THE TRIANGULATION OF LOCALLY TRIANGULABLE SPACES[1]

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We shall be concerned with the following triangulation problem: "May a space which can be triangulated locally be triangulated in the large?" A locally triangulable space is a separable metric space every point of which has a neighborhood which is homeomorphic to an open subset of some finite polyhedron. Special cases of this problem include the triangulation problems for manifolds, for manifolds with boundary, and for differentiable manifolds; these problems have been attacked in the past, with varying degrees of success. In 1935, G. Nöbeling published an argument for the triangulation theorem for manifolds [7], but it contained an essential error [9]. Subsequently, S. S. Cairns proved triangulability for differentiable manifolds [3], [4], and T. Radó triangulated the general manifold of dimension two [8]. More recently, E. E. Moise proved that three-dimensional manifolds are triangulable [5], and both Moise and R. H. Bing extended this to three-dimensional manifolds with boundary [6], [2]. In the present paper, the author uses some of these results to prove triangulability for locally triangulable spaces of dimension three or less.

We attack the general triangulation problem by attempting to reduce it to the triangulation problem for n-manifolds with boundary. This approach proves successful for n not greater than three. The proofs, while complicated, are elementary in the sense that they involve no algebraic topology.

In Chapter I certain basic definitions and lemmas are given. A new definition of locally polyhedral space is introduced in Chapter II, and some of the implications of this definition are studied. In Chapter III a space X^*, called the composition space of X, is defined; it is in some respects the opposite of a decomposition space. The technical device of passing from a space to its composition space enables us, in Chapter IV, to treat the general triangulation problem.

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CHAPTER I

Introduction

1. Throughout this paper, every space is Hausdorff and every map is continuous, unless specifically stated otherwise. The closure of a subset A of X is denoted by either $\bar{A}$ or $Cl(A)$. The *frontier* of A is the set $Cl(A) \cap Cl(X - A)$; it is denoted by $Fr(A)$.

The word *simplex* means "open simplex", and *complex* means "locally finite simplicial complex". The *boundary* of a simplex s is the set $\bar{s} - s$; it is denoted by $Bd(s)$. The *polytope* $|K|$ of a complex K is the space which is the union of the simplices of K; where no confusion will result we do not distinguish between K and $|K|$. The *k-skeleton* of K is the complex consisting of all simplices of K having dimension k or less; it is denoted by $K^{(k)}$. A *subdivision* of K is a complex L such that $|L| = |K|$ and $|K^{(k)}| \subset |L^{(k)}|$ for every k. It is a basic proposition that two subdivisions of the same complex have a common subdivision. A *polyhedron* in K is a set which is the polytope $|L|$ of a subcomplex L of some subdivision of K. If A is a subset of $|K|$, then the *star* of A in K is the union of those simplices s of K such that some face of s is contained in A; it is denoted by $St(A)$.

A linear map of one complex K into another L is a map f of $|K|$ into $|L|$ which maps each simplex s of K into a simplex t of L in such a fashion that the map is linear with respect to the barycentric coordinates of s and t. If f is a homeomorphism of $|K|$ onto $|L|$ and both f and f^{-1} are linear maps, then f is a *linear isomorphism.*

An *n-cell* is a homeomorph of an n-simplex s; a *closed n-cell* is a homeomorph of $\bar{s}$. An *n-manifold* is a separable metric space M such that each point of M has a neighborhood which is an n-cell. An *n-manifold with boundary* is a separable metric space M such that each point of M has a neighborhood whose closure is a closed n-cell. The *boundary* of an n-manifold with boundary consists of those points x of M such that x has no neighborhood which is an n-cell; it is denoted by $Bd(M)$. (Do not confuse $Bd(M)$ with $Bd(s)$, where s is a simplex.) The set $M - Bd(M)$ is called the *interior* of M and is denoted by $Int(M)$.

1.1. LEMMA. *Let K be a complex and let L be a subcomplex of K. Let L' be a subdivision of L. This subdivision may be extended to a subdivision of K without affecting any simplex of K outside $St|L|$.*

Proof. Let $K^{(m-1)}$ be so subdivided that $L' \cup K^{(m-1)}$ is a complex. Let s be an m-simplex of $K - L$. If Bds has been subdivided, we may extend this subdivision to s by means of radial lines from the barycenter of s; otherwise, we need not subdivide s. The subdivision of K defined in this manner satisfies the demands of the lemma.

2. Triangulations. A *triangulation* of a space X is a complex K along with a homeomorphism h carrying $|K|$ onto X. We sometimes refer to "simplices of X"; by this we mean those subsets of X which are carried into simplices of K by h^{-1}. A *subdivision* of the triangulation (K, h) is a triangulation (L, k) of X such that $h^{-1}k$ is a linear map of L onto K. If J is a subcomplex of K, let $A = h(|J|)$, and let $g = h|J$ (the restriction of h to $|J|$). Then (J, g) is the triangulation of A *induced by* (K, h).

Two triangulations (K, h) and (L, k) of the space X are said to be *equivalent* if $h^{-1}k$ is a linear isomorphism of L onto K. They are said to be *compatible* if they have a common subdivision.

If (K, h) and (L, k) are triangulations of the subsets A and B of X, respectively, such that (K, h) and (L, k) induce triangulations of $Y \subset A \cap B$ which are compatible, then (K, h) and (L, k) are said to be *compatible on* Y. It is clear (using 1.1) that there are subdivisions K' and L' of K and L, respectively, such that the triangulations of Y induced by (K', h) and (L', k) are equivalent.

2.1. DEFINITION. Let f be a mapping of A onto B. f is *strongly continuous* if it satisfies the following condition: If $U \subset B$ and $f^{-1}(U)$ is open in A, then U is open in B. A continuous map f induces (1) a decomposition space C whose points are the sets $f^{-1}(y)$, and (2) a map F of C onto B which is continuous and $1-1$; if f is strongly continuous, F is a homeomorphism ([1], p. 65).

Let f be a linear map of the complex L into the complex K. Then f is *strongly linear* if it maps each simplex onto one of the same dimension.

2.2. LEMMA. *Let L be a complex and let f be a strongly continuous map of $|L|$ onto X satisfying the following conditions*:

(1) *f is $1-1$ on each closed simplex of L.*

(2) *For each x in $|L|$, $f^{-1}(f(x))$ is finite.*

(3) *Let s_1 and s_2 be any two simplices of L, and let $f_i = f|\bar{s}_i$.* (a) *If the sets $f(s_1)$ and $f(s_2)$ intersect, then $f_2^{-1}f_1$ maps $\bar{s}_1$ linearly onto $\bar{s}_2$.* (b) *If f maps the vertices of s_1 and those of s_2 onto the same set, then $f_2^{-1}f_1$ maps $\bar{s}_1$ linearly onto $\bar{s}_2$.*

Then there is a triangulation (K, h) of X and a strongly linear map p of L onto K, such that $hp = f$.

Proof. We define an abstract complex J. If v and w are vertices of K, let v be defined to be equivalent to w if $f(v) = f(w)$. If v is a vertex of K, let v' denote its equivalence class. The set of vertices of J will consist of these equivalence classes.

Let $s = v_0 \dots v_n$ be a simplex of L. Since f is a homeomorphism on $\bar{s}$, the classes $v_0', \dots, v_n'$ are distinct. We define the set $\{v_0', \dots, v_n'\}$ to be a simplex of J. Because of (2), J

is a locally finite abstract complex. Let K denote a geometric realization of J; if v' is a vertex of J, let v^* denote the corresponding vertex of K. There is a natural linear map p of L onto K, defined by mapping the vertex v into v^* and extending linearly; p is strongly linear.

It follows that p is strongly continuous. We observe first that a subset U of $|L|$ is open if for every simplex s of L, $U \cap \bar{s}$ is open in $\bar{s}$. (Then $\bar{s} - U$ is closed, so that for every vertex v of L, $Cl(Stv) - U$ is closed. Hence $U \cap Stv$ is open for each v, so that U is open.) Then strong continuity follows from strong linearity, since each closed simplex of L is the homeomorphic image of a closed simplex of K.

Now $p(x) = p(y)$ if and only if $f(x) = f(y)$: Let x and y be points of $|L|$ contained in the simplices s_1 and s_2, respectively. Let $p_i = p | s_i$. If $f(x) = f(y)$, it follows from condition (3a) that $f_2^{-1} f_1 = p_2^{-1} p_1$ on s_1 (so that $p(x) = p(y)$); if $p(x) = p(y)$, the same result follows from (3b) (so that $f(x) = f(y)$). Hence the decomposition space C of $|L|$ induced by f is the same as that induced by p. Since f and p are strongly continuous, they induce homeomorphisms F and P, respectively, of C onto X and $|K|$, respectively. FP^{-1} is the desired homeomorphism h.

2.3. Lemma. *Let L and f be as in* 2.2, *except that condition* (3b) *may fail. Let L_1 be the first barycentric subdivision of L; then the pair (L_1, f) satisfies all the hypotheses of* 2.2.

2.4. Proposition. *Let A and B be closed subsets of the space X; let $X = A \cup B$. Let (K, h) and (L, k) be triangulations of A and B, respectively, which are compatible on $A \cap B$. Then there is a triangulation of X which induces triangulations of A and B which are subdivisions of the given ones.*

Proof. Let K' and L' be subdivisions such that the triangulations (K', h) and (L', k) induce equivalent triangulations of $A \cap B$. Let J denote the complex which is the disjoint union of K' and L'; let f be the map of $|J|$ onto X which equals h on $|K|$ and equals k on $|L|$. Then f is strongly continuous: Let U be a subset of X such that $f^{-1}(U)$ is open in $|J|$. Then $f^{-1}(U) \cap |K|$ is open in $|K|$, so that $U \cap A$ is open in A (h is a homeomorphism). Similarly, $U \cap B$ is open in B. Then $A - U$ and $B - U$ are closed, so that $X - U$ is closed and U is open.

It is easy to check that (J, f) satisfies conditions (1), (2), and (3a) of 2.2. Let J_1 denote the first barycentric subdivision of J. Then there is a triangulation $(\mathbf{K}, \mathbf{h})$ of X and a strongly linear map p of J_1 onto $\mathbf{K}$ such $\mathbf{h}p = f$. It follows from this equation that $(\mathbf{K}, \mathbf{h})$ induces triangulations of A and B which are subdivisions of (K, h) and (L, k), respectively.

