SUBANALYTIC SETS IN THE CALCULUS OF VARIATION

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0. Introduction

In this paper we shall study a type of analytic extreme value problems depending on parameters. More precisely, the purpose is to study the singularities of the extreme value as a function of these parameters. The key to all that follows is the concept of subanalytic functions. These are functions whose graphs are subanalytic in the sense of Hironaka [5]. In fact, in Section 3.2 of this paper, we shall see that under rather general circumstances, extreme value functions are subanalytic, hence their singularities are amenable to the rather detailed analysis in Chapter 2. As a by-product, we obtain some results in analytic geometry, for example that the singular set of a subanalytic set is subanalytic.

The main motivation for this work however, depends on the fact that the abstract machine can be applied in different areas of mathematics to give interesting results. In the last two sections of Chapter 3 we shall very breifly encounter such applications. However, the emphasis in the presentation here is on illustrating the abstract machine rather than on exploiting special consequences that can be drawn in these cases. These and other applications will recieve a much more thorough treatment in their own right in another paper.

Recently the author learned that similar ideas (to use the theory of subanalytic sets in connection with problems in the calculus of variation) have appeared also in control theory (see Sussmann [12]). However, the applications made there appear to be of essentially finite dimensional nature. Also they can be viewed as special concequences of the infinite dimensional theory in Section 3.2.

It is a pleasure for me to express my gratitude to the people who helped me during this work in one way or another. First of all I want to thank professor L. Carleson. Without his constant encouragement and friendly advice, this paper would never have been initiated, much less completed. I also want to thank professors J.-E. Björk and B. Malgrange for invaluable help in connection with the development in Chapter 2, and A. M. Gabrielov for a helpful suggestion in connection with the stationarity claim in Theorem 2.3.3. With their help the originally very messy proofs of the author have simplified tremendously.

1. Results in real analytic geometry

1.1. Semi-analytic sets

In this section we recall the basic facts about semi-analytic sets. Throughout this paper, M and N will denote finite dimensional real analytic manifolds. For convenience, the manifolds will be assumed second countable. However, we allow different components to have different dimensions. Also, in most cases the manifolds are allowed to have boundary, even if this is not mentioned explicitly.

Definition 1.1.1. A subset A of M is called *semi-analytic* iff, for every x in M, we can find a neighbourhood U of x in M and $2pq$ real analytic functions g_{ij} and h_{ij} ($1 \leqslant i \leqslant p$ and $1 \leqslant j \leqslant q$) such that

$$A \cap U = \bigcup_{i=1}^{p} \{y \in U \colon g_{ij}(y) = 0 \quad \text{and} \quad h_{ij}(y) > 0 \quad \text{for } j = 1, \ldots, q\}.$$

We let SEM (M) denote the family of semi-analytic subsets of M. This definition generalizes the notion of real analytic set (take all h_{ij} constant). The basic properties of semi-analytic sets are summarized in the following:

THEOREM 1.1.2.

(i) SEM (M) *is closed under locally finite union and intersection, and under set theoretic difference.*

(ii) *If* $A \in$ SEM (M) *and* $B \in$ SEM (N), *then* $A \times B \in$ SEM ($M \times N$).

(iii) *If* $A \in$ SEM (M), *then* $\bar{A}$, A° *and* $\partial A \in$ SEM (M), *and* $\dim \bar{A} = \dim A$. *Furthermore, each topological component of* A *belongs to* SEM (M).

(iv) *For any locally finite family* $\{A_\tau\}_{\tau \in T}$ *of sets in* SEM (M), *we can find a semi-analytic stratification of* $\bigcup_{\tau \in T} A_\tau$, *compatible with each* A_τ. *In fact, the stratification can even be chosen as a triangulation with each simplex semi-analytic.*

(v) *If* $A \in$ SEM (M), *then the regular set* $r(A)$ *and the singular set* $s(A)$ *are both semi-analytic. Moreover,* $\dim s(A) \leqslant \dim A - 1$ (*unless* A *is empty*).

Remarks 1.1.3.

(i) By a *stratification* of a set A we mean a locally finite decomposition $A = \bigcup_{\alpha \in I} \Gamma_\alpha$ into a disjoint union of connected real analytic submanifolds such that if $\bar{\Gamma}_\alpha \cap \Gamma_\beta \neq \emptyset$, then $\Gamma_\beta \subset \bar{\Gamma}_\alpha$, and $\dim \Gamma_\beta \leqslant \dim \Gamma_\alpha - 1$ whenever $\alpha \neq \beta$. The stratification is *semi-analytic* if $\Gamma_\alpha \in$ SEM (M) for all $\alpha \in I$. It is *compatible* with $B \subset M$ iff for each $\alpha \in I$, either $\Gamma_\alpha \subset B$ or $\Gamma_\alpha \cap B = \emptyset$. (In this paper, submanifolds are always imbedded.)

(ii) The dimension in (iii) and (v) above is the Hausdorff dimension. However, it is easily seen that for any stratifiable subset A of M, this dimension is an integer and equals the maximum dimension of the strata in any stratification of A.

For the proof of Theorem 1.1.2, see Lojasiewicz [8], [9].

The local theory of semi-analytic sets is in many respects similar to the (global) theory of semi-algebraic sets. However, at one very important point, this analogy breaks down; semi-analytic sets do not in general behave well under maps. More precisely, if $A \in$ SEM(N) and $\varphi: N \to M$ is a proper analytic map, then it does not necessarily follow that $\varphi(A) \in$ SEM (M) (see for instance [5] or [8]). This should be compared with the Tarski–Seidenberg theorem ([11], [13]) in the algebraic case. This fact severely restricts the usefulness of semi-analytic sets in many situations. However, it turns out that these difficulties often disappear completely if one replaces SEM (M) by a slightly larger family. This will be discussed in the next section.

1.2. Subanalytic sets

The purpose of this section is to define and establish properties of the smallest extension of SEM (M) which is closed under the usual set theoretic operations, and with respect to taking images under proper analytic maps. Sets in this family are called, following Hironaka [5], subanalytic. More precisely:

Definition 1.2.1. A subset A of M is called *subanalytic* iff, for every x in M, we can find a neighbourhood U of x in M and $2p$ pairs $(\varphi_i^\delta, A_i^\delta)$ $(1 \leqslant i \leqslant p$ and $\delta = 1, 2)$, where $A_i^\delta \in \mathrm{SEM}\,(M_i^\delta)$ for some real analytic manifolds M_i^δ, and where the maps $\varphi_i^\delta: M_i^\delta \to M$ are proper analytic, such that

$$A \cap U = \bigcup_{i=1}^{p} (\varphi_i^1(A_i^1) \setminus \varphi_i^2(A_i^2)).$$

We let SUB (M) denote the family of subanalytic subsets of M. Clearly every semi-analytic set is also subanalytic. As we shall see, Definition 1.2.1 is much better adapted for dealing with maps than is Definition 1.1.1. The following theorem, analogous to Theorem 1.1.2, shows that in fact very little is lost when passing from semi- to subanalyticity.

THEOREM 1.2.2.

(i) SUB (M) *is closed under locally finite union and intersect on, and under set theoretic difference.*

(ii) *If* $A \in \mathrm{SUB}\,(M)$ *and* $B \in \mathrm{SUB}\,(N)$, *then* $A \times B \in \mathrm{SUB}\,(M \times N)$.

(iii) *If* $A \in \mathrm{SUB}\,(M)$, *then* $\bar{A}$, A° *and* $\partial A \in \mathrm{SUB}\,(M)$, *and* $\dim \bar{A} = \dim A$. *Furthermore, each topological component of* A *belongs to* SUB (M).

(iv) *For any locally finite family* $\{A_\tau\}_{\tau \in T}$ *of sets in* SUB (M), *we can find a subanalytic stratification of* $\bigcup_{\tau \in T} A_\tau$, *compatible with each* A_τ. *In fact, the stratification can even be chosen as as a triangulation with each simplex subanalytic.*

(v) *If* $A \in \mathrm{SUB}\,(M)$, *then the regular set* $r(A)$ *and the singular set* $s(A)$ *are both subanalytic. Moreover,* $\dim s(A) \leqslant \dim A - 1$ *(unless* A *is empty).*

In addition, we now also get the following useful property:

(vi) *Let* $\varphi: N \to M$ *be an analytic map and let* $A \in \mathrm{SUB}\,(M)$ *be a set such that* $\varphi: \bar{A} \to M$ *is proper. Then* $\varphi(A) \in \mathrm{SUB}\,(M)$. *Also, for any* $B \in \mathrm{SUB}\,(M)$, $\varphi^{-1}(B) \in \mathrm{SUB}\,(N)$.

Proof. (i), (ii) and (vi) are simple manipulations with the definitions. (iii) follows easily from (iv). The proof of (iv) can be found in Hardt [4] (for an alternative proof using desingularization, see Hironaka [5], [6]). (v) finally, will be proved in Section 2.4 of this paper.

There are many links between the theory of desingularization and subanalytic sets. In Section 2.2, we shall, for instance, make use of the following tool, the proof of which can be found in Hironaka [5].

THEOREM 1.2.3. *Let $A \in \mathrm{SUB}(M)$ be a relatively compact set. Then we can find a compact analytic manifold $\check{M}$ and an analytic map $\varphi: \check{M} \to M$ such that $\varphi(\check{M}) = \bar{A}$.*

1.3. Subanalytic functions

Definition 1.3.1. A continuous map $\varphi: N \to M$ is said to be *subanalytic* iff its graph $G_\varphi \subset N \times M$ belongs to SUB $(N \times M)$.

We denote the family of subanalytic maps from N to M by $\mathcal{SUB}(N, M)$. $\mathcal{SUB}(N, M)$ is a natural set of morphisms to study in connection with subanalytic sets. For instance, we have:

PROPOSITION 1.3.2.

(i) *If $\varphi \in \mathcal{SUB}(N, M)$ and $A \in \mathrm{SUB}(M)$, then $\varphi^{-1}(A) \in \mathrm{SUB}(N)$.*

(ii) *If $\varphi \in \mathcal{SUB}(N, M)$, $A \in \mathrm{SUB}(N)$ and $\varphi: \bar{A} \to M$ is proper, then $\varphi(A) \in \mathrm{SUB}(M)$.*

Proof. This follows from Theorem 1.2.2 if we note that $\varphi^{-1}(A) = \pi_N(G_\varphi \cap (N \times A))$ in (i) and that $\varphi(A) = \pi_M(G_\varphi \cap (A \times M))$ in (ii) where π_N and π_M are the projections.

When working with $\mathcal{SUB}(N, M)$, it is natural to extend the idea of stratification from sets to maps:

Definition 1.3.3. A *stratification* of a map $\varphi: N \to M$ on a set $\Gamma \subset N$ is a simultaneous stratification of $\Gamma = \bigcup_{\mu \in I} \Gamma_\mu$ and $M = \bigcup_{\nu \in J} \Delta_\nu$ such that

(i) For each $\mu \in I$, $\varphi|_{\Gamma_\mu}$ is real analytic and has constant rank.

(ii) For each $\mu \in I$, there is a (unique) $\nu(\mu) \in J$ such that $\varphi(\Gamma_\mu) = \Delta_{\nu(\mu)}$. The stratification is *simple* iff $\varphi|_{\Delta_\mu}$ is one-to-one for every $\mu \in I$ such that $\dim \Gamma_\mu = \dim \Delta_{\nu(\mu)}$. It is *subanalytic* iff all Γ_μ and Δ_ν are.

One can now prove the following result:

THEOREM 1.3.4 (Hardt). *Let $\varphi \in \mathcal{SUB}(N, M)$ satisfy the condition that $\varphi: \bar{\Gamma} \to M$ is proper, where $\Gamma \in \mathrm{SUB}(N)$. Then φ has a simple subanalytic stratification on Γ. Moreover, the corresponding stratifications of Γ and M can be chosen compatible with any locally finite families of sets in* SUB (N) *and* SUB (M).

Proof. This follows immediately from Corollary 4.4 in Hardt [4] if we choose an arbitary open subanalytic subset L of N containing $\bar{\Gamma}$ such that $\varphi: L \to M$ is still proper.

We now restrict attention to subanalytic functions. This is the central concept in this paper.

Let $\overline{\mathbf{R}}=\mathbf{R}\cup\{\pm\infty\}$ be the extended real line. $\overline{\mathbf{R}}$ carries a natural structure of an analytic manifold with boundary (diffeomorphic to $[-1,+1]$). Furthermore, for any set X, let $\mathfrak{F}(X)$ be the set of extended real valued functions on X, i.e. maps from X to $\overline{\mathbf{R}}$.

Definition 1.3.5. Let $f\in\mathfrak{F}(M)$. Then we say that f is *subanalytic* iff its graph $G_f\subset M\times\overline{\mathbf{R}}$ belongs to SUB $(M\times\overline{\mathbf{R}})$.

We denote the class of subanalytic functions on M by $\mathcal{S}\mathfrak{F}(M)$. Clearly, $\mathcal{S}\mathfrak{F}(M)=\mathcal{SUB}(M,\overline{\mathbf{R}})$. Of special interest is the class of *locally bounded* subanalytic functions, $\mathcal{S}\mathfrak{F}^{\mathrm{loc}}(M)(=\mathcal{S}\mathfrak{F}(M)\cap L^{\mathrm{loc}}_\infty(M))$, and the class of *continuous* subanalytic functions, $\mathcal{S}\mathfrak{F}^0(M)(=\mathcal{S}\mathfrak{F}(M)\cap C(M))$.

