BOUNDARY BEHAVIOUR AND NORMAL MEROMORPHIC FUNCTIONS

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Introduction

1. This paper deals with the boundary behaviour of meromorphic functions. The considerations lead in a natural manner to a conformally invariant class of meromorphic functions, distinguished by a number of interesting properties, which we call *normal* meromorphic functions. Their definition reads as follows: If $f(z)$ is meromorphic in a simply connected domain G, then $f(z)$ is normal if and only if the family $\{f(S(z))\}$, where $z' = S(z)$ denotes an arbitrary one-one conformal mapping of G onto itself, is normal in the sense of Montel. In multiply connected domains $f(z)$ is said to be normal if it is normal on the universal covering surface.

Normal meromorphic functions admit the following characterization in terms of the spherical derivative: A non-constant meromorphic $f(z)$ is normal in a domain G, which then necessarily is of hyperbolic type, if and only if there exists a finite constant C so that

$$\frac{|f'(z)|\,|dz|}{1+|f(z)|^2} \leqq C\, d\sigma(z), \tag{1}$$

where $d\sigma(z)$ denotes the element of length in the hyperbolic metric of G.

It follows from the definition that e.g. bounded functions, schlicht functions, and, more generally, functions omitting at least three values, are always normal. On the other hand, all functions of bounded type are not normal.

2. In order to arrive in as natural a way as possible at the concept of a normal function, we devote § 1 to a systematic study of the situation that a meromorphic $f(z)$ possesses an asymptotic value α at a boundary point P but has not the angular limit α at this point. In this case there exists, roughly speaking, a zone containing curves with endpoint at P on which $f(z)$ tends to the limit α. This zone, however, is sharply limited, i.e., there exist curves terminating at P on which $f(z)$ has not the limit α, such that the hyperbolic distance of these curves to an α-path can be made arbitrarily small (Theorem 1).

This first result is then used in § 2 to finding conditions under which the existence of an asymptotic value implies the existence of the angular limit. It is here that the family $\{f(S(z))\}$ enters in a most natural manner, and it follows easily that the angular limit certainly exists, if $\{f(S(z))\}$ is a normal family (Theorem 2). This gives rise to the above definition of normal meromorphic functions as functions generating a normal family $\{f(S(z))\}$.

By studying conformally invariant normal families we arrive in § 3 at the condition (1) (Theorem 3). Theorem 2 is then restated in terms of the spherical derivative. Although the condition ensuring the existence of the angular limit is not necessary, the theorem seems to belong to the best general results in this direction. This derives from the fact that, owing to the nature of the problem, non-trivial necessary *and* sufficient conditions can scarcely be given. Besides, if the situation is specialized, sharp results can immediately be established. For instance, assuming that a meromorphic $f(z)$ tends to a limit as z approaches a boundary point P in an arbitrary manner along the boundary, we give a necessary and sufficient condition under which $f(z)$ then uniformly tends to this limit as z approaches P in the closure of G (Theorem 4). As a second conclusion, we give a necessary and sufficient condition concerning the existence of the angular limit, if the corresponding radial limit exists (Theorem 5).

At the end of § 3, the relation of normal functions to functions of bounded type is briefly discussed. In both classes the growth of the functions is restricted by a condition involving the spherical derivative. The boundary behaviour, however, is quite different. For normal functions asymptotic values imply the corresponding angular limits, whereas this is not true for all functions of bounded type. On the other hand, while functions of bounded type always possess angular limits almost everywhere, there exist normal functions with no asymptotic values at all.

Starting from the relation (1), we derive in § 4 sharp estimates of a more special kind for normal functions. As a fundamental result we first establish an improved version of the classical Two Constants Theorem, yielding an inequality not only for regular $f(z)$ but also for functions possessing poles (Theorems 6 and 7). This result can be applied in several directions. First, we obtain a sharp theorem of the Phragmén-Lindelöf type for the boundary values of normal meromorphic functions (Theorem 8). Secondly, we can readily prove that a meromorphic function cannot be normal in any neighbourhood of an isolated essential singularity (Theorem 9), which result is a generalization of Picard's classical theorem. Finally, it follows that if a sequence of normal functions, satisfying (1) with a fixed C, uniformly tends to zero on a boundary arc, then the sequence uniformly converges towards zero in every compact part of the domain G (Theorem 10). As an immediate corollary of this general result we find anew Theorem 2 that for normal functions the existence of an asymptotic value α at a boundary point implies the existence of the angular limit α at this point.

§ 1. Asymptotic paths of meromorphic functions

3. Let $f(z)$ be a meromorphic function in a simply connected domain G bounded by a Jordan curve. In this section we study the behaviour of $f(z)$ in the neighbourhood of a boundary point.

For convenience of notations, we write for every boundary point $z = z_0$,

$$|f(z_0)| = \limsup_{z \to z_0} |f(z)|.$$

If $|f(z_0)|$ tends to zero as z_0 on the boundary approaches a point P, we say that $f(z)$ has the limit zero at P along the boundary. If so, there always exists a Jordan curve in G with endpoint at P on which $f(z)$ tends to zero as $z \to P$.

We call an *angle* with vertex at P a domain A defined as follows: If Q is some other boundary point and $\omega(z)$ the harmonic measure in G of one of the arcs PQ, then A is a domain whose points z satisfy a condition $\varepsilon < \omega(z) < 1 - \varepsilon$, $\varepsilon > 0$. If $f(z)$ uniformly tends to a limit α as $z \to P$ inside every angle A of the above kind, we say that $f(z)$ possesses the *angular limit* α at the point P.

Concerning the behaviour of $f(z)$ in the neighbourhood of P, we make the following assumptions: We suppose that there exists a Jordan curve Γ, terminating at P and lying in the closure of G, such that $f(z)$ tends to zero as $z \to P$ along this curve. Besides, we suppose that $f(z)$ does *not* possess the angular limit zero at P.

