A^{\cdot}(X)$$

its total Chern class. Then the special case we are considering is where

$$c = j^*\bar{c}, \quad \bar{c} \in A^{\cdot}(Z).$$

THEOREM 4.2. *With the above hypotheses, the class of* X_k *is given by*

$$[X_k] = \prod_{j=1}^{k} \left(p_j^*([X]) - [D^{j,k}] \left\{ \frac{\bar{c}}{1+[D^{j,k}]} \right\}_{n-m-1} \right),$$

where $n = \dim Z$, $m = \dim X$ and $\{\ \}_a$ *denotes the homogeneous component of degree* a.

In the general case one is forced to give a stronger, though less explicit, result, namely a formula for the Gysin map. Let

$$r_k : X_k \hookrightarrow (p^k_{1,k-1})^{-1}(X_{k-1})$$

be the inclusion. In view of Lemma 4.1, the Fulton-MacPherson formula yields:

THEOREM 4.3. *With the above notations, the Gysin map is given by*

$$(r_k)_*((p_k)^*(\beta)) = (p_k)^*(j_*(\beta)) - \sum_{i=1}^{k-1} p_i^*(\beta)\, [D^k_{i,k}] \left\{ \frac{p_i^*(c)}{1+[D^{k,k}]} \right\}_{n-m-1}$$

for $\beta \in A^{\cdot}(X)$.

Note that r_k is $A^{\cdot}(Z_k) + (p^k_{1,k-1})^*(X_{k-1})$-linear, so by applying (4.3) repeatedly we can compute the Gysin map j_{k*}, restricted to the subring of $A(X_k)$ generated by the $p_i^*(A(X))$, and the $D^k_{i_1,i_2}$. In particular, we can compute $[X_k]$, but finer classes as well: for example if $B \subset X_{k-1}$ is a subvariety we can compute $[X_k \cap p^{-1}_{1,k-1}(B)]$, i.e. the class of the "part of X_k lying over B".

Another consequence of Lemma 4.1 is an exact sequence for the normal bundle of X_k in Z_k, namely

$$0 \to (p_k)^*(N) \otimes \mathcal{O}(-D^{k,k}) \to N_{X_k} \xrightarrow{dp_{1,k-1}} (p_{1,k-1})^* N_{X_{k-1}} \to 0. \tag{4.4}$$

An analogous formula, in the context of the curvilinear Hilbert scheme, was also given by Le Barz [16] who had, in the same context, essentially computed several special cases of Theorem 4.2 for X a curve or a surface in $Z=\mathbf{P}^N$, [14], [15].

Proof of Lemma 4.1. By induction, it clearly suffices to prove

$$(p_{2,k}^k)^*(\mathcal{I}_{X_{k-1}}) \equiv \mathcal{I}_{X_k} \cdot \mathcal{I}_{D_{1,k}^k} \quad \operatorname{mod} (p_{1,k-1}^k)^*(\mathcal{I}_{X_{k-1}}).$$

But in view of the self-regenerating nature of Z_k noted at the end of § 1, it suffices to prove (4.5) for $k=2$, in which case it is nearly obvious, but we will write the details anyway.(1) Take "local coordinates" $x^1, \ldots, x^m$, $y^1, \ldots, y^{n-m}$ on Z relative to S, so that $y^1, \ldots, y^{n-m}$ define X, and let x_j^i, y_j^i, $j=1,2$, be the corresponding functions on Z_2. Then local coordinates on a typical open set in Z_2 are given by

$$x_1^1, \ldots, x_1^m, \ldots, y_1^1, \ldots, y_1^{n-m}, x_2^1, u_2, \ldots, u_m, v_1, \ldots, v_{n-m}$$

with

$$x_2^i = x_1^i + u_i(x_2^1 - x_1^1), \quad y_2^i = y_1^i + v_i(x_2^1 - x_1^1).$$

Then $(p_1^2)^{-1}(X)$ (resp. $(p_2^2)^{-1}(X)$) is defined locally by $y_1^i=0$ (resp. $y_2^i=0$) for $i=1, \ldots, n-m$, X_2 is defined by $y_1^i=v_i=0$, $i=1, \ldots, n-m$, and $D_{1,2}^2$ is defined by $x_2^1=x_1^1$. This makes Lemma 4.1 obvious.

