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A global calculus of parameter-dependent pseudodifferential boundary problems in L_p Sobolev spaces

by

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Introduction

The theory of pseudodifferential boundary problems has been developed to provide a larger framework for the study of differential boundary value problems, allowing algebraic manipulations with the operators (reflected in a symbolic calculus) and allowing the inclusion of non-local terms. The elliptic calculus has its origin in works of Vishik, Èskin and Boutet de Monvel (cf. [V-E], [E], [BM1] and in particular the Acta article [BM2]) and was further developed e.g. in Rempel and Schulze [R-S1] and Grubb [G1]. The scope of the theory was enlarged by the consideration of systems *depending on a parameter* (running in a noncompact set), which can be for example a spectral parameter $\lambda \in \mathbf{C}$ (allowing functional calculus), a time dependence (for parabolic problems) or a small parameter $\varepsilon > 0$ (entering in singular perturbation problems). For operators in L_2 spaces, such a theory was worked out in the book [G2], and further developed for parabolic problems by Grubb and Solonnikov [G-S1], who applied it to give new results on fully nonhomogeneous Navier–Stokes problems (cf. [G-S2] and its references). Let us also mention the treatment in [R-S2] of resolvent estimates and complex powers for systems without the so-called transmission property.

The purpose of the present work is to extend the parameter-dependent calculus to the L_p setting, $1 < p < \infty$, and to a suitable class of unbounded manifolds, including exterior domains (complements of smooth compact sets) in $\mathbf{R}^n$ and $\overline{\mathbf{R}}^n_+$.

A fundamental difficulty in the study of parameter-dependent elliptic pseudodifferential problems, depending e.g. on a spectral parameter λ on a ray in $\mathbf{C}$, is the following: Without the parameter, the singularity of the homogeneous symbols at $\xi = 0$ is harmless (since it is felt in a compact set only), but when λ is adjoined, the singularity has an important influence as $|\lambda|\to\infty$. This is reflected in a complicated structure of the symbol classes, noticeable already for the pseudodifferential operators P in the theory, but even more so for the boundary terms (the trace operators T, Poisson operators K, singular Green operators G).

The book [G2] handles this problem by introducing a version of the Boutet de Monvel calculus with symbols depending on a parameter μ (a suitable power of λ), where μ has much the same role as the cotangent variables in the ps.d.o. calculus, but enters with less regular estimates. The symbol seminorms for operators on $\mathbf{R}^n_+$ are formulated by use of $L_2(\mathbf{R}_+)$ estimates, where $x_n \in \mathbf{R}_+$ is the variable normal to the boundary (here one finds that e.g. systems of supremum norms, that are equivalent with the present choice in the non-parametrized case, are far from equally efficient when the parameter is included); and the full calculus is established in this framework. L_2 Sobolev space estimates of the operators are obtained by use of convenient properties of the Fourier transform.

For L_p results one might expect that it would be necessary to work the symbolic calculus over again with $L_p(\mathbf{R}_+)$ estimates, but we found that this is neither necessary nor very useful. In fact, the results can very well be obtained on the basis of the development in [G2]. Mapping properties in $L_p(\mathbf{R}^n_+)$ Sobolev spaces are obtained by use of multiplier theorems of Mihlin and Lizorkin with respect to the tangential variable x', valued in *Hilbert spaces* of functions of x_n. (The theorems do not hold generally in Banach spaces, which is why $L_p(\mathbf{R}_+)$ symbol estimates are not used.) Our point here is that we can get the desired $L_p(\mathbf{R}^n_+)$ estimates on the basis of certain weighted $L_2(\mathbf{R}_+)$ and $H^s_2(\mathbf{R}_+)$ estimates derived from the calculus of [G2], by use of an interpolation result from Gilbert [Gi] together with other results on Bessel-potential and Besov spaces (cf. [T1]).

In the process, we have revised the results of [G2] and worked out some precisions of that calculus; notably, a certain "loss of regularity" occurring there has been almost eliminated. We have also found a simplification in the use of "order-reducing operators", namely that the simple version $(\langle D'\rangle \pm iD_n)^t$ composes in a very good way with the boundary operator types K, T and G, and hence can be used in many places where one would think it necessary to use the more refined version of [BM2]. Moreover, we extend the calculus to unbounded manifolds with finitely many conical ends, by working with a special type of symbols that are uniformly estimated in x (as in Hörmander [H3, 18.1] for ps.d.o.s without μ-dependence); and systems of negative class are included (generalizing [G3]). — Some of our results were announced in [G-K].

Contents. Section 1 introduces the appropriate Bessel-potential spaces $H^{s,\mu}_p(\overline{\Omega})$ and Besov spaces $B^{s,\mu}_p(\overline{\Omega})$ depending on a parameter $\mu \in \overline{\mathbf{R}}_+$; and the expected properties (interpolation, duality, imbedding and trace theorems) are established, uniformly in the parameter. The admissible unbounded manifolds are defined. The needed variable-coefficient version of Mihlin's and Lizorkin's theorems is presented, and we show some particular interpolation results to be used later.

In Section 2, we introduce a stricter, globally estimated version of the symbol classes in [G2], based on [H3, 18.1] (results for symbol classes with local estimates can be recovered from this). An advantage of the global calculus (besides allowing unbounded manifolds) is that it gives more precise results: the so-called negligible operators are included in the operator spaces instead of forming residual classes, and the compositions and inverses on $\overline{\mathbf{R}}^n_+$ (when they exist) are defined by precise symbols in the calculus.

In Section 3, we show how a certain projection estimate from [G2] can be sharpened, allowing any $\varepsilon>0$ instead of $\varepsilon=\frac{1}{2}$ for (0.2) below. We introduce negative classes and study the composition of boundary operators with operators $(\langle D'\rangle \pm iD_n)^t$, and we prove some delicate estimates of the boundary symbol operators in weighted L_2 spaces, crucial for the later L_p estimates.

In Section 4, we prove the main theorem on continuity in $H^{s,\mu}_p$ and $B^{s,\mu}_p$ spaces. It is found, roughly speaking, that for boundary operators of regularity ν, the continuity holds with an $O(\langle\mu\rangle^{-\nu+|1/2-1/p|}+1)$ estimate of the operator norm, and for ps.d.o.s it holds with an $O(\langle\mu\rangle^{-\nu}+1)$ estimate (see the precise statements in Theorem 4.1 below; for the necessity, see Remark 4.2). In particular, if $\nu\geqslant\frac{1}{2}$, the estimates are uniform in μ for each $p\in\,]1,\infty[$.

Finally, Section 5 gives the full proof of composition rules for the present operator classes, showing that when $\mathcal{A}_\mu$ and $\mathcal{A}'_\mu$ are Green operators of order d resp. d', class r resp. r' and regularity ν resp. ν',

$$\mathcal{A}_\mu=\begin{pmatrix} P_{\mu,+}+G_\mu & K_\mu \\ T_\mu & S_\mu \end{pmatrix},\quad \mathcal{A}'_\mu=\begin{pmatrix} P'_{\mu,+}+G'_\mu & K'_\mu \\ T'_\mu & S'_\mu \end{pmatrix}; \tag{0.1}$$

then (when the matrix dimensions match) $\mathcal{A}_\mu\mathcal{A}'_\mu$ is again a Green operator:

$$\mathcal{A}''_\mu=\mathcal{A}_\mu\mathcal{A}'_\mu=\begin{pmatrix} P''_{\mu,+}-L(P_\mu,P'_\mu)+G'''_\mu & K''_\mu \\ T''_\mu & S''_\mu \end{pmatrix}, \tag{0.2}$$

of order $d''=d+d'$ and class $r''=\max\{r',r+d'\}$, with all terms of regularity $\nu''=\min(\nu,\nu',\nu+\nu')$ except $L(P_\mu,P'_\mu)$, that is of regularity $\nu''-\varepsilon$, any $\varepsilon>0$.

Applications. The above gives the fundamental steps necessary for establishing the calculus in the desired generality. In sequels [G5] to this paper, polyhomogeneous *parameter-elliptic* Green operators $\mathcal{A}_\mu$ are considered; it is shown that parameter-elliptic operators are invertible (within the calculus) for sufficiently large μ, and that an inverse, when it exists, belongs to the calculus; and the consequences for *parabolic* pseudodifferential boundary problems (in anisotropic Bessel-potential and Besov spaces) are developed.

Moreover, a concrete application of the anisotropic L_p Sobolev space results to the time-dependent nonhomogeneous Navier–Stokes problem along the lines of [G-S2] is worked out. Further consequences for Navier–Stokes problems with initial data of low regularity have been derived in [G6].

1. Spaces and manifolds

1.1. Bessel-potential and Besov spaces with a parameter

There are many good explanations of the various generalizations of Sobolev spaces in the L_p situation, cf. e.g. Stein [St], Nikolskiĭ [Ni], Bergh–Löfström [B-L], Triebel [T1, 2]. In the present paper we shall use the terminology summed up in [G3] (based to a large extent on [T1]), without repeating the full details here, where we shall focus on the information on parameter-dependent spaces necessary for our presentation.

Let $1<p<\infty$. We recall that a constant-coefficient pseudodifferential operator Q with symbol $q(\xi)$,

$$Qu=\mathrm{OP}(q(\xi))u=q(D)u=\mathcal{F}^{-1}(q\mathcal{F}u)$$

(where $\mathcal{F}$ is the Fourier transform), is continuous in L_p, if one of the following criteria (derived from the Marcinkiewics multiplier theorem) holds:

$$\begin{aligned} C(q)&\equiv \sup_{|\alpha|\leqslant[n/2]+1,\,\xi\in\mathbf{R}^n}|\xi|^{|\alpha|}|D_\xi^\alpha q(\xi)|<\infty \quad \text{(Mihlin [M])},\\ C'(q)&\equiv \sup_{\alpha_i=0,1,\,\xi\in\mathbf{R}^n}|\xi^\alpha D_\xi^\alpha q(\xi)|<\infty \quad \text{(Lizorkin [L])}; \end{aligned} \tag{1.1}$$

and the norm is $\leqslant C_pC(q)$ resp. $C_pC'(q)$, where C_p is independent of q. Lizorkin's criterion covers a different type of symbols than Mihlin's, since "mixed" expressions $\xi_i D_{\xi_j} q$ with $i\neq j$ do not enter. (The total order is higher there, when $n\geqslant 3$, but in many situations that is no problem.)

In order to state estimates of parameter-dependent Green operators in Bessel-potential (H_p^s) and Besov (B_p^s) spaces, we introduce for each of these spaces a family of equivalent norms depending on a parameter μ. The parametrized norms are introduced first for spaces over $\mathbf{R}^n$, then for spaces over $\overline{\mathbf{R}}^n_+=\{x\in\mathbf{R}^n\mid x_n\geqslant 0\}$, and finally for spaces over suitable manifolds with boundary.

Consider first the spaces over $\mathbf{R}^n$. Let $\langle\xi\rangle=(1+|\xi|^2)^{1/2}$, and denote $\mathrm{OP}(\langle\xi\rangle^s)=\langle D\rangle^s$. Recall that the *Bessel-potential space* $H_p^s(\mathbf{R}^n)$ is defined as the space of $f\in\mathcal{S}'(\mathbf{R}^n)$ for which $\langle D\rangle^s f\in L_p(\mathbf{R}^n)$, with norm

$$\|f\|_{H_p^s(\mathbf{R}^n)}=\|\langle D\rangle^s f\|_{L_p(\mathbf{R}^n)}. \tag{1.2}$$

For $s \in \mathbf{N}$, $H_p^s(\mathbf{R}^n)$ equals the Sobolev space $W_p^s(\mathbf{R}^n)$ that consists of the functions f with $D^\alpha f \in L_p(\mathbf{R}^n)$ for $|\alpha| \leqslant s$. For $s \in \mathbf{R}_+ \setminus \mathbf{N}$, another definition is used for the Sobolev–Slobodetskiĭ space $W_p^s(\mathbf{R}^n)$; it is taken equal to the *Besov space* $B_p^s(\mathbf{R}^n)$. Here $B_p^s(\mathbf{R}^n)$ can be described either in terms of an integral formula:

$$\begin{aligned} &\text{for } s \in]0,1[, \quad f \in B_p^s(\mathbf{R}^n) \iff \|f\|_{L_p} + \int_{\mathbf{R}^{2n}} \frac{|f(x)-f(y)|^p}{|x-y|^{n+ps}}\, dx\, dy < \infty, \\ &\text{for } s,t \in \mathbf{R}, \quad f \in B_p^{s+t}(\mathbf{R}^n) \iff \langle D\rangle^t f \in B_p^s(\mathbf{R}^n); \end{aligned} \tag{1.3}$$

or in terms of a norm defined by a partition of unity as follows (cf. e.g. [B-L, Lemma 6.1.7]): Let $\varphi \in C_0^\infty(\mathbf{R}^n)$ have the properties

$$\begin{gathered} \operatorname{supp}\varphi = \{\xi \mid 2^{-1} \leqslant |\xi| \leqslant 2\}, \\ \varphi(\xi) > 0 \quad \text{when } 2^{-1} < |\xi| < 2, \\ \sum_{k=-\infty}^{\infty} \varphi(2^{-k}\xi) = 1 \quad \text{when } |\xi| \neq 0, \end{gathered}$$

and define $\varphi_0 \in C_0^\infty(\mathbf{R}^n)$ by

$$\varphi_0(\xi) = 1 - \sum_{1 \leqslant k < \infty} \varphi(2^{-k}\xi).$$

Then $B_p^s(\mathbf{R}^n) = B_{p,p}^s(\mathbf{R}^n)$, where, for any $s \in \mathbf{R}$ and $0 < q < \infty$, the space $B_{p,q}^s(\mathbf{R}^n)$ is defined as the space of tempered distributions f with finite norm

$$\begin{aligned} &\|f\|_{B_{p,q}^s(\mathbf{R}^n)} = \Big(\|\varphi_0(D)f\|_{L_p(\mathbf{R}^n)}^q + \sum_{1 \leqslant k < \infty} (2^{sk} \|\varphi(2^{-k}D)f\|_{L_p(\mathbf{R}^n)})^q \Big)^{1/q}; \\ &\|f\|_{B_p^s(\mathbf{R}^n)} = \|f\|_{B_{p,p}^s(\mathbf{R}^n)}. \end{aligned} \tag{1.4}$$

Parameter-dependent versions are now defined as follows: Let $\mu \in \overline{\mathbf{R}}_+$ (we could equally well let $\mu \in \mathbf{R}$). For the Bessel-potential spaces, we simply take the norms

$$\begin{gathered} \|f\|_{H_p^{s,\mu}(\mathbf{R}^n)} = \|\langle D,\mu\rangle^s f\|_{L_p(\mathbf{R}^n)}, \quad \text{where} \\ \langle D,\mu\rangle^s = \operatorname{OP}(\langle \xi,\mu\rangle^s) = \Xi_\mu^s, \quad \langle \xi,\mu\rangle = (1+|\xi|^2+\mu^2)^{1/2}. \end{gathered} \tag{1.5}$$

(Also (2.2) or (2.3) can be used.) For each fixed μ, this is equivalent to $\|f\|_{H_p^s(\mathbf{R}^n)}$, by Mihlin's criterion (the functions $\langle\xi\rangle\langle\xi,\mu\rangle^{-1}$ and $\langle\xi,\mu\rangle\langle\xi\rangle^{-1}$ satisfy (1.1)), but with estimates depending on μ. For $s \geqslant 0$, the functions $\langle\xi,\mu\rangle^s/(\langle\xi\rangle^s + \langle\mu\rangle^s)$ and $(\langle\xi\rangle^s + \langle\mu\rangle^s)/\langle\xi,\mu\rangle^s$ satisfy Mihlin's criterion with constants independent of μ, so

$$\begin{aligned} \|f\|_{H_p^{s,\mu}(\mathbf{R}^n)} &= \|\langle D,\mu\rangle^s f\|_{L_p(\mathbf{R}^n)} = \|\operatorname{OP}(\langle\xi,\mu\rangle^s/(\langle\xi\rangle^s + \langle\mu\rangle^s))\operatorname{OP}(\langle\xi\rangle^s + \langle\mu\rangle^s) f\|_{L_p(\mathbf{R}^n)} \\ &\leqslant C_1(\|\langle D\rangle^s f\|_{L_p(\mathbf{R}^n)} + \langle\mu\rangle^s \|f\|_{L_p(\mathbf{R}^n)}); \end{aligned}$$

and similarly $\|\langle D\rangle^s f\|_{L_p}\leqslant C_2\|\langle D,\mu\rangle^s f\|_{L_p}$ and $\|\langle\mu\rangle^s f\|_{L_p}\leqslant C_3\|\langle D,\mu\rangle^s f\|_{L_p}$; altogether one has a simple comparison

$$C^{-1}\|f\|_{H_p^{s,\mu}(\mathbf{R}^n)}\leqslant\|f\|_{H_p^s(\mathbf{R}^n)}+\langle\mu\rangle^s\|f\|_{L_p(\mathbf{R}^n)}\leqslant C\|f\|_{H_p^{s,\mu}(\mathbf{R}^n)},\quad\text{for } s\geqslant 0,\tag{1.6}$$

with a constant C independent of μ (and depending on s). Observe that with the choice of norm (1.5),

$$\langle D,\mu\rangle^t=\Xi_\mu^t\colon H_p^{s,\mu}(\mathbf{R}^n)\xrightarrow{\sim}H_p^{s-t,\mu}(\mathbf{R}^n)\quad\text{isometrically},\tag{1.7}$$

for all $s,t\in\mathbf{R}$, since $\Xi_\mu^t\Xi_\mu^{t'}=\Xi_\mu^{t+t'}$ in general.

To motivate the definition of μ-dependent Besov norms, let us rewrite (1.5) a little. Let M_μ be the homeomorphism of $\mathcal{S}'(\mathbf{R}^n)$ (and $H_p^s(\mathbf{R}^n)$ and $B_p^s(\mathbf{R}^n)$ etc.) given on $\mathcal{S}(\mathbf{R}^n)$ by

$$(M_\mu v)(x)=v(\langle\mu\rangle^{-1}x).\tag{1.8}$$

Then $DM_\mu^{-1}=M_\mu^{-1}\langle\mu\rangle D$, and hence

$$\langle(D,\mu)\rangle^s=M_\mu^{-1}\langle(\langle\mu\rangle D,\mu)\rangle^s M_\mu=\langle\mu\rangle^s M_\mu^{-1}\langle D\rangle^s M_\mu,$$

from which follows a simple, precise formula:

$$\|f\|_{H_p^{s,\mu}(\mathbf{R}^n)}=\langle\mu\rangle^{-n/p+s}\|M_\mu f\|_{H_p^s(\mathbf{R}^n)}.\tag{1.9}$$

This allows us to make a very convenient choice of the parameter-dependent Besov norms, namely, by analogy:

$$\|f\|_{B_p^{s,\mu}(\mathbf{R}^n)}=\langle\mu\rangle^{-n/p+s}\|M_\mu f\|_{B_p^s(\mathbf{R}^n)}.\tag{1.10}$$

We conclude from the standard imbedding results that for $s\in\mathbf{R}$, $\varepsilon>0$,

$$\begin{aligned}&B_p^{s,\mu}(\mathbf{R}^n)\subset H_p^{s,\mu}(\mathbf{R}^n)\subset B_p^{s-\varepsilon,\mu}(\mathbf{R}^n)\quad\text{for } p\leqslant 2,\\&H_p^{s,\mu}(\mathbf{R}^n)\subset B_p^{s,\mu}(\mathbf{R}^n)\subset H_p^{s-\varepsilon,\mu}(\mathbf{R}^n)\quad\text{for } p\geqslant 2,\end{aligned}\tag{1.11}$$

uniformly in μ, with $H_p^{s,\mu}(\mathbf{R}^n)=B_p^{s,\mu}(\mathbf{R}^n)$ if and only if $p=2$. Moreover (cf. e.g. [T1, 2.8.1]), one has for s and $s_1\in\mathbf{R}$, p and $p_1\in\,]1,\infty[$, $m\in\mathbf{N}$,

$$\begin{aligned}H_p^{s,\mu}(\mathbf{R}^n)+B_p^{s,\mu}(\mathbf{R}^n)\subset H_{p_1}^{s_1,\mu}(\mathbf{R}^n)\cap B_{p_1}^{s_1,\mu}(\mathbf{R}^n),&\quad\text{when } s-\tfrac{n}{p}\geqslant s_1-\tfrac{n}{p_1},\ p_1>p;\\H_p^{s,\mu}(\mathbf{R}^n)+B_p^{s,\mu}(\mathbf{R}^n)\subset C_*^m(\mathbf{R}^n),&\quad\text{when } s>\tfrac{n}{p}+m,\end{aligned}\tag{1.12}$$

uniformly in μ; here $C_*^m(\mathbf{R}^n)$ is the space of m times continuously differentiable functions u with $D^\alpha u(x)\to 0$ for $|x|\to\infty$, $|\alpha|\leqslant m$. ($\mathcal{S}(\mathbf{R}^n)$ is dense in all these spaces.)

Observe also the following difference: For $s=0$,

$$\|\cdot\|_{H_p^{0,\mu}(\mathbf{R}^n)} = \|\cdot\|_{L_p(\mathbf{R}^n)} \quad \text{for all } \mu\in\overline{\mathbf{R}}_+,$$

whereas $\|\cdot\|_{B_p^{0,\mu}(\mathbf{R}^n)}$ depends in an essential way on μ when $p\neq 2$. In fact, we have for all $f\in\mathcal{S}(\mathbf{R}^n)$ in view of (1.4) and (1.10),

$$\|f\|_{B_p^{0,\mu}(\mathbf{R}^n)} \to \|f\|_{L_p(\mathbf{R}^n)} \quad \text{as } \mu\to\infty, \tag{1.13}$$

where $L_p(\mathbf{R}^n)=H_p^0(\mathbf{R}^n)\neq B_p^0(\mathbf{R}^n)$ when $p\neq 2$.

We need to know that the interpolation, duality and trace theorems valid for the usual Bessel-potential and Besov spaces have counterparts for these parameter-dependent spaces, with estimates uniform in μ. The statements and proofs will be straightforward when we use (1.9) and (1.10). To give an example, recall that real interpolation $(\cdot,\cdot)_{\theta,p}$ (cf. [B-L, Theorem 6.2.4]) of the Bessel-potential spaces $H_p^{s_0}(\mathbf{R}^n)$ and $H_p^{s_1}(\mathbf{R}^n)$ with $0<\theta<1$ and $s_0\neq s_1$, gives the Besov space $B_p^s(\mathbf{R}^n)$, where $s=(1-\theta)s_0+\theta s_1$. In particular, for some constant $C>0$ independent of f we have when $f\in B_p^s(\mathbf{R}^n)$:

$$C^{-1}\|f\|_{B_p^s(\mathbf{R}^n)} \leqslant \|f\|_{(H_p^{s_0}(\mathbf{R}^n),H_p^{s_1}(\mathbf{R}^n))_{\theta,p;K}} \leqslant C\|f\|_{B_p^s(\mathbf{R}^n)}. \tag{1.14}$$

(The precise value of the interpolation norm depends on the method; here we refer to the K-method, to fix the ideas.) Since M_μ is a homeomorphism in all these spaces, and the exponent $-\frac{n}{p}+s$ in (1.9) and (1.10) is an affine function of s, we get *with the same* C:

$$C^{-1}\|f\|_{B_p^{s,\mu}(\mathbf{R}^n)} \leqslant \|f\|_{(H_p^{s_0,\mu}(\mathbf{R}^n),H_p^{s_1,\mu}(\mathbf{R}^n))_{\theta,p;K}} \leqslant C\|f\|_{B_p^{s,\mu}(\mathbf{R}^n)}. \tag{1.15}$$

We formulate this result as follows:

$$(H_p^{s_0,\mu}(\mathbf{R}^n),H_p^{s_1,\mu}(\mathbf{R}^n))_{\theta,p}\simeq B_p^{s,\mu}(\mathbf{R}^n), \quad \text{uniformly in } \mu, \text{ with } s=(1-\theta)s_0+\theta s_1. \tag{1.16}$$

Recall that for f in the smallest of the spaces,

$$\|f\|_{B_p^{s,\mu}(\mathbf{R}^n)} \leqslant C\|f\|^{1-\theta}_{H_p^{s_0,\mu}(\mathbf{R}^n)}\|f\|^{\theta}_{H_p^{s_1,\mu}(\mathbf{R}^n)}. \tag{1.17}$$

As an application, we see that for each $t,s\in\mathbf{R}$,

$$\langle D,\mu\rangle^t=\Xi_\mu^t\colon B_p^{s,\mu}(\mathbf{R}^n)\xrightarrow{\sim} B_p^{s-t,\mu}(\mathbf{R}^n), \quad \text{uniformly in } \mu \tag{1.18}$$

(in the sense that the operators $\langle D,\mu\rangle^t$ and $\langle D,\mu\rangle^{-t}$ between these spaces have bounds independent of μ). Because $-\frac{n}{p}+s$ is affine in $\frac{1}{p}$ too, all the other interpolation and

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imbedding results of [B-L, 6.2 and 6.4] generalize immediately to μ-uniform results for the parameter-dependent spaces. The same is true for the duality results, since (1.10) means that $\langle\mu\rangle^{-n/p+s}M_\mu$ is an isometry of $B_p^{s,\mu}(\mathbf{R}^n)$ onto $B_p^s(\mathbf{R}^n)$, and thus

$$((\langle\mu\rangle^{-n/p+s}M_\mu)^*)^{-1}=\langle\mu\rangle^{n/p-s}\langle\mu\rangle^{-n}M_\mu=\langle\mu\rangle^{n/p'-s}M_\mu$$

is an isometry of $B_p^{s,\mu}(\mathbf{R}^n)^*$ onto $B_p^s(\mathbf{R}^n)^*=B_{p'}^{-s}(\mathbf{R}^n)$, where

$$\tfrac{1}{p}+\tfrac{1}{p'}=1. \tag{1.19}$$

(The notation (1.19) will be used throughout.) Thus for all $s\in\mathbf{R}$,

$$B_p^{s,\mu}(\mathbf{R}^n)^*\simeq B_{p'}^{-s,\mu}(\mathbf{R}^n),\quad \text{uniformly in } \mu. \tag{1.20}$$

Finally, we have for the restriction operator $\tilde\gamma_0\colon u(x)\mapsto u(x',0)$ (from $\mathbf{R}^n$ to $\mathbf{R}^{n-1}$), since $-\frac{n}{p}+s=-\frac{n-1}{p}+\left(s-\frac{1}{p}\right)$ and $\tilde\gamma_0(M_\mu u)=M_\mu\tilde\gamma_0 u$, that the usual trace theorem (cf. e.g. [B-L, Theorem 6.6.1]) implies:

$$\begin{aligned}&\tilde\gamma_0\colon H_p^{s,\mu}(\mathbf{R}^n)\to B_p^{s-1/p,\mu}(\mathbf{R}^{n-1})\quad \text{and}\\ &\tilde\gamma_0\colon B_p^{s,\mu}(\mathbf{R}^n)\to B_p^{s-1/p,\mu}(\mathbf{R}^{n-1})\quad \text{uniformly in } \mu \text{ for } s>\tfrac{1}{p},\end{aligned} \tag{1.21}$$

and s cannot be taken $\leqslant\frac{1}{p}$ (cf. e.g. [G3, Lemma 2.2]).

Concerning spaces defined relative to $\overline{\mathbf{R}}_+^n$, one has the usual two variants. On one hand there are the closed subspaces of distributions supported in $\overline{\mathbf{R}}_+^n$,

$$\begin{aligned}H_{p;0}^{s,\mu}(\overline{\mathbf{R}}_+^n)&=\{u\in H_p^{s,\mu}(\mathbf{R}^n)\mid \operatorname{supp}u\subset\overline{\mathbf{R}}_+^n\};\\ B_{p;0}^{s,\mu}(\overline{\mathbf{R}}_+^n)&=\{u\in B_p^{s,\mu}(\mathbf{R}^n)\mid \operatorname{supp}u\subset\overline{\mathbf{R}}_+^n\};\end{aligned} \tag{1.22}$$

that inherit the norm from the full spaces; and on the other hand, there are the distribution spaces over $\mathbf{R}_+^n$ obtained by restriction:

$$H_p^{s,\mu}(\overline{\mathbf{R}}_+^n)=r^+H_p^{s,\mu}(\mathbf{R}^n),\quad B_p^{s,\mu}(\overline{\mathbf{R}}_+^n)=r^+B_p^{s,\mu}(\mathbf{R}^n); \tag{1.23}$$

they are provided with the infimum norms

$$\|f\|_{H_p^{s,\mu}(\overline{\mathbf{R}}_+^n)}=\inf\{\,\|g\|_{H_p^{s,\mu}(\mathbf{R}^n)}\mid g\in H_p^{s,\mu}(\mathbf{R}^n),\ r^+g=f\,\},\quad \text{etc.} \tag{1.24}$$

Here $r^\pm$ denotes restriction to $\mathbf{R}_\pm^n$ (and we later also use $e^\pm$, extending functions on $\mathbf{R}_\pm^n$ by zero on $\mathbf{R}^n\setminus\mathbf{R}_\pm^n$). The spaces

$$\mathcal{S}_0(\overline{\mathbf{R}}_+^n)=\{u\in\mathcal{S}(\mathbf{R}^n)\mid \operatorname{supp}u\subset\overline{\mathbf{R}}_+^n\}\quad \text{resp.}\quad \mathcal{S}(\overline{\mathbf{R}}_+^n)=r^+\mathcal{S}(\mathbf{R}^n) \tag{1.25}$$

are dense in $H^{s,\mu}_{p;0}(\overline{\mathbf{R}}^n_+)$ and $B^{s,\mu}_{p;0}(\overline{\mathbf{R}}^n_+)$ resp. $H^{s,\mu}_p(\overline{\mathbf{R}}^n_+)$ and $B^{s,\mu}_p(\overline{\mathbf{R}}^n_+)$; and the norms in the latter equal the quotient norms when we identify

$$H^{s,\mu}_p(\overline{\mathbf{R}}^n_+) \simeq H^{s,\mu}_p(\mathbf{R}^n)/H^{s,\mu}_{p;0}(\overline{\mathbf{R}}^n_-), \quad \text{etc.} \tag{1.26}$$

(There is a detailed analysis e.g. in Triebel [T1, 2.8, 2.9] of the spaces in the μ-independent case; $H^s_{p;0}(\overline{\mathbf{R}}^n_+)$ and $B^s_{p;0}(\overline{\mathbf{R}}^n_+)$ are called $\widetilde{H}^s_p(\mathbf{R}^+_n)$ resp. $\widetilde{B}^s_{p,p}(\mathbf{R}^+_n)$ there.) Now the crucial observation is, that since the dilation $x \mapsto \langle\mu\rangle x$ preserves $\overline{\mathbf{R}}^n_+$, we can conclude that the norms are related to the μ-independent norms in the same way as for the spaces over $\mathbf{R}^n$:

$$\begin{gathered}\|f\|_{X^{s,\mu}} = \langle\mu\rangle^{-n/p+s}\|M_\mu f\|_{X^s} \quad \text{for } s\in\mathbf{R}, \text{ with}\\ X = H_p(\overline{\mathbf{R}}^n_+),\ H_{p;0}(\overline{\mathbf{R}}^n_+),\ B_p(\overline{\mathbf{R}}^n_+) \text{ or } B_{p;0}(\overline{\mathbf{R}}^n_+).\end{gathered} \tag{1.27}$$

Uniform interpolation, imbedding, duality, trace and extension theorems then follow from the case without a parameter just as above. (For an overview of relevant non-parametrized statements, see [G3].) Let us collect some of the resulting statements in a theorem.

THEOREM 1.1. *The following identifications hold, uniformly in μ:*

$$\begin{aligned}
&H^{s,\mu}_p(\overline{\mathbf{R}}^n_+) \simeq H^{s,\mu}_{p;0}(\overline{\mathbf{R}}^n_+) \text{ and } B^{s,\mu}_p(\overline{\mathbf{R}}^n_+) \simeq B^{s,\mu}_{p;0}(\overline{\mathbf{R}}^n_+), \quad \text{when } \tfrac{1}{p}-1 < s < \tfrac{1}{p};\\
&H^{s,\mu}_p(\overline{\mathbf{R}}^n_+)^* \simeq H^{-s,\mu}_{p';0}(\overline{\mathbf{R}}^n_+) \text{ and } B^{s,\mu}_p(\overline{\mathbf{R}}^n_+)^* \simeq B^{-s,\mu}_{p';0}(\overline{\mathbf{R}}^n_+), \quad \text{for all } s\in\mathbf{R};\\
&(H^{s,\mu}_p(\overline{\mathbf{R}}^n_+), H^{t,\mu}_p(\overline{\mathbf{R}}^n_+))_{\theta,p} \simeq (H^{s,\mu}_p(\overline{\mathbf{R}}^n_+), B^{t,\mu}_p(\overline{\mathbf{R}}^n_+))_{\theta,p}\\
&\qquad \simeq (B^{s,\mu}_p(\overline{\mathbf{R}}^n_+), B^{t,\mu}_p(\overline{\mathbf{R}}^n_+))_{\theta,p} \simeq B^{s_1,\mu}_p(\overline{\mathbf{R}}^n_+),\\
&(H^{s,\mu}_p(\overline{\mathbf{R}}^n_+), H^{s,\mu}_q(\overline{\mathbf{R}}^n_+))_{\theta,p_1} \simeq H^{s,\mu}_{p_1}(\overline{\mathbf{R}}^n_+),\\
&[B^{s,\mu}_p(\overline{\mathbf{R}}^n_+), B^{t,\mu}_p(\overline{\mathbf{R}}^n_+)]_\theta \simeq B^{s_1,\mu}_p(\overline{\mathbf{R}}^n_+),\\
&[H^{s,\mu}_p(\overline{\mathbf{R}}^n_+), H^{t,\mu}_p(\overline{\mathbf{R}}^n_+)]_\theta \simeq H^{s_1,\mu}_p(\overline{\mathbf{R}}^n_+); \quad \text{where}\\
&s,t\in\mathbf{R},\ s\neq t,\ p,q\in\,]1,\infty[,\ \theta\in\,]0,1[,\ s_1=(1-\theta)s+\theta t,\ \tfrac{1}{p_1}=\tfrac{1-\theta}{p}+\tfrac{\theta}{q}.
\end{aligned} \tag{1.28}$$

Here the dual spaces are defined with respect to an extension of the sesquilinear duality $(u,v)_{\mathbf{R}^n_+} = \int_{\mathbf{R}^n_+} u(x)\bar{v}(x)\,dx$; and the interpolations that are used are real interpolation $(\cdot,\cdot)_{\theta,p}$ resp. complex interpolation $[\cdot,\cdot]_\theta$. The interpolation statements hold also with H_p and B_p replaced by $H_{p;0}$ and $B_{p;0}$, and the norms satisfy generalizations of (1.17).

The trace operator $\gamma_j\colon u(x)\mapsto D^j_{x_n}u(x',0)$ is continuous for $s>j+\frac{1}{p}$,

$$\gamma_j\colon H^{s,\mu}_p(\overline{\mathbf{R}}^n_+)+B^{s,\mu}_p(\overline{\mathbf{R}}^n_+) \to B^{s-j-1/p,\mu}_p(\mathbf{R}^{n-1}), \quad \text{uniformly in } \mu. \tag{1.29}$$

Note also that in view of (1.6),

$$C^{-1}\|f\|_{H^{s,\mu}_p(\overline{\mathbf{R}}^n_+)} \leqslant \|f\|_{H^s_p(\overline{\mathbf{R}}^n_+)} + \langle\mu\rangle^s\|f\|_{L_p(\mathbf{R}^n_+)} \leqslant C\|f\|_{H^{s,\mu}_p(\overline{\mathbf{R}}^n_+)}, \quad \text{for each } s\geqslant 0, \tag{1.30}$$

with C independent of μ.