The results which we shall obtain on the boundary behaviour of $f(z)$ will be expressed in conformally invariant form. In what follows we may, therefore, freely perform one-one conformal mappings of G onto other suitably chosen domains.

4. We shall prove that under the above conditions there are certain "last" curves around Γ on which $f(z)$ still tends to zero. More precisely, we establish the following theorem, which in a slightly weaker form will be used in the subsequent considerations.

THEOREM 1. *Let the function $f(z)$, meromorphic in G, have the asymptotic value zero at a boundary point P along a Jordan curve lying in the closure of G. If $f(z)$ has not the nagular limit zero at P, there exist for any given $\varepsilon > 0$ two curves in G with endpoints at P,* *so that $f(z)$ tends to zero on one curve but not on the other, and so that the hyperbolic distance of these curves is less than ε.*

Proof. It proves convenient to choose as domain G the right angle $0 < \arg z < \pi/2$, and suppose that the boundary point P lies at $z = \infty$. By the above, there is no loss of generality to assume that the asymptotic path Γ, along which $f(z)$ tends to zero, lies entirely in G. We suppose that Γ starts at $z = 0$ so that it divides G into two distinct parts G_1 and G_2; let G_1 denote the part of G bounded by Γ and the imaginary axis.

Because $f(z)$ does not uniformly tend to zero in every angle, there exists an angle A: $\delta < \arg z < \pi/2 - 2\delta$, $\delta > 0$, containing an infinite number of points which cluster at infinity and at which $f(z)$ has not the limit zero. The same is thus also true at least in one of the intersections $G_1 \cap A$ and $G_2 \cap A$; we assume in the following that it is true in $G_1 \cap A$.

In order to avoid difficulties arising from the possible complicated structure of the asymptotic path Γ, we perform an auxiliary conformal mapping $w = w(z)$. We map G_1 again onto the right angle $0 < \arg w < \pi/2$, and normalize the mapping by keeping fixed the boundary points 0 and ∞. In this mapping, the curve Γ is mapped on the positive real axis. Moreover, the images of $G_1 \cap A$ lie in the angle $\arg w < \pi/2 - 2\delta$, as follows immediately if we apply the maximum principle to the harmonic measures of G and G_1, vanishing on the imaginary axis and equal to 1 on the real axis and on Γ, respectively.

In the w-angle we thus have the following situation. The transformed function $f(w)$ tends to zero on the positive real axis as $w \to \infty$, whereas there exists in $\arg w < \pi/2 - 2\delta$ a point set on which $f(z)$ has not the limit zero. From this it follows that given any three non-zero values a, b, c, there is in $\arg w < \pi/2 - \delta$ an infinite number of points, clustering at infinity, at which $f(z)$ takes at least one of the values a, b, c. For if not, $f(w)$ would omit the values a, b, c in $(\arg w < \pi/2 - \delta) \cap (|w| > R)$ for a sufficiently large R. By a well-known generalization of Lindelöf's Theorem,[1] $f(w)$ would then uniformly tend to zero in $\arg w < \pi/2 - 2\delta$ as $w \to \infty$, thus contradicting the hypothesis.

After these preliminary considerations, we introduce a family of similar triangles Δ, defined as follows: The base of Δ lies on the real axis, the two other sides are equal, and the vertex angle equals $\frac{1}{2}\delta$. Given three non-zero values a, b, c, we construct all triangles Δ of the above kind containing no points at which $f(w)$ takes one of these values. A component of the union of all these triangles is an unbounded simply connected strip domain bounded by the coordinate axis and a polygonal curve. If needed, we cut the tops of the latter curve so as to ascertain its entirely lying in the angle $\arg w < \pi/2 - \delta$, and denote the curve so obtained by C and the corresponding strip domain by D (Fig. 1).

[1] First proved by W. Gross, Über die Singularitäten analytischer Funktionen, *Monatshefte für Mathematik und Physik*, 29 (1918).

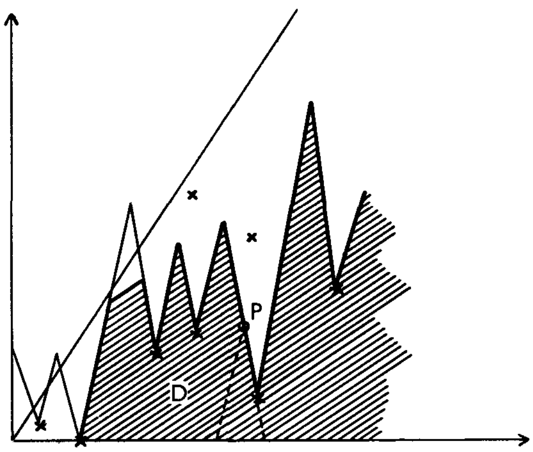

Fig. 1.

In D, $f(w)$ omits the three values a, b, c, whereas on C there is an infinite number of points at which $f(w)$ takes some of these values.

In the domain D we again apply the generalized Lindelöf Theorem to $f(w)$ and conclude that $f(w)$ has the angular limit zero at infinity. In other words, if $\omega(w, D)$ denotes the harmonic measure of D which vanishes on the real axis and is equal to 1 on the rest of the boundary, then $f(w)$ tends to zero on every level curve $\omega(w, D) = \lambda$, $0 < \lambda < 1$. We shall now prove that these level curves have a bounded hyperbolic distance from the polygonal curve C and that the bound obtained tends to zero as $\lambda \to 1$.