CHAPTER II

Locally Polyhedral Spaces

3. Locally polyhedral structures. In this section we define the concept of locally polyhedral space. The reasons for introducing this notion are utilitarian; it is essential to our approach to the general triangulation theorem.

3.1. DEFINITION. Let U be an open subset of X, let L be a finite complex, and let h be a homeomorphism of U into $|L|$ such that $|L| = Cl(h(U))$ and $|L| - h(U)$ is the polytope of a subcomplex of L. Then the triple (U, h, L) is said to be a *polyhedral neighborhood* on X. We sometimes refer to a "simplex of U", meaning a subset s of U such that $h(\bar{s})$ is a simplex of L.

3.2. DEFINITION. Let A be a covering of X by polyhedral neighborhoods. If $x \in X$, the *index of x relative to A*, denoted by $I_A(x)$, is the maximum of the following set of integers:

$$\{\dim s \mid (U, h, L) \in A,\ s \text{ is a simplex of } L, \text{ and } h(x) \in s\}.$$

For example, let X consist of the coordinate planes in E_3, and let A be the collection of all polyhedral neighborhoods on X. Then for the origin, $I_A = 0$, for each point of the coordinate axes, $I_A = 1$, and for every other point, $I_A = 2$.

3.3. DEFINITION. Let B be a covering of X by polyhedral neighborhoods satisfying the following condition: If $(U, h, L) \in B$, and x and y belong to the same simplex of U, then $I_B(x) = I_B(y)$. Given B, let A denote the collection of all polyhedral neighborhoods (U, h, L) on X which satisfy the following condition: If x and y are two points of the simplex s of U, then $I_B(x) = I_B(y) \geq \dim s$. Then A is a *locally polyhedral structure* on X, and B is a *basis* for this structure. Note that $B \subset A$, and that $I_B = I_A$.

A *locally polyhedral space* is a separable metric space X, provided with a fixed locally polyhedral structure A. If X is a locally polyhedral space (the structure A, although fixed, is not usually mentioned), a polyhedral neighborhood on X will always mean an element of the structure A, and the index $I(x)$ of $x \in X$ will mean $I_A(x)$.

3.4. LEMMA. *Let A be a locally polyhedral structure on X; let $(U, h, L) \in A$. Let V be open in U, and let J be a subcomplex of some subdivision of L such that $Cl(h(V)) = |J|$ and $|J| - h(V)$ is a subcomplex of J. Then $(V, h|V, J) \in A$.*

3.5. LEMMA. *Let K be a finite complex; let s be an m-simplex of K; let $x \in s$. Let $\lambda(K)$ denote the maximum diameter of a simplex of K. Given $\varepsilon > 0$, there is a subdivision K' of K such that x lies in an m-simplex of K' and $\lambda(K') < \varepsilon$.*

Proof. The first barycentric subdivision K_1 of K is defined inductively, extending to $K^{(n)}$ the subdivison of $K^{(n-1)}$ by means of radial lines from the barycenters of the n-simplices of K. We modify this subdivision slightly. First subdivide $K^{(m-1)}$ barycentrically. Let $y \in s$; consider the ray beginning at y and going through x. This ray intersects Bds in a point belonging to some simplex of the subdivision of Bds; let the dimension of this simplex be denoted by $\beta(y, x)$. Let z denote the barycenter of s; if $\beta(z, x) = m-1$, subdivide s barycentrically. Otherwise, choose a point y close to z such that $\beta(y, x) = m-1$, and use radial lines from y instead of from z to subdivide s.

Subdivide the rest of K barycentrically; let this subdivision be denoted by K''. Then x lies in an m-simplex of K''; by choosing y close enough to z, we may make $\lambda(K'')$ as close to $\lambda(K_1)$ as we wish. Hence repeated applications of this method of subdivision will give us the desired subdivision K' of K.

3.6. Corollary. *Let X be a locally polyhedral space. If $x \in X$ and $I(x) = m$, there are arbitrarily small polyhedral neighborhoods of x (i.e., elements of the structure A) in which x lies in an m-simplex.*

3.7. Lemma. *Let x and y be two points of the simplex s of the complex K. There is a homeomorphism of $St\, s$ onto itself which carries x into y.*

Proof. Let f be the map of $Cl(Sts)$ onto itself defined as follows: (1) f is the identity on $Fr(Sts)$, (2) f maps x into y, and (3) f is extended to Sts by means of radial lines from x and y, respectively. Then $h = f|Sts$ is the required homeomorphism.

3.8. Proposition. *Let X be a metric space. If X may be covered by polyhedral neighborhoods, and A is the collection of all polyhedral neighborhoods on X, then A is a locally polyhedral structure on X.*

Proof. We need only to show that A is a basis for a structure. Given $\varepsilon > 0$, let $A(\varepsilon)$ be the subset of A consisting of polyhedral neighborhoods (U, h, L) such that U has diameter less than ε. It follows from 3.5 that $I_A = I_{A(\varepsilon)}$. Let x and y belong to the simplex s of U, where $(U, h, L) \in A$. By 3.7, there are neighborhoods of x and y which are homeomorphic, the homeomorphism g carrying x into y. Then $I_{A(\varepsilon)}(x) \leq I_A(y)$, for some ε, and the proposition follows, by symmetry.

3.9. Definition. If X and A are defined as in 3.8, A is called the *trivial* locally polyhedral structure on X. There are structures which are not trivial. Let K be a finite complex and let i be the identity map of $|K|$ onto itself. The covering consisting of the single polyhedral neighborhood $(|K|, i, K)$ is a basis for a locally polyhedral structure.

If, for example, K consists of an n-simplex and its faces ($n > 1$), this structure is not the trivial one.

3.10. Definition. A map f of one locally polyhedral space X into another Y is said to *preserve index* if for each $x \in X$, $I(x) = I(f(x))$.

3.11. Definition. A subspace Y of X is said to be *admissible* if for each polyhedral neighborhood (U, h, L) on X, Y contains every simplex of U which it intersects.

3.12. Proposition. *Let X be a locally polyhedral space. Let Y be a closed admissible subspace of X. Then Y has a naturally induced locally polyhedral structure relative to which the inclusion map of Y into X preserves index.*

Proof. Let A be the structure on X. We define a basis B for a structure on Y. Let $(U, h, L) \in A$. $h(U \cap Y)$ contains every simplex of L which it intersects, so that its closure is the polytope of a subcomplex J of L. We show that $|J| - h(U \cap Y)$ is the polytope of a subcomplex of J. Let s be a simplex of J contained in $|J| - h(U \cap Y)$; let s' be a face of s. Suppose $s' \subset h(U \cap Y)$. Then since $h(U)$ is open in $|L|$, $s \subset h(U)$. By definition of J, s is a face of some simplex of L contained in $h(U \cap Y)$, and since $h(U \cap Y)$ is closed in $h(U)$, $s \subset h(U \cap Y)$. This is a contradiction. Hence $s' \subset |J| - h(U \cap Y)$.

Then $(U \cap Y, h | Y, J)$ is a polyhedral neighborhood on Y; let this neighborhood belong to B. It is clear that if $x \in Y$, $I_B(x) = I_A(x)$. If x and y belong to the same simplex of $U \cap Y$, they belong to the same simplex of U, so that $I_A(x) = I_A(y)$. Then $I_B(x) = I_B(y)$. Since B obviously covers Y, B is a basis for a locally polyhedral structure on Y. Relative to this structure, the inclusion map preserves index.

4. The pseudo skeleton. The pseudo skeletons of a locally polyhedral space X are closed subsets of X which break X up into pieces which are manifolds in much the same way as the skeletons of a complex K break $|K|$ up into cells.

4.1. Definition. If X is a locally polyhedral space, the *pseudo k-skeleton* of X is the subset of X consisting of all points x such that $I(x) \leq k$. It will be denoted by X^k (see [1], p. 400); we show in 4.2 that it is a closed subset of X. In general, X^k depends on the particular structure A; only when A is the trivial structure is X^k topologically determined. I.e., in the example of 3.9, the k-skeleton and pseudo k-skeleton of $|K|$ coincide. Note that X^k is an admissible subspace of X, by 3.3, so that by 3.12, X^k has a naturally induced locally polyhedral structure, relative to which the inclusion map preserves index. Whenever we consider X^k as a locally polyhedral space, we will suppose it is provided with this structure.

A word about dimension is in order. The dimension of X is well-defined, since X is separable and metric. One shows easily that the locally polyhedral space X has dimension n if and only if (1) for every polyhedral neighborhood (U, h, L) on X, $\dim L \leq n$, and (2) for some such neighborhood, $\dim L = n$. It follows that X^k has dimension not greater than k.

4.2. PROPOSITION. *Let X be a locally polyhedral space. Then the set X^m is closed, and $X^m - X^{m-1}$ is an m-manifold.*

Proof. Let $x \in X - X^m$; then $I(x) = k > m$. Let (U, h, L) be a polyhedral neighborhood such that x lies in a k-simplex s of U. Now $St\, s$ contains no simplices of dimension less than k, so that $St\, s \subset X - X^m$. Since $St\, s$ is open in X, X^m is closed.

Let $x \in X^m - X^{m-1}$. Let (U, h, L) be a polyhedral neighborhood such that x lies in the m-simplex t of U. Now $St\, t$ is open in X, so that $(St\, t) \cap X^m$ is open in X^m. But $(St\, t) \cap X^m = t$, by 3.3, and t is an m-cell neighborhood of x in $X^m - X^{m-1}$. Hence $X^m - X^{m-1}$ is an m-manifold.

4.3. LEMMA. *Let X be a locally polyhedral space, let B a basis for the structure on X, and let $\alpha(x)$ be any function on X such that α is constant on every simplex of every polyhedral neighborhood of B. If $I(x) = j$, then there is a neighborhood V of x in X^j such that α is constant on V.*

Proof. Let (U, h, L) be an element of B such that x lies in the j-simplex s of U. Let $V = s$; then α is constant on V. Since $V = St\, s \cap X^j$, V is open in X^j.

4.4. COROLLARY. *Assume the hypotheses of 4.3. Let $Y \subset X$ be a connected set on which $I(x)$ is constant. Then $\alpha(x)$ is also constant on Y.*

Proof. Let $I(Y) = j$. Then $Y \subset X^j$. The subsets of Y on which $\alpha(x)$ is constant are open in Y, by 4.3. Since Y is connected, there is only one such subset.

