A NOTE ON RESIDUAL INTERSECTIONS AND THE DOUBLE POINT FORMULA

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In this note we show how to extend Laksov's proof [L] of the residual intersection formula in the preceding paper to allow the case where there are imbedded components. This gives a proof of the double point formula for a general mapping provided only that an appropriate double point scheme has the right dimension.

§ 1. The theorems

1.1. Residual Intersections

Let $f: X^n \to Y^m$ be a morphism from an n-dimensional variety X to a non-singular quasi-projective variety Y. Let Z be a non-singular closed subscheme of Y of codimension d, and let N be the normal bundle of Z in Y.

Assume that there is a Cartier divisor R on X which is a subscheme of the inverse image scheme $f^{-1}(Z)$. Laksov defines the *residual* subscheme W by the equation

$$I(f^{-1}(Z)) = I(R)\cdot I(W)$$

relating their ideal sheaves. Let $r: R \to X$ be the inclusion, $h: R \to Z$ the map induced by f. Since W is locally defined by d equations, codim $(W, X) \leqslant d$.

THEOREM 1. *Assume* codim $(W, X) = d$. *Then*

$$f^*[Z] = [W] + r_*(\eta \frown [R])$$

in $A_{n-d}X$, *where*

$$\eta = \sum_{i+j=d-1} r^*c_1(I(R))^i \cdot h^*c_j(N)$$

(*We denote the group of k-cycles modulo rational equivalence on X by A_kX; for properties of Chern classes, cap products, etc., see* [$F1$]).

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1.2. Double points

Let $f\colon X^n \to Y^m$ be a morphism of non-singular varieties. Let $\pi\colon \widetilde{X\times X} \to X\times X$ be the blowing-up of $X\times X$ along the diagonal; the exceptional divisor $P(X)$ can be identified with the projectivized tangent bundle of X. Let $F\colon \widetilde{X\times X} \to Y\times Y$ be the composition of π with $f\times f$. Laksov defines the *double point scheme* $Z(f)$ to be the subscheme of $\widetilde{X\times X}$ which is residual to $P(X)$ in $F^{-1}(\Delta_Y)$, i.e.

$$I(F^{-1}(\Delta_Y)) = I(P(X))\cdot I(Z(f))$$

He showed that the underlying set of $Z(f)$ consists of pairs (x, x') with $x \neq x'$ and $f(x) = f(x')$, together with points of $P(X)$ which correspond to tangent lines that are mapped to zero by df.

If $Z(f)$ has the expected codimension m, then it determines a cycle $\mathbf{Z}(f)$ of dimension $2n-m$ on $\widetilde{X\times X}$. The *double point cycle* $\mathbf{D}(f)$ is then the $(2n-m)$-cycle on X defined by $\mathbf{D}(f) = \mathrm{pr}_{1*}\pi_*\mathbf{Z}(f)$, where pr_1 is the first projection of $X\times X$ to X.

THEOREM 2. *Assume* codim $Z(f) = m$. *Then*

$$[\mathbf{D}(f)] = f^*f_*[X] - c_{m-n}(\nu_f)\frown[X]$$

in $A_{2n-m}X$, *where* $\nu_f = f^*T_Y - T_X$ *is the virtual normal bundle of* f.

1.3. Curves

Let $f\colon X^1 \to Y^2$ be a morphism from a non-singular curve to a non-singular surface. Assume that f maps X birationally onto its image $\bar{X}$. The conductor of $X \to \bar{X}$ is an ideal sheaf on X, so it determines a zero cycle C on X.

THEOREM 3. *The double point cycle* $\mathbf{D}(f)$ *is equal to* C.

1.4. Divisors

Let C and D be effective Cartier divisors on a purely n-dimensional variety X. Then

$$c_1(O(C))\frown[D] = c_1(O(D))\frown[C]$$

in $A_{n-2}X$, as follows easily from the facts that $[C] = c_1(O(C))\frown[X]$, $[D] = c_1(O(D))\frown[X]$, and the Chern classes commute. Both sides of this formula live naturally in $A_{n-2}E$, where $E = C\cup D$, and we claim that equality holds there. Theorems 1 and 2 depend on this theorem.