The basic property of subanalytic functions which makes them useful in the calculus of variation is contained in Proposition 1.3.7 below.

Given a map $\varphi\colon N\to M$ and a subset $A\subset N$, we define two maps, $\check{\varphi}_A$ and $\hat{\varphi}_A$ from $\mathfrak{F}(N)$ to $\mathfrak{F}(M)$ by

$$\left.\begin{aligned}\check{\varphi}_A(f)(x) &\stackrel{\text{def}}{=} \inf\{f(u)\colon u\in\varphi^{-1}(x)\cap A\}\\ \hat{\varphi}_A(f)(x) &\stackrel{\text{def}}{=} \sup\{f(u)\colon u\in\varphi^{-1}(x)\cap A\}.\end{aligned}\right.\quad (f\in\mathfrak{F}(N),\ x\in M\text{ and }u\in N) \tag{1.3.6}$$

PROPOSITION 1.3.7. *Suppose that $\varphi\in\mathcal{SUB}(N,M)$, $A\in\mathcal{SUB}(N)$ and that $\varphi\colon\bar{A}\to M$ is proper. Then $\check{\varphi}_A$ and $\hat{\varphi}_A$ both map $\mathcal{S}\mathfrak{F}(N)$ into $\mathcal{S}\mathfrak{F}(M)$.*

COROLLARY 1.3.8. *For any finite set $\{f_j\}_{j=1}^p$ of functions in $\mathcal{S}\mathfrak{F}(M)$ (or in $\mathcal{S}\mathfrak{F}^{\mathrm{loc}}(M)$ or $\mathcal{S}\mathfrak{F}^0(M)$) $\max_{1\leqslant j\leqslant p} f_j$ and $\min_{1\leqslant j\leqslant p} f_j\in\mathcal{S}\mathfrak{F}(M)$ $(\mathcal{S}\mathfrak{F}^{\mathrm{loc}}(M)$ or $\mathcal{S}\mathfrak{F}^0(M))$.*

Proof of Corollary 1.3.8. This follows immediately from the proposition if we take $A=N=M\times\{1,\ldots,p\}$, $\varphi=$ projection $M\times\{1,\ldots,p\}\to M$ and apply $\check{\varphi}_A$ and $\hat{\varphi}_A$ to the function f on N, defined by $f(x,j)=f_j(x)$.

Proof of Proposition 1.3.7. Since $\hat{\varphi}_A(f)=-\check{\varphi}_A(-f)$, it follows that it is enough to consider $\check{\varphi}_A$. Consider the set

$$B\stackrel{\text{def}}{=}\{(u,t,t')\in N\times\overline{\mathbf{R}}\times\overline{\mathbf{R}}\colon t=f(u),\quad u\in A\text{ and }t'\geqslant t\}. \tag{1.3.9}$$

Clearly B is subanalytic, hence so are the sets B_0 and $B_1\subset M\times\overline{\mathbf{R}}$ defined by $B_0\stackrel{\text{def}}{=}\psi(B)$ and $B_1\stackrel{\text{def}}{=}(M\times\overline{\mathbf{R}})\setminus B_0$ by Proposition 1.3.2 and Theorem 1.2.2, where ψ is the map defined by $\psi((u,t,t'))=(\varphi(u),t')$. Moreover, define B_2 and $B_g\subset M\times\bar{R}$ by $B_2=\pi(B_0')$ and $B_3=\pi(B_1')$ where

$$B_0'\stackrel{\text{def}}{=}\{(x,t',t'')\in M\times\overline{\mathbf{R}}\times\overline{\mathbf{R}}\colon(x,t')\in B_0\text{ and }t''>t'\}$$

and

$$B_1'\stackrel{\text{def}}{=}\{(x,t',t'')\in M\times\overline{\mathbf{R}}\times\overline{\mathbf{R}}\colon(x,t')\in B_1\text{ and }t''<t'\}$$

and where π is the projection $\pi((x, t', t''))=(x, t'')$. Again it follows that B_2 and B_3 are subanalytic. Therefore the proposition follows from the following formula for the graph of $\check{\varphi}_A(f)$.

$$G_{\check{\varphi}_A(f)} = (B_0 \setminus B_2) \cup (B_1 \setminus B_3) \tag{1.3.10}$$

To prove (1.3.10), we need only note that $(B_0 \setminus B_2)$ consists exactly of those points on $G_{\check{\varphi}_A(f)}$ where the infimum is actually attained, whereas $(B_1 \setminus B_3)$ consists of those points where it is not attained.

2. Singularities of subanalytic functions

2.1. Singular supports of subanalytic functions, the C^k-case

Definition 2.1.1. Let $f \in \mathfrak{J}(M)$ and let k be a non-negative integer. *The singular support of f* mod C^k (denoted $\mathrm{Sing}_k \operatorname{supp}(f)$) is the complement of the set of points x in M such that for some neighbourhood U of x in M, the restriction of f to U is of class C^k, i.e. finitely valued and k times continuously differentiable.

Note that by definition $\mathrm{Sing}_k \operatorname{supp}(f)$ is closed. In general, this is as much as can be said. However, for functions in $\mathcal{S}\mathfrak{J}(M)$, we can prove the following

THEOREM 2.1.2. *Suppose that* $f \in \mathcal{S}\mathfrak{J}(M)$. *Then* $\mathrm{Sing}_k \operatorname{supp}(f) \in \mathrm{SUB}(M)$ *for every* $k=0, 1, \ldots$.

Proof. Let us first consider the case $k=0$. In this case, the theorem follows trivially from Theorem 1.2.2 as soon as we have recalled that a function is continuous iff its graph is closed. In fact, from this it follows that $x \notin \mathrm{Sing}_0 \operatorname{supp}(f)$ iff for some neighbourhood U of x, f is finitely valued on U and $G_f \cap (U \times \mathbf{R}) = \bar{G}_f \cap (U \times \mathbf{R})$ where G_f is the graph of f. Hence, if we let W_f be the set of points in M where $f = \pm\infty$ (which is clearly subanalytic), then

$$\mathrm{Sing}_0 \operatorname{supp}(f) = \overline{W}_f \cup \overline{\pi(\bar{G}_f \setminus G_f)} \tag{2.1.3}$$

where $\pi: M \times \overline{\mathbf{R}} \to M$ is the projection. Therefore $\mathrm{Sing}_0 \operatorname{supp}(f)$ is subanalytic by Theorem 1.2.2.

The general case will now be proved by induction on k, using essentially the above simple idea applied to each derivative of f. Note that since subanalyticity is a local property, we can without loss of generality assume that M is an open subset of $\mathbf{R}^m$.

Assume now inductively that we have proved that $\mathrm{Sing}_{k-1} \operatorname{supp}(f)$ belongs to SUB (M), and moreover that for each multi-index α with $|\alpha| \leqslant k-1$, the graph of $D^\alpha f$ on the set

$M \setminus \mathrm{Sing}_{k-1} \operatorname{supp}(f)$ belongs to SUB $(M \times \overline{\mathbf{R}})$. We shall prove the same statement for k instead of $k-1$.

Consider for each α, $|\alpha| = k-1$, and each j, $1 \leqslant j \leqslant m$, the set $H_f^{\alpha,j} \subset M \times \overline{\mathbf{R}} \times \mathbf{R}$ defined by

$$H_f^{\alpha,j} \stackrel{\text{def}}{=} \left\{(x, t, u) \colon t = \frac{1}{u} \cdot (D^\alpha f(x + u \cdot \bar{e}_j) - D^\alpha f(x)),\ u \neq 0 \text{ and } x, x + u \cdot \bar{e}_j \in M \setminus \mathrm{Sing}_{k-1} \operatorname{supp}(f)\right\} \tag{2.1.4}$$

where $\bar{e}_1, \ldots, \bar{e}_m$ is the standard basis in $\mathbf{R}^m$. It then follows easily from Theorem 1.2.2 and the induction hypothesis that $H_f^{\alpha,j} \in$ SUB $(M \times \overline{\mathbf{R}} \times \mathbf{R})$. Therefore also $G_f^{\alpha,j} \stackrel{\text{def}}{=} \overline{H_f^{\alpha,j}} \cap (M \times \overline{\mathbf{R}} \times \{0\}) \in$ SUB $(M \times \overline{\mathbf{R}})$. Clearly $D^\alpha f$ is continuously differentiable in the $\bar{e}_j$-direction at $x \in M \setminus \mathrm{Sing}_{k-1} \operatorname{supp}(f)$ iff x has a neighbourhood U such that $G_f^{\alpha,j} \cap (U \times \overline{\mathbf{R}})$ is the graph of a continuous function on U. If we let $A_f^{\alpha,j}$ be the complement of the set of points with this property, then it follows that

$$\mathrm{Sing}_k \operatorname{supp}(f) = \mathrm{Sing}_{k-1} \operatorname{supp}(f) \cup \bigcup_\alpha \bigcup_{j=1}^m A_f^{\alpha,j}. \tag{2.1.5}$$

Hence Theorem 2.1.2 follows immediately from

PROPOSITION 2.1.6. *Let $G \in$ SUB $(M \times \overline{\mathbf{R}})$. Then the set A of points $x \in M$ such that for no neighbourhood U of x in M, $G \cap (U \times \overline{\mathbf{R}})$ is the graph of a continuous function on U, belongs to* SUB (M).

Proof of Proposition 2.1.6. Let $W = (G \cap (M \times \{\infty\})) \cup (G \cap (M \times \{-\infty\}))$ and $B = \{x \in M \colon \operatorname{card} \{\pi^{-1}(x) \cap G\} > 1\}$. Then it is easy to see that

$$A = \overline{\pi(W)} \cup \bar{B} \cup \overline{\pi(\bar{G} \setminus G)} \tag{2.1.7}$$

In fact, if $x \notin \bar{B}$ then on some neighbourhood U of x, G is a graph, which can moreover be assumed finitely valued if $x \notin \overline{\pi(W)}$. Finally, if $x \notin \overline{\pi(\bar{G} \setminus G)}$ then this function can even be assumed continuous. This proves the inclusion $\subset$, and the other one is even more trivial.

Since W is clearly subanalytic, it is enough to prove that $B \in$ SUB (M) by Theorem 1.2.2. To see this, consider the set

$$D \stackrel{\text{def}}{=} \{(x, t, u) \in M \times \overline{\mathbf{R}} \times \overline{\mathbf{R}} \colon (x, t), (x, u) \in G \text{ and } u > t\}. \tag{2.1.8}$$

Let $C \stackrel{\text{def}}{=} \pi_1(D)$ where π_1 is the projection which sends (x, t, u) to (x, u). Then $B = \pi(C)$ as is easily seen. This completes the proof.

2.2. Theory of graphic points

This section is mainly a preparation for the study of the analytic singular supports of functions in $\mathcal{SF}$ (M) in the next section.

Let $\mathcal{N}$ and $\mathcal{M}$ be complex analytic, second countable manifolds. Assume that $\mathcal{M}$ is connected and has dimension m. Let G: $\mathcal{N} \to \mathcal{M}$ be an analytic map and let $\mathcal{Z}$ be the critical set of G, i.e. the set of points in $\mathcal{N}$ where the rank of G is strictly less than m. We assume that $\mathcal{Z}$ is nowhere dense in $\mathcal{N}$, hence has (local) codimension at least one everywhere, since $\mathcal{Z}$ is an analytic set.

Finally, let H be a meromorphic function on N with pole-set $\mathcal{W}$ such that the map $\Phi = (G, H)$: $\mathcal{N} \setminus \mathcal{W} \to \mathcal{M} \times \mathbf{C}$ has rank $\leqslant m$ everywhere.

Definition 2.2.1. A point $u \in \mathcal{N}$ is said to be *graphic* (with respect to $\Phi = (G, H)$) iff, there exists a germ of a holomorphic function F^u at $G(u)$ such that $H_u \equiv F^u \circ G_u$ (the lower indices denote induced germs). The set of non-graphic points (with respect to Φ) will be denoted $\mathcal{E} (= \mathcal{E}_\Phi)$ in the following.

THEOREM 2.2.2 (Malgrange). *$\mathcal{E}$ is an analytic subset of $\mathcal{N}$. Moreover, $\mathcal{W} \subset \mathcal{E} \subset \mathcal{Z} \cup \mathcal{W}$.*

It is easy to see that u is graphic iff the image under Φ of some small neighbourhood of u is contained in the graph of some holomorphic function, defined in a neighbourhood of $G(u)$; hence the name "graphic". Moreover, the germ F^u is easily seen to be unique whenever it exists. Therefore we have a well-defined map $u \mapsto F^u$ from the open set $\mathcal{N} \setminus \mathcal{E}$ into the sheaf of holomorphic germs on $\mathcal{M}$.

Proof. It is clearly no loss of generality to assume that $\mathcal{N}$ is connected. In this case, we shall actually prove that $\mathcal{E}$ is a union of $\mathcal{W}$ and certain other irreducible components of $\mathcal{W}'$, each having codimension one in $\mathcal{N}$, where we have put $\mathcal{W}' \overset{\text{def}}{=} \mathcal{Z} \cup \mathcal{W}$.