4.5. DEFINITION. Let X be a locally polyhedral space. A triangulation (K, h) of X is said to be a *proper triangulation* if (K, h) induces a triangulation of X^k, for each k. This is equivalent to the requirement that $I(x)$ should be constant on each simplex of X under this triangulation. If the structure on X is not the trivial one, there may exist non-proper triangulations. Consider the example of 3.9. Any homeomorphism g of $|K|$ onto itself defines a triangulation (K, g) of $|K|$; but g must map each simplex onto one of the same dimension if (K, g) is to be proper.

4.6. PROPOSITION. *Let X be a locally polyhedral space of dimension n; let $M = X - X^{n-1}$. Then*

(1) *$\bar{M}$ is a locally polyhedral space; it has a naturally induced locally polyhedral structure relative to which the inclusion map of $\bar{M}$ into X preserves index.*

(2) $\bar{M}^{n-1} = \bar{M} - M = Fr\, M$.

(3) *If* X^{n-1} *is properly triangulated, this triangulation induces a proper triangulation of* $\bar{M}^{n-1}$.

Proof. Let $\alpha(x)$ denote the largest integer m such that $x \in Cl(X^m - X^{m-1})$. Then $\alpha(x)$ is constant on every simplex of every polyhedral neighborhood on X. $\bar{M}$ consists precisely of those points of X for which $\alpha(x) = n$. Hence $\bar{M}$ is an admissible subspace of X, and (1) follows from 3.12. It is clear that (2) holds. To prove (3), let X^{n-1} be properly triangulated. By 4.4 and 4.5, each simplex of X^{n-1} which intersects $Fr\, M$ must lie in $Fr\, M$. Since $Fr\, M$ is closed, it is the polytope of a subcomplex of X^{n-1}. This induced triangulation of $Fr\, M$ is proper, since the inclusion map of $Fr\, M$ into X^{n-1} preserves index.

4.7. Remark. The hypothesis that the triangulation of X^{n-1} is proper is needed in the preceding proposition. Consider the following example: Take two copies of E_2 and identify them along the closed half-line $x \geq 0$, $y = 0$. Then take another closed half-line and identify its end point with the end point of the previous half-line. Consider the resulting space X as a locally polyhedral space under the trivial structure. X^1 is the homeomorph of a straight line, and X^0 is a single point dividing the line into two parts, one of which is $Fr\, M$. Obviously, if a triangulation of X^1 is to induce a triangulation of $Fr\, M$, the point X^0 must be a vertex in this triangulation. Hence the triangulation of X^1 must be proper.

Less trivial is the following example: Take two copies of E_3 and identify them on the closed half-plane $x \geq 0$, $z = 0$. Take another copy of this closed half-plane and identify it with the previous half-plane along their y-axes. Take the trivial locally polyhedral structure for this space X. X^2 is a plane, and X^1 is a line running across this plane, dividing it into two half-planes. $Fr\, M$ is one of these closed half-planes. In the previous example, $Fr\, M$ was always a *polyhedron* in $Y = X^{n-1}$, whether the triangulation of Y was proper or not. In the present example, this is not the case; there are clearly triangulations of Y relative to which $Fr\, M$ is not a polyhedron.

Our approach to the triangulation problem will be by induction on the dimension of the space. We first choose a triangulation of Y and seek to extend this triangulation to a triangulation of X. If we should happen to choose a triangulation of Y relative to which $Fr\, M$ is not a polyhedron, this would not be possible. Hence we shall require that the triangulation of Y be a proper one.

5. Singularity of a point. The preceding proposition reduces our problem to that of "extending" a proper triangulation of $Fr\, M$ to $\bar{M}$; the advantage is that $\bar{M}$ is a manifold with boundary except possibly at points of $Fr\, M$. Let us study these points more closely.

5.1. DEFINITION. Let X be a locally polyhedral space; let $x \in X$. Then x is said to have *singularity k with respect to X^m and X* if k is the smallest number such that x has arbitrarily small neighborhoods U such that $U - X^m$ has k components. We obtain in 5.5 an equivalent definition which is more usable. It will follow that every point of X has finite singularity.

Let X have dimension n. Usually we are concerned with the singularity of a point with respect to X^{n-1} and X. If we say "x has singularity k", the phrase "with respect to X^{n-1} and X" will be understood. Further, x will be said to be *non-singular* if its singularity is less than 2; otherwise it is *singular*.

Let $M = X - X^{n-1}$. The singular points x of X lie in $Fr\ M$: If x is in M, then x has a connected neighborhood which does not intersect X^{n-1}, so that its singularity is 1. If x is not in $\bar{M}$, it has a neighborhood which does not intersect M, so that x has singularity 0.

5.2. LEMMA. *Let K be a complex in E_n, let L be a subcomplex of K, and let x be a point of the simplex s of K. Let S be a spherical neighborhood of x in $|K|$, lying in $St\ s$. Then each component of $St\ s - |L|$ contains exactly one component of $S - |L|$.*

Proof. Project $St\ s$ onto S by means of rays from x. This homeomorphism carries each simplex into itself, so that it maps $St\ s - |L|$ onto $S - |L|$. The lemma follows.

5.3. COROLLARY. *Assume the hypotheses of the preceding lemma. If U is any neighborhood of x in $|K|$ which lies in $St\ s$, then $U - |L|$ has at least as many components as $St\ s - |L|$.*

5.4. REMARK. Let X be a locally polyhedral space; let (U, h, L) be a polyhedral neighborhood on X and let x belong to the simplex s of U. $St\ s$ is open in U, and $Cl(h(St s))$ is the polytope of a subcomplex J of L. Moreover, $|J| - h(St s)$ is a subcomplex of J. By 3.4, $(St\ s, h | St\ s, J)$ is a polyhedral neighborhood on X. The statement "Let $(St s, k, J)$ be a polyhedral neighborhood of x" means that it is derived in this way and that x is in s.

5.5. PROPOSITION. *Let X be a locally polyhedral space; let $(St s, k, J)$ be a polyhedral neighborhood of $x \in X$. Let $St s - X^m$ have k components. Then x has singularity k with respect to X^m and X.*

Proof. Given $\varepsilon > 0$, let S be a spherical neighborhood of $h(x)$ having diameter less than ε and lying in $St(h(s))$. Then $S - h(X^m)$ has k components, by 5.2, and so does $h^{-1}(S) - X^m$. By definition, the singularity of x with respect to X^m and X is not greater than k.

On the other hand, if U is any neighborhood of x which is small enough to lie in $St s$, then by 5.3, $U - X^m$ must have at least k components. Hence the singularity of x with respect to X^m and X is not less than k.

5.6. COROLLARY. *Let X be a locally polyhedral space, and let $\alpha(x)$ equal the singularity of x with respect to X^m and X. Then α is constant on every simplex of every polyhedral neighborhood on X.*

5.7. LEMMA. *Let X be a locally polyhedral space of dimension n such that $X = Cl(X - X^{n-1})$. Let $j \leq n-1$, and suppose X has no singular points of index greater than j. If $x \in X$ has singularity k (with respect to X^{n-1} and X), then it has singularity k with respect to X^j and X.*

Proof. Let $(St s, h, L)$ be a polyhedral neighborhood of x (5.4). Then $St s - X^{n-1}$ has k components; we show that $St s - X^j$ also has k components. Note that $St s$ consists entirely of n-simplices and their faces, since $X = Cl(X - X^{n-1})$. Let $y \in St s \cap (X^{n-1} - X^j)$. Then y lies in some proper face t of an n-simplex of $St\, s$. Since y has index greater than j, y is non-singular, so that $St\, t - X^{n-1}$ has exactly one component, which must be contained in some component of $St s - X^{n-1}$. Since $St\, t$ is open in X, y is a limit point of only one component of $St s - X^{n-1}$.

Let the components of $St s - X^{n-1}$ be denoted by C_i $(i = 1, \ldots, k)$. Let $D_i = \bar{C}_i \cap (St s - X^j)$. Each set D_i is connected, consisting of C_i plus some of its limit points. As just shown, the sets D_i are disjoint and their union is $St s - X^j$. Hence they are the components of $St s - X^j$, and there are exactly k of them.

5.8. PROPOSITION. *Let X be a locally polyhedral space of dimension n such that $X = Cl(X - X^{n-1})$; suppose X has no singular points. If $x \in X$ has index $n-1$ or $n-2$, it has a neighborhood whose closure is a closed n-cell, while if x has index $n-1$, it has no neighborhood which is an n-cell.*

Proof. Let $(St s, k, L)$ be a polyhedral neighborhood of x such that $\dim s = I(x)$. Let t denote the simplex $k(s)$ of L; then $L = Cl(St\, t)$, and L consists entirely of n-simplices and their faces.

Suppose x has index $n-1$. L contains only one n-simplex, since otherwise $St\, t - t = St\, t - k(X^{n-1})$ would not be connected. Then $|L|$ is a closed n-simplex and $k(x)$ lies on the boundary of this closed n-simplex.

Suppose x has index $n-2$. Consider, for each n-simplex t_i of L, the face e_i of t_i opposite t. Let N denote the collection of these 1-simplices e_i and their vertices. Then $|N|$ is a connected 1-manifold with boundary: If $|N|$ were not connected, then the n-simplices of L could be divided into two disjoint sets such that no simplex of the first set has an $(n-1)$-dimensional face in common with a simplex of the second set. Then $St\, t - t = St\, t - k(X^{n-2})$ would not be connected, contradicting 5.7. If $|N|$ is not a 1-manifold with boundary, then three 1-simplices of N meet at a vertex v. If e_i and e_j meet at v, then t_i and t_j have an

$(n-1)$-dimensional face r in common; r is the face spanned by v and t. If the third n-simplex t_k has r as a face, then the points of $h^{-1}(r)$ have index $n-1$, so that $St\,r - r$ equals $St\,r - k(X^{n-1})$. But $St\,r - r$ is not connected, contradicting the fact that X has no singular points.

Then $|N|$ is a homeomorph of a closed line segment or a circle. From this, it is readily shown by combinatorial means that $|L|$ is a closed n-cell. In the former case, x lies on its boundary; in the latter case, x lies in its interior.

CHAPTER III

The Composition Space

The preceding proposition indicates that $\bar{M}$ would be more like a manifold with boundary if one could dispose of the singular points of $Fr\,M$. The accomplishment of this task occupies the present chapter. We form a new space by "pulling $\bar{M}$ apart" along the singular parts of its frontier. The new space has fewer singular points than $\bar{M}$ (7.9), but it is equivalent to $\bar{M}$ as far as triangulability is concerned (8.5).

