RESIDUAL INTERSECTIONS AND TODDS FORMULA FOR THE DOUBLE LOCUS OF A MORPHISM

BY

DAN LAKSOV

M.I.T., Cambridge Mass., USA

§ 1. Introduction

The main purpose of the following article is to present a proof of a formula of J. A. Todd for the rational equivalence class of the double locus of a morphism. In our proof, the main ingredient is a formula for the rational equivalence classes of residual intersections without embedded components which is of considerable interest in itself. The latter formula we shall derive from an important special case called the "formule clef", conjectured by Grothendieck ([3], exposé 0, 1957) and proved by A. T. Lascu, D. Mumford and D. B. Scott [8]. In his article [10], Todd obtained his formula for the double locus and a formula closely related to our formula for residual intersections simultaneously, by an inductive argument. It is interesting to notice that while we can prove the latter formula under mild conditions on the intersections involved, the formula for the double locus is shown to be true only under restrictive transversality conditions on the morphisms in question. We shall however, give a weaker version of Todd's formula for the double locus, which follows immediately from the formula for residual intersections and which holds under correspondingly mild transversality conditions.

In a previous article [7] we treated the particular case of Todd's formula for the double locus when the target variety was a projective space. We could in that case, define the scheme of double points as the scheme of zeroes of a section of a locally free sheaf. As a consequence, when the section intersected the zero section property, we obtained an easy proof of the weak version of Todd's formula, avoiding the "formule clef". We also proved that when the morphism in question was induced by a generic projection (possibly after a twisting of the embedding involved) the morphism satisfied the restrictive conditions under which Todd's formula holds. As a consequence of our results we obtained generalizations of results of A. Holme [5] and of C. A. M. Peters and J. Simonis [9] about secants of projective schemes. These results are generalized further below.

The arrangement of this article is as follows. In sections two and three we derive the formula for residual intersections from the "formule clef". Then, in section four, we derive the weaker version of Todd's formula from the formula for residual intersections. In section five, we first prove a criterion for a smooth scheme to be embedded into another smooth scheme which generalizes a result given in our previous article [7] (§ 5, Theorem 20). Then we prove that, when the morphisms satisfy suitable transversality conditions, Todd's formula follows from the weaker version of section four.

§ 2. Residual intersections

We shall in the following, exclusively consider schemes that are algebraic over an algebraically closed field k. In sections one and two, we shall fix the following notation and assumptions. We denote by X, Y and Z schemes that are connected and smooth over k and by $f: X \to Y$ a proper morphism. Assume that Z is a closed subscheme of Y and that the scheme theoretic inverse image $V = f^{-1}(Z)$ of Z by f contains a divisor R in X. Denote by $I(V)$ and $I(R)$ the ideals in O_X defining V and R. The ideal $I(R)$ is invertible and the product $I(V) \cdot I(R)^{-1}$ defines a closed subscheme W of X. Then $I(V) = I(W)I(R)$, where $I(W)$ is the ideal defining W.

In the above situation, when the scheme W is of pure codimension codim (Z, Y) *in X we shall say that the inverse image $V = f^{-1}(Z)$ is residual and we shall call R the residual divisor and W the proper part of the inverse image.*

Throughout section two and three we shall assume that $f^{-1}(Z)$ is residual and that no component of W is contained in R.

LEMMA 1. *Under the above assumptions the scheme W is locally a complete intersection in X.*

Proof. By assumption, the schemes Y and Z are smooth. Hence, Z is locally a complete intersection in Y. It follows that the ideal $I(V)$ defining the inverse image of Z by f is generated locally by codim (Z, Y) elements. The ideal $I(R)$ is locally principal. Consequently the ideal $I(V)I(R)^{-1}$ defining W is also generated locally by codim (Z, Y) equations. However, by assumption, W is of pure codimension codim (Z, Y) in X. Consequently W is locally a complete intersection in X.

We denote the monoidal transformation of X (resp. Y) with center V (resp. Z) by X' (resp. Y'). The structure morphism of X' (resp. Y') over X (resp. Y) we denote by ψ (resp. ϕ) and the exceptional divisor in X' (resp. Y') by V' (resp. Z'). Moreover, we let $\delta: V' \to V$ (resp. $\gamma: Z' \to Z$) be the morphisms induced by ψ (resp. ϕ). Finally, we denote by j and j' (resp. i and i') the inclusions of V in X and V' in X' (resp. of Z in Y and Z' in Y').

By the naturality of monoidal transformations, there is a canonical morphism $f'\colon X' \to Y'$ such that $\phi f' = f\psi$ and such that $(f')^{-1}(Z') = V'$. We let $g\colon V \to Z$ and $g'\colon V' \to Z'$ denote the morphisms induced by f and f'. The morphisms and schemes defined above make up the following commutative diagram.

$$\begin{tikzcd}
X' \arrow[rrr, "f'"] \arrow[ddd, "\psi"'] & & & Y' \arrow[ddd, "\phi"] \\
& V' \arrow[ul, "j'"] \arrow[r, "g'"] \arrow[d, "\delta"'] & Z' \arrow[ur, "i'"'] \arrow[d, "\gamma"] & \\
& V \arrow[r, "g"'] \arrow[dl, "j"'] & Z \arrow[dr, "i"] & \\
X \arrow[rrr, "f"'] & & & Y
\end{tikzcd}$$

Lemma 2. *Denote by X'' the monoidal transformation of X with center W. Then the schemes X'' and X' are canonically X-isomorphic.*

Proof. The ideal $I(R)$ is invertible. Hence, the natural maps $I(W)^\nu \otimes I(R)^\nu \to I(W)^\nu \cdot I(R)^\nu$ are isomorphisms. Moreover, $I(V) = I(W)I(R)$ and consequently $I(V)^\nu = I(W)^\nu I(R)^\nu$. We conclude that $X' = \operatorname{Proj}\,(\oplus_{\nu=0}^{\infty} I(V)^\nu) = \operatorname{Proj}\,(\oplus_{\nu=0}^{\infty} I(W)^\nu \otimes I(R)^\nu)$. By definition $X'' = \operatorname{Proj}\,(\oplus_{\nu=0}^{\infty} I(W)^\nu)$. Hence, the isomorphism of the lemma is the canonical isomorphism.

$\operatorname{Proj}\,(\oplus_{\nu=0}^{\infty} I(W)^\nu) \to \operatorname{Proj}\,(\oplus_{\nu=0}^{\infty} I(W)^\nu \otimes I(R)^\nu)$ associated to the invertible sheaf $I(R)$. ([4] Chapter II, Proposition (3.1.8) (iii), p. 52.)