THEOREM 4. *Let* $k\colon E\to X$ *be the inclusion. Then*

$$c_1(k^*O(C))\frown[D] = c_1(k^*O(D))\frown[C]$$

in $A_{n-2}E$.

§ 2. Remarks

2.1. Theorem 1 is Laksov's Theorem 8 [L], except that we do not need to assume that W has no components in R. Similarly Theorem 2 is Laksov's Theorem 15, but without the assumption that the double point scheme has no components in the exceptional divisor. Thus entire components of $Z(f)$ can consist of ramification points, provided the dimensions are correct. When we map a curve to a surface, this means we may allow cusps and higher singularities as well as ordinary multiple points (cf. Theorem 3).

2.2. By following the ideas of K. Johnson [J], the double point formula may be extended to mappings $f\colon X \to Y$ when X is singular. See [F–L] for a report that includes this generalization.

2.3. The *double point locus* $D(f)$ of a morphism $f\colon X \to Y$ is the set of points $x \in X$ such that either there is an $x' \neq x$ with $f(x') = f(x)$, or f is ramified at x. As a set, $D(f)$ is the image of $Z(f)$ under $\mathrm{pr}_1 \circ \pi$. Artin and Nagata [A–N] have shown that if X is a local complete intersection and Y is non-singular, then each irreducible component of $\pi(Z(f))$ has dimension at least $2n - m$. It follows that each component of $D(f)$ has dimension at least $2n - m$. (This is easy to see for components which do not consist entirely of ramification points, and for those that do it follows from the fact that pr_1 maps the diagonal isomorphically onto X.)

In particular, if X is a local complete intersection, and $Z(f)$ has dimension $2n - m$, then $\mathbf{D}(f)$ is a positive $(2n-m)$-cycle whose support is exactly the double point locus.

2.4. Given a morphism $f\colon X^n \to Y^m$, it is desirable to find a subscheme of X whose support is the double point locus, and whose cycle is $\mathbf{D}(f)$, assuming the dimensions are correct. Laksov [L] studies the image scheme of $Z(f)$ under the projection $\mathrm{pr}_1 \circ \pi\colon \widetilde{X \times X} \to X$ as a possible solution; he shows that if f is sufficiently generic it answers the question. The generic assumption doesn't allow for higher order multiplicites, however; e.g. for a map from a curve to a surface it allows only nodes, while we have seen that the conductor works in general here.

For the general codimension one case $X^n \to Y^{n+1}$ Kleiman [K] has showed that the divisor C defined by the conductor satisfies the desired formula $[C] = f^* f_*[X] - c_1(\nu_f)$. There should certainly be a proof that the C and $\mathbf{D}(f)$ are the same divisor in this case.

When $m > n+1$, however, the conductor does not give the right answer. A counterexample (cf. [A–N], § 5.8) is given by taking X^2 to be the disjoint union of three projective planes, Y^4 = projective 4-space, and $f\colon X \to Y$ mapping X to three planes that intersect (pairwise transversally) in one point Q. In this case the conductor consists of functions that vanish, together with their linear parts, at the three points that map to Q. *So* Deg $C=9$, while Deg $\mathbf{D}(f)=6$, as may be calculated from Theorem 2.

2.5. It is clear in general that the class $f^*f_*[X]-c_{m-n}(\nu_f)$ restricts to zero on the complement of the double point locus $D(f)$ and therefore ([F], § 1.9) there is a class in $A_{2n-m}(D(f))$ that maps to this class in $A_{2n-m}X$. A double point formula may be thought of as a construction of such an element in $A_{2n-m}(D(f))$.