5. Secant bundles

If A is any space parametrizing k-tuples on a variety Z, and E is a vector bundle on Z, Schwarzenberger [22] has introduced the notion of secant bundle associated to A and E, which is nothing but $(\mathrm{pr}_1)_*(\mathrm{pr}_2^*(E)|_B)$ where $B \subset A \times Z$ is the natural subscheme. Applying this construction to Z_k, we get a vector bundle defined (at least) over the open subset Z_k^0, but not over all of Z_k. In this section we will show, however, that the secant bundle over Z_k^0 extends to a vector bundle E_k over Z_k and we will give an inductive recipe for computing the Chern classes of E_k, somewhat analogous to a formula of MacDonald [17] in the case of curves. This result will then be applied to describe multisecant lines of projective varieties, and multisecant spaces of curves. The latter case will then be applied, in the usual manner, to special divisors.

The bundles E_k are constructed as follows: $E_1=E$; E_2 is defined by

$$0 \to E_2 \xrightarrow{(\lambda_2^1, \lambda_2^2)} (p_1^2)^*(E_1) \oplus (p_2^2)^*(E_1) \xrightarrow{\varrho_2} (p_1^2)^*(E_1)\Big|_{D_{1,2}^2} \to 0$$

where $\varrho_2(a,b)=\operatorname{res}(a)-\operatorname{res}(b)$.

We note that E_2 is in fact a bundle: quite generally, if Z is a variety, $D \subset Z$ a Cartier divisor, E (resp. F) a vector bundle on Z (resp. D), and $\alpha: E \to F$, a surjection, then $\ker \alpha$ is locally free, as follows by an easy argument from Nakayama's lemma.

By induction, suppose E_j is defined for all $j \leq k$, as are maps

$$\lambda_j^1: E_j \to (p_{1,j-1}^j)^* (E_{j-1}), \quad \lambda_j^2: E_j \to (p_{2,j}^j)^* (E_{j-1}).$$

Define a vector bundle K_{k+1} by

$$0 \to K_{k+1} \xrightarrow{(\varkappa_{k+1}^1, \varkappa_{k+1}^2)} (p_{1,k}^{k+1})^* (E_k) \oplus (p_{2,k}^{k+1})^* (E_k) \xrightarrow{\tau_{k+1}} (p_{2,k}^{k+1})^* (E_{k-1}) \to 0$$

where

$$\tau_{k+1} = ((p_{1,k}^{k+1})^* (\lambda_k^2) - (p_{2,k+1}^{k+1})^* (\lambda_k^1)),$$

and E_{k+1} by

$$0 \to E_{k+1} \xrightarrow{(\lambda_{k+1}^1, \lambda_{k+1}^2)} K_{k+1} \xrightarrow{\theta_{k+1}} (p_{1,k}^{k+1})^* E_k \Big|_{D_{1,k+1}^{k+1}} \to 0$$

where

$$\theta_{k+1} = \mathrm{res} \circ \varkappa_{k+1}^1 - \mathrm{res} \circ \varkappa_{k+1}^2.$$

Note the exact sequence

$$0 \to (p_{k+1}^{k+1})^* (E) (-D^{k+1,k+1}) \to E_{k+1} \to (p_{1,k}^{k+1})^* (E_k) \to 0 \tag{5.1}$$

from which one can conveniently compute by induction the Chern classes of E_k. In the case of curves, analogous formulas for the Chern classes were already given by MacDonald [17] and Mattuck [18].

PROPOSITION 5.2. (i) *Any section s of E induces a section s_k of E_k and if the zero-set $X=(s)_0$ is smooth then $X_k=(s_k)_0$.*

(ii) *The restriction of E_k to Z_k^0 coincides with Schwarzenberger's bundle.*

(iii) *If N is the normal bundle of X in Z then N_k is the normal bundle of X_k in Z_k.*

Proof. Cf. Lemma 4.1, (4.4) and Proposition 3.1.

In the context of the curvilinear Hilbert scheme the analogue of (iii) was pointed out by G. Larry.