Lemma 3. *Let $s\colon T \to U$ be a morphism from an equidimensional scheme T, without embedded components, to a scheme U. Moreover, let S be a Cartier divisor in U defined by an ideal $I \subseteq O_U$. If the subscheme $s^{-1}(S)$ does not contain any component of T then s^*I is the ideal in O_T defining the subscheme $s^{-1}(S)$ of T.*

Proof. Let Spec B be an open affine subset of U where the subscheme S is defined by a single equation $b \in B$. Moreover, let Spec A be an open affine subset of T which is mapped into Spec A by s. Then over the scheme Spec B the inclusion $I \subseteq O_U$ can be represented by the injective homomorphism $B \xrightarrow{b} B$ which sends c to cb. The pull back $s^*I \to O_T$ to O_T of the above injection is then represented over Spec A by the homomorphism $A \xrightarrow{b} A$ sending a to ba. However, b is not a zero divisor in A. Indeed, by assumption A has no embedded components and $s^{-1}(S) \cap \operatorname{Spec} A = \operatorname{Spec} A/bA$ is of dimension strictly less than Spec A. Consequently, b is in no associated prime of A.

It follows that the homomorphism $s^*I \to O_T$ is injective.

LEMMA 4. *Denote by R' the subscheme $\psi^{-1}(R)$ of X' and by $I(Z')$ (resp. $I(V')$) the ideals defining the subscheme Z' of Y' (resp. the subscheme V' of X'). Then $(f')^*I(Z')=I(V')$ and $\psi^*I(R)$ is the ideal $I(R')$ defining the subscheme R' of X'.*

Proof. The scheme X' is integral because it is a monoidal transformation of the integral scheme X. Hence, the assertions of the lemma are immediate consequences of Lemma 3 applied to the morphisms f' and ψ.

LEMMA 5. *The ideal $I(W)O_{X'}$ defining the subscheme $W'=\psi^{-1}(W)$ of X' is invertible. Moreover, the sheaf $I(W)O_{X'}\otimes\psi^*I(R)$ is the invertible ideal $I(V')$ defining the subscheme V' of X'.*

Proof. By Lemma 2 the scheme $\psi^{-1}(W)$ is a Cartier divisor in X'. Hence, $I(W)O_{X'}$ is invertible.

To prove the second assertion, recall that V is defined by the ideal $I(V)=I(W)I(R)$ in O_X. Hence, V' is defined by the ideal $I(W)I(R)O_{X'}$ in $O_{X'}$. By Lemma 4, the latter ideal is equal to $I(W)O_{X'}\cdot\psi^*I(R)$ and since $I(W)O_{X'}$ and $\psi^*I(R)$ both are invertible by the first part of Lemma 5 and Lemma 4, the natural map

$$I(W)O_{X'}\otimes\psi^*I(R)\simeq I(W)O_{X'}\cdot\psi^*I(R)$$

is an isomorphism.

LEMMA 6. *Denote by m and n the immersions of W' and R' in V'. Moreover, denote by $I(W')$ and $I(R')$ the ideals defining the subschemes W' and R' of X'. Then $(j'm)^*I(R')$ and $(j'n)^*I(W')$ are the ideals in $O_{W'}$ and $O_{R'}$ defining the scheme $W'\cap R'$ as a subscheme of W' and R'.*

Proof. Each component of the scheme W' intersect each component of the scheme R' properly, because the same is true for the components of the schemes W and R. Moreover, the scheme W' is a Cohen-Macaulay scheme. Indeed, by Lemma 1, the scheme W is Cohen-Macaulay and the conormal sheaf $I(W)/I(W)^2$ is a locally free O_W-module ([1], VI-1, Theorem (1.8), p.104 and VII-5, Theorem (5.1), p. 147). Hence, the projective bundle $W'=\mathbf{P}(I(W)/I(W)^2)$ is Cohen-Macaulay ([4], Chapter IV_2, Corollary (6.3.5) (ii), p. 140). However, the scheme W' is a Cartier divisor, and the scheme X' is integral being a monoidal transform of an integral scheme. Consequently, X' is a Cohen-Macaulay scheme and the same is true for the Cartier divisor R' in X'. In particular, both the schemes W' and R' are equidimensional without imbedded component.

Both assertions of the lemma now follow from Lemma 3 applied to the morphisms $j'm$ and $j'n$.

§ 3. The rational equivalence classes of residual intersections

We shall next recall some basic definitions and results from the theory of rational equivalence on schemes that are not necessarily smooth. Our source of reference is W. Fulton's article [2].

In the remainder of the article we shall fix the following notation. Let U be a scheme. The Chow homology group of U ([2], § 1.8) we denote by $A.(U)$ and the class of a closed subscheme S of U by $[S]$ (if S is also a subscheme of a scheme T we also denote the class of S in $A.(T)$ by $[S]$. It will be clear from the context in which group we consider the class). The direct image map $A.(T) \to A.(U)$ associated to a morphism $s: T \to U$ ([2], § 1.9) we denote by s_*. If U is smooth there is also a Gysin map $s^*: A.(U) \to A.(T)$ associated to s.

The Chow cohomology ring of U ([2], § 3.1,) we denote by $A^{\cdot}(U)$. Associated to the morphism $s: T \to U$ there is a Gysin map $s^*: A^{\cdot}(U) \to A^{\cdot}(T)$ ([2], § 3.1). The group $A.(U)$ is a $A^{\cdot}(U)$ module in a natural way ([2], § 3.1). We denote the product of two elements $a \in A^{\cdot}(U)$ and $b \in A.(U)$ by $a \frown b$. If U is smooth, the Gysin maps satisfy the following relation $h^*(a \frown b) = h^*a \frown h^*b$.

The top square of diagram (1.1) above is Cartesian. Moreover, the morphisms i' and j' are proper and the schemes Y' and Z' are smooth. Consequently, ([2], § 2.2, Lemma (4)) the following equality holds in $A.(X')$(1),

$$(f')^* i'_* z' = j'_* (g')^* z' \tag{2.1}$$

for all $z' \in A.(Z')$.

The Chern classes in $A^{\cdot}(U)$ associated to a locally free O_U-module E we denote by $c_i(E)$ and the corresponding Chern polynomial $\sum_{\nu=0}^{\infty} c_\nu(E) t^\nu$ by $c_t(E)$. Let F be another locally free O_U-module. The coefficient of t^ν in the polynomials $c_t(F)c_t(E)^{-1}$ and $c_t(E)^{-1}$ we denote by $c_\nu(F-E)$ and $c_\nu(-E)$. *We let* $s_\nu(U) = c_\nu(-(\Omega_U^1)^\vee)$, *where* $(\Omega_U^1)^\vee$ *is the dual module of the module of one Kähler differentials on* U *and let* $t_\nu(s) = c_\nu(s^*(\Omega_V^1)^\vee - (\Omega_T^1)^\vee)$. *We call the classes* $s_\nu(U)$ *the Segre classes of* U *and the classes* $t_\nu(s)$ *the Todd classes of the morphism* s. *Clearly* $t_\nu(s) = \sum_{\mu=0}^{\nu} c_\mu(s^*(\Omega_U^1)^\vee) s_{\nu-\mu}(T)$.