It is obvious that $\mathcal{W} \subset \mathcal{E}$. To see that $\mathcal{E} \subset \mathcal{W}'$, simply observe that if $u \notin \mathcal{W}'$, then on some neighbourhood U of u, Φ is analytic and rank $(\Phi) = $rank $(G) = m$. Hence by the rank theorem, $\Phi(U)$ is an analytic submanifold of $\mathcal{M} \times \mathbf{C}$ (for U sufficiently small), and since $G = \pi \circ \Phi$ (π is the projection), the chain rule and the implicit function theorem gives that π: $\Phi(U) \to G(U)$ is a diffeomorphism (again possible shrinking U). Hence $\Phi(U)$ is a graph and the claim follows.

To prove the rest of the theorem, first note that the problem is local. We can therefore without loss of generality assume that $\mathcal{M}$ and $\mathcal{N}$ are open subsets of $\mathbf{C}^m$ and $\mathbf{C}^n$ respectively.

Now let $\alpha = (\alpha_1, ..., \alpha_m)$ be a multi-index, and let as usual $D^\alpha = D_1^{\alpha_1} \dots D_m^{\alpha_m}$. For each α, we define a function H^α on $\mathcal{N} \setminus \mathcal{E}$ by

$$H^\alpha(u) \overset{\text{def}}{=} D^\alpha F^u(G(u)). \tag{2.2.3}$$

It is easy to see that H^α is analytic on $\mathcal{N} \setminus \mathcal{E}$, because for any $u \in \mathcal{N} \setminus \mathcal{E}$, there is a neighbourhood U of u such that for all $v \in U$, F^v is induced by the same holomorphic function F (i.e. $F^v = F_{G(v)}$ for all $v \in U$). For $\alpha = 0$, H^α equals H, hence extends to a well-defined meromorphic function on all of $\mathcal{N}$. For $\alpha \neq 0$, this is also true:

LEMMA 2.2.4. *For each $\alpha \neq 0$, H^α extends to a meromorphic function on $\mathcal{N}$. More precisely, H^α can be expressed as a quotient of two global holomorphic functions on $\mathcal{N}$, and the denominator may always be chosen as a power of the function $\Delta(u) \overset{\text{def}}{=} \det \{(DG)_u \circ {}^t(DG)_{G(u)}\}$ which is zero iff $u \in \mathcal{Z}$.*

Proof of Lemma 2.2.4. First we shall for each $i = 1, \ldots, m$ find a certain meromorphic vector field X_i on $\mathcal{N}$ such that for all $u \in \mathcal{N} \setminus \mathcal{Z}$

$$(DG)_u(X_i(u)) = D_i(G(u)) \tag{2.2.5}$$

(we identify D_i with a vector field). In fact, for every u we may attempt to solve equation (2.2.5) for $X_i(u)$. However, the solution need not in general be unique. To resolve this problem, we require $X_i(u)$ to be orthogonal to the kernel of $(DG)_u$, or equivalently, we look for a solution of the form $X_i(u) = {}^t(DG)_{G(u)}(Y_i(u))$, where the new undetermined Y_i: $\mathcal{N} \to T(\mathcal{M})$ is a vector field along G. (2.2.5) then becomes

$$(DG)_u \circ {}^t(DG)_{G(u)}(Y_i(u)) = D_i(G(u)). \tag{2.2.6}$$

By Cramer's rule, this equation has (for $u \in \mathcal{N} \setminus \mathcal{Z}$) a unique solution of the form $Y_i(u) = \Delta(u)^{-1} \cdot Y_i^0$ where Y_i^0 is a holomorphic vector field along G (in fact, a linear combination of $D_i(G(u))$'s with holomorphic coefficients). Therefore $X_i(u) \overset{\text{def}}{=} {}^t(DG)_{G(u)}(Y_i(u)) = \Delta(u)^{-1} \cdot {}^t(DG)_{G(u)}(Y_i^0(u)$ is a meromorphic vector field (with poles in $\mathcal{Z}$) solving (2.2.5) as required.

We can now prove the lemma by induction over $|\alpha| \overset{\text{def}}{=} \sum_i \alpha_i$. For $\alpha = 0$, the result is clear since $H^0 = H$ on $\mathcal{N} \setminus \mathcal{E}$ as previously remarked. For an arbitary $\alpha \neq 0$, we can write $D^\alpha = D_i \circ D^{\alpha'}$ for some i where $|\alpha'| = |\alpha| - 1$. Using (2.2.5) we obtain after a short verification that on $\mathcal{N} \setminus \mathcal{Z}$,

$$H^\alpha(u) = X_i(H^{\alpha'})(u). \tag{2.2.7}$$

Using the previously derived explicit form of X_i, the lemma follows by induction.

For each positive integer k, let $\mathcal{D}_k$ be the set of points in $\mathcal{N}$ where some H^α with $|\alpha| \leqslant k$ has a pole, and let $\mathcal{D} \overset{\text{def}}{=} \bigcup_{k=1}^\infty \mathcal{D}_k$. We claim that $\mathcal{D} = \mathcal{E}$. In fact, the inclusion $\mathcal{D} \subset \mathcal{E}$ is trivial (from (2.2.3) and the remark following it). The other inclusion is an immediate consequence of the following result:

PROPOSITION 2.2.8. *u does not belong to $\mathcal{E}$ if (and only if) there exists a sequence $\{u_j\}_{j=1}^{\infty}$ in $\mathcal{N}\setminus\mathcal{W}'$ converging to u such that for every multi-index α, $\{H^{\alpha}(u_j)\}_{j=1}^{\infty}$ converge to a finite limit.*

Now if $u \notin \mathcal{D}$, then all H^{α}'s are holomorphic at u, hence a fortiori continuous. Therefore $u \notin \mathcal{E}$ by the proposition which shows that $\mathcal{D} \supset \mathcal{E}$.

We are now reduced to showing that $\mathcal{D}$ is a union of irreducible components of $\mathcal{W}'$ of codimension one in $\mathcal{N}$.

But from classical theory of meromorphic functions, one knows that each $\mathcal{D}_k$ is an analytic set of *pure* codimension one. Since $\mathcal{D}_k \subset \mathcal{W}'$ and the codimension of $\mathcal{W}'$ is at least one, it follows that each $\mathcal{D}_k$ must be a union of irreducible components of $\mathcal{W}'$. Moreover, since there are locally only finitely many components of $\mathcal{W}'$, it follows that the increasing sequence of sets

$$\mathcal{D}_1 \subset \mathcal{D}_2 \subset \dots \quad \dots \subset \mathcal{D}_k \subset \dots \tag{2.2.9}$$

must be locally stationary, i.e. for each compact $K \subset \mathcal{N}$, $\mathcal{D}_k = \mathcal{D}$ on K if k is chosen large enough. Therefore also $\mathcal{D}$ is analytic and a union of components of $\mathcal{W}'$ (of codimension one).

Proof of Proposition 2.2.8. A *formal germ* at a point in $\mathbf{C}^m$ is defined by an element in the ring of formal power series. In particular, each holomorphic germ gives rise to a formal germ (its Taylor series).

It now makes sense to say that a point $u \in \mathcal{N}$ is formally graphic iff $H_u \equiv \eta^u \circ G_u$ for some formal germ η^u at $G(u)$. In fact, this identity means that all the relations obtained by derivation of the equation $H = \eta \circ G$,

$$\begin{aligned} &H(u) = \eta(G(u)) \\ &D_i H(u) = \sum_{v=1}^{m} (D_v \eta)(G(u)) \cdot D_i G_v(u), \quad i = 1, \dots, n \\ &\vdots \end{aligned} \tag{2.2.10}$$

should be satisfied if we substitute the corresponding formal Taylor coefficients η_α of η^u for $D^{\alpha}\eta(G(u))$.

THEOREM 2.2.11. *$u \in M$ is graphic (in the sense of Definition 2.2.1) iff u is formally graphic.*

For the proof, the reader is refered to Malgrange [10] or Gabrielov [3].

In view of this result, we need only check that u is formally graphic. Let η^u be the formal germ at $F(u)$ defined by $\eta_\alpha = \lim_{j\to\infty} H^{\alpha}(u_j)$. We claim that $H_u \equiv \eta^u \circ G_u$. In fact, since $\mathcal{E} \subset \mathcal{W}'$, each point u_j is graphic. Therefore we obtain by differentiating the formula $H_{u_j} \equiv F^{u_j} \circ G_{u_j}$,

$$\begin{aligned} &H(u_j) = F^{u_j}(G(u_j)) \\ &D_i H(u_j) = \sum_{v=1}^{m} D_v F^{u_j}(G(u_j)) \cdot D_i G_v(u_j), \quad i = 1, \ldots, n \\ &\vdots \end{aligned} \tag{2.2.12}$$

If we now let $j \to \infty$, we see that all the equations (2.2.10) are satisfied. This finishes the proof of the proposition, and hence also that of Theorem 2.2.2.

Let us now consider the analogous real analytic situation. Hence, let W and N be real analytic manifolds. Assume that M is connected and has dimension m. Let $g: N \to M$ be an analytic map and let Z be the critical set of g, i.e. the set of points in N where the rank of g is strictly less than m. We assume that Z is nowhere dense in N, hence has (local) codimension at least one everywhere. Finally, let h be a real meromorphic function on N (i.e. it is locally expressible as a quotient of two real analytic functions with the denominator not identically zero) with pole set (i.e. set of points where it is not analytic) W. We suppose that the map $\varphi = (g, h): N \setminus W \to M \times \mathbf{R}$ has rank $\leqslant m$ everywhere.

Definition 2.2.13. A point $u \in N$ is called *graphic* (with respect to $\varphi = (g, h)$) iff there exists a germ of a real analytic function f^u at $g(u)$ such that $h_u \equiv f^u \circ G_u$ (lower indices denote induced germs). The set of non-graphic points (with respect to φ) is denoted by $E(=E_\varphi)$.

As in the complex case, f^u is necessarily unique if it exists.

COROLLARY 2.2.14. *E is a real analytic subset of N. Moreover, $W \subset E \subset Z \cup W$.*

Proof. Let $G: \mathcal{N} \to \mathcal{M}$ and H meromorphic on $\mathcal{N}$ be complexifications of $g: N \to M$ and h respectively. Then we have a commutative diagram

$$\begin{array}{ccc} \mathcal{N} \setminus \mathcal{W} & \longrightarrow & \mathcal{M} \times \mathbf{C} \\ \uparrow & & \uparrow \\ N \setminus W & \xrightarrow{\ \varphi\ } & M \times \mathbf{R} \end{array} \tag{2.2.15}$$

and $Z = N \cap \mathcal{Z}$, $W = N \cap \mathcal{W}$ and $E = N \cap \mathcal{E}$ where $\mathcal{Z}$, $\mathcal{W}$ and $\mathcal{E}$ are as before. The claim now follows immediately from Theorem 2.2.2.

Of course, it need no longer hold that E has pure codimension one.

Remark 2.2.16. We shall only apply this corollary in a slightly more special situation suppose that $\varphi = (g, h): N \to M \times \overline{\mathbf{R}}$ is an analytic map (where $\overline{\mathbf{R}}$ is considered as a manifold with boundary), and suppose that the set $W \overset{\text{def}}{=} \{x \in N: h(x) = \pm\infty\}$ is nowhere dense in N. Then h may be considered as a real meromorphic function on N with pole set W. This follows from the fact that near infinity, the function $1/t$ may be considered as a coordinate map on $\overline{\mathbf{R}}$. Hence the analyticity of φ implies that $h_0 \overset{\text{def}}{=} 1/h$ is an analytic function, which in turn shows that $h = 1/h_0$ is meromorphic. Therefore, if only the conditions that rank φ: $N \setminus W \to M \times \mathbf{R}$ is everywhere less than m, and $Z = \{x \in N: \operatorname{rank} g \leqslant m\}$ is nowhere dense in N hold, we may apply Corollary 2.2.14.

The key to the proof of Theorem 2.2.2 is the fact that the increasing sequence of analytic sets in (2.2.9) is stationary. In the next section, we will use Theorem 2.2.2 to prove that $\operatorname{Sing}_\omega \operatorname{supp}(f) \in \operatorname{SUB}(M)$ for all $f \in \mathcal{SF}(M)$. However, to prove the stationarity of singular supports themselves (i.e. that $\operatorname{Sing}_\omega \operatorname{supp}(f) = \operatorname{Sing}_k \operatorname{supp}(f)$ on compacts for large enough k) is somewhat more delicate, in particular it does not follow in any non-trivial way directly from (2.2.9). To bridge this gap, we shall end this section by proving two other stationarity results which together will give all we need.

For each integer $k > 0$, let

$$E_k \overset{\text{def}}{=} \{u \in N: \exists \text{ analytic germ } \eta \text{ at } g(u) \text{ such that } (\eta \circ g - h)_u \in \mathfrak{M}_u^k\} \tag{2.2.17}$$

where $\mathfrak{M}_u^k$ is the kth power of the maximal ideal $\mathfrak{M}_u$ in the ring of analytic germs at u. Clearly $E_1 \subset E_2 \subset \ldots$ is an increasing sequence of sets.