The following assumption holds throughout the chapter: X is a locally polyhedral space of dimension n such that (1) $X = Cl(X - X^{n-1})$, and (2) X has no singular points of index greater than the fixed number j ($j \leq n-1$).

6. Defining the space. Let A_k denote the set of all points of X having index j and singularity k, $k > 1$. Let $A = \bigcup_k A_k$. We note certain important facts:

(1) If (U, h, L) is a polyhedral neighborhood on X, then A_k contains every simplex of U which it intersects, by 5.6.

(2) The sets A_k are disjoint open subsets of X^j, by 4.3.

(3) $\bar{A} = \bigcup \bar{A}_k$. If x is a limit point of A, each polyhedral neighborhood of x intersects only a finite number of the sets A_k, by (1). Then x is a limit point of some A_k.

(4) Let $x \in A_k$, and let (Sts, h, J) be a polyhedral neighborhood of x such that $\dim s = j$. Then $Sts - X^{n-1}$ has k components, by 5.5, and so does $Sts - X^j$, by 5.7. Also, $Sts - X^j = Sts - s = Sts - A_k$.

6.1. LEMMA. *Let $x \in A_k$. Let (Sts_i, k_i, J_i) be polyhedral neighborhoods of x ($i = 1, 2, 3$). If C_1 is a component of $Sts_1 - X^j$, there is exactly one component C_2 of $Sts_2 - X^j$ such that $C_1 \cap C_2$ has x as a limit point. If C_3 is the component of $Sts_3 - X^j$ such that $C_1 \cap C_3$ has x as a limit point, then so does $C_2 \cap C_3$.*

Proof. This follows immediately from 5.2.

6.2. DEFINITION. We define the *composition space* X^*. Its points consist of one copy of each point of $X - A$, along with k copies of each point of A_k ($k = 2, 3, \ldots$).

We define a function ω. Choose for each point x of A, a fixed polyhedral neighborhood (Sts_x, k_x, J_x) of x. For each distinct component C_x^i of $Sts_x - X^j$, let $\omega(C_x^i, x)$ denote a different one of the copies of x in X^* (there are k such components, and k copies of x). Then if (Sts, k, J) is any polyhedral neighborhood of x and C is any component of $Sts - X^j$, there is exactly one component C_x^i of $Sts_x - X^j$ such that $C \cap C_x^i$ has x as a limit point. Define $\omega(C, x) = \omega(C_x^i, x)$. By 6.1, ω has the following important property: If (Sts_i, k_i, J_i) is a polyhedral neighborhood of $x \in A$ and C_i is a component of $Sts_i - X^j$ ($i = 1, 2$), then $\omega(C_1, x) = \omega(C_2, x)$ if and only if $C_1 \cap C_2$ has x as a limit point.

Let π be the natural projection of X^* onto X. We define a basis for open sets in X^*:

(A) If U is open in X, $\pi^{-1}(U)$ is a basis element. (Hence π is continuous.)

(B) Let $x \in A$ and let (Sts, k, J) be any polyhedral neighborhood of x *such that* $\dim s = j$. Let C be any component of $Sts - s$. Since $s \subset A_k$, $\omega(C, z)$ is defined for each point z of s. Let the set

$$W(C) = \pi^{-1}(C) \cup \{\omega(C, z) \mid z \in s\}$$

be a basis element.

These neighborhoods will be referred to as neighborhoods of type (A) and of type (B), respectively.

6.3. PROPOSITION. *X^* is a Hausdorff space.*

Proof. We must show that if y is contained in the intersection of two basis elements, so is a basis element containing y. Suppose the basis elements are of type (B), say $W(C_i)$ ($i = 1, 2$), where C_i is a component of $Sts_i - s_i$. (Similar arguments apply in the other cases.) If $y \in \pi^{-1}(C_1)$, then $y \in \pi^{-1}(C_2)$ as well. C_i is open in X, since X is locally connected and C_i is a component of the open set $Sts_i - s_i$. Then $\pi^{-1}(C_1 \cap C_2)$ is the required basis element.

Otherwise, $y = \omega(C_1, x) = \omega(C_2, x)$, for some x in $s_1 \cap s_2$. Let (Sts_3, k_3, J_3) be a polyhedral neighborhood of x such that $Sts_3 \subset Sts_1 \cap Sts_2$ and $\dim s_3 = j$ (by 3.6). Let C_3 be the component of $Sts_3 - s_3$ such that $\omega(C_3, x) = y$. Then $W(C_3) \subset W(C_1) \cap W(C_2)$.

Hence X^* is a topological space. To show it is Hausdorff, let $x, y \in X^*$ and let $\pi(x) = \pi(y) = z$ (the other case is trivial). Let (Sts, k, J) be a polyhedral neighborhood of z such that $\dim s = j$. Then $x = \omega(C_1, z)$ and $y = \omega(C_2, z)$, where C_1 and C_2 are distinct components of $Sts - s$. $W(C_1)$ and $W(C_2)$ are disjoint neighborhoods of x and y, respectively.

6.4. LEMMA. *Let $y \in X^*$, and let U be a neighborhood of y. If $\pi(y) \in X - A$, U contains a neighborhood of y of type (A); if $\pi(y) \in A$, one of type (B).*

Proof. Let $\pi(y) \in X - A$. If U contains no neighborhood of y of type (A), there is a neighborhood $W(C)$ of y contained in U. Then $y \in \pi^{-1}(C)$, since $\pi(y)$ is not in A, so that $\pi^{-1}(C) \subset U$ is a neighborhood of y of type (A), contrary to hypothesis.

Let $\pi(y) \in A_k$. If U contains no neighborhood of y of type (B), there is a neighborhood $\pi^{-1}(V)$ of y of type (A) contained in U. There is a polyhedral neighborhood $(St\, s, k, J)$ of $\pi(y)$ such that $\dim s = j$ and $St\, s \subset V$. For some component D of $St\, s - s$, $\omega(D, \pi(y)) = y$. Then $W(D) \subset U$ is a neighborhood of y of type (B), contrary to hypothesis.

6.5. Proposition. *X^* has a countable basis.*

Proof. Let $U_1, U_2, \ldots$ be a countable basis for X. Then the collection $\{\pi^{-1}(U_i)\}$ contains arbitrarily small neighborhoods of every point in $\pi^{-1}(X - A)$, by 6.4. On the other hand, there is a countable collection $\{(St\, s_{ik}, g_{ik}, J_{ik})\}$ of polyhedral neighborhoods on X such that (1) $\dim s_{ik} = j$, (2) $s_{ik} \subset A$, and (3) the collection contains arbitrarily small neighborhoods of each point of A. This collection is defined by choosing a polyhedral neighborhood $(St\, s_{ik}, g_{ik}, J_{ik})$ satisfying (1) and (2) such that $U_i \subset St\, s_{ik} \subset U_k$, for each pair of indices i, k for which such a polyhedral neighborhood exists. From each polyhedral neighborhood in this collection may be derived a finite number of neighborhoods in X^* of type (B); the resulting derived collection will still be countable. Using 6.4, one may prove that this derived collection contains arbitrarily small neighborhoods of each point of $\pi^{-1}(A)$. The union of these two collections of open sets is a countable basis for X^*.

6.6. Proposition. *The map π is strongly continuous.*

Proof. Let U be a subset of X such that $\pi^{-1}(U)$ is open in X^*. Let $x \in U$. If $x \in X - A$, $\pi^{-1}(x)$ has a neighborhood $\pi^{-1}(V)$ of type (A) contained in $\pi^{-1}(U)$. Then V is a neighborhood of x contained in U.

If $x \in A_k$, let $y_1, \ldots, y_k$ be the points of $\pi^{-1}(x)$. There is a neighborhood $W(C_i)$ of y_i of type (B) contained in $\pi^{-1}(U)$, derived from the polyhedral neighborhood $(St\, s_i, k_i, J_i)$ of x. There is a polyhedral neighborhood $(St\, s, k, J)$ of x such that $\dim s = j$ and $St\, s \subset \bigcap St\, s_i$. Let D_i be the component of $St\, s - s$ such that $\omega(D_i, x) = y_i$. Then $W(D_i) \subset W(C_i) \subset \pi^{-1}(U)$. The set $\pi^{-1}(St\, s) = \bigcup W(D_i)$ is contained in $\pi^{-1}(U)$, so that $St\, s$ is a neighborhood of x contained in U.

Hence U is open, and π is strongly continuous.

7. Imposing a locally polyhedral structure. We recall some facts about covering maps. If p maps E into B and f maps Y into B, then a map f^* of the subset Z of Y into E is said to be a *lifting* of f over Z if for every $z \in Z$, $pf^*(z) = f(z)$. A map p of E onto B is said to be a *covering map* if each point x of B has a neighborhood V such that p maps each of the components of $p^{-1}(V)$ homeomorphically onto V. It is a basic proposition on covering maps that if p is a covering map and f is a map of the simply connected, arcwise connected, locally arcwise connected space Y into B, then a lifting of f over a single point of Y may be uniquely extended to a lifting of f over all of Y.

7.1. LEMMA. *Let f map Z into X and let f^* be a (not necessarily continuous) map of Z into X^* which is a lifting of f over Z. Then f^* is continuous at each point of $f^{-1}(X-A)$.*

Proof. This follows immediately from 6.4.

7.2. LEMMA. *Let $W(C)$ be a neighborhood on X^* of type (B), derived from $(St\,s, h, L)$. Then π maps $W(C)$ homeomorphically onto $C \cup s$.*

Proof. Clearly $\pi | W(C)$ is continuous and $1-1$; let g denote the inverse map. By 7.1, g is continuous at each point of C. Let $x \in s$. Let U be a neighborhood of $g(x)=y=\omega(C, x)$ in $W(C)$. Then U is open in X^* and contains a neighborhood $W(D)$ of y of type (B), derived from $(St\,t, k, J)$. Let $V = D \cup t$; then $V \subset C \cup s$, since $D \subset C$ and $t \subset s$. Each component of $St\,t - t$ is contained in a different component of $St\,s - s$, so that $V = (C \cup s) \cap (St\,t)$. Hence V is open in $C \cup s$. Since $g(V) \subset U$, g is continuous at x.