If we start with a non-singular hypersurface $\bar{X}^n$ in a non-singular variety Y^{n+1}, and let $X^n \to \bar{X}^n$ be the blow-up of $\bar{X}$ along a non-singular subvariety, then the induced map $f: X^n \to Y^{n+1}$ has the exceptional divisor D for its double point locus. Note that in this case D has the correct dimension, but $Z(f)$ is too big. And in fact a simple calculation shows that

$$f^*f_*[X]-c_1(\nu_f) = -[D].$$

2.6. Theorem 3 is proved by a straight-forward calculation, comparing the given map to one obtained by blowing up a point in the image. I have learned that M. Fischer, M. Gusein-Zade and B. Tessier have, independently, done essentially the same thing. (See [T] for this and interesting uses of the double point scheme in the study of equisingularity.) We include an elementary proof in § 4 which doesn't use the relation between the conductor and the arithmetic genus p_a. This relation can then be deduced from Theorems 2 and 3. Theorem 2 says

$$\mathbf{D}(f) = f^*[\bar{X}] - f^*c_1(Y) + c_1(X)$$

If we let K_X, K_Y denote canonical divisors, we have

$$\begin{aligned}\deg \mathbf{D}(f) &= X\cdot(X+K_Y) - \deg K_X\\ &= 2p_a\bar{X} - 2 - (2p_aX - 2)\end{aligned}$$

by Riemann-Roch. Combining this with Theorem 3 we recover the formula

$$\deg C = 2(p_a\bar{X} - p_aX).$$

2.7. Theorem 4 can also be proved by using MacPherson's graph construction (cf. [F–M]). In fact, one can show that both sides of the equation live naturally and are equal in $A_{n-2}(C \cap D)$. The proof given in § 5 is more elementary.

2.8. The result here, and in [L], make use of rational equivalence on singular varieties. In particular we use Lemma (4) of § 2.2 in [F]. Unfortunately the essential hypothesis that the two maps f and g are Tor-independent, i.e. that $\mathrm{Tor}_i^{\mathcal{O}_Y}(\mathcal{O}_{Y'}, \mathcal{O}_X)=0$ for $i>0$, was left out of the statement there. This hypothesis does hold in the places necessary for our results. See [F–M] for a detailed proof.

2.9. I would like to thank K. Johnson, S. Kleiman, A. Landman, R. MacPherson, and C. McCrory for helpful conversations concerning intersection problems and double point formulae.

§ 3. Proof of Theorems 1 and 2

The only place where essential change is needed in Laksov's argument in [L] is in his proof of Lemma 13. With notations as in § 1.1, let $\psi: X' \to X$ be the blow-up of X along $V = f^{-1}(Z)$, and construct the following Cartesian diagram:

$$\begin{array}{ccccc} W' & \xrightarrow{k'} & V' & \xrightarrow{j'} & X' \\ \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\delta} & & \downarrow{\scriptstyle\psi} \\ W & \xrightarrow{k} & V & \xrightarrow{j} & X \end{array}$$

Let $R' = \psi^{-1}(R)$. The assertion of Lemma 13 in [L] follows immediately from the following two formulae, valid for any $c_i \in A^i V$.

(i) $$\psi_* j'_*\{\delta^*(c_i) j'^*(c_1(I(W'))^l c_1(I(R'))^m) \frown [W']\} = 0$$

for any $l \leqslant d-2$.

(ii) $$\psi_* j'_*\{\delta^*(c_i) j'^*(c_1(I(W'))^l c_1(I(R'))^m) \frown [R']\} = 0$$

for any $l \leqslant d-1$.

Since $I(R') = \psi^* I(R)$, the left side of (i) equals

$$\begin{aligned} c_1(I(R))^m \frown \psi_* j'_*\{\delta^*(c_i) j'^* c_1(I(W'))^l \frown [W']\} \\ = c_1(I(R))^m \frown j_* \delta_*\{\delta^*(c_i) k'_*(k'^* j'^* c_1(I(W'))^l \frown [W'])\} \\ = c_1(I(R))^m \frown j_*\{c_i k_* \alpha_*(c_1((j'k')^* I(W'))^l \frown [W'])\} \end{aligned}$$

But W' has pure dimension $n-1$, and W has dimension $n-d$, so $\alpha_*(c^l \frown [W']) = 0$ for any $c \in A^1 W'$, $l \leqslant d-2$, by reasons of dimension.