LEMMA 2.2.18. *For each $k > 0$, E_k is locally a finite union of differences of analytic sets, i.e. sets of the form $A_1 \setminus A_2$ where A_1, A_2 are real analytic. Moreover, $\bigcup_{k=1}^\infty E_k = E$.*

Proof. The condition that $(\eta \circ g - h)_u \in \mathfrak{M}_u^k$ is clearly a condition only on the Taylor coefficients of η up to order $k-1$. Hence if we consider some coordinate neighbourhood $U \subset N$, then a necessary and sufficient condition for $u \in U$ to belong to E_k, is the solvability of the system of linear equations for these coefficients, obtained by putting $D^\alpha(\eta \circ g - h)(u) = 0$ for all α, $|\alpha| \leqslant k-1$, and using Leibniz' rule. Hence the set E_k is determined in U by the equality of two ranks of matrices, the entries of which are analytic functions of u (in fact, polynomials in g, h and their derivatives). From this it clearly follows that E_k is of the requested type.

To prove the last statement, first observe that the inclusion $\bigcup_{k=1}^\infty E_k \subset E$ is trivial. To prove the other direction, assume that $u \notin \bigcup_{k=1}^\infty E_k$. In view of Theorem 2.2.11, we need only check that u is formally graphic. This is equivalent to proving that we can find formal Taylor coefficients $\{\eta_\alpha\}$ which satisfy the infinite system of equations, obtained by putting $D^\alpha(\eta\circ g-h)(u)=0$ for all α. However, since we know that each finite subsystem has a solution, this is just standard linear algebra.

COROLLARY 2.2.19. *Suppose that $K\subset N$ is a compact set. Then we can find an integer k_0 such that for all $k\geqslant k_0$, $E_k\cap K=E\cap K$.*

Proof. Each analytic set can be written as a locally finite union of connected analytic manifolds (follows from Theorem 1.1.2 for instance). It is therefore enough prove that for any such manifold $\Gamma\subset E$, there exists an integer k' such that $\Gamma\subset E_{k'}$. Now from the first part of the lemma it follows easily that for each k, either $\dim(E_k\cap\Gamma)\leqslant\dim\Gamma-1$ or $\dim(\Gamma\setminus E_k)\leqslant\dim\Gamma-1$. Since $\Gamma\subset\bigcup_{k=1}^\infty E_k$ by the second part of the lemma, and since Γ can not be a countable union of sets of lower dimension, the second alternative must occur for some integer k_1. We can now apply the same argument to the set $\Gamma_1=\Gamma\setminus E_{k_1}$, again writing it as a locally finite union of manifolds. For each such manifold we thus get an integer, and if we let $k_2\geqslant k_1$ be larger than the maximum of all these integers, then it follows that $\dim(\Gamma\setminus E_{k_2})\leqslant\dim\Gamma_1-1\leqslant\dim\Gamma-2$. Proceeding in this way inductively, we obtain after a finite number of steps that $\dim(\Gamma\setminus E_{k'})<0$ for some large enough k'. Hence $\Gamma\subset E_{k'}$ which is what we wanted to prove.

Observe that the proof of the stationarity in Corollary 2.2.19 depends on the fact that we already known that E is an analytic set.

We now restrict attention to the case when M is an open subset of $\mathbf{R}^m$ and the map $g: N\to M$ is proper. Also, recall that $h^\alpha(u)=D^\alpha f^u(g(u))$ is meromorphic (compare Lemma 2.2.4). Let

$$B_k\stackrel{\text{def}}{=}\{x\in M\setminus g(E)\colon \exists u,\ v\in g^{-1}(x)\quad\text{and}\quad \alpha,|\alpha|\leqslant k,\text{ such that } h^\alpha(u)\neq h^\alpha(v)\}.\tag{2.2.20}$$

for each integer $k>0$, and moreover let

$$B\stackrel{\text{def}}{=}\{x\in M\setminus g(E)\colon \exists u,\ v\in g^{-1}(x)\quad\text{and}\quad \alpha,\text{ such that } h^\alpha(u)\neq h^\alpha(v)\}.\tag{2.2.21}$$

Then we clearly have a increasing chain of sets $B_1\subset B_2\subset\ldots\ \ldots\subset B$.

LEMMA 2.2.22. *$B\in\mathrm{SUB}(M)$. Moreover, for each compact $K\subset M$, we can find an integer k_1 such that for $k\geqslant k_1$, $B_k\cap K=B\cap K$.*

Proof. For each $k>0$, let us define

$$A_k\stackrel{\text{def}}{=}\{(u,v)\in(N\setminus E)\times(N\setminus E)\colon g(u)=g(v)\quad\text{and}\quad h^\alpha(u)=h^\alpha(v),\quad|\alpha|\leqslant k\}\tag{2.2.23}$$

and

$$A\stackrel{\text{def}}{=}\{(u,v)\in(N\setminus E)\times(N\setminus E)\colon g(u)=g(v)\quad\text{and}\quad h^\alpha(u)=h^\alpha(v),\quad\forall\alpha\}.\tag{2.2.24}$$

Then it is easily verified that each A_k is an analytic subset of $N \times N$, and that we have a decreasing sequence $A_1 \supset A_2 \supset \ldots \ldots \supset A$. Now every such sequence must be locally stationary by Frisch's theorem (see Frisch [2]). Hence, by the properness of g, we can find an integer k_1 such that for $k \geqslant k_1$, $K_1 \cap A = K_1 \cap A_k$, where K_1 is the compact $g^{-1}(K) \times g^{-1}(K)$. Let $D_g \overset{\text{def}}{=} \{(u, v) \in (N \setminus E) \times (N \setminus E): g(u) = g(v)\}$ and let $\pi: N \times N \to N$ be the projection onto the first factor. Then it is easy to see that $B_k = g \circ \pi(D_g \setminus A_k)$ and $B = g \circ \pi(D_g \setminus A)$. Since the map $g \circ \pi: \bar{D}_g \to N$ is clearly proper, the lemma now follows from Theorem 1.2.2(vi).

2.3. Singular supports of subanalytic functions, the analytic case

Definition 2.3.1. Let $f \in \mathcal{F}(M)$. *The analytic singular support of f* (denoted by $\operatorname{Sing}_\omega \operatorname{supp}(f)$) is the complement of the set of points x in M with the property that for some neighbourhood U of x, the restriction of f to U is real analytic.

By definition, $\operatorname{Sing}_\omega \operatorname{supp}(f)$ is closed. Now observe that for every f we have an increasing sequence of sets

$$\operatorname{Sing}_0 \operatorname{supp}(f) \subset \operatorname{Sing}_1 \operatorname{supp}(f) \subset \ldots \quad \ldots \subset \operatorname{Sing}_k \operatorname{supp}(f) \subset \ldots \quad \ldots \subset \operatorname{Sing}_\omega \operatorname{supp}(f). \tag{2.3.2}$$

THEOREM 2.3.3.

(i) *Suppose that $f \in S\mathcal{F}(M)$. Then $\operatorname{Sing}_\omega \operatorname{supp}(f) \in \mathrm{SUB}(M)$. Moreover, for any compact $K \subset M$, we can find an integer k' such that for $k \geqslant k'$*

$$\operatorname{Sing}_k \operatorname{supp}(f) \cap K = \operatorname{Sing}_\omega \operatorname{supp}(f) \cap K.$$

(ii) *If we, more restrictively, assume that $f \in S\mathcal{F}^{\mathrm{loc}}(M)$, then in addition we can say that* $\dim(\operatorname{Sing}_\omega \operatorname{supp}(f)) \leqslant \dim M - 1$.

Remark 2.3.4. The following proof is based on Theorem 1.2.3, hence makes strong use of the theory of desingularization. An alternative proof could be based on the more elementary stratification theory in Hardt [4] (see for instance Theorem 4.2 of that paper), thus pulling the problem back to semi-analytic sets instead of smooth manifolds. The general outline of the proof would be the same, although the details would be somewhat more complicated.

Proof of Theorem 2.3.3. It is enough to prove the theorem locally. Hence we may assume that M is an open subset of $\mathbf{R}^m$, and restrict the attention to some open, relatively compact, subanalytic subset $U \subset M$. Let G_f be the graph of f on U. Then G_f is a relatively compact, subanalytic subset of $M\times\overline{\mathbf{R}}$. By Theorem 1.2.3 we can then find a compact real analytic manifold $\check{M}$ and an analytic map $\varphi=(g,h)\colon \check{M}\to M\times\overline{\mathbf{R}}$ such that $\varphi(\check{M})=\overline{G}_f$. Theorem 1.2.2(iii) implies that rank $\varphi\leqslant m$.

Let $N\subset\check{M}$ be the union of all components N_0 of $\check{M}$ such that

(a) The analytic subset $W_0\overset{\text{def}}{=}\{u\in N_0\colon h(u)=\pm\infty\}$ is nowhere dense in N_0.

(b) The set $Z_0\overset{\text{def}}{=}\{u\in N_0\colon \operatorname{rank} g<m\}$ is nowhere dense in N_0.

Then $\varphi\colon N\to M\times\overline{\mathbf{R}}$ is a map to which we can apply the theory and terminology of Section 2.2 (see Remark 2.2.16). Hence we have defined sets Z, E, E_k, B, $B_k\subset N$, real meromorphic functions h^α etc.

LEMMA 2.3.5. *Let $u\in N\setminus Z$ and suppose that $x\overset{\text{def}}{=}g(u)\notin\operatorname{Sing}_0\operatorname{supp}(f)$. Then actually $x\notin\operatorname{Sing}_\omega\operatorname{supp}(f)$, and $D^\alpha f(x)=h^\alpha(u)$ for all α. Moreover,*

$$(\operatorname{Sing}_0\operatorname{supp}(f)\cap U)\cup g(E)\cup B\overset{\mathrm{I}}{\supset}\operatorname{Sing}_\omega\operatorname{supp}(f)\cap U\overset{\mathrm{II}}{\supset}g(Z)\cup g(\check{M}\setminus N).$$

Assuming the lemma, the theorem is proved as follows; let $k'\overset{\text{def}}{=}\max\{k_0,k_1\}$ where k_0 and k_1 are constructed as in Corollary 2.2.19 and Lemma 2.2.22 respectively (with $K=N$). Hence, for each $k\geqslant k'$, we have that $E_k=E$ and $B_k=B$. It follows that

$$\begin{aligned}\operatorname{Sing}_\omega\operatorname{supp}(f)\cap U&\subset(\operatorname{Sing}_0\operatorname{supp}(f)\cap U)\cup g(E)\cup B\\&=(\operatorname{Sing}_0\operatorname{supp}(f)\cap U)\cup g(E_k)\cup B_k\subset\operatorname{Sing}_k\operatorname{supp}(f)\cap U.\end{aligned}\tag{2.3.6}$$

To see the last inclusion, first observe that if $x\notin\operatorname{Sing}_k\operatorname{supp}(f)$, then the Taylor polynomial of f of order k defines an analytic germ η at x such that for any $u\in N$ with $g(u)=x$, $(\eta\circ g-h)_u\in\mathfrak{M}_u^k$. Using the definition (2.2.17) we see that $x\notin g(E_k)$.

On the other hand, if $x\in B_k$ then there exists u, v in N such that $g(u)=g(v)=x$ and $h^\alpha(u)\neq h^\alpha(v)$ for some α, $|\alpha|\leqslant k$. Therefore we can find sequences $u_j\to u$ and $v_j\to v$ in $N\setminus Z$ so that $\lim h^\alpha(u_j)\neq\lim h^\alpha(v_j)$. If $x\notin\operatorname{Sing}_0\operatorname{supp}(f)$, then we can take the images of these sequences under g, and use the first part of the lemma to prove that $D^\alpha f$ can not be continuous at x. This gives that $x\in\operatorname{Sing}_k\operatorname{supp}(f)$ which proves (2.3.6). Combining this with the trivial inclusion $\operatorname{Sing}_k\operatorname{supp}(f)\subset\operatorname{Sing}_\omega\operatorname{supp}(f)$, we obtain that $\operatorname{Sing}_\omega\operatorname{supp}(f)\cap U=\operatorname{Sing}_k\operatorname{supp}(f)\cap U$ which proves part (i) of the theorem in view of Theorem 2.1.2 (or alternatively, using Theorem 1.2.2 and the fact that the first inclusion in (2.3.6) is an equality).

To prove part (ii), it is clearly enough to prove that if $f\in\mathcal{SF}^{\mathrm{loc}}(M)$, then $\dim(g(Z)\cup g(\check{M}\setminus N))\leqslant\dim M-1$ (by the lemma). But since in this case W is always empty, condition (a) above on N_0 is trivially satisfied for every component $N_0\subset\check{M}$. Therefore (b) implies that rank $g<m$ on all of $\check{M}\setminus N$, and the same holds on Z by its definition. This implies the claim.