To this end, we consider an arbitrary point P $(w = u + iv)$ on C. Let $\omega(w, \Delta)$ denote the harmonic measure of the triangle Δ with vertex at P, which vanishes on the base and is equal to 1 on the remaining boundary. Since the triangle Δ is contained in D (cf. Fig. 1), it follows from the maximum principle that $\omega(w, \Delta) > \omega(w, D)$. Hence, the Euclidean distance of the level curve $\omega(w, D) = \lambda$ from P is less than the corresponding distance of the curve $\omega(w, \Delta) = \lambda$.

As regards the corresponding hyperbolic distances, we conclude as follows. Let $Q_1(u + iv_1)$ and $Q_2(u + iv_2)$ denote points at which the curves $\omega(w, D) = \lambda$ and $\omega(w, \Delta) = \lambda$ bisect the straight line $w = u$. If $\sigma(P, Q)$ designates the hyperbolic distance between P and Q, we have

$$\sigma(P, Q_1) = \int_{Q_1}^{P} \frac{1}{2} \sqrt{\frac{1}{u^2} + \frac{1}{v^2}}\, |d w| < \frac{1}{2} \frac{1}{\sin \delta} \int_{Q_1}^{P} \frac{d v}{v}$$

$$= \frac{1}{2} \frac{1}{\sin \delta} \log \frac{v}{v_1} < \frac{1}{2} \frac{1}{\sin \delta} \log \frac{v}{v_2}.$$

But since the triangles Δ are similar, the ratio $v/v_2 = k(\lambda)$ is independent of the choice of the point P and depends on λ only. Evidently, $k(\lambda) \to 1$ as $\lambda \to 1$. Accordingly, the hyperbolic distance between the curve $\omega(w, D) = \lambda$, on which $f(w)$ tends to zero, and the curve C, on which $f(w)$ has not the limit zero, is less than the constant $\frac{1}{2}(1/\sin\delta)\log k(\lambda)$, which tends to zero as $\lambda \to 1$.

Now the hyperbolic metric is invariant with respect to one-one conformal transformations. On returning to z-angle, we thus infer the existence of two curves, stretching to ∞, so that $f(z)$ tends to zero on one curve but not on the other, and so that the hyperbolic distance of these curves is less than a constant which can be chosen arbitrarily small. The hyperbolic metric is then defined with respect to the image of the w-angle, i.e. with respect to the domain G_1. By the principle of the hyperbolic measure, however, the hyperbolic measure with respect to G_1 is greater than the hyperbolic measure with respect to the whole angle. Our result thus holds *a fortiori* in the hyperbolic measure of the angle, and the theorem is completely proved.

§ 2. Normal meromorphic functions

5. Making use of Theorem 1, we shall now derive a condition under which the existence of an asymptotic value zero at P implies the existence of the angular limit zero at P. The result obtained contains as special case Gross's above-mentioned generalization of Lindelöf's Theorem.

Preserving the situation of Theorem 1, we suppose that $f(z)$ has the asymptotic value zero at P along a Jordan curve lying in the closure of G and that $f(z)$ does not possess the angular limit zero at P. By Theorem 1, there then exists a Jordan curve L in G with end-point at P, on which $f(z)$ tends to zero, and a sequence of points z_n, $n = 1, 2, \ldots$, which converge towards P and at which $f(z_n) = a \neq 0$, such that the points z_n have a bounded hyperbolic distance ($< M$) from the curve L.

We fix an arbitrary point z_0 in G, and associate with the points z_n conformal mappings $z' = S_n(z)$, which are defined as follows: $S_n(z)$ is the function which gives a one-one mapping of G onto itself, keeps the boundary point P invariant, and satisfies the condition $S_n(z_0) = z_n$.

Let K denote the hyperbolic circle whose centre lies at $z = z_0$ and whose radius, in the hyperbolic metric, equals $M + 1$. Because of the invariance of the hyperbolic metric with respect to one-one conformal mappings, every inverse transformation $z = S_n^{-1}(z')$ maps one or several arcs of the curve L inside K. For large values of n, the functions $f(S_n(z))$ are small on these image arcs, since $f(z)$ tends to zero on L. On the other hand, $f(S_n(z_0)) = a \neq 0$ for every n.

6. Let us now impose a new condition on the function $f(z)$: we assume that the family of functions $f(S_n(z))$ is *normal*. By definition, a family of meromorphic functions is said to be normal in a domain, if every sequence of its functions contains a subsequence which converges uniformly in every compact part of the domain. Since the functions may have poles, convergence must be defined in the *spherical* metric.

As remarked above, for large values of n the functions $f(S_n(z))$ are small on certain arcs in K. Hence, they cannot uniformly tend to ∞, and we conclude from the normality the existence of a subsequence $f(S_{n_k}(z))$, whose functions uniformly converge towards a meromorphic limit function $\varphi(z)$ in K.

The images of the arcs of L mapped into K by the functions $z = S_{n_k}^{-1}(z')$ clearly possess at least one accumulation continuum c. Because $f(z) \to 0$ on L, it follows that on c, $\varphi(z) = 0$. Hence, the limit function $\varphi(z)$ vanishes identically. On the other hand, for every k, $f(S_{n_k}(z_0)) = a \neq 0$, so that also $\varphi(z_0) \neq 0$. We have thus arrived at a contradiction, and it follows that *if $f(z)$ does not possess the angular limit zero, the family $\{f(S_n(z))\}$ cannot be normal.*

7. We now introduce the definition mentioned in the Introduction: A meromorphic function $f(z)$ is called *normal* in a simply connected domain G, if the family $\{f(S(z))\}$ is normal, where $z' = S(z)$ denotes an arbitrary one-one mapping of G onto itself. In a multiply connected domain $f(z)$ is said to be normal if it is normal on the universal covering surface of the domain.

In view of this definition, the above result on the boundary behaviour of $f(z)$ can be expressed as follows.