7.3. LEMMA. *Let s be a simplex; let h map $\bar{s}$ homeomorphically into X, mapping s into A_k. Let s' denote $\bar{s} \cap h^{-1}(A_k)$. Then $\pi^{-1}h(s')$ has k components, each of whose closures is mapped homeomorphically onto $h(\bar{s})$ by π.*

Proof. We show that π is a covering map of the space $\pi^{-1}(A)$ onto A. Let $x \in A_k$, and let $(St\,t, k, J)$ be a polyhedral neighborhood of x such that $\dim t = j$. Then t is a neighborhood of x in A. Let C_i be a component of $St\,t - t$, and let s_i denote $W(C_i) \cap \pi^{-1}(A)$. Then π maps s_i homeomorphically onto t, by 7.2, and the sets s_i are merely the components of $\pi^{-1}(t)$.

Let z be a fixed point of s', and let $y_1, \ldots, y_k$ be the points of $\pi^{-1}h(z)$. Define $h_i(z)=y_i$; then h_i is a lifting of h over z, so that it may be uniquely extended to a lifting h_i of h over all of s'. The sets $h_i(s')$ are disjoint, for if $h_i(s')$ and $h_m(s')$ had the point q in common, then both h_i and h_m would be the unique extension to s' of that lifting of h over $h^{-1}\pi(q)$ which carries this point into q. But they disagree at z. Since the sets $h_i(s')$ are k in number, their union is $\pi^{-1}h(s')$.

Now h_i is defined on s'; if $w \in \bar{s} - s'$, define $h_i(w) = \pi^{-1}h(w)$. By 7.1, h_i is continuous at each point of $\bar{s} - s'$; since s' is open in $\bar{s}$, h_i is still continuous at each point of s'. The components of $\pi^{-1}h(s')$ are the sets $h_i(s')$; since $\bar{s}$ is compact, their closures are the sets $h_i(\bar{s})$. Finally, π is a homeomorphism of $h_i(\bar{s})$ onto $h(\bar{s})$, since $h_i(\bar{s})$ is compact.

7.4. LEMMA. *Let s be a simplex; let h map $\bar{s}$ homeomorphically into X, mapping s into $X - A$. Then there is a unique lifting of h to a homeomorphism h^* of $\bar{s}$ into X^*.*

Proof. Let s' denote the set $\bar{s} - h^{-1}(A)$. If $z \in s'$, we must define $h^*(z) = \pi^{-1} h(z)$.

Let $z \in \bar{s} - s'$, and let $x = h(z)$. Choose a polyhedral neighborhood $(St\, t, k, J)$ of x such that $\dim t = j$. There is one and only one component C of $St\, t - t$ such that $C \cap h(s')$ has x as a limit point: Since x is a limit point of $(St\, t \cap h(s')) \subset (St\, t - t)$, there is at least one such component C. On the other hand, if U_ε is the ε-neighborhood of z in $\bar{s}$ (where ε is small enough that $h(U_\varepsilon) \subset St\, t$), then $h(U_\varepsilon \cap s')$, being connected, is contained in some component C of $St\, t - t$. Hence there is only one such component C.

Define $h^*(z) = \omega(C, x)$. If h^* is to be continuous, $h^*(z)$ must be so defined. For $h^*(z)$ must be a limit point of $h^*(U_\varepsilon \cap s') \subset \pi^{-1}(C) \subset W(C)$, and $\omega(C, x)$ is only the point of $\pi^{-1}(x)$ which can be a limit point of $W(C)$.

This definition of $h^*(z)$ is independent of the choice of $(St\, t, k, J)$. Let $(St\, t', k', J')$ be a polyhedral neighborhood of x such that $\dim t' = j$. x is a limit point of $(St\, t) \cap (St\, t') \cap h(s')$, so that x is a limit point of $C \cap C' \cap h(s')$, for some component C' of $St\, t' - t'$. If $(St\, t', k', J')$ were used to define $h^*(z)$, we would define $h^*(z) = \omega(C', x)$. But $\omega(C', x) = \omega(C, x)$.

Finally, we show that h^* is continuous at z. Let $W(D)$ be a neighborhood of $h^*(z)$ of type (B), derived from $(St\, r, g, L)$. Then $\omega(D, x) = h^*(z)$. Let ε be small enough that $h(U_\varepsilon) \subset St\, r$. We show that $h^*(U_\varepsilon) \subset W(D)$. First let $z' \in U_\varepsilon - s'$. Now $h(U_\varepsilon \cap s') \subset D$, so that D is the only component of $St\, r - r$ such that z' could be a limit point of $D \cap h(s')$. By the preceding paragraph, we must have $h^*(z') = \omega(D, h(z)) \subset W(D)$. Hence $h^*(U_\varepsilon - s') \subset W(D)$. Since also $h^*(U_\varepsilon \cap s') \subset \pi^{-1}(D) \subset W(D)$, it follows that $h^*(U_\varepsilon) \subset W(D)$.

h^* is automatically continuous at each point of s', so that it is continuous on $\bar{s}$. It is $1 - 1$, since h is, so that it is a homeomorphism on $\bar{s}$. We have already shown that h^* is unique.

7.5. DEFINITION. Let K' be a subdivision of K such that for each simplex s of K', the simplex of K containing s contains a vertex of s. Then K' is called an *admissible* subdivision. Examples of such subdivisions include the barycentric subdivisions of K and the "modified" barycentric subdivisions defined in 3.5.

7.6. LEMMA. *Let f be a map of the complex K into X; consider the following conditions on the pair (K, f):*

(1) *Both $f^{-1}(\bar{A}_k)$ and $f^{-1}(A_k)$ contain every simplex of K which they intersect, for each k.*

(2) *If the vertices of a simplex of K lie in $f^{-1}(\bar{A}_k)$, so does the simplex.*

(3) *If a simplex lies in $f^{-1}(A_k)$, so does at least one of its vertices.*

Let (K, f) *satisfy condition* (1), *and let* K' *be an admissible subdivision of* K. *Then* (K', f) *satisfies all three conditions.*

7.7. Lemma. *Let* h *be a homeomorphism of* K *into* X *such that* (K, h) *satisfies conditions* (1) *to* (3) *of* 7.6. *Then there is a triangulation* (L, g) *of* $\pi^{-1}h(|K|)$ *and a strongly linear map* p *of* L *onto* K *such that* $hp = \pi g$ (see 2.1).

Proof. We define an abstract complex J as follows: Let V denote the set of vertices of K. Let W be a set which is in one-to-one correspondence with the set $\pi^{-1}h(V)$; if $y \in \pi^{-1}h(V)$, let y' denote the corresponding point of W. The vertices of J are the points of W. The simplices of J are defined as follows:

(a) Let $s = v_0 \ldots v_m$ be a simplex of K lying in $h^{-1}(A_k)$. Then $\pi^{-1}h(s)$ consists of k disjoint homeomorphs of s, by 7.3. Let t be one of the components of $\pi^{-1}h(s)$; then $\bar{t}$ contains precisely one point y_i of the set $\pi^{-1}h(v_i)$ for each i. Let $\{y_0', \ldots, y_m'\}$ be a simplex of J. There are k such distinct simplices, by condition (3).

(b) Let $s = v_0 \ldots v_m$ be a simplex of K lying in $h^{-1}(X - A)$. There is a unique lifting of the map h of $\bar{s}$ into X into a map h^* of $\bar{s}$ into X^*. $h^*(\bar{s})$ contains exactly one point y_i of the set $\pi^{-1}h(v_i)$, for each i. Let $\{y_0', \ldots, y_m'\}$ be a simplex in J. Each such simplex is distinct from one of the previous type (if the vertices of a simplex s lie in $h^{-1}(\bar{A}_k)$ and some vertex lies in $h^{-1}(A_k)$, then $s \subset h^{-1}(A_k)$).

Each face of a simplex of J is also a simplex of J (the fact that h^* is unique is essential here). There is a natural projection of J onto K which carries y' into $h^{-1}\pi(y)$; it is strongly linear and maps at most a finite number of simplices of J onto any one simplex of K. Hence J is a locally finite abstract complex.

Let L be a geometric realization of J; let y^* denote the vertex of L corresponding to the vertex y' of J. Let p be the linear extension to L of the map which carries the vertex y^* into $h^{-1}\pi(y)$. We define the homeomorphism g. Let $s = y_0^* \ldots y_m^*$ be a simplex of L. If $p(s)$ lies in $h^{-1}(A_k)$, then $\pi^{-1}hp(s)$ consists of k components, to one of whose closures $y_0, \ldots, y_m$ belong. π is a homeomorphism of this closure onto $hp(\bar{s})$, so that $\pi^{-1}hp = g$ maps $\bar{s}$ homeomorphically onto this closure. On the other hand, if $p(s)$ lies in $h^{-1}(X - A)$, there is a unique lifting of the map h of $p(\bar{s})$ into X to a map h^* of $p(\bar{s})$ into X^*. Let $g = h^*p$ on $\bar{s}$.

The map g is defined on each closed simplex of L; it is readily verified that these definitions agree on common faces, so that g is continuous. It is also readily verified that g is $1-1$, that it is a lifting of hp, and that it maps $|L|$ *onto* $\pi^{-1}h(|K|)$.

It follows that g is a homeomorphism: Let v be a vertex of K. The map g is a homeomorphism on $Cl(St(p^{-1}(v)))$, since this closure is compact. Hence it suffices to show that

$g(St(p^{-1}(v)))$ is open in $g(|L|)$. But this set equals $\pi^{-1}h(Stv)$, which is open in $\pi^{-1}h(|K|) = g(|L|)$.

7.8. PROPOSITION. *X^* is a locally polyhedral space of dimension n; it has a naturally induced locally polyhedral structure relative to which π preserves index and $X^* = Cl(X^* - (X^*)^{n-1})$.*

Proof. We define a basis B for a locally polyhedral structure on X^*. Let (U, k, J) be a polyhedral neighborhood on X; let V be an open set such that $\bar{V} \subset U$ and $|J| - k(V)$ is a polyhedron in J. Consider a subdivision J' of J in which $|J| - k(V)$ is a subcomplex; then $k(\bar{V})$ is a subcomplex H of J'. The pair $(H, k^{-1}|H)$ satisfies condition (1) of 7.6. Let H' be any admissible subdivision of H. Then there is a triangulation (L, g) of $\pi^{-1}(\bar{V})$ and a strongly linear map p of L onto H' such that $k^{-1}p = \pi g$. The set $\pi^{-1}(V)$ is open in X^*, and $Cl(g^{-1}\pi^{-1}(V)) = |L|$. Moreover, $|L| - g^{-1}\pi^{-1}(V)$ is a subcomplex of L, because $|H'| - k(V)$ is a subcomplex of H'. Define

$$(\pi^{-1}(V), g^{-1}, L)$$

to be an element of B. This polyhedral neighborhood on X^* is said to be *derived from* (U, k, J). Obviously, B covers X^*.