For the spaces over $\overline{\mathbf{R}}^n_+$, the following operators are more convenient than the Ξ^t_μ: Denoting $(\xi_1, \dots, \xi_{n-1}) = \xi'$, we let

$$\begin{gathered}\chi^t_\pm(\xi, \mu) = (\langle \xi', \mu \rangle \pm i\xi_n)^t, \quad t \in \mathbf{R},\\ \chi^t_\pm(\xi', \mu, D_n) = \mathrm{OP}_{x_n}(\chi^t_\pm(\xi, \mu)) = (\langle \xi', \mu \rangle \pm iD_n)^t,\\ \Xi^t_{\pm,\mu} = \mathrm{OP}(\chi^t_\pm(\xi, \mu)) = (\langle D', \mu \rangle \pm iD_n)^t;\end{gathered} \tag{1.31}$$

then one has for all $t, s \in \mathbf{R}$:

$$\Xi^t_{\pm,\mu} \Xi^s_{\pm,\mu} = \Xi^{t+s}_{\pm,\mu}; \quad (\Xi^t_{\pm,\mu})^* = \Xi^t_{\mp,\mu}. \tag{1.32}$$

Moreover, the funtions $\langle \xi, \mu \rangle^{-t} (\langle \xi', \mu \rangle \pm i\xi_n)^t$ satisfy Lizorkin's criterion (1.1) uniformly in μ for each $t \in \mathbf{R}$, and hence the operators

$$\Xi^{s-t}_\mu \Xi^t_{\pm,\mu} \Xi^{-s}_\mu = \mathrm{OP}(\langle \xi, \mu \rangle^{-t} (\langle \xi', \mu \rangle \pm i\xi_n)^t)$$

are bounded in $L_p(\mathbf{R}^n)$, uniformly in μ, for any s and t, which implies

$$\Xi^t_{\pm,\mu} \colon H^{s,\mu}_p(\mathbf{R}^n) \xrightarrow{\sim} H^{s-t,\mu}_p(\mathbf{R}^n), \quad \text{uniformly in } \mu, \tag{1.33}$$

with similar statements for $B^{s,\mu}_p$ spaces. We define as usual

$$P_{\mathbf{R}^n_+} = r^+ P e^+, \quad \text{also written for short as } P_+.$$

By the Paley–Wiener theorem, the operators $\Xi^t_{+,\mu}$ map $\mathcal{S}_0(\overline{\mathbf{R}}^n_+)$ onto itself, and $\Xi^t_{+,\mu} \Xi^{t'}_{+,\mu}$ equals $\Xi^{t+t'}_{+,\mu}$ there, so by extension by continuity and by duality one gets the homeomorphism properties:

THEOREM 1.2. *For all $s, t \in \mathbf{R}$ one has, uniformly in μ,*

$$\begin{aligned}\Xi^t_{+,\mu} \colon H^{s,\mu}_{p;0}(\overline{\mathbf{R}}^n_+) &\xrightarrow{\sim} H^{s-t,\mu}_{p;0}(\overline{\mathbf{R}}^n_+), \quad \textit{with } \Xi^t_{+,\mu,\mathbf{R}^n_+} \Xi^s_{+,\mu,\mathbf{R}^n_+} = \Xi^{t+s}_{+,\mu,\mathbf{R}^n_+};\\ \Xi^t_{-,\mu,\mathbf{R}^n_+} \colon H^{s,\mu}_p(\overline{\mathbf{R}}^n_+) &\xrightarrow{\sim} H^{s-t,\mu}_p(\overline{\mathbf{R}}^n_+), \quad \textit{with } \Xi^t_{-,\mu,\mathbf{R}^n_+} \Xi^s_{-,\mu,\mathbf{R}^n_+} = \Xi^{t+s}_{-,\mu,\mathbf{R}^n_+}.\end{aligned} \tag{1.34}$$

(On distribution spaces where e^+ is not defined, $\Xi^t_{-,\mu,\mathbf{R}^n_+}$ is taken as the continuous extension from $\mathcal{S}(\overline{\mathbf{R}}^n_+)$.) The analogous statements hold with H_p replaced by B_p.

Here the results in Besov spaces follow from the results in Bessel-potential spaces by interpolation, using (1.28). We note furthermore that the mapping $\langle \mu \rangle \Xi^{-1}_{-,\mu,\mathbf{R}^n_+}$ is bounded in $H^{s,\mu}_p(\overline{\mathbf{R}}^n_+)$ and in $B^{s,\mu}_p(\overline{\mathbf{R}}^n_+)$, uniformly in μ, so that for each $s \in \mathbf{R}$,

$$\|f\|_{H^{s-1,\mu}_p(\overline{\mathbf{R}}^n_+)} \leqslant C_s \langle \mu \rangle^{-1} \|f\|_{H^{s,\mu}_p(\overline{\mathbf{R}}^n_+)}, \quad \|f\|_{B^{s-1,\mu}_p(\overline{\mathbf{R}}^n_+)} \leqslant C_s \langle \mu \rangle^{-1} \|f\|_{B^{s,\mu}_p(\overline{\mathbf{R}}^n_+)}. \tag{1.35}$$

We observe the following simple property of the spaces:

LEMMA 1.3. *Let m be an integer >0. For each $\beta=(\beta_0,\beta_1,\dots,\beta_n)$ with length $\beta_0+\beta_1+\dots+\beta_n=m$ there is a linear operator $A_{\beta,\mu}$ that is continuous, uniformly in μ, from $H_p^{s,\mu}(\overline{\mathbf{R}}^n_+)$ to $H_p^{s+m,\mu}(\overline{\mathbf{R}}^n_+)$ and from $B_p^{s,\mu}(\overline{\mathbf{R}}^n_+)$ to $B_p^{s+m,\mu}(\overline{\mathbf{R}}^n_+)$, for all $s\in\mathbf{R}$, $1<p<\infty$, such that any $u\in H_p^{s,\mu}(\overline{\mathbf{R}}^n_+)+B_p^{s,\mu}(\overline{\mathbf{R}}^n_+)$ can be represented as the sum over β*

$$u=\sum_{\beta_0+\beta_1+\dots+\beta_n=m}\langle\mu\rangle^{\beta_0}D_1^{\beta_1}\dots D_n^{\beta_n}A_{\beta,\mu}u. \tag{1.36}$$

Proof. Let $m=1$. Then we can write $u=\Xi^1_{-,\mu,\mathbf{R}^n_+}\Xi^{-1}_{-,\mu,\mathbf{R}^n_+}u=\langle D',\mu\rangle v+iD_nv$, with $v=\Xi^{-1}_{-,\mu,\mathbf{R}^n_+}u$. Here we note that

$$\langle D',\mu\rangle v=(\langle\mu\rangle^2+D_1^2+\dots+D_{n-1}^2)\langle D',\mu\rangle^{-1}v=\langle\mu\rangle v_0+D_1v_1+\dots+D_{n-1}v_{n-1},$$

where $v_0=\langle\mu\rangle\langle D',\mu\rangle^{-1}v$, $v_j=D_j\langle D',\mu\rangle^{-1}v$ for $1\leqslant j\leqslant n-1$. Using Lizorkin's criterion (1.1), it is seen that the operators $\langle\mu\rangle\langle D',\mu\rangle^{-1}$ and $D_j\langle D',\mu\rangle^{-1}$, $j\leqslant n-1$, are uniformly bounded in $H_p^{s,\mu}(\mathbf{R}^n)$ for all s and p, and they act similarly on $H_p^{s,\mu}(\overline{\mathbf{R}}^n_+)$ since they preserve the property of being supported in $\overline{\mathbf{R}}^n_\pm$. Then in view of (1.34), we have the desired decomposition for $m=1$:

$$u=\langle\mu\rangle A_0u+\sum_{j=1}^nD_jA_ju,$$

with $A_0=\langle\mu\rangle\langle D',\mu\rangle^{-1}\Xi^{-1}_{-,\mu,\mathbf{R}^n_+}$, $A_j=D_j\langle D',\mu\rangle^{-1}\Xi^{-1}_{-,\mu,\mathbf{R}^n_+}$ for $1\leqslant j\leqslant n-1$, $A_n=i\,\Xi^{-1}_{-,\mu,\mathbf{R}^n_+}$. The general result follows by iteration and interpolation. □

In one dimension, we often replace μ by $\varkappa=(1+|\xi'|^2+\mu^2)^{1/2}$ containing the $(n-1)$-dimensional parameter ξ'; in particular we consider

$$H_2^{t,\varkappa}(\overline{\mathbf{R}}_+)\simeq(\varkappa-iD_{x_n})^{-t}_{\mathbf{R}_+}L_2(\mathbf{R}_+). \tag{1.37}$$

Remark 1.4. The operators $\Xi^t_{\pm,\mu}$ are used e.g. in Èskin [E], Rempel–Schulze [R-S2], and other works on ps.d.o.s of a general kind. An inconvenience of the functions $\chi^t_\pm$ in connection with the Boutet de Monvel calculus is that they are *not* pseudodifferential symbols belonging to the $S_{1,0}$ symbol spaces over $\mathbf{R}^n$, because the higher ξ'-derivatives do not behave well enough (do not fall off for $|\xi|\to\infty$ in the way required for such symbols). This difficulty has been overcome by the introduction of operators $\Lambda^t_{\pm,\mu}$ defined from more refined symbols $\lambda^t_\pm(\xi,\mu)$ for $t\in\mathbf{Z}$, cf. [BM2], [R-S1], [G2, 3, 4], [F2], such as

$$\begin{gathered}\lambda^t_\pm(\xi,\mu)=[\sigma(|(\xi',\mu)|)\psi(\xi_n/a\sigma(|(\xi',\mu)|))\pm i\xi_n]^t,\\ \text{where }\psi(t)\in\mathcal{S}(\mathbf{R})\text{ with }\psi(0)=1,\quad\operatorname{supp}\mathcal{F}^{-1}\psi\subset\overline{\mathbf{R}}_\pm,\\ \sigma\in C^\infty(\mathbf{R};\mathbf{R}_+),\ \sigma(r)=|r|\text{ for }|r|\geqslant1,\ \sigma(r)=\tfrac12\text{ for }|r|\leqslant\tfrac12;\end{gathered} \tag{1.38}$$

and $a>0$ is taken so large that $\sigma\psi(\xi_n/a\sigma)\pm i\xi_n\neq 0$ for all values of σ and ξ_n (e.g. $a\geqslant 2\sup_t|\psi'(t)|$). These symbols are truly pseudodifferential, elliptic, of regularity $+\infty$ in the terminology of [G2], and have the transmission property. Moreover, they have the same mapping properties as the $\Xi^t_{\pm,\mu}$ listed in (1.34) ($t\in\mathbf{Z}$), shown in a similar way, see also [G3, Theorem 5.1]. (The hypothesis $\operatorname{supp}\mathcal{F}^{-1}\psi\subset\overline{\mathbf{R}}_\pm$ was not always included, it assures the extension of the mapping properties as in (1.34) to negative s, cf. [F2], [G3].)

But in fact we shall show below in Section 2 that the operators in (1.34) are, after all, useful in the $S_{1,0}$ pseudodifferential boundary operator calculus, as long as they are *only composed with the "singular" operators*, i.e. with Poisson, trace and singular Green operators, where they do *preserve the operator classes.* This observation will allow us to make some calculations in a simpler way than if the $\lambda^t_\pm$ were to be used. We shall even discuss some estimates where noninteger powers t are involved. (When $t\notin\mathbf{Z}$, neither the $\lambda^t_\pm$ nor the $\chi^t_\pm$ have the two-sided transmission property that plays an essential role in the Boutet de Monvel calculus, so they are rarely considered then.)

1.2. Admissible manifolds

Now let us say a few words about corresponding spaces defined over more general subsets of $\mathbf{R}^n$ and manifolds. Here we shall use the following two fundamental observations: (i) When $\psi\in C^\infty(\mathbf{R}^n)$ is bounded and all its derivatives are bounded, the mapping $f\mapsto\psi f$ is continuous from $H^{s,\mu}_p(\mathbf{R}^n)$ resp. $B^{s,\mu}_p(\mathbf{R}^n)$ to itself, uniformly in μ for any $s\in\mathbf{R}$, $1<p<\infty$. (ii) A diffeomorphism $\varkappa$ from $\mathbf{R}^n$ to $\mathbf{R}^n$, for which all derivatives of $\varkappa$ and $\varkappa^{-1}$ are bounded (we call this a bounded diffeomorphism), induces a mapping $f\mapsto f\circ\varkappa$ that sends $H^{s,\mu}_p(\mathbf{R}^n)$ resp. $B^{s,\mu}_p(\mathbf{R}^n)$ homeomorphically onto itself, uniformly in μ, for any $s\in\mathbf{R}$, $1<p<\infty$. These mapping properties are easily shown for $H^{s,\mu}_p(\mathbf{R}^n)$ when $s\in 2\mathbf{N}$, where (1.5) gives an elementary definition of the spaces; and they follow for the other spaces by interpolation and duality. In a similar way one gets (i) and (ii) for the spaces defined relative to $\overline{\mathbf{R}}^n_+$ (here $\varkappa$ is a bounded diffeomorphism from $\overline{\mathbf{R}}^n_+$ to $\overline{\mathbf{R}}^n_+$, and it extends to a bounded diffeomorphism from $\mathbf{R}^n$ to $\mathbf{R}^n$). Note that a bounded diffeomorphism $\varkappa\colon\overline{\mathbf{R}}^n_+\to\overline{\mathbf{R}}^n_+$ has the property

$$C_1|x-y|\leqslant|\varkappa(x)-\varkappa(y)|\leqslant C_2|x-y|, \tag{1.39}$$

with positive constants C_1 and C_2 (for C_2 take e.g. $\sup|\varkappa'(x)|$ and for C_1^{-1} take e.g. $\sup|(\varkappa^{-1})'(x)|$). In the following, $\{1,\dots,m\}\times A$ denotes a disjoint union of m copies of A, for an integer $m>0$. When $\varkappa$ goes from U to V, relatively open subsets of $\{1,\dots,m\}\times\overline{\mathbf{R}}^n_+$, we say that $\varkappa$ is an *admissible diffeomorphism* if it is a bounded diffeomorphism and (1.39) holds when x and y lie in the same copy of $\overline{\mathbf{R}}^n_+$.

These observations allow us to define $H_p^{s,\mu}$ and $B_p^{s,\mu}$ spaces over C^∞ manifolds $\bar\Omega$ (with interior Ω and boundary $\Gamma=\partial\Omega$) that are formed of a compact piece and finitely many conical pieces. More precisely, we assume that $\bar\Omega$, in addition to being an n-dimensional C^∞ manifold with boundary, is the union $\bar\Omega=\Omega_{\mathrm{co},1}\cup\Omega_{\mathrm{bd}}$ of a conical piece $\Omega_{\mathrm{co},1}$ and a bounded piece Ω_{bd} with $\bar\Omega_{\mathrm{bd}}$ compact; here, with $\mathbf{B}^n=\{x\in\mathbf{R}^n \mid |x|<1\}$, $S^{n-1}=\partial\mathbf{B}^n$,

$$\Omega_{\mathrm{co},r}=\{x=tx_0 \mid t>r,\ x_0\in M\subset\{1,\dots,m\}\times S^{n-1}\},\quad \text{for } r>0, \tag{1.40}$$

where M denotes a closed smooth $(n-1)$-dimensional submanifold of $\{1,\dots,m\}\times S^{n-1}$ with interior M° and boundary ∂M; and we assume that $\Omega_{\mathrm{co},1}\cap\Omega_{\mathrm{bd}}$ equals $\Omega_{\mathrm{co},1}\setminus\bar\Omega_{\mathrm{co},2}$ (i.e. the set where $1<t<2$). Using local coordinates for Ω_{bd} and for M in a standard way, we can describe the structure of $\bar\Omega$ as follows: There is a finite cover of $\bar\Omega$ by open sets $\Omega_i\subset\bar\Omega$ $(i=1,\dots,i_0)$ with associated C^∞ bijections $\varkappa_i\colon\Omega_i\tilde{\rightarrow}\Xi_i$ (coordinate mappings), where the Ω_i and Ξ_i are finite unions of (relatively) open subsets U of $\bar\Omega$, resp. relatively open subsets V of $\{1,\dots,m\}\times\bar{\mathbf{R}}^n_+$, of the following types:

$$\begin{aligned}
&U_1\subset\Omega_{\mathrm{bd}}\cap\Omega,\quad V_1\subset\{1,\dots,m\}\times\mathbf{R}^n_+,\quad U_2\subset\Omega_{\mathrm{bd}},\quad V_2\subset\{1,\dots,m\}\times\bar{\mathbf{R}}^n_+,\\
&U_3=\{x=tx_0 \mid t>r,\ x_0\in\omega \text{ open } \subset M^\circ\},\\
&V_3=\{x=tx_0 \mid t>r,\ x_0\in\omega' \text{ rel. open } \subset\{1,\dots,m\}\times(S^{n-1}\cap\mathbf{R}^n_+)\},\\
&U_4=\{x=tx_0 \mid t>r,\ x_0\in\omega \text{ rel. open } \subset M\},\\
&V_4=\{x=tx_0 \mid t>r,\ x_0\in\omega' \text{ rel. open } \subset\{1,\dots,m\}\times(S^{n-1}\cap\bar{\mathbf{R}}^n_+)\};
\end{aligned}\tag{1.41}$$

such that $U_2\cap\partial\Omega$ and $U_4\cap\partial\Omega$ are mapped into $\{1,\dots,m\}\times\mathbf{R}^{n-1}$; and the mappings $\varkappa_i\circ\varkappa_j^{-1}$ are admissible diffeomorphisms from $\varkappa_j(\Omega_i\cap\Omega_j)$ to $\varkappa_i(\Omega_i\cap\Omega_j)$ for all i,j. The mappings from sets of the types U_3,U_4 to V_3,V_4 can be taken of the form $\varkappa\colon tx_0\mapsto\alpha(x_0)t\lambda(x_0)$, where α is a positive scalar C^∞ function and λ is a C^∞ bijection from ω to ω'. We can obtain (by use of linear transformations in $\bar{\mathbf{R}}^n_+$) that the Ξ_i are mutually disjoint. ($\bar\Omega$ can also be described by other finite systems $\tilde\varkappa_j\colon\tilde\Omega_j\tilde{\rightarrow}\tilde\Xi_j$, $j=1,\dots,j_0$, with these properties, and we say that two systems are equivalent when the $\varkappa_i\circ\tilde\varkappa_j^{-1}$ are admissible diffeomorphisms from $\tilde\varkappa_j(\Omega_i\cap\tilde\Omega_j)$ to $\varkappa_i(\Omega_i\cap\tilde\Omega_j)$ for all i,j.)

For short, we call such $\bar\Omega$ *admissible manifolds*, and we call the finite systems of local coordinates describing them as above *admissible*. By restriction to Γ they give admissible systems of local coordinates for Γ.

We can assume that each Ω_i has an open subset Ω_i' such that $\varkappa_i(\Omega_i')=\Xi_i'$ with $\operatorname{dist}(\Xi_i',\complement\Xi_i)>0$ and $\{\varkappa_i\colon\Omega_i'\to\Xi_i'\}_{i=1}^{i_0}$ is already an admissible system of coordinates.

Following [G2, Appendix A.5] (to which we refer for more details), we assume moreover that $\bar\Omega$ is a closed smooth subset of a neighboring open n-dimensional admissible

C^∞ manifold Σ, and that a normal coordinate x_n has been chosen on a neighborhood Σ_2' of Γ in Σ (and fits together with an admissible system of local coordinates), such that Σ_2' is identified with $\Gamma\times[-2,2]$, with points (x',x_n), $x'\in\Gamma$, $x_n\in[-2,2]$. (One can obtain by a suitable choice that x_n is a multiple of the usual x_n-coordinate in the patches of the type V_2 and V_4.) Then the trace operators $\gamma_j\colon u\mapsto D^j_{x_n}u(x',0)$ ($j\in\mathbf{N}$) are well-defined.

We furthermore consider C^∞ vector bundles E over $\bar\Omega$, described by finite systems of local trivializations, where the part of the coordinates concerning the manifold form an admissible system. Similar requirements are made for vector bundles F over Γ. We assume that E is the restriction to $\bar\Omega$ of a bundle $\widetilde E$ given on Σ, and that $\widetilde E|_{\Sigma_2'}$ is the lifting of $E|_\Gamma$ to $\Gamma\times[-2,2]$. For short, we shall call such bundles E and F (and the mentioned local trivializations) *admissible.*

Now the spaces $H_p^{s,\mu}(\bar\Omega)$ and, more generally, the spaces of sections of E, called $H_p^{s,\mu}(\bar\Omega,E)$ or just $H_p^{s,\mu}(E)$, are defined by use of a partition of unity as described in [G2, A.5] (with the obvious modifications), and so are the $B_p^{s,\mu}$ spaces. The various uniform mapping properties shown above for spaces over $\mathbf{R}^n$ and $\overline{\mathbf{R}}^n_+$ ((1.6), (1.16), (1.20), (1.21), (1.28–30)) carry over to this situation. Also (1.33) and (1.34) can be generalized, cf. [G2, 3, 5] and [R-S2]; and (1.35) extends.

A C^∞ function on $\bar\Omega$ or Γ will be called an admissible C^∞ function when it is bounded with bounded derivatives (when considered in admissible local coordinates); the multiplication by such a function is continuous in the spaces we are considering.

The admissible manifolds include compact manifolds and "exterior domains" and "exterior halfspaces" (complements in $\mathbf{R}^n$ or $\overline{\mathbf{R}}^n_+$ of smooth compact subsets), as well as more complicated manifolds.

In Schrohe [Sc1, 2] there is introduced a concept of unbounded manifold called SG-compatible manifolds, that contains the present type. We observe however that our definitions of *function spaces* (and later *symbols*) are very different from those in [Sc1, 2], since ours do not involve a weight function for $|x|\to\infty$ as in [Sc1, 2]. Furthermore, we shall not play on compactness or Fredholm properties in the study of elliptic problems [G5]; instead invertibility is obtained for sufficiently large values of the parameter μ.

When $\{\varkappa_i\colon\Omega_i\xrightarrow{\sim}\Xi_i\}_{i=1}^{i_0}$ is an admissible system of local coordinates, there exists an associated partition of unity $\{\varphi_i\}_{i=1}^{i_0}$, where the φ_i are admissible C^∞ functions with $\operatorname{supp}\varphi_i\subset\Omega_i$ and $\sum_{i=1}^{i_0}\varphi_i=1$ on $\bar\Omega$. It will be convenient to have a more refined partition of unity (that is just *sub*-ordinate to the cover Ω_i), as used in Seeley [Se3].

LEMMA 1.5. *When $\bar\Omega$ is an admissible manifold, there exists an admissible system of local coordinates $\{\varkappa_i\colon\Omega_i\xrightarrow{\sim}\Xi_i\}_{i=1}^{i_0}$ for which there is a subordinate partition of unity $\{\varrho_j\}_{j=1}^{j_0}$ consisting of admissible C^∞ functions ϱ_j with $\sum_{j\le j_0}\varrho_j=1$, such that for any four integers $j_1,j_2,j_3,j_4\le j_0$ there is an $i\le i_0$ (a function $i=i(j_1,j_2,j_3,j_4)$) of the four* *variables) satisfying* $\operatorname{supp}\varrho_{j_1}\cup\operatorname{supp}\varrho_{j_2}\cup\operatorname{supp}\varrho_{j_3}\cup\operatorname{supp}\varrho_{j_4}\subset\Omega_i$.

For given admissible vector bundles E and E' over $\bar\Omega$, F and F' over Γ, one can likewise choose a system of local coordinates $\{\varkappa_i:\Omega_i\xrightarrow{\sim}\Xi_i\}_{i=1}^{i_0}$ and trivializations $\psi_i: E|_{\Omega_i}\to\Xi_i\times\mathbf{C}^N$, etc., such that there is a subordinate partition of unity $\{\varrho_j\}_{j=1}^{j_0}$ with the above property.

Proof. Consider first the case where $\bar\Omega$ is a compact manifold; we can assume that it is provided with a Riemannian metric. Consider an admissible system of coordinates $\{\varkappa_i:\Omega_i\xrightarrow{\sim}\Xi_i\}_{i=1}^{i_0}$; we assume that the patches Ξ_i are disjoint, and note that they need not be connected sets. By the compactness of $\bar\Omega$, there is a number $\delta>0$ such that any subset of $\bar\Omega$ with geodesic diameter $\leqslant\delta$ is contained in one of the sets Ω_i. Now cover $\bar\Omega$ by a finite system of open balls B_j, $j=1,...,j_0$, of radius $\leqslant\delta/8$. This system has the following 4-*cluster property*: Any four sets B_{j_1}, B_{j_2}, B_{j_3}, B_{j_4} can be grouped in clusters that are mutually disjoint and where each cluster lies in one of the sets Ω_i.

The 4-cluster property is seen as follows: Let $j_1,j_2,j_3,j_4\leqslant j_0$ be given. First adjoin to B_{j_1} those of the B_{j_k}, $k=2,3,4$, that it intersects with; next adjoin to this union those of the remaining sets that it intersects with, and finally do it once more; this gives the first cluster. If any sets are left, repeat the procedure with these (at most three). Now the procedure is repeated with the remaining sets, and so on; this ends after at most four steps. The clusters are clearly mutually disjoint, and by construction, each cluster has diameter $\leqslant\delta$, hence lies in a set Ω_i. (One could similarly obtain covers with an N-cluster property, taking balls of radius $\leqslant\delta/2N$.)

To the original coordinate mappings we now adjoin the following new ones: Assume that B_{j_1}, B_{j_2}, B_{j_3}, B_{j_4} gave rise to the disjoint clusters $U',U'',...$, where $U'\subset\Omega_{i'}$, $U''\subset\Omega_{i''},...$. Then use $\varkappa_{i'}$ on U', $\varkappa_{i''}$ on U'',... (if necessary followed by linear tranformations Φ'',... to separate the images) to define the mapping $\varkappa: U'\cup U''\cup...\xrightarrow{\sim}\varkappa_{i'}(U')\cup\Phi''\varkappa_{i''}(U'')\cup...$. This gives a new (admissible) coordinate mapping, for which $B_{j_1}\cup B_{j_2}\cup B_{j_3}\cup B_{j_4}$ equals the initial set $U'\cup U''\cup...$. In this way, finitely many new coordinate mappings, say $\{\varkappa_i:\Omega_i\xrightarrow{\sim}\Xi_i\}_{i=i_0+1}^{i_1}$, are adjoined to the original ones, and we have established a mapping $(j_1,j_2,j_3,j_4)\mapsto i=i(j_1,j_2,j_3,j_4)$ for which

$$B_{j_1}\cup B_{j_2}\cup B_{j_3}\cup B_{j_4}\subset\Omega_{i(j_1,j_2,j_3,j_4)}.$$

Let $\{\varrho_j\}_{j=1}^{j_0}$ be a partition of unity associated with the cover $\{B_j\}_{j=1}^{j_0}$, then it has the desired property with respect to the system $\{\varkappa_i:\Omega_i\xrightarrow{\sim}\Xi_i\}_{i=1}^{i_1}$.

Now consider a general manifold $\bar\Omega$. Here we can assume that the system of coordinates $\{\varkappa_i:\Omega_i\xrightarrow{\sim}\Xi_i\}_{i=1}^{i_0}$ is such that the Ω_i for $i=1,...,i_0'$, say, lie in $\Omega_{\mathrm{co},1}$ and form a cover of $\bar\Omega_{\mathrm{co},5/4}$, and the Ω_i for $i=i_0'+1,...,i_0$ lie in $\Omega_{\mathrm{bd}}\setminus\Omega_{\mathrm{co},7/4}$ and form a cover of

$\bar{\Omega}_{\rm bd}\setminus\Omega_{{\rm co},6/4}$. Let $\Omega'_i=\Omega_i\cap\bar{\Omega}_{\rm bd}$ for $i=1,\dots,i'_0$. Now since $\{\Omega'_i\}_{i=1}^{i'_0}\cup\{\Omega_i\}_{i=i'_0+1}^{i_0}$ is a relatively open cover of the compact set $\bar{\Omega}_{\rm bd}$, we can choose a cover with small balls $\{B_j\}_{j=1}^{j_0}$ exactly as above, having the 4-cluster property. Here δ can be taken so small that $\bar{\Omega}_{{\rm co},2}\cap\bar{\Omega}_{\rm bd}$ is covered by those B_j that lie in $\Omega_{{\rm co},7/4}$; let us rearrange so that they have the indices $j=1,\dots,j'_0$. For $j\leqslant j'_0$ we define $\tilde{B}_j=\{tx_0\mid t\geqslant t_0,\, t_0x_0\in B_j\}$, cf. (1.40). Now $\{\tilde{B}_j\}_{j=1}^{j'_0}\cup\{B_j\}_{j=j'_0+1}^{j_0}$ is a cover of $\bar{\Omega}$ with the 4-cluster property relative to $\{\Omega_i\}_{i=1}^{i_0}$, so when we adjoin more coordinate mappings as above, a partition of unity associated with $\{\tilde{B}_j\}_{j=1}^{j'_0}\cup\{B_j\}_{j=j'_0+1}^{j_0}$ will be subordinate to $\{\Omega_i\}_{i=1}^{i_1}$ in the desired way. □

1.3. L_p mapping properties of ps.d.o.s

A basic tool in the theory is the generalization of the theorems on L_p continuity (1.1) to variable-coefficient pseudodifferential operators. Recall the general formula for the definition of a ps.d.o. with C^∞ symbol $q(x,\xi)$:

$$(Qu)(x)=(2\pi)^{-n}\int e^{ix\cdot\xi}q(x,\xi)\hat{u}(\xi)\,d\xi=\mathrm{OP}(q(x,\xi))u(x)=q(x,D)u(x), \qquad (1.42)$$

(defined at least for $u\in\mathcal{F}^{-1}C_0^\infty$). Under suitable hypotheses on the symbol, one can define ps.d.o.s with a symbol depending on (y,ξ) instead (briefly called ps.d.o.s "in y-form"), as

$$\mathrm{OP}(q(y,\xi))u(x)=(2\pi)^{-n}\int e^{i(x-y)\cdot\xi}q(y,\xi)u(y)\,dy\,d\xi=\mathrm{OP}(\bar{q}(x,\xi))^*u(x). \qquad (1.43)$$

The relevant operator classes are defined in detail in Section 2 below.

THEOREM 1.6. *There exist finite index sets J and $J'\subset\mathbf{N}^{2n}$ such that the following holds:*

Let H_0 and H_1 be Hilbert spaces, and let $q(x,\xi)$ be an operator valued symbol in $C^\infty(\mathbf{R}^n\times\mathbf{R}^n;\mathcal{L}(H_0,H_1))$ satisfying

$$C(q)\equiv\sup_{\{\alpha,\beta\}\in J,\,x,\xi\in\mathbf{R}^n}\|\langle\xi\rangle^{|\alpha|}D_\xi^\alpha D_x^\beta q(x,\xi)\|_{\mathcal{L}(H_0,H_1)}<\infty. \qquad (1.44)$$

For each $p\in\,]1,\infty[$ there is a constant C_p (independent of q) such that $q(x,D)=\mathrm{OP}(q(x,\xi))$ is continuous from $L_p(\mathbf{R}^n;H_0)$ to $L_p(\mathbf{R}^n;H_1)$ with norm $\leqslant C_pC(q)$.

A similar result is valid with $C(q)<\infty$ replaced by

$$C'(q)\equiv\sup_{\{\alpha,\beta\}\in J',\,x,\xi\in\mathbf{R}^n}\|\langle\xi_1\rangle^{\alpha_1}D_{\xi_1}^{\alpha_1}\dots\langle\xi_n\rangle^{\alpha_n}D_{\xi_n}^{\alpha_n}D_x^\beta q(x,\xi)\|_{\mathcal{L}(H_0,H_1)}<\infty. \qquad (1.45)$$

The results hold also for operators given in y-form $\mathrm{OP}(q(y,\xi))$, *when* $q(y,\xi)$ *satisfies* (1.44) *resp.* (1.45) *with* x *replaced by* y.

Indications of proof. (1.44) and (1.45) are generalizations of, respectively, Mihlin's and Lizorkin's criteria (1.1). For (1.44), the proof of Hörmander [H1, Theorem 3.5], although formulated for the L_2 case, applies to the abovementioned L_p case by use of Mihlin's theorem in a Hilbert space valued variant (cf. [B-L, Theorem 6.1.6], [T1, 2.2.4]); this shows that one can take $J=\{\{\alpha,\beta\}\in\mathbf{N}^{2n}\mid |\alpha|,|\beta|\leqslant n+1\}$. Statements requiring fewer values of β (and of α, when $p\geqslant 2$) can be found e.g. in Nagase [Na]. With J replaced by $\mathbf{N}^{2n}$, the L_p continuity of polyhomogeneous ps.d.o.s is shown in Seeley [Se2], and Beals [B] includes the $S^0_{1,0}$ case. — For (1.45), let us first observe that Lizorkin's constant coefficient result generalizes to Hilbert space valued operators e.g. by the argument given in Shamir [Sh] for reducing Lizorkin's result to a version of Mihlin's result. The present authors have worked out a generalization of the proof of [H1, Theorem 3.5], that shows that one can take $J'=\{\{\alpha,\beta\}\in\mathbf{N}^{2n}\mid\alpha_j\leqslant 2 \text{ for } j=1,\dots,n,\ |\beta|\leqslant n+2\}$ in (1.45) (for lack of space, we shall not include the details here). Yamazaki [Y] shows that one can take $J'=\{\{\alpha,\beta\}\in\mathbf{N}^{2n}\mid\alpha=\{\delta_{ij}\alpha_j\}_{i=1,\dots,n} \text{ with } \alpha_j\leqslant n+1, \text{ for } j=1,\dots,n,\ |\beta|\leqslant 1\}$.

Finally, when an operator is given in y-form $\mathrm{OP}(q(y,\xi))$, we simply use that its adjoint satisfies $\mathrm{OP}(q(y,\xi))^*=\mathrm{OP}(\bar{q}(x,\xi))$ (cf. (1.43)), where $\bar{q}(x,\xi)$ has the required estimates (1.44) resp. (1.45) as an operator from H_1 to H_0 for each (x,ξ). Then, by the preceding results, $\mathrm{OP}(\bar{q}(x,\xi))$ is continuous from $L_p(\mathbf{R}^n;H_1)$ to $L_p(\mathbf{R}^n;H_0)$ for all $p\in\,]1,\infty[$ with a bound $C_pC(q)$ resp. $C_pC'(q)$, so it follows by duality that $\mathrm{OP}(q(y,\xi))$ has the asserted continuity properties. □

Remark 1.7. The theorem does not extend to the general case where H_0 and H_1 are replaced by Banach spaces. As a simple illustration, we mention the fact that the Poisson operator K with symbol $\langle\xi'\rangle^{1/p}(\langle\xi'\rangle+i\xi_n)^{-1}$ (and symbol-kernel $\langle\xi'\rangle^{1/p}H(x_n)e^{-\langle\xi'\rangle x_n}$, $H(x_n)=1_{\{x_n>0\}}$) is *not* continuous from $L_p(\mathbf{R}^{n-1})$ to $L_p(\mathbf{R}^n_+)$ when $p<2$ and $n>1$, just continuous from the strict subspace $B^0_p(\mathbf{R}^{n-1})$ to $L_p(\mathbf{R}^n_+)$, cf. [G3, Remark 3.3]. For $n=1$, K is continuous from $\mathbf{C}$ to $L_p(\mathbf{R}_+)$, and if Theorem 1.6 (in dimension $n-1$) could be used for the mapping $q(x',\xi')\colon v\mapsto\langle\xi'\rangle^{1/p}e^{-\langle\xi'\rangle x_n}v$, with $H_0=\mathbf{C}$, $H_1=L_p(\mathbf{R}_+)$, the L_p-continuity of K for $n>1$ would follow.