We apply this set-up to the following situation: let Z, E be as above with $\operatorname{rk} E=1$, let F be a vector bundle on the base scheme S, and $\lambda: p^*F \to E$ a surjection; note that this induces a map

$$f: Z \to \mathbf{P}(F^*) \quad \text{with} \quad E = f^*\mathcal{O}_{\mathbf{P}(E)}(1).$$

Let $G=G(l,F)$ denote a Grassmann bundle of F and T the tautological subbundle on G. Put

$$X = \{(z,L) \in Z \underset{S}{\times} G: \lambda(L)(z) = 0\} = \text{0-locus of the natural section of } T^* \otimes E \text{ on } Z \underset{S}{\times} G.$$

(We omit symbols such as pr_i^* when this causes no confusion.) Note the natural embedding

$$Z_k \underset{S}{\times} G \hookrightarrow (Z \underset{S}{\times} G)_k$$

and the natural isomorphism, for any sheaf A on Z, B on G,

$$(A \times B)_k|_{Z_k \times G} \simeq A_k \otimes B.$$

Put

$$J_{k,l} = X_k \cap Z_k \times G.$$

Applying (5.1) and Proposition 5.2, we get

PROPOSITION 5.3. *If $J_{k,l}$ has codimension kl in $Z_k \times G$, then*

$$[J_{k,l}] = c_{kl}(T^* \otimes E_k) = \prod_{j=1}^{k} c_l(T^* \otimes E^j \otimes \mathcal{O}(-D^{j,k}))$$

where $E^j = (p_j^k)^*(E)$.

Applications. (i) *Multisecant lines*. Suppose $l = \operatorname{rk} E - 2$ and f is an immersion (more generally, we could suppose $\bar{S}_2(f)=\varnothing$). Note that in this case $J_{k,l} \subseteq Z_k^0 \times G$, and hence $J_{k,l}$ can be identified with the set of pairs $(\mathbf{z}, M)$ where $\mathbf{z}$ is an ordered length-k subscheme of Z and M is a projective line containing $f(\mathbf{z})$; similarly $J_{k,l} \cap V_{\mathbf{k}}$, for any partition $\mathbf{k}$, parametrizes such pairs where $\mathbf{z}$ is of type $\mathbf{k}$. Thus Proposition 5.3 describes completely (up to explicit computation) the enumeration geometry of multisecant and multitangent lines, or more precisely aligned k-tuples, for projective varieties. Note that by pushing forward to G (resp. to Z_k) we obtain formulas for the loci of the lines themselves (resp. for their contact loci).

(ii) *The case of curves.* Now assume $\dim_S Z=1$, l arbitrary. Thus $Z_k=Z_k^0$=the Cartesian product. Then Proposition 5.3 yields formulas for the multisecant spaces of a family of curves. In particular, when $F=p_*E$, $J_{k,l}$ parametrizes, essentially, effective divisors $\mathbf{z}$ on Z with $h^0(E(-\mathbf{z}))\geq l$, considering $\mathbf{z}$ as a divisor on Z. In particular, we can take E to be the canonical divisor $K_{Z/S}$, in which case $h^0(K(-\mathbf{z}))=h^0(\mathcal{O}(\mathbf{z}))-1-k+g$, by Riemann-Roch, so the image $\bar{J}_{k,l}$ of $J_{k,l}$ in Z_k parametrizes precisely those effective divisors which move in a linear series g_k^r, where $r=l+k-g$, and we can compute the fundamental class $[\bar{J}_{k,l}]$ as

$$[J_{k,l}]=(p_1^r)_*c_{kl}(T^*\otimes K_k),$$

this being valid whenever $J_{k,l}$ has its expected dimension or, equivalently, whenever $\dim \bar{J}_{k,l}=\dim S+\varrho+r$ and a general divisor $\mathbf{z}\in\bar{J}_{k,l}$ has $h^0(\mathcal{O}(\mathbf{z}))=r+1$; here ϱ is the "Brill-Noether number" $\varrho=g-(r+1)(k-d+r)$.