Let $y \in X^*$; let $x = \pi(y)$. Then $I_B(y) \leq I(x)$, since each element of B is derived from a polyhedral neighborhood on X. Let (U, k, J) be a polyhedral neighborhood of x. If $k(x)$ lies in an m-simplex of J, there is a complex H' defined as in the preceding paragraph such that $k(x)$ lies in an m-simplex of H' (by 3.5). Then y lies in an m-simplex of the derived neighborhood $(\pi^{-1}(V), g^{-1}, L)$. Hence $I_B(y) \geq I(x)$.

Since $I_B(y) = I(x)$, I_B is constant on each simplex of an element of B. Hence B is a basis for a structure, and relative to this structure, π preserves index.

To show that X^* is metrizable, note that it is a Hausdorff space with a countable basis; being locally compact, it is also regular. Every regular space with a countable basis is metrizable ([1], p. 81). Hence X^* is a locally polyhedral space. $X^* = Cl(X^* - (X^*)^{n-1})$, since every neighborhood of type (A) or (B) (see 6.2) must intersect $\pi^{-1}(X - X^{n-1})$.

7.9. PROPOSITION. *X^* has no singular points of index greater than $j - 1$.*

Proof. Let y be a point of X^* having index greater than $j - 1$; let $x = \pi(y)$. First, suppose $x \in X - \bar{A}$. Then x is non-singular. Let V be a neighborhood of y such that $\bar{V}$ is compact and does not intersect $\pi^{-1}(\bar{A})$. Then π is a homeomorphism on $\bar{V}$. Since $V = \pi^{-1}(\pi(V))$ and π is strongly continuous, $\pi(V)$ is open in X. Now π maps V homeomorphically onto $\pi(V)$ and carries $(X^*)^{n-1}$ onto X^{n-1}, so that x and y have the same singularity (by 5.1). Hence y is non-singular.

Second, suppose $x \in \bar{A}$. Then $x \in \bar{A}_k$ for some k, since $\bar{A} = \bigcup \bar{A}_k$; and $I(x) = j$. Some neighborhood U of x in X^j consists entirely of points having the same singularity and index as x (4.3); since U intersects A_k, $x \in A_k$. Let $W(C)$ be a neighborhood of y of type (B), derived from the polyhedral neighborhood (Sts, h, L) of x. Now π maps $W(C)$ homeomorphically onto $C \cup s$, mapping $W(C) - (X^*)^j$ onto $(C \cup s) - X^j = C$, which is connected. In the preceding paragraph, we showed that X^* has no singular points of index greater than j (since $\bar{A} \subset X^j$). Then by 5.7 and 6.4, y is non-singular.

8. Triangulating the space. The following two definitions hold for any locally polyhedral space X, without the assumption stated at the beginning of this chapter.

8.1. Definition. Let (K, h) be a triangulation of some subset of X, and let (U, k, J) be a polyhedral neighborhood of x. Then (K, h) and (U, k, J) are said to be *locally compatible at x* if there is a neighborhood V of $h^{-1}(x)$ in $|K|$ such that (1) $h(\bar{V}) \subset U$, and (2) there is a subcomplex H of some subdivision of K such that $|H| = \bar{V}$ and kh is a linear map of H into J. The set $kh(\bar{V})$ is clearly a polyhedron in J; it follows from 1.1 that there are subdivisions H' and J' of H and J respectively, such that kh a linear isomorphism of H' onto a subcomplex of J'.

8.2. Definition. The locally polyhedral space X is said to possess *property T* if the following holds:

Let the points of X^0 be denoted by $x_1, x_2, \ldots$ Given a polyhedral neighborhood (U_i, k_i, J_i) of x_i, for each x_i, and a proper triangulation of X^{n-1} which is locally compatible with (U_i, k_i, J_i) at x_i, there exists a proper triangulation of X which is compatible with the given triangulation of X^{n-1} and locally compatible with (U_i, k_i, J_i) at x_i, for each i. (The purpose of the "local compatibility" part of this definition will become clear later—9.11.)

8.3. Lemma. *Let p be a strongly linear map of the complex L onto K; let $p^{-1}(x)$ be finite for each x. If L' is a subdivision of L, there are subdivisions L'' and K'' of L' and K respectively, such that p is a strongly linear map of L'' onto K''.*

Proof. If K'' is any subdivision of K, then K'' naturally induces a subdivision L'' of L such that p is a strongly linear map of L'' onto K''. We must so choose the subdivision K'' that the induced subdivision L'' is a subdivision of L'. We proceed as follows: For each simplex s of K, there are a finite number of simplices $t_1, \ldots, t_k$ of L which p maps homeomorphically onto s. L' induces a subdivision of each set $\bar{t}_i$; then L' induces several subdivisions of $\bar{s}$. Order the simplices of K: $s_1, s_2, \ldots$ Let $K_0 = K$; suppose a subdivision K_{m-1} of K is given. Now L' induces several subdivisions of $\bar{s}_m$, and K_{m-1} induces another subdivision of $\bar{s}_m$. Take a subdivision of $\bar{s}_m$ which is a common subdivision of all these; it may be extended to a subdivision K_m of K_{m-1} without affecting any simplex outside the star of $\bar{s}_m$ in K_{m-1}. If K is not finite, the sequence $K_1, K_2, \ldots$ is infinite. But each simplex of K is subdivided only a finite number of times, so that a limiting complex K'' is obtained.

8.4. LEMMA. *Let (K, h) be a triangulation of $Y \subset X$, and let (L, g) be a triangulation of $\pi^{-1}(Y) \subset X^*$. Let p be a strongly linear map of L onto K, such that $hp = \pi g$* (see 7.7). *Let (U, k, J) be a polyhedral neighborhood of $x \in Y$. Then (U, k, J) and (K, h) are locally compatible at x if and only if there is a polyhedral neighborhood on X^* derived from (U, k, J) which is locally compatible with (L, g) at each point of $\pi^{-1}(x)$.*

Proof. Let us recall how a polyhedral neighborhood on X^* is derived from (U, k, J) (7.8). Let V be a suitable open set in X and H' a suitable subcomplex of a subdivision of J. There is a triangulation $(\mathbf{L}, \mathbf{g})$ of $\pi^{-1}(\bar{V})$, and a strongly linear map $\mathbf{p}$ of $\mathbf{L}$ onto H' such that $k^{-1}\mathbf{p} = \pi\mathbf{g}$. Then $(\pi^{-1}(V), \mathbf{g}^{-1}, \mathbf{L})$ is a polyhedral neighborhood on X^*.

Suppose that (U, k, J) and (K, h) are locally compatible at x. Then V and H' may so be chosen that there is a neighborhood W of $h^{-1}(x)$ in $|K|$ such that $\overline{W}$ is the polytope of a subcomplex of a subdivision K' of K and kh is a linear isomorphism of this subcomplex into H'. Let L' be the subdivision of L induced by K'.

$$\begin{tikzcd} L' \arrow[r, "g"] \arrow[d, "p"'] & X^* \arrow[d, "\pi"] & \mathbf{L} \arrow[l, "\mathbf{g}"'] \arrow[d, "\mathbf{p}"] \\ K' \arrow[r, "h"] & X & H' \arrow[l, "k^{-1}"'] \end{tikzcd}$$

Now $p^{-1}(\overline{W})$ is a subcomplex of L', and $p^{-1}(W)$ is a neighborhood of $g^{-1}(\pi^{-1}(x))$ in $|L|$. It is easily seen that $\mathbf{g}^{-1}g$ is linear on $p^{-1}(\overline{W})$, since it equals $\mathbf{p}^{-1}khp$ on each simplex of $p^{-1}(\overline{W})$. Then the derived neighborhood $(\pi^{-1}(V), \mathbf{g}^{-1}, \mathbf{L})$ and (L, g) are locally compatible at each point of $\pi^{-1}(x)$.

Conversely, suppose that $(\pi^{-1}(V), \mathbf{g}^{-1}, \mathbf{L})$ and (L, g) are locally compatible at each point y_i of $\pi^{-1}(x)$. Then there is a neighborhood U_i of $g^{-1}(y_i)$ in $|L|$ such that (1) $g(\overline{U}_i) \subset \pi^{-1}(V)$, and (2) there is a subdivision L_i of L such that $\overline{U}_i$ is the polytope of a subcomplex of L_i and $\mathbf{g}^{-1}g$ is a linear map of this subcomplex into $\mathbf{L}$. Let L' be a common subdivision of the L_i; in addition, let $g^{-1}(y_1)$ be a vertex of L'. By 8.3, there are subdivisions L'' and K'' of L' and K respectively such that p is a strongly linear map of L'' onto K''. Now $h^{-1}(x)$ is a vertex in K''; let V denote the star of $h^{-1}(x)$ in K''. Then kh is linear on every simplex s of K'' contained in $\bar{V}$, since it equals $\mathbf{p}\mathbf{g}^{-1}gp^{-1}$ on s. Hence (K, h) and (U, k, J) are locally compatible at x.

8.5. Proposition. *If X^* has property T, so does X.*

Proof. Let the points of X^0 be denoted by $x_1, x_2, \ldots$. Let (U_i, k_i, J_i) be a polyhedral neighborhood of x_i, for each i, and let (K, h) be a proper triangulation of X^{n-1} which is locally compatible with (U_i, k_i, J_i) at x_i, for each i. By 4.4 and 4.5, $h^{-1}(A_k)$ contains each simplex of K which it intersects. Then $h^{-1}(\bar{A}_k)$ does as well, so that (K, h) satisfies condition (1) of 7.6. Let K_1 be the first barycentric subdivision of K. By 7.7, there is a triangulation (L, g) of $\pi^{-1}h(|K|) = (X^*)^{n-1}$, and a strongly linear map p of L onto K_1, such that $hp = \pi g$. The triangulation (L, g) is proper, since p is strongly linear and π preserves index. By 8.4, for each i there is a polyhedral neighborhood of $\pi^{-1}(x_i)$ derived from (U_i, k_i, J_i) such that (L, g) is locally compatible with this derived neighborhood at each point of $\pi^{-1}(x_i)$. Since X^* has property T, by hypothesis, there is a triangulation $(\mathbf{L}, \mathbf{g})$ of X^* which is compatible with (L, g) on $(X^*)^{n-1}$ and locally compatible with this derived polyhedral neighborhood at each point of $\pi^{-1}(x_i)$.