In our approach to L_p estimates for the boundary operators in the parameter-dependent calculus, we apply the above theorem to symbols depending on the tangential variables, valued in spaces of functions (or distributions) in the normal variable. A fundamental idea is to choose these spaces, that must be Hilbert spaces, in such a way that they give spaces contained in L_p by interpolation. Here we use the L_2 Sobolev spaces over $\mathbf{R}_+$, and the weighted L_2 spaces:

$$L_2(\mathbf{R}_+, x_n^\delta) = \{v \in \mathcal{D}'(\mathbf{R}_+) \mid x_n^\delta v(x_n) \in L_2(\mathbf{R}_+)\}, \quad \delta \in \mathbf{R}. \tag{1.46}$$

Note that the dual space of $L_2(\mathbf{R}_+, x_n^\delta)$ (in relation to the sesquilinear duality $\int_{\mathbf{R}_+} u(x_n)\bar{v}(x_n)\,dx_n$) is

$$L_2(\mathbf{R}_+, x_n^\delta)^* = L_2(\mathbf{R}_+, x_n^{-\delta}). \tag{1.47}$$

More precisely, we shall use:

THEOREM 1.8. (1) *If* $1<p\leqslant 2$ *and* $\delta' < \frac{1}{p} - \frac{1}{2} < \delta$, *then with* $\theta = (\frac{1}{p} - \frac{1}{2} - \delta')/(\delta - \delta')$,

$$\begin{aligned} &(L_2(\mathbf{R}_+, x_n^{\delta'}), L_2(\mathbf{R}_+, x_n^\delta))_{\theta,p} \subset L_p(\mathbf{R}_+), \\ &(H_{2;0}^{-\delta'}(\overline{\mathbf{R}}_+), H_{2;0}^{-\delta}(\overline{\mathbf{R}}_+))_{\theta,p} \supset L_p(\mathbf{R}_+). \end{aligned} \tag{1.48}$$

(2) *If* $2\leqslant p<\infty$ *and* $\delta' < \frac{1}{2} - \frac{1}{p} < \delta$, *then with* $\theta = (\frac{1}{2} - \frac{1}{p} - \delta')/(\delta - \delta')$,

$$\begin{aligned} &(H_2^{\delta'}(\overline{\mathbf{R}}_+), H_2^{\delta}(\overline{\mathbf{R}}_+))_{\theta,p} \subset L_p(\mathbf{R}_+), \\ &(L_2(\mathbf{R}_+, x_n^{-\delta'}), L_2(\mathbf{R}_+, x_n^{-\delta}))_{\theta,p} \supset L_p(\mathbf{R}_+). \end{aligned} \tag{1.49}$$

(*The inclusions are identities when* $p=2$.)

Proof. The identities are classical for $p=2$; it is the case $p\neq 2$ that demands a special argumentation.

The first line in (1) follows for general δ' from Gilbert [Gi]. By Hölder's inequality,

$$\|f\|_{L_p(\mathbf{R}_+)}^p = \sum_{k\in\mathbf{Z}} \int_{2^{k-1}\leqslant x_n^{\delta'-\delta}\leqslant 2^k} x_n^{-p\delta'} |x_n^{\delta'} f(x_n)|^p\,dx_n$$

$$\leqslant \sum_{k\in\mathbf{Z}} \Big[\int_{2^{k-1}\leqslant x_n^{\delta'-\delta}\leqslant 2^k} x_n^{-2p\delta'/(2-p)}\,dx_n\Big]^{(2-p)/2} \Big[\int_{2^{k-1}\leqslant x_n^{\delta'-\delta}\leqslant 2^k} |x_n^{\delta'} f(x_n)|^2\,dx_n\Big]^{p/2}$$

$$\leqslant c \sum_{k\in\mathbf{Z}} \Big[2^{-k\theta}\Big(\int_{2^{k-1}\leqslant x_n^{\delta'-\delta}\leqslant 2^k} |x_n^{\delta'} f(x_n)|^2\,dx_n\Big)^{1/2}\Big]^p.$$

The last expression gives by [Gi, Theorem (3.7)(i)] a norm on the interpolation space on the left hand side of the first line in (1.48), and hence this inclusion follows. (We observe that in the result of Gilbert, $2^{k-1} < \omega(x) \leqslant 2^k$ should have been written $2^{k-1} < \omega(x)^{-1} \leqslant 2^k$, cf. [Gi, Theorem (3.3)].) The second line in (1.49) follows by duality. When $\delta'=0$, a simpler argument suffices, where $\mathbf{R}_+$ is just divided into $]0,a[$ and $[a,\infty[$, Hölder's inequality is applied, and a is chosen so that the two terms balance.

To show the first line in (1.49), we use that

$$(H_2^{\delta'}(\overline{\mathbf{R}}_+), H_2^{\delta}(\overline{\mathbf{R}}_+))_{\theta,p} = B_{2,p}^{1/2-1/p}(\overline{\mathbf{R}}_+) \subset H_p^0(\overline{\mathbf{R}}_+) = L_p(\mathbf{R}_+),$$

where the first identity is accounted for in [T1, 2.10.1], and the inclusion (which is an identity for $p=2$) follows for $p>2$ from [T1, 2.8.1 (18)], cf. also [T1, 4.6.1]. The second line in (1.48) follows by duality. □

As an immediate consequence we get (using e.g. [B-L, 5.8.6]):

COROLLARY 1.9. (1) *If* $1<p\leqslant 2$ *and* $\delta'<\frac{1}{p}-\frac{1}{2}<\delta$, *then with* $\theta=(\frac{1}{p}-\frac{1}{2}-\delta')/(\delta-\delta')$,

$$\begin{gathered}(L_p(\mathbf{R}^{n-1};L_2(\mathbf{R}_+,x_n^{\delta'})),\,L_p(\mathbf{R}^{n-1};L_2(\mathbf{R}_+,x_n^{\delta})))_{\theta,p}\subset L_p(\mathbf{R}^n_+),\\ (L_p(\mathbf{R}^{n-1};H_{2;0}^{-\delta'}(\overline{\mathbf{R}}_+)),\,L_p(\mathbf{R}^{n-1};H_{2;0}^{-\delta}(\overline{\mathbf{R}}_+)))_{\theta,p}\supset L_p(\mathbf{R}^n_+).\end{gathered}\tag{1.50}$$

(2) *If* $2\leqslant p<\infty$ *and* $\delta'<\frac{1}{2}-\frac{1}{p}<\delta$, *then with* $\theta=(\frac{1}{2}-\frac{1}{p}-\delta')/(\delta-\delta')$,

$$\begin{gathered}(L_p(\mathbf{R}^{n-1};H_2^{\delta'}(\overline{\mathbf{R}}_+)),\,L_p(\mathbf{R}^{n-1};H_2^{\delta}(\overline{\mathbf{R}}_+)))_{\theta,p}\subset L_p(\mathbf{R}^n_+),\\ (L_p(\mathbf{R}^{n-1};L_2(\mathbf{R}_+,x_n^{-\delta'})),\,L_p(\mathbf{R}^{n-1};L_2(\mathbf{R}_+,x_n^{-\delta})))_{\theta,p}\supset L_p(\mathbf{R}^n_+).\end{gathered}\tag{1.51}$$

(*The inclusions are identities when* $p=2$.)

2. Uniformly estimated symbols

2.1. Symbol spaces

We shall now supplement the symbol spaces introduced in [G2] with spaces of symbols satisfying estimates that are uniform in x resp. x', following the pattern set up for ps.d.o.s in Hörmander [H3, Chapter 18]. Besides that this leads to operators with convenient continuity properties in global Sobolev spaces relative to $\mathbf{R}^n_+$, this has the advantage that the rules of calculus can be stated in a more exact form than for the symbols satisfying local estimates: To take an adjoint or to compose two operators leads to an operator with a precisely defined symbol in the appropriate space, not just defined modulo so-called negligible operators. (Recall from [G2] that there are many different kinds of negligible operators involved in the local parameter-dependent calculus: one for each regularity number and for each operator type, and some additional ones.)

The symbols will depend on a space variable generally called $X\in\mathbf{R}^{n_1}$ and later specialized to be $x\in\mathbf{R}^n$, $x'\in\mathbf{R}^{n-1}$, $(x,y)\in\mathbf{R}^{2n}$ or $(x',y')\in\mathbf{R}^{2n-2}$, etc.

To avoid lengthy repetitions, we simply use the notation from [G2] and refer the reader to the details there, recalling only a few conventions, such as

$$\begin{gathered}D_{x_j}=-i\partial_{x_j};\quad \overline{D}_{x_j}=+i\partial_{x_j};\\ (\mathcal{F}f)(\xi)=\hat{f}(\xi)=\int e^{-ix\cdot\xi}f(x)\,dx;\quad (\overline{\mathcal{F}}f)(\xi)=\int e^{+ix\cdot\xi}f(x)\,dx;\\ a_\pm=\max\{\pm a,0\};\quad \langle\xi,\mu\rangle=(|\xi|^2+\mu^2+1)^{1/2},\\ \varrho(\xi,\mu)=\langle\xi\rangle/\langle\xi,\mu\rangle,\quad \varrho(\xi',\mu)=\langle\xi'\rangle/\langle\xi',\mu\rangle,\quad \varkappa(\xi',\mu)=\langle\xi',\mu\rangle.\end{gathered}\tag{2.1}$$

Occasionally, it can be useful to redefine $\langle\xi,\mu\rangle$ as

$$\langle\xi,\mu\rangle=(|\xi|^{2d}+\mu^{2d}+1)^{1/2d} \tag{2.2}$$

or

$$\langle\xi,\mu\rangle=(|\xi|^{2d}+\mu^{2d})^{1/2d} \quad \text{for } |(\xi,\mu)|\geqslant 1 \tag{2.3}$$

(extended to be smooth positive), where d is a fixed positive integer; these are useful for holomorphic extensions in μ and homogeneous symbols, e.g. in connection with resolvent studies ([G2, 5]). Observe also:

$$\langle\xi\rangle^{\nu}\langle\xi,\mu\rangle^{d-\nu}+\langle\xi,\mu\rangle^{d}=(\varrho(\xi,\mu)^{\nu}+1)\langle\xi,\mu\rangle^{d}\simeq\begin{cases}\langle\xi,\mu\rangle^{d} & \text{when } \nu\geqslant 0,\\ \langle\xi\rangle^{\nu}\langle\xi,\mu\rangle^{d-\nu} & \text{when } \nu\leqslant 0.\end{cases} \tag{2.4}$$

We shall use the conventions: $\dot{\leqslant}$ (resp. $\dot{\geqslant}$) means "$\leqslant$ (resp. $\geqslant$) a constant (independent of the space variable) times". Moreover, $\doteq$ means that both $\dot{\leqslant}$ and $\dot{\geqslant}$ hold. The constants vary from case to case.

Recall (e.g. from [G2, 2.2]) that for $r\in\mathbf{Z}$, $\mathcal{H}_{r-1}$ is the space of functions $f(\xi_n)\in C^{\infty}(\mathbf{R})$ that have an asymptotic expansion $\sum_{-\infty<j\leqslant r-1}s_j\xi_n^j$ for $|\xi_n|\to\infty$; and $\mathcal{H}=\bigcup_{r\in\mathbf{Z}}\mathcal{H}_{r-1}$. Moreover, $\mathcal{H}^+$ is the subspace of $\mathcal{H}_{-1}$ consisting of functions extending holomorphically into $\mathbf{C}_-=\{\xi_n\in\mathbf{C}\,|\,\mathrm{Im}\,\xi_n<0\}$ with the same asymptotics there, and $\mathcal{H}^-$ is the direct sum of the space of complex conjugates of $\mathcal{H}^+$ and the space $\mathbf{C}[\xi_n]$ of polynomials. For a simple formulation of symbols of negative class it is convenient to introduce the notation

$$\mathcal{H}^-_{r-1}=\mathcal{H}^-\cap\mathcal{H}_{r-1}, \quad \mathcal{H}^+_{r-1}=\mathcal{H}^+\cap\mathcal{H}_{r-1}, \quad \text{for } r\in\mathbf{Z}. \tag{2.5}$$

We recall that $\mathcal{H}=\mathcal{H}_{-1}\dot{+}\mathbf{C}[\xi_n]$, $\mathcal{H}=\mathcal{H}^+\dot{+}\mathcal{H}^-$ (direct sums) and $\mathcal{H}_{-1}=\mathcal{H}^+\oplus\mathcal{H}^-_{-1}$ (orthogonal direct sum in $L_2(\mathbf{R})$); the projections onto $\mathcal{H}_{-1}$, $\mathcal{H}^+$, $\mathcal{H}^-$ and $\mathcal{H}^-_{-1}$ being denoted h_{-1}, h^+, h^- resp. h^-_{-1}.

We denote by $S^d_{1,0}(\mathbf{R}^n\times\mathbf{R}^n)$ the space called S^d or $S^d(\mathbf{R}^n\times\mathbf{R}^n)$ in [H3, Chapter 18], consisting of symbols $p(x,\xi)\in C^{\infty}(\mathbf{R}^n\times\mathbf{R}^n)$ satisfying the estimates (on $\mathbf{R}^n\times\mathbf{R}^n$):

$$|D_x^{\beta}D_{\xi}^{\alpha}p(x,\xi)|\,\dot{\leqslant}\,\langle\xi\rangle^{d-|\alpha|} \quad \text{for all } \alpha,\beta\in\mathbf{N}^n. \tag{2.6}$$

The subspace of polyhomogeneous symbols (called $S^d_{\mathrm{phg}}(\mathbf{R}^n\times\mathbf{R}^n)$ in [H3, Chapter 18]) will here be denoted $S^d(\mathbf{R}^n\times\mathbf{R}^n)$. The definitions generalize immediately to the case where $x\in\mathbf{R}^n$ is replaced by by $X\in\mathbf{R}^{n_1}$. Now let us define parameter-dependent symbols:

Definition 2.1. *Let d and $\nu\in\mathbf{R}$.*

(1) *The space* $S^{d,\nu}_{1,0}(\mathbf{R}^{n_1}\times\overline{\mathbf{R}}^{n+1}_+)$ *of X-uniformly estimated (or just: uniformly estimated) pseudodifferential symbols of order d and regularity* ν *consists of the* C^∞ *functions* $p(X,\xi,\mu)$ *on* $\overline{\mathbf{R}}^{n_1+n+1}_+$ *satisfying the estimates*

$$
\begin{aligned}
|D^\beta_X D^\alpha_\xi D^j_\mu p(X,\xi,\mu)| &\dot{\leqslant} (\langle\xi\rangle^{\nu-|\alpha|}\langle\xi,\mu\rangle^{d-\nu-j}+\langle\xi,\mu\rangle^{d-|\alpha|-j}) \\
&= (\varrho(\xi,\mu)^{\nu-|\alpha|}+1)\langle\xi,\mu\rangle^{d-|\alpha|-j},
\end{aligned}
\tag{2.7}
$$

for all indices $\alpha\in\mathbf{N}^n$, $\beta\in\mathbf{N}^{n_1}$, $j\in\mathbf{N}$.

(2) *The space* $S^{d,\nu}(\mathbf{R}^{n_1}\times\overline{\mathbf{R}}^{n+1}_+)$ *of uniformly estimated polyhomogeneous ps.d.o. symbols of order d and regularity* ν *consists of the symbols* $p\in S^{d,\nu}_{1,0}(\mathbf{R}^{n_1}\times\overline{\mathbf{R}}^{n+1}_+)$ *that moreover have asymptotic expansions* $p\sim\sum_{l\in\mathbf{N}}p_{d-l}$, *in the sense that* $p-\sum_{l<N}p_{d-l}$ *belongs to* $S^{d-N,\nu-N}_{1,0}(\mathbf{R}^{n_1}\times\overline{\mathbf{R}}^{n+1}_+)$ *for all* $N\in\mathbf{N}$, *and* $p_{d-l}(X,\xi,\mu)$ *is homogeneous of degree* $d-l$ *in* (ξ,μ) *for* $|\xi|\geqslant 1$.

(3) *A symbol* $p(x,y,\xi,\mu)\in S^{d,\nu}_{1,0}(\mathbf{R}^{2n}\times\overline{\mathbf{R}}^{n+1}_+)$ *is said to satisfy the uniform two-sided transmission condition at* $x_n=y_n=0$, *when p and its derivatives at* $x_n=y_n=0$ *are in* $\mathcal{H}$ *as functions of* ξ_n, *in such a way that the estimates required in connection with the asymptotic expansions in* ξ_n *(cf.* [G2, Definition 2.2.7]*) hold with constants independent of* (x',y'). *This can also be expressed as the property that for all indices* $j,N,N'\in\mathbf{N}$ *and* $\alpha,\beta,\gamma\in\mathbf{N}^n$, *the restrictions to* $z_n\in\mathbf{R}_\pm$ *of the inverse Fourier transforms,*

$$
r^\pm_{z_n} z_n^N D^{N'}_{z_n}\mathcal{F}^{-1}_{\xi_n\to z_n} D^\beta_x D^\gamma_y D^\alpha_\xi D^j_\mu p(x',0,y',0,\xi',\xi_n,\mu),
\tag{2.8}
$$

are bounded functions of $(x',y',z_n)\in\mathbf{R}^{n-1}\times\mathbf{R}^{n-1}\times\mathbf{R}_\pm$ *(hence extend to* C^∞ *functions of* $(x',y',z_n)\in\overline{\mathbf{R}}^{2n-1}_\pm$*). The space of such symbols is denoted* $S^{d,\nu}_{1,0,\mathrm{uttr}}(\mathbf{R}^{2n}\times\overline{\mathbf{R}}^{n+1}_+)$, *and the subspace of symbols that moreover are polyhomogeneous is denoted* $S^{d,\nu}_{\mathrm{uttr}}(\mathbf{R}^{2n}\times\overline{\mathbf{R}}^{n+1}_+)$. *Similar definitions are made when p is independent of y or x (then* $\mathbf{R}^{2n}$ *is replaced by* $\mathbf{R}^n$*).*

The definitions (1) and (2) are straightforward generalizations of the definitions in [G2, Sections 2.1 and 2.2]. As for the transmission condition, a further analysis is given in Grubb and Hörmander [G-H] and [G4, 1.3]. In order for a ps.d.o. $P=\mathrm{OP}(p(x,\xi))$ to have what was originally called the transmission property (on $\mathbf{R}^n_+$), namely that $P_{\mathbf{R}^n_+}$ preserves the property of being C^∞ on $\overline{\mathbf{R}}^n_+$, it is necessary and sufficient that $r^+_{z_n} z_n^N\mathcal{F}^{-1}_{\xi_n\to z_n}D^\beta_x D^\alpha_\xi p(x',0,\xi)$ is C^∞ for $(x',z_n)\in\mathbf{R}^{n-1}\times\overline{\mathbf{R}}_+$, for all indices (no requirement is needed for $z_n<0$). However, the full pseudodifferential boundary operator calculus needs the preservation of C^∞ property for both P and its adjoint, or equivalently, the preservation of C^∞ property for P on both $\mathbf{R}^n_+$ and $\mathbf{R}^n_-$. But here, when d is integer and P is polyhomogeneous (the case mainly considered in [G2]), one can show that the two-sided preservation of C^∞ property (on $\mathbf{R}^n_+$ and $\mathbf{R}^n_-$) is equivalent with the one-sided preservation of C^∞ property (on $\mathbf{R}^n_+$). The transmission condition used in [G2] was indicated there by a subscript 'tr'. In [G-H], spaces of symbols with the one-sided (resp. one-sided uniform) transmission property were indicated by a subscript 'tr' (resp. 'utr'). In the present work, we allow noninteger d in the study of mapping properties, and we indicate the spaces of symbols satisfying the *uniform two-sided transmission condition* by the subscript 'uttr', to avoid conflict with earlier conventions. Since this is the condition used from now on, we often just say that such symbols satisfy "the uniform transmission condition".

For fixed μ, Definition 2.1 (3) gives the space $S^d_{1,0,\mathrm{uttr}}(\mathbf{R}^{2n}\times\mathbf{R}^n)$ of parameter-independent ps.d.o. symbols of order d satisfying the uniform two-sided transmission condition.

Now let us define the uniform versions of the boundary operator symbol spaces.

DEFINITION 2.2. *Let $d,\nu\in\mathbf{R}$ and $r\in\mathbf{Z}$. The spaces $S^{d,\nu}_{1,0}(\mathbf{R}^{n_1}\times\overline{\mathbf{R}}^n_+,\mathcal{K})$ ($\mathcal{K}=\mathcal{H}^+$, $\mathcal{H}^-_{r-1}$ resp. $\mathcal{H}^+\hat{\otimes}\mathcal{H}^-_{r-1}$) of X-uniformly estimated Poisson, trace resp. singular Green symbols of degree d, class r and regularity ν consist of the symbols in $S^{d,\nu}_{1,0}(\mathbf{R}^{n_1},\overline{\mathbf{R}}^n_+,\mathcal{K})$ (cf. [G2, Section 2.3]) satisfying the estimates listed there with constants independent of $X\in\mathbf{R}^{n_1}$.*

Expressed in detail, with $\varrho=\varrho(\xi',\mu)$, $\varkappa=\langle\xi',\mu\rangle$ (cf. (2.1)):

(1) *The space $S^{d,\nu}_{1,0}(\mathbf{R}^{n_1}\times\overline{\mathbf{R}}^n_+,\mathcal{H}^+)$ of uniformly estimated Poisson symbols of order $d+1$ (degree d) and regularity ν consists of the functions $k(X,\xi,\mu)$ lying in $\mathcal{H}^+$ with respect to ξ_n and satisfying estimates*

$$\|h^+D^\beta_X D^\alpha_{\xi'} D^m_{\xi_n}\xi_n^{m'} D^j_\mu k(X,\xi,\mu)\|_{L_{2,\xi_n}} \dot{\leqslant} (\varrho^{\nu-|\alpha|-[m-m']_+}+1)\varkappa^{d+1/2-|\alpha|-m+m'-j}, \quad (2.9)$$

for all indices.

(2) *The space $S^{d,\nu}_{1,0}(\mathbf{R}^{n_1}\times\overline{\mathbf{R}}^n_+,\mathcal{H}^-_{r-1})$ of uniformly estimated trace symbols of order and degree d, regularity ν and class r consists of the functions $t(X,\xi,\mu)$ of the form*

$$t(X,\xi,\mu)=\sum_{0\leqslant j<r_+} s_j(X,\xi',\mu)\xi_n^j+t'(X,\xi,\mu), \quad (2.10)$$

where $s_j(X,\xi',\mu)\in S^{d-j,\nu}_{1,0}(\mathbf{R}^{n_1}\times\overline{\mathbf{R}}^n_+)$ and $t'(X,\xi,\mu)$ is in $\mathcal{H}^-_{\min\{-1,r-1\}}$ as a function of ξ_n, with estimates

$$\|h^-_{-1}D^\beta_X D^\alpha_{\xi'} D^m_{\xi_n}\xi_n^{m'} D^j_\mu t'(X,\xi,\mu)\|_{L_{2,\xi_n}} \dot{\leqslant} (\varrho^{\nu-|\alpha|-[m-m']_+}+1)\varkappa^{d+1/2-|\alpha|-m+m'-j}, \quad (2.11)$$

for all indices. (The space $S^{d,\nu}_{1,0}(\mathbf{R}^{n_1}\times\overline{\mathbf{R}}^n_+,\mathcal{H}^+_{r-1})$ is defined similarly.)

(3) *The space $S^{d,\nu}_{1,0}(\mathbf{R}^{n_1}\times\overline{\mathbf{R}}^n_+,\mathcal{H}^+\hat{\otimes}\mathcal{H}^-_{r-1})$ of uniformly estimated singular Green symbols of order $d+1$ (degree d), regularity ν and class r consists of the functions*

$g(X,\xi,\eta_n,\mu)$ *of the form*

$$g(X,\xi,\eta_n,\mu)=\sum_{0\leqslant j<r_+} k_j(X,\xi,\mu)\eta_n^j+g'(X,\xi,\eta_n,\mu), \tag{2.12}$$

where $k_j(X,\xi,\mu)\in S_{1,0}^{d-j,\nu}(\mathbf{R}^{n_1}\times\overline{\mathbf{R}}^n_+,\mathcal{H}^+)$ *and* $g'(X,\xi,\eta_n,\mu)$ *is in* $\mathcal{H}^+\widehat{\otimes}\mathcal{H}^-_{\min\{-1,r-1\}}$ *as a function of* (ξ_n,η_n)*, with estimates*

$$\begin{aligned}\|h^+_{\xi_n}h^-_{-1,\eta_n}D^\beta_X D^\alpha_{\xi'}D^k_{\xi_n}\xi_n^{k'}D^m_{\eta_n}\eta_n^{m'}D^j_\mu g'(X,\xi,\eta_n,\mu)\|_{L_{2,\xi_n,\eta_n}(\mathbf{R}^2)}\\ \dot{\leqslant}\,(\varrho^{\nu-|\alpha|-[k-k']_+-[m-m']_+}+1)\varkappa^{d+1/2-|\alpha|-k+k'-m+m'-j},\end{aligned} \tag{2.13}$$

for all indices.

(4) *The spaces* $S^{d,\nu}(\mathbf{R}^{n_1}\times\overline{\mathbf{R}}^n_+,\mathcal{K})$ *of uniformly estimated polyhomogeneous ps.d.o. symbols of degree* d *and regularity* ν *consist of the symbols* $f\in S^{d,\nu}_{1,0}(\mathbf{R}^{n_1}\times\overline{\mathbf{R}}^n_+,\mathcal{K})$ *that moreover have asymptotic expansions* $f\sim\sum_{l\in\mathbf{N}}f_{d-l}$*, in the sense that* $f-\sum_{l<N}f_{d-l}\in S^{d-N,\nu-N}_{1,0}(\mathbf{R}^{n_1}\times\overline{\mathbf{R}}^n_+,\mathcal{K})$ *for all* $N\in\mathbf{N}$*, and* $f_{d-l}(X,\xi,\mu)$ *resp.* $f_{d-l}(X,\xi,\eta_n,\mu)$ *is homogeneous of degree* $d-l$ *in* (ξ,μ) *resp.* (ξ,η_n,μ) *for* $|\xi'|\geqslant 1$.

(5) *There is a similar convention for the associated spaces of symbol-kernels* $\tilde{k}=\mathcal{F}^{-1}_{\xi_n\to x_n}k$, $\tilde{t}=\overline{\mathcal{F}}^{-1}_{\xi_n\to x_n}t$ *and* $\tilde{g}=\mathcal{F}^{-1}_{\xi_n\to x_n}\overline{\mathcal{F}}^{-1}_{\eta_n\to y_n}g$ *(cf.* (2.1)*) when* $r\leqslant 0$*; here* $\mathcal{K}=\mathcal{H}^+$, $\mathcal{H}^-_{-1}$ *resp.* $\mathcal{H}^+\widehat{\otimes}\mathcal{H}^-_{-1}$ *is replaced by* $\mathcal{S}(\overline{\mathbf{R}}_+)$, $\mathcal{S}(\overline{\mathbf{R}}_+)$ *resp.* $\mathcal{S}(\overline{\mathbf{R}}^2_{++})$ *(with* $\mathbf{R}^2_{++}=\mathbf{R}_+\times\mathbf{R}_+$*), and the estimates are replaced by the following, where* $\tilde{f}=\tilde{k}$ *or* $\tilde{t}'$*,*

$$\begin{aligned}&\|D^\beta_X x_n^m D^{m'}_{x_n}D^\alpha_{\xi'}D^j_\mu\tilde{f}(X,x_n,\xi',\mu)\|_{L_{2,x_n}(\mathbf{R}_+)}\,\dot{\leqslant}\,(\varrho^{\nu-[m-m']_+-|\alpha|}+1)\varkappa^{d+1/2-m+m'-|\alpha|-j},\\ &\|D^\beta_X x_n^k D^{k'}_{x_n}y_n^m D^{m'}_{y_n}D^\alpha_{\xi'}D^j_\mu\tilde{g}'(X,x_n,y_n,\xi',\mu)\|_{L_{2,x_n,y_n}(\mathbf{R}^2_{++})}\\ &\qquad\dot{\leqslant}\,(\varrho^{\nu-[k-k']_+-[m-m']_+-|\alpha|}+1)\varkappa^{d+1-k+k'-m+m'-|\alpha|-j}.\end{aligned} \tag{2.14}$$

The parameter-independent boundary symbol spaces (generalizing [BM2]) have a simpler definition, without the powers of ϱ and without reference to a ν. To save space we just write:

DEFINITION 2.3. *The spaces of uniformly* X*-estimated boundary symbols of degree* d *and class* r *are defined as in Definition* 2.2*, considered for a fixed* μ.

For the uniformly estimated symbols there are the same inclusions of the parameter-independent spaces in the parameter-dependent spaces as described in [G2, Proposition 2.3.14]. The definition of symbols of negative class is consistent with that of [G3, 4].

We now have to show that the calculus for operators defined from these symbols works in a precise way, as indicated in the introduction. In principle, this requires going through the whole development of [G2, Chapter 2] for the new symbol classes; however, much of what has to be done is so close to the development there that it suffices to give some indications, drawing also on the presentation in [H3].

2.2. Composition rules for ps.d.o.s

Let $p(x,\xi,\mu)\in S^{d,\nu}_{1,0}(\mathbf{R}^n\times\overline{\mathbf{R}}^{n+1}_+)$. For each $\mu\in\overline{\mathbf{R}}_+$ it follows from [H3, 18.1] that $P_\mu=\mathrm{OP}(p(x,\xi,\mu))=p(x,D,\mu)$, defined by

$$P_\mu u(x)=\mathrm{OP}(p(x,\xi,\mu))u(x)=(2\pi)^{-n}\int e^{ix\cdot\xi}p(x,\xi,\mu)\hat{u}(\xi)\,d\xi, \tag{2.15}$$

maps $\mathcal{S}(\mathbf{R}^n)$ into itself, extending to a continuous mapping of $\mathcal{S}'(\mathbf{R}^n)$ into itself, with Schwartz kernel $\mathcal{K}_P(x,y,\mu)$ related to $p(x,\xi,\mu)$ by

$$\begin{gathered}\mathcal{K}_P(x,y,\mu)=(2\pi)^{-n}\int e^{i(x-y)\cdot\xi}p(x,\xi,\mu)\,d\xi \\ \text{(identifiable with } \mathcal{F}^{-1}_{\xi\to z}p(x,\xi,\mu)|_{z=x-y});\end{gathered} \tag{2.16}$$

$$p(x,\xi,\mu)=\mathcal{F}_{z\to\xi}\mathcal{K}_P(x,x-z,\mu)\ \Big(=\int e^{-iz\cdot\xi}\mathcal{K}_P(x,x-z,\mu)\,dz\Big). \tag{2.17}$$

The integral in (2.16) is an oscillatory integral, cf. [H3, 7.8], it is the limit for $\varepsilon\to 0$ of the expressions obtained by multiplying the integrand by a function $\chi(\varepsilon\xi)$, $\chi\in C_0^\infty(\mathbf{R}^n)$ with $\chi=1$ on a neighborhood of 0 (cf. also (2.22) below). Operators and symbols of order $-\infty$ will be called negligible.

Recall from [G2] the simple inequalities:

$$\langle\xi\rangle^\nu\langle\xi,\mu\rangle^{-\nu'}\ \dot{\leqslant}\ \langle\xi\rangle^{\nu+|\nu'|}\langle\mu\rangle^{-\nu'};\quad \langle\xi\rangle^\sigma\langle\mu\rangle^{-\nu'}\ \dot{\leqslant}\ \langle\xi\rangle^{\sigma+|\nu'|}\langle\xi,\mu\rangle^{-\nu'}; \tag{2.18}$$

showing how a decrease for $|\mu|\to\infty$ carries over to a decrease for $|(\xi,\mu)|\to\infty$ and vice versa (with a certain loss of precision in the ξ-variable). They can be used, together with elementary properties of the Fourier transform, to show the following characterization of the present negligible symbols of regularity ν' in terms of kernel properties:

LEMMA 2.4. *If* $p(x,\xi,\mu)\in S^{-\infty,\nu'-\infty}(\mathbf{R}^n\times\overline{\mathbf{R}}^{n+1}_+)=\bigcap_N S^{-N,\nu'-N}_{1,0}(\mathbf{R}^n\times\overline{\mathbf{R}}^{n+1}_+)$, *i.e.,*

$$|D^\beta_x D^\alpha_\xi D^j_\mu p(x,\xi,\mu)|\ \dot{\leqslant}\ \langle\xi\rangle^{-N}\langle\mu\rangle^{-\nu'-j}\quad \textit{for all } \alpha,\beta\in\mathbf{N}^n,\ j,N\in\mathbf{N}, \tag{2.19}$$

then

$$|D^\beta_x D^\gamma_z D^j_\mu\mathcal{K}_P(x,x-z,\mu)|\ \dot{\leqslant}\ \langle z\rangle^{-N'}\langle\mu\rangle^{-\nu'-j}\quad \textit{for all } \gamma,\beta\in\mathbf{N}^n,\ j,N'\in\mathbf{N}. \tag{2.20}$$

Conversely, if $\mathcal{K}_P(x,y,\mu)$ *is a function satisfying* (2.20), *then the function* p *derived from it by* (2.17) *satisfies* (2.19), *hence is a symbol in* $S^{-\infty,\nu'-\infty}(\mathbf{R}^n\times\overline{\mathbf{R}}^{n+1}_+)$ *defining the operator with kernel* $\mathcal{K}_P$. *(For fixed* μ*, this characterizes the negligible ps.d.o.s in the standard uniform calculus.)*

The proof is straightforward and consists of showing that each estimate in (2.20) follows from a certain finite set of estimates (2.19) and vice versa, where the number of estimates grows with the size of the indices. Note the advantage of the present calculus: The operators with C^∞ kernels (satisfying suitable estimates in x, y and μ) belong to the calculus, and do not have to be treated as residual classes as in [G2].

Proposition 18.1.3 in [H3] extends readily to the parameter-dependent situation, showing that for any sequence of symbols $p_{d-l}\in S^{d-l,\nu-l}_{1,0}(\mathbf{R}^{n_1}\times\overline{\mathbf{R}}^{n+1}_+)$ ($l\in\mathbf{N}$) there exists a symbol $p\in S^{d,\nu}_{1,0}(\mathbf{R}^{n_1}\times\overline{\mathbf{R}}^{n+1}_+)$ such that $p-\sum_{l<M}p_{d-l}\in S^{d-M,\nu-M}_{1,0}(\mathbf{R}^{n_1}\times\overline{\mathbf{R}}^{n+1}_+)$ for any $M\in\mathbf{N}$. We then say that $p\sim\sum_{l\in\mathbf{N}}p_{d-l}$.

Now it is also of interest to consider ps.d.o.s represented by formulas (as in [H2])

$$P_\mu u(x)=(2\pi)^{-n}\iint e^{i(x-y)\cdot\xi}p(x,y,\xi,\mu)u(y)\,dy\,d\xi=\mathrm{OP}(p(x,y,\xi,\mu))u(x), \tag{2.21}$$

$u\in\mathcal{S}(\mathbf{R}^n)$, where $p(x,y,\xi,\mu)\in S^{d,\nu}(\mathbf{R}^{2n}\times\overline{\mathbf{R}}^{n+1}_+)$. When p is not integrable in ξ, this is interpreted, via oscillatory integrals (cf. [H3, 7.8]), by a weak definition as follows: For $u,v\in\mathcal{S}(\mathbf{R}^n)$ we set

$$\begin{aligned}(P_\mu u,v)&=\lim_{\varepsilon\to0}(2\pi)^{-n}\iiint e^{i(x-y)\cdot\xi}p(x,y,\xi,\mu)u(y)\bar v(x)\chi(\varepsilon\xi)\,dx\,dy\,d\xi\\&=\lim_{\varepsilon\to0}(P_{\mu,\varepsilon}u,v),\quad P_{\mu,\varepsilon}=\mathrm{OP}(p(x,y,\xi,\mu)\chi(\varepsilon\xi));\end{aligned} \tag{2.22}$$

this defines P_μ as an operator from $\mathcal{S}(\mathbf{R}^n)$ to $\mathcal{S}'(\mathbf{R}^n)$. The formulas imply

$$P_{\mu,\varepsilon}=\mathrm{OP}(q_\varepsilon(x,\xi,\mu)),\quad\text{where}\quad q_\varepsilon(x,\xi,\mu)=e^{iD_y\cdot D_\xi}(p(x,y,\xi,\mu)\chi(\varepsilon\xi))|_{y=x},$$

as defined in [H3] before Theorem 18.1.7; and the results on convergence of symbols there ([H3, Theorem 18.1.7 and Remark]) show that $q_\varepsilon(x,\xi,\mu)$ converges weakly in $S^d_{1,0}(\mathbf{R}^n\times\mathbf{R}^n)$ for each μ to

$$\begin{aligned}q(x,\xi,\mu)=e^{iD_y\cdot D_\xi}p(x,y,\xi,\mu)|_{y=x}&\sim\sum_{\alpha\in\mathbf{N}^n}\tfrac{1}{\alpha!}\partial_y^\alpha D_\xi^\alpha p(x,y,\xi,\mu)|_{y=x},\\ \text{with }\ \mathrm{OP}(p(x,y,\xi,\mu))&=\mathrm{OP}(q(x,\xi,\mu)).\end{aligned} \tag{2.23}$$

That $q_\varepsilon(x,\xi)\to0$ weakly in $S^d_{1,0}(\mathbf{R}^n\times\mathbf{R}^n)$ for $\varepsilon\to0$, $\varepsilon\in\,]0,1]$, means that the set $\{q_\varepsilon\}_{\varepsilon\in\,]0,1]}$ is bounded in $S^d_{1,0}(\mathbf{R}^n\times\mathbf{R}^n)$ and $q_\varepsilon\to0$ in $\mathcal{D}'(\mathbf{R}^{2n})$; the latter convergence can be sharpened to uniform convergence of $D_x^\beta D_\xi^\alpha q_\varepsilon(x,\xi)$ for all $\alpha,\beta\in\mathbf{N}^n$ on the compact subsets of $\mathbf{R}^{2n}$.