For the proof of (ii), we apply Theorem 4 to know that

$$j'^* c_1(I(W')) \frown [R'] = j'^* c_1(I(R')) \frown [W']$$

in $A_{n-2} V'$, and therefore

$$j'^*(c_1(I(W'))^l \cdot c_1(I(R'))^m) \frown [R'] = j'^*(c_1(I(W'))^{l-1} c_1(I(R'))^{m+1}) \frown [W']$$

for $l>0$. Substituting this last identity into formula (ii) reduces it to one of type (i), and concludes the proof.

§ 4. Proof of Theorem 3

By extending the ground field, we may assume we are working over an algebraically closed field k. Let $\mathfrak{C}$ be the conductor of $X\to\bar{X}$. We must show that $\mathrm{ord}_P\,\mathfrak{C}=\mathrm{ord}_P\,\mathbf{D}(f)$ at each point P in X.

Let $f(P)=\bar{P}$, and let $\delta\colon Y'\to Y$ be the blow-up of Y at $\bar{P}$, and let $f'\colon X\to Y'$ be the induced morphism, so $\delta\circ f'=f$, and let $\bar{X}'=f'(X)$, $\bar{P}'=f'(P)$, and let $\mathfrak{C}'$ be the conductor of $X\to\bar{X}'$. The result is obvious if f is an imbedding at P, and this is realized by enough blowing up, so it suffices to show

$$(*)\qquad \mathrm{ord}_P\,\mathbf{D}(f)-\mathrm{ord}_P\,\mathbf{D}(f')=\mathrm{ord}_P\,\mathfrak{C}-\mathrm{ord}_P\,\mathfrak{C}'$$

This depends only on the completions of the local rings, so we may take uniformizing parameters (x, y) for Y at $\bar{P}$ so $\bar{X}$ is given by an equation $F(x,y)=0$, where the y-axis is not tangent to $\bar{X}$ at $\bar{P}=(0,0)$. Let m be the multiplicity of $\bar{X}$ at $\bar{P}$. Write $k(Q)=\mathrm{ord}_Q\,(X)$ for any $Q\in f^{-1}(\bar{P})$, and for a zero-dimensional subscheme Z of $X\times X$, write $\mathrm{ord}_{(P,Q)}Z$ for the length of O_Z at (P,Q). Since

$$\mathrm{ord}_P\,\mathbf{D}(f)\sum_{f(Q)=\bar{P}}\mathrm{ord}_{(P,Q)}\,Z(f)$$

the formula (*) follows from the following four formulae:

$$\text{(i)}\qquad m=\sum_{f(Q)=\bar{P}}k(Q)$$

$$\text{(ii)}\qquad \mathrm{ord}_{(P,P)}\,Z(f)-\mathrm{ord}_{(P,P)}\,Z(f')=k(P)\,(k(P)-1)$$

$$\text{(iii)}\qquad \mathrm{ord}_{(P,Q)}\,Z(f)-\mathrm{ord}_{(P,Q)}\,Z(f')=k(P)\,k(Q)\quad\text{if}\quad P\neq Q.$$

$$\text{(iv)}\qquad \mathrm{ord}_P\,\mathfrak{C}-\mathrm{ord}_P\,\mathfrak{C}'=(m-1)\,k(P)$$

Formula (i) follows from the fact that m is the intersection number of $\bar{X}$ with the y-axis.

To prove formula (ii), let t be a uniformizing parameter for X at P, so f is given by $f(t)=(x(t), y(t))$ for power series $x(t)$, $y(t)$, and f' is given by $f'(t)=(x(t), z(t))$ where $y(t)=x(t)\,z(t)$.

For any power-series $w(t)$ define the power series $\tilde{w}(t,h)$ in two variables by the equation $w(t+h)-w(t)=h\tilde{w}(t,h)$.