Proof of Lemma 2.3.5. If $x \notin \mathrm{Sing}_0 \operatorname{supp}(f)$, then it follows that for some small enough neighbourhood V of x, $G_f \cap (V \times \overline{\mathbf{R}}) = \bar{G}_f \cap (V \times \overline{\mathbf{R}}) = \varphi(\check{M}) \cap (V \times \overline{\mathbf{R}})$. Moreover, the fact that $u \in N \setminus Z$ and the argument in the beginning of the proof of Theorem 2.2.2 shows that some small neighbourhood of u is mapped onto the graph of some analytic function f^u on V (possibly by shrinking V further). It follows that $G_{f^u} \cap (V \times \overline{\mathbf{R}}) \subset G_f \cap (V \times \overline{\mathbf{R}})$, hence we must have equality and $f^u \equiv f$ on V since both sides are graphs of functions. This proves that f is analytic at x, and the fact that $D^\alpha f(x) = h^\alpha(u)$ is just the definition of h^α. It remains to verify the inclusions.

I. Let $x \in U \setminus (\mathrm{Sing}_0 \operatorname{supp}(f) \cup g(E) \cup B)$. Then as above, we can find a neighbourhood V of x such that $G_f \cap (V \times \overline{\mathbf{R}}) = \varphi(\check{M}) \cap (V \times \overline{\mathbf{R}}) = \varphi(N) \cap (V \times \overline{\mathbf{R}})$, where the last equality follows from compactness of N, the continuity of f on V and the fact that $\dim\,(\varphi(\check{M} \setminus N) \cap (V \times \mathbf{R})) \leqslant m-1$. (This is exactly the same argument as the concluding part of the proof of Theorem 2.3.3.) Since $x \notin g(E)$, for each $u \in g^{-1}(x)$ we have an analytic germ f^u at x. We claim that any two such germs must be equal. In fact, since $x \notin B$ we have for any α and $u, v \in g^{-1}(x)$ that

$$D^\alpha f^u(g(u)) \stackrel{\text{def}}{=} h^\alpha(u) = h^\alpha(v) \stackrel{\text{def}}{=} D^\alpha f^v(g(v)). \tag{2.3.7}$$

If we let f_x denote this unique germ, then it follows (with V sufficiently small) that $G_f \cap (V \times \overline{\mathbf{R}}) = \varphi(N) \cap (V \times \overline{\mathbf{R}}) = \varphi(\Omega) \cap (V \times \overline{\mathbf{R}}) = G_{f_x} \cap (V \times \overline{\mathbf{R}})$ where Ω is a small neighbourhood of $g^{-1}(u)$. Hence $f \equiv f_x$ on V which proves that f is analytic at x, i.e. $x \notin \mathrm{Sing}_\omega \operatorname{supp}(f)$.

II. This proof is similar. In fact, if $x \notin g(Z) \cup g(\check{M} \setminus N)$ then it again follows as in the beginning of the proof of Theorem 2.2.2 that for each $u \in g^{-1}(x)$, some small neighbourhood of u is mapped by φ onto the graph of some meromorphic function (in fact, of some analytic function except in the special case when $h(u) = \pm\infty$). We easily conclude that

$$\bar{G}_f \cap (V \times \overline{\mathbf{R}}) = \varphi(N) \cap (V \times \overline{\mathbf{R}}) = \bigcup_i G_{f_i} \tag{2.3.8}$$

where each f_i is meromorphic on V. However, Theorem 1.2.2 (iii) implies (since G_f is a graph) that this is possible only if all f_i's are equal to $f|_V$, which in turn implies that f is analytic on V.

Hence the proof of Theorem 2.3.3 is complete.

COROLLARY 2.3.9. *Suppose $f \in \mathcal{SJ}(M)$. Then we can find a simultaneous stratification of* $\mathrm{Sing}_\omega \operatorname{supp}(f)$ *and* $\mathrm{Sing}_k \operatorname{supp}(f)$ *for all $k \geqslant 0$. More precisely, we can find a stratification of* $\mathrm{Sing}_\omega \operatorname{supp}(f) = \bigcup_{\alpha \in I} \Gamma_\alpha$ *and an increasing sequence of subsets of I, $I_0 \subset I_1 \subset \dots \dots \subset I_{k'} = I_{k'+1} = \dots \dots = I$, such that for each k,* $\mathrm{Sing}_k \operatorname{supp}(f) = \bigcup_{\alpha \in I_k} \Gamma_\alpha$.

Proof. By the stationarity claim in Theorem 2.3.3, the collection of sets, $A_0 = \mathrm{Sing}_0 \operatorname{supp}(f)$, $A_1 = \mathrm{Sing}_1 \operatorname{supp}(f) \setminus \mathrm{Sing}_0 \operatorname{supp}(f)$, ..., $A_k = \mathrm{Sing}_k \operatorname{supp}(f) \setminus \mathrm{Sing}_{k-1} \operatorname{supp}(f)$, ... is locally finite. Moreover, by Theorem 2.1.2, each A_k is subanalytic. Hence by Theorem 1.2.2 (iv), we can find a subanalytic stratification of $A = \bigcup_{k=1}^{\infty} A_k = \mathrm{Sing}_\omega \operatorname{supp}(f)$, compatible with each A_k. This implies the claim.

2.4. The singular set of a subanalytic set

In this section, we shall prove the earlier announced result (Theorem 1.2.2 (v)) about the regular and singular sets of a subanalytic set.

Definition 2.4.1. Let $A \subset M$ be a set. For each integer q, $0 \leqslant q \leqslant m = \dim NM$, *the set of* (*analytic*) *q-regular points of* A (denoted by $r^q(A)$) is the set of all $x \in A$ such that for some neighbourhood U of x in M, $A \cap U$ is a q-dimensional analytic submanifold of M. *The set of regular points of* A (denoted by $r(A)$) is the union $\bigcup_{q=0}^{m} r^q(A)$. *The singular set of* A (denoted $s(A)$) is the complement of $r(A)$ in A.

Clearly, $r(A)$ and $r^q(A)$ are analytic submanifolds of M.

Similarly, one may define regular and singular sets mod C^k for each integer $k > 0$ (denoted $r_k^q(A)$, $r_k(A)$ and $s_k(A)$) in analogy with Definition 2.4.1, by considering the sets of points in A around which A is or is not a submanifold of class C^k. We shall, however, be very little concerned with these sets, which motivates the simple notation $r(A)$, ... instead of the perhaps more adequate $r_\omega(A)$, ...

THEOREM 2.4.2. *Suppose* $A \in \mathrm{SUB}(M)$, *then also* $r(A)$, $r^q(A)$ (*for* $0 \leqslant q \leqslant m$) *and* $s(A)$ *belong to* $\mathrm{SUB}(M)$. *Moreover*, $\dim s(A) \leqslant \dim A - 1$ (*unless* $A = \varnothing$).

Remark 2.4.3. The same results hold for $r_k(A)$, $r_k^q(A)$ and $s_k(A)$, and the proof is identical, except that we must use Theorem 2.1.2 instead of Theorem 2.3.3. Moreover, it can be proved that for any compact $K \subset M$, $r_k(A) \cap K = r(A) \cap K$ for large enough k and so on.

Proof. It is enough to prove that $r^q(A) \in \mathrm{SUB}(M)$ for each q, since $r(A)$ and $s(A)$ are obtained from these sets by Boolean operations. Also observe that the claim is entirely local; hence it is no loss of generality to assume that M is an open subset of $\mathbf{R}^m$, and to restrict attention to a small open cube $Q \subset \subset M$ with sides parallel to the coordinate hyperplanes (i.e. the hyperplanes spanned by subsets of the standard basis in $\mathbf{R}^m$).

For each coordinate plane E of dimension q, let $r(A; E)$ be the set of all points x in $A \setminus Q$ such that, for some neighbourhood U of x in M, $A \cap U$ is an analytic submanifold, and the orthogonal projection $\pi: \mathbf{R}^m \to E$ induces a diffeomorphism of $A \cap U$ onto an open subset of E. It is then easy to see that

$$r^q(A) \cap Q = \bigcup_E r(A; E) \tag{2.4.4}$$

where the union is over all possible coordinate planes E (there are $\binom{m}{q}$ of them). In fact, if $x \in r^q(A)$, then there must be a coordinate plane E such that $D\pi: T_x(A) \to E$ is an isomorphism, which implies the claim by the implicit function theorem.

It is therefore enough to prove that for each E, $r(A; E) \in \text{SUB}(M)$. The idea of the proof is now to use the fact that $r(A; E)$ is, roughly speaking, locally like the graph of some section of π from E to $\mathbf{R}^m$, and to apply Theorem 2.3.3 to each component of this section.

At this point, it is very convenient to make use of Theorem 1.3.4. Hence, choose a simple subanalytic stratification of $\pi: \mathbf{R}^m \to E$ on $A \cap Q$, say $A \cap Q = \bigcup_{\alpha \in I} \Gamma_\alpha$ and $E = \bigcup_{\beta \in J} \Delta_\beta$. Over each stratum Δ_β, there may lie any finite number of stratas Γ_α such that $\pi(\Gamma_\alpha) = \Delta_\beta$. However, if we let $\mathcal{J}$ be the family of all possible sections σ of π on $\pi(A \cap Q)$ such that $\sigma(\pi(A \cap Q))$ is a union of strata Γ_α, then $\mathcal{J}$ is finite by the relative compactness of Q and the local finiteness of the stratifications. In the following, we consider each σ to be extended by $\sigma = +\infty$ on $E \setminus \pi(A \cap Q)$. Now, it is trivial to see that $r(A; E) \subset \bigcup_{\sigma \in \mathcal{J}} G_\sigma$, where $G_\sigma = \sigma(E)$. Moreover, if we let $G_\sigma^r \overset{\text{def}}{=} r(A; E) \cap G_\sigma$, then the following conditions are easily seen to be both necessary and sufficient for $x \in G_\sigma$ to belong to G_σ^r:

(1) For some neighbourhood U of x, $A \cap U = G_\sigma \cap U$.

(2) For some neighbourhood V of $\pi(x)$ in E, $\sigma: V \to \mathbf{R}^m$ is analytic.

If we denote by G_σ^1 and G_σ^2 the sets of points in G_σ satisfying (1) and (2) respectively, then clearly $r(A: E) = \bigcup_{\sigma \in \mathcal{J}} G_\sigma^r = \bigcup_{\sigma \in \mathcal{J}} (G_\sigma^1 \cap G_\sigma^2)$. Hence it suffices to prove that G_σ^1 and G_σ^2 belong to SUB $(\mathbf{R}^m)$ for each σ.

The first statement is an immediate consequence of the trivial topological formula

$$G_\sigma^1 = G_\sigma \cap (\mathbf{R} \setminus \overline{(A \setminus G_\sigma)}) \cap A \in \text{SUB}(\mathbf{R}^m). \tag{2.4.5}$$

The second is a consequence of Theorem 2.3.3; in fact, σ is analytic at $\pi(x)$ iff each component σ_i of $\sigma = (\sigma_1, ..., \sigma_m)$ is analytic at $\pi(x)$. Let $S_\sigma \overset{\text{def}}{=} \bigcup_{i=1}^m \text{Sing}_\omega \text{ supp}(\sigma_i)$. Then we see that

$$G_\sigma^2 = \sigma(E \setminus S_\sigma). \tag{2.4.6}$$

Now the graph of σ is subanalytic by construction, hence so is the graph of each σ_i since it is obtained from the graph of σ by projection. Therefore Theorem 2.3.3 implies that $\text{Sing}_\omega \text{ supp}(\sigma_i) \in \text{SUB}(E)$ for each i, which implies that also $G_\sigma^2 \in \text{SUB}(\mathbf{R}^m)$ by (2.4.6) and Theorem 1.2.2.

The dimension statement finally, is a trivial consequence of Theorem 1.2.2 (iv). In fact, if we consider any stratification of $A = \bigcup_{\alpha \in I} \Gamma_\alpha$, then any Γ_α of maximal dimension (i.e. such that $\dim \Gamma_\alpha = \dim A$) must belong to $r(A)$ by the topological properties of stratifications. Hence $s(A)$ is contained in a union of Γ_α's of dimension at most $\dim A - 1$. Consequently $s(A)$ itself can have dimension at most $\dim A - 1$ which completes the proof of the theorem.

In addition to being interesting in its own right, Theorem 2.4.2 can be used to give a more precise description of the singularities of functions in $S\mathfrak{J}(M)$ as follows; for each function $f \in \mathfrak{J}(M)$, we define inductively sets $S_i(f)$, $i = 1, 2, \ldots$, by

$$S_1(f) \stackrel{\text{def}}{=} \operatorname{Sing}_\omega \operatorname{supp}(f) \tag{2.4.7}$$

$$S_i(f) \stackrel{\text{def}}{=} \operatorname{Sing}_\omega \operatorname{supp}(f|_{r(S_{i-1}(f))}) \quad \text{for } i \geq 2. \tag{2.4.8}$$

These definitions clearly make sense, since for any set $A \subset M$, $r(A)$ is an analytic (sub-) manifold. We now have the following

THEOREM 2.4.9.

(i) *Suppose that* $f \in S\mathfrak{J}(M)$. *Then for all* i, $S_i(f) \in \mathrm{SUB}(M)$.

(ii) *If* $f \in S\mathfrak{J}^{\mathrm{loc}}(M)$, *then in addition we can say that for* $i \leq \dim M$, $\dim S_i(f) \leq \dim M - i$ *and for* $i > \dim M$, $S_i(f) = \emptyset$.