LEMMA 7. *Let E be a locally free O_U-module of rank e and let $P = \mathbf{P}(E)$ be the projective bundle over the scheme U, associated to E. Denote by α the structure morphism of the projective bundle P and by $O_P(1)$ the universal invertible quotient sheaf of $\alpha^* E$. Then the following formula holds in $A.(U)$,*

$$a_*(c_1(O_P(1))^\nu \frown [P]) = c_{\nu-e+1}(-E^\vee) \frown [U]$$

for $\nu = 0, 1, \ldots$

(1) See Remark 2.8 in W. Fulton, "A note on residual intersetions ...". Acta. Math., this volume.

Proof. Choose an embedding ε of U into a smooth scheme G such that E is the pull back by ε of a locally free O_G-module F ([2], § 3.2, Lemma (1)). Let $Q = \mathbf{P}(F)$ be the projective bundle associated to F and denote by β the structure morphism of the projective bundle Q. The formulas of the lemma are well known when the base scheme is smooth. Hence, the following formulas hold in $A.(G)$,

$$\beta_* c_1(O_Q(1))^\nu \frown [Q]) = c_{\nu-e+1}(-F^\vee) \frown [G] \tag{2.2}$$

for $\nu = 0, 1, \ldots$ Clearly

$$\varepsilon^*(c_{\nu-e+1}(-F^\vee) \frown [G]) = c_{\nu-e+1}(-E^\vee) \frown [U].$$

Consequently, we obtain from (2.2) the formulas

$$\varepsilon^* \beta_*(c_1(O_Q(1))^\nu \frown [Q]) = c_{\nu-e+1}(-E^\vee) \frown [U] \tag{2.3}$$

for $\nu = 0, 1, \ldots$

Denoted by ζ the natural morphism $P \to Q$ defined by the surjection $\alpha^* \varepsilon^* F = \alpha^* E \to O_P(1)$ Then the commutative diagram

$$\begin{tikzcd} P \arrow[r, "\zeta"] \arrow[d, "\alpha"'] & Q \arrow[d, "\beta"] \\ U \arrow[r, "\varepsilon"'] & G \end{tikzcd}$$

is Cartesian, the morphisms α and β are projective and G and Q are smooth schemes. Hence the equality $\varepsilon^* \beta_* = \alpha_* \xi^*$ holds ([2], § 2.2, Lemma (4)). We obtain from (2.3) the formulas

$$\alpha_* \zeta^*(c_1(O_Q(1))^\nu \frown [Q]) = c_{\nu-e+1}(-E^\vee) \frown [U] \tag{2.4}$$

for $\nu = 0, 1, \ldots$ The lemma now follows from (2.4) and the equality $\alpha_* \zeta^*(c_1(O_Q(1))^\nu \frown [Q]) = \alpha_*(c_1(O_P(1))^\nu \frown [P])$.

Denote by $I(Z)$ the ideal defining the subscheme Z of Y. The schemes Y and Z are smooth. Hence, $I(Z)/I(Z)^2$ is a locally free O_Z-module of rank codim (Z, Y) ([1], VI-1, Theorem (1.8), p. 104 and VII-5, Theorem (5.1), p. 147) and consequently, $Z' = \mathbf{P}(I(Z)/I(Z)^2)$. The structure morphism of the projective bundle $\mathbf{P}(I(Z)/I(Z)^2)$ over Z is the morphism $\gamma: Z' \to Z$ of diagram (1.1). Moreover, the universal invertible quotient sheaf of $\gamma^* I(Z)/I(Z)^2$ is $(i')^* I(Z')$ where $I(Z')$ is the ideal defining the subscheme Z' of Y'. Denote by K the kernel of the surjection $\gamma^* I(Z)/I(Z)^2 \to (i')^* I(Z)$. Then the following fundamental formula of intersection theory, called the "formule clef", holds in $A.(Y')$ for all $z \in A.(Z)$,

$$\phi^* i_* z = i'_*(c_{c-1}(K^\vee) \frown \gamma^*(z))$$

where $c = \operatorname{codim}(Z, Y)$ ([8], § 4, Theorem 2, p. 122). We use the equation

$$c_t(K^\vee) = c_t(\gamma^*(I(Z)/I(Z)^2)^\vee)\, c_t((i')^* I(Z')^\vee)^{-1}$$

to rewrite the "formule clef" in the following form,

$$\phi^* i_* z \sum_{\mu+\nu=c-1} i'_*\{c_\mu(\gamma^* N_{Z/Y})\, c_1((i')^* I(Z'))^\nu \frown \gamma^*(Z)\}, \tag{2.5}$$

where $N_{Z/Y} = (I(Z)/I(Z)^2)^\vee$ is the normal sheaf of Z in Y.

The purpose of section three is to deduce the following formula from the "formule clef".

THEOREM 8. *Let X, Y and Z be smooth, connected and projective schemes and let $f\colon X \to Y$ be a morphism. Assume that Z is a subscheme of Y and that the inverse image $f^{-1}(Z)$ is residual with a proper part W and a residual divisor R. Moreover, assume that no component of W is contained in R. Denote by r the inclusion of R in X and by $h\colon R \to Z$ the morphism induced by f. Then the following formula holds in $A.(X)$,*

$$f^*[Z] = [W] + \sum_{\mu+\nu=c-1} c_1(I(R))^\nu r_*(h^* c_\mu(N_{Z/Y}) \frown [R]).$$

Here $c = \operatorname{codim}(Z, Y)$ and $N_{Z/Y}$ is the normal bundle of Z in Y.

We shall break up the computations needed in the proof of Theorem 8 into a series of lemmas.

LEMMA 9. *With the notation of Lemma 6 the following formula holds in $A.(X')$,*

$$(f')^*\phi^*[Z] = j'_* \sum_{\mu+\nu=c-1} \sum_{l=0}^{\nu} \binom{\nu}{l} (j')^*\{c_1(I(W')^l c_1(I(R'))^{\nu-l}\}\, c_\mu((g\delta)^* N_{Z/Y}) \frown ([W'] + [R'])$$

where $c = \operatorname{codim}(Z, Y)$.