Let $t_i=t\circ\mathrm{pr}_i$, $i=1,2$. Then $u=t_1$, $v=t_2-t_1$ may be taken as uniformizing parameters for $X\times X$ at (P,P), and the ideal of $(f\times f)^{-1}(\Delta_Y)$ is generated at (P,P) by the functions $x(t_1)-x(t_2)$ and $y(t_1)-y(t_2)$, or by $x(t_1)-x(t_2)$ and $y(t_1)\cdot(z(t_1)-z(t_2))$, i.e. by $v\tilde{x}(u, v)$ and $vx(u)\tilde{z}(u, v)$. Therefore $I(Z(f))=(\tilde{x}(u, v),\ x(u)\tilde{z}(u, v))$ at (P, P). Similarly $I(Z(f'))=(\tilde{x}(u, v),\ \tilde{z}(u, v))$. It follows (by the bilinarity of intersection numbers) that

$$\operatorname{ord}_{(P,P)} Z(f)-\operatorname{ord}_{(P,P)} Z(f') = \text{length } (k[[u, v]]/(\tilde{x}(u, v), x(u)))$$

Since the "curves" $x(u)$ and $\tilde{x}(u, v)$ are not tangent, this length is the product of their multiplicities, which proves (ii).

The proof of (iii) is similar but easier, so we leave it to the reader.

To prove (iv), let $\bar{O}$, $\bar{O}'$, and O be the completions of the local rings of $\bar{X}$ at $\bar{P}$, $\bar{X}'$ at $\bar{P}'$, and X at P respectively, and regard $\mathfrak{C}$ (resp. $\mathfrak{C}'$) as the conductor of O over $\bar{O}$ (resp. O over $\bar{O}'$). Then (iv) is equivalent to showing that $\mathfrak{C}=x^{m-1}\mathfrak{C}'$. This follows from the fact that the conductor of $\bar{O}'$ over $\bar{O}$ is $x^{m-1}\bar{O}'$. And this last fact can be seen by writing $\bar{O}=\sum_{i=0}^{m-1} k[[x]]y^i$, $\bar{O}'=\sum_{i=0}^{m-1} k[[x]]z^i$; By induction on r, one calculates that for $a_0, \ldots, a_{m-1}\in k[[x]]$, we have $(\Sigma a_i z^i)\cdot z^j \in \bar{O}$ for $0\leqslant j\leqslant r\leqslant m-1$, if and only if $x^{m-1}|a_{m-1-j}$ for $0\leqslant j\leqslant r$.

§ 5. Proof of Theorem 4

Reduction step. Let $\pi\colon X'\to X$ be a birational morphism, $C'=\pi^*C$, $D'=\pi^*C$. Then it is enough to prove the theorem for C' and D' on X'.

For if $E'=C'\cup D'$, $k'\colon E'\to X'$ the inclusion, $\eta\colon E'\to E$ the induced morphism, then $[D]=\pi_*[D']=\eta_*[D']$ ([F1], § 1.5 Prop. 1(2)), so

$$\begin{aligned} k^*(c_1 O(C))\frown[D]) &= \eta_*(\eta^* k^* c_1 O(C)\frown[D']) \\ &= \eta_*(k'^* c_1(O(C'))\frown[D']) \end{aligned}$$

Then we interchange C' and D' in this last formula and reverse the argument.

In general let $[C]=\Sigma m_P(C)[P]$, $[D]=\Sigma m_P(D)[P]$ be the Weil divisors of C and D, the sums being over the codimension one subvarieties P of X which are components of E. Define $m(C, D)=\Sigma m_P(C)m_P(D)$, which measures the excess intersection of C and D.

Case 1. The divisors C and D intersect properly, i.e. $m(C, D)=0$.

We claim that both sides of the required formula are equal to

$$\sum_{=0}^{1}(-1)^i Z_{n-2}(\operatorname{Tor}_i^{O_X}(O_C, O_D)),$$

where we use the notation $Z_k(\mathcal{J})$ for the k-cycle defined by a sheaf $\mathcal{J}$ whose support has dimension at most k.