THEOREM 2. *Let $f(z)$ be meromorphic and normal in G, and let $f(z)$ have an asymptotic value α at a boundary point P along a Jordan curve lying in the closure of G. Then $f(z)$ possesses the angular limit α at the point P.*

Remark. It follows immediately from the above considerations that if the asymptotic path Γ lies on the boundary, a normal $f(z)$ does not only possess the limit α in every angle A, but it also uniformly tends to α in the part of G lying between A and the curve Γ.

8. On the basis of the definition, we can immediately conclude the normality of certain meromorphic functions. For instance, if $f(z)$ omits three values in G, all functions $f(S(z))$ omit the same three values. Hence, $\{f(S(z))\}$ is a normal family (Montel's Theorem), and thus $f(z)$ is normal. In particular, all bounded functions are normal as well as all schlicht functions in domains of hyperbolic type.

It is clear that if $f(z)$ is normal, then so is $f(z) + g(z)$, if $g(z)$ is bounded. Likewise, if $f(z)$ is normal, then also all powers $f(z)^{\mu}$, μ real, are normal. (If μ is not integer, we have to suppose $f(z) \neq 0, \infty$ in order that $f(z)^{\mu}$ will be single-valued.) It is also readily seen that with $f(z)$, every rational function $R(f(z))$ of $f(z)$ is normal.

If $f(z)$ omits less than three values, the subordination principle cannot be applied, and it is often difficult to judge whether $f(z)$ is normal or not. However, in certain simple cases this can readily be done. For instance, suppose that $f(z) \neq 0, \infty$ and that $f(z)$ takes some third value only a finite number of times, say $n-1$ times. Then $f(z)^{1/n}$ is single-valued and omits at least three values. Hence, $f(z)^{1/n}$ is normal, and thus also $f(z) = (f(z)^{1/n})^n$. It is not difficult to establish the more general result that $f(z)$ is normal if it takes three values only a finite number of times.

§ 3. Spherical derivative of normal functions

9. We shall now study what conditions on the growth of $f(z)$ are imposed by the requirement that $f(z)$ is normal. To begin with, we introduce the spherical derivative

$$\varrho(f(z)) = \frac{|f'(z)|}{1+|f(z)|^2}$$

of $f(z)$, and start from the known result that a family $\mathfrak{F}$ of meromorphic functions is normal in a domain G if and only if

$$\sup_{f\in\mathfrak{F}} \varrho(f(z)) < \infty \tag{2}$$

in every compact part of G.[1] This condition, however, assumes a much sharper form as applied to the family $\{f(S(z))\}$, which is conformally invariant.

10. We call a family $\mathfrak{F}$ of meromorphic, not necessarily normal functions in a simply connected domain *conformally invariant*, if $f(z)\in\mathfrak{F}$ always implies $f(S(z))\in\mathfrak{F}$.

Let us suppose, for a moment, that the domain G is the unit disc $|z|<1$. The mapping function $z' = S(z)$ can then be written

$$S(z) = e^{i\alpha}\frac{z+\zeta}{1+\bar{\zeta}z} \qquad (\alpha \text{ real}, \ |\zeta|<1)$$

so that
$$\varrho(f(S(z))) = \frac{\left|\dfrac{d f(z')}{d z'}\right|}{1+|f(z')|^2}\cdot\frac{1-|\zeta|^2}{|1+\bar{\zeta}z|^2}.$$

For $z=0$, this yields
$$\varrho(f(S(0))) = (1-|\zeta|^2)\,\varrho(f(e^{i\alpha}\zeta)). \tag{3}$$

[1] See e. g. L. Ahlfors, *Complex Analysis*, McGraw-Hill Book Company, Inc. (1953), p. 169.

From this we conclude that in a family $\mathfrak{F}$, conformally invariant in $|z| < 1$,

$$\sup_{f \in \mathfrak{F}} \varrho(f(z)) = \frac{1}{1-|z|^2} \sup_{f \in \mathfrak{F}} \varrho(f(0)).$$

Introducing the notation $$d\sigma(z) = \frac{|dz|}{1-|z|^2}$$

for the hyperbolic element of length, this can also be written

$$\sup_{f \in \mathfrak{F}} \varrho(f(z))\,|dz| = \sup_{f \in \mathfrak{F}} \varrho(f(0))\, d\sigma(z). \tag{4}$$

This relation holds in the special domain $|z| < 1$. Since, however, both $\varrho(f(z))|dz|$ and $d\sigma(z)$ are conformal invariants, the relation can immediately be extended by conformal mapping to every domain G of hyperbolic type.

11. We remark that $$\sup_{f \in \mathfrak{F}} \varrho(f(0)) = \infty$$

in domains G of elliptic or parabolic type, if the conformally invariant family $\mathfrak{F}$ contains non-constant functions. For we can then suppose that G is either the whole extended z-plane or the punctured plane $z \neq \infty$. In both cases, $S(z) = az + b$ gives a one-one mapping of G onto itself, where $a \neq 0$ and b are arbitrary complex numbers. Now, if $f(z)$ is meromorphic in G,

$$\varrho(f(S(0))) = |a|\varrho(f(b)),$$

whence the assertion follows.

12. Considering the criterion (2) for normal families, we infer from (4) the validity of the following condition: *A conformally invariant class $\mathfrak{F}$ containing non-constant functions is a normal family in a domain G if and only if its functions satisfy an inequality*

$$\varrho(f(z))\,|dz| \leqq C\, d\sigma(z), \tag{5}$$

where C is a fixed finite constant.