To avoid repetitions, we have a μ in the notation here, but up until Theorem 2.7, it should be considered as a fixed number.

For the special case of (2.21) where p is independent of x, (2.23) implies

$$\begin{aligned}&\mathrm{OP}(p(y,\xi,\mu))=\mathrm{OP}(q_1(x,\xi,\mu)),\quad\text{with}\\ &q_1(x,\xi,\mu)=e^{iD_y\cdot D_\xi}p(y,\xi,\mu)|_{y=x}\sim\sum_{\alpha\in\mathbf{N}^n}\tfrac{1}{\alpha!}\partial_y^\alpha D_\xi^\alpha p(y,\xi,\mu)|_{y=x}.\end{aligned} \tag{2.24}$$

We observe furthermore that, by (2.22), the adjoint of $P_\mu = \mathrm{OP}(p(x,\xi,\mu))$ is $P_\mu^* = \mathrm{OP}(\bar p(y,\xi,\mu))$, which may be reduced by (2.24), so that we get:

$$\begin{aligned} &\mathrm{OP}(p(x,\xi,\mu))^* = \mathrm{OP}(\bar p(y,\xi,\mu)) = \mathrm{OP}(q(x,\xi,\mu)), \quad \text{with} \\ &q(x,\xi,\mu) = e^{iD_x\cdot D_\xi}\bar p(x,\xi,\mu) \sim \sum_{\alpha\in\mathbf{N}^n} \tfrac{1}{\alpha!}\partial_x^\alpha D_\xi^\alpha \bar p(x,\xi,\mu); \end{aligned} \tag{2.25}$$

cf. also [H3, (18.1.9)]. Note that then $P_\mu = \mathrm{OP}(q(x,\xi,\mu))^* = \mathrm{OP}(\bar q(y,\xi,\mu))$, so that we also find (by conjugation of the formula for q, cf. (2.1) for $\bar D_\xi$):

$$\begin{aligned} &\mathrm{OP}(p(x,\xi,\mu)) = \mathrm{OP}(q_2(y,\xi,\mu)), \quad \text{with} \\ &q_2(y,\xi,\mu) = e^{-iD_x\cdot D_\xi} p(x,\xi,\mu)|_{x=y} \sim \sum_{\alpha\in\mathbf{N}^n} \tfrac{1}{\alpha!}\partial_x^\alpha \bar D_\xi^\alpha p(x,\xi,\mu)|_{x=y}. \end{aligned} \tag{2.26}$$

This prepares for the formula for the composition of two operators, cf. [H3, Theorem 18.1.8]:

$$\begin{aligned} &\mathrm{OP}(p_1(x,\xi,\mu))\,\mathrm{OP}(p_2(x,\xi,\mu)) = \mathrm{OP}(q(x,\xi,\mu)), \quad \text{with} \\ &q(x,\xi,\mu) = e^{iD_y\cdot D_\eta}(p_1(x,\eta,\mu)p_2(y,\xi,\mu))|_{y=x,\eta=\xi} \\ &\qquad \sim \sum_{\alpha\in\mathbf{N}^n} \tfrac{1}{\alpha!}\partial_y^\alpha D_\eta^\alpha (p_1(x,\eta,\mu)p_2(y,\xi,\mu))|_{y=x,\eta=\xi}, \\ &\text{denoted } q(x,\xi,\mu) = p_1(x,\xi,\mu)\circ p_2(x,\xi,\mu). \end{aligned} \tag{2.27}$$

The formula can (as in [H2] and [G2]) be derived from the preceding rules, by first using (2.26) to construct $q_2(y,\xi,\mu)$ such that $\mathrm{OP}(p_2(x,\xi,\mu)) = \mathrm{OP}(q_2(y,\xi,\mu))$, next observing that one has the simple product formula

$$\mathrm{OP}(p_1(x,\xi,\mu))\,\mathrm{OP}(q_2(y,\xi,\mu)) = \mathrm{OP}(p_1(x,\xi,\mu)q_2(y,\xi,\mu)),$$

and finally reducing $p_1(x,\xi,\mu)q_2(y,\xi,\mu)$ to $q(x,\xi,\mu)$ by use of (2.23) and a certain formula for simplification of binomial expressions.

Finally, we have to discuss coordinate changes. Here [H3, Theorem 18.1.17] is only formulated for symbols with compactly supported kernels, but the hypothesis of compactness can be removed when admissible diffeomorphisms are considered, cf. Section 1.2, as follows: Let U and $U_\varkappa$ be open sets in $\mathbf{R}^n$, and let $\varkappa\colon U\to U_\varkappa$ be an admissible diffeomorphism. Let $P_\mu = \mathrm{OP}(p(x,\xi,\mu))$ be a ps.d.o. with distribution kernel supported in a subset of $U\times U$ with positive distance from $\partial(U\times U)$. Then it induces a ps.d.o. on $U_\varkappa$ by

$$\begin{aligned} &(P_{\mu,\varkappa}u)\circ\varkappa = P_\mu(u\circ\varkappa) \quad \text{for } u\in C_0^\infty(U_\varkappa), \quad \text{with} \\ &p_\varkappa(\varkappa(x),\eta,\mu) = e^{-i\varkappa(x)\cdot\eta}P_\mu(e^{i\varkappa(x)\cdot\eta}) \\ &\qquad \sim \sum_{\alpha\in\mathbf{N}^n} \tfrac{1}{\alpha!}D_\eta^\alpha p(x,{}^t\varkappa'(x)\eta,\mu)\partial_y^\alpha e^{i\varrho_x(y)\cdot\eta}|_{y=x}, \end{aligned} \tag{2.28}$$

where $\varrho_x(y)=\varkappa(y)-\varkappa(x)-\varkappa'(x)(y-x)$. This is seen by a generalization of the proof of [H3, Theorem 18.1.17]: Let $\chi(x)\in C_0^\infty(\mathbf{R}^n)$ with $\chi=1$ near 0, and write the distribution kernel $\mathcal{K}_P(x,y,\mu)$ of P_μ as a sum of two terms $\mathcal{K}^1=(1-\chi(x-y))\mathcal{K}_P$ and $\mathcal{K}^2=\chi(x-y)\mathcal{K}_P$. Here $\mathcal{K}^1$ is the kernel of an operator of order $-\infty$, and the estimates assuring this ((2.20) with fixed μ) carry over to the corresponding kernel $\mathcal{K}^1_\varkappa$ in view of (1.39). Concerning $\mathcal{K}^2_\varkappa$, we note that the estimates in [H3, Theorem 18.1.17] are valid, *uniformly in* $a\in\mathbf{R}^n$, when $\mathcal{K}^2$ is replaced by $\varphi(x+a)\psi(y+a)\mathcal{K}^2$ with φ and $\psi\in C_0^\infty$. Taking φ and ψ equal to 1 on a sufficiently large neighborhood of 0, we can obtain that for any $(x,y)\in\operatorname{supp}\chi(x-y)$ there is an a so that $\varphi(x+a)\psi(y+a)=1$ there, so the desired estimates hold for $\mathcal{K}^2_\varkappa$. (These arguments were kindly supplied by L. Hörmander in a personal communication.)

We say as in [G2], with a somewhat loose terminology, that P_μ is in x-form, (x,y)-form, resp. y-form, when it is defined as in (2.15), (2.21), resp. (2.21) with $p(x,y,\xi,\mu)$ replaced by $p(y,\xi,\mu)$. We shall also sometimes need a mixture of these concepts; e.g. an operator defined by (2.21) with $p(x,y,\xi,\mu)$ replaced by $p(x',y_n,\xi,\mu)$ is said to be in (x',y_n)-form. One changes an x-form to an (x',y_n)-form by a variant of (2.26), namely

$$\begin{gathered}\mathrm{OP}(p(x,\xi,\mu))=\mathrm{OP}(q(x',y_n,\xi,\mu)),\quad\text{with}\\ q(x',y_n,\xi,\mu)=e^{-iD_{x_n}D_{\xi_n}}p(x,\xi,\mu)\sim\sum_{j\in\mathbf{N}}\tfrac{1}{j!}\partial^j_{x_n}\bar{D}^j_{\xi_n}p(x,\xi,\mu)|_{x_n=y_n}.\end{gathered}\tag{2.29}$$

The above considerations describe the basic rules of calculus for ps.d.o.s when μ is fixed. For the parameter-dependent calculus, we moreover have to justify the asymptotic expansions (2.23) etc. with respect to the symbol spaces $S^{d,\nu}_{1,0}(\mathbf{R}^n\times\bar{\mathbf{R}}^{n+1}_+)$. This will be based on the following general rule inferred from [H3, Theorem 18.1.7 ff., Theorem 18.4.11]:

LEMMA 2.5. *Let B be a Banach space, and let $S^d_{1,0}(\mathbf{R}^n\times\mathbf{R}^n)\otimes B$ be the space of C^∞ functions $a(x,\xi)$ from $\mathbf{R}^n\times\mathbf{R}^n$ to B satisfying $\|D^\beta_x D^\alpha_\xi a(x,\xi)\|_B\dot{\leqslant}\langle\xi\rangle^{d-|\alpha|}$, for all $\alpha,\beta\in\mathbf{N}^n$. Then $a(x,\xi)\mapsto e^{iD_x\cdot D_\xi}a(x,\xi)$ extends from a mapping in $\mathcal{S}(\mathbf{R}^n\times\mathbf{R}^n)\otimes B$ to a weakly continuous mapping from $S^d_{1,0}(\mathbf{R}^n\times\mathbf{R}^n)\otimes B$ to $S^d_{1,0}(\mathbf{R}^n\times\mathbf{R}^n)\otimes B$; and for each $k\in\mathbf{N}$, the mapping R_k defined by*

$$R_k\colon a(x,\xi)\mapsto e^{iD_x\cdot D_\xi}a(x,\xi)-\sum_{|\alpha|<k}\tfrac{1}{\alpha!}\partial^\alpha_x D^\alpha_\xi a(x,\xi)$$

is weakly continuous from $S^d_{1,0}(\mathbf{R}^n\times\mathbf{R}^n)\otimes B$ to $S^{d-k}_{1,0}(\mathbf{R}^n\times\mathbf{R}^n)\otimes B$.

Remark 2.6. For $A\in\mathcal{S}(\mathbf{R}^n\times\mathbf{R}^n)\otimes B$, the Fourier transform $\hat{A}=\mathcal{F}_{(x,\xi)\to(\hat{x},\hat{\xi})}A$ is defined by the usual formula and $\hat{A}\in\mathcal{S}(\mathbf{R}^n\times\mathbf{R}^n)\otimes B$ by the standard proof. This allows us to define $e^{iD_x\cdot D_\xi}A(x,\xi)\in\mathcal{S}(\mathbf{R}^n\times\mathbf{R}^n)\otimes B$ in the usual way, cf. [H3, Volume I, p. 207].

— As in [H3, Definition 18.4.9], we say that a mapping from $S^{d_1}_{1,0}\otimes B$ to $S^{d_2}_{1,0}\otimes B$ is weakly continuous when it is continuous in the ordinary sense (with respect to the Fréchet topologies) and, in addition, the restriction to a bounded subset is continuous in the C^∞ topology.

Proof of Lemma 2.5. For $A\in\mathcal{S}(\mathbf{R}^n\times\mathbf{R}^n)\otimes B$, $b^*\in B^*$, one has that $\langle b^*,A\rangle\in\mathcal{S}$ and

$$\langle b^*, e^{iD_x\cdot D_\xi}A(x,\xi)\rangle = e^{iD_x\cdot D_\xi}\langle b^*, A(x,\xi)\rangle. \tag{2.30}$$

Define the seminorms on $S^d_{1,0}\otimes B$, $|A|_{k,B}=\sup_{|\alpha|+|\beta|\leqslant k}\sup_{x,\xi}\|\langle\xi\rangle^{|\alpha|-d}D^\beta_x D^\alpha_\xi A(x,\xi)\|_B$. (This can also be reformulated in the notation of [H3, 18.4].) The proof of [H3, (18.4.17)] gives for $a\in S^d_{1,0}$ (with the special partition of unity $\varphi_\nu\in C_0^\infty$ used there): For any N there is an M and a C (depending only on the dimension and the choice of metric and partition of unity) such that

$$|e^{iD_x\cdot D_\xi}(\varphi_\nu a)(x,\xi)|\leqslant C(1+d_\nu(x,\xi))^{-N}\langle\xi\rangle^d|a|_{M,\mathbf{C}}.$$

From this we get by use of (2.30) for $A\in S^d_{1,0}\otimes B$, *with the same constants*:

$$\|e^{iD_x\cdot D_\xi}(\varphi_\nu A)(x,\xi)\|_B\leqslant C(1+d_\nu(x,\xi))^{-N}\langle\xi\rangle^d|A|_{M,B}.$$

Hence we have for $A\in S^d_{1,0}\otimes B$:

$$\sum_\nu\|e^{iD_x\cdot D_\xi}(\varphi_\nu A)(x,\xi)\|_B\leqslant C'\langle\xi\rangle^d|A|_{M,B}.$$

We can therefore define $(e^{iD_x\cdot D_\xi}A)(x,\xi)=\sum_\nu e^{iD_x\cdot D_\xi}(\varphi_\nu A)(x,\xi)$ as a weakly continuous mapping from $S^d_{1,0}\otimes B$ to B (cf. [H3, Theorem 18.4.10]). Estimates of derivatives and remainders follow easily, and give [H3, Theorem 18.4.11] for $S^d_{1,0}\otimes B$. □

The lemma will be used with $B=\mathcal{L}(H_1,H_2)$ for Hilbert spaces H_1 and H_2 in the study of ps.d. boundary operators. For the ps.d.o.s we use it with $B=\mathbf{C}$ to obtain:

THEOREM 2.7. *For $p(x,y,\xi,\mu)\in S^{d,\nu}_{1,0}(\mathbf{R}^{2n}\times\overline{\mathbf{R}}^{n+1}_+)$, the asymptotic expansion* (2.23) *holds in $S^{d,\nu}_{1,0}(\mathbf{R}^n\times\overline{\mathbf{R}}^{n+1}_+)$. Moreover, when $p(x,\xi,\mu)\in S^{d,\nu}_{1,0}(\mathbf{R}^n\times\overline{\mathbf{R}}^{n+1}_+)$, the asymptotic expansions* (2.24), (2.25) *and* (2.26) *hold in $S^{d,\nu}_{1,0}(\mathbf{R}^n\times\overline{\mathbf{R}}^{n+1}_+)$.*

When $p_i(x,\xi,\mu)\in S^{d_i,\nu_i}_{1,0}(\mathbf{R}^n\times\overline{\mathbf{R}}^{n+1}_+)$ for $i=1,2$, the asymptotic expansion (2.27) *holds in $S^{d_1+d_2,m(\nu_1,\nu_2)}_{1,0}(\mathbf{R}^n\times\overline{\mathbf{R}}^{n+1}_+)$, where*

$$m(\nu_1,\nu_2)=\min\{\nu_1,\nu_2,\nu_1+\nu_2\}. \tag{2.31}$$

Hence when $P_{i,\mu}$ $(i=1,2)$ are parameter-dependent ps.d.o.s of order d_i and regularity ν_i, with uniformly estimated symbols, then the adjoint $P^*_{1,\mu}$ and the composition $Q_\mu=$

$P_{1,\mu}P_{2,\mu}$ are parameter-dependent ps.d.o.s with uniformly estimated symbols, of order d_1 resp. d_1+d_2 and regularity ν_1 resp. $m(\nu_1,\nu_2)$. Here Q_μ satisfies the uniform transmission condition when $P_{1,\mu}$ and $P_{2,\mu}$ do so.

The operator classes are invariant under admissible coordinate changes, satisfying rules as in (2.28).

Proof. We first treat (2.23), from which the statements on (2.24), (2.25) and (2.26) follow. Let $q'(x,y,\xi,\mu)=e^{iD_y\cdot D_\xi}p(x,y,\xi,\mu)$, then $q(x,\xi,\mu)=q'(x,x,\xi,\mu)$. Since $(x,\mu)\mapsto p(x,\cdot,\cdot,\mu)$ is C^∞ with values in $S^d_{1,0}(\mathbf{R}^n\times\mathbf{R}^n)$, the same is true for $(x,\mu)\mapsto q'(x,\cdot,\cdot,\mu)$. In particular, q' and q are C^∞.

Now we shall show that q satisfies the first seminorm estimate for the space $S^{d,\nu}_{1,0}(\mathbf{R}^{2n}\times\overline{\mathbf{R}}^{n+1}_+)$:

$$|q(x,\xi,\mu)|\mathrel{\dot{\leqslant}}\langle\xi,\mu\rangle^d+\langle\xi,\mu\rangle^{d-\nu}\langle\xi\rangle^\nu. \tag{2.32}$$

Note that for large γ ($|\gamma|>\nu$) and general α,β one has that

$$\begin{aligned}|D^\beta_y D^{\alpha+\gamma}_\xi p(x,y,\xi,\mu)|&\mathrel{\dot{\leqslant}}\langle\xi,\mu\rangle^{d-|\alpha|-|\gamma|}+\langle\xi,\mu\rangle^{d-\nu}\langle\xi\rangle^{\nu-|\alpha|-|\gamma|}\\&\mathrel{\dot{\leqslant}}\langle\mu\rangle^{d-\nu}\langle\xi\rangle^{\nu+|d-\nu|-|\alpha|-|\gamma|};\end{aligned}$$

cf. (2.18). It follows that $(y,\xi)\mapsto D^\gamma_\xi p(x,y,\xi,\mu)$ belongs to $S^{\nu+|d-\nu|-|\gamma|}_{1,0}(\mathbf{R}^n\times\mathbf{R}^n)$ with seminorms that are $O(\langle\mu\rangle^{d-\nu})$ uniformly in x. According to Lemma 2.5, we have for $k\in\mathbf{N}$ if $|\gamma|>\nu$,

$$\Big|D^\gamma_\xi\Big(q'(x,y,\xi,\mu)-\sum_{|\alpha|<k}\tfrac{1}{\alpha!}\partial^\alpha_y D^\alpha_\xi p(x,y,\xi,\mu)\Big)\Big|\mathrel{\dot{\leqslant}}\langle\mu\rangle^{d-\nu}\langle\xi\rangle^{\nu+|d-\nu|-|\gamma|-k}, \tag{2.33}$$

but also for general β if $k>d$,

$$\begin{aligned}\Big|D^\beta_\xi\Big(q'(x,y,\xi,\mu)-\sum_{|\alpha|<k}\tfrac{1}{\alpha!}\partial^\alpha_y D^\alpha_\xi p(x,y,\xi,\mu)\Big)\Big|&\mathrel{\dot{\leqslant}}C(\mu)\langle\xi\rangle^{d-k-|\beta|}\\&\to 0\quad\text{as }|\xi|\to\infty.\end{aligned} \tag{2.34}$$

Choosing $k>2|d|+2|\nu|+1$, we get by $|\gamma|$ integrations of the radial derivatives of q' along rays from ∞, using (2.33) and (2.34), that

$$\begin{aligned}\Big|q'(x,y,\xi,\mu)-\sum_{|\alpha|<k}\tfrac{1}{\alpha!}\partial^\alpha_y D^\alpha_\xi p(x,y,\xi,\mu)\Big|&\mathrel{\dot{\leqslant}}\langle\mu\rangle^{d-\nu}\langle\xi\rangle^{\nu+|d-\nu|-k}\\&\mathrel{\dot{\leqslant}}\langle\xi,\mu\rangle^{d-\nu}\langle\xi\rangle^{\nu+2|d-\nu|-k}\mathrel{\dot{\leqslant}}\langle\xi,\mu\rangle^{d-\nu}\langle\xi\rangle^{\nu},\end{aligned}$$

which implies (2.32).

Since $D_{x,y,\xi,\mu}$ commutes with $e^{iD_y\cdot D_\xi}$ and $\partial_y^\alpha D_\xi^\alpha$, all other estimates of q in $S_{1,0}^{d,\nu}(\mathbf{R}^n\times\overline{\mathbf{R}}_+^{n+1})$ follow immediately, and the expansion in the full asymptotic series (2.23) follows by taking k even larger in (2.33). This implies the first part of the theorem.

For the second part we proceed as indicated after (2.27) above. The rule for (2.26) proved above implies that the expansion of $q_2(y,\xi,\mu)$ holds in $S_{1,0}^{d_2,\nu_2}(\mathbf{R}^n\times\overline{\mathbf{R}}_+^{n+1})$. Now [G2, Proposition 2.1.5] shows that $p_1(x,\xi,\mu)q_2(y,\xi,\mu)\in S_{1,0}^{d_1+d_2,m(\nu_1,\nu_2)}(\mathbf{R}^{2n}\times\overline{\mathbf{R}}_+^{n+1})$ with $m(\nu_1,\nu_2)$ defined by (2.31). Then by the rule for (2.23), the expansion of $q(x,\xi,\mu)$ holds in $S_{1,0}^{d_1+d_2,m(\nu_1,\nu_2)}(\mathbf{R}^n\times\overline{\mathbf{R}}_+^{n+1})$.

The third statement is an immediate consequence, and the rules for coordinate changes follow by an extension of the proof described after (2.28) including the μ-dependence. □

2.3. Boundary operators

Recall from [G2] the defining formulas for a Poisson operator K_μ, a trace operator T'_μ of class 0 and a singular Green operator G'_μ of class 0, in terms of symbol-kernels (cf. Definition 2.2 (5)):

$$\begin{aligned}
K_\mu v(x) &= \mathrm{OPK}\,(k(x',\xi,\mu))v(x) = \mathrm{OPK}\,(\tilde{k}(x,\xi',\mu))v(x)\\
&= (2\pi)^{1-n}\int_{\mathbf{R}^{n-1}} e^{ix'\cdot\xi'}\,\tilde{k}(x',x_n,\xi',\mu)\,\hat{v}(\xi')\,d\xi',\\
T'_\mu u(x') &= \mathrm{OPT}\,(t'(x',\xi,\mu))u(x') = \mathrm{OPT}\,(\tilde{t}'(x,\xi',\mu))u(x')\\
&= (2\pi)^{1-n}\int_{\mathbf{R}^{n-1}} e^{ix'\cdot\xi'}\int_0^\infty \tilde{t}'(x',x_n,\xi',\mu)\,\acute{u}(\xi',x_n)\,dx_n\,d\xi',\\
G'_\mu u(x) &= \mathrm{OPG}(g'(x',\xi,\eta_n,\mu))u(x) = \mathrm{OPG}(\tilde{g}'(x,y_n,\xi',\mu))u(x)\\
&= (2\pi)^{1-n}\int_{\mathbf{R}^{n-1}} e^{ix'\cdot\xi'}\int_0^\infty \tilde{g}'(x',x_n,y_n,\xi',\mu)\,\acute{u}(\xi',y_n)\,dy_n\,d\xi';
\end{aligned}\tag{2.35}$$

here $\hat{u}(\xi)=\mathcal{F}_{x\to\xi}u(x)$, $\hat{v}(\xi')=\mathcal{F}_{x'\to\xi'}v(x')$, and $\acute{u}(\xi',x_n)=\mathcal{F}_{x'\to\xi'}u(x',x_n)$. (See e.g. [G2, 2.4] for the definitions in terms of symbols.) When the class r is $\geqslant 0$, the trace and singular Green symbols are of the form

$$\begin{aligned}
t(x',\xi,\mu) &= \sum_{0\leqslant j\leqslant r-1} s_j(x',\xi',\mu)\xi_n^j + t'(x',\xi,\mu)\\
g(x',\xi,\eta_n,\mu) &= \sum_{0\leqslant j\leqslant r-1} k_j(x',\xi,\mu)\eta_n^j + g'(x',\xi,\eta_n,\mu),
\end{aligned}\tag{2.36}$$

cf. (2.10), (2.12). Then we define $T_\mu=\mathrm{OPT}(t(x',\xi,\mu))$ resp. $G_\mu=\mathrm{OPG}(g(x',\xi,\eta_n,\mu))$, by

$$T_\mu = \sum_{0\leqslant j\leqslant r-1} S_{j,\mu}\gamma_j + T'_\mu, \quad \text{resp.} \quad G_\mu = \sum_{0\leqslant j\leqslant r-1} K_{j,\mu}\gamma_j + G'_\mu, \tag{2.37}$$

where $S_{\mu,j}=\mathrm{OP}'(s_j(x',\xi',\mu))$ and $K_{j,\mu}=\mathrm{OPK}(k_j(x',\xi,\mu))$.

When the operator definitions are applied with respect to the x_n-variable only, the operators are denoted OPK_n etc. and called called *boundary symbol operators* (one can also use a notation where ξ_n or (ξ_n,η_n) is replaced by the indication D_n). We use OP' to denote application of the pseudo-differential definition w.r.t. the x'-variable. OP may be written OP_x when we want to underline that it is applied w.r.t. the x-variable.

The main object now is to show continuity properties and composition rules for all the operator types, but before we go on to that, we insert a section with some technical improvements of results in [G2].

3. Refinement of L_2 symbol estimates

3.1. An improved estimate for ps.d.o.s

In [G2], there were shown a number of estimates in L_2 Sobolev spaces for pseudodifferential boundary operators depending on a parameter μ, where the behavior with respect to μ is expressed in terms of the so-called regularity number ν. In some cases, the regularity was lowered in the passage from an operator to a derived operator. We show in the following how this loss of regularity can be avoided in most cases (or even improved), by more delicate applications of estimates from [G2]. Next, we study the compositions with the simple order-reducing operators $(\langle D',\mu\rangle\pm iD_{x_n})^t_{\mathbf{R}^n_+}$, $t\in\mathbf{Z}$, and include operators of negative class in the calculus. Finally, we analyze the effect of applying $(\langle\xi',\mu\rangle-iD_{x_n})^t_+$ and x_n^δ to a symbol-kernel, for t and $\delta\in\,]0,1[$, in preparation for the L_p estimates in Section 4. From here on, we write $P_{\mathbf{R}^n_+}$ and $P_{\mathbf{R}_+}$ as P_+ (e.g. $\Xi^t_{\pm,\mu,\mathbf{R}^n_+}=\Xi^t_{\pm,\mu,+}$).

By a refinement of the proof of [G2, Theorem 2.2.8] we shall show an improvement of the regularity by essentially $\frac{1}{2}$, in the application of the projection h_{-1}.

PROPOSITION 3.1. *Let d and $\nu\in\mathbf{R}$, and let $p(x',\xi,\mu)$ be an x_n-independent pseudodifferential symbol belonging to the space $S^{d,\nu}_{1,0,\mathrm{uttr}}(\mathbf{R}^n\times\overline{\mathbf{R}}^{n+1}_+)$. Then $h_{-1}p$ satisfies*

$$\|h_{-1}p\|_{L_{2,\xi_n}}\dot{\leqslant}\begin{cases}(\varrho^{\nu+1/2}+1)\varkappa^{d+1/2} & \text{if } \nu\neq-\frac{1}{2};\\ (|\log\varrho|^{1/2}+1)\varkappa^{d+1/2} & \text{if } \nu=-\frac{1}{2}.\end{cases}\tag{3.1}$$

Similarly, the Laguerre series estimates in [G2, Theorem 2.2.8] *can be improved by a replacement of ϱ^ν by $\varrho^{\nu+1/2}$ if $\nu\neq-\frac{1}{2}$, and by $|\log\varrho|^{1/2}$ if $\nu=-\frac{1}{2}$.*

Proof. The proof of [G2, Theorem 2.2.5] shows that

$$|h_{-1}p(x',\xi,\mu)|\dot{\leqslant}\begin{cases}\langle\xi',\mu\rangle^{d+1}|\xi_n|^{-1} & \text{when } |\xi_n|\geqslant\langle\xi',\mu\rangle;\\ \langle\xi\rangle^{\nu}\langle\xi',\mu\rangle^{d-\nu}+\langle\xi,\mu\rangle^{d} & \text{when } |\xi_n|\leqslant\langle\xi',\mu\rangle.\end{cases}\tag{3.2}$$

Thus if $\nu\neq -\frac{1}{2}$, we get

$$\|h_{-1}p\|_{L_{2,\xi_n}} \mathrel{\dot{\leqslant}} \Big\{ \int_{|\xi_n|\leqslant\langle\xi'\rangle} (\langle\xi'\rangle^{\nu}\langle\xi',\mu\rangle^{d-\nu}+\langle\xi',\mu\rangle^{d})^2\,d\xi_n$$

$$+\int_{\langle\xi'\rangle\leqslant|\xi_n|\leqslant\langle\xi',\mu\rangle} (|\xi_n|^{2\nu}\langle\xi',\mu\rangle^{2d-2\nu}+\langle\xi',\mu\rangle^{2d})\,d\xi_n+\int_{\langle\xi',\mu\rangle\leqslant|\xi_n|}\langle\xi',\mu\rangle^{2d+2}|\xi_n|^{-2}\,d\xi_n\Big\}^{1/2}$$

$$\mathrel{\dot{\leqslant}} \langle\xi'\rangle^{\nu+1/2}\langle\xi',\mu\rangle^{d-\nu}+\langle\xi',\mu\rangle^{d+1/2}=(\varrho^{\nu+1/2}+1)\varkappa^{d+1/2},$$

which is (3.1) in this case. The case $\nu=-\frac{1}{2}$ follows easily from this computation too. □

With analogous considerations for the derivatives of p, one finds the following improved version of [G2, Corollary 2.3.5]:

THEOREM 3.2. *Let $p(x',\xi,\mu)$ be as in Proposition 3.1. If $\nu\notin -\frac{1}{2}+\mathbf{N}$, then h^+p satisfies, for all α and $\beta\in\mathbf{N}^{n-1}$, m, m' and $j\in\mathbf{N}$,*

$$\|h^+D^{\beta}_{x'}D^{\alpha}_{\xi'}D^{m}_{\xi_n}\xi_n^{m'}D^{j}_{\mu}h^+p(x',\xi,\mu)\|_{L_{2,\xi_n}} \mathrel{\dot{\leqslant}} (\varrho^{\nu+1/2-|\alpha|-m+m'}+1)\varkappa^{d+1/2-|\alpha|-m+m'-j}; \tag{3.3}$$

in particular, $h^+p\in S^{d,\nu+1/2}_{1,0}(\mathbf{R}^{n-1}\times\overline{\mathbf{R}}^n_+,\mathcal{H}^+)$. If $\nu\in -\frac{1}{2}+\mathbf{N}$, the estimates (3.3) hold when $m<m'$ or $|\alpha|+m-m'\neq\nu+\frac{1}{2}$, and in the remaining cases one has:

$$\|h^+D^{\beta}_{x'}D^{\alpha}_{\xi'}D^{m}_{\xi_n}\xi_n^{m'}D^{j}_{\mu}h^+p(x',\xi,\mu)\|_{L_{2,\xi_n}} \mathrel{\dot{\leqslant}} (|\log\varrho|^{1/2}+1)\varkappa^{d+1/2-|\alpha|-m+m'-j}; \tag{3.4}$$

so $h^+p\in S^{d,\nu+1/2-\varepsilon}_{1,0}(\mathbf{R}^{n-1}\times\overline{\mathbf{R}}^n_+,\mathcal{H}^+)$ for any $\varepsilon>0$. Similar statements hold for $h^-_{-1}p$.

In applications of the theory, the ps.d.o.s where ν is integer are of primary interest, so we shall not analyze the ps.d.o.s with half-integer ν any further.

In the above results, and also in the results below concerning the (x_n,ξ_n)-behavior, the parameter x' can of course be replaced by a more general parameter X running in a space $\mathbf{R}^{n_1}$ (in particular by $y'\in\mathbf{R}^{n-1}$).

3.2. Composition of boundary operators with simple order-reducing operators; negative class

We shall now show how the order-reducing operators defined in (1.31) can be used in compositions with boundary operators. This will first be formulated in terms of the one-dimensional variant $(\varkappa\pm iD_{x_n})^t=\chi^t_{\pm}(\xi',\mu,D_{x_n})$, cf. also (1.37).