Remark 2.4.10. There are of course analogous results which involve $\operatorname{Sing}_k \operatorname{supp}(f)$ (and $r_k(A)$) which the reader may wish to formulate for himself.

PROPOSITION 2.4.11. *Let* Ω *be an analytic submanifold of* M *which also belongs to* $\mathrm{SUB}(M)$. *Then we have that*

(i) *Suppose* $f \in S\mathfrak{J}(M)$. *Then* $\operatorname{Sing}_\omega \operatorname{supp}(f|_\Omega) \in \mathrm{SUB}(M)$.

(ii) *If* $f \in S\mathfrak{J}^{\mathrm{loc}}(M)$, *then in addition we get that* $\dim(\operatorname{Sing}_\omega \operatorname{supp}(f|_\Omega)) \leq \dim \Omega - 1$.

It is clear that Proposition 2.4.11 together with Theorem 2.4.2 implies Theorem 2.4.9 by induction. Hence we need only prove Proposition 2.4.11.

Proof of Proposition 2.4.11. The dimension statement (ii) is an immediate consequence of Theorem 2.3.3 (ii). Hence we are left with (i).

The proof of (i) is in many respects similar to that of Theorem 2.4.2. Again we may assume that $M \subset \mathbf{R}^m$ and that we are working in some open cube Q.

In the present situation, $r(\Omega) = r^q(\Omega) = \Omega$, where $q = \dim \Omega$, since Ω was assumed regular. Formula (2.4.4) now becomes

$$\Omega \cap Q = \bigcup_E r(\Omega; E) \tag{2.4.12}$$

in the notation of the previous proof. Let $r(f|_{\Omega}) \overset{\text{def}}{=} \Omega \setminus \operatorname{Sing}_{\omega} \operatorname{supp}(f|_{\Omega})$. Clearly it is enough to prove that $r(f|_{\Omega}) \in \operatorname{SUB}(\mathbf{R}^m)$. We now have that

$$r(f|_{\Omega}) \cap Q = \bigcup_{E} P(f; E) \tag{2.4.13}$$

where $P(f; E)$ by definition is the set of points $x \in r(\Omega; E)$ such that for some small neighbourhood U of x, $\pi\colon \Omega \cap U \to E$ gives a homeomorphism of $\Omega \cap U$ onto some open subset of E, and furthermore, $\pi(x) \notin \operatorname{Sing}_{\omega} \operatorname{supp}(f \circ (\pi|_{\Omega \cap U})^{-1})$. Hence we need only prove that each $P(f; E) \in \operatorname{SUB}(R^m)$. By the previous proof we have that $r(\Omega; E) = \bigcup_{\sigma \in \mathcal{J}} G_{\sigma}^{r}$. It follows that $P(f; E) = \bigcup_{\sigma \in \mathcal{J}} H_{\sigma}$ where $H_{\sigma} \subset G_{\sigma}^{r}$ by definition is the set of points x such that $\pi(x) \notin \operatorname{Sing}_{\omega} \operatorname{supp}(f \circ \sigma)$, i.e.

$$H_{\sigma} = G_{\sigma}^{r} \setminus \sigma\,(\operatorname{Sing}_{\omega} \operatorname{supp}(f \circ \sigma)). \tag{2.4.14}$$

From the proof of Theorem 2.4.2 we know that $G_{\sigma}^{r} \in \operatorname{SUB}(\mathbf{R}^m)$. Moreover, it is easily verified that $f \circ \sigma$ is a subanalytic function. Hence by Theorems 1.2.2 and 2.3.3, we see that $H_{\sigma} \in \operatorname{SUB}(\mathbf{R}^m)$ which finishes the proof.

3. Calculus of variation

3.1. Definitions in infinite dimensional differential geometry

The purpose of this section is to introduce a class of infinite dimensional real analytic manifolds and some related concepts, which are suited for our applications in the calculus of variation later on.

A *norm-space* is a topological vector space with a distinguished continuous norm defined on it. The *weak topology* on a norm-space is the topology defined by this norm. Hence, if $(V, \|\cdot\|)$ is a norm-space, we can define the concept of *weak Cauchy sequence* in V, and form the completion V^w of V. If $(V, \|\cdot\|)$ and $(V', \|\cdot\|')$ are norm-spaces, then we let $L_r^w(V, V')$ be the space of all (weakly) continuous r-linear maps from $(V^w)^r (= V^w \times V^w \times \ldots \times V^w)$ to V'^w. $L_r^w(V, V')$ becomes a new norm-space with the norm

$$\|T\|_r \overset{\text{def}}{=} \sup\,\{\|T(\eta_1, \ldots, \eta_r)\|' \colon \|\eta_j\| \leqslant 1, \quad j = 1, \ldots, r\}. \tag{3.1.1}$$

Now let $\Phi\colon \Omega \to \Omega'$ be a map, where $\Omega \subset V$ and $\Omega' \subset V'$ are open (in the usual topology). We assume that Φ is weakly continuous.

Definition 3.1.2. Φ is said to be *weakly differentiable* iff

(i) For each $\xi \in \Omega$, there is a (unique) map $D\Phi_{\xi}$ in $L_1^w(V, V')$ such that $\|\Phi(\zeta) - \Phi(\xi) - D\Phi_{\xi}(\zeta - \xi)\|' = o(\|\zeta - \xi\|)$ when $\zeta \in \Omega$ tends weakly to ξ.

(ii) The map $D\Phi\colon \Omega \to L_1^w(V, V')$ defined by $\xi \mapsto D\Phi_{\xi}$ is weakly continuous.

Inductively we define Φ to be *r times weakly differentiable* iff Φ is weakly differentiable, and $D\Phi: \Omega \to L_1^w(V, V')$ is $r-1$ times weakly differentiable. Also, the rth weak differential of Φ at ξ is defined to be the element in $L_r^w(V, V')$ given by $D^r\Phi_\xi(\eta_1, \ldots, \eta_r) \overset{\text{def}}{=} (D^1(D^{r-1}\Phi)_\xi(\eta_1))(\eta_2, \ldots, \eta_r)$ where $D^{r-1}\Phi: \Omega \to L_{r-1}^w(V, V')$ is already defined.

Definition 3.1.3. $\Phi: \Omega \to \Omega'$ is said to be *weakly analytic* iff

(i) Φ is infinitely weakly differentiable.

(ii) For each $\xi \in \Omega$, we can find a neighbourhood U of ζ in Ω and a constant C such that for all $\zeta \in U$,

$$\|D^r\Phi_\zeta\|_r \leqslant C(Cr)^r, \quad r = 1, 2, \ldots .$$

Remark 3.1.4.

(i) It can be verified that the composition of two weakly analytic maps is weakly analytic.

(ii) Also, if both V and V' are Hilbert spaces, then weak analyticity is equivalent to analyticity in the usual sense.

Let us from now on restrict our attention to *inner product spaces*, i.e. norm-spaces where the distinguished norm is given by a continuous non-degenerate scalar product.

We then define a *weakly analytic inner product manifold* (WI-manifold for short) to be a topological Hausdorff space with an atlas $\{(U_\tau, \varphi_\tau)\}_{\tau \in I}$ of coordinate systems, where each φ_τ is a homeomorphism of the corresponding U_τ onto some open subset of an inner product space, such that all compositions $\varphi_\sigma \circ \varphi_\tau^{-1}$ are weakly analytic. Moreover, it is clear what should be ment by a weakly analytic map between WI-manifolds.

Let Ξ be a WI-manifold, and let $\Omega \subset \Xi$. A sequence $\{\xi_j\}_{j=1}^\infty$ in is said to be *dominated by* Ω iff we can find Ω_0 such that $\overline{\Omega}_0 \subset \Omega$ and integer j_0 such that $\xi_j \in \Omega_0$ for $j \geqslant j_0$.

A sequence is a *weak Cauchy sequence* iff it is dominated by some coordinate neighbourhood U_τ, and the image of the sequence under the corresponding coordinate map φ_τ is a weak Cauchy sequence (with respect to the scalar product norm).

A WI-manifold can in some sense be thought of as a "pre-Hilbert manifold". However, one should beware of taking this analogy to far. For instance, it is not true in general that WI-manifolds can be completed to Hilbert manifolds in any natural way with respect to their weak topologies. The completion will in general be just some topological space (let for example $\Xi = \{f \in C(S^1): f > 0\}$ with the scalar product $\langle f, g \rangle = \int f \cdot g\, dt$ on $C(S^1)$. Then the completion is $\{f \in L^2(S^1): f > 0\}$).

Definition 3.1.5. Let Ξ and Ξ' be WI-manifolds, and let $\Phi: \Xi \to \Xi'$ be a continuous map. Φ is a *submersion* iff for every $\xi \in \Xi$, we can find neighbourhoods U and U' of ξ and

$\Phi(\xi)$ respectively, and weakly bi-analytic homeomorphisms φ and ψ of U and U' onto open subsets Ω and Ω' of $V \oplus W$ and V, where V and W are inner product spaces, such that $\psi^{-1} \circ \Phi \circ \varphi$ is the natural (orthogonal) projection. φ and ψ are said to give a *trivialization* of Φ on U.

In the following definitions, we consider a fixed (weakly analytic) submersion Φ: $\Xi \to \Xi'$.

Definition 3.1.6. Let F: $\Xi \to R$ be a function. F is said to be *gradientiable* (with respect to Φ) at $\xi \in \Xi$ iff we can find a trivialization φ, ψ of Φ on some neighbourhood U of ξ as in Definition 3.1.5, and a weakly analytic map $\nabla_\Omega F$: $\Omega \to W$ such that for all $\zeta \in \Omega$ and $\eta \in W$, $D(F_\Omega)_\zeta(\eta) = \langle \nabla_\Omega F(\zeta), \eta \rangle$, where $F_\Omega \overset{\text{def}}{=} F \circ \varphi^{-1}$.

Remark 3.1.7. Clearly the gradient ∇F is unique if it exists. Also observe that if Ξ and Ξ' are open subsets of Hilbert spaces, and Φ is a linear map, then ∇F is given by the classical formula $\nabla F(\zeta) = \sum_{\alpha \in I} DF(e_\alpha) \cdot e_\alpha$ where $\{e_\alpha\}_{\alpha \in I}$ is some orthonormal basis for $\ker(\Phi)$.

Definition 3.1.8. F: $\Xi \to R$ is said to be *regularizing* (with respect to Φ) at $\xi \in \Xi$ iff F is gradientiable and, in addition, for some trivialization as in Definition 3.1.6, every weak solution in Ω of the inhomogeneous Euler–Lagrange equation $\nabla_\Omega F(\cdot) = \eta$, $\eta \in W$, converges to an element in Ω.

A *weak solution of* $\nabla_\Omega F(\cdot) = \eta$ in Ω is a weak Cauchy sequence $\{\xi_j\}_{j=1}^\infty$ which is dominated by Ω such that for every $\varrho \in W$, $\lim_{j\to\infty} \langle \nabla_\Omega F(\xi_j), \varrho \rangle = \langle \eta, \varrho \rangle$.

Definition 3.1.9. F: $\Xi \in R$ is said to be *non-degenerate* (with respect to Φ) at $\xi \in \Xi$ iff, in addition to what is said in Remark 3.1.4 and Definition 3.1.6 above, the Hessian of F at ξ (i.e. D^2F_ξ considered as a quadratic form) is non-degenerate on W. By this we mean that there is an orthogonal decomposition of $W = T' \oplus T''$, and a constant $c > 0$ such that (i) T' is finite dimensional, (ii) for all $\eta \in T''$, $|D^2F_\xi(\eta, \eta)| \geqslant c \cdot \langle \eta, \eta \rangle$.

With all these definitions at hand, we are finally ready to proceed to the main results of this paper in the next section.

3.2. Abstract calculus of variation

In this section, we shall study a general class of extreme value problems depending on parameters, and prove that the corresponding extreme value functions are subanalytic. The theorems will only be formulated in terms of infima, but the reader may formulate the completely analogous results for suprema himself.

Let Ξ be a WI-manifold, and let Φ: $\Xi \to M$ be a surjective (weakly analytic) submersion onto the finite dimensional real analytic manifold M. Moreover, let $F\colon \Xi \to R$ be continuous and weakly analytic. Then we can define an extreme value function f on M by

$$f(x) \overset{\text{def}}{=} \inf \{F(\xi)\colon \xi \in \Phi^{-1}(x)\}, \quad x \in M, \xi \in \Xi. \tag{3.2.1}$$

Also, let

$$E \overset{\text{def}}{=} \{\xi \in \Xi\colon F(\xi) = f(\Phi(\xi))\}. \tag{3.2.2}$$

Hence, E is the set of points in Ξ where the infimum in (3.2.1) is actually attained. We now have the following

THEOREM 3.2.3. *Suppose in addition that*

(i) *at each point $\xi \in E$, F is gradientiable, regularizing and non-degenerate (with respect to Φ).*

(ii) *The restricted map $\Phi\colon E \to M$ is proper and surjective. Then $f \in \mathcal{S}\mathcal{F}^0(M)$.*

Except for the continuity of f, this theorem is actually a special case of a more general result which we shall now formulate.