For we first conclude from the above that if $\mathfrak{F}$ is a normal family, the domain G must be of hyperbolic type, and thus a hyperbolic metric can be introduced. If $\mathfrak{F}$ is normal in $|z| < 1$, then $\sup \varrho f(0)) < \infty$ by (2), and hence by (4), (5) is satisfied for $C = \sup \varrho(f(0))$. The validity of (5) in arbitrary domains follows hereafter immediately by conformal mapping.

Conversely, if (5) holds, then $\sup \varrho(f(z)) \leqq C(d\sigma(z)/|dz|) < \infty$ in every compact part of G, and by (2), $\mathfrak{F}$ is a normal family.

13. By the above, we get the following characterization of normal functions in terms of the spherical derivative and hyperbolic metric.

THEOREM 3. *A non-constant $f(z)$, meromorphic in a domain G, is normal if and only if the condition* (5) *is satisfied at every point of G.*

For it follows from (3) that if (5) is valid, $\varrho(f(S(0))) \leqq C$. Hence, by (4),

$$\varrho(f(S(z)))\,|dz| \leqq C\,d\sigma(z),$$

and the above normality criterion applies to the conformally invariant family $\{f(S(z))\}$.

14. Considering Theorem 3, Theorem 2 can also be expressed in the following form:

THEOREM 2'. *Let $f(z)$ be meromorphic in G and have an asymptotic value α at a boundary point P along a Jordan curve lying in the closure of G. If*

$$\limsup_{z\to P} \frac{\varrho(f(z))\,|dz|}{d\sigma(z)} < \infty, \tag{6}$$

then $f(z)$ possesses the angular limit α at the point P.

For if (6) is valid, we have a finite C such that $\varrho(f(z))\,|dz| \leqq C\,d\sigma(z)$ in a G-neighbourhood of P. From the principle of hyperbolic measure it then follows, by Theorem 3, that $f(z)$ is normal in this neighbourhood.

15. The inequality (6) is of course no *necessary* condition, since Theorem 2' has to hold for *all* asymptotic paths. However, as soon as the asymptotic path is fixed, (6) can be modified so as to yield *precise* conditions for the existence of the angular limit.

As a first example, we establish the following generalization of a well known boundary theorem of Lindelöf.

THEOREM 4. *Let $f(z)$ be meromorphic in G and approach a limit α as z tends to P in an arbitrary manner along the boundary. Then $f(z)$ uniformly tends to α as z tends to P in the closure of G, if and only if the condition* (6) *is fulfilled.*

Proof. If (6) holds, we first conclude as in proving Theorem 2' that $f(z)$ is normal in a neighbourhood of P. Hence, by Theorem 2 and the subsequent Remark, $f(z)$ uniformly tends to α, no matter how z approaches P in the closure of G. Thus condition (6) is sufficient.

In order to establish the converse part of the Theorem, we assume, for simplifying the computations, that G is the upper half-plane and P the point $z=0$. If $d\sigma_r(z)$ denotes the hyperbolic element of length with respect to the semicircle $(|z|<r) \cap (Im(z)>0)$, we can show by an elementary computation that $d\sigma_r(z)/d\sigma(z)$ is bounded in every smaller semicircle $|z| < r - \delta$, $\delta > 0$. Hence, if (6) is not fulfilled, it follows from Theorem 3 that $f(z)$ cannot be normal in any semicircle $|z| < r$. But then $f(z)$ cannot omit more than two values in any neighbourhood of P, and the uniform convergence towards α is impossible.

16. With only formal modifications to the above proof we can also establish

THEOREM 5. *Let $f(z)$ be meromorphic in $|z| < 1$, and let it possess the radial limit α at the point $z = e^{i\varphi}$. Then $f(z)$ possesses the angular limit α at $z = e^{i\varphi}$, if and only if*

$$\varrho(f(z)) = \frac{O(1)}{1-|z|}$$

in every angle $|\arg(1 - ze^{-i\varphi})| < \pi/2 - \delta$, $\delta > 0$.

17. By means of Theorem 3, we can draw certain general conclusions concerning normal functions. In Theorems 2′, 4 and 5, we already made use of the important property, ensuing from the principle of hyperbolic measure, that if $f(z)$ is normal in G, it is also normal in every subdomain of G. It may be noted that this is not as easily seen if the original definition of normality with the help of normal families is used.

By the principle of hyperbolic measure, we can also establish the following result: Let $f(w)$ be normal in a simply or multiply connected domain G_w, and $w = \varphi(z)$ meromorphic in a simply connected G_z with values lying in G_w. Then $f(\varphi(z))$, which is single-valued by the monodromy theorem, is normal in G_z. For since $f(w)$ is normal, we have

$$\varrho(f(\varphi(z))) = \varrho(f(w))\,|\varphi'(z)| \leq C\,\frac{d\,\sigma_w(w)}{|d\,w|}\,|\varphi'(z)| = C\,\frac{d\,\sigma_w(\varphi(z))}{|d\,z|}.$$

By the principle of hyperbolic measure, $d\sigma_w(\varphi(z)) \leq d\sigma_z(z)$, so that finally $\varrho(f(\varphi(z)))\,|dz| \leq C\,d\sigma(z)$, whence the normality of $f(\varphi(z))$ follows.

18. It may be of interest to compare normal functions with meromorphic functions of bounded type. It is well known that there exist functions of bounded type possessing more than one asymptotic value at a boundary point (e.g. $f(z) = ze^{-z}$ is of bounded type in the right half-plane and possesses the asymptotic values 0 and ∞ at $z = \infty$). Hence, all functions of bounded type are not normal. On the other hand, the elliptic modular function, which omitting three values is normal, is not of bounded type. The two function classes thus overlap each other.