Proof. From the "formule clef" in the form (2.5) and with $z = [Z]$, together with formula (2.1), we obtain the following formula,

$$(f')^*\phi^* i_*[Z] = j'_* \sum_{\mu+\nu=c-1} (g')^*\{c_\mu(\gamma^* N_{Z/Y})\, c_1((i')^* I(Z')^\nu) \frown \gamma^*[Z]\}.$$

The latter formula we can rewrite in the following way,

$$(f')^*\phi^* i_*[Z] = j'_* \sum_{\mu+\nu=c-1} c_\mu((\gamma g')^* N_{Z/Y})\, c_1((i'g')^* I(Z'))^\nu \frown (\gamma g')^*[Z]. \tag{2.6}$$

However, $(i'g')^* = (f'j')^*$ and by Lemma 5 and Lemma 6 the following equalities hold

$$c_1((f')^* I(Z')) = c_1(I(W) O_{X'} \otimes \psi^* I(R)) = c_1(I(W')) + c_1(I(R')).$$

Moreover, $(\gamma g')^* = (g\delta)^*$ and $(\gamma g')^*[Z] = [V'] = [W'] + [R']$. Consequently, formula (2.6) can be rewritten in the following form,

$$(f')^*\phi^* i_*[Z] = j'_* \sum_{\mu+\nu=c-1} c_\mu((g\delta)^* N_{Z/Y})(j')^*\{c_1(I(W')) + c_1(I(R'))\}^\nu \frown ([W'] + [R']).$$

Rearranging the latter formula we obtain the formula of the lemma.

Lemma 10. *Keep the notation of Lemma 9.*

Let
$$A = j'_* \sum_{\mu+\nu=c-1} (j')^* c_1(I(W'))^\nu \delta^* c_\mu(g^* N_{Z/Y}) \frown [W'],$$

$$B = j'_* \sum_{\mu+\nu=c-1} (j')^* c_1(I(R'))^\nu \delta^* c_\mu(g^* N_{Z/Y}) \frown [R'],$$

and
$$C = (f')^*\phi^*[Z] - A - B,$$

where $c = \operatorname{codim}(Z, Y)$. *Then the element C in $A.(X')$ is in the image of $A.(R' \frown W')$ by the direct image map associated to the inclusion of $R' \frown W'$ in X'.*

Proof. It follows from Lemma 9 that C is a sum of terms of the form

$$j'_*\left\{\binom{\nu}{l}(j')^*\{c_1(I(W'))^l c_1(I(R'))^{\nu-l}\} c_\mu((g\delta)^* N_{Z/Y}) \frown ([W'] + [R'])\right\}$$

with $0 < l < \nu$. Consequently, to prove the lemma, it is sufficient to prove that the elements $(j')^* c_1(I(R')) \frown [W']$ and $(j')^* c_1(I(W')) \frown [R']$ in $A.(X')$ are in the image of $A.(R' \frown W')$ ([2], § 2.4). However, with the notation of Lemma 6, we have $m_*[W'] = [W']$ and $n_*[R'] = [R']$ and using the projection formula we obtain the formulas $(j')^* c_1(I(R')) \frown [W'] = m_*(m^*(j')^* c_1(I(R')) \frown [W']$ and $(j')^* c_1(I(W')) \frown [R'] = n_*(n^*(j')^* c_1(I(W')) \frown [R'])$. By Lemma 6, the sheaves $(j'm)^* c_1(I(R'))$ and $(j'n)^* c_1(I(W'))$ are the ideals defining $W' \frown R'$ as a subscheme of W' and R'. Consequently we have that $(j'm)^* c_1(I(R')) \frown [W'] = [W' \frown R']$ in $A.(W')$, and $(j'n)^* c_1(I(W')) \frown [R']) = [W' \frown R']$ in $A.(R')$ ([2], § 3.2, Proposition (3)). Hence both $(j')^* c_1(I(R')) \frown [W']$ and $(j')^* c_1(I(W')) \frown [R']$ are in the image of $A.(W' \frown R')$.

Lemma 11. *With the notation of Lemma 10, the equality $\psi_* A = [W]$ holds in $A.(X)$.*

Proof. We have that $\psi j' = j\delta$. Consequently, we have the equation,

$$\psi_* A = j_* \delta_* \sum_{\mu+\nu=c-1} (j')^* c_1(I(W'))^\nu \delta^* c_\mu(g^* N_{Z/Y}) \frown [W'].$$

From the latter equation, we obtain, using the projection formula, the equation

$$\psi_* A = j_* \sum_{\mu+\nu=c-1} c_\mu(g^* N_{Z/Y})\, \delta_*\{(j')^* c_1(I(W'))^\nu \frown [W']\}. \tag{2.7}$$

By Lemma 1, the scheme W is a complete intersection in X. Hence, the normal bundle $N_{W/X}=(I(W)/I(W)^2)^\vee$ is a locally free O_W-module of rank c ([1], VI-1, Theorem (1.8), p. 104 and VIII-5, Theorem (5.1), p. 147) and $W'=\mathbf{P}(I(W)/I(W)^2)$. The structure morphism ϱ of the projective bundle $\mathbf{P}(I(W)/I(W)^2)$ over W is the morphism induced by δ: $V'\to V$. Moreover, $(j'm)^*I(W')$ is the universal invertible quotient sheaf of $\varrho^*I(W)/I(W)^2$. Consequently, it follows from Lemma 7 that the expression

$$\varrho_*(c_1((j'm)^*I(W'))^\nu \frown [W'])$$

is zero when $0\leqslant \nu<c-1$ and is equal to $[W]$ when $\nu=c-1$. The same is therefore true for the expression $\delta_* m_*(c_1((j'm)^*I(W'))^\nu \frown [W'])$. However, by the projection formula the latter expression is equal to $\delta_*(c_1((j')^*I(W'))^\nu \frown [W'])$. The only nonzero term in the right hand side of the equation (2.7) is therefore the term

$c_0(g^*N_{Z/Y})\delta_*((j')^*c_1(I(W'))^{c-1}\frown[W'])$ and this term is equal to $[W]$.

LEMMA 12. *With the notation of Lemma 10, the following equality holds in $A.(X)$,*

$$\psi_* B=\sum_{\mu+\nu=c-1} c_1(I(R))^\nu j_*(c_\mu(g^*N_{Z/Y})\frown[R])$$

Proof. By Lemma 4 we have that $I(R')=\psi^*I(R)$ and consequently that $c_1(I(R'))=\psi^*c_1(I(R))$. Moreover, we have that $\psi j'=j\delta$. Consequently, we obtain the equation,

$$\psi_* B=j_*\delta_* \sum_{\mu+\nu=c-1} \delta^* j^* c_1(I(R))\,\delta^* c_\mu(g^*N_{Z/Y})\frown[R'].$$

Using the projection formula we can rewrite the latter equation in the following form,

$$\psi_* B=\sum_{\mu+\nu=c-1} c_1(I(R))\, j_*\{c_\mu(g^*N_{Z/Y})\frown\delta_*[R']\}. \tag{2.8}$$

However, we have that $\delta_*[R']=[R]$. Indeed, ψ is an isomorphism outside of W' by Lemma 2. Consequently, δ induces a morphism from R' onto R which sends each component of R' birationally onto exactly one component of R. The equation of the lemma now follows from (2.8).