Consider in particular the case where $\varrho\leq 0$. Put

$$S_k^r=p^k(\bar{J}_k^r)=\{s\in S\colon Z_s \text{ has a } g_k^r\}.$$

To compute the (expected) class of S_k^r, note that each g_k^r contributes an r-dimensional subvariety to the fibre of p^k; to cut this down, we can require that the first r points belong to specified canonical divisors. Thus:

PROPOSITION 5.4. *Under the above hypotheses, we have*

$$[S_k^r]=\frac{1}{(2g-2)^r}p_*^k([J_k^r][K^1]\dots[K^r])\in A^{-\varrho}(S).$$

Now as in Mumford [19], put $\varkappa_i=p_*[K]^{i+1}$, $\lambda_i=c_i(F)$. Mumford shows that the λ_i and $\varkappa_i$ are polynomials in $\varkappa_1,\dots,\varkappa_{g-2}$. Now it is clear from Proposition 5.3 and Proposition 5.4 that $[S_k^r]$ must be a polynomial in the λ_i and $\varkappa_i$, hence

COROLLARY 5.5. *If $J_{k,l}$ has its expected codimension then $[S_k^r]$ is a polynomial in $\varkappa_1,\dots,\varkappa_{g-2}$.*

(Actually this result already follows from the relative version of MacDonald's formula.) Replacing $J_{k,l}$ by $J_{k,l}\cap V_{\mathbf{k}}$ for a partition $\mathbf{k}$, for analogous results hold for the analogous subvarieties $S_{\mathbf{k}}^r$ of curves carrying a g_k^r with ramifications of type $\mathbf{k}$.

By general results of Mumford [19] and Gillet [7] the above results carry over to the moduli space itself, even though this has no universal family and is singular. Thus denote by $\mathfrak{M}_g$ the moduli space of curves of genus g, and define natural subvarieties, for every partition $\mathbf{k}$ of k

$$\mathfrak{M}^r_{g,\mathbf{k}} = \{[C] \in \mathfrak{M}_g : C \text{ has a } g^r_k \text{ containing a divisor } D = \sum k_i p_i, p_i \in C\}.$$

Mumford shows that the classes λ_i and $\varkappa_j$ above, and their relations, come from analogous classes and relations in $A^{\cdot}(\mathfrak{M}_g)$, Fulton's Chow ring [3]. Gillet shows that "universal" formulas as in Proposition 5.4 and Corollary 5.5 yield analogous formulas in $A^{\cdot}(\mathfrak{M}_g)$. Thus we have:

THEOREM 5.6. *Under similar dimension hypothesis, the class of* $\mathfrak{M}^r_{g,\mathbf{k}}$ *in* $A^{\cdot}(\mathfrak{M}_g)$ *is a polynomial in* $\varkappa_1, \ldots, \varkappa_{g-2}$.

By a result of Harer [7], it is known that $\varkappa_1$ generates $A^1(\mathfrak{M}_g)$. But in higher codimensions, it is unknown whether the $\varkappa_j$ generate $A^{\cdot}(\mathfrak{M}_g)$.

Examples 5.7. Many examples of such formulas were known before, see e.g. Mumford [19]. Without giving details, we will write down a few more.

(a) The hyperelliptics: the recipe above gives the following formula, already given by Mumford:

$$[\mathfrak{M}^1_{g,2}] = \frac{1}{2g-2} \sum_i (-1)^i \lambda_i \left(\sum_j \varkappa_j \varkappa_{g-2-i-j} - 2^{g-i} - g + i - 1)\varkappa_{g-2-i} \right)$$

where $\varkappa_0 = 2g-2$, $\varkappa_i = 0$ for $i<0$.