However, for normal functions the characteristic function cannot grow very rapidly. In fact, in the unit disc we can write

$$T(r) = \frac{1}{\pi}\int_0^r \frac{d\,r}{r} \int\int_{|z|<r} \varrho(f(z))^2\,d\,x\,d\,y.$$

Making use of Theorem 3, we get from this by an elementary computation

$$T(r) \leqq \frac{C^2}{2} \log \frac{1}{1-r^2}.$$

19. A function $f(z)$ is of bounded type, if the *average* growth of $\varrho(f(z))$ is not too large. In this case, the existence of an asymptotic value does not necessarily imply the existence of the corresponding angular limit. Instead, it does follow that angular limits exist almost everywhere. On the other hand, a function $f(z)$ is normal, if $\varrho(f(z))$ *itself* does not grow too rapidly. Asymptotic values then always imply angular limits, but we cannot say anything about the *existence* of asymptotic values. In fact, there exist normal functions which possess no asymptotic values at all.

In order to prove this we make use of a theorem of Lohwater and Piranian,[1] which states that if E is an arbitrary denumerable set on $|z| = 1$, there exists a bounded function in $|z| < 1$ possessing a radial limit at every point outside E and failing to have a limit at any point of E.

In particular, if $f(z)$ is a modular function which possesses radial limits only in a denumerable set, we have a bounded $g(z)$ such that the normal function $f(z) + g(z)$ possesses radial limits nowhere. It cannot then have any asymptotic values either.

§ 4. Boundary behaviour of normal functions

20. By aid of the simple metrical condition for normal functions, it is possible to derive sharp estimates of a more special kind for the modulus of $f(z)$, and thus accurately describe the boundary behaviour of a normal function. We first establish an improved version of Two Constants Theorem, and formulate it in view of the applications as follows.

THEOREM 6. *Let a meromorphic $f(z)$ be normal in a domain G, $\varrho(f(z))\,|dz| \leqq C d\sigma(z)$, and satisfy an inequality $|f(z)| \leqq m$ on a boundary arc γ.*[2] *Let G^* be a subdomain of G with boundary $\gamma \cup \gamma'$, where γ' is an analytic curve. If $|f(z)| \leqq M$ in G^* and if there is an inner point Q on γ' at which $|f(z)| = M$, then*

$$m \geqq M e^{-C\lambda_Q (M+1/M)}. \tag{7}$$

Here

$$\lambda_Q = \left(\frac{d\sigma(z)}{|dz|} \Big/ \frac{\partial \omega(z)}{\partial n}\right)_{z=Q},$$

the hyperbolic metric being defined with respect to G and $\partial\omega/\partial n$ denoting the derivative in the

[1] To appear in *Ann. Acad. Sci. Fennicæ.*

[2] We recall that for boundary points z, $|f(z)|$ refers to $\limsup |f(z)|$.

direction of the inner normal of γ' *of the harmonic measure* $\omega(z)$ *of* G^*, *which is equal to* 1 *on* γ *and vanishes on* γ'.

Proof. The function $\log (M/|f(z)|)$ is positive in G^* and harmonic except for possible logarithmic singularities. On γ, $\log (M/|f(z)|) \geqq \log (M/m)$, while on γ', $\log (M/|f(z)|) \geqq 0$. Hence, by the maximum principle,

$$\log \frac{M}{|f(z)|} \geqq \omega(z) \log \frac{M}{m},$$

or

$$|f(z)| \leqq M \left(\frac{m}{M}\right)^{\omega(z)}. \tag{8}$$

An inequality for the modulus of $f(z)$ in the opposite direction is obtained by taking into consideration that $f(z)$ is normal. The spherical distance of M and $|f(z)|$ equals

$$s(M, |f(z)|) = \int_{|f(z)|}^{M} \frac{dt}{1+t^2} = \operatorname{arc\,tg} \frac{M-|f(z)|}{1+M|f(z)|}, \tag{9}$$

where the branch of arc tg of modulus less than $\pi/2$ must be chosen.

Because $f(z)$ is normal, we have, on the other hand,

$$s(M, |f(z)|) \leqq \int_z^Q \varrho(f(z)) |dz| \leqq C \int_z^Q d\sigma(z). \tag{10}$$

Since we are interested in points z near Q, we can suppose that the right-hand majorant is less than $\pi/2$. Combining (9) and (10), we then get

$$|f(z)| \geqq \frac{M - \operatorname{tg}\left(C \int_z^Q d\sigma(z)\right)}{1 + M \operatorname{tg}\left(C \int_z^Q d\sigma(z)\right)}. \tag{11}$$

In (8) and (11) we have a double inequality for $|f(z)|$. For $z = Q$, the bounds coincide. Hence, at Q the normal derivative of the majorant in (8) cannot be less than the normal derivative of the minorant in (11). Performing the derivation, we obtain the desired inequality (7).

21. Because the minorant in (7) is conformally invariant, we can assume, without loss of generality, that G is the upper half-plane. As domain G^* we choose a circular segment T_α containing the angle α and having as chord a segment of line of the real axis, which we take as γ.

In this case, the exact value of λ_Q can easily be computed. If γ is the segment of line with endpoints $z = \pm r$, then

$$\omega(z) = \frac{1}{\pi - \alpha}\left(\arg\left(\frac{z-r}{z+r}\right) - \alpha\right),$$

and it follows by an elementary computation that

$$\lambda = \frac{\pi - \alpha}{2 \sin \alpha}.$$

Consequently, λ is independent of r and z and depends on α only. With respect to α, λ is monotonic decreasing.

In this special case (7) assumes the form

$$m \geqq M\, e^{-\frac{\pi-\alpha}{2\sin\alpha} C \left(M + \frac{1}{M}\right)}. \tag{12}$$

22. If $M \to \infty$, the minorant in (12) tends to zero, and the inequality becomes trivial. This is due to the fact that for very large M, the two inequalities (8) and (11) yield a better estimate than (12) if applied in some inner point of G^*. However, we need not apply (12) for large values of M at all. For if (12) holds for a certain $M = M_0$, we can show that it holds for every $M < M_0$.