Let us consider a sequence of WI-manifolds and surjective submersions

$$\Xi_k \xrightarrow{\Phi_k} \Xi_{k-1} \xrightarrow{\Phi_{k-1}} \dots \quad \dots \xrightarrow{\Phi_1} \Xi_0 \xrightarrow{\Phi_0} M, \tag{3.2.4}$$

where as before M is finite dimensional. Moreover, suppose that for each $i = 0, 1, \dots, k$, we are given a continuous, weakly analytic function $F_i\colon \Xi_i \to R$. We can then inductively define extreme value functions f_i on M and sets $E_i \subset \Xi_i$ by

$$f_0(x) \overset{\text{def}}{=} \inf \{F_0(\xi)\colon \xi \in \Phi_0^{-1}(x)\} \tag{3.2.5}$$

$$E_0 \overset{\text{def}}{=} \{\xi \in \Xi_0\colon F_0(\xi) = f_0 \circ \Phi_0(\xi)\} \tag{3.2.6}$$

$$f_i(x) \overset{\text{def}}{=} \inf \{F_i(\xi)\colon \xi \in (\Phi_0 \circ \Phi_1 \circ \dots \circ \Phi_i)^{-1}(x) \quad \text{and} \quad \Phi_i(\xi) \in E_{i-1}\} \tag{3.2.7}$$

$$E_i \overset{\text{def}}{=} \{\xi \in \Xi_i\colon F_i(\xi) = f_i \circ \Phi_0 \circ \Phi_1 \circ \dots \circ \Phi_i(\xi) \quad \text{and} \quad \Phi_i(\xi) \in E_{i-1}\} \tag{3.2.8}$$

Hence, f_i is the infimum of F_i over the fibers of $\Phi_0 \circ \Phi_1 \circ \dots \circ \Phi_i$ under the condition that, losely speaking, all previous F_i's are also minimized.

THEOREM 3.2.9. *Suppose in addition that for each $i = 0, 1, \dots, k$,*

(i) *at each point $\xi \in E_i$, F_i is gradientiable, regularizing and non-degenerate (with respect to Φ).*

(ii) *The restricted map $\Phi_0 \circ \Phi_1 \circ \dots \circ \Phi_i\colon E_i \to M$ is proper and surjective.*

Then $f_0, f_1, \dots, f_k$ all belong to $\mathcal{S}\mathcal{F}^{\text{loc}}(M)$.

Remark 3.2.10. If Ξ and $\Xi_0, \Xi_1, ..., \Xi_k$ in Theorems 3.2.3 and 3.2.9 are all finite dimensional, then conditions 3.2.3(i) and 3.2.9(i) are trivially satisfied. Furthermore, in this case both theorems are rather straight forward consequences of Proposition 1.3.7.

Remark 3.2.11. The proof of the continuity in Theorem 3.2.3 is a standard verification which we shall omit. It uses only the continuity of F and Φ, assumption 3.2.3(ii) and the fact that Φ factors locally as a projection. However, it is easily seen from examples that the f_i's in Theorem 3.2.9 need not be continuous for $i \geqslant 1$.

Proof of Theorem 3.2.9. The fact that the f_i's are locally bounded is a trivial consequence of condition 3.2.9(ii). In fact, if $K \subset M$ is a compact, then f_i is clearly bounded from above and below on K by the sup and the inf respectively, of F_i on the compact $E_i \cap \chi_i^{-1}(K)$, where we have introduced the notation $\chi_i \overset{\text{def}}{=} \Phi_0 \circ \Phi_1 \circ ... \circ \Phi_i$. Hence it is enough to prove that the f_i's belong to $\mathcal{SF}(M)$.

Next we shall reduce the problem to a local situation.

LEMMA 3.2.12. *Suppose $\xi \in E_i$. Then we can find a neighbourhood Ω of ζ in Ξ_i and a subset $S \subset \Omega$ such that*

(i) $E_i \cap \Omega \subset S$.

(ii) $g(x) \overset{\text{def}}{=} \inf \{F_i(\zeta) \colon \zeta \in \chi_i^{-1}(x) \cap S \quad \text{and} \quad \Phi_i(\zeta) \in E_{i-1}\} \in \mathcal{SF}(M)$.

If we assume the lemma, Theorem 3.2.9 can be proved as follows: It is enough to prove that $f_i|_U \in \mathcal{SF}(U)$ for some neighbourhood U of an arbitary point in M, since subanalyticity is a local property. For any relatively compact open set $U \subset M$, we can cover $\chi_i^{-1}(U) \cap E_i$ by finitely many open neighbourhoods Ω_α, $\alpha \in J$, as in the lemma (by condition 3.2.9(ii)). Let g_α, $\alpha \in J$, be the corresponding functions as in Lemma 3.2.12(ii). Then it is easy to see that on U, $f_i = \inf \{g_\alpha \colon \alpha \in J\}$. In fact, it is trivial that $g_\alpha \geqslant f_i$ for each $\alpha \in J$. On the other hand, for each $x \in U$ there is (again by Lemma 3.2.12(ii)) a $\zeta \in E_i$ such that $\chi_i(\zeta) = x$. If we choose $\alpha \in J$ such that $\zeta \in \Omega_\alpha$, then the definition of E_i together with Lemma 3.2.12(i) and (ii) imply that $g_\alpha(x) = f_i(x)$. This proves that we have equality, and the theorem now follows from Corollary 1.3.8.

Proof of Lemma 3.2.12. Since we are now in a local situation, we see that since all the Φ_i's are submersions, it is no loss of generality to assume that M and the Ξ_i's are open subsets of direct sums of inner product spaces, $M \subset \mathbf{R}^m$ and $\Xi_i \subset \mathbf{R}^m \oplus V_0 \oplus V_1 \oplus ... \oplus V_i$, and that the Φ_i's are the projections. Moreover, we can let $\xi = 0$.

Now from the non-degeneracy condition 3.2.9(i) it follows that for each $n = 0, 1, ..., i$, we can find a decomposition $V_n = V'_n \oplus V''_n$ such that V'_n is finite dimensional and $D^2(F_n)_0$ is non-degenerate on V''_n. We shall now choose Ω as a product neighbourhood

$$\Omega = U \times U'_0 \times U''_0 \times U'_1 \times U''_1 \times \ldots \times U''_i, \quad \bar{\Omega} \subset \Xi_i, \tag{3.2.13}$$

where $U \subset M$, $U'_n \subset V'_n$ and $U''_n \subset V''_n$, such that the following conditions are fulfilled:

(CI) $U \in \mathrm{SUB}\,(M)$ and $U'_n \in \mathrm{SUB}\,(V'_n)$ for each $n = 0, 1, \ldots, i$. (This makes sense since the V_n's are finite dimensional vector spaces.)

(CII) On some neighbourhood of $\bar{\Omega}_n$, we have a weakly analytic gradient

$$\nabla F_n = \theta_n = (\theta'_n, \theta''_n): \Omega_n \to V_n = V'_n \oplus V''_n, \quad n = 0, 1, \ldots, i,$$

with the notation $\Omega_n \overset{\text{def}}{=} U \times U'_0 \times U''_0 \times \ldots \times U''_n$. Moreover every weak solution of $\nabla F_n(\cdot) = \eta$, $\eta \in V_n$, in Ω_n actually belongs to Ω_n.

(CIII) For each $n = 0, 1, \ldots, i$, we have that $D^2 F_n$ is uniformly strictly positive definite on V''_n, i.e. for all $\zeta \in \Omega_n$ and $\eta \in V''_n$, $D^2(F_n)_\zeta(\eta, \eta) \geqslant c \cdot \langle \eta, \eta \rangle$ where c is independent of ζ, η. Moreover, we can assume that for every point a in $\Sigma_n \overset{\text{def}}{=} \Omega_{n-1} \times U'_n$

$$\inf \{F_n(a, u): u \in U''_n\} < \inf \{F_n(a, u): u \in \partial U''_n\}. \tag{3.2.14}$$

To see that this is possible, first observe that we can choose the factors in (3.2.13) in the order $U''_i, U'_i, U''_{i-1}, \ldots, U$. Now the condition (CIII) is clearly satisfied at $a = 0$ for a suitable choice of U''_i, since the non-degeneracy assumption implies that $F_i|_{U''_i}$ is a convex function. Hence it also holds if we let a vary in some small neighbourhood of 0. We can now go on to choose U''_{i-1} and so on. (CII) will follow as soon as we choose Ω small enough, by the assumption that the F_n's are gradientiable and regularizing. To guarantee (CI) finally, we simply observe that we can choose U and the U'_n's as small spheres, for instance.

We can now define S as follows:

$$S \overset{\text{def}}{=} \{\zeta \in \Omega: \theta_n \circ \Phi_{n+1} \circ \Phi_{n+2} \circ \ldots \circ \Phi_i(\zeta) = 0 \quad \text{for } n = 0, 1, \ldots, i\}. \tag{3.2.15}$$

(In the case $n = i$, the equation is simply $\theta_i(\zeta) = 0$.) With this definition, the inclusion $E_i \cap \Omega \subset S$ is simply a consequence of the well-known lemma in the calculus of variation which states that a point where the extreme value is attained is stationary. In fact, if $\zeta \in E_i$, then for each $n = 0, 1, \ldots, i$, $\zeta_n \in E_n$ where $\zeta_n = \Phi_{n+1} \circ \ldots \circ \Phi_i$, as follows from the definition (3.2.8), which implies that $DF_{\zeta_n}(\eta) = 0$ for all $\eta \in V_n$. Hence $\theta_n(\zeta_n) = \nabla F_n(\zeta_n) = 0$ which implies the claim.

It remains to prove that g, defined as in Lemma 3.2.12(ii), belongs to $\mathcal{S}\mathcal{J}(M)$. The idea of the proof is now, roughly speaking, to throw away all the variables corresponding to the spaces V_n'', hence obtaining a finite dimensional problem on $W \overset{\text{def}}{=} U \times U_0' \times U_1' \times \ldots \times U_i'$ to which we can apply Proposition 1.3.7.

To make this precise, we shall need to prove the following two claims:

Claim I. For each $n=0, 1, \ldots, i$, we can find a weakly analytic map α_n from some neighbourhood of $\bar{\Sigma}_n$ to U_n'' such that $\theta_n''(\sigma, \alpha_n(\sigma))=0$ for all $\sigma \in \Sigma_n$, and

$$\{\zeta \in \Omega_n: \theta_n(\zeta)=0\} = \{\zeta=(\sigma, \tau) \in \Sigma_n \times U_n'': \tau=\alpha_n(\sigma) \quad \textit{and} \quad \theta_n'(\sigma)=0\}.$$

(Here $\Sigma_n = U \times U_0' \times U_0'' \times \ldots \times U_n'$.) We can now define functions $G_n: W = \mathbf{R}$ by ($w = (u, u_0, u_1, \ldots, u_i)$)

$$G_n(w) \overset{\text{def}}{=} F_n(u, u_0, \alpha_0(u, u_0), u_1, \alpha_1(u, u_0, \alpha_0(u, u_0), u_1), u_2, \ldots, \alpha_n(\ldots)). \tag{3.2.16}$$

Similarly, we define maps $H_n: W \to V_n'$ by

$$H_n(w) \overset{\text{def}}{=} \theta_n'(u, u_0, \alpha_0(u, u_0), u_1, \alpha_1(u, u_0, \alpha_0(u, u_0), u_1), u_2, \ldots, \alpha_n(\ldots)). \tag{3.2.17}$$

Finally, let $\pi: \mathbf{R}^m \oplus V_0' \oplus V_1' \oplus \ldots \oplus V_i' \to \mathbf{R}^m$ be the projection, and let $T \subset W$ be defined by

$$T \overset{\text{def}}{=} \{w \in W: H_n(w)=0 \quad \text{for } n=0, 1, \ldots, i \quad \text{and} \quad G_n(w)=f_n(\pi(w)) \quad \text{for } n=0, 1, \ldots, i-1\}. \tag{3.2.18}$$

Claim II. $g(x) = \inf \{G_i(w): w \in \pi^{-1}(x) \cap T\}$.

Assuming claims I and II, Lemma 3.2.12 is proved as follows:

Inductively we may assume that we have proved that $f_n \in \mathcal{SF}(M)$ for $n=0, 1, \ldots, i-1$. Since the F_n's, the θ_n''s and the α_n's (by claim I) are weakly analytic, and since weak analyticity in finite dimension reduces to usual analyticity, it follows that the functions G_n and the maps H_n are real analytic on a *neighbourhood* of W. Therefore $T \in$ SUB (W), since on any finite dimensional manifold, a set which is defined by a finite number of equations $h_j = g_j$, $j=1, \ldots, q$, where the h_j's and g_j's are subanalytic functions, is subanalytic as is easily seen. Moreover, if we (re-)define G_i to be $+\infty$ outside W, then $G_i \in \mathcal{SF}(M)$, and the formula in claim II is still valid. Combining claim II with Proposition 1.3.7 now immediately gives the lemma. Therefore it is enough to prove claims I and II.

Proof of claim I. First we shall prove the existence of a map $\alpha_n: \Sigma_n \to U_n''$ such that for each $\sigma \in \Sigma_n$, $\alpha_n(\sigma)$ is the *unique* solution in U_n'' of the equation $\theta_n''(\sigma, \tau)=0$. (Hence $\theta_n''(\sigma, \tau)=0$ is equivalent to $\tau = \alpha_n(\sigma)$ which implies the last part of the lemma.)