Instead of cutting down by requiring the first point to be in a fixed canonical divisor, we can instead require the divisor to be non-reduced, i.e., to contain one of the branch points of the g^1_2 of which there are $2g+2$. This yields

$$[\mathfrak{M}^1_{g,2}] = \frac{1}{2g+2} \sum_i (-1)^i (2^{g-i} - 1) \lambda_i \varkappa_{g-2-i}$$

(b) The trigonals: as above, we get

$$[\mathfrak{M}^1_{g,3}] = \frac{1}{4g-4} \sum_i (-1)^i \lambda_i \left(\sum_{i,j} \varkappa_j \varkappa_k \varkappa_{g-4-i-j-k} - \sum_j (2^{j+4} - 3j - 10) \varkappa_j \varkappa_{g-4-i-j} \right)$$
$$+ \left(\frac{1}{2}(3^{g-1}+1) - 2^{g+2-i} + (g-i-1)(g-i+2) + (g-i+5)\varkappa_{g-4-i} \right)$$

$$=\frac{1}{2g+4}\sum_i(-1)^i\lambda_i\left(\sum_i(2^{j+2}-1)\varkappa_j\varkappa_{g-4-i-j}-\left(\frac{1}{2}(3^{g-i}+1)-2^{j+1-i}\right.\right.$$
$$\left.\left.+g-i+1\right)\varkappa_{g-4-i}\right).$$

Finally, we will mention a slightly different type of application, pertaining to the case $\varrho=0$ and due originally to Beauville [1] and which also follows from one of Harer's results [7]. First we recall that, according to the fundamental result of Griffiths and Harris, when $\varrho=0$ a generic cirve of genus g possesses only *finitely many* g_k^r's.

PROPOSITION (Beauville). *If* $\varrho=0$ *then the sum of all divisor classes of* g_k^r*'s on an generic curve of genus g is a multiple of the canonical class.*

Proof. Take a family of curves Z/S parametrized by a suitable open subset of $\mathfrak{M}_g$. Instead of pushing J_k^r to S as above, push it only down to $Z_1=Z$, then restrict on a generic fibre Z_s. By Proposition 5.4 the class in question must be a multiple of the canonical divisor class K. On the other hand, the locus in question will consist in a certain number, say a, of divisors from each g_k^r. Hence if D is the sum of all classes of g_k^r's, then aD is a multiple of K. Since the Picard group of the generic curve over $\mathfrak{M}_g$ is torsion-free, it follows that D is a multiple of K, as claimed.

6. United-set formula

Given a morphism of varieties

$$f: X\rightarrow Y,$$

a *united set* or *united k-tuple* for f is, roughly speaking, a k-tuple of points

$$x_1,\dots,x_k\in X,$$

say distinct, such that

$$f(x_1)=\dots=f(x_k).$$

The object of this section is to give an enumerative formula for the united k-tuples of a map. We define a cycle which parametrizes these k-tuples, and give a formula for the class of this cycle in a suitable Chow ring (namely $A^{\cdot}(X_k\times Y)$, in fact).

For the rest of this section, we fix the following notation: X, Y are smooth varieties of dimensions m, r, respectively, and $f: X \to Y$ is an arbitrary morphism. We recall from § 3 the natural (functorial) map $\Phi_k: X_k^0 \to \mathrm{Hilb}^k(X/S)$.

Definition 6.1. The *united k-tuple locus* of f, as a set, is

$$U_k = U_k(f) = \{(\mathbf{x}, y) \in X_k^0: f(\Phi_k(\mathbf{x})) = y \quad \text{(as schemes)}\}.$$

We denote $\bar{U}_k$ the closure of U_k in $X_k \times Y$.

We are going to define a scheme structure on U_k and, in particular, make it into a cycle. To this end, a crucial thing to observe is the natural embedding

$$\alpha_k: X_k \times Y \hookrightarrow (X \times Y)_k,$$

as in § 5. Let

$$\Gamma = \Gamma_f \subset X \times Y$$

denote the graph of f and note that the normal bundle $N_\Gamma \simeq f^* T_Y$, via $\mathrm{id} \times f: X \xrightarrow{\sim} \Gamma$. Now define

$$\tilde{U}_k = \tilde{U}_k(f) = (\Gamma)_k \cap X_k \times Y \subset (X \times Y)_k$$

with the natural scheme structure as intersection.

LEMMA 6.2. *As sets, $\tilde{U}_k \cap (X_k^0 \times Y) = U_k$.*

Proof. By construction, we clearly have $\alpha_k(X_k^0 \times Y) \subset (X \times Y)_k^0$ and in fact, by functoriality of Φ_k,

$$\alpha_k(X_k^0 \times Y) = \{z \in (X \times Y)_k^0: \mathrm{pr}_2(\Phi_k(z)) \text{ has length } 1\}.$$

Restricting this on Γ_k yields the lemma.