For $M \leqq m$, this is trivial. If $M = M_1$ satisfies the inequalities $m < M_1 < M_0$, we consider the closed point set E on which $|f(z)| \geqq M_1$. This set has a positive distance from γ. From among all domains T_α bounded by a part of γ we then select one which has at least one boundary point in common with E but does not contain any interior points of E. In such a T_α, $|f(z)| < M_1$, and $|f(z)| = M_1$ at some point of γ'. Hence, (12) holds for $M = M_1$.

Applying the same reasoning we see that (12) remains valid also if $|f(z)| > M$ at some point of T_α.

The minorant in (12) attains its largest value for

$$M = M(\alpha, C) = \frac{1 + \left(1 + \left(\frac{\pi - \alpha}{\sin \alpha} C\right)^2\right)^{\frac{1}{2}}}{\frac{\pi - \alpha}{\sin \alpha} C}.$$

If $\sup |f(z)| \geqq M(\alpha, C)$ in T_α, (12) thus yields the best estimate, if $M = M(\alpha, C)$.

Summarizing the results, we obtain

THEOREM 7. *Let $f(z)$ be meromorphic and normal in the upper half-plane, and let $|f(z)| \leqq m$ on a segment of line γ of the real axis. Then*

$$m \geqq M\, e^{-C \frac{\pi-\alpha}{2\sin\alpha}\left(M+\frac{1}{M}\right)},$$

where M is an arbitrary positive number satisfying the condition

$$M \leqq \sup_{z \in T_\alpha} |f(z)|, \quad \textit{if} \quad \sup_{z \in T_\alpha} |f(z)| < M(\alpha, C).$$

The best estimates are obtained for

$$M = \sup_{z \in T_\alpha} |f(z)|, \quad \textit{if} \quad \sup_{z \in T_\alpha} |f(z)| < M(\alpha, C),$$

and for

$$M = M(\alpha, C), \quad \textit{if} \quad \sup_{z \in T_\alpha} |f(z)| \geqq M(\alpha, C).$$

23. In the special case that $|f(z)| \leqq m$ on every finite segment of line γ of the real axis, Theorem 7 gives the following accurate description about the boundary values.

THEOREM 8. *Let $f(z)$ be meromorphic and normal in the upper $(z = x + iy)$-half-plane,*

$$\sup \frac{\varrho(f(z))\,|dz|}{d\sigma(z)} = C < \infty, \tag{13}$$

and denote $m = \sup_{-\infty < x < \infty} |f(x)|$. Then

$$m \geqq \frac{1+\sqrt{1+C^2}}{C}\, e^{-\sqrt{1+C^2}}, \tag{14}$$

unless $f(z)$ is bounded, in which case

$$m \geqq C.$$

Both bounds are sharp.

Proof. The case that $f(z)$ is bounded is readily established. By the maximum principle, we then have $|f(z)| \leqq m$ in the whole half-plane. Hence, by Schwarz's Lemma,

$$|f'(z)|\,|dz| \leqq m d\sigma(z),$$

with equality at $z = z_0$ if and only if $w = f(z)$ gives a one-one mapping of the half-plane onto $|w| < m$ and $f(z_0) = 0$. Hence, *a fortiori,*

$$\varrho(f(z))\,|dz| \leqq m d\sigma(z),$$

i.e.,

$$C \leqq m,$$

with equality for the above mapping functions.

Let us hereafter suppose that $f(z)$ is unbounded. For an arbitrary $M > m$, we consider the set E_M on which $|f(z)| \geqq M$. For every α, $0 < \alpha < \pi$, we obviously have a domain T_α such that $T_\alpha \cap E_M$ is void, while T_α and E_M have at least one boundary point in common. Hence, Theorem 7 holds for *every* α, $0 < \alpha < \pi$. Letting $\alpha \to \pi$, we obtain (14).

In order to prove that (14) is sharp we consider a function

$$f(z) = A\, e^{-i(bz+c)}, \tag{15}$$

where $b > 0$ and c are arbitrary real numbers. If $A > 0$ is determined by the condition

$$\max_{y>0} \frac{2\,A\,y\,e^y}{1 + A^2\,e^{2y}} = C,$$

we see immediately that the condition (13) is fulfilled. Besides, it follows by an easy computation that

$$A = \frac{1 + \sqrt{1 + C^2}}{C}\, e^{-\sqrt{1+C^2}}.$$

Because $m = A$ for the functions (15), (14) holds for them as an equality.

24. From Theorem 8 we also immediately get information on $\mu = \inf\limits_{-\infty < x < \infty} |f(x)|$. In fact,

$$\frac{1}{\mu} = \sup_{-\infty < x < \infty} \frac{1}{|f(x)|},$$

and since $\varrho(f(z)) = \varrho(1/f(z))$, it follows that

$$\mu \leqq \frac{C}{1 + \sqrt{1 + C^2}}\, e^{\sqrt{1+C^2}},$$

unless $1/f(z)$ is bounded, in which case $\mu \leqq 1/C$. Both bounds are again sharp.

In the general case that both $f(z)$ and $1/f(z)$ are unbounded, $m \to \infty$ and $\mu \to 0$ as $C \to 0$. Hence, the "more normal" such a function is, the more its modulus has to oscillate on the boundary.

25. As another application of Theorem 7, we prove the following generalization of Picard's classical Theorem.

THEOREM 9. *A meromorphic function cannot be normal in any neighbourhood of an isolated essential singularity.*

Proof. Let the isolated singularity lie at $z = 0$. Owing to the monotonicity property of normal functions with respect to the domain, there is no restriction involved in supposing that the neighbourhood G considered is the unit disc punctured at $z = 0$.