As proved in [F2] there is an inclusion $i: X \to X'$ with X' non-singular, and a line bundle E' on X' with a section s' of E' transversal to the zero-section C' of E' such that $i^*E' = O(C)$ and $i^*(s')$ is a section whose zero-scheme is C. Then $c_1(E') = [C']$, and $k^*c_1(O(C)) \frown [D]$ is represented by the cycle

$$[D] \cdot_i [C'] = \sum_{i=0}^{1} (-1)^i Z_{n-2}(\operatorname{Tor}_i^{O_{X'}}(O_{C'}, O_D)).$$

Now the claim follows from the isomorphism

$$\operatorname{Tor}_i^{O_{X'}}(O_{C'}, O_D) \cong \operatorname{Tor}_i^{O_X}(O_C, O_D)$$

Case 2. All the codimension one subvarieties of X which are contained in $C \cap D$ are simple on X.

The proof is by induction on $m(C, D)$, starting with $m(C, D) = 0$, covered by case 1. Let $J = I(C) + I(D)$, and let $\pi: X' \to X$ be the blow-up of X along J. Then $JO_{X'} = I(B')$, where B' is the exceptional divisor on X'. Let $C' = \pi^*C$, $D' = \pi^*D$, $E' = C' \cup D'$, $k': E' \to X'$ the inclusion. Then $I(B') = I(C') + I(D')$, so effective divisors C'' and D'' can be defined on X' by the equations $I(C') = I(B')I(C'')$ and $I(D') = I(B')I(D'')$. Then $I(B') = I(B') \cdot (I(C'') + I(D''))$. Therefore we have written

$$C' = B' + C'', \quad D' = B' + D''$$

with C'' and D'' disjoint. By the reduction step it is enough to prove the formula for C' and D' on X', and by linearity it is enough to prove it for each pair of divisors (B', B'), (C'', D''), (C'', B') and (B', D''). The first two of these are obvious, and the fourth is the same as the third, so we consider C'' and B'.

We claim first that π maps the set of codimension one subvarieties of X' contained in $C'' \cap B'$ one-to-one into the set of codimension one subvarieties of X contained in $C \cap D$, and if $\pi(P') = P$ then $O_P(X) \simeq O_{P'}(X')$, $m_P(C) = m_{P'}(C')$, and $m_P(D) = m_{P'}(D')$.

This claim is local on X, so assume C and D are defined by functions u and v respectively. Then X' can be identified with a subscheme of $X \times \mathbf{P}^1$, where if T_0 and T_1 are the homogeneous coordinates on $\mathbf{P}^1$, then $uT_0 - vT_1$ vanishes on X'. Since $(u) = T_1 \cdot (u, v)$ on X', we see that $I(C)O_{X'} = T_1 J O_{X'}$, so C'' is defined by the equation $T_1 = 0$ on X'. Similarly T_0 defines D'' on X'.

The map $\pi: X' \to X$ is induced by the projection $p: X \times \mathbf{P}^1 \to X$, so we deduce from the above that p maps C'' and D'' isomorphically into X. In particular any codimension one subvariety P' of X' lying in C'' maps to a variety P of codimension one in X; and if P' is also contained in B', then P is in $C \cap D$, and therefore $O_P(X) \simeq O_{P'}(X')$ since by our assumption in this case $O_P(X)$ is a discrete valuation ring. The equality of multiplicites of divisors follows immediately from the isomorphism of the local rings.

To finish the proof in case 2 it is enough to show $m(C'', B') < m(C, D)$ if $m(C, D) > 0$.

By the claim $m(C, D) \geqslant \Sigma m_{P'}(C') m_{P'}(D')$, the sum taken over codimension one subvarieties P' of $C'' \cap B'$. Now substitute the equations $m_{P'}(C') = m_{P'}(B') + m_{P'}(C'')$ and $m_{P'}(D') = m_{P'}(B') + m_{P'}(D'')$ into this inequality to finish.

General case. Let $\pi: X' \to X$ be the normalization of X, and apply the reduction step to reduce to the case where X is normal. Then case 2 applies.

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