LEMMA 13. *With the notation of Lemma 10 we have that $\psi_* C=0$.*

Proof. On the one hand, Lemma 10 asserts that the element C is in the image of the map $A.(W'\frown R')\to A.(X')$ associated to the inclusion $W'\frown R'\to X'$ and on the other hand, the composite map $W'\frown R'\to X$ of the inclusion $W'\frown R'\to X'$ with ψ factors via the inclusion $W\frown R\to X$. Consequently, the element $\psi_* C$ in $A.(X)$ is in the image of $A.(W\frown R)$ by the map associated to the inclusion $W\frown R\to X$. However, the element C is in the graded piece $A_d(X')$ of cycles of dimension $d=\dim X'-c=\dim W$. Hence, $\psi_* C$ is in $A_d(E)$ and consequently, is in the image of an element in $A_d(W\frown R)$. However, by assumption, no component of W is contained in R. Consequently, $\dim(W\frown R)<\dim W=d$ and $A_d(W\frown R)=0$.

Proof of Theorem 8. It follows from Lemma 10 and Lemma 13 that $\psi_*(f')^*\phi^*[Z]=\psi_* A+\psi_* B$. However, $\psi_*\psi^*=id_{A(X)}$, because the morphism ψ is birational and X is smooth. Moreover, $(f')^*\phi^*=\psi^*f^*$. Hence, $\psi_*(f')^*\phi^*[Z]=f^*[Z]$ and the equation $f^*[Z]=\psi_* A+\psi_* B$ holds. From the latter equation together with Lemma 11 and Lemma 12 we obtain the following formula

$$f^*[Z]=[W]+\sum_{\mu+\nu=c-1} c_1(I(R))^\nu j_*(c_\mu(g^*N_{Z/Y})\frown[R]). \tag{2.8}$$

Moreover, from the projection formula applied to the inclusion of R in V it follows that $j_*(c_\mu(g^*N_{Z/Y})\frown[R])=r_*(c_\mu(h^*N_{Z/Y})\frown[R])$. Hence, the formula of Theorem 8 follows from (2.8).

§ 4. The rational equivalence class of the scheme of double points of a morphism

We shall in the remainder of this article fix the following notation. Let $f\colon X\to Y$ be a proper morphism of smooth schemes over k. We denote by $(X\times X)'$ the monoidal transformation of the product scheme $X\times X$ with center on the diagonal $\delta_X(X)$ and by π the structure morphism of the monoidal transformation. The conormal sheaf of the subscheme $\delta_X(X)$ of $X\times X$ is isomorphic to the sheaf Ω_X^1 of one Kähler differentials on X ([1], VI-1, Pro p.(1.13), p. 106). Hence, the exceptional locus of the monoidal transformation is isomorphic to the projective bundle $\mathbf{P}(\Omega_X^1)$ over X. We denote this bundle by $T(X)$ and its structure morphism by τ. The inclusion of $T(X)$ in $(X\times X)'$ we denote by t. Finally, we denote by $(X\times_Y X)'$ the scheme $((f\times f)\pi)^{-1}(\delta_Y(Y))=\pi^{-1}(X\times_Y X)$. The scheme $X\times_Y X$ contains $\delta_X(X)$. Consequently, the scheme $(X\times_Y X)'$ contains $T(X)$. We denote by δ and δ' the embeddings of X in $(X\times_Y X)$ and of $T(X)$ in $(X\times_Y X)'$. The schemes and morphisms defined above make up the following commutative diagram,

$$\begin{tikzcd}
(X\times X)' \arrow[rrr, "\pi"] & & & X\times X \arrow[r, "f\times f"] & Y\times Y \\
& (X\times_Y X)' \arrow[ul] \arrow[r] & X\times_Y X \arrow[ur] \arrow[rr] & & Y \arrow[u, "\delta_y"'] \\
T(X) \arrow[uu, "t"] \arrow[ur, "\delta'"'] \arrow[rrr, "\tau"'] & & & X \arrow[uu, "\delta_x"'] \arrow[ul, "\delta"] \arrow[ur, "f"'] &
\end{tikzcd} \tag{3.1}$$

Denote by $I((X\times_Y X)')$ and $I(T(X))$ the ideals defining the subschemes $(X\times_Y X)'$ and $T(X)$ of $(X\times X)'$. Then $I(T(X))$ is an invertible sheaf and $I((X\times_Y X)')I(T(X))^{-1}$

is an ideal which defines a closed subscheme $Z(f)$ of X. *We call* $Z(f)$ *the scheme of double points of* f. *The scheme theoretic image of* $Z(f)$ *by the morphism* $pr_1\pi$ *we denote by* $D(f)$ *and call the double locus of the morphism* f.

Remark 14. The set $X(k)$ of rational points of $D(f)$ is the union of the set $\{x \in X(k) \mid \exists x' \in X(k),\, x' \neq x \text{ and } f(x') = f(x)\}$, or ordinary double points of f, and the set $\{x \in X(k) \mid f \text{ is ramified at } x\}$, of ramification points of f. More precisely, the following two assertions hold,

(i) Let x and x' be two different rational points of X. Then $\pi^{-1}(x, x')$ is contained in $Z(f)$ if and only if $f(x) = f(x')$.

(ii) Let u be a rational point of $T(X)$ and put $\tau(u) = x$. Then u is contained in $Z(f)$ if and only if the tangent vector t_u in $T_X(x) = \mathrm{Hom}_{k(x)}(\Omega^1_X, k(x))$, associated to u, is mapped to zero by the homomorphism $df\colon T_X(x) \to T_Y(f(x))$ associated to f.

Proof. Assertion (i) is immediate from the definition of $Z(f)$.

To prove assertion (ii), we first note that the conormal sheaves of the diagonals in $X \times X$ and $Y \times Y$ are isomorphic to Ω^1_X and Ω^1_X ([1], VI-1, Proposition (1.13), p. 106) and that the conormal sheaf of $T(X)$ in $(X \times X)'$ is $t^*I(T(X))$. We denote the conormal bundle of $T(X)$ in $(X \times_Y X)'$ by N_1 and the conormal bundles of $X \times_Y X$ and $(X \times_Y X)'$ in $X \times X$ and $(X \times X)'$ by N_2 and N_3. Corresponding to diagram (3.1) we obtain a commutative diagram of conormal sheaves,

$$\begin{tikzcd}
0 & & \\
N_1 \arrow[u] & & \\
t^*I(T(X)) \arrow[u] & \tau^*\Omega^1_X \arrow[l] & \\
(\delta')^*N_3 \arrow[u, "\varkappa"] & \tau^*\delta^*N_2 \arrow[l] \arrow[u, "l"'] & f^*\Omega^1_Y \arrow[l]
\end{tikzcd} \tag{3.2}$$