Let $t\in\mathbf{Z}$. It is easily verified that $\chi^t_{\pm}\in S^{t,\infty}(\mathbf{R}^{n-1}\times\overline{\mathbf{R}}^n_+,\mathcal{H}^{\pm}_t)$; note also that for $t<0$, $\chi^t_{+}\in S^{t,\infty}(\mathbf{R}^{n-1}\times\overline{\mathbf{R}}^n_+,\mathcal{H}^{+}_t)\subset S^{t,\infty}(\mathbf{R}^{n-1}\times\overline{\mathbf{R}}^n_+,\mathcal{H}^{+})$. The $\chi^t_{\pm}$ do not lie in our *pseudo-differential* symbol spaces $S^t_{1,0}$ over $\mathbf{R}^n$, since $\langle\xi',\mu\rangle$ does not lie there (does not fall off in the ξ_n-direction when derived w.r.t. ξ'); so a composition of our ps.d.o.s with $\mathrm{OP}(\chi^t_\pm)$ will generally lead outside of our ps.d.o. classes. However, the crucial observation we shall make now is that the $\chi^t_\pm$ do satisfy, not only the L_2 estimates for boundary symbol classes derived in [G2, Lemma 2.3.9], but also the sup norm estimates that were shown to be valid for decompositions of pseudodifferential $S_{1,0}$ symbols with the transmission property in [G2, Theorem 2.2.5]:

LEMMA 3.3. *Let $t\in\mathbf{Z}$. For any $\alpha\in\mathbf{N}^n$, k and $j\in\mathbf{N}$,*

$$\begin{aligned}\xi_n^k D_\xi^\alpha D_\mu^j \chi^t_\pm = \sum_{0\leqslant l\leqslant t+k-|\alpha|-j} s_{l,k,\alpha,j}(\xi',\mu)\xi_n^l + h_{-1}(\xi_n^k D_\xi^\alpha D_\mu^j \chi^t_\pm);\\ \text{where } |s_{l,k,\alpha,j}| \dot\leqslant \langle\xi',\mu\rangle^{t-l+k-|\alpha|-j},\\ |h_{-1}(\xi_n^k D_\xi^\alpha D_\mu^j \chi^t_\pm)| \dot\leqslant \langle\xi',\mu\rangle^{t+1+k-|\alpha|-j}\langle\xi,\mu\rangle^{-1}.\end{aligned}\tag{3.5}$$

Proof. For $t\geqslant 0$, there is no h_{-1} part, and the estimates of the coefficients $s_{l,k,\alpha,j}$ are straightforward to see. When $t<0$, we write the functions $\xi_n^k D_\xi^\alpha D_\mu^j(\langle\xi',\mu\rangle\pm i\xi_n)^t$ as sums of terms of the form

$$c'\xi'^{\beta'}\mu^{j'}\langle\xi',\mu\rangle^{t'}\xi_n^l \quad\text{or}\quad c''\xi'^{\gamma'}\mu^{j''}\langle\xi',\mu\rangle^{t''}(\langle\xi',\mu\rangle\pm i\xi_n)^{t'''},$$

where

$$\begin{aligned} j', l\geqslant 0,\quad |\beta'|+j'+t'+l = t+k-|\alpha|-j,\quad \text{resp.}\\ j''\geqslant 0,\quad t'''<0,\quad |\gamma'|+j''+t''+t''' = t+k-|\alpha|-j;\end{aligned}$$

these terms obviously satisfy the desired estimates. It is used here that terms containing $\xi_n^m(\langle\xi',\mu\rangle\pm i\xi_n)^s$ with $m>0>s$ can be reduced away by insertion of formulas

$$\xi_n^m = (\pm i)^m(\langle\xi',\mu\rangle\pm i\xi_n - \langle\xi',\mu\rangle)^m = \sum_{m'} c_{m,m'}(\langle\xi',\mu\rangle\pm i\xi_n)^{m'}\langle\xi',\mu\rangle^{m-m'}.$$

(The estimates of the $s_{l,k,\alpha,j}$ follow also from [G2, Lemma 2.3.9].) □

This leads to the following results on products:

LEMMA 3.4. *Let t and $t'\in\mathbf{Z}$, and let $r\in\mathbf{N}$.*
(1) *When $k(x',\xi,\mu)\in S^{d-1,\nu}_{1,0}(\mathbf{R}^{n-1}\times\overline{\mathbf{R}}^n_+,\mathcal{H}^+)$, then*

$$h^+(\chi^t_\pm k(x',\xi,\mu))\in S^{d+t-1,\nu}_{1,0}(\mathbf{R}^{n-1}\times\overline{\mathbf{R}}^n_+,\mathcal{H}^+).\tag{3.6}$$

(2) *When $t(x',\xi,\mu)\in S^{d,\nu}_{1,0}(\mathbf{R}^{n-1}\times\overline{\mathbf{R}}^n_+,\mathcal{H}^-_{r-1})$, then*

$$h^-(t(x',\xi,\mu)\,\chi^{t'}_\pm)\in S^{d+t',\nu}_{1,0}(\mathbf{R}^{n-1}\times\overline{\mathbf{R}}^n_+,\mathcal{H}^-_{[r+t']_+-1}).\tag{3.7}$$

(3) *When* $g(x',\xi,\eta_n,\mu)\in S_{1,0}^{d-1,\nu}(\mathbf{R}^{n-1}\times\overline{\mathbf{R}}_+^n,\mathcal{H}^+\hat{\otimes}\mathcal{H}_{r-1}^-)$, *then* (*with all combinations of choices of* $+$ *and* $-$ *for* $\pm$)

$$h_{\xi_n}^+ h_{\eta_n}^-(\chi_\pm^t(\xi,\mu)\,g(x',\xi,\eta_n,\mu)\,\chi_\pm^{t'}(\xi',\eta_n,\mu))\in S_{1,0}^{d+t+t'-1,\nu}(\mathbf{R}^{n-1}\times\overline{\mathbf{R}}_+^n,\mathcal{H}^+\hat{\otimes}\mathcal{H}_{[r+t']_+-1}^-). \tag{3.8}$$

The projections h^+ *and* h^- *can be omitted when* $\chi_\pm^t$ *is taken as* χ_+^t *with* $t\leqslant 0$ *and* $\chi_\pm^{t'}$ *is taken as* $\chi_-^{t'}$; *then moreover, the indication* $\mathcal{H}^+$ *resp.* $\mathcal{H}_{[r+t']_+-1}^-$ *can be replaced by* $\mathcal{H}_t^+$ *resp.* $\mathcal{H}_{r+t'-1}^-$.

Proof. This is seen by carrying out the proof of [G2, Lemmas 2.6.2 and 2.6.3] with $\chi_\pm^t(\xi,\mu)$ playing the role of p there, using the estimates from Lemma 3.3 above, followed by an application of [G2, Lemmas 2.3.9 and 2.3.10]. □

The lemma implies that the full operators $K_\mu=\mathrm{OPK}(k)$, $T_\mu=\mathrm{OPT}(t)$ and $G_\mu=\mathrm{OPG}(g)$, by composition with operators $\Xi_{\pm,\mu,+}^t$ to the left and right in a similar way, give Poisson, trace resp. singular Green operators *belonging to our calculus* and having the same regularity. This is seen first for the cases where K_μ is given in y'-form, T_μ is given in x'-form, and G_μ is given in y'-form resp. x'-form for compositions with $\Xi_{\pm,\mu,+}^t$ to the left resp. to the right of G_μ only. In each of these cases it follows from the operator definition that the full symbol of the resulting operator equals the projected product in formulas (3.6)–(3.8), without further terms. For more general symbols, we get the result by changing from x'-form to y'-form or vice versa (cf. (2.26) and [G2, Theorem 2.4.6]); in the treatment of G_μ, one composes first on one side and afterwards on the other side using these techniques.

Let us also observe that for $t\in\mathbf{R}$, the operators $\mathrm{OP}(\langle\xi',\mu\rangle^t)=\langle D',\mu\rangle^t$, although they are not ps.d.o.s with symbol in $S_{1,0}^t(\mathbf{R}^n\times\mathbf{R}_+^{n+1})$, compose very well with Poisson operators, trace operators and s.g.o.s. More precisely, the restrictions $\mathrm{OP}(\langle\xi',\mu\rangle^t)_+=\langle D',\mu\rangle_+^t=r^+\langle D',\mu\rangle^t e^+$ are well-defined on $\mathcal{S}(\overline{\mathbf{R}}_+^n)$ since the $\langle D',\mu\rangle^t$ preserve the property of being supported in $\overline{\mathbf{R}}_\pm^n$ (but their mapping properties in $H_p^{s,\mu}(\overline{\mathbf{R}}_+^n)$ spaces are not convenient for $t<0$). Now, when $T_\mu=\mathrm{OPT}(t(x',\xi,\mu))$ and $G_\mu=\mathrm{OPG}(g(x',\xi,\eta_n,\mu))$ are of order d, class r and regularity ν, then a direct calculation shows that

$$\begin{aligned} T_\mu\langle D',\mu\rangle_+^t &= \mathrm{OPT}(t(x',\xi,\mu)\langle\xi',\mu\rangle^t),\\ G_\mu\langle D',\mu\rangle_+^t &= \mathrm{OPG}(g(x',\xi,\eta_n,\mu)\langle\xi',\mu\rangle^t); \end{aligned} \tag{3.9}$$

these are operators belonging to the calculus, of order $d+t$, class r and regularity ν. $\langle D',\mu\rangle_+^t G_\mu$ and $\langle D',\mu\rangle_+^t K_\mu$ are likewise of order $d+t$ and regularity ν when G_μ and K_μ are given of order d and regularity ν, with slightly more complicated symbol formulas.

Before collecting all this in a theorem, let us extend the results to operators of negative class (cf. Definition 2.2 (2)–(3) and (2.36)). Note that one has for any $r\in\mathbf{Z}$, that $T_\mu = \mathrm{OPT}(t(x',\xi,\mu))$ is of class r precisely when its symbol $t(x',\xi,\mu)$ is $O(\langle\xi_n\rangle^{r-1})$ for each (x',ξ',μ), and $G_\mu = \mathrm{OPG}(g(x',\xi,\eta_n,\mu))$ is of class r precisely when its symbol $g(x',\xi,\eta_n,\mu)$ is $O(\langle\eta_n\rangle^{r-1})$ for each (x',ξ,μ). Since for $m\in\mathbf{N}$,

$$\mathrm{OPT}(t(x',\xi,\mu))D_{x_n}^m = \mathrm{OPT}(t(x',\xi,\mu)\xi_n^m),$$
$$\mathrm{OPG}(g(x',\xi,\eta_n,\mu))D_{x_n}^m = \mathrm{OPG}(g(x',\xi,\eta_n,\mu)\eta_n^m),$$

we find in particular that

$$\begin{aligned} &T_\mu \text{ is of class } -m \iff T_\mu D_{x_n}^m \text{ is of class } 0,\\ &G_\mu \text{ is of class } -m \iff G_\mu D_{x_n}^m \text{ is of class } 0. \end{aligned} \tag{3.10}$$

For fixed μ we get the corresponding observations for parameter-independent operators.

The class concept is important for the mapping properties the operators on Sobolev spaces: T_μ resp. G_μ is well-defined on $H_2^r(\overline{\mathbf{R}}_+^n)$ (for each fixed μ) if and only if it is of class $\leqslant r$. This was shown for $r\geqslant 0$ in [BM2], [G2], and the negative classes were included in [F1, 2], [G3, 4]. [G3, 4] moreover introduced the class concept for operators $P_+ + G$, leading to very complete results on the mapping properties of elliptic systems. Since P_+ is well-defined on $L_2(\mathbf{R}_+^n)$, it is generally assigned the class 0; and when G is of class $r\geqslant 0$, $P_+ + G$ is said to be of class r. The same holds for parameter-dependent operators. We define negative class as in (3.10) and show below that it is consistent with [G3, 4]:

Definition 3.5. *Let $m\in\mathbf{N}$. Then $P_{\mu,+}+G_\mu$ is said to be of class $-m$, when $(P_{\mu,+}+G_\mu)D_{x_n}^m$ is of class 0.*

For fixed μ, this defines the concept for parameter-independent operators. The definition applies in particular to G_μ or $P_{\mu,+}$ alone. Operators that are of class $-m$ for all $m\in\mathbf{N}$ are said to be of class $-\infty$.

Let us see how the class of $P_{\mu,+}+G_\mu$ is reflected in the symbols of P_μ and G_μ. Since there are generally two different types of symbols involved, it cannot be quite as simple as the condition $O(\langle\xi_n\rangle^{r-1})$ mentioned above. In the sequel we drop the explicit mention of μ-dependence. Recall from [G4, 2.4] and [G3] the formulas valid for general T, P and G:

$$\begin{aligned} TD_{x_n}^j &= \sum_{k=0}^{j-1} S_T^{(k)}\gamma_{j-1-k} + T^{(j)};\\ (P_+ + G)D_{x_n}^j &= \sum_{k=0}^{j-1}(K_P^{(k)}+K_G^{(k)})\gamma_{j-1-k} + P_+^{(j)} + G^{(j)}; \end{aligned} \tag{3.11}$$

where

$$T^{(j)} = \mathrm{OPT}(h_{-1}^-[\xi_n^j t(x',\xi)]),\quad S_T^{(k)} = \mathrm{OP}'\big(i\lim_{x_n\to+0}\overline{D}_{x_n}^k\overline{\mathcal{F}}_{\xi_n\to x_n}^{-1}t(x',\xi)\big),$$
$$P^{(j)} = \mathrm{OP}(\xi_n^j p(x,\xi)),\quad K_P^{(k)}v(x') = ir^+PD_{x_n}^k(v(x')\otimes\delta(x_n)),$$
$$G^{(j)} = \mathrm{OPG}(h_{-1,\eta_n}^-[\eta_n^j g(x',\xi,\eta_n)]),\quad K_G^{(k)} = \mathrm{OPK}\big(i\lim_{y_n\to+0}\overline{D}_{y_n}^k\overline{\mathcal{F}}_{\eta_n\to y_n}^{-1}g(x',\xi,\eta_n)\big),$$

cf. (2.1). We see from the first line in (3.11) that $TD^m_{x_n}=\mathrm{OPT}(t)D^m_{x_n}$ is of class 0 if and only if $S^{(0)}_T=\ldots=S^{(m-1)}_T=0$; and that when this holds, $TD^j_{x_n}$ is of class 0 for $j=0,\ldots,m-2$ also (the latter properties need not be mentioned explicitly in the condition for T being of class $-m$, as in [G3, 4]). It then follows also that $TD^\alpha_x=TD^{\alpha_n}_{x_n}D^{\alpha'}_{x'}$ is of class 0 for $|\alpha|\leqslant m$, since $D^{\alpha'}_{x'}$ does not interfere with the class. The analogous considerations apply to P_++G. Then we find altogether, for $m\in\mathbf{N}$,

$$\begin{aligned} T \text{ is of class } -m &\iff S^{(0)}_T=\ldots=S^{(m-1)}_T=0\\ &\iff TD^\alpha_x \text{ is of class 0 for all } |\alpha|\leqslant m;\\ P_++G \text{ is of class } -m &\iff K^{(0)}_P+K^{(0)}_G=\ldots=K^{(m-1)}_P+K^{(m-1)}_G=0\\ &\iff (P_++G)D^\alpha_x \text{ is of class 0 for all } |\alpha|\leqslant m. \end{aligned} \tag{3.12}$$

Let us moreover note that in view of (3.10) and (3.12) one has *for any* $r\in\mathbf{Z}$, that T_μ is of class $r \iff T_\mu\Xi^{-r}_{-,\mu,+}$ is of class 0; and G_μ is of class $r\iff G_\mu\Xi^{-r}_{-,\mu,+}$ is of class 0. For $P_{\mu,+}+G_\mu$ one can show that it is of class r if and only if its composition with $\Lambda^{-r}_{-,\mu,+}$ (cf. Remark 1.4) is of class 0.

We recall from [G3, Theorem 3.10] that when an operator is of class r for some $r\in\mathbf{Z}$, then it is defined on $H^s_p(\overline{\mathbf{R}}^n_+)$ and $B^s_p(\overline{\mathbf{R}}^n_+)$ for $s>r+\frac{1}{p}-1$, but it cannot be well-defined on $H^s_p(\overline{\mathbf{R}}^n_+)$ or $B^s_p(\overline{\mathbf{R}}^n_+)$ for an $s\leqslant r+\frac{1}{p}-1$ unless it is actually of class $r-1$.

Now we can sum up the results on compositions with $\Xi^t_{\pm,\mu,+}$ and $\langle D',\mu\rangle^t$:

THEOREM 3.6. *Let K_μ, T_μ and G_μ be Poisson, trace and singular Green operators of order $d\in\mathbf{R}$, regularity $\nu\in\mathbf{R}$ and class $r\in\mathbf{Z}$ (for T_μ and G_μ), and let t and $t'\in\mathbf{Z}$.*

When $r=0$, then $\Xi^t_{\pm,\mu,+}K_\mu$ is a Poisson operator of order $d+t$ and regularity ν, $T_\mu\,\Xi^{t'}_{\pm,\mu,+}$ is a trace operator of order $d+t'$, regularity ν and class $[t'+1]_+-1$, and $\Xi^t_{\pm,\mu,+}G_\mu\,\Xi^{t'}_{\pm,\mu,+}$ is a singular Green operator of order $d+t+t'$, regularity ν and class $[t'+1]_+-1$ (with all combinations of choices of $+$ or $-$ for $\pm$).

When $r\in\mathbf{Z}$, $T_\mu\,\Xi^{t'}_{-,\mu,+}$ and $\Xi^t_{\pm,\mu,+}G_\mu\,\Xi^{t'}_{-,\mu,+}$ are trace resp. singular Green operators of order $d+t'$ resp. $d+t+t'$, regularity ν and class $r+t'$.

For any $s\in\mathbf{R}$, $T_\mu\langle D',\mu\rangle^s_+$, $\langle D',\mu\rangle^s_+K_\mu$, $\langle D',\mu\rangle^s_+G_\mu$ and $G_\mu\langle D',\mu\rangle^s_+$ are trace, Poisson resp. singular Green operators of order $d+s$, regularity ν and class r.

The symbols are determined by the usual formulas for composition of the operators with ps.d.o.s.

3.3. Symbol-kernel estimates in $H^{t,\varkappa}_2(\overline{\mathbf{R}}_+)$ and $L_2(\mathbf{R}_+,x^\delta_n)$ spaces

When t is not integer, the symbols $\chi^t_\pm$ are not in $\mathcal{H}$ (although they do have the property of extending holomorpically for $\operatorname{Im}\xi_n<0$ resp. $\operatorname{Im}\xi_n>0$). Nevertheless, we can obtain some special results concerning application of $(\varkappa-iD_{x_n})^t_+$ to the symbol-kernels (recall Definition 2.2 (5)), that show how some compositions with $\Xi^t_{-,\mu,+}$ have good properties. This implies estimates of the corresponding boundary symbol operators in $H_2^{t,\varkappa}(\overline{\mathbf{R}}_+)$ spaces, which will be useful in the study of L_p estimates later. The estimates are convenient since there is no loss of regularity when one applies D_{x_n}.

THEOREM 3.7. *Let d and $\nu\in\mathbf{R}$, let t,t' and $t''\in\overline{\mathbf{R}}_+$, and let $\alpha\in\mathbf{N}^{n-1}$. When*

$$k(x',\xi,\mu)\in S_{1,0}^{d-1,\nu}(\mathbf{R}^{n-1}\times\overline{\mathbf{R}}^n_+,\mathcal{H}^+) \quad and \quad g(x',\xi,\eta_n,\mu)\in S_{1,0}^{d-1,\nu}(\mathbf{R}^{n-1}\times\overline{\mathbf{R}}^n_+,\mathcal{H}^+\hat{\otimes}\mathcal{H}^-_{-1}),$$

then the associated symbol-kernels and boundary symbol operators satisfy:

$$\begin{aligned}\|(\varkappa-iD_{x_n})^t_+\langle\xi'\rangle^{|\alpha|}D^\alpha_{\xi'}[(\varkappa-iD_{x_n})^{t'}_+\tilde{k}(x',x_n,\xi',\mu)]\|_{L_{2,x_n}(\mathbf{R}_+)}&\\ =\|\operatorname{OPK}_n(\langle\xi'\rangle^{|\alpha|}D^\alpha_{\xi'}(\varkappa-iD_{x_n})^{t'}_+\tilde{k})\|_{\mathcal{L}(\mathbf{C},H_2^{t,\varkappa}(\overline{\mathbf{R}}_+))}&\\ \dot{\leqslant}(\varrho^\nu+1)\varkappa^{d+t+t'-1/2};&\end{aligned}\tag{3.13}$$

$$\begin{aligned}\|(\varkappa-iD_{x_n})^t_+(\varkappa-iD_{y_n})^{t''}_+\langle\xi'\rangle^{|\alpha|}D^\alpha_{\xi'}(\varkappa-iD_{x_n})^{t'}_+\tilde{g}(x,y_n,\xi',\mu)\|_{L_{2,x_n,y_n}(\mathbf{R}^2_{++})}&\\ \dot{\leqslant}(\varrho^\nu+1)\varkappa^{d+t+t'+t''-1/2};&\end{aligned}\tag{3.14}$$

$$\begin{aligned}\|\operatorname{OPG}_n(\langle\xi'\rangle^{|\alpha|}D^\alpha_{\xi'}(\varkappa-iD_{x_n})^{t'}_+g)\|_{\mathcal{L}((H_{2;0}^{-t'',\varkappa}(\overline{\mathbf{R}}_+),H_2^{t,\varkappa}(\overline{\mathbf{R}}_+))}&\\ \dot{\leqslant}(\varrho^\nu+1)\varkappa^{d+t+t'+t''-1/2}.&\end{aligned}\tag{3.15}$$

(3.14) *and* (3.15) *hold also with $(\varkappa-iD_{x_n})^{t'}_+$ replaced by $(\varkappa-iD_{y_n})^{t'}_+$.*

Proof. Consider (3.13) in the case $\alpha=0$. The operator $\operatorname{OPK}_n(k)\colon\mathbf{C}\to\mathcal{S}(\overline{\mathbf{R}}_+)$ acts simply as multiplication of $v\in\mathbf{C}$ by the symbol-kernel $\tilde{k}(x',x_n,\xi',\mu)$; it is also called $\operatorname{OPK}_n(\tilde{k})$, and the application of $(\varkappa-iD_{x_n})^t_+$ to $\tilde{k}$ corresponds to composition of $(\varkappa-iD_{x_n})^t_+$ with $\operatorname{OPK}_n(k)$. (We recall that $\tilde{k}=\mathcal{F}^{-1}_{\xi_n\to x_n}k$ is in $e^+\mathcal{S}(\overline{\mathbf{R}}_+)$ as a function of x_n, usually identified with its restriction to $\{x_n>0\}$.) Hence, in view of (1.34) and (1.37),

$$\|(\varkappa-iD_{x_n})^{t'}_+\operatorname{OPK}_n(k)v\|_{H_2^{t,\varkappa}(\overline{\mathbf{R}}_+)}=\|(\varkappa-iD_{x_n})^{t+t'}_+\tilde{k}\|_{L_2(\mathbf{R}_+)}|v|=\|\tilde{k}\|_{H_2^{t+t',\varkappa}(\overline{\mathbf{R}}_+)}|v|,$$

and we may assume that $t'=0$. The symbol estimates for $\tilde{k}$ show that for all $m\in\mathbf{N}$,

$$\|\tilde{k}\|_{H_2^{m,\varkappa}(\overline{\mathbf{R}}_+)}=\|(\varkappa-iD_{x_n})^m\tilde{k}\|_{L_2(\mathbf{R}_+)}\dot{\leqslant}(\varrho^\nu+1)\varkappa^{d+m-1/2}.$$

Applying this with some $m>t$ and with m replaced by 0, we find by a simple interpolation, taking $1-\theta=t/m$ (cf. (1.17)):

$$\begin{aligned}\|\tilde{k}\|_{H_2^{t,\varkappa}(\overline{\mathbf{R}}_+)}&\dot{\leqslant}\|\tilde{k}\|^{1-\theta}_{H_2^{m,\varkappa}(\overline{\mathbf{R}}_+)}\|\tilde{k}\|^{\theta}_{H_2^{0,\varkappa}(\overline{\mathbf{R}}_+)}\\ &\dot{\leqslant}((\varrho^\nu+1)\varkappa^{d+m-1/2})^{t/m}((\varrho^\nu+1)\varkappa^{d-1/2})^{1-t/m}\dot{\leqslant}(\varrho^\nu+1)\varkappa^{d+t-1/2},\end{aligned}$$

showing (3.13) in this case. For $|\alpha|=1$, say $\alpha=(1,0,\dots,0)$, we note that

$$\begin{aligned}\langle\xi'\rangle D_{\xi_1}[r^+(\varkappa-iD_{x_n})^{t'}e^+\tilde{k}(x',x_n,\xi',\mu)]\\ &=\langle\xi'\rangle r^+\,\mathrm{OP}_n(t'(\varkappa-i\xi_n)^{t'-1}D_{\xi_1}\varkappa)e^+\tilde{k}+\langle\xi'\rangle(\varkappa-iD_{x_n})_+^{t'}D_{\xi_1}\tilde{k}\\ &=\langle\xi'\rangle t'(D_{\xi_1}\varkappa)(\varkappa-iD_{x_n})_+^{t'}(\varkappa-iD_{x_n})_+^{-1}\tilde{k}+\langle\xi'\rangle(\varkappa-iD_{x_n})_+^{t'}D_{\xi_1}\tilde{k},\end{aligned}$$

cf. (1.34). The latter two expressions are similar to those we have already treated, for $(\varkappa-iD_{x_n})_+^{-1}\tilde{k}$ is a Poisson symbol-kernel of order $d-1$ and regularity ν:

$$\mathcal{F}_{x_n\to\xi_n}r^+(\varkappa-iD_{x_n})^{-1}e^+\tilde{k}=h^+[(\varkappa-i\xi_n)^{-1}k(x',\xi,\mu)]\in S^{d-2,\nu}(\mathbf{R}^{n-1}\times\overline{\mathbf{R}}^n_+,\mathcal{H}^+)$$

by Lemma 3.4, and $D_{\xi_j}\tilde{k}$ is a Poisson symbol-kernel of order $d-1$ and regularity $\nu-1$; moreover, the factor $\langle\xi'\rangle D_{\xi_1}\varkappa$ is $\dot{\leqslant}\langle\xi'\rangle$. Since one has in general (cf. (2.1))

$$\langle\xi'\rangle^{|\alpha|}(\varrho^{\nu-|\alpha|}+1)\varkappa^{\sigma-|\alpha|}\,\dot{\leqslant}\,(\varrho^\nu+1)\varkappa^\sigma, \tag{3.16}$$

we conclude that (3.13) holds for $\alpha=(1,0,\dots,0)$. The general estimate (3.13) follows by iteration of this argument.

The proof of (3.14) is very similar. Again we begin with the case $\alpha=0$, and use that $(\varkappa-iD_{x_n})_+^t(\varkappa-iD_{x_n})_+^{t'}$ equals $(\varkappa-iD_{x_n})_+^{t+t'}$, so we may assume $t'=0$. In view of the symbol rules, we only have to account for the cases where t and $t''\in[0,1]$. Here we depart from the inequalities, valid by the definition of $S^{d-1,\nu}$,

$$\begin{aligned}\|\tilde{g}(x',x_n,y_n,\xi',\mu)\|_{L_{2,x_n,y_n}(\mathbf{R}^2_{++})}&\,\dot{\leqslant}\,(\varrho^\nu+1)\varkappa^d,\\ \|(\varkappa-iD_{x_n})\tilde{g}\|_{L_2(\mathbf{R}^2_{++})}+\|(\varkappa-iD_{y_n})\tilde{g}\|_{L_2(\mathbf{R}^2_{++})}&\,\dot{\leqslant}\,(\varrho^\nu+1)\varkappa^{d+1},\\ \|(\varkappa-iD_{x_n})(\varkappa-iD_{y_n})\tilde{g}\|_{L_2(\mathbf{R}^2_{++})}&\,\dot{\leqslant}\,(\varrho^\nu+1)\varkappa^{d+2},\end{aligned}\tag{3.17}$$

which show (3.14) for t and t'' equal to 0 or 1 (and $t'=0$, $\alpha=0$). Since one has for functions $f\colon\mathbf{R}_+\to X$ when X is a Hilbert space,

$$\|(\varkappa-iD_{x_n})_+^s f(x_n)\|_{L_2(\mathbf{R}_+;X)}=\|f(x_n)\|_{H_2^{s,\varkappa}(\overline{\mathbf{R}}_+;X)},$$

we can read the statements in (3.17) as Sobolev space estimates of vector valued functions. Then we obtain (3.14) (with $t'=0$, $\alpha=0$) by application of interpolation, first between the spaces $L_{2,x_n}(\mathbf{R}_+;Y)$ and $H_2^{1,\varkappa}(\overline{\mathbf{R}}_+;Y)$, with Y equal to $L_{2,y_n}(\mathbf{R}_+)$ or $H_2^{1,\varkappa}(\overline{\mathbf{R}}_+)$, and thereafter between the spaces $L_{2,y_n}(\mathbf{R}_+;X)$ and $H_2^{1,\varkappa}(\overline{\mathbf{R}}_+;X)$ with $X=H_2^{t,\varkappa}(\overline{\mathbf{R}}_+)$. The factors $\langle\xi'\rangle^{|\alpha|}D_{\xi'}^\alpha$ are included in a similar way as for $\tilde{k}$, by use of the Leibniz formula and Lemma 3.4.

For the explanation of (3.15), we let $\alpha=0$; the general case can afterwards be included by a calculation using the Leibniz formula. Then we can also assume that $t'=0$. The estimates (3.14) can be read as Hilbert–Schmidt norm estimates of the family of operators $\mathrm{OPG}_n(g')$ from $L_2(\mathbf{R}_+)$ to $L_2(\mathbf{R}_+)$ (parametrized by (x',ξ',μ)) with the integral operator kernels

$$\tilde{g}'(x',x_n,y_n,\xi',\mu)=(\varkappa-iD_{x_n})_+^t(\varkappa-iD_{y_n})_+^{t''}\tilde{g}(x',x_n,y_n,\xi',\mu).$$

One has a fortiori that the operator norms of $\mathrm{OPG}_n(g')$ in $\mathcal{L}(L_2(\mathbf{R}_+),L_2(\mathbf{R}_+))$ are bounded with the same bound as in (3.14). This carries over to Sobolev space estimates of $\mathrm{OPG}_n(g)$ itself, as follows: Let $u\in H_{2;0}^{-t'',\varkappa}(\overline{\mathbf{R}}_+)$ and set $v=(\varkappa+iD_{x_n})^{-t''}\bar{u}$. In view of (1.34), v identifies with a function in $L_2(\mathbf{R}_+)$, and moreover, by duality,

$$\begin{aligned}(\varkappa-iD_{x_n})_+^t\,\mathrm{OPG}_n(g)u&=(\varkappa-iD_{x_n})_+^t\int_0^\infty \tilde{g}(x',x_n,y_n,\xi',\mu)\overline{(\varkappa+iD_{y_n})^{t''}v(y_n)}\,dy_n\\&=\int_0^\infty[(\varkappa-iD_{x_n})_+^t(\varkappa-iD_{y_n})_+^{t''}\tilde{g}(x',x_n,y_n,\xi',\mu)]\bar{v}(y_n)\,dy_n\\&=\mathrm{OPG}_n(g')\bar{v}.\end{aligned}$$

Then the L_2 operator norm estimates of $\mathrm{OPG}_n(g')$ imply

$$\|\mathrm{OPG}_n(g)\|_{\mathcal{L}(H_{2;0}^{-t'',\varkappa}(\overline{\mathbf{R}}_+),H_2^{t,\varkappa}(\overline{\mathbf{R}}_+))}\dot{\leqslant}(\varrho^\nu+1)\varkappa^{d+t+t''}. \tag{3.18}$$

For the last assertion, one uses that the adjoint $\mathrm{OPG}_n(\tilde{g}(x',x_n,y_n,\xi',\mu))^*$ equals $\mathrm{OPG}_n(\bar{\tilde{g}}(x',y_n,x_n,\xi',\mu))$, which is of the same kind as above. □

We shall also need to analyze the effect of multiplication of the symbol-kernels by a fractional power of x_n, or rather, the study of the norms of boundary symbol operators in $L_2(\mathbf{R}_+,x_n^\delta)$ spaces, cf. (1.46). This is more delicate than the preceding study, since the multiplication by x_n lowers the regularity by 1, and the regularity concept does not always interpolate well.

Let $k(x',\xi,\mu)\in S^{d-1,\nu}(\mathbf{R}^{n-1}\times\overline{\mathbf{R}}_+^n,\mathcal{H}^+)$; then $\tilde{k}(x,\xi',\mu)$ satisfies, by definition,

$$\|\tilde{k}\|_{L_{2,x_n}(\mathbf{R}_+)}\dot{\leqslant}(\varrho^\nu+1)\varkappa^{d-1/2},\quad \|x_n\tilde{k}\|_{L_{2,x_n}(\mathbf{R}_+)}\dot{\leqslant}(\varrho^{\nu-1}+1)\varkappa^{d-3/2}. \tag{3.19}$$

Let $0<\delta<1$, and recall the interpolation inequality

$$\|x_n^\delta f(x_n)\|_{L_2(\mathbf{R}_+)}\dot{\leqslant}\|f(x_n)\|_{L_2(\mathbf{R}_+)}^{1-\delta}\|x_nf(x_n)\|_{L_2(\mathbf{R}_+)}^\delta; \tag{3.20}$$

it gives for $\tilde{k}$:

$$\begin{aligned}\|x_n^\delta\tilde{k}\|_{L_{2,x_n}}&\leqslant\|\tilde{k}\|_{L_{2,x_n}}^{1-\delta}\|x_n\tilde{k}\|_{L_{2,x_n}}^\delta\dot{\leqslant}(\varrho^\nu+1)^{1-\delta}(\varrho^{\nu-1}+1)^\delta\varkappa^{d-1/2-\delta}\\&\dot{\leqslant}(1+\varrho^{\nu(1-\delta)}+\varrho^{(\nu-1)\delta}+\varrho^{\nu-\delta})\varkappa^{d-1/2-\delta}.\end{aligned} \tag{3.21}$$

Here $\min\{\nu(1-\delta),(\nu-1)\delta,\nu-\delta\}=\nu-\delta$ if $\nu\leqslant 0$ and is $\geqslant 0$ if $\nu\geqslant 1$, so that (3.21) implies (recall that $\varrho^\sigma+1\doteq 1$ when $\sigma\geqslant 0$, cf. (2.1))

$$\|x_n^\delta\tilde{k}\|_{L_{2,x_n}}\mathrel{\dot{\leqslant}}(\varrho^{\nu-\delta}+1)\varkappa^{d-1/2-\delta}\quad\text{when }\nu\in\mathbf{R}\setminus\,]0,1[\,;\tag{3.22}$$

which is as good as one could hope for; note that ν and $\nu-\delta$ have the same sign. But when $\nu\in\,]0,1[$, we only get

$$\|x_n^\delta\tilde{k}\|_{L_{2,x_n}}\mathrel{\dot{\leqslant}}(\varrho^{\delta\nu-\delta}+1)\varkappa^{d-1/2-\delta},\quad\nu\in\,]0,1[\,,\tag{3.23}$$

where $\delta\nu-\delta=\nu-\delta-(1-\delta)\nu$ is <0 and $<\nu-\delta$. Note however that (3.21) is fine in the region $|\xi'|\geqslant\langle\mu\rangle$, for here $\langle\xi'\rangle\sim\langle\xi',\mu\rangle$, so (3.21) is equivalent with

$$\|x_n^\delta\tilde{k}\|_{L_{2,x_n}}\mathrel{\dot{\leqslant}}\varkappa^{d-1/2-\delta},\quad\text{when }|\xi'|\geqslant\langle\mu\rangle.\tag{3.24}$$

When $n>1$ (the case of principal interest for applications), we can use (3.24) together with a consideration of derivatives to extend (3.22) to all $\nu\neq\delta$, with just a logarithmic loss when $\nu=\delta$. The proof is given in the following theorem, where we also collect facts shown above.