The rest of $\tilde{U}_k$ is, as one may expect, not so well behaved, but we can still say a little bit about it. Put

$$\bar{S}_2(f) = \{x \in X: \mathrm{rk}(df_x) \leq m-1\},$$

$$\tilde{S}_{2,k}(f) = \tilde{U}_k(f) \cap \left(\bigcup_{i_1 < i_2 < i_3} \left(p_{i_1}^{-1}(\bar{S}_2(f)) \cap D_{i_1, i_2} \cap D_{i_2, i_2}\right) \right)$$

where $P_i: X_k \times Y \to X$ are projections and $D_{i,j}^k$ are pullbacks of divisors on X_k.

LEMMA 6.3. *We have* $\tilde{U}_k \cap (X_k^2 \times Y) \subseteq \tilde{S}_{2,k}(f)$.

Proof. Let $(\mathbf{x}, y) \in \tilde{U}_k \cap X_k^2 \times Y$. Without loss of generality, we may assume $\mathbf{x} \in D_{1,2}^k \cap D_{2,3}^k$ and consequently also $k=3$. By functoriality of the map $\Phi_3 : V \to \mathrm{Hilb}$ for $V=X$, Γ and $X \times Y$, it follows as in Lemma 6.2 that $f(\Phi_3(\mathbf{x}))=y$ as schemes. Since $\Phi_3(\mathbf{x})$ has embedding dimension 2, we get $\mathrm{rk}(df_x) \leq m-2$ where $x=\mathrm{supp}(\mathbf{x})$. Thus $(\mathbf{x}, y) \in \tilde{S}_{2,k}(f)$.

Now set

$$u_k = \prod_{j=1}^{k} \left((p_j \times q)^* ([\Gamma]) - [D^{j,k}] \left\{ \frac{q^*(c(T_Y))}{1+[D^{j,k}]} \right\}_{r-1} \right) \in A^{\cdot}(X_k \times Y)$$

where $q: X_k \times Y \to Y$ is the projection.

LEMMA 6.4. *Every component of* $\tilde{U}_k$ *has dimension* $\geq km-(k-1)r$; *if* $\tilde{U}_k$ *has dimension* $\leq km-(k-1)r$, *then*

$$[\tilde{U}_k] = u_k.$$

Proof. The first assertion is standard, and the second follows directly from Theorem 4.2.

We can state the main result of this section as follows.

THEOREM 6.5. (*a*) *The scheme* U_k *defined above parametrizes precisely the pairs* $(\mathbf{x}, y)$ *where* $\mathbf{x}$ *is an ordered curvilinear length-k subscheme of* X *such that* $f(\mathbf{x})=y$ *as schemes.*

(b) *Every component of* U_k *has dimension* $\geq km-(k-1)r$.

(c) *If* U_k *has dimension* $km-(k-1)r$ *or is empty, then the fundamental class of its closure*

$$[\bar{U}_k] \equiv u_k \in A^{\cdot}(X_k \times Y) \quad \textit{modulo}\, A^{\cdot}(\tilde{S}_{2,k}(f)).$$

Proof. In view of Lemmas 6.2 and 6.3, $\tilde{U}_k$ consists of $\bar{U}_k$ plus (possibly) a cycle supported on $\tilde{S}(f)$, so we are done.

For $k=2$, a result equivalent to Theorem 6.5 was given by Fulton and Laksov [4] (note that $\tilde{S}_{2,2}=\Phi$).

COROLLARY 6.6. *If* $u_k \not\equiv 0 \bmod A^{\cdot}(\tilde{S}(f))$, *then* $U_k \neq \varnothing$.

Remark 6.7. Note in particular that if $\dim(\tilde{S}_{2,k}(f)) < km-(k-1)r$, then one may replace congruence by equality in Theorem 6.5 (c) and Corollary 6.6. This will happen, in particular, if f is finite and $\dim(\bar{S}_2(f))<km-(k-1)r$. Since every component of $\bar{S}_2(f)$ has dimension $\geq 3m-2r-4$, the latter can happen only if either $\bar{S}_2(f)=0$, or k, m, r are the following range: $k\leq 3$, or $k\leq 4$ and $r\leq m+3$, or $k\leq 6$ and $r\leq m+1$.