Let L_1 and $\mathbf{L}_1$ be subdivisions of L and $\mathbf{L}$ respectively which induce equivalent triangulations of $(X^*)^{n-1}$. Then $\alpha = \mathbf{g}^{-1}g$ is a linear isomorphism of L_1 onto a subcomplex of $\mathbf{L}_1$. By 8.3, there are subdivisions L_2 and K_2 of L_1 and K_1 respectively, such that p is a strongly linear map of L_2 onto K_2. Now (L_2, α) induces a subdivision of a subcomplex of $\mathbf{L}_1$, which may be extended to a subdivision $\mathbf{L}_2$ of $\mathbf{L}_1$, by 1.1. Then α is a linear isomorphism of L_2 into $\mathbf{L}_2$. If we replace K_2, L_2, and $\mathbf{L}_2$ by their barycentric subdivisions of order $m-2$, K_m, L_m, and $\mathbf{L}_m$ respectively, then α is still a linear isomorphism and p is still strongly linear.

$$
\begin{tikzcd}
\mathbf{L}_m \arrow[rrr, "\mathbf{g}"] & & & X^* \arrow[ddd, "\pi"] \\
& L_m \arrow[ul, "\alpha"] \arrow[r, "g"] \arrow[d, "p"] & (X^*)^{n-1} \arrow[ur, "i"] \arrow[d, "\pi"] & \\
& K_m \arrow[r, "h"] & X^{n-1} \arrow[dr, "i"] & \\
& & & X
\end{tikzcd}
$$

Let f denote the map $\pi\mathbf{g}$ of $\mathbf{L}_2$ onto X. Then $(\mathbf{L}_2, f)$ satisfies condition (1) of 7.6; this follows from the fact that (K_2, h) satisfies this condition, so that $(L_2, \pi g)$ does as well. Then $(\mathbf{L}_3, f)$ satisfies all three conditions of the lemma. Using this fact, we show that $(\mathbf{L}_3, f)$ satisfies all the conditions of 2.2, except (3b). First, f is strongly continuous, because $\mathbf{g}$ is a homeomorphism and π is strongly continuous. Second, if s is a simplex of $\mathbf{L}_3$, then f is $1-1$ on $\bar{s}$: If s in $\alpha(L_3)$, f restricted to $\bar{s}$ equals $ihp\alpha^{-1}$ restricted to $\bar{s}$. If s is not in $\alpha(L_3)$, and $f(x) = f(y)$ for $x \neq y$, then $f(x) \in A_k$ for some k. Now $\bar{s} \cap f^{-1}(A_k)$ consists of some of the faces $s_1, \ldots, s_m$ of s. All vertices of $s_1, \ldots, s_m$ lie in $f^{-1}(\bar{A}_k)$, and some vertex of one of them lies in $f^{-1}(A_k)$. Let t be the face spanned by all the vertices of the s_i; then $t \subset f^{-1}(A_k)$ and $x, y \in \bar{t}$. This contradicts the fact that t is in $\alpha(L_3)$, so that f is $1-1$ on $\bar{t}$.

Condition (2) is clearly satisfied. Finally, let s_1 and s_2 be two simplices of $\mathbf{L}_3$ such that $f(s_1)$ and $f(s_2)$ intersect. s_1 and s_2 must lie in $\alpha(L_3)$; and $p\alpha^{-1}$ maps them linearly onto the same simplex of K_3. Condition (3a) follows at once.

By 2.2 and 2.3, there is a triangulation $(\mathbf{K}, \mathbf{h})$ of X and a strongly linear map $\mathbf{p}$ of $\mathbf{L}_4$ onto $\mathbf{K}$ such that $\mathbf{h}\mathbf{p} = f = \pi g$. Note that $\mathbf{p}$ maps two points of $\mathbf{L}_4$ into the same point if and only if they are in $\alpha(L_4)$ and $p\alpha^{-1}$ maps them into the same point. Hence α induces a linear isomorphism β of K_4 into $\mathbf{K}$ such that $\mathbf{p}\alpha = \beta p$. (Then $\mathbf{K}$, $\mathbf{h}$, $\mathbf{p}$, and β "fill in" the missing corner of the preceding diagram.) It follows that $\mathbf{h}\beta = ih$, so that $(\mathbf{K}, \mathbf{h})$ and (K_4, h) induce equivalent triangulations of X^{n-1}. Then $(\mathbf{K}, \mathbf{h})$ is compatible with (K, h) on X^{n-1}; and $(\mathbf{K}, \mathbf{h})$ is proper. Finally, it follows from 8.4 that $(\mathbf{K}, \mathbf{h})$ and (U_i, k_i, J_i) are locally compatible at x_i, for each i.

Chapter IV

The General Triangulation Theorem

9. The following proposition summarizes the results of the preceding chapters.

9.1. Proposition. *In order to prove that every locally polyhedral space of dimension n has property T, it is sufficient to prove that property T is possessed by those spaces X of dimension n such that* (1) *X has no singular points, and* (2) $X = Cl(X - X^{n-1})$.

Proof. Let X be any locally polyhedral space of dimension n. Let $M = X - X^{n-1}$. $\bar{M}$ is a locally polyhedral space of dimension n; if $\bar{M}$ has property T, so does X: Let the points of X^0 be denoted by $x_1, x_2, \ldots$; let (U_i, k_i, J_i) be a polyhedral neighborhood of x_i; and let (K, h) be a proper triangulation of X^{n-1} which is locally compatible with (U_i, k_i, J_i) at x_i, for each i. Now (K, h) induces a proper triangulation of $Fr\, M = \bar{M}^{n-1}$ (by 4.6). If x_i is in $\bar{M}$, then $(U_i \cap \bar{M}, k_i, L_i)$ is a polyhedral neighborhood of x_i in $\bar{M}$, where L_i is a subcomplex of J_i. If $\bar{M}$ has property T, there is a triangulation (L, g) of $\bar{M}$ which is compatible with (K, h) on $Fr\, M$ and locally compatible with $(U_i \cap \bar{M}, k_i, L_i)$ at x_i. By 2.4, there is a triangulation $(\mathbf{K}, \mathbf{h})$ of X such that the induced triangulations of X^{n-1} and $\bar{M}$ are subdivisions of (K, h) and (L, g), respectively. Hence $(\mathbf{K}, \mathbf{h})$ is proper, and it is easily shown that $(\mathbf{K}, \mathbf{h})$ is locally compatible with (U_i, k_i, J_i) at x_i.

Let $\bar{M}$ be denoted by Y_n; Y_n is a locally polyhedral space satisfying the hypotheses of Chapter III, for $j = n - 1$. Let Y_{n-1} denote the composition space $(Y_n)^*$; then Y_{n-1} is a locally polyhedral space which also satisfies the hypotheses of Chapter III, for $j = n - 2$ (by 7.8 and 7.9). In general, let $Y_k = (Y_{k+1})^*$; Y_k has no singular points of index greater than $k - 1$. Consider the space Y_0; it satisfies (1) and (2) of the present proposition. If Y_0 has property T, it follows from a finite number of applications of 8.5 that $Y_n = \bar{M}$ has property T also. Then X has property T as well.

9.2. THEOREM. *Let M be a 2-manifold with boundary. Given a triangulation of $Bd\, M$, there is a triangulation of M which is compatible with the given triangulation of $Bd\, M$.*

Proof. Let (L, g) be a triangulation of $Bd\, M$. There exists a triangulation (K, h) of M ([6], p. 167). Clearly there are subdivisions L', K' of L, K respectively, such that $h^{-1}g$ carries each simplex of L' onto a simplex of K' (not necessarily linearly). Let k be the linear map of L' into K' which agrees with $h^{-1}g$ on the vertices of L', and let $f = kg^{-1}h$ on $Bd|K|$. Define f as the identity on the remainder of the 1-skeleton of K, and extend f to $|K|$ by means of radial lines from the barycenters of the 2-simplices of K'. Let $\mathbf{h} = hf^{-1}$. Then $(K', \mathbf{h})$ is a triangulation of M which is compatible with (L, g) on $Bd\, M$.

9.3. DEFINITION. If X is a space, a subset A of X is said to be *weakly locally tame* in X if for each point $x \in A$, there is a neighborhood U of x, a complex K_x and a homeomorphism h of $\bar{U}$ onto a polyhedron in K_x such that $h(\bar{U} \cap A)$ is also a polyhedron in K_x. If it is required that $X = |K|$ for some complex K and $K_x \equiv K$, then A is said to be *locally tame* in K ([2], p. 146).

9.4. THEOREM. *If M is a 3-manifold with boundary, then M may be triangulated. If M is triangulated and A is a locally tame closed subset of M, let there be given a triangulation of $A \cup Bd\, M$ in which A is a polyhedron. Then there is a triangulation of M which is compatible on $A \cup Bd\, M$ with the given one.*

Proof. This is a theorem of Bing ([2], p. 156).

9.5. LEMMA. *Let X be a locally polyhedral space of dimension n ($n \leq 3$), such that X has no singular points and $X = Cl(X - X^{n-1})$. Let $x \in X^0$, let $(St\, x, h, L)$ be a polyhedral neighborhood on X, and let $v = h(x)$. Then $Fr(St\, v)$ is a connected $(n-1)$-manifold with boundary.*

Proof. $Fr(St\, v)$ is a complex J consisting entirely of $(n-1)$-simplices and their faces. Every $(n-1)$-simplex t of $St\, v$ is a face of exactly one or two n-simplices of $St\, v$, since otherwise its points would have index $n-1$ and $St\, t - t$ would not be connected. As a result, every $(n-2)$-simplex of J is a face of exactly one or two $(n-1)$-simplices of J.

$|J|$ is obviously connected, since otherwise $St\, v - v$ would not be connected and x would be a singular point of X (by 5.7). These facts are sufficient, for $n \leq 2$, to ensure that $|J|$ is a connected $(n-1)$-manifold with boundary.

