Correction to

On elliptic systems in $\mathbf{R}^n$

by

ROBERT B. LOCKHART and ROBERT C. McOWEN

University of Notre Dame
Notre Dame, IN, U.S.A.

Northeastern University
Boston, MA, U.S.A.

(Acta Math. 150 (1983), 125–135)

This note is to announce an error in the statement (and proof) of Theorem 4 in [2], namely the equality of the Fredholm index of the variable coefficient elliptic system A and the constant coefficient elliptic system is false. Thus Theorem 4 should read

THEOREM 4. *If* (1.7) *and* (1.8) *hold with* $C^{ij}_{\alpha\beta}=0$ *for all* $|\alpha|\leq t_j-s$, $|\beta|\leq s_i$, *and* $i,j=1,\ldots,k$ *then* (††) *is Fredholm if and only if* (1.9) *holds.*

The error in the proof occurs on page 135 where the homotopy A_τ is discontinuous at $\tau=0$. To complete the proof it is necessary to construct a Fredholm inverse for $A_\infty+\varphi_R Q$ which may be done by patching together a parametrix in $|x|\leq 3R$ with a Fredholm inverse for A_∞ in $|x|>2R$, thereby showing that (4.5) is finite.

The error was carried over from [1] where the same homotopy was used to assert the equality of the indices for scalar operators A and A_∞ (as in theorem 2). Though the proof in [1] also fails, the result for scalars can be proved by studying the symbol homomorphism as in [4], so Theorem 2 is true.

For the special case of classically elliptic systems (as in [4]) the symbol homomorphism may also be used to compute index $(A)-$index (A_∞), and in particular to obtain a counterexample to index $(A)=$index (A_∞). In fact, in $\mathbf{R}^2$ consider the 2×2 system

$$A=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\frac{\partial}{\partial x}+\begin{pmatrix}i\cos r & e^{i\theta}\sin r\\ -e^{-i\theta}\sin r & -i\cos r\end{pmatrix}\frac{\partial}{\partial y}$$

where $0<r=\sqrt{x^2+y^2}\leq\pi$, $0\leq\theta=\arctan y/x<2\pi$, and extend A to $r>\pi$ by the constant coefficient operator

$$A_\infty=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\frac{\partial}{\partial x}+\begin{pmatrix}-i & 0\\ 0 & i\end{pmatrix}\frac{\partial}{\partial y}$$

Let us fix $-2/p<\delta<-1+2/p'$ and observe that

$$A_\infty\colon W^p_{1,\delta}\to W^p_{0,\delta+1} \tag{1}$$

is an isomorphism. We may realize $H=AA_\infty^{-1}$ as an elliptic singular integral operator and

$$H\colon W^p_{0,\delta+1}\to W^p_{0,\delta+1} \tag{2}$$

$$A\colon W^p_{1,\delta}\to W^p_{0,\delta+1} \tag{3}$$

have the same Fredholm index. But using the results of [4], [5], and [6] we find that the index of (2) is given by the degree of the mapping $p\circ\sigma_H\colon S^2\times S^1\to S^3$ (where $p\circ\sigma_H$ is the 1st column vector of σ_H) which is 2. So index $(A)\neq$index (A_∞).

In the general case of Douglis-Nirenberg ellipticity a little more can be said than Theorem 4, namely in [3] it is shown that the Fredholm index of (††) and that of $(\dagger\dagger)_\infty$ differ by a constant which is independent of $\delta\in\mathbf{R}$.

Finally, the authors wish to acknowledge M. Murata for pointing out the error in the proof of Theorem 4, and C. Taubes for suggesting the above counterexample.

References

[1] LOCKHART, R., Fredholm properties of a class of elliptic operators on non-compact manifolds. *Duke Math. J.*, 48 (1981), 289–312.
[2] LOCKHART, R. & MCOWEN, R., On elliptic systems in $\mathbf{R}^n$. *Acta Math.*, 150 (1983), 125–135.
[3] — Elliptic differential operators on non-compact manifolds. To appear.
[4] MCOWEN, R., On elliptic operators in $\mathbf{R}^n$. *Comm. Partial Differential Equations*, 5 (1980), 913–933.
[5] SEELEY, R. T., The index of elliptic systems of singular integral operators. *J. Math. Anal. Appl.*, 7 (1963), 289–309.
[6] — Integro-differential operators on vector bundles. *Trans. Amer. Math. Soc.*, 117 (1965), 167–204.