([4], Chap. IV_4 (16.2.1), p. 10). The left vertical sequence of (3.2) is exact ([4], Chap. IV_4 Proposition (16.2.7), p. 13) and the two bottom horizontal maps are surjective ([4], Chap. IV_4 Proposition (16.2.2) (iii), p. 10). Moreover, the top horizontal map is the universal quotient map of the projective bundle $T(X) = \mathbf{P}(\Omega^1_X)$. It follows by an easy diagram chase that the composite map

$$\tau^*f^*\Omega^1_Y \to t^*I(T(X)), \tag{3.3}$$

obtained from diagram (3.2), is surjective at the point u, if and only if the map $\varkappa$ of diagram (3.2) is surjective at u. However, on the one hand the map $\Omega^1_Y(f(x)) \to k(x)$ obtained from (3.3) at the point u is the tangent vector $df(t_u)$. On the other hand, $\varkappa$ is surjective if and only if $N_1(u)$ is zero. Hence, $df(t_u)$ is different from zero if and only if $N_1(u)$ is zero. However, $N_1(u)$ is zero if and only if u is not in $Z(f)$. Indeed, by Nakayamas lemma $N_1(u)$ is zero if and only if the immersion δ' is open at the point u. However, δ' is an open immersion if and only if the inclusion $I((X \times_Y X)') = I(T(X))\, I(Z(f)) \subseteq I(T(X))$ is surjective at u. Twisting by $I(T(X))^{-1}$ we see that this happens if and only if $I(Z(f))$ is trivial at u. That is, if and only if u is not contained in $Z(f)$.

Note. It is not hard to prove that the schemes $Z(f)$ and $D(f)$ defined above are the same as the corresponding schemes defined in [7], Section 4, in the case when the scheme Y is isomorphic to a projective space $\mathbf{P}^n_k$. Most of the results proved below are generalizations of the results of the article [7] to the case when Y is an arbitrary smooth, connected and projective scheme over k. The following result generalizes Theorem 10, § 4, of [7].

THEOREM 15. *Let X and Y be smooth connected projective schemes and let $f: X \to Y$ be a morphism. Assume that the scheme $Z(f)$ of double points of f is of pure dimension equal to $2 \dim (X) - \dim (Y)$ and that no component of $Z(f)$ is contained in $T(X)$. Then the following formula holds in $A.(X)$,*

$$(pr_1)_* \pi_* [Z(f)] = f^* f_* [X] - t_c(f) \frown [X],$$

where $c = \dim (Y) - \dim (X)$ and $t_c(f)$ is the c'th Todd class of f.

We shall break up the computations needed in the proof of Theorem 15 into a series of lemmas.

LEMMA 16. *With the notation and assumptions of Theorem 15, the following formula holds in $A.((X \times X)')$,*

$$\pi^*(f \times f)^*[\delta_Y(Y)] = [Z(f)] + t_* \sum_{\mu+\nu=\dim(Y)-1} c_1(t^* I(T(X))^\nu c_\mu((f\tau)^*(\Omega^1_Y)^\vee) \frown [T(X)].$$

Proof. The morphism $(f \times f)\pi$ is a proper morphism of smooth, projective and connected schemes and $\delta_Y(Y)$ is a smooth connected subscheme of $Y \times Y$. By assumption, the inverse image of $\delta_Y(Y)$ is residual with residual divisor $T(X)$ and with a proper part $Z(f)$ and no component of $Z(f)$ is contained in $T(X)$. Consequently, by Theorem 8, the following formula holds in $A.((X \times X)')$,

$$\pi^*(f\times f)^*[\delta_Y)] = [Z(f)] + \sum_{\mu+\nu=\dim(Y)-1} c_1(I(T(X))^\nu t_*\{c_\mu((f\tau)^*(\Omega^1_Y)^\vee) \frown [T(X)]\}.$$

The formula of the lemma follows from this formula and the projection formula.

LEMMA 17. *With the notation and assumptions of Theorem 15, the following formula holds in* $A.(X\times X)$,

$$(f\times f)^*[\delta_Y(Y)] = \pi_*[Z(f)] + (\delta_X)_* \sum_{\mu+\nu=\dim(Y)-1} c_\mu(f^*(\Omega^1_Y)^\vee)\, \tau_*\{c_1(t^*I(T(X))^\nu \frown [T(X)]\}$$

Proof. The morphism π is birational and the scheme $X\times X$ is smooth. Consequently, we have that $\pi_*\pi^* = id_{A(X\times X)}$. Moreover, $\pi_* t_* = (\delta_X)_*\tau_*$. Hence, mapping both sides of the formula of Lemma 16 into $A.(X\times X)$ by π_*, we obtain the formula

$$(f\times f)^*[\delta_Y(Y)] = \pi_*[Z(f)] + (\delta_X)_*\tau_* \sum_{\mu+\nu=\dim Y-1} c_1(t^*I(T(X))^\nu c_\mu((f\tau)^*(\Omega^1_Y)^\vee) \frown [T(X)]$$

The formula of the lemma follows from this formula and the projection formula.

LEMMA 18. *With the notation and assumptions of Theorem 15, the following formula holds in* $A.(X)$,

$$(pr_1)_*(f\times f)^*[\delta_Y(Y)] = (pr_1)_*\pi_*[Z(f)] + t_c(f) \frown [X].$$

Proof. The invertible sheaf $t^*I(T(X))$ is the universal sheaf of the projective bundle $T(X) = \mathbf{P}(\Omega^1_X)$ over X. Consequently, it follows from Lemma 7 that

$$\tau_*\{t^*c_1(I(T(X)))^\nu \frown [T(X)]\} = s_{\nu-\dim X+1}(X) \frown [X].$$

Hence, the formula of Lemma 17 can be written in the following form,

$$(f\times f)^*[\delta_Y(Y)] = \pi_*[Z(f)] + (\delta_X)_* \sum_{\mu+\nu=\dim Y-1} c_\mu(f^*(\Omega^1_Y)^\vee)\, s_{\nu-\dim X+1}(X) \frown [X]. \tag{3.4}$$

Here

$$\sum_{\mu+\nu=\dim Y-1} c_\mu(f^*(\Omega^1_Y)^\vee)\, s_{\nu-\dim X+1}(X) = \sum_{\mu+\nu=c} c_\mu(f^*(\Omega^1_Y)^\vee)\, s_\nu(X) = t_c(f).$$

Hence, mapping each side of equation (3.4) into $A.(X)$ by $(pr_1)_*$ and taking into account that $pr_1\delta = id_X$ we get the formula of the lemma.

LEMMA 19. *The following two formulas hold in* $A.(X\times X)$,

(i) $(id_Y\times f)^*[\delta_Y(Y)] = [\Gamma_f]$

(ii) $f^*(pr_1)_*[\Gamma_f] = pr_1^*(f\times id_X)^*[\Gamma_f]$

where $\Gamma_f = Y\times Y$ *is the graph of the morphism* f.