THEOREM 3.8. *Let d and $\nu\in\mathbf{R}$, $\delta\in\,]0,1[$, $\alpha\in\mathbf{N}^{n-1}$, and assume that $n>1$ if $\nu\in\,]0,1[$. Let $k(x',\xi,\mu)$ be a Poisson symbol in $S^{d-1,\nu}(\mathbf{R}^{n-1}\times\overline{\mathbf{R}}^n_+,\mathcal{H}^+)$. Then*

$$\|x_n^\delta\langle\xi'\rangle^{|\alpha|}D_{\xi'}^\alpha\tilde{k}(x',x_n,\xi',\mu)\|_{L_{2,x_n}(\mathbf{R}_+)}\mathrel{\dot{\leqslant}}\begin{cases}(\varrho^{\nu-\delta}+1)\varkappa^{d-1/2-\delta} & \textit{when }\nu\neq\delta,\\ (|\log\varrho|^{1/2}+1)\varkappa^{d-1/2-\delta} & \textit{when }\nu=\delta.\end{cases}\tag{3.25}$$

Proof. Consider first the case $\alpha=0$. The result is proved above when $\nu\in\mathbf{R}\setminus\,]0,1[$, so we let $\nu\in\,]0,1[$. For $|\xi'|\geqslant\langle\mu\rangle$ we have (3.24) which shows the desired estimate in that region; it remains to consider $|\xi'|<\langle\mu\rangle$. Let $\xi'\neq 0$, then the basic trick is to integrate the following inequalities along the ray in ξ'-space from ξ' to $(\langle\mu\rangle/|\xi'|)\xi'$, using (3.24) at one endpoint. For $1\leqslant j\leqslant n-1$ we have (with norms in $L_{2,x_n}(\mathbf{R}_+)$):

$$\begin{aligned}\big|D_{\xi_j}\|x_n^\delta\tilde{k}\|^2\big|&\leqslant 2\int_0^\infty x_n^{2\delta}|\tilde{k}(x,\xi',\mu)D_{\xi_j}\tilde{k}(x,\xi',\mu)|\,dx_n\\&\leqslant 2\|x_n\tilde{k}\|\,\|x_n^{2\delta-1}D_{\xi_j}\tilde{k}\|\quad\text{for }\delta\geqslant\tfrac12,\end{aligned}\tag{3.26}$$

$$\big|D_{\xi_j}\|x_n^\delta\tilde{k}\|^2\big|\leqslant 2\|x_n^{2\delta}\tilde{k}\|\,\|D_{\xi_j}\tilde{k}\|\quad\text{for }\delta>0.\tag{3.27}$$

First we treat the values ν,δ with $\delta>\nu$. Here we prove by induction on l that for $l=0,1,\dots,l_0$, where $2^{-l_0-1}\leqslant\nu$,

$$\|x_n^\delta\tilde{k}\|\mathrel{\dot{\leqslant}}(\varrho^{\nu-\delta}+1)\varkappa^{d-1/2-\delta},\quad\text{when }\delta>\nu\text{ and }2^{-l-1}\leqslant\delta<2^{-l}.\tag{3.28}$$

The induction start is based on (3.26). When $\frac{1}{2}\leqslant\delta<1$, then $0\leqslant 2\delta-1<1$ and we can apply (3.22) (with ν and δ replaced by $\nu-1$ and $2\delta-1$) to the second factor in (3.26), which gives

$$\begin{aligned} \left|D_{\xi_j}\|x_n^\delta\tilde{k}\|^2\right| &\leqslant (\varrho^{\nu-1}+1)\varkappa^{d-3/2}(\varrho^{\nu-2\delta}+1)\varkappa^{d-3/2-(2\delta-1)} \\ &\dot{\leqslant}\, \varrho^{2\nu-2\delta-1}\varkappa^{2d-2-2\delta}; \end{aligned}$$

in the last inequality we used that $\delta\geqslant\nu$. It follows that

$$\begin{aligned} \|x_n^\delta\tilde{k}\|^2 &\dot{\leqslant}\, \varkappa^{2d-1-2\delta}+\int_{|\xi'|}^{\langle\mu\rangle}\langle t\rangle^{2\nu-2\delta-1}\langle\mu\rangle^{2d-2\nu-1}\,dt \\ &\dot{\leqslant}\, \langle\xi'\rangle^{2\nu-2\delta}\langle\mu\rangle^{2d-1-2\nu}+\langle\mu\rangle^{2d-1-2\delta}\,\dot{\leqslant}\,(\varrho^{\nu-\delta}\varkappa^{d-1/2-\delta})^2, \end{aligned} \tag{3.29}$$

which proves (3.28) for $l=0$. Since $\tilde{k}$ is smooth and the estimate is uniform in (ξ',μ) for $\xi'\neq0$, it extends to all $(\xi',\mu)\in\overline{\mathbf{R}}^n_+$. Now we have when $\delta>\nu$ and $2^{-(l'+1)-1}\leqslant\delta<2^{-(l'+1)}$ that $2\delta>\nu$ and $2^{-(l'+1)}\leqslant 2\delta<2^{-l'}$. So if (3.28) is proved for $l=l'$, (3.29) gives

$$\left|D_{\xi_j}\|x_n^\delta\tilde{k}\|^2\right|\,\dot{\leqslant}\,\varrho^{\nu-2\delta}\varkappa^{d-1/2-2\delta}\varrho^{\nu-1}\varkappa^{d-3/2}\,\dot{\leqslant}\,\varrho^{2\nu-2\delta-1}\varkappa^{2d-2-2\delta}, \tag{3.30}$$

and then (3.28) follows for $l=l'+1$ by integration. By induction (up to $l=l_0$), (3.25) is obtained for any $\delta>\nu$. An inspection of the proof shows that the case $\delta=\nu$ can be included, giving (3.25), when the estimate of the integral in (3.29) is replaced by a logarithmic estimate.

Finally, the values $\delta<\nu$ are treated as follows: If $0<\nu/2<\delta<\nu'$, then $2\delta>\nu$ and $2\nu-2\delta-1>-1$, hence (3.30) holds and gives (3.25) by integration. The remaining $\delta\in\,]0,\nu/2]$ can be included by interpolation of the estimates for $\delta=0$ and $\delta=3\nu/4$, much as in (3.20) ff.

When $\alpha\neq0$, $\langle\xi'\rangle^{|\alpha|}D^\alpha_{\xi'}\tilde{k}$ is of order d and regularity $\min\{\nu,|\alpha|\}$ (cf. [G2, Lemma 2.1.6]), so the preceding result applied to $\langle\xi'\rangle^{|\alpha|}D^\alpha_{\xi'}\tilde{k}$ implies (3.25). □

Note that the trick in Theorem 3.8 also improves [G2, (2.2.90)] when $n>1$.

We shall now consider singular Green symbols. Here one can study the effect of multiplication of the symbol-kernel with x_n^δ as well as y_n^δ, corresponding to letting $\mathrm{OPG}_n(g)$ end in or start in a weighted L_2 space over $\mathbf{R}_+$. What we shall need later is a mixture of this with the applications of $(\varkappa-iD_{x_n})^t_+$ and $\langle\xi'\rangle^{|\alpha|}D^\alpha_{\xi'}$ studied in Theorem 3.7, and we go directly to the needed result. Recall the notation (1.46).

THEOREM 3.9. *Let d and $\nu\in\mathbf{R}$, $\delta\in[0,1]$, t and $t'\in\overline{\mathbf{R}}_+$, $\alpha\in\mathbf{N}^{n-1}$, and assume that $n>1$ if $\nu\in\,]0,1[$. Let $g(x',\xi,\eta_n,\mu)\in S^{d-1,\nu}(\mathbf{R}^{n-1}\times\overline{\mathbf{R}}^n_+,\mathcal{H}^+\hat{\otimes}\mathcal{H}^-_{-1})$. Then the associated* *symbol-kernel* $\tilde g(x',x_n,y_n,\xi',\mu)$ *satisfies:*

$$\begin{aligned}&\|x_n^\delta(\varkappa-iD_{y_n})_+^t\langle\xi'\rangle^{|\alpha|}D_{\xi'}^\alpha(\varkappa-iD_{y_n})_+^{t'}\tilde g\|_{L_{2,x_n,y_n}(\mathbf{R}^2_{++})} \text{ and}\\ &\|y_n^\delta(\varkappa-iD_{x_n})_+^t\langle\xi'\rangle^{|\alpha|}D_{\xi'}^\alpha(\varkappa-iD_{x_n})_+^{t'}\tilde g\|_{L_{2,x_n,y_n}(\mathbf{R}^2_{++})} \text{ are}\\ &\qquad\dot\leqslant\begin{cases}(\varrho^{\nu-\delta}+1)\varkappa^{d-\delta+t+t'} & \text{when } \nu\neq\delta, \text{ or } \delta=0 \text{ or } 1;\\ (|\log\varrho|^{1/2}+1)\varkappa^{d-\delta+t+t'} & \text{when } \nu=\delta\in\,]0,1[\,.\end{cases}\end{aligned}\tag{3.31}$$

The operator $\mathrm{OPG}_n(g')$ *with symbol-kernel* $\tilde g'=\langle\xi'\rangle^{|\alpha|}D_{\xi'}^\alpha(\varkappa-iD_{x_n})_+^{t'}\tilde g$ *satisfies:*

$$\begin{aligned}&\|\mathrm{OPG}_n(g')\|_{\mathcal{L}(H_{2;0}^{-t,\varkappa}(\overline{\mathbf{R}}_+),L_2(\mathbf{R}_+,x_n^\delta))} \text{ and } \|\mathrm{OPG}_n(g')\|_{\mathcal{L}(L_2(\mathbf{R}_+,x_n^{-\delta}),H_2^{t,\varkappa}(\overline{\mathbf{R}}_+))} \text{ are}\\ &\qquad\dot\leqslant\begin{cases}(\varrho^{\nu-\delta}+1)\varkappa^{d-\delta+t} & \text{when } \nu\neq\delta, \text{ or } \delta=0 \text{ or } 1;\\ (|\log\varrho|^{1/2}+1)\varkappa^{d-\delta+t} & \text{when } \nu=\delta\in\,]0,1[\,.\end{cases}\end{aligned}\tag{3.32}$$

Proof. Consider the estimate of the first norm in (3.31). Since $x_n\tilde g$ is of order $d-1$ and regularity $\nu-1$, it follows from Theorem 3.7 and considerations like (3.11) that the estimate is valid when δ is 0 or 1. Then we apply a version of the preceding considerations on $\tilde k$ (the interpolation in (3.20) and the lifting from lower regularities in the proof of Theorem 3.8), now for vector valued functions. This gives the estimate of the first norm in (3.31), and the estimate of the first norm in (3.32) follows from this as in the passage from (3.14) to (3.15) in Theorem 3.7.

For the estimate of the second norm in (3.31), we just have to interchange the roles of x_n and y_n, then the estimate of the second norm in (3.32) follows by variant of the explanation in Theorem 3.7, where we use the duality (1.47). □

Remark 3.10. We shall not burden the exposition with more detailed conclusions based on (3.23) for $\nu\in\,]0,1[$ and $n=1$, that can be worked out when needed. Instead, let us mention another technique giving adequate resolvent estimates:

In the resolvent construction for normal elliptic boundary problems, the symbols of the boundary operators are of integer or half-integer regularity (cf. [G2, 3.3]), so it is only symbols of regularity $\frac12$ that are not covered by (3.22). But here one has additional information, namely e.g. that the Poisson symbol-kernels of regularity $\frac12$ and order d satisfy estimates

$$\|(L_{\varkappa,+})^\delta\tilde k(x',x_n,\xi',\mu)\|_{L^2_{x_n}(\mathbf{R}_+)}\dot\leqslant\varkappa^{d-1/2},\quad \text{for } \delta\in\big[0,\tfrac12\big[\,,\tag{3.33}$$

where $L_{\varkappa,+}$ is the Laguerre operator on $\mathbf{R}_+$, $L_{\varkappa,+}=-\varkappa^{-1}\partial_{x_n}x_n\partial_{x_n}+\varkappa x_n+1$. We recall from [G2, (2.2.15)] that $\|L^\delta_{\varkappa,+}\tilde k\|_{L^2(\mathbf{R}_+)}$ is equivalent to the ℓ^2_δ norm of the Laguerre expansion of $\tilde{k}$. Estimates like (3.33) hold both for the terms in the direct operator $\mathcal{A}_{\mu,\theta}$ and its inverse $\mathcal{B}_{\mu,\theta}$; see [G2, Remark 2.6.16, Theorems 3.2.3 3° and 3.3.1]. Now

$$(L_{\varkappa,+}u,u)_{L^2(\mathbf{R}_+)}=\varkappa^{-1}(x_n\partial_{x_n}u,\partial_{x_n}u)+\varkappa(x_nu,u)+\|u\|^2,$$

for $u\in\mathcal{S}(\overline{\mathbf{R}}_+)$, since $\gamma_0(x_n\partial_{x_n}u)=0$. The terms are $\geqslant 0$, so one gets the operator inequality

$$L_{\varkappa,+}\geqslant \varkappa x_n+1,\quad \text{in the sense that } (L_{\varkappa,+})^{-1}\leqslant(\varkappa x_n+1)^{-1}.$$

By an operator-theoretic monotonicity theorem (cf. e.g. Donoghue [D]), this implies $(L_{\varkappa,+})^{-a}\leqslant(\varkappa x_n+1)^{-a}$ for $a\in\,]0,1[$, and hence, by (3.33),

$$\begin{aligned}\|(\varkappa x_n+1)^{\delta}\tilde{k}\|_{L^2(\mathbf{R}_+)}&=((\varkappa x_n+1)^{2\delta}\tilde{k},\tilde{k})^{1/2}\\&\leqslant((L_{\varkappa,+})^{2\delta}\tilde{k},\tilde{k})^{1/2}=\|L^{\delta}_{\varkappa,+}\tilde{k}\|\,\dot{\leqslant}\,\varkappa^{d-1/2},\quad\text{for }\delta\in\left[0,\tfrac12\right[.\end{aligned}\tag{3.34}$$

Thus (3.25) *holds for* $\delta\in\left[0,\frac12\right[$ *and* $n=1$ *also, in this case.* Similar estimates hold for the trace symbol-kernels of class 0 entering in the resolvent construction. The arguments likewise apply to the singular Green symbol-kernels $\tilde{g}$ entering into the resolvent construction, of class 0, regularity $\frac12$ and order d, say. Here one finds by application of the above method for the x_n-variable and straightforward Sobolev space interpolation for the y_n-variable:

$$\begin{aligned}\|(\varkappa x_n+1)^{\delta}(\varkappa-iD_{y_n})^t_+\tilde{g}\|_{L_{2,x_n,y_n}(\mathbf{R}^2_{++})}&\leqslant\|(L_{\varkappa,+})^{\delta}(\varkappa-iD_{y_n})^t_+\tilde{g}\|_{L_2(\mathbf{R}^2_{++})}\\&\dot{\leqslant}\,\varkappa^{d+t}\quad\text{for }\delta\in\left[0,\tfrac12\right[,\ t\geqslant0.\end{aligned}\tag{3.35}$$

Such estimates hold also with x_n and y_n interchanged, so (3.31)–(3.32) *holds for* $\delta\in\left[0,\frac12\right[$ *and* $n=1$ *in this case.*

The methods also allow an improvement of [G2, Theorem 2.6.6], concerning the symbol $g^+(p)$ of the special s.g.o. $G^+(P_\mu)=r_+P_\mu e^-J$ (recall that $J\colon u(x',x_n)\mapsto u(x',-x_n)$).

THEOREM 3.11. *Let p be as in Proposition 3.1, with $\nu\notin-\frac12+\mathbf{N}$, and let $n>1$ if $\nu>-\frac12$. If $\nu\notin\mathbf{N}$, then $\tilde{g}^+(p)$ satisfies for all indices $\alpha,\beta\in\mathbf{N}^{n-1}$, $k,k',m,m',j\in\mathbf{N}$:*

$$\begin{aligned}&\|D^{\beta}_{x'}D^{\alpha}_{\xi'}x_n^kD^{k'}_{x_n}y_n^mD^{m'}_{y_n}D^j_\mu\tilde{g}^+(p)(x',x_n,y_n,\xi',\mu)\|_{L_{2,x_n,y_n}(\mathbf{R}^2_{++})}\\&\qquad\dot{\leqslant}\,(\varrho^{\nu-|\alpha|-k+k'-m+m'}+1)\varkappa^{d-|\alpha|-k+k'-m+m'-j};\end{aligned}\tag{3.36}$$

in particular, $g^+(p)$ belongs to $S^{d-1,\nu}(\mathbf{R}^{n-1}\times\overline{\mathbf{R}}^n_+,\mathcal{H}^+\hat{\otimes}\mathcal{H}^-_{-1})$.

If $\nu\in\mathbf{N}$, (3.36) *holds for all indices except those satisfying*

$$k\geqslant k',\quad m\geqslant m',\quad |\alpha|+k-k'+m-m'=\nu,\tag{3.37}$$

for which one has

$$\begin{aligned}\|D^{\beta}_{x'}D^{\alpha}_{\xi'}x_n^k D^{k'}_{x_n} y_n^m D^{m'}_{y_n} D^j_{\mu}\tilde{g}^+(p)(x',x_n,y_n,\xi',\mu)\|_{L_{2,x_n,y_n}(\mathbf{R}^2_{++})} \\ \dot{\leqslant} (|\log\varrho|^{1/2}+1)\varkappa^{d-|\alpha|-k+k'-m+m'-j};\end{aligned} \tag{3.38}$$

so $g^+(p)\in S^{d-1,\nu-\varepsilon}(\mathbf{R}^{n-1}\times\overline{\mathbf{R}}^n_+, \mathcal{H}^+\hat{\otimes}\mathcal{H}^-_{-1})$ *for any* $\varepsilon>0$.

Similar estimates hold for $g^-(p)$.

Proof. It is shown in [G2, Theorem 2.6.6] that $\tilde{g}^+(p)=(\mathcal{F}^{-1}_{\xi_n\to z_n}h^+p)|_{z_n=x_n+y_n}$ for $x_n, y_n>0$. By Theorem 3.2, the regularity of h^+p is $\nu+\frac{1}{2}$. Then we get by application of (3.25) with $\delta=\frac{1}{2}$ to $\tilde{p}^+=\mathcal{F}^{-1}_{\xi_n\to z_n}h^+p$:

$$\begin{aligned}\|\tilde{g}^+(p)(x',x_n,y_n,\xi',\mu)\|_{L_2(\mathbf{R}^2_{++})} &= \left(\int_{\mathbf{R}^2_{++}} |\tilde{p}^+(x',x_n+y_n,\xi',\mu)|^2\,dx_n\,dy_n\right)^{1/2} \\ &= \left(2\int_{\mathbf{R}_+} z_n|\tilde{p}^+(x',z_n,\xi',\mu)|^2\,dz_n\right)^{1/2} \\ &\dot{\leqslant} (\varrho^{\nu+1/2-1/2}+1)\varkappa^d = (\varrho^{\nu}+1)\varkappa^d,\end{aligned}$$

where ϱ^{ν} is replaced by $|\log\varrho|^{1/2}$ if $\nu=0$. In a similar way, we get improvements of the estimates in [G2, Theorem 2.6.6] of the derived symbols. For example we have, since $D_{\xi_n}p$ is of order $d-1$ and regularity $\nu-1$:

$$\begin{aligned}\|x_n\tilde{g}^+(p)(x',x_n,y_n,\xi',\mu)\|_{L_2(\mathbf{R}^2_{++})} &\leqslant \left(\int_{\mathbf{R}^2_{++}} (x_n+y_n)^2|\tilde{p}^+(x',x_n+y_n,\xi',\mu)|^2\,dx_n\,dy_n\right)^{1/2} \\ &= \left(2\int_{\mathbf{R}_+} z_n|\mathcal{F}^{-1}_{\xi_n\to z_n}h^+(D_{\xi_n}p)|^2\,dz_n\right)^{1/2} \\ &\dot{\leqslant} (\varrho^{\nu-1+1/2-1/2}+1)\varkappa^{d-1} = (\varrho^{\nu-1}+1)\varkappa^{d-1},\end{aligned}$$

with $\varrho^{\nu-1}$ replaced by $|\log\varrho|^{1/2}$ if $\nu=1$. — The result for $g^-(p)$ follows since $g^-(p)=g^+(p^*)^*$. □

This result permits an improvement of the regularity numbers in the general parametrix construction from [G2, 3.2], see [G5].

Remark 3.12. The logarithmic loss in case $\nu\in\mathbf{N}$ in Theorem 3.11 can be removed in some special cases, for example when p is a rational function of ξ and μ of the form $(f(x',\xi)g(x',\xi)^{-1}+\mu^l)^{-1}$ with f and g polynomial in ξ of degree $k+l$ resp. k (here k and $l\in\mathbf{N}$, and $d=-l$). The proof is somewhat long, and will be left out since it is not necessary for the resolvent estimates taken up [G5], where the loss of regularity is eliminated by other methods established in [G2].

Remark 3.13. Let us note that when $\nu=0$, and $d=0$, say, then the $L_2(\mathbf{R}_+)$ operator norm of $\mathrm{OPG}_n(g^+(p))$ is $\dot{\leqslant}1$ (without a logarithmic loss), simply because the operator equals $r^+\,\mathrm{OP}_n(p)e^-J$, where the $L_2(\mathbf{R})$ operator norm of $\mathrm{OP}_n(p)$ is $\dot{\leqslant}1$. One can extend this to uniform bounds in $\mathcal{L}(H_2^{t,\varkappa}(\overline{\mathbf{R}}_+), H_2^{t,\varkappa}(\overline{\mathbf{R}}_+))$ and $\mathcal{L}(L_2(\mathbf{R}_+, x_n^\delta), L_2(\mathbf{R}_+, x_n^\delta))$ by use of formulas such as (4.25) below and a related version containing the multiplication by x_n. But these estimates are far from the generality covered by Theorem 3.9, and we do not see how to obtain the estimates in Theorem 3.9 for $g^+(p)$ with general independent values of δ, t and t' (or just those values that enter in the proof of Theorem 4.1 below) without using Hilbert–Schmidt norm estimates, such as the estimates of the symbol-kernel in Theorem 3.11.

4. L_p estimates of parameter-dependent operators

4.1. Estimates on Euclidean space

We shall now extend the L_2 mapping properties shown in [G2, Section 2.5] to L_p spaces, $1<p<\infty$. The basic result is the following theorem on mapping properties for operators with x-uniformly resp. x'-uniformly estimated symbols (cf. Definition 2.1).

THEOREM 4.1. *Let d and $\nu\in\mathbf{R}$, let $p\in\,]1,\infty[$, and let $r\in\mathbf{Z}$.*

(1) *Let $p(x,\xi,\mu)\in S_{1,0}^{d,\nu}(\mathbf{R}^n\times\overline{\mathbf{R}}_+^{n+1})$. Then $P_\mu=\mathrm{OP}(p(x,\xi,\mu))$ is continuous for all $s\in\mathbf{R}$:*

$$\begin{aligned}&P_\mu\colon H_p^{s,\mu}(\mathbf{R}^n)\to H_p^{s-d,\mu}(\mathbf{R}^n)\quad \textit{and}\\ &P_\mu\colon B_p^{s,\mu}(\mathbf{R}^n)\to B_p^{s-d,\mu}(\mathbf{R}^n),\quad \textit{with norm } O(\langle\mu\rangle^{-\nu}+1).\end{aligned}\tag{4.1}$$

(2) *Let $k(x',\xi,\mu)\in S_{1,0}^{d-1,\nu}(\mathbf{R}^{n-1}\times\overline{\mathbf{R}}_+^n,\mathcal{H}^+)$, with $n>1$ if $\nu\in\,]0,1[$ and $p<2$. Then $K_\mu=\mathrm{OPK}(k)$ is continuous for all $s\in\mathbf{R}$:*

$$\begin{aligned}K_\mu\colon B_p^{s-1/p,\mu}(\mathbf{R}^{n-1})\to H_p^{s-d,\mu}(\overline{\mathbf{R}}_+^n)\cap B_p^{s-d,\mu}(\overline{\mathbf{R}}_+^n),\\ \textit{with norm } O(\langle\mu\rangle^{-\nu+[1/p-1/2]_+}+1),\end{aligned}\tag{4.2}$$

unless $0<\nu=\frac{1}{p}-\frac{1}{2}$, where the norm is $O(\langle\mu\rangle^\varepsilon)$ for any $\varepsilon>0$.

(3) *Let $t(x',\xi,\mu)\in S_{1,0}^{d,\nu}(\mathbf{R}^{n-1}\times\overline{\mathbf{R}}_+^n,\mathcal{H}_{r-1}^-)$ (of class r), with $n>1$ if $\nu\in\,]0,1[$ and $p>2$. Then $T_\mu=\mathrm{OPT}(t)$ is continuous for all $s>r+\frac{1}{p}-1$:*

$$\begin{aligned}T_\mu\colon H_p^{s,\mu}(\overline{\mathbf{R}}_+^n)+B_p^{s,\mu}(\overline{\mathbf{R}}_+^n)\to B_p^{s-d-1/p,\mu}(\mathbf{R}^{n-1}),\\ \textit{with norm } O(\langle\mu\rangle^{-\nu+[1/2-1/p]_+}+1),\end{aligned}\tag{4.3}$$

unless $0<\nu=\frac{1}{2}-\frac{1}{p}$, where the norm is $O(\langle\mu\rangle^\varepsilon)$ for any $\varepsilon>0$.

(4) *Let* $p(x,\xi,\mu)\in S^{d,\nu}_{1,0,\mathrm{uttr}}(\mathbf{R}^n\times\overline{\mathbf{R}}^{n+1}_+)$, *with* $n>1$ *if* $\nu\in\left]-\frac{1}{2},\frac{1}{2}\right[$ *and* $p<2$. *Then* $P_{\mu,+}=\mathrm{OP}(p(x,\xi,\mu))_+$ *is continuous for all* $s>\frac{1}{p}-1$:

$$\begin{aligned}&P_{\mu,+}\colon H^{s,\mu}_p(\overline{\mathbf{R}}^n_+)\to H^{s-d,\mu}_p(\overline{\mathbf{R}}^n_+) \quad \text{and}\\ &P_{\mu,+}\colon B^{s,\mu}_p(\overline{\mathbf{R}}^n_+)\to B^{s-d,\mu}_p(\overline{\mathbf{R}}^n_+), \quad \text{with norm } O(\langle\mu\rangle^{-\nu}+1).\end{aligned} \tag{4.4}$$

(5) *Let* $g(x',\xi,\eta_n,\mu)\in S^{d-1,\nu}_{1,0}(\mathbf{R}^{n-1}\times\overline{\mathbf{R}}^n_+,\mathcal{H}^+\hat{\otimes}\mathcal{H}^-_{r-1})$ *(of class r), with* $n>1$ *if* $\nu\in\left]0,1\right[$ *and* $p\neq 2$. *Then* $G_\mu=\mathrm{OPG}(g)$ *is continuous for all* $s>r+\frac{1}{p}-1$:

$$\begin{aligned}&G_\mu\colon H^{s,\mu}_p(\overline{\mathbf{R}}^n_+)\to H^{s-d,\mu}_p(\overline{\mathbf{R}}^n_+) \quad \text{and}\\ &G_\mu\colon B^{s,\mu}_p(\overline{\mathbf{R}}^n_+)\to B^{s-d,\mu}_p(\overline{\mathbf{R}}^n_+), \quad \text{with norm } O(\langle\mu\rangle^{-\nu+|1/p-1/2|}+1),\end{aligned} \tag{4.5}$$

unless $0<\nu=\left|\frac{1}{p}-\frac{1}{2}\right|$, *where the norm is* $O(\langle\mu\rangle^{\varepsilon})$ *for any* $\varepsilon>0$. *Moreover, if* $P_{\mu,+}$ *is as in* (4) *and* $P_{\mu,+}+G_\mu$ *is of class* $-m$ *for some* $m\in\mathbf{N}$, *then* (4.5) *holds with* G_μ *replaced by* $P_{\mu,+}+G_\mu$, *for all* $s>-m+\frac{1}{p}-1$.

(6) *Let* $p(x,\xi,\mu)$ *be as in* (4). *Then the s.g.o.* $G^+(P_\mu)$ *is continuous for all* $s>\frac{1}{p}-1$:

$$\begin{aligned}&G^+(P_\mu)\colon H^{s,\mu}_p(\overline{\mathbf{R}}^n_+)\to H^{s-d,\mu}_p(\overline{\mathbf{R}}^n_+) \quad \text{and}\\ &G^+(P_\mu)\colon B^{s,\mu}_p(\overline{\mathbf{R}}^n_+)\to B^{s-d,\mu}_p(\overline{\mathbf{R}}^n_+), \quad \text{with norm } O(\langle\mu\rangle^{-\nu}+1).\end{aligned} \tag{4.6}$$

Proof. (1) We shall apply Theorem 1.6. Since symbols $p(x,\xi,\mu)\in S^{0,\nu}_{1,0}(\mathbf{R}^n\times\overline{\mathbf{R}}^{n+1}_+)$ satisfy (1.44) with $C(p)=O(\langle\mu\rangle^{-\nu}+1)$ ($H_0=H_1=\mathbf{C}$), the first statement in (4.1) follows immediately for the case $d=s=0$. When s and d are general, one reduces to this case, cf. (1.5), by replacing P_μ by $\langle D,\mu\rangle^{s-d}P_\mu\langle D,\mu\rangle^{-s}$, which is a ps.d.o with symbol in $S^{0,\nu}_{1,0}$, hence is bounded in $L_p(\mathbf{R}^n)$, uniformly in μ. The statement for $B^{s,\mu}_p$ spaces follows by interpolation, cf. (1.16). This proves (1).

(2) For the boundary operators, results for fixed μ were proved in [G3] and [F1, 2]; and we shall to some extent follow [G3] in the argumentation. In view of the formula (2.35), K_μ can be considered as a vector valued μ-dependent ps.d.o. $Q(x',D',\mu)$ in the x' variable, such that the symbol takes values in a space of linear mappings $c\mapsto\tilde{k}(x',x_n,\xi',\mu)c$ from $\mathbf{C}$ to a function space over $\mathbf{R}_+$. Here $\tilde{k}(x',x_n,\xi',\mu)$ is in $\mathcal{S}(\overline{\mathbf{R}}_+)$ as a function of x_n, on which many different norms can be used.

For one thing, $\tilde{k}$ satisfies, by hypothesis, the estimates for all α and $\beta\in\mathbf{N}^{n-1}$ (recall that $\varkappa=\langle\xi',\mu\rangle$):

$$\begin{gathered}\|\langle\xi'\rangle^{|\alpha|}D^\alpha_{\xi'}D^\beta_{x'}(\tilde{k}(x',x_n,\xi',\mu)\varkappa^{-d+1/2})\|_{\mathcal{L}(\mathbf{C},L_2(\mathbf{R}_+))}\dot{\leqslant}\langle\mu\rangle^{-\nu}+1,\\ \text{for } x',\xi'\in\mathbf{R}^{n-1},\ \mu\geqslant 0,\end{gathered} \tag{4.7}$$

Then since $\mathrm{OP}_{x'}(\varkappa^{d-1/2})$ is an isometry of $H_p^{d-1/2,\mu}(\mathbf{R}^{n-1})$ onto $L_p(\mathbf{R}^{n-1})$ (cf. (1.5)), we find by application of Theorem 1.6:

$$K_\mu\colon H_p^{d-1/2,\mu}(\mathbf{R}^{n-1})\to L_p(\mathbf{R}^{n-1};L_2(\mathbf{R}_+)),\quad \text{with norm } O(\langle\mu\rangle^{-\nu}+1). \tag{4.8}$$

When $2\leqslant p<\infty$, we also use that

$$\|\langle\xi'\rangle^{|\alpha|}D_{\xi'}^{\alpha}D_{x'}^{\beta}D_{x_n}(\tilde{k}(x',x_n,\xi',\mu)\varkappa^{-d-1/2})\|_{\mathcal{L}(\mathbf{C};L_2(\mathbf{R}_+))}\mathrel{\dot{\leqslant}}\langle\mu\rangle^{-\nu}+1,$$

which gives, by use of Theorem 1.6 and (4.8),

$$K_\mu\colon H_p^{d+1/2,\mu}(\mathbf{R}^{n-1})\to L_p(\mathbf{R}^{n-1};H_2^1(\overline{\mathbf{R}}_+)),\quad \text{with norm } O(\langle\mu\rangle^{-\nu}+1).$$

Using Corollary 1.9 (2) we then find by application of real interpolation $(\cdot,\cdot)_{1/2-1/p,p}$ (cf. (1.16)) the boundedness of:

$$K_\mu\colon B_p^{d-1/p,\mu}(\mathbf{R}^{n-1})\to L_p(\mathbf{R}_+^n),\quad \text{with norm } O(\langle\mu\rangle^{-\nu}+1).$$

By composition to the left with operators $\Xi_{-,\mu,+}^{t}$, $t\in\mathbf{Z}$ (cf. (1.31–34)) we get moreover, since $\Xi_{-,\mu,+}^{t}K_\mu$ is a Poisson operator of order $d+t$ and regularity ν by Theorem 3.6,

$$K_\mu=\Xi_{-,\mu,+}^{-t}\,\Xi_{-,\mu,+}^{t}\,K_\mu\colon B_p^{t+d-1/p,\mu}(\mathbf{R}^{n-1})\to H_p^{t,\mu}(\overline{\mathbf{R}}_+^n),\quad \text{with norm } O(\langle\mu\rangle^{-\nu}+1).$$

The mapping property is extended to arbitrary $t\in\mathbf{R}$ by complex interpolation (cf. (1.28)); and by real interpolation we furthermore get

$$K_\mu\colon B_p^{t+d-1/p,\mu}(\mathbf{R}^{n-1})\to B_p^{t,\mu}(\overline{\mathbf{R}}_+^n),\quad \text{with norm } O(\langle\mu\rangle^{-\nu}+1).$$

This shows (4.2) for $p\geqslant 2$.

When $1<p<2$, we choose a $\delta\in\,]\frac{1}{p}-\frac{1}{2},\frac{1}{2}[$, $\delta\neq\nu$, and apply Theorem 1.6 to $\tilde{k}(x',x_n,\xi',\mu)\varkappa^{-d-1/2+\delta}$, considered as a ps.d.o. symbol in (x',ξ') for each μ with values in $\mathcal{L}(\mathbf{C},L_2(\mathbf{R}_+,x_n^\delta))$; it acts as in (4.7) with right hand side $O(\langle\mu\rangle^{-\nu+\delta}+1)$ by Theorem 3.8. This gives

$$K_\mu\colon H_p^{d+1/2-\delta,\mu}(\mathbf{R}^{n-1})\to L_p(\mathbf{R}^{n-1};L_2(\mathbf{R}_+,x_n^\delta)),\quad \text{with norm } O(\langle\mu\rangle^{-\nu+\delta}+1). \tag{4.9}$$

Here the interpolation $(\cdot,\cdot)_{(1/p-1/2)/\delta,\,p}$ applied to (4.8) and (4.9) gives, by use of Corollary 1.9 (1), (1.16) and (1.17):

$$\begin{gathered}K_\mu\colon B_p^{d-1/p,\mu}(\mathbf{R}^{n-1})\to L_p(\mathbf{R}_+^n),\\ \text{with norm } O((\langle\mu\rangle^{-\nu}+1)^{1-(1/p-1/2)/\delta}(\langle\mu\rangle^{-\nu+\delta}+1)^{(1/p-1/2)/\delta}).\end{gathered} \tag{4.10}$$

When $\nu\leqslant 0$, this is estimated by

$$O(\langle\mu\rangle^{|\nu|(1-(1/p-1/2)/\delta)+(|\nu|+\delta)(1/p-1/2)/\delta})=O(\langle\mu\rangle^{|\nu|+1/p-1/2}), \tag{4.11}$$

and when $\nu>\frac{1}{p}-\frac{1}{2}$, we can take $\delta<\nu$, getting $O(1)$; this shows the desired result if $\nu\leqslant 0$ or $\nu>\frac{1}{p}-\frac{1}{2}$. If $\nu=\frac{1}{p}-\frac{1}{2}$, we get the estimate $O(\langle\mu\rangle^{\varepsilon})$ by taking δ arbitrarily close to ν. Finally, if $\nu\in\left]0,\frac{1}{p}-\frac{1}{2}\right[$, we replace the above interpolation argument by one where we interpolate between two versions of (4.9) with δ' resp. δ, where $\nu<\delta'<\frac{1}{p}-\frac{1}{2}<\delta$, and use (1.50); this gives the desired estimate. $L_p(\mathbf{R}^n_+)$ is replaced by $H^{t,\mu}_p$ spaces and $B^{t,\mu}_p$ spaces in the same way as in the case $p\geqslant 2$; and this completes the proof of (2).

(3) If $r\geqslant 0$, we are dealing with operators and symbols as in (2.37), (2.36). For the terms $S_{j,\mu}\gamma_j$, the result follows from (1.29) and (1). For the term T'_μ of class 0, the result follows, for $\frac{1}{p}-1<s<\frac{1}{p}$, from (2) by duality, using that T'^*_μ is a Poisson operator of order $d+1$ and regularity ν; here $\frac{1}{p}-\frac{1}{2}$ is replaced by $\frac{1}{2}-\frac{1}{p}$, since the Poisson operator is considered in $L_{p'}$ spaces with $\frac{1}{p'}=1-\frac{1}{p}$. By composition to the right with operators $\Xi^t_{-,\mu,+}$ one gets (4.3) for $t+\frac{1}{p}-1<s<t+\frac{1}{p}$, $t\in\mathbf{N}$, and the exceptional values $s\in\frac{1}{p}+\mathbf{N}$ are included by complex resp. real interpolation. If $r=-m<0$ (cf. Section 2.2), we can write

$$T_\mu=T_\mu\Xi^m_{-,\mu,+}\Xi^{-m}_{-,\mu,+},$$

where $T_\mu\Xi^m_{-,\mu,+}$ is a trace operator of class 0, order $d+m$ and regularity ν according to Theorem 3.6. Then in view of (1.34), the region where (4.3) holds extends down to $s>-m+\frac{1}{p}-1$. This proves (3).