This is equivalent to showing that for each fixed $\sigma \in \Sigma_n$, the function $\tau \mapsto F_n(\sigma, \tau)$ has a unique stationary point in U_n''. The uniqueness follows easily from the convexity property in (CIII) above. To prove the existence, note that (CIII) also implies that if we choose a sequence $\{\tau_j\}_{j=1}^{\infty}$ in U_j'' such that $\lim_{j\to\infty} F_n(\sigma, \tau_j) = \inf\{F_n(\sigma, \tau): \tau \in U_n'\}$ then $\{\tau_j\}_{j=1}^{\infty}$ is a weak Cauchy sequence. Moreover, since V_n' is finite dimensional, we can assume that $\{\theta_n'(\tau_j)\}_{j=1}^{\infty}$ converges to some vector η in V_n', and the fact that the sequence $F_n(\sigma, \tau_j)$ tends to the infimum implies that $\lim_{j\to\infty}\langle\theta_n''(\tau_j), \varphi\rangle = 0$ for every $\varphi \in V_n''$. It follows easily that the sequence $\{(\sigma, \tau_j)\}_{j=1}^{\infty}$ defines a weak solution in Ω_n of the equation $\nabla F_n(\cdot) = (0, \eta)$. Hence by (CII), the sequence actually converges to an element on Ω_n which gives the existence.

It remains to prove that α_n is weakly analytic on (some neighbourhood of) Σ_n. To see this, we differentiate the formula $\langle\nabla F_n(\zeta), \eta\rangle = D(F_n)_\zeta(\eta)$ with respect to ζ and obtain

$$\langle D(\nabla F_n)_\zeta(\eta'), \eta\rangle = D^2(F_n)_\zeta(\eta, \eta'). \tag{3.2.19}$$

The uniform positive definiteness of D^2F_n on V_n'' now implies that for each $\zeta \in \Omega_n$, $D(\nabla F_n)_\zeta = D(\theta_n'')_\zeta$: $V_n'' \to V_n''$ is invertible as a map in $L_1^w(V_n'', V_n'')$, and the norms of these inverses are uniformly bounded. If we now differentiate the previously obtained formula $\theta_n''(\sigma, \alpha_n(\sigma)) = 0$ and argue as in the usual proof of the implicit function theorem, we easily obtain the estimate (ii) in Definition 3.1.3 for α_n which proves claim I.

Remark 3.2.20. Observe that we need the assumption that F_n is regularizing in order to guarantee the existence of α_n, since the implicit function theorem is certainly not valid in general for arbitary inner product spaces.

Proof of claim II. This is essentially just a checking of the definitions. In fact, suppose that $\xi \in S \subset \Omega$. Then it follows from claim I that there is a unique point w in W, $w = (u, u_0, ..., u_i)$ such that $\xi = (u, u_0, \alpha_0(u, u_0), u_1, \alpha_1(u, u_0, \alpha_0(u, u_0), u_1), u_2, ..., \alpha_i(...))$ and that this point must satisfy the equation $H_n(w) = 0$ for $n = 0, 1, ..., i$. Moreover, clearly $G_i(w) = F_i(\xi)$. Finally, from the definitions (3.2.6) and (3.2.8), we see that the condition that $\Phi_i(\xi) \in E_{i-1}$ is equivalent to requiering that $F_n \circ \Phi_{n+1} \circ ... \circ \Phi_i(\xi) = f_n(\chi_i(\xi))$ for $n = 0, 1, ..., i-1$, which is equivalent to $G_n(w) = f_n(\pi(w))$ for $n = 0, 1, ..., i-1$. Combining this with the definition in Lemma 3.2.12(ii), we obtain the claim.

3.3. The general program

The terminology and statements of the previous sections are quite involved. In this section we shall therefore ask, in a less formal way, what conditions have to be imposed on an extreme value problem in order to apply the previous results.

Suppose that we are given some "sufficiently analytic" extreme value problem depending on parameters, with extreme value function

$$f(x) \overset{\text{def}}{=} \inf\{F(\xi): \xi \in \Phi^{-1}(x)\}, \quad x \in M, \ \xi \in \Xi, \tag{3.3.1}$$

F: $\Xi \to R$, Φ: $\Xi \to M$, as in the previous section. Suppose furthermore that we are interested in a description of the singularities of f. In order to conclude that f is subanalytic, hence that its singularities are amenable to the rather detailed analysis in Chapter 2 (notably Theorem 2.3.3 and Corollary 2.3.9) ,we have to verify that the conditions in Theorem 3.2.3 are satisfied. If F is gradientiable (which is a rather natural condition in view of Remark 3.1.7), we are left with the verification of the following four conditions:

(R1) Φ: $E \to M$ is surjective.

(R2) Φ: $E \to M$ is proper.

(R3) F is regularizing on E.

(R4) F is non-degenerate on E.

For practical purposes however, these four conditions can be replaced by a methodological recipe in three steps:

The general program.

(GP1) To prove an existence theorem.

(GP2) To prove a regularity theorem.

(GP3) To prove an index theorem.

Here (GP1) means to verify that the extreme value is actually attained, which is another way to formulate (R1). For example, in the case of Riemannian geometry, which we shall study in Section 3.5, this corresponds to verifying that in a connected, complete Riemannian manifold there is always a geodesic of shortest length between two arbitary points.

(GP2) means to prove a (rather weak) regularity theorem for the Euler-Lagrange equation, corresponding to our extreme value problem. This will in particular take care of the requirement (R3). Also it will take care of (R2). In fact, the set of extreme values E can essentially be computed from the Euler–Lagrange equation, hence if we know that all solutions of this equation have some regularity, then we can usually prove (R2) using standard arguments in analysis (compare for instance, the Rellich lemma in the theory of Sobolev spaces). In the Riemannian case already mentioned above, it is for instance enough to prove that each geodesic has three continuous derivatives.

(GP3) finally, means to prove an index theorem for the Hessian of F. This is essentially just a reformulation of (R1). However, we note that if we again take Riemannian geometry as example, then (GP3) can be interpreted as (a special case of) the Morse Index-Theorem. In fact, this theorem states, among other things, that there are at most finitely many directions along which we can vary a geodesic without making it longer (namely the Jacobi vectorfield).

Remarks 3.3.2.

(i) The same general program applies to the more general Theorem 3.2.9 (with obvious modifications).

(ii) In case we are dealing with Hilbert manifolds, the situation simplifies somewhat. Not only is the gradientiability more or less automatic, and weak analyticity reduces to analyticity in the usual sense, but also the condition (R3) is unnecessary.

In the last two sections of this paper, we shall consider (without proofs) some examples of situations where these general ideas can be applied.

3.4. Example I; A special class of variational problems

Let X be a compact Hausdorff space, λ be a positive finite measure supported on X and let M be a finite dimensional real analytic manifold. Moreover, let U and S be real-valued functions on $M \times X \times X$ and $M \times [0, \infty[$ respectively, such that the following conditions are fulfilled:

(E1) U is a continuous function on $M \times X \times X$ (which without loss of generality can be assumed symmetric in the two X-variables), which is analytic when considered as a map from M to the Banach space $L_\infty(X \times X)$.

(E2) S is analytic on $M \times]0, \infty[$. Moreover, S is continuous on $M \times [0, \infty[$ and $S(u, 0) = 0$ for all $u \in M$.

(E3) S is a strictly convex function of t (i.e. $(\partial^2/\partial t^2) S(u, t) > 0$ for all $(u, t) \in M \times]0, \infty[$).

(E4) The following two limits hold uniformly for u varying over compact subsets of M:

(a) $\lim_{t \to 0^+} t^{-1} \cdot S(u, t) = -\infty$

(b) $\lim_{t \to \infty} t^{-1} \cdot S(u, t) = +\infty$.

With this set up, we can define a functional F on $M \times L_\infty^+(X)$, where $L_\infty^+(X)$ is the set of non-negative functions in $L_\infty(X)$, by

$$F(u, \varphi) \stackrel{\text{def}}{=} \iint_{X \times X} U(u, x, y)\, \varphi(x)\, \varphi(y)\, d\lambda(x)\, d\lambda(y) + \int_X S(u, \varphi(x))\, d\lambda(x). \tag{3.4.1}$$

Moreover, we define a function f on M by

$$f(u) \stackrel{\text{def}}{=} \inf \left\{ F(u, \varphi) \colon \varphi \in L_\infty^+(X) \quad \text{and} \quad \int_X \varphi\, d\lambda = 1 \right\}. \tag{3.4.2}$$

THEOREM 3.4.3. $f \in \mathcal{SF}^0(M)$. *Hence*, Sing_ω supp $(f) \in$ SUB (M).

This can be seen to be a consequence of Theorems 2.3.3 and 3.2.3; we let Ξ be the product $M \times \Xi_0$, where Ξ_0 is the space of functions in $L_\infty(X)$ with strictly positive lower bounds and integral one, and let Φ: $M \times \Xi_0 \to M$ be the projection. However, the verifications are rather lengthy.

Remark 3.4.4. Conditions (E1)–(E4) are in particular satisfied if we choose X to be T^n (the n-dimensional torus withits usual measure), $M = \mathbf{R}^2_+$ (the first quadrant), $u = (\beta, \varrho)$, $U(u, x, y) = \varrho^2 \beta U(x-y)$ and $S(u, t) = \varrho t \log t$. This example has a certain interest in statistical mechanics (in the so called Van der Waals model). In particular, Sing_ω supp (f) can be interpreted as the set of phase transitions.

3.5. Example II; Cut loci in Riemann an geometry

In this section we shall assume that M is a finite dimensional, connected, complete, real analytic Riemannian manifold.

Let Ξ be the space of continuously differentiable (directed) curves on M and let Φ: $\Xi \to M \times M$ be the map which sends each curve γ onto the pair of its endpoints, (γ_0, γ_1). Moreover, let L be the function on Ξ which on each curve γ evaluates its length. Then we can define the *distance function* d on $M \times M$ by $d \overset{\text{def}}{=} (d_2)^{1/2}$ where

$$d_2(p, q) \overset{\text{def}}{=} \inf \{L^2(\gamma): \gamma \in \Phi^{-1}(p, q)\}. \tag{3.5.1}$$

It can now again be verified that this is the kind of extreme value problem to which we can apply Theorem 3.2.3. Since the square-root of a subanalytic function is easily seen to be subanalytic, we have

THEOREM 3.5.2. $d \in \mathcal{SF}^0(M \times M)$.

For each $p \in M$, we may also consider the *distance function* on M *with base point* p, defined by $d_p \overset{\text{def}}{=} d(p, \cdot)$. Combining with Theorem 2.3.3 we get

COROLLARY 3.5.3. *For each* $p \in M$, $d_p \in \mathcal{SF}^0$ (M). *Hence* Sing_ω supp $(d_p) \in$ SUB (M).

Remark 3.5.4. Sing_ω supp (d_p) can be interpreted as the *cut locus* of p (see [7]). Hence, we have proved that the cut locus of p is a subanalytic set (in particular it is stratifiable and triangulable by Theorem 1.2.2(iv). (This has previously been proved by Buchner (see [1]), using a more direct geometric method.)

It is tempting to try to generalize these results to higher dimensions, using the more general Theorem 3.2.9. For simplicity we assume that M is simply connected, and consider only the case of two-simplices.

Let $D\subset M\times M\times M$ be the set of *geodesically dependent points*, i.e. triples (p, q, r) such that one of the points lies on a minimal geodesic segment, joining the other two. Let $N=M\times M\times M\setminus D$. Moreover, let Ξ_1 be the space of continuously differentiable two-simplices in M with ordered vertices in N, and let Ξ_0 be the subspace of $\Xi\times\Xi\times\Xi$, consisting of triples $(\gamma', \gamma'', \gamma''')$ such that $(\gamma')_1=(\gamma'')_0$, $(\gamma'')_1=(\gamma''')_0$ and $(\gamma''')_1=(\gamma')_0$. We then get a sequence

$$\Xi_1 \xrightarrow{\Phi_1} \Xi_0 \xrightarrow{\Phi_0} N \tag{3.5.5}$$

where Φ_1 and Φ_0 are defined in the obvious way, sending each simplex onto its sides and the sides onto the corners. Finally, we have natural functions F_1 and F_0 on Ξ_1 and Ξ_0 respectively which represent area-evaluation and evaluation of the sum of the length of the sides. Hence, in this situation the extreme value function f_1 (see (3.2.7)) is the function which to each triple (p, q, r) in N associate the area of the minimal geodesic triangle which they span.

THEOREM 3.5.6. $f_1\in \mathcal{S}\mathcal{F}^{\mathrm{loc}}(N)$. *Consequently,* $\mathrm{Sing}_\omega\,\mathrm{supp}\,(f_1)\in \mathrm{SUB}\,(N)$.

Remark 3.5.7. $\mathrm{Sing}_\omega\,\mathrm{supp}\,(f_1)$ can be thought of as the second order cut locus of M. Hence Theorems 3.2.9 and 2.3.3 together give a method to prove triangulability of higher order cut-loci.

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