Proof. Consider the two commutative diagrams,

$$\begin{tikzcd} X \times X \arrow[r, "f \times id_X"] \arrow[d, "pr_1"'] & Y \times X \arrow[d, "pr_1"] \\ X \arrow[r, "f"'] & Y \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} \Gamma_f \arrow[r, "q"] \arrow[d, "\varepsilon"'] & Y \arrow[d, "\delta_Y"] \\ Y \times X \arrow[r, "id_Y \times f"'] & Y \times Y \end{tikzcd}$$

where ε is the embedding of Γ_f in $Y \times X$ and q is the morphism induced by $id_Y \times f$. Clearly the two diagrams are cartesian. Moreover, the schemes $Y \times Y$ and $Y \times X$ are smooth and the morphisms pr_1, ε and q_Y are proper. Consequently, the following two equations hold ([2], § 2.2, Lemma (4)), $(id_Y \times f)^*(\delta_Y)_*[Y] = \varepsilon_* q^*[Y]$ and $f^*(pr_1)_*[\Gamma_f] = (pr_1)_*(f \times id_X)^*[\Gamma_f]$. The formula of the lemma follows from these equations together with the obvious equalities $(\delta_Y)_*[Y] = [\delta_Y(Y)]$ and $\varepsilon_* q^*[Y] = \varepsilon_*[\Gamma_f] = [\Gamma_f]$.

LEMMA 20. *The following formula holds in* $A.(X)$,

$$f^* f_*[X] = (pr_1)_*(f \times f)^*[\delta_Y(Y)]$$

Proof. As a consequence of formula (i) and (ii) of Lemma 19 we obtain the formula, $f^*(pr_1)_*[\Gamma_f] = pr^*(f \times id_X)^*(id_Y \times f)^*[\Gamma_f]$. The formula of the lemma is a consequence of this formula together with the obvious equalities $(pr_1)_*[\Gamma_f] = f_*[X]$ and $(id_Y \times f)(f \times id_X) = f \times f$.

The formula of Theorem 15 is an immediate consequence of Lemma 18 and Lemma 20.

§ 5. The rational equivalence class of the double locus of a morphism

In section three, we defined the double locus $D(f)$, of a morphism f, as the direct image of the scheme of double points $Z(f)$ by the morphism $(pr_1\pi)$. Moreover, we gave (Theorem 15) a formula for the rational equivalence class $(pr_1\pi)_*[Z(f)]$. Consequently, when the relation

$$(pr_1\pi)_*[Z(f)] = [D(f)] \tag{4.1}$$

holds we obtain a formula for the rational equivalence class of $D(f)$. However, let $Z_1(f), \ldots, Z_q(f)$ be the irreducible components of $D(f)$ and let $(pr_1\pi)(Z_\nu(f)) = D_\nu(f)$ be the corresponding irreducible components of $D(f)$. Then $(pr_1\pi)_*[Z_\nu(f)] = d_\nu[D_\nu(f)]$ where $d_\nu \geqslant 0$ is the degree of the morphism $\varrho_\nu \colon Z_\nu(f) \to D_\nu(f)$ and consequently, in general, the following relation holds

$$(pr_1\pi)_*[Z(f)] = \sum_{\nu=1}^{q} d_\nu[D_\nu(f)].$$

We see that a necessary condition for the relation (4.1) to hold is that all the morphisms

$\varrho_1, \ldots, \varrho_q$ are birational and that all the components $D_1(f), \ldots, D_q(f)$ are different, in other words, that the morphism $\varrho: Z(f) \to D(f)$, induced by $pr_1\pi$, is birational The main object of section four is to show that when the morphism f satisfies the following three conditions, then the morphism ϱ is birational,

(G_1) The morphism $f: X \to Y$ is finite.

(G_2) Let Y'' be the (closed) subset of $Y' = f(X)$ over which f does not induce an isomorphism. Then Y'' is empty or of pure dimension $2 \dim (X) - \dim (Y)$ and for each point y in an open dense subset of Y'' the fiber $f^{-1}(y)$ consists of exactly two different reduced points.

(G_3) The scheme $Z(f)$ of double points of f is of pure codimension $\dim (Y)$ in $(X \times X)'$ and no component of $Z(f)$ is contained in $T(X)$.

Note. The conditions (G_1), (G_2) and (G_3) were introduced in [7] (§ 4, (4.12)) in the case when Y is isomorphic to a projective space. We proved there ([7], Proposition 17) that when there is a sufficiently twisted embedding (see [7], Proposition 17 for details) of X into a projective space $\mathbf{P}^N$, such that the morphism f is induced by a generic projection, then f satisfies the conditions (G_1), (G_2) and (G_3). We also proved that this was true, even without twisting the embedding, in certain important cases. As a consequence of these results and a formula of J. A. Todd ([7], Theorem 13) we obtained ([7], Theorem 19 and Theorem 20) generalizations of some recent result of A. Holme and of C. A. M. Peters and J. Simonis about secants of projective schemes ([5], II, Theorem 4.2 and [9], Theorem (3.4)). See also [6], Theorem 20 and Theorem 21). The main reason for choosing the conditions (G_1), (G_2) and (G_3) to other possible transversality conditions on f, is that they are easy to verify for morphism induced by projections.

We shall below generalize Theorem 22 and Todd's formula (Theorem 13) of [7] to the case when Y is an arbitrary smooth projective scheme.

PROPOSITION 21. *Let x be a rational point of $D(f)$. Then the complement of x in the fiber $f^{-1}(f(x))$ is isomorphic to the fiber over x of the morphism $(Z(f) - Z(f) \cap T(X)) \to D(h)$ induced by $pr_1\pi$.*

Proof. The morphism π induces an isomorphism from the scheme $(X \times X)' - T(X)$ onto the scheme $(X \times X) - \delta_X(X)$ and under this isomorphism the scheme $Z(f) - Z(f) \cap T(X)$ is mapped isomorphically onto the inverse image D of $\delta_Y(Y)$ by the morphism $(f \times f) | (X \times X - \delta_X(X))$. Consequently, Proposition 21 asserts that the fiber of the morphism $p: D \to X$, induced by pr_1, over the point x is isomorphic to the scheme $(f^{-1}f(x) - x)$. However, $p^{-1}(x) = (x \times X) \cap D$, and $(x \times X) \cap D$ is the inverse image of $\delta_Y(Y)$ by the morphism

$(x \times f)$: $(x \times X - (x, x)) \to Y \times Y$ induced by $f \times f$. Moreover, the morphism $(x \times f)$ factors via the inclusion $f(x) \times Y \to Y \times Y$ and the inverse image of $\delta_Y(Y)$ by the latter morphism is the point $(f(x), f(x))$. Consequently, $(x \times D) \cap X$ is the pull back $(f^{-1}f(x) - x)$ of the point $(f(x), f(x))$ by the morphism $x \times X \to f(x) \times Y$.