(4) Here we use the scheme of [BM1] in the same way as in [G3, Theorem 3.4], so the explanation can be brief. In view of the formulas

$$\begin{aligned} D_{x_n}e^{\pm}u&=e^{\pm}D_{x_n}u\mp i(\gamma_0u)(x')\otimes\delta(x_n),\\ D_{x_j}e^{\pm}u&=e^{\pm}D_{x_j}u,\quad \text{for } j<n; \end{aligned} \tag{4.12}$$

one has, setting $K_\mu v=r^+P_\mu(v(x')\otimes\delta(x_n))$;

$$\begin{aligned} D_{x_n}P_{\mu,+}u&=P_{\mu,+}D_{x_n}u+[D_{x_n},P_\mu]_+u-iK_\mu\gamma_0u,\\ D_{x_j}P_{\mu,+}u&=P_{\mu,+}D_{x_j}u+[D_{x_j},P_\mu]_+u,\quad \text{for } j<n. \end{aligned} \tag{4.13}$$

K_μ is a Poisson operator of order $d+1$, and its regularity is $\nu+\frac{1}{2}$ except when $\nu\in-\frac{1}{2}+\mathbf{N}$, where the regularity of K_μ is $\nu+\frac{1}{2}-\varepsilon$, any $\varepsilon>0$; these assertions follow from Theorem 3.2, when we use that P_μ can be written on (x',y_n)-form (cf. (2.29)): $P_\mu=\mathrm{OP}(q(x',y_n,\xi,\mu))$, and then the symbol of K_μ equals $h^+q(x',0,\xi,\mu)$.

When $\frac{1}{p}-1<s<\frac{1}{p}$, (4.4) follows directly from (1) in view of (1.28), first line. Next, it is proved for $t+\frac{1}{p}-1<s<t+\frac{1}{p}$ by induction in $t\in\mathbf{N}$ using (4.13). Here we apply (2)

to K_μ, noting that the norm in the various spaces will be $O(\langle\mu\rangle^{-\nu-1/2+\varepsilon+[1/p-1/2]_+}+1)$, which is $O(\langle\mu\rangle^{-\nu}+1)$ for small ε. The full range of s, and the generalization to Besov spaces, is obtained by interpolation as in [G3, Theorem 3.4]. This shows (4).

(5) If $r\geqslant 0$, we are considering operators and symbols as in (2.37), (2.36), with G'_μ of class 0. The terms $K_{j,\mu}\gamma_j$ are handled by (1.29) and (2). The result for a case $r=-m<0$ is derived from the case $r=0$ by writing $G_\mu=G_\mu\Xi^m_{-,\mu,+}\Xi^{-m}_{-,\mu,+}$, similarly to the treatment of T_μ in the proof of (3); here $G_\mu\Xi^m_{-,\mu,+}$ is of class 0. Thus we may assume that the class r is zero.

In view of (2.35), G_μ can be considered as a vector valued μ-dependent ps.d.o. $Q(x',D',\mu)$ in the x' variable, with the symbol valued in a space of linear mappings from a function space over $\mathbf{R}_+$ to another, $v(x_n)\mapsto(g(x',\xi',\mu,D_n)v)(x_n)$. Here $g(x',\xi',\mu,D_n)$ is an integral operator with kernel $\tilde g(x',x_n,y_n,\xi',\mu)$ lying in $\mathcal{S}(\overline{\mathbf{R}}^2_{++})$ as a function of x_n,y_n, so many different norms can be used.

Let $p\geqslant 2$. Consider first the case $d=0$. Here we have by Theorem 3.9 with $t=\delta$, $t'=0$, that for any $\delta\in[0,1]$ such that $\delta\neq\nu$ if $\nu\in\,]0,1[$, and any $\alpha,\beta\in\mathbf{N}^{n-1}$,

$$\begin{aligned}\|\langle\xi'\rangle^{|\alpha|}D^\alpha_{\xi'}D^\beta_{x'}g(x',\xi',\mu,D_n)\|_{\mathcal{L}(L_2(\mathbf{R}_+,x_n^{-\delta}),H_2^\delta(\overline{\mathbf{R}}_+))}\\ \leqslant\|\langle\xi'\rangle^{|\alpha|}D^\alpha_{\xi'}D^\beta_{x'}g(x',\xi',\mu,D_n)\|_{\mathcal{L}(L_2(\mathbf{R}_+,x_n^{-\delta}),H_2^{\delta,\varkappa}(\overline{\mathbf{R}}_+))}\\ \dot\leqslant(\langle\xi'\rangle/\langle\xi',\mu\rangle)^{\nu-\delta}+1\,\dot\leqslant\,\langle\mu\rangle^{-\nu+\delta}+1,\end{aligned}\tag{4.15}$$

since $\|v\|_{H_2^\delta(\overline{\mathbf{R}}_+)}\leqslant\|v\|_{H_2^{\delta,\varkappa}(\overline{\mathbf{R}}_+)}$. It then follows from Theorem 1.6 that G_μ is continuous

$$\begin{gathered}G_\mu\colon L_p(\mathbf{R}^{n-1};L_2(\mathbf{R}_+,x_n^{-\delta}))\to L_p(\mathbf{R}^{n-1};H_2^\delta(\overline{\mathbf{R}}_+)),\\ \text{with norm } O(\langle\mu\rangle^{-\nu+\delta}+1).\end{gathered}\tag{4.16}$$

If $p=2$, we get the continuity of $G_\mu\colon L_2(\mathbf{R}^n_+)\to L_2(\mathbf{R}^n_+)$, with norm $O(\langle\mu\rangle^{-\nu}+1)$ directly from this, taking $\delta=0$. Here in fact the estimates (4.15–16) follow directly from the symbol properties, and one need not appeal to Theorem 3.9.

Now let $p>2$. When $\nu\neq\frac12-\frac1p$, we use (4.16) with $\delta=\delta_1$ and $\delta=\delta_2$ satisfying $0\leqslant\delta_1<\frac12-\frac1p<\delta_2<\frac12$ and $\nu\notin[\delta_1,\delta_2]$; then we get by interpolation using Corollary 1.9 that

$$G_\mu\colon L_p(\mathbf{R}^n_+)\to L_p(\mathbf{R}^n_+),\quad\text{with norm } O(\langle\mu\rangle^{-\nu+1/2-1/p}+1).\tag{4.17}$$

When $\nu=\frac12-\frac1p$, we get the estimates with a loss of $\varepsilon>0$ as in (2).

If d is integer $\neq 0$, we apply the same treatment to $\Xi^{-d}_{-,\mu,+}G_\mu$, which is a s.g.o. of order and class 0, and regularity ν, by Theorem 3.6. Then we get in view of (1.34),

$$G_\mu\colon L_p(\mathbf{R}^n_+)\to H_p^{-d,\mu}(\overline{\mathbf{R}}^n_+),\quad\text{with norm}\tag{4.18}$$

$$O(\langle\mu\rangle^{-\nu+1/2-1/p}+1)\text{ for }\nu\neq\tfrac12-\tfrac1p,\quad O(\langle\mu\rangle^{\varepsilon})\text{ for }\nu=\tfrac12-\tfrac1p>0.\tag{4.19}$$

When d is noninteger, $\Xi_{-,\mu,+}^{-d} G_\mu$ is not a standard s.g.o., but some formulas can be used anyway. Assuming as we may that G_μ is given in y'-form, $G_\mu = \mathrm{OPG}(g(y',\xi,\eta_n,\mu))$, we can write

$$\Xi_{-,\mu,+}^{-d} G_\mu = \Xi_{-,\mu,+}^{t'} \Xi_{-,\mu,+}^{k} G_\mu = \Xi_{-,\mu,+}^{t'} G'_\mu = G''_\mu, \tag{4.20}$$

where $t'>0$, $k\in\mathbf{Z}$, G'_μ is the s.g.o. $\Xi_{-,\mu,+}^{k} G_\mu$ of order $d+k$, class 0 and regularity ν by Theorem 3.6, and G''_μ is the generalized s.g.o.

$$G''_\mu = \mathrm{OPG}(g'') = \mathrm{OP}_{x'}(g''(y',\xi',\mu,D_n)), \quad \text{where}$$
$$\tilde g''(y',x_n,y_n,\xi',\mu) = (\varkappa - iD_n)^{t'}_{\mathbf{R}_+} \tilde g'(y',x_n,y_n,\xi',\mu)$$
$$\tilde g'(y',x_n,y_n,\xi',\mu) = (\varkappa - iD_n)^{k}_{\mathbf{R}_+} \tilde g(y',x_n,y_n,\xi',\mu).$$

Theorem 3.9 applied to g' shows that $g''(y',\xi',\mu,D_n)$ satisfies

$$\begin{aligned} &\|\langle\xi'\rangle^{|\alpha|} D^\alpha_{\xi'} D^\beta_{x'} g''(y',\xi',\mu,D_n)\|_{\mathcal{L}(L_2(\mathbf{R}_+,x_n^{-\delta}),H_2^\delta(\overline{\mathbf{R}}_+))} \\ &\quad\leqslant \|\langle\xi'\rangle^{|\alpha|} D^\alpha_{\xi'} D^\beta_{x'} g''(y',\xi',\mu,D_n)\|_{\mathcal{L}(L_2(\mathbf{R}_+,x_n^{-\delta}),H_2^{\delta,\varkappa}(\overline{\mathbf{R}}_+))} \dot{\leqslant} \langle\mu\rangle^{-\nu+\delta}+1, \end{aligned} \tag{4.21}$$

for all $\delta\in[0,1]$ with $\delta\neq\nu$ if $\nu\in]0,1[$, all α and $\beta\in\mathbf{N}^{n-1}$, and hence it follows as in the first part of the proof that

$$G''_\mu : L_p(\mathbf{R}^n_+) \to L_p(\mathbf{R}^n_+), \quad \text{with norm (4.19).}$$

Then by (4.20), we get (4.18) using (1.34).

We have hereby obtained (4.18)–(4.19) for all $d\in\mathbf{R}$, when $p\geqslant 2$.

In order to generalize this easily to more general $H_p^{t,\mu}$ spaces and to $p<2$, we shall prove an auxiliary result that may be of some independent interest. The result is that the restriction of G_μ to $r^+\mathcal{S}_0(\overline{\mathbf{R}}^n_+)$ (cf. (1.25)) has a continuous extension

$$\begin{aligned} &\overline{G}_\mu : H^{t,\mu}_{p;0}(\overline{\mathbf{R}}^n_+) \to H_p^{t-d,\mu}(\overline{\mathbf{R}}^n_+), \quad \textit{for all } t\in\mathbf{R},\ 1<p<\infty, \text{ with norm} \\ &O(\langle\mu\rangle^{-\nu+|1/2-1/p|}+1) \text{ for } \nu\neq\left|\tfrac12-\tfrac1p\right|, \quad O(\langle\mu\rangle^{\varepsilon}) \text{ for } \nu=\left|\tfrac12-\tfrac1p\right|>0. \end{aligned} \tag{4.22}$$

We have up to now shown (4.22) for $t=0$ and $p\geqslant 2$. When $m\in\mathbf{Z}$, $G_\mu\Xi^{-m}_{+,\mu,+}$ can be written in a unique way as

$$G_\mu \Xi^{-m}_{+,\mu,+} = G'_\mu + \sum_{j=0}^{[-m]_+-1} K^{(m)}_{j,\mu}\gamma_j$$

with G'_μ s.g.o. of class 0 and suitable Poisson operators $K^{(m)}_{j,\mu}$, and then (cf. (1.25))

$$G_\mu \Xi^{-m}_{+,\mu,+}|_{r^+\mathcal{S}_0(\overline{\mathbf{R}}^n_+)} = G'_\mu|_{r^+\mathcal{S}_0(\overline{\mathbf{R}}^n_+)}.$$

Now $\Xi^m_{+,\mu,+}$ maps $r^+\mathcal{S}_0(\overline{\mathbf{R}}^n_+)$ homeomorphically onto $r^+\mathcal{S}_0(\overline{\mathbf{R}}^n_+)$ (see the statements before Theorem 1.2), so when $p\geqslant 2$, an application of (4.18) to G'_μ shows in view of (1.34) that for $f\in r^+\mathcal{S}_0(\overline{\mathbf{R}}^n_+)$,

$$\begin{aligned}\|G_\mu f\|_{H^{m-d,\mu}_p(\overline{\mathbf{R}}^n_+)} &= \|G'_\mu \Xi^m_{+,\mu,+} f\|_{H^{m-d,\mu}_p(\overline{\mathbf{R}}^n_+)}\\ &\leqslant \|G'_\mu\|_{\mathcal{L}(L_p(\mathbf{R}^n_+),H^{m-d,\mu}_p(\overline{\mathbf{R}}^n_+))}\|\Xi^m_{+,\mu,+}f\|_{L_p(\mathbf{R}^n_+)}\\ &\doteq \|G'_\mu\|_{\mathcal{L}(L_p(\mathbf{R}^n_+),H^{m-d,\mu}_p(\overline{\mathbf{R}}^n_+))}\|f\|_{H^{r,\mu}_{p;0}(\overline{\mathbf{R}}^n_+)}.\end{aligned}$$

Thus G_μ on $r^+\mathcal{S}_0(\overline{\mathbf{R}}^n_+)$ has a continuous extension $\overline{G}_\mu$ satisfying (4.22) when $t\in\mathbf{Z}$ and $2\leqslant p<\infty$. The validity for general $t\in\mathbf{R}$ follows by complex interpolation. The remaining values of p are included by duality, when we use (1.28), second line, and the fact that the adjoint of G_μ is another s.g.o. of order d, class 0 and regularity ν.

The estimate (4.22) implies in view of (1.28), first line, that when $\frac{1}{p}-1<t<\frac{1}{p}$,

$$G_\mu\colon H^{t,\mu}_p(\overline{\mathbf{R}}^n_+)\to H^{t-d,\mu}_p(\overline{\mathbf{R}}^n_+),\quad \text{with norm as in (4.22)}, \tag{4.23}$$

for any p. Writing G_μ as

$$G_\mu = G_\mu \Xi^{-k}_{-,\mu,+}\Xi^k_{-,\mu,+},\quad k\in\mathbf{N}, \tag{4.24}$$

where $G_\mu\Xi^{-k}_{-,\mu,+}$ is a s.g.o. of order $d-k$, class 0 and regularity ν according to Theorem 3.6, we extend the validity of (4.23) to all $t>\frac{1}{p}-1$ with $t-\frac{1}{p}\notin\mathbf{N}$, by use of (1.34). The exceptional values of $t\geqslant\frac{1}{p}$ are included by complex interpolation, cf. (1.28). Finally, a version of (4.23) with H_p replaced by B_p follows by real interpolation, cf. (1.28).

For the statement with $P_{\mu,+}+G_\mu$ of class $-m<0$ we observe that in view of Lemma 1.3, we just have to prove that $(P_{\mu,+}+G_\mu)\langle\mu\rangle^{\beta_0}D_1^{\beta_1}\dots D_n^{\beta_n}$ is continuous from $H^{s,\mu}_p(\overline{\mathbf{R}}^n_+)$ to $H^{s-d-m,\mu}_p(\overline{\mathbf{R}}^n_+)$ for $s>\frac{1}{p}-1$, with norm as asserted, for all $\beta_0+\beta_1+\dots+\beta_n=m$. But here

$$(P_{\mu,+}+G_\mu)D_1^{\beta_1}\dots D_n^{\beta_n}=(P_\mu D_1^{\beta_1}\dots D_n^{\beta_n})_+ + G_\mu^{(\beta_1,\dots,\beta_n)}$$

with $G_\mu^{(\beta_1,\dots,\beta_n)}$ of order $d+\beta_1+\dots+\beta_n$, class 0 and regularity ν (since the other terms are so), and multiplication by $\langle\mu\rangle^{\beta_0}$ is uniformly bounded from $H^{t,\mu}_p(\overline{\mathbf{R}}^n_+)$ to $H^{t-\beta_0,\mu}_p(\overline{\mathbf{R}}^n_+)$ for any t (cf. (1.35)), so the result follows from the case already treated and (4).

This completes the proof of (5).

(6) These estimates are generally better in their dependence on μ than what (5) would give, and can be obtained by treating $G^+(P_\mu)$ analogously to $P_{\mu,+}$ in (4). In fact, since

$$G^+(P_\mu)u=r^+P_\mu e^-Ju,$$

one has by (4.12),

$$\begin{aligned} D_n G^+(P_\mu)u &= r^+ D_n P_\mu e^- Ju \\ &= -G^+(P_\mu)D_n u + G^+([D_n, P_\mu])u + iK_\mu \gamma_0 u, \end{aligned} \tag{4.25}$$

with the same Poisson operator K_μ as in (4); and one also has

$$D_{x_j} G^+(P_\mu)u = -G^+(P_\mu)D_{x_j}u + G^+([D_{x_j}, P_\mu])u, \quad \text{for } j<n. \tag{4.26}$$

Then $G^+(P_\mu)$ has the desired continuity property for $\frac{1}{p}-1<s<\frac{1}{p}$ in view of (1), and this is lifted to all $s>\frac{1}{p}-1$ by induction (using (4.25–26)) and interpolation as in (4). □

Remark 4.2. The augmentation of $-\nu$ in (4.2) for $p<2$ and in (4.3) for $p>2$, hence in (4.5) for $p\neq 2$, cannot be avoided in general. For example, let $p<2$ and $\sigma\leqslant\frac{1}{2}$, and consider the Poisson operator K with symbol $k=\langle\xi'\rangle^\sigma(\langle\xi'\rangle+i\xi_n)^{-1}$; as a μ-dependent symbol it belongs to $S^{\sigma-1,\sigma-1/2}(\mathbf{R}^{n-1}\times\overline{\mathbf{R}}^n_+,\mathcal{H}^+)$ (see [G2, (2.3.51)] or make a simple direct calculation). Since $-\sigma+\frac{1}{2}+\frac{1}{p}-\frac{1}{2}=-\sigma+\frac{1}{p}>0$, $\langle\mu\rangle^{-\sigma+1/p}+1\sim\mu^{-\sigma+1/p}$ for $\mu\geqslant 1$. Here we have for $v\in\mathcal{S}(\mathbf{R}^{n-1})\setminus\{0\}$:

$$\begin{aligned} \frac{\|K\|_{\mathcal{L}(B_p^{\sigma-1/p,\mu}(\mathbf{R}^{n-1}),L_p(\mathbf{R}^n_+))}}{\mu^{-\sigma+1/p}} &\geqslant c\frac{\|K\langle D',\mu\rangle^{1/p-\sigma}v\|_{L_p(\mathbf{R}^n_+)}}{\mu^{1/p-\sigma}\|v\|_{B_p^{0,\mu}(\mathbf{R}^{n-1})}} \\ &= c\frac{\|K(\langle D',\mu\rangle/\mu)^{1/p-\sigma}v\|_{L_p(\mathbf{R}^n_+)}}{\|v\|_{B_p^{0,\mu}(\mathbf{R}^{n-1})}} \\ &\to c\frac{\|Kv\|_{L_p(\mathbf{R}^n_+)}}{\|v\|_{L_p(\mathbf{R}^{n-1})}}>0 \quad \text{for } \mu\to\infty, \end{aligned}$$

cf. (1.13), so the exponent on $\langle\mu\rangle$ in (4.2) cannot be lowered in this case.

For arbitrary p, the exponent on $\langle\mu\rangle$ cannot in general be less than $-\nu$. As a simple example for Poisson operators one can take K_μ with symbol

$$k(x',\xi,\mu)=\langle\xi'\rangle^\nu(\langle\xi',\mu\rangle+i\xi_n)^{-1}\in S^{\nu-1,\nu}(\mathbf{R}^{n-1}\times\overline{\mathbf{R}}^n_+,\mathcal{H}^+);$$

the symbol-kernel is $\tilde{k}=\langle\xi'\rangle^\nu e^{-\langle\xi',\mu\rangle x_n}$. Here we have for $v\in\mathcal{S}(\mathbf{R}^{n-1})\setminus\{0\}$, when $\mu\to\infty$:

$$\begin{aligned} \frac{\|K_\mu\|_{\mathcal{L}(B_p^{\nu-1/p,\mu}(\mathbf{R}^{n-1}),L_p(\mathbf{R}^n_+))}}{\mu^{-\nu}} &\geqslant c\frac{\left(\int|\int\langle\xi'\rangle^\nu e^{ix'\cdot\xi'-\varkappa x_n}\varkappa^{1/p-\nu}\hat{v}(\xi')\,d\xi'|^p\,dx\right)^{1/p}}{\mu^{-\nu}\|v\|_{B_p^{0,\mu}(\mathbf{R}^{n-1})}} \\ &= c\frac{\left(\int|\int\langle\xi'\rangle^\nu e^{ix'\cdot\xi'-\varkappa x_n/\mu}(\varkappa/\mu)^{1/p-\nu}\hat{v}(\xi')\,d\xi'|^p\,dx\right)^{1/p}}{\|v\|_{B_p^{0,\mu}(\mathbf{R}^{n-1})}} \\ &\to c\frac{\left(\int|\int\langle\xi'\rangle^\nu e^{ix'\cdot\xi'-x_n}\hat{v}(\xi')\,d\xi'|^p\,dx\right)^{1/p}}{\|v\|_{L_p(\mathbf{R}^{n-1})}} \\ &= c'\frac{\|\langle D'\rangle^\nu v\|_{L_p(\mathbf{R}^{n-1})}}{\|v\|_{L_p(\mathbf{R}^{n-1})}}>0. \end{aligned}$$

For trace operators, the related example $T_\mu = \langle D'\rangle^\nu \langle D',\mu\rangle^{1-1/p-\nu}\gamma_0$ with symbol $t(x',\xi,\mu)=\langle\xi'\rangle^\nu\langle\xi',\mu\rangle^{1-1/p-\nu}$ of order $1-\frac{1}{p}$, class 1 and regularity ν, is seen in a similar way to have the property

$$\|T_\mu\|_{\mathcal{L}(H_p^{1,\mu}(\overline{\mathbf{R}}_+^n),B_p^{0,\mu}(\mathbf{R}^{n-1}))} \geqslant c\mu^{-\nu} \quad \text{for } \mu\geqslant 1,$$

with $c>0$.

The analysis in [G3] of the parameter-independent case implies that the estimates (4.3) and (4.5) cannot be extended to lower values of s unless T_μ resp. G_μ is of a lower class than r. Therefore the information on the extension $\overline{G}_\mu$ in (4.22) is of interest; and we can add that (4.22) is valid also for operators of class >0, since terms of the form $K_{j,\mu}\gamma_j$ vanish on $r^+\mathcal{S}_0(\overline{\mathbf{R}}_+^n)$. There is a similar result for trace operators, derived by duality from (4.2), applied to the adjoint of the part of T_μ of class 0. Altogether, we have

COROLLARY 4.3. *Let T_μ and G_μ be as in Theorem 4.1. Their restrictions to $r^+\mathcal{S}_0(\overline{\mathbf{R}}_+^n)$ have continuous extensions $\overline{T}_\mu$ and $\overline{G}_\mu$ with the continuity properties, for all $t\in\mathbf{R}$:*

$$\overline{T}_\mu\colon H_{p;0}^{t,\mu}(\overline{\mathbf{R}}_+^n)\to B_p^{t-d-1/p,\mu}(\mathbf{R}^{n-1}), \tag{4.27}$$

with norm $O(\langle\mu\rangle^{-\nu+[1/2-1/p]_+}+1)$ for $\nu\neq\frac{1}{2}-\frac{1}{p}$ or $\nu\leqslant 0$; $O(\langle\mu\rangle^{\varepsilon})$ for $\nu=\frac{1}{2}-\frac{1}{p}>0$;

$$\overline{G}_\mu\colon H_{p;0}^{t,\mu}(\overline{\mathbf{R}}_+^n)\to H_p^{t-d,\mu}(\overline{\mathbf{R}}_+^n), \tag{4.28}$$

with norm $O(\langle\mu\rangle^{-\nu+|1/2-1/p|}+1)$ for $\nu\neq\left|\frac{1}{2}-\frac{1}{p}\right|$, $O(\langle\mu\rangle^{\varepsilon})$ for $\nu=\left|\frac{1}{2}-\frac{1}{p}\right|>0$. (Here H_p can be replaced by B_p.)

Remark 4.4. In some cases in (2)–(6) we had to assume $n>1$ when $\nu\in\,]0,1[$ in order to apply Theorems 3.8 and 3.9 in the proofs. Note however that for $n=1$, the estimates needed in the proofs for K_μ, T_μ and G_μ are furnished by Remark 3.10 in the cases with regularity $\frac{1}{2}$ and certain Laguerre estimates described there. Thus Theorem 4.1 extends to $n=1$ in these cases. This suffices to get uniform mapping properties for all $n\geqslant 1$ for the operators entering in the resolvent construction (cf. [G5]), where the ps.d.o. part has regularity $\geqslant 1$ and the other operators have regularity $\geqslant\frac{1}{2}$ and satisfy the needed Laguerre estimates. (For the remaining cases we note that regularity in $]0,1[$ implies of course regularity 0; the results may be improved slightly by a discussion departing from (3.23).)

4.2. Estimates of operators on manifolds

Let us first make some observations on negligible operators. By (2.18) with ξ replaced by ξ' (cf. also [G2, Lemma 2.4.3]), a singular Green symbol-kernel $\tilde{g}(x',x_n,y_n,\xi',\mu)$ belongs to

$$S_{1,0}^{-\infty,\nu'+1-\infty}(\mathbf{R}^{n-1}\times\overline{\mathbf{R}}^n_+,\mathcal{S}(\overline{\mathbf{R}}^2_{++}))=\bigcap_{d\in\mathbf{Z}}S_{1,0}^{d-1,\nu'+d}(\mathbf{R}^{n-1}\times\overline{\mathbf{R}}^n_+,\mathcal{S}(\overline{\mathbf{R}}^2_{++}))$$

if and only if

$$\begin{aligned}\|D^\beta_{x'}x_n^k D^{k'}_{x_n}y_n^m D^{m'}_{y_n}D^\alpha_{\xi'}D^j_\mu\tilde{g}(x',x_n,y_n,\xi',\mu)\|_{L_{2,x_n,y_n}(\mathbf{R}^2_{++})}\\ \dot{\leqslant}\langle\xi'\rangle^{-N}\langle\mu\rangle^{-\nu'+[k-k']_-+[m-m']_--j}\\ \text{for all indices }\alpha,\beta\in\mathbf{N}^{n-1},\ k,k',m,m',j,N\in\mathbf{N};\end{aligned}\tag{4.29}$$

with constants depending on the indices and on $\tilde{g}$. These symbol-kernels (and corresponding operators and symbols) will be said to be *negligible of regularity* $\nu'+1$ (within the uniform symbol spaces); they are of degree $-N$, order $1-N$ and regularity $\nu'+1-N$ for any N. The operator kernels corresponding to these symbol-kernels,

$$\mathcal{K}_G(x,y,\mu)=\mathcal{F}^{-1}_{\xi'\to z'}\tilde{g}(x',x_n,y_n,\xi',\mu)|_{z'=x'-y'},\tag{4.30}$$

are characterized by the conditions:

$$\begin{aligned}\|D^\beta_{x'}D^\gamma_{z'}x_n^k D^{k'}_{x_n}y_n^m D^{m'}_{y_n}D^j_\mu\mathcal{K}_G(x',x_n,x'-z',y_n,\mu)\|_{L_{2,x_n,y_n}(\mathbf{R}^2_{++})}\\ \dot{\leqslant}\langle z'\rangle^{-N'}\langle\mu\rangle^{-\nu'+[k-k']_-+[m-m']_--j}\\ \text{for all indices }\gamma,\beta\in\mathbf{N}^{n-1},\ k,k',m,m',j,N'\in\mathbf{N}.\end{aligned}\tag{4.31}$$

The passage between the set of estimates (4.29) and the set of estimates (4.31) is worked out in a similar way as in the study of ps.d.o.s in Lemma 2.4. As usual, symbol sequences can be assigned symbols in the appropriate asymptotic sense.

For the μ-independent case, a similar (simpler) characterization holds with μ omitted from the formulas (see (4.34 i) below).

When P_μ is a negligible ps.d.o. of regularity ν', the kernel $\mathcal{K}_{P,+}(x,y,\mu)$ of $P_{\mu,+}$ satisfies, cf. (2.20),

$$\begin{aligned}|\langle x-y\rangle^{N'}D^\beta_x D^\gamma_y D^j_\mu\mathcal{K}_{P,+}(x,y,\mu)|\ \dot{\leqslant}\ \langle\mu\rangle^{-\nu'-j}\\ \text{for all }\gamma,\beta\in\mathbf{N}^n,\ N',j\in\mathbf{N},\ \text{ when }x,y\in\overline{\mathbf{R}}^n_+.\end{aligned}\tag{4.32}$$

Conversely, let $\mathcal{K}(x,y,\mu)$ be a function in $C^\infty(\overline{\mathbf{R}}^n_+\times\overline{\mathbf{R}}^n_+\times\overline{\mathbf{R}}_+)$ satisfying (4.32). Then we observe that in the region $\{x,y\in\overline{\mathbf{R}}^n_+\mid x_n\leqslant 1 \text{ or } y_n\leqslant 1\}$, (4.32) implies in fact

$$\begin{aligned}|x_n^k y_n^m\langle x'-y'\rangle^{N'}D^\beta_x D^\gamma_y D^j_\mu\mathcal{K}(x,y,\mu)|\ \dot{\leqslant}\ \langle\mu\rangle^{-\nu'-j}\\ \text{for all }\gamma,\beta\in\mathbf{N}^n,\ N',k,m,j\in\mathbf{N}.\end{aligned}\tag{4.33}$$

Hence, by the extension procedure described by Seeley in [Se1], used first from $x_n \in \overline{\mathbf{R}}_+$ to $x_n \in \mathbf{R}$ and next from $y_n \in \overline{\mathbf{R}}_+$ to $y_n \in \mathbf{R}$, we can find a C^∞ function $\mathcal{K}_1(x,y,\mu)$ on $\mathbf{R}^n \times \mathbf{R}^n \times \overline{\mathbf{R}}_+$ such that (4.32) holds in the whole space. Then by Lemma 2.4, $\mathcal{K}_1$ is the kernel of a ps.d.o. P_μ that is negligible of regularity ν', and $\mathcal{K}(x,y,\mu) = \mathcal{K}_{P,+}(x,y,\mu)$. Thus (4.32) characterizes the kernels of negligible operators $P_{\mu,+}$ of regularity ν'.

Similar considerations hold for μ-independent symbols, where we drop all reference to μ and to regularities.

Note in particular that for μ-independent operators, the negligible singular Green operators G resp. negligible pseudo-differential operators P_+ in the uniform calculus are those with kernel $\mathcal{K}_G$ resp. $\mathcal{K}_{P,+}$ satisfying, on $\overline{\mathbf{R}}^n_+ \times \overline{\mathbf{R}}^n_+$,

$$\sup_{x',y' \in \mathbf{R}^{n-1}} \|\langle x'-y'\rangle^{N'} x_n^k y_n^m D^\alpha_{x,y} \mathcal{K}_G(x,y)\|_{L_{2,x_n,y_n}(\mathbf{R}^2_{++})} \dot{\leqslant} 1 \quad \text{for all indices,} \tag{4.34 i}$$

$$\text{resp.} \quad \sup_{x,y \in \overline{\mathbf{R}}^n_+} |\langle x-y\rangle^{N'} D^\alpha_{x,y} \mathcal{K}_{P,+}(x,y)| \dot{\leqslant} 1 \quad \text{for all indices.} \tag{4.34 ii}$$

Here the set of estimates (4.34 i) implies the set of estimates (4.34 ii), since

$$\sup_{x_n,y_n} |f(x_n,y_n)|^4 \dot{\leqslant} \|f\| \, \|D_{x_n} f\| \, \|D_{y_n} f\| \, \|D_{x_n} D_{y_n} f\|,$$

for $f \in \mathcal{S}(\overline{\mathbf{R}}^2_{++})$, with norms in $L_2(\mathbf{R}^2_{++})$. (One cannot conclude the other way.)

Remark 4.5. In the local parameter-independent calculus (of [BM2]), there is no distinction between the two types of integral operators $\mathcal{K}_G$ and $\mathcal{K}_{P,+}$. In the local μ-dependent calculus of [G2], the negligible ps.d.o.s of regularity ν' are a subset of the negligible s.g.o.s of regularity $\nu'+1$, since sup-norm estimates over bounded sets imply L_2 estimates; whereas the above shows that a converse inclusion holds in the μ-independent uniformly estimated case. In the cases with both μ-dependence and uniform estimates, the two types of integral operators differ in a complicated way, so one should avoid using them at the same time.

Negligible trace and Poisson operators are characterized in similar ways. For example, in the μ-independent uniform calculus, the negligible trace operators of class $r \geqslant 0$ are of the form $T = \sum_{0 \leqslant j < r} S_j \gamma_j + T'$, where the S_j are negligible ps.d.o.s on $\mathbf{R}^{n-1}$ and T' has a kernel $\mathcal{K}_{T'}(x',y)$ satisfying

$$\sup_{x',y' \in \mathbf{R}^{n-1}} \|\langle x'-y'\rangle^{N'} y_n^m D^\beta_{x',y} \mathcal{K}_{T'}(x',y)\|_{L_{2,y_n}(\mathbf{R}_+)} \dot{\leqslant} 1 \quad \text{for all indices;} \tag{4.35}$$

and the negligible Poisson operators have similar kernels, only with y_n replaced by x_n. In the μ-dependent calculus, (4.35) is replaced by

$$\begin{aligned} \sup_{x',y' \in \mathbf{R}^{n-1}} & \|\langle x'-y'\rangle^{N'} D^\beta_{x',y'} y^m D^{m'}_{y_n} D^j_\mu \mathcal{K}_{T'}(x',y,\mu)\|_{L_{2,y_n}(\mathbf{R}_+)} \\ & \dot{\leqslant} \langle\mu\rangle^{-\nu'+[m-m']_+-j} \quad \text{for all indices.} \end{aligned} \tag{4.36}$$

We find as in [G2]:

THEOREM 4.6. *The spaces of parameter-dependent uniformly estimated Poisson, trace and singular Green operators of a given order, class and regularity, are invariant under admissible coordinate changes that preserve the set* $\{x_n=0\}$.

Proof. One goes through the proof of [G2, Theorem 2.4.11], now with global estimates, using the arguments given around (2.28) for the coordinate changes in x', y'-variables, and using the above characterizations of negligible operators. For the x_n, y_n-variables, one passes via symbols of the form in [G2, Remark 2.4.9]. □

Also the uniform transmission condition for ps.d.o.s is preserved under admissible coordinate changes.

For an admissible manifold $\bar\Omega$ (cf. Section 1.2), the various types of operators are now generalized to mappings between sections of vector bundles over $\bar\Omega$ and Γ in the way explained sketchily in [G2, end of Section 2.4]. For precision, we assume that a system of local trivializations has been chosen with the properties listed in Lemma 1.5, for given admissible vector bundles E and E_1 over $\bar\Omega$, F and F_1 over Γ. Then, for example, $G_\mu: C^\infty_{(0)}(\bar\Omega, E)\to C^\infty(\bar\Omega, E_1)$ is a uniformly estimated, parameter-dependent singular Green operator, when each term in the decomposition $G_\mu=\sum_{j_1,j_2\leqslant j_0}\varrho_{j_1}G_\mu\varrho_{j_2}$ gives such an operator in the local coordinates, i.e., when (cf. also [G2, A.5])

$$(\varrho_{j_1}G_\mu\varrho_{j_2})_{*,i}=\varphi_{1,i}\circ(\varrho_{j_1}G_\mu\varrho_{j_2})\circ\varphi_i^{-1} \tag{4.37}$$

is an $N_1\times N$-matrix formed s.g.o. on $\{1,\dots,m\}\times\bar{\mathbf{R}}^n_+$, where $\varphi_i: E|_{\Omega_i}\to\Xi_i\times\mathbf{C}^N$ and $\varphi_{1,i}: E_1|_{\Omega_i}\to\Xi_i\times\mathbf{C}^{N_1}$ are the local trivializations associated with a coordinate set Ω_i containing $\operatorname{supp}\varrho_{j_1}\cup\operatorname{supp}\varrho_{j_2}$. The other types of operators are similarly described.