If $n = 3$, we must also show that if w is a vertex of J and $St\, w$ denotes its star in J, then $St\, w - w$ is connected. This is equivalent to the statement that for every 1-simplex s of $St\, v$, $St\, s - s$ is connected. If the points of s have index 1, this follows from 5.7. If they have index 2, $St\, s - s$ consists of the connected set $St\, s - h(X^2)$ plus some of its limit points, so that $St\, s - s$ is connected. If the points of s have index 3, so does every point of $St\, s$, so that every 2-simplex of $St\, s$ is a face of exactly two 3-simplices of $St\, s$. Accordingly, those closed edges of the 3-simplices of $St\, s$ which are opposite to s form a finite number of homeomorphs of a circle. Since each point of s has a 3-cell neighborhood, there is only one such circle. Then $St\, s - s$ is connected.

9.6. THEOREM. *Let X be a locally polyhedral space of dimension n ($n \leq 3$). Then X has property T.*

Proof. By 9.1, we may assume that X has no singular points and that $X = Cl(X - X^{n-1})$. Let $x_1, x_2, \ldots$ denote the points of X^0 and let (U_i, k_i, J_i) be a polyhedral neighborhood of x_i. Let (K, h) be a proper triangulation of X^{n-1} which is locally compatible with (U_i, k_i, J_i) at x_i, for each i. We assume that the sets $\overline{U}_i$ are disjoint. We also assume that for each i, there is a neighborhood V_i of $h^{-1}(x_i)$ in $|K|$ such that $\overline{V}_i$ is a subcomplex of K and the map $k_i h$ is a linear isomorphism of this subcomplex into J_i. This involves no loss of generality, since we may obtain this situation by passing to appropriate subdivisions of the complexes $K, J_1, J_2, \ldots$.

Let y_i denote the vertex $k_i(x_i)$ of J_i, and let L_i denote the subcomplex $Cl(St\, y_i)$ of J_i. Let L denote the complex which is the disjoint union of the complexes L_i, and let g be the map of $|L|$ into X defined by setting g equal to k_i^{-1} on $|L_i|$. Then (L, g) is a triangulation of a subset of X. It induces a triangulation on $g(L) \cap X^{n-1}$ which is equivalent to the one induced by (K, h). By 2.4, there is a triangulation $(\mathbf{K}, \mathbf{h})$ of $g(L) \cup X^{n-1}$ which induces subdivisions of (L, g) and (K, h) on $g(L)$ and X^{n-1} respectively.

Let z_i denote the vertex $\mathbf{h}^{-1}(x_i)$ of $\mathbf{K}$, let $\mathbf{K}'$ denote the first barycentric subdivision of K, and let $St\, z_i$ and $St'\, z_i$ denote the stars of z_i in $\mathbf{K}$ and $\mathbf{K}'$ respectively. Let $\mathbf{L}$ denote the subcomplex

$$\mathbf{h}^{-1}(X^{n-1}) \cup (\textstyle\bigcup_i Cl(St'\, z_i))$$

of $\mathbf{K}'$. Let $Y = \mathbf{h}(\mathbf{L})$. Let $N = X - \bigcup_i \mathbf{h}(St'\, z_i)$. Then N and Y are closed subsets of X, and

$$N \cap Y = Z = Y - \textstyle\bigcup_i \mathbf{h}(St'\, z_i) = (N \cap X^{n-1}) \cup (\bigcup_i Fr(\mathbf{h}(St'\, z_i))).$$

Now $(\mathbf{L}, \mathbf{h})$ induces a triangulation of Z; we seek a triangulation of N which is compatible with $(\mathbf{L}, \mathbf{h})$ on Z. These triangulations of N and Y will fit together (by 2.4) to give a triangulation of X, which is then automatically compatible with (K, h) on $\dot{X}^{n-1}$ and locally compatible with (U_i, k_i, J_i) at x_i, for each i. Our theorem will then be proved.

The crucial fact here is that for $n \leq 3$, N is an n-manifold with boundary, and $Bd\,N \subset Z$. It is clear that each point of N which is not in any set $Fr(\mathbf{h}(St'\, z_i))$ has a neighborhood whose closure is a closed n-cell, by 5.8 (using the fact that no point of N belongs to X^0). Points of N in $Fr(\mathbf{h}(St'\, z_i))$ require special consideration. The set $Fr(\mathbf{h}(St'\, z_i))$ is entirely contained in $\mathbf{h}(St\, z_i)$, so that the problem is reduced to considering neighborhoods of points of $Fr(St'\, z_i)$ in $St\, z_i - St'\, z_i$. (This is the reason we "deleted" the neighborhoods $\mathbf{h}(St'\, z_i)$ from X to form N rather than the neighborhoods $\mathbf{h}(St\, z_i)$.) But by 9.5, $Cl(St\, z_i)$ is merely the cone over an $(n-1)$-manifold with boundary. It is easy to show by direct combinatorial means that each point of $Fr(St'\, z_i)$ has a neighborhood in $St\, z_i - St'\, z_i$ whose closure is a closed n-cell, and that it has no neighborhood which is itself an n-cell. Hence N is an n-manifold with boundary, and $Fr(\mathbf{h}(St'\, z_i)) \subset Bd\, N$. If $x \in Bd\,N - \bigcup_i Fr(\mathbf{h}(St'\, z_i))$, then $x \in X^{n-1}$. Hence $Bd\, N \subset Z$.

For $n = 1$ and $n = 2$, it is also true that $Bd\, N = Z$. (This follows from the fact that $N \cap X^{n-1}$ lies in $Bd\, N$, by 5.8.) Then by 9.2, there is a triangulation of N which is compatible with the given triangulation of $Bd\, N = Y$, and our theorem follows. (If $n = 1$, 9.2 does not apply, but this case is trivial.)

If $n = 3$, there is more work to do. The difficulty here is that it need not be true that $Z = Bd\, N$, for $N \cap X^{n-1}$ may contain points of $Int\, N$. The points of $(Int\, N) \cap X^{n-1}$ have index 1, by 5.8. Let A denote the set $X^1 \cap N$; then $Z = A \cup Bd\, N$. There is a triangulation of N, by 9.4; we shall prove that A is a locally tame subset of N. Every point of A has index 1, for it cannot have index 0 and lie in N. Let x be in A, and let x not be in any set $Fr(\mathbf{h}(St'\, z_i))$. Let $(St\, s, k, H)$ be a polyhedral neighborhood of x such that $\dim s = 1$. By the argument used in 5.8, the closed edges of H opposite $k(s)$ must form a homeomorph of a line segment or a circle, and $A \cap St\, s$ is merely the set s. It is easy to see that $St\, s$ is a neighborhood of x satisfying the hypotheses for A to be locally tame at x. On the other hand, suppose x is in $Fr(\mathbf{h}(St'\, z_i))$. Then $\mathbf{h}^{-1}(x)$ is a point of some 1-simplex of $St\, z_i$, and $\mathbf{h}^{-1}(X^1) \cap St\, z_i$ consists of some of the 1-simplices of $St\, z_i$. Using these facts, one may readily show by combinatorial means that there is a neighborhood of $\mathbf{h}^{-1}(x)$ in $St\, z_i - St'\, z_i$ which satisfies the hypotheses for local tameness of A at x.

A is a subcomplex in the triangulation of Z induced by $(\mathbf{L}, \mathbf{h})$, since the original triangulation (K, h) of X^2 was proper. Then by 9.4, there is a triangulation of N which is compatible with the triangulation of $A \cup Bd\, N = Z$ induced by $(\mathbf{L}, \mathbf{h})$. This completes the proof of the theorem for $n = 3$.

9.7. Corollary. *Let X be a locally polyhedral space of dimension n ($n \leq 3$). Given polyhedral neighborhoods (U_i, k_i, J_i) of each $x_i \in X^0$, there is a proper triangulation of X which is locally compatible with (U_i, k_i, J_i) at x_i, for each i.*

Proof. We proceed by induction on n. X^{n-1} is a locally polyhedral space of dimension less than n, so by the induction hypothesis there is a proper triangulation (K, h) of X^{n-1} which is locally compatible with the polyhedral neighborhood $(U_i \cap X^{n-1}, k_i, L_i)$ at x_i, for each $x_i \in (X^{n-1})^0 = X^0$ (L_i is a subcomplex of J_i). Then (K, h) is also locally compatible with (U_i, k_i, J_i) at x_i, so that 9.6 applies.

9.8. Theorem. *If X is a locally polyhedral space of dimension n ($n \leq 3$), X may be triangulated.*

9.9. Lemma. *Let X be a locally polyhedral space, and let Y be a closed admissible subspace of X. Any proper triangulation of X induces a triangulation of Y.*

Proof. Let $\alpha(x) = 1$ if $x \in Y$, and $\alpha(x) = 0$ otherwise. Since Y is admissible, α is constant on each simplex of every polyhedral neighborhood on X. By 4.4 and 4.5, given a proper triangulation of X, α is constant on each simplex of X under this triangulation. Since Y is closed, it is the polytope of a subcomplex of X.

9.10. Theorem. *Let X be a locally triangulable space of dimension n ($n \leq 3$), and let Y be a weakly locally tame closed subset of X. Then there is a triangulation of X under which Y is a polyhedron.*

Proof. This generalizes Theorem 8 of [2]. Let B be the collection of all polyhedral neighborhoods (U, h, L) on X such that Y contains every simplex of U which it intersects. From the definition of weak local tameness, it is clear that B covers X. To show that I_B is constant on every simplex of an element of B, one uses the argument of 3.8 (with the additional fact that the homeomorphism g there defined maps Y into itself). Then B is a basis for a structure. Let $\alpha(x) = 1$ if $x \in Y$, otherwise let $\alpha(x) = 0$. Then α satisfies the hypotheses of 4.3, so that by 4.4, α is constant on every simplex of each polyhedral neighborhood of the structure on X. Hence Y is an admissible subspace of X, and 9.9 applies.

9.11. Remark. These results do not imply that the triangulability of the general locally polyhedral space of dimension n would follow from a triangulation theorem for the general n-manifold with boundary, or from a stronger theorem involving local tameness, like 9.4. Removing the singular points from X makes X locally a manifold with boundary only at points of index n, $n-1$, and $n-2$. If $n = 3$, the points at which X fails to be a manifold with boundary are isolated, and one can use polyhedral neighborhoods to triangulate a neighborhood of this set. After deleting these neighborhoods, one has left a manifold with boundary to triangulate. If $n > 3$, the set of points where X fails to be a manifold with boundary has positive dimension, and it is not clear how one could proceed in this case.

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