LEMMA 22. *Denote by ϱ: $Z(f) \to D(f)$ the morphism induced by $(pr_1 \pi)$. Assume that the morphism f satisfies the conditions (G_1) and (G_3). Then the restriction of ϱ to each component of $Z(f)$ is generically finite.*

Proof. Denote by Z_0 the open subscheme $(Z(f) - Z(f) \cap T(x))$ of $Z(f)$. Then by (G_3) the scheme Z_0 is dense in $Z(f)$. Let $Z_\nu(f)$ be a component of $Z(f)$ and put $Z_1 = Z_0 \cap Z_\nu(f)$. It follows from Proposition 21 and (G_1) that the restriction $\varrho | Z_1$ is quasi finite. However, $\varrho | Z_\nu(f)$ is proper. Hence, $\varrho | Z_\nu(f)$ is generically finite.

As a consequence of Lemma 22 and Theorem 15, we obtain a criterion for a morphism satisfying the conditions of Corollary 22 to be an embedding. The criterion we give generalizes an earlier improvement ([7], § 5, Theorem 20) of a result of A. Holme ([5], II, Theorem 4.2). We first need the following well-known lemma.

LEMMA 23. *The morphism f is an embedding if and only if $Z(f)$ is empty.*

Proof. We may assume that the morphism f is finite. Indeed, by Remark 14 (i), $D(f)$ contains every fiber of f of dimension at least equal to one. Moreover, by Remark 14 (i) and (ii), the scheme $Z(f)$ is empty if f is an embedding.

Conversely, assume that $Z(f)$ is empty. Let Spec A be an open affine subset of X which is mapped into an open affine subset Spec B of Y, by the morphism f. The corresponding ring homomorphism $B \to A$ makes A into a finite B-module. By Remark 14 (i) there is only one maximal ideal M_A of A lying over a maximal ideal M_B of B. Moreover, by Remark 14 (ii) the morphism f is unramified at the point corresponding to M_A. Consequently, the natural homomorphism $M_B/M_B^2 \to M_A/M_A^2$ is surjective ([1], VI, Prop. (3.6) (i), p. 114). It follows by Nakayama's lemma that $M_A = M_B A$. However, since the ground field is algebraically closed, the map $B/M_B \to A/M_A = A/M_B A$ is surjective. Consequently, by Nakayamas lemma, the map $B \to A$ is surjective. That is, f is a closed embedding.

THEOREM 24. *Let X and Y be smooth, connected and projective schemes. Assume that the morphism f: $X \to Y$ satisfies the above conditions (G_1) and (G_3). Then f is a closed embedding if and only if the following relation holds in $A.(X)$*

$$f^* f_* [X] = t_c(f) \frown [X].$$

Here $c = \dim(Y) - \dim(X)$ and $t_c(f)$ is the c'th Todd class of f.

Proof. By Lemma 23 the morphism f is a closed embedding if and only if $Z(f)$ is empty. However, the scheme $(X \times X)'$ is projective. Hence, $Z(f)$ is empty if and only if $[Z(f)]$ is zero in $A.((X \times X)')$. By Lemma 22, the element $[Z(f)]$ is zero if and only if $(pr_1\pi)_*[Z(f)]$ is zero in $A.(X)$. Consequently, the assertion of Theorem 24 follows from Theorem 15.

PROPOSITION 25. *Assume that the morphism $f\colon X \to Y$ satisfies the above conditions (G_1), (G_2) and (G_3). Then the morphism $\varrho\colon Z(f) \to D(f)$ induced by $(pr_1\pi)$ is birational.*

Proof. Denote by Z_0 the open subscheme of $(Z(f) - Z(f) \cap T(X))$ of $Z(f)$. Then by (G_3) the scheme Z_0 is dense in $Z(f)$. Moreover, by Lemma 22 the set $\varrho(Z_0)$ is dense in $D(f)$ and $\dim (D(f)) = \dim (Z_0)$. By (G_3) $\dim (Z_0) = 2 \dim (X) - \dim (Y)$.

Let z be a rational point of Z_0 and put $x = \varrho(z)$. Then $z = \pi^{-1}(x, x')$ where x and x' are different rational points of X and $f(x) = f(x')$. Hence, $f(x) \in Y''$. We conclude that $\varrho(Z_0) = Y''$ and since $\varrho(Z_0)$ is dense in $D(f)$ we have that $D(f) \subseteq f^{-1}(Y'')$. However, we have seen that $\dim (D(f)) = 2 \dim (X) - \dim (Y)$ and from (G_2) it follows that $\dim f^{-1}(Y'') \leqslant 2 \dim (X) - \dim (Y)$. Consequently, each component of $D(f)$ is also a component of $f^{-1}(Y'')$.

Denote by U the open dense subset of Y'' over which the fibers of f consists of two reduced points and put $U_0 = f^{-1}(U) \cap D(f)$. Then U_0 is an open dense subset of $D(f)$. Indeed, since the components of $D(f)$ are also components of $f^{-1}(Y'')$ and f is finite, each component of $D(f)$ dominates a component of Y''. Let x be a rational point of U_0. Then x is a reduced point of the fiber $f^{-1}f(x)$ and consequently f is not ramified at the point x ([1], VI, Proposition (3.6) (ii), p. 114). It follows from Remark 14 (ii) that $\varrho^{-1}(x) \in Z_0$. Moreover, the fiber $f^{-1}f(x)$ consists of exactly one more reduced point. We conclude from Proposition 21 that the fiber $\varrho^{-1}(x)$ consists of exactly one reduced point. However, the morphism ϱ is proper and $D(f)$ is the scheme theoretic image of $Z(f)$ by $(pr_1\pi)$. We conclude that the morphism ϱ is an isomorphism in a neighborhood of the point $\varrho^{-1}(x)$. Consequently, ϱ induces an isomorphism over the open dense subset U_0 of $D(f)$. Finally, the open set $\varrho^{-1}(U)$ is dense in $Z(f)$. Indeed, we have noted that $\varrho | Z_0$ is quasi-finite and that $\varrho(Z_0)$ is dense in $D(f)$. Consequently, $\varrho^{-1}(U_0) \cap Z_0$ is dense in Z_0 and hence in $Z(f)$.

In view of the discussion at the beginning of this section the main result of section five is an immediate consequence of Theorem 15 and Proposition 25.

THEOREM 26. (*J. A. Todd* [10], § 7.1, *Theorem* (B_k), *p.* 224). *Let X and Y be smooth, connected and projective schemes and let $f\colon X \to Y$ be a morphism satisfying the above conditions (G_1), (G_2) and (G_3). Then the following formula, for the rational equivalence class of the double locus $D(f)$ of f, holds in $A.(X)$,*

$$[D(f)] = f^*f_*[X] - t_c(f) \frown [X],$$

where $c = \dim (Y) - \dim (X)$ and $t_c(f)$ is the c'th Todd class of f.

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