Then the continuity properties for scalar operators on $\bar{\mathbf{R}}^n_+$ imply the continuity properties for general operators:

COROLLARY 4.7. *Let $\bar\Omega$ be an admissible (cf. Section 1.2) manifold of dimension $n>1$ with boundary Γ, and let E and E_1, F and F_1 be admissible vector bundles over $\bar\Omega$ resp. Γ. Let P_μ be a ps.d.o. going from a bundle $\tilde E$ to another $\tilde E_1$, where $\tilde E$ and $\tilde E_1$ extend E resp. E_1 to a an admissible neighboring boundaryless manifold Σ, P_μ having the transmission property at Γ; and let K_μ be a Poisson operator going from F to E_1, T_μ a trace operator going from E to F_1, G_μ a singular Green operator going from E to E_1, and S_μ a ps.d.o. going from F to F_1, all operators being parameter-dependent with uniformly x-estimated symbols when considered in the local coordinates, of order $d\in\mathbf{R}$, class $r\in\mathbf{Z}$ and regularity $\nu\in\mathbf{R}$. Then the mapping properties in Theorem 4.1 hold with the $H^{s,\mu}_p$ and $B^{s,\mu}_p$ spaces over $\bar{\mathbf{R}}^n_+$ and $\mathbf{R}^{n-1}$ replaced by $H^{s,\mu}_p$ and $B^{s,\mu}_p$ spaces of sections of the bundles over $\bar\Omega$ resp. Γ.*

For $n=1$, the result also holds when the symbols in local coordinates have the properties described in Remark 4.4.

5. Composition of Green operators

For the operators other than the ps.d.o.s, the establishing of rules of calculus for the uniformly estimated symbol classes is new even in the parameter-independent case. (In the studies of L_p mapping properties in [G3] and [F2], symbol classes with uniform estimates were considered, but the general composition rules were only used for operators on compact manifolds, where the rules from [BM2], [G2], [R-S1] suffice.) So here we must include full explanations of the parameter-independent case. Sometimes this will be in the form of a specialization of the (usually more complicated) parameter-dependent case, to save space.

Theorem 4.1 shows the continuity of systems $\mathcal{A}_\mu$ (also called Green operators):

$$\mathcal{A}_\mu = \begin{pmatrix} P_{\mu,+}+G_\mu & K_\mu \\ T_\mu & S_\mu \end{pmatrix} : \begin{matrix} H_p^{s,\mu}(\overline{\mathbf{R}}_+^n)^N \\ \times \\ B_p^{s-1/p,\mu}(\mathbf{R}^{n-1})^M \end{matrix} \rightarrow \begin{matrix} H_p^{s-d,\mu}(\overline{\mathbf{R}}_+^n)^{N_1} \\ \times \\ B_p^{s-d-1/p,\mu}(\mathbf{R}^{n-1})^{M_1} \end{matrix}, \tag{5.1}$$

$$\text{with norm } O(\langle\mu\rangle^{-\nu+|1/p-1/2|})+1, \quad \text{for } s>r+\tfrac{1}{p}-1,$$

when P_μ, G_μ, K_μ, T_μ and S_μ, are, respectively, a ps.d.o. in $\mathbf{R}^n$ satisfying the uniform transmission condition, a singular Green operator on $\mathbf{R}_+^n$, a Poisson operator from $\mathbf{R}^{n-1}$ to $\mathbf{R}_+^n$, a trace operator from $\mathbf{R}_+^n$ to $\mathbf{R}^{n-1}$, and a ps.d.o. in $\mathbf{R}^{n-1}$, all parameter-dependent with uniformly x-estimated symbols, of order $d\in\mathbf{R}$, class $r\in\mathbf{Z}$ and regularity $\nu\in\mathbf{R}$. Here $\mathcal{A}_\mu$ can be composed with another Green operator $\mathcal{A}'_\mu$ to the right, when the dimensions N and M fit with the range dimensions for $\mathcal{A}'_\mu$. (No precautions concerning compact support are needed, as in the local calculus.)

The composition rules we shall show, are summed up in the following theorem:

THEOREM 5.1. *Let $\mathcal{A}_\mu$ and $\mathcal{A}'_\mu$ be Green operators as in* (5.1), *of order d resp. d', class r resp. r' and regularity ν resp. ν',*

$$\mathcal{A}_\mu = \begin{pmatrix} P_{\mu,+}+G_\mu & K_\mu \\ T_\mu & S_\mu \end{pmatrix}, \quad \mathcal{A}'_\mu = \begin{pmatrix} P'_{\mu,+}+G'_\mu & K'_\mu \\ T'_\mu & S'_\mu \end{pmatrix}. \tag{5.2}$$

Then (*when the matrix dimensions match*) *$\mathcal{A}_\mu\mathcal{A}'_\mu$ is again a Green operator*:

$$\begin{aligned} &\mathcal{A}''_\mu = \mathcal{A}_\mu\mathcal{A}'_\mu = \begin{pmatrix} P''_{\mu,+}+G''_\mu & K''_\mu \\ T''_\mu & S''_\mu \end{pmatrix}, \quad \textit{where } P''_\mu = P_\mu P'_\mu, \\ &G''_\mu = -L(P_\mu, P'_\mu)+P_{\mu,+}G'_\mu+G_\mu P'_{\mu,+}+G_\mu G'_\mu+K_\mu T'_\mu, \\ &K''_\mu = P_{\mu,+}K'_\mu+G_\mu K'_\mu+K_\mu S'_\mu, \\ &T''_\mu = T_\mu P'_{\mu,+}+T_\mu G'_\mu+S_\mu T'_\mu, \quad S''_\mu = T_\mu K'_\mu+S_\mu S'_\mu, \end{aligned} \tag{5.3}$$

of order $d''=d+d'$ and class $r''=\max\{r', r+d'\}$, with $L(P_\mu, P'_\mu)$ of regularity $m(\nu,\nu')-\varepsilon$ (any $\varepsilon>0$) and all the other terms of regularity $m(\nu,\nu')$ (cf. (2.31)). The same asymptotic symbol formulas as in [G2, 2.7] *are valid.*

The proof is continued through the major part of this section.

For the composed terms without P_μ or P'_μ, the proof is a straightforward generalization of that in [G2, 2.6–2.7], where the rules for the boundary symbol operators shown in [G2, 2.6] are combined with the pseudodifferential arguments established in Theorem 2.7 above, applied (in suitable vector valued form) with respect to the x' variable.

For the operator $(P_\mu P'_\mu)_+$, the result follows directly from Theorem 2.7.

It remains to treat $L(P_\mu, P'_\mu)=(P_\mu P'_\mu)_+ - P_{\mu,+}P'_{\mu,+}$ and the compositions

$$P_{\mu,+}G'_\mu, \quad G_\mu P'_{\mu,+}, \quad P_{\mu,+}K'_\mu, \quad T_\mu P'_{\mu,+}. \tag{5.4}$$

For $L(P_\mu, P'_\mu)$ we recall the formula, shown in [G2, (2.6.23)],

$$\begin{aligned} L(P_\mu, P'_\mu) &= \sum_{0\leqslant m<d'} K_{m,\mu}\gamma_m + G^+(P_\mu)G^-(P'_\mu), \quad \text{with} \\ G^+(P_\mu) &= r^+P_\mu e^- J, \quad G^-(P'_\mu) = Jr^-P'_\mu e^+ = [G^+(P'^*_\mu)]^*, \\ K_{m,\mu}v &= -i\sum_{l=m+1}^{d'} r^+P_\mu S'_{l,\mu}D_n^{l-1-m}(v(x')\otimes\delta(x_n)); \end{aligned} \tag{5.5}$$

where $Ju(x',x_n)=u(x',-x_n)$, and the $S'_{l,\mu}$ are tangential differential operators such that the symbol of $P'_\mu - \sum_{l=0}^{d'} S'_{l,\mu}(x,D')D_n^l$ is $O(\langle\xi_n\rangle^{-1})$. For the Poisson operators $K_{m,\mu}$ one gets the desired statement by a straightforward generalization of [G2, Lemma 2.6.4 and Theorem 2.7.5]; in the present situation we have the advantage that $P_\mu S'_{l,\mu}$ can be transformed to y-form in an exact way, so that Lemma 2.6.4 applies directly. For the term with G^+, we make the following considerations.

When $P_\mu=\mathrm{OP}(p(x',\xi,\mu))$ is of order d and regularity ν, with symbol independent of x_n and satisfying the transmission condition, we have from [G2, Theorem 2.6.6] that the symbol-kernel of $G^+(P_\mu)$ equals

$$\tilde g^+(p)(x',x_n,y_n,\xi',\mu) = [\mathcal{F}^{-1}_{\xi_n\to z_n} p(x',\xi,\mu)]|_{z_n=x_n+y_n} \quad \text{for } x_n, y_n>0, \tag{5.6}$$

and hence (with $H(x_n)=1_{\{x_n>0\}}$)

$$\begin{aligned} g^+(p)(x',\xi,\eta_n,\mu) &= \mathcal{F}_{x_n\to\xi_n}\overline{\mathcal{F}}_{y_n\to\eta_n} H(x_n)H(y_n)[\mathcal{F}^{-1}_{\zeta_n\to z_n} p(x',\xi',\zeta_n,\mu)]|_{z_n=x_n+y_n} \\ &= \tfrac{1}{2\pi}\int e^{-ix_n\xi_n+iy_n\eta_n+i(x_n+y_n)\zeta_n} H(x_n)H(y_n)p(x',\xi',\zeta_n,\mu)\,d\zeta_n\,d\xi_n\,d\eta_n; \end{aligned} \tag{5.7}$$

and we see from Theorem 3.11 above that it belongs to $S^{d-1,\nu-\varepsilon}(\mathbf{R}^{n-1}\times\overline{\mathbf{R}}^n_+,\mathcal{H}^+\hat{\otimes}\mathcal{H}^-_{-1})$, with $\varepsilon=0$ in some cases.

Now let $p(x,\xi,\mu)$ be a general element of $S^{d,\nu}_{1,0,\mathrm{uttr}}(\mathbf{R}^n\times\overline{\mathbf{R}}^{n+1}_+)$. It is seen as in [G2, (2.3.6)] that for $u\in\mathcal{S}(\overline{\mathbf{R}}_+)$,

$$\begin{aligned} r^+\operatorname{OP}_n(p)e^-Ju &= \int_0^\infty \tilde g^+(p)(x,y_n,\xi',\mu)u(y_n)\,dy_n, \quad \text{where}\\ \tilde g^+(p)(x,y_n,\xi',\mu) &= \tilde p(x,z_n,\xi',\mu)|_{z_n=x_n+y_n} \quad \text{for } x_n,y_n>0;\\ &\text{with } \tilde p(x,z_n,\xi',\mu)=\mathcal{F}^{-1}_{\xi_n\to z_n}p(x,\xi,\mu); \end{aligned} \tag{5.8}$$

here $r^+\tilde p(x,z_n,\xi',\mu)$ is in $\mathcal{S}(\overline{\mathbf{R}}_+)$ as a function of z_n, in view of the transmission property. Then we can show the following precise version of [G2, Theorem 2.7.6]:

THEOREM 5.2. *Let $p(x,\xi,\mu)\in S^{d,\nu}_{1,0,\mathrm{uttr}}(\mathbf{R}^n\times\overline{\mathbf{R}}^{n+1}_+)$, and let $P_\mu=\operatorname{OP}(p)$. Then $\tilde g^+(p)$ defined by (5.8) belongs to $S^{d-1,\nu'}_{1,0}(\mathbf{R}^{n-1}\times\overline{\mathbf{R}}^n_+,\mathcal{S}(\overline{\mathbf{R}}_+))$, with $\nu'=\nu$ if $d-\frac12$ and ν are non-integer, and with $\nu'=\nu-\varepsilon$ (any $\varepsilon>0$) in general; and $G^+(P_\mu)=r^+P_\mu e^-J$ is the singular Green operator with symbol-kernel $\tilde g^+(p)$. It has the asymptotic expansion:*

$$\tilde g^+(p)(x,y_n,\xi',\mu)\sim\sum_{j\in\mathbf{N}}\tfrac{1}{j!}x_n^j[\partial^j_{x_n}\tilde p(x',0,z_n,\xi',\mu)]|_{z_n=x_n+y_n}; \tag{5.9}$$

and the associated symbol has the asymptotic expansion

$$g^+(p)(x',\xi,\eta_n,\mu)\sim\sum_{j\in\mathbf{N}}\tfrac{1}{j!}\overline{D}^j_{\xi_n}g^+(\partial^j_{x_n}p(x',0,\xi,\mu)), \tag{5.10}$$

where (5.7) is applied in each term.

Proof. Consider a Taylor expansion of $p(x,\xi,\mu)$,

$$\begin{aligned} p(x,\xi,\mu) &= \sum_{j<M}\tfrac{1}{j!}x_n^j\partial^j_{x_n}p(x',0,\xi,\mu)+x_n^M r_M(x,\xi,\mu), \quad \text{where}\\ r_M(x,\xi,\mu) &= \tfrac{1}{(M-1)!}\int_0^1(1-h)^{M-1}\partial^M_{x_n}p(x',hx_n,\xi,\mu)\,dh; \end{aligned} \tag{5.11}$$

clearly $r_M\in S^{d,\nu}_{1,0,\mathrm{uttr}}(\mathbf{R}^n\times\overline{\mathbf{R}}^{n+1}_+)$. For each of the terms $\frac{1}{j!}x_n^j\partial^j_{x_n}p(x',0,\xi,\mu)$, the procedures of [G2, 2.6], improved as in Theorem 3.11 above, apply directly to show that

$$G^+\big(\tfrac{1}{j!}x_n^j\operatorname{OP}(\partial^j_{x_n}(p(x',0,\xi,\mu)))\big)=\operatorname{OPG}\big(\tfrac{1}{j!}\overline{D}^j_{\xi_n}g^+(\partial^j_{x_n}p(x',0,\xi,\mu))\big), \tag{5.12}$$

with symbol in $S^{d-1,\nu'}_{1,0}(\mathbf{R}^{n-1}\times\overline{\mathbf{R}}^n_+,\mathcal{H}^+\hat{\otimes}\mathcal{H}^-_{-1})$; it corresponds to the symbol-kernel

$$\tilde g^+_{d-1-j}(x,y_n,\xi',\mu)=\tfrac{1}{j!}x_n^j\tilde g^+(\partial^j_{x_n}p(x',0,\xi,\mu)),$$

defined as in (5.6).

Now consider the remainder. Here we have as in (5.8) that

$$r^+ x_n^M \operatorname{OP}_n(r_M(x,\xi,\mu))e^- Ju = \int_0^\infty x_n^M \tilde r_M(x, x_n+y_n,\xi',\mu)u(y_n)\,dy_n,$$

for $u\in\mathcal{S}(\overline{\mathbf{R}}_+)$, with $r^+_{z_n}\tilde r_M = r^+_{z_n}\mathcal{F}^{-1}_{\xi_n\to z_n} r_M$ in $\mathcal{S}(\overline{\mathbf{R}}_+)$ as a function of z_n. Let $\tilde g_{(M)} = x_n^M \tilde r_M(x, x_n+y_n, \xi',\mu)$ for $x_n, y_n>0$; then we shall show that it satisfies better estimates, the larger M is taken. First we note that for $x_n, y_n>0$,

$$\begin{aligned}|\tilde g_{(M)}(x', x_n, y_n, \xi',\mu)| &= |x_n^M \mathcal{F}_{\xi_n\to z_n} r_M(x,\xi,\mu)|_{z_n=x_n+y_n}|\\ &\leqslant |(1+x_n^2)^{-1}(x_n+y_n)^M(1+(x_n+y_n)^2)\mathcal{F}_{\xi_n\to z_n} r_M(x,\xi,\mu)|_{z_n=x_n+y_n}|\\ &= |(1+x_n^2)^{-1}\mathcal{F}^{-1}_{\xi_n\to z_n}\overline{D}^M_{\xi_n}(1+\overline{D}^2_{\xi_n})r_M(x,\xi,\mu)|_{z_n=x_n+y_n}|.\end{aligned}$$

Since r_M is in $S^{d,\nu}_{1,0}$, $r'_M = \overline{D}^M_{\xi_n}(1+\overline{D}^2_{\xi_n})r_M$ belongs to $S^{d-M,\nu-M}_{1,0}$, and hence

$$\begin{aligned}&\|\tilde g_{(M)}(x', x_n, y_n, \xi',\mu)\|^2_{L_2(\mathbf{R}^2_{++})}\\ &\quad\leqslant \iint |(1+x_n^2)^{-1}\mathcal{F}^{-1}_{\xi_n\to z_n} r'_M(x,\xi,\mu)|_{z_n=x_n+y_n}|^2\,dx_n\,dy_n\\ &\quad= \int_0^\infty (1+x_n^2)^{-2}\,dx_n \int_{x_n}^\infty |\mathcal{F}^{-1}_{\xi_n\to z_n} r'_M(x,\xi,\mu)|^2\,dz_n\\ &\quad\dot\leqslant \sup_x \int_{\mathbf{R}} |\mathcal{F}^{-1}_{\xi_n\to z_n} r'_M(x,\xi,\mu)|^2\,dz_n \doteq \sup_x \|r'_M(x,\xi,\mu)\|^2_{L_{2,\xi_n}(\mathbf{R})}\\ &\quad\dot\leqslant \int_{\mathbf{R}} (\langle\xi\rangle^{\nu-M}\langle\xi,\mu\rangle^{d-\nu})^2 d\xi_n \,\dot\leqslant \int_{\mathbf{R}} (\langle\xi\rangle^{\nu+|d-\nu|-M}\langle\mu\rangle^{d-\nu})^2\,d\xi_n\\ &\quad\dot\leqslant (\langle\xi'\rangle^{\nu+|d-\nu|-M+1/2}\langle\mu\rangle^{d-\nu})^2, \quad \text{when } M\geqslant \nu+|d-\nu|+1.\end{aligned}$$

For the derived functions $D^\beta_{x'}x_n^k D^{k'}_{x_n} y_n^m D^{m'}_{y_n} D^\alpha_{\xi'} D^j_\mu \tilde g_{(M)}$, one finds in a similar way (more details are given in the calculation [G2, (2.7.28)]) that for large M,

$$\|D^\beta_{x'}x_n^k D^{k'}_{x_n} y_n^m D^{m'}_{y_n} D^\alpha_{\xi'} D^j_\mu \tilde g_{(M)}\|_{L_2(\mathbf{R}^2_{++})} \dot\leqslant \langle\xi'\rangle^{-M'}\langle\mu\rangle^{d-\nu},$$

where M' for each fixed set of indices α,β,k,k',m,m',j goes to ∞ for $M\to\infty$.

Now let $N\in\mathbf{N}$. For $M\geqslant N+\nu+2|d-\nu|+1$, $\|\tilde g_{(M)}\|\dot\leqslant\langle\xi'\rangle^{\nu-N}\langle\xi',\mu\rangle^{d-\nu}$ in view of (2.18). Then we find, using the information on the symbols in (5.12) for $j\leqslant M$, that

$$\Big\|\tilde g^+(p)(x, y_n,\xi',\mu) - \sum_{j<N}\tilde g^+_{d-1-j}(x, y_n,\xi',\mu)\Big\|_{L_2(\mathbf{R}^2_{++})} \dot\leqslant \langle\xi'\rangle^{\nu'-N}\langle\xi',\mu\rangle^{d-\nu'},$$

for any $N\in\mathbf{N}$. The derived functions are similarly estimated, with the correct orders, assuring that $\tilde g^+(p)$ does indeed belong to $S^{d-1,\nu'}_{1,0}(\mathbf{R}^{n-1}\times\overline{\mathbf{R}}^n_+, \mathcal{S}(\overline{\mathbf{R}}_+))$ with asymptotic expansion (5.9). It follows moreover that $\operatorname{OPG}(g^+(p)) = \operatorname{OP}'\operatorname{OPG}_n(g^+(p))$ is well-defined, and equals $G^+(P_\mu)$ in view of (5.8). This completes the proof. □

$G^-(P'_\mu)$ can be treated in a similar way, or one can use the above together with the following result on adjoints, that is shown as in [G2, Theorem 2.4.6] (now based on the present Theorem 2.7):

THEOREM 5.3. *One has, with symbol formulas as in* [G2, Theorem 2.4.6]:

(1) *The adjoint of a uniformly estimated singular Green operator of order d, class* 0 *and regularity* ν, *is a uniformly estimated singular Green operator of order d, class* 0 *and regularity* ν.

(2) *The adjoints of uniformly estimated Poisson operators of order d and regularity* ν *are precisely the uniformly estimated trace operators of order* $d-1$, *class* 0 *and regularity* ν.

Using the composition rule for s.g.o.s one then finds the desired statement for $L(P_\mu, P'_\mu)$.

A typical case of the remaning terms to be treated (cf. (5.4)) is the following one.

THEOREM 5.4. *Let* p *and* k' *be symbols with* $p(x,\xi,\mu)\in S^{d,\nu}_{1,0,\mathrm{uttr}}(\mathbf{R}^n\times\overline{\mathbf{R}}^{n+1}_+)$ *and* $k'(y',\xi,\mu)\in S^{d'-1,\nu'}_{1,0}(\mathbf{R}^{n-1}\times\overline{\mathbf{R}}^n_+,\mathcal{H}^+)$. *Then*

$$\mathrm{OP}_n(p(x,\xi,\mu))_+\,\mathrm{OPK}_n(k'(y',\xi,\mu))=\mathrm{OPK}_n(k''(x',y',\xi,\mu)), \tag{5.13}$$

where $k''\in S^{d''-1,\nu''}_{1,0}(\mathbf{R}^{2n-2}\times\overline{\mathbf{R}}^n_+,\mathcal{H}^+)$, *with* $d''=d+d'$, $\nu''=m(\nu,\nu')$; *and* k'' *and the associated symbol-kernel* $\tilde{k}''$ *have the asymptotic expansions*

$$\begin{aligned} k''(x',y',\xi,\mu)&\sim\sum_{j\in\mathbf{N}}\tfrac{1}{j!}\overline{D}^j_{\xi_n}h^+_{\xi_n}(\partial^j_{x_n}p(x',0,\xi,\mu)k'(y',\xi,\mu)),\\ \tilde{k}''(x,y',\xi',\mu)&\sim\sum_{j\in\mathbf{N}}\tfrac{1}{j!}x_n^j[r^+_{z_n}(\partial^j_{x_n}\tilde{p}(x',0,z_n,\xi',\mu)\tilde{k}'(y',z_n,\xi',\mu))]|_{z_n=x_n}. \end{aligned} \tag{5.14}$$

It follows that $K''=\mathrm{OPK}(k'')=\mathrm{OP}(p)_+\,\mathrm{OPK}(k')$ *is a uniformly estimated Poisson operator of order* d'' *and regularity* $m(\nu,\nu')$.

Proof. Consider the Taylor expansion (5.11) of p. For the jth term we have, according to [G2, Theorem 2.6.1, Lemma 2.4.2],

$$\begin{aligned} &\mathrm{OP}_n\big(\tfrac{1}{j!}\partial^j_{x_n}p(x',0,\xi,\mu)\big)_+\,\mathrm{OPK}_n(k'(y',\xi,\mu))=\mathrm{OPK}_n(k''_{d-1-j}(x',y',\xi,\mu)),\\ &\qquad\text{where } k''_{d-1-j}(x',y',\xi,\mu)=\tfrac{1}{j!}\overline{D}^j_{\xi_n}h^+_{\xi_n}(\partial^j_{x_n}p(x',0,\xi,\mu)k'(y',\xi,\mu)) \end{aligned} \tag{5.15}$$

belongs to $S^{d''-1-j,\nu''-j}_{1,0}(\mathbf{R}^{2n-2}\times\overline{\mathbf{R}}^n_+,\mathcal{H}^+)$.

Now consider the contribution from the remainder $x_n^M r_M$. We omit the dependence on x' and y' from the notation. Since a Poisson operator in the x_n-variable acts simply as the multiplication of $c\in\mathbf{R}$ by the symbol-kernel, we can write (cf. [G2, (2.7.5)]):

$$\begin{aligned} \mathrm{OP}_n(x_n^M r_M(x_n,\xi,\mu))_+\,\mathrm{OPK}_n(k'(\xi,\mu))c&=x_n^M r_M(x_n,\xi',\mu,D_n)_+(\tilde{k}'(x_n,\xi',\mu)c)\\ &=\sum_{0\leqslant l\leqslant M}\tbinom{M}{l}\,\mathrm{OP}_n(\overline{D}^l_{\xi_n}r_M(x_n,\xi,\mu))_+(x_n^{M-l}\tilde{k}'(x_n,\xi',\mu)c). \end{aligned}$$

Assume that $M \geqslant d$, and let $N(d,l)$ be the smallest integer $\geqslant \max\{d-l,0\}$, then $M-l \geqslant N(d,l)$ for all $l \in [0,M]$, and hence $e^+ x_n^{M-l}\tilde{k}' \in H_2^{N(d,l)}(\mathbf{R})$. Since $\overline{D}_{\xi_n}^l r_M$ is of order $d-l$ and regularity $\nu-l$, it satisfies (cf. (2.1))

$$|D_{x_n}^m \langle \xi_n \rangle^{m'} D_{\xi_n}^{m'} (\overline{D}_{\xi_n}^l r_M(x_n,\xi,\mu) \langle \xi_n, \varkappa \rangle^{l-d})| \leqslant \varrho^{\nu-l}+1,$$

for all $m, m' \in \mathbf{N}$. Then we have in view of (5.1), that $\mathrm{OP}_n(\overline{D}_{\xi_n}^l r_M)$ maps $H_2^{d-l,\varkappa}(\mathbf{R})$ into $L_2(\mathbf{R})$ with norm $O(\varrho^{\nu-l}+1)$. This gives:

$$\begin{aligned}
\|r^+ \mathrm{OP}_n(\overline{D}_{\xi_n}^l r_M)(e^+ x_n^{M-l}\tilde{k}')\|_{L_2(\mathbf{R}_+)} &\dot{\leqslant} (\varrho^{\nu-l}+1)\|e^+ x_n^{M-l}\tilde{k}'\|_{H_2^{d-l,\varkappa}(\mathbf{R})} \\
&\doteq (\varrho^{\nu-l}+1)\|\langle \xi_n, \varkappa\rangle^{d-l-N(d,l)} \langle \xi_n, \varkappa\rangle^{N(d,l)} \overline{D}_{\xi_n}^{M-l} k'(\xi,\mu)\|_{L_2(\mathbf{R})} \\
&\dot{\leqslant} (\varrho^{\nu-l}+1)\varkappa^{d-l-N(d,l)} \sum_{j=0}^{N(d,l)} \varkappa^{N(d,l)-j} \|D_{x_n}^j (x_n^{M-l}\tilde{k}')\|_{L_2(\mathbf{R}_+)} \\
&\dot{\leqslant} (\varrho^{\nu-l}+1)\varkappa^{d-l} \sum_j \varkappa^{-j} (\varrho^{\nu'-[M-l-j]_+}+1)\varkappa^{d'-M+l+j-1/2} \\
&\dot{\leqslant} (\varrho^{\nu-l}+1)(\varrho^{\nu'-M+l}+1)\varkappa^{d''-M-1/2} \dot{\leqslant} (\varrho^{\nu''-M}+1)\varkappa^{d''-M-1/2}.
\end{aligned}$$

Application of $D_{x',y'}^\beta x_n^m D_{x_n}^{m'} D_{\xi'}^\alpha D_\mu^j$ gives expressions that can be treated in the same way, showing that for M sufficiently large (depending on the indices), one gets the correct estimate. The proof of (5.14) is completed by taking this together with the information on the symbols in (5.15). The composition rule for the full operators now follows as in [G2, Theorem 2.7.9]. □

This proof is a little simpler than those in [G2, Lemmas 2.7.1–2, Theorem 2.7.3], because we can use the global properties of p. When Theorem 5.4 is used to treat the term $P_{\mu,+}K'_\mu$ in Theorem 5.1, one first brings K'_μ on y'-form.

The other compositions in (5.4) are treated in a very similar way, and this ends the proof of Theorem 5.1.

By restriction to a fixed value of μ, we get the corresponding (simpler) rules for compositions and adjoints of parameter-independent systems:

COROLLARY 5.5. *For μ-independent systems with x-uniformly estimated symbols, the statements of Theorems* 5.1 *and* 5.3 *are valid when μ and the regularity numbers are disregarded.*

The rules are easily carried over to the situation of operators between admissible vector bundles over manifolds, by use of the coordinate systems and partition of unity described in Lemma 1.5. In fact, when we write

$$\mathcal{A}_\mu \mathcal{A}'_\mu = \sum_{j_1,j_2,j_3,j_4 \leqslant j_0} \tilde{\varrho}_{j_1} \mathcal{A}_\mu \tilde{\varrho}_{j_2} \tilde{\varrho}_{j_3} \mathcal{A}'_\mu \tilde{\varrho}_{j_4}, \quad \text{where } \tilde{\varrho}_j = \begin{pmatrix} \varrho_j & 0 \\ 0 & \varrho_j^0 \end{pmatrix}, \ \varrho_j^0 = \varrho_j|_\Gamma, \qquad (5.16)$$

the study of each term carries over to a study of compositions of operators on $\{1, ..., m\} \times \overline{\mathbf{R}}^n_+$, where Theorem 5.1 can be applied. Likewise, taking adjoints can be localized in this way. Thus we can conclude:

COROLLARY 5.6. *Theorems* 5.1 *and* 5.3 *extend to operators on admissible manifolds.*

The calculus is hereby ready for the consideration of parameter-elliptic Green operators in global L_p spaces, the consequences for parabolic problems, and the applications to Navier–Stokes problems, established in continuations of this article [G5, 6].

References

[B] BEALS, R., L^p and Hölder estimates for pseudodifferential operators: sufficient conditions. *Ann. Inst. Fourier*, 24 (1979), 239–260.

[B-L] BERGH, J. & LÖFSTRÖM, J., *Interpolation Spaces.* Springer-Verlag, Berlin–New York, 1976.

[BM1] BOUTET DE MONVEL, L., Comportement d'un opérateur pseudo-différentiel sur une variété à bord, I–II. *J. Analyse Math.*, 17 (1966), 241–304.

[BM2] — Boundary problems for pseudo-differential operators. *Acta Math.*, 126 (1971), 11–51.

[D] DONOGHUE JR., W. F., *Monotone Matrix Functions.* Grundlehren, 207. Springer-Verlag, Berlin–New York, 1970.

[E] ÈSKIN, G., *Boundary Value Problems for Elliptic Pseudodifferential Equations.* AMS, Providence, R. I., 1981.

[F1] FRANKE, J., Besov–Triebel–Lizorkin spaces and boundary value problems, in *Seminar Analysis, 1984/85*, pp. 89–104. Karl Weierstrass Institut für Math., Berlin, DDR, 1985.

[F2] — *Elliptische Randwertprobleme in Besov–Triebel–Lizorkin-Raümen.* Dissertation, Friedrich Schiller Universität, Jena, 1986.

[Gi] GILBERT, J. E., Interpolation between weighted L^p spaces. *Ark. Mat.*, 10 (1972), 235–249.

[G1] GRUBB, G., Singular Green operators and their spectral asymptotics. *Duke Math. J.*, 51 (1984), 477–528.

[G2] — *Functional Calculus of Pseudo-Differential Boundary Problems.* Progr. Math., 65. Birkhäuser, Boston, 1986.

[G3] — Pseudo-differential boundary problems in L_p spaces. *Comm. Partial Differential Equations*, 15 (1990), 289–340.

[G4] — Parabolic pseudo-differential boundary problems and applications, in *Microlocal Analysis and Applications (Proceedings CIME, Montecatini Terme, 1989)*, pp. 46–117. Lecture Notes in Math., 1495; by J. M. Bony, G. Grubb, L. Hörmander, H. Komatsu and J. Sjöstrand (L. Cattabriga and L. Rodino, eds.). Springer-Verlag, Heidelberg, 1991.

[G5] — 1. Parameter-elliptic and parabolic pseudodifferential boundary problems in global L_p Sobolev spaces. To appear in *Math. Z.* 2. Nonhomogeneous time-dependent Navier–Stokes problems in L_p Sobolev spaces. To appear; a preliminary version is given in Preprint no. 20, Copenhagen Univ. Math. Dept. Preprint Series, 1991.

[G6] — Initial value problems for the Navier–Stokes equations with Neumann conditions, in *The Navier–Stokes Equations II—Theory and Numerical Methods (Proceedings, Oberwolfach, 1991)* (J. G. Heywood et al., eds.), pp. 262–283. Lecture Notes in Math., 1530. Springer-Verlag, Heidelberg, 1992.

[G-H] GRUBB, G. & HÖRMANDER, L., The transmission property. *Math. Scand.*, 67 (1990), 273–289.

[G-K] GRUBB, G. & KOKHOLM, N. J., Calcul symbolique et estimations L_p pour les problèmes aux limites pseudo-différentiels paramétrisés. *C. R. Acad. Sci. Paris Sér. I Math.*, 312 (1991), 223–226.

[G-S1] GRUBB, G. & SOLONNIKOV, V. A., Solution of parabolic pseudo-differential initial-boundary value problems. *J. Differential Equations*, 87 (1990), 256–304.

[G-S2] — Boundary value problems for the nonstationary Navier–Stokes equations treated by pseudo-differential methods. *Math. Scand.*, 69 (1991), 217–290.

[H1] HÖRMANDER, L., Pseudo-differential operators and hypoelliptic equations. *Proc. Sympos. Pure Math.*, 10 (1966), 138–183.

[H2] — Fourier integral operators I. *Acta Math.*, 127 (1971), 79–183.

[H3] — *The Analysis of Linear Partial Differential Operators.* Springer-Verlag, Berlin–New York, 1983–85.

[L] LIZORKIN, P. I., Multipliers of Fourier integrals in the space L_p. *Trudy Mat. Inst. Steklov.*, 89 (1967), 247–267; English translation in *Proc. Steklov Inst. Math.*, 89 (1967), 269–290.

[M] MIHLIN, S. G., On the multipliers of Fourier integrals. *Dokl. Akad. Nauk SSSR*, 109 (1956), 701–703.

[Na] NAGASE, M., The L^p boundedness of pseudo-differential operators with non-regular symbols. *Comm. Partial Differential Equations*, 2 (1977), 1045–1061.

[Ni] NIKOLSKIĬ, S. M., *Approximation of Functions of Several Variables and Imbedding Theorems.* Springer-Verlag, Berlin, 1975.

[R-S1] REMPEL, S. & SCHULZE, B.-W., *Index Theory of Elliptic Boundary Problems.* Akademie Verlag, Berlin, 1982.

[R-S2] — Complex powers for pseudo-differential boundary problems II. *Math. Nachr.*, 116 (1984), 269–314.

[Sc1] SCHROHE, E., Spaces of weighted symbols and weighted Sobolev spaces on manifolds, in *Pseudo-Differential Operators*, pp. 360–377. Lecture Notes in Math., 1256. Springer-Verlag, Heidelberg, 1987.

[Sc2] — A pseudodifferential calculus for weighted symbols and a Fredholm criterion for boundary value problems on noncompact manifolds. Habilitationsschrift, University of Mainz, 1992.

[Se1] SEELEY, R. T., Extension of C^∞ functions defined in a half space. *Proc. Amer. Math. Soc.*, 15 (1964), 625–626.

[Se2] — Refinement of the functional calculus of Calderón and Zygmund. *Nederl. Akad. Wetensch. Proc. Ser. A*, 68 (1965), 521–531.

[Se3] — Topics in pseudo-differential operators, in *CIME Conference on Pseudo-Differential Operators, 1968*, pp. 169–305. Edizioni Cremonese, Roma, 1969.

[Sh] SHAMIR, E., A remark on Mihlin–Hörmander multipliers theorem. *J. Math. Anal. Appl.*, 16 (1966), 104–107.

[St] STEIN, E. M., *Singular Integrals and Differentiability Properties of Functions.* Princeton University Press, Princeton, N. J., 1970.

[T1] TRIEBEL, H., *Interpolation Theory, Function Spaces, Differential Operators.* North-Holland Publ. Co., Amsterdam–New York, 1978.

[T2] — *Theory of Function Spaces.* Birkhäuser, Basel, 1983.

[V-E] VISHIK, M. I. & ÈSKIN, G. I., Elliptic equations in convolution in a bounded domain and their applications. *Uspekhi Mat. Nauk*, 22 (1967), 15–76; English translation in *Russian Math. Surveys*, 22 (1967), 13–75.

[Y] M. YAMAZAKI, The L^p-boundedness of pseudodifferential operators with estimates of parabolic type and product type. *J. Math. Soc. Japan*, 38 (1986), 199–225.

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