Multiple points. In conclusion, we will compare the results above with Kleiman's multiple-point formulas [9]. First, we define multiple points.

Definition. $x\in X$ is a *k-fold point* for f if x is contained in a length-k subscheme of a fibre of f.

Given a proper morphism $f\colon X\to Y$ (X and Y may be singular, as long as f is a "locally complete intersection" morphism, Kleiman defines a cycle M_k on X, image of a certain associated space X_k under a natural projection, which is presumably—though this is nowhere stated explicitly—supposed to parametrize the k-fold points of f; he also gives an algorithm for computing a certain class $m_k\in A^{\cdot}(X)$. Then his main result may be stated as follows.

THEOREM (Kleiman). *If X_j for $j=2, \ldots, k$ have their expected dimensions, namely $jm-(j-1)r$ or -1, then $[M_k]=m_k$.*

Remarks. (a) It can happen that X_k has its expected dimension without the same being true of X_{k-1}: X_{k-1} may have an "excessively large component" which contributes nothing to X_k. In particular, an analogue of Corollary 6.6 does not seem to be accessible by Kleiman's method.

(b) As noted by Kleiman ([9], (5.2)) the assumption that X_k have its expected dimension implies that either $\bar{S}_2(f)=\varnothing$, or $\bar{S}_2(f)\neq\varnothing$ but $\dim(\bar{S}_2(f))<km-(k-1)r$ (where the later can happen only in the range listed in Remark 6.7). Hence no information is lost by replacing the equality in the Theorem by congruence modulo $A^{\cdot}(S_2'(f))$, where $S_2'(f)=f^{-1}(f(\bar{S}_2(f)))$, the smallest closed subset of X, on whose complement, X', the restriction of f is both proper and without S_2-singularities.

(c) Kleiman does not claim, nor is it in fact quite true, that M_k parametrizes the k-fold points, which makes the significance of his formula somewhat obscure (however, see the discussion in [11] or below).

Now comparing Kleiman's construction and ours, one sees that they are merely two ways of looking at the same thing. Thus we have, at least if X, Y are smooth:

(i) $X_k=\tilde{U}_k$, naturally.

(ii) $M_k\cap X'=\{k\text{-fold points of } f\}\cap X'=\{k\text{-fold points of } f|_{X'}\}$.

(iii) If X is complete, then $m_k=(1/k!)\,(p_1)_*\,u_k$.

Thus we obtain the following refinement of Kleiman's theorem for smooth, complete varieties.

THEOREM 6.8. *If X is complete and X_k has its expected dimension, then $[M_k] \equiv m_k$ modulo $A^{\cdot}(S_2'(f))$, and this formula validly enumerates the k-fold points of f, away from $S_2'(f)$.*

COROLLARY 6.9. *If X is complete, Kleiman's explicit formulas (e.g.* [9] (5.7), (5.10), (5.11)) *are valid, modulo $A^{\cdot}(S_2'(f))$, provided only that X_k has its expected dimension.*

Using the calculus of § 2, it is of course possible to obtain these formulas, and formulas for other m_k's directly. But I haven't set about making such calculations systematically.

Intersecting $\tilde{U}_k$ with $V_{\mathbf{k}}$ for various partitions $\mathbf{k}$, one obtains formulas for the united sets of type $\mathbf{k}$ of f, valid under similar hypotheses. In the context of Kleiman's theory, an algorithm for computing $p_{1*}[\tilde{U}_k \cdot V_{\mathbf{k}}]$ was independently given by S. Colley, and carried out by her in a number of cases (M.I.T. thesis, 1983). Similar remarks as above apply to her results.

Finally, we mention a corollary about the class m_2 which follows from the present approach.

COROLLARY 6.10. *m_2 is an integral class and $f_* m_2$ is divisible by 2.*

The proof is described by Fulton in [3], p. 171.

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