Let us make the antithesis that $f(z)$ is normal in G. Since the universal covering surface of G is mapped onto $|w| < 1$ by $z = e^{(w+1)/(w-1)}$, we get by an easy computation

$$\frac{d\sigma(z)}{|dz|} = \frac{1}{2|z|\log\frac{1}{|z|}}.$$

From the normality of $f(z)$ it thus follows that[1]

$$\varrho(f(z)) \leq \frac{C}{2|z|\log\frac{1}{|z|}}.$$

Because $z = 0$ is an isolated essential singularity, $f(z)$ takes all values except at most two in every neighbourhood of $z = 0$. Without any essential restriction we can therefore assume that there exists a sequence of points $z_1, z_2, \ldots$ with $|z_1| \geqq |z_2| \geqq \ldots$, $\lim_{n\to\infty} |z_n| = 0$, such that $f(z_n) = 0$, $n = 1, 2, \ldots$

On the circle $|z| = |z_n|$, we have for the spherical distance of $f(z)$ and 0,

$$s(f(z), 0) = \text{arc tg } |f(z)| \leq \int_{z_n}^{z} \varrho(f(z))\,|dz|$$

$$\leq \frac{C}{2}\int_{z_n}^{z} \frac{|dz|}{|z|\log\frac{1}{|z|}} \leq \frac{\pi C}{\log\frac{1}{|z_n|}}.$$

For sufficiently large n, it thus follows that on $|z| = |z_n|$,

$$|f(z)| \leqq \text{tg}\left(\frac{\pi C}{\log\frac{1}{|z_n|}}\right) < 1.$$

Let now $r < |z_n|$ be so chosen that $|f(z)| < 1$ in $r < |z| < |z_n|$ and that $|f(z)| = 1$ for some z of modulus r. We apply Theorem 6 to $f(z)$ by taking as domain G the disc $|z| < |z_n|$ punctured at $z = 0$ and as G^* the ring $r < |z| < |z_n|$. Then

$$\frac{1}{\lambda} = 2r\log\frac{|z_n|}{r}\cdot\frac{\partial}{\partial n}\left(\frac{\log\frac{|z|}{r}}{\log\frac{|z_n|}{r}}\right)_{|z|=r} = 2,$$

[1] It can be proved that only $\varrho(f) = o(|z|^{-1})$ is actually needed. We shall discuss this problem in a forthcoming paper.

and since the hyperbolic measure of $(|z| < |z_n|) \cap z \neq 0$ is greater than the measure in the punctured unit disc, it follows that

$$\operatorname{tg}\left(\frac{\pi C}{\log \frac{1}{|z_n|}}\right) \geq e^{-C}.$$

This, however, leads to a contradiction as $n \to \infty$, and we conclude that $f(z)$ cannot be normal.

26. Theorem 6, or its sharpened specialization Theorem 7, can also conveniently be used for the study of the boundary convergence of normal meromorphic functions. We can easily establish the important property of normal functions that convergence towards a constant on a boundary arc implies convergence towards this constant in the whole domain.

THEOREM 10. *Let $f_n(z)$, $n = 1, 2, \ldots$, be a sequence of meromorphic functions normal in G, $\varrho(f_n(z))|dz| \leq C d\sigma(z)$, where C is independent of n. Let further $\lim_{n\to\infty} |f_n(z)| = 0$ uniformly on a boundary arc γ. Then the sequence $f_n(z)$ tends uniformly to zero in every compact part of G.*

Proof. Let us first suppose that G is the upper half-plane. Let B be an arbitrary compact part of G, and let us consider a domain T_α with γ as chord which contains B.

By Theorem 7, we have in T_α,

$$\sup_{z\in\gamma} |f_n(z)| \geq M e^{-C\frac{\pi-\alpha}{2\sin\alpha}\left(M + \frac{1}{M}\right)}, \tag{16}$$

where M is equal to $\sup_{z\in T_\alpha} |f_n(z)|$ or $M(\alpha, C)$, according as $\sup_{z\in T_\alpha} |f_n(z)| < M(\alpha, C)$ or $\sup_{z\in T_\alpha} |f_n(z)| \geq M(\alpha, C)$.

By hypothesis, the left-hand side of (16) tends to zero, as $n \to \infty$. Hence, from a certain n on, the alternative that (16) holds for the fixed $M = M(\alpha, C)$ is impossible. With the left-hand side tending to zero, $M = \sup_{z\in T_\alpha} |f_n(z)| \geq \max_{z\in B} |f_n(z)|$ also has to approach zero. Hence, the sequence $f_n(z)$ converges uniformly towards zero in B.

By conformal mapping, the result is extended for more general domains G.

27. We finally remark that Theorem 2 (or Theorem 2′), our generalization of Lindelöf's Theorem, can also be derived as a direct consequence of Theorem 10.

Let us suppose that a normal $f(z)$ has an asymptotic value α at a boundary point P along a Jordan curve Γ lying in the closure of G. We make the antithesis that there exists a point set $z_1, z_2, \ldots$ in an angle with vertex at P such that $z_n \to P$ and that $f(z_n)$ does not tend to α.

In proving Theorem 2, we introduced mappings $z' = S_n(z)$ of G onto itself, which left P invariant and were normalized by the requirement $S_n(z_0) = z_n$, where z_0 was a fixed point in G. The proof was essentially based on the result of Theorem 1 according to which there exists an asymptotic α-path L such that under the inverse mappings $z = S_n(z')$ certain arcs of L had an accumulation continuum *inside* G. In possession of Theorem 10, we can, however, draw the same conclusions also if the continuum lies on the *boundary*, and Theorem 2 follows more directly.