Viscosity solutions of fully nonlinear second-order equations and optimal stochastic control in infinite dimensions. Part I: the case of bounded stochastic evolutions

by

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I. Introduction

We study here fully nonlinear second-order degenerate elliptic equations of the following form

$$F(D^2u, Du, u, x) = 0 \quad \text{in } H \tag{1}$$

where H is a separable Hilbert space, x denotes a generic point in H, u—the unknown—is a function from H into $\mathbf{R}$, Du and D^2u denote the first and second Fréchet differentials that we identify respectively with elements of H, and symmetric bounded bilinear forms over H or indifferently bounded symmetric operators on H. We will denote by $L'(H)$ the space of all symmetric bounded bilinear forms over H and we will always assume at least that

$$F \text{ is bounded, uniformly continuous on bounded sets of } L'(H)\times H\times \mathbf{R}\times H. \tag{2}$$

By degenerate ellipticity we mean that F satisfies

$$F(A,p,t,x) \leq F(B,p,t,x), \quad \forall A \geq B, \quad \forall p \in H, \quad \forall t \in \mathbf{R}, \quad \forall x \in H \tag{3}$$

where $A \geq B$ is defined by the partial ordering of the quadratic forms associated with A, B i.e.

$$A \geq B \quad \text{if and only if} \quad (Ax,x) \geq (Bx,x) \quad \text{for all} \quad x \in H \tag{4}$$

denoting by (x,y), $|x|$ respectively the scalar product and the norm of H.

The main motivation for studying such equations is the study of optimal stochastic control problems and their associated Hamilton-Jacobi-Bellman equations (HJB in short). We will explain in section III the precise infinite dimensional stochastic control problems we consider here. Let us only mention at this stage that it is well-known that a powerful approach to optimal stochastic control problems is the so-called dynamic programming method—initially due to R. Bellman—which, in particular, indicates that the value function (or minimum cost function) of general control problems should be "the solution" of an equation of the form (1) namely the HJB equation—see for more details W. H. Fleming and R. Rishel [12], A. Bensoussan [1], N. V. Krylov [22], P. L. Lions [25]. The essential feature of HJB equations in the general context of equations (1) is that F is convex with respect to D^2u (in fact (D^2u, Du, u)) and a typical form is

$$\sup_{\alpha \in A} \left[-\sum_{i,j=1}^{\infty} a_{ij}^{\alpha}(x)\, \partial_{ij} u - \sum_{i=1}^{\infty} b_i^{\alpha}(x)\, \partial_i u + c^{\alpha}(x)\, u - f^{\alpha}(x) \right] = 0 \quad \text{in } H \tag{5}$$

with appropriate conditions on the coefficients a_{ij}^{α}, b_i^{α}, c^{α}, f^{α}, where A is a fixed set (of values of controls), where we identified x with $(x_1, x_2, x_3, \ldots) \in l^2$ choosing an orthonormal basis $(e_1, e_2, e_3, \ldots)$ of H and where $\partial_{ij} u$, $\partial_i u$ denote the partial derivatives of u.

In section II below, we present a notion of weak solutions of (1) that we call viscosity solutions since this notion is clearly adapted from the notion introduced by M. G. Crandall and P. L. Lions [4], [5] for finite-dimensional problems or infinite-dimensional first-order problems. We also explain how a few "classical" properties of viscosity solutions may be carried out in this infinite-dimensional setting and we refer to [4], [5], P. L. Lions [26], [27], M. G. Crandall, L. C. Evans and P. L. Lions [7] for more detailed properties in the "standard cases".

Then, in section III, we introduce the class of stochastic control problems in infinite dimensions we will be studying. And we will show various properties of the value function such as regularity properties. In some vague sense, these results are the infinite-dimensional analogues of those obtained in P. L. Lions [27].

Next, in section IV, we check that the value function is the unique viscosity solution of the associated HJB equation. This verification theorem will not be obtained by a purely PDE argument (even if it is possible to "translate" it into purely PDE steps . . .) and is more in the spirit of the results obtained by P. L. Lions [27] for finite-dimensional problems.

Since we will be studying in sections III and IV model problems (with severe restrictions on the coefficients) we briefly explain in section V how to weaken some of the assumptions required in the preceding sections.

At this stage, we would like to point out that even if the results presented here are somewhat analogous to those known in finite dimensions, the methods for proving them are quite different and many considerable "technical" difficulties appear.

Let us also mention that various attempts to use dynamic programming arguments for infinite dimensional stochastic control problems have been already made, leading essentially to the construction of nonlinear semigroups (equivalent formulations of the optimality principle) and we refer, for instance, to A. Bensoussan [2], W. H. Fleming [13], Y. Fujita [14], Y. Fujita and M. Nisio [15], M. Kohlmann [21], G. Da Prato [8, 9]. Most of these works deal with the particular case of the optimal control of certain stochastic partial differential equations: a very important particular case since it contains the optimal control of Zakaï's equation which is the basic object of interest for the classical optimal control of stochastic differential equations with partial observations. However, such situations introduce the additional difficulty of unbounded terms in the HJB equations, terms that require appropriate modifications of the arguments. For deterministic problems, similar difficulties were solved in M. G. Crandall and P. L. Lions [6]. Therefore, in order to keep the ideas clear, we will treat such cases in Part II ([30]).

We would like to conclude this introduction by a few comments on the structure of proofs concerning uniqueness results of viscosity solutions of second-order equations. In finite dimensions, except for [27] which is the guide line for our analysis here, general uniqueness results for second-order equations have been recently obtained by R. Jensen [19]; R. Jensen, P. L. Lions and P. E. Souganidis [20]; P. L. Lions and P. E. Souganidis [31]; H. Ishii [16]; H. Ishii and P. L. Lions [17]. All these proofs use in a fundamental way the existence of second-order expansions at almost all points for convex or concave functions on $\mathbf{R}^N$ ($N<\infty$): a classical result due to Alexandrov, whose counterpart in infinite dimensions is not clear and this seems to prevent a straightforward adaptation of these arguments to infinite dimensions. However, some of the arguments that we use in the next sections indicate that a rather weak version of this differentiability result is needed. We hope to come back on this point in a future publication.

II. Viscosity solutions for second-order equations in infinite dimensions

The notion of viscosity solutions of (1) will be adapted from the corresponding notions in finite dimensions. The main difference will be in the choice of test functions: we will work with the following space of functions

$$\begin{aligned} X = \{\varphi \in C^1(H;\mathbf{R});\ & D\varphi \text{ is Lipschitz on bounded sets of } H; \\ & \text{for all } h,k\in H,\ \lim_{t\to 0+} (1/t)(D\varphi(x+tk)-D\varphi(x),h) \text{ exists} \\ & \text{and is uniformly continuous on bounded sets of } H\}. \end{aligned} \tag{6}$$

By elementary differential calculus considerations, one checks easily that if $\varphi\in X$ then we have

$$\begin{aligned} \lim_{t\to 0+} \frac{1}{t}(D\varphi(x+tk)-D\varphi(x),h) &= \lim_{\substack{t\to 0\\ t\neq 0}} \frac{1}{t}(D\varphi(x+tk)-D\varphi(x),h) \\ &= (A(x)h,k), \quad \forall x,h,k\in H \end{aligned}$$

where $A(x)\in L'(H)$, $\|A(x)\|$ is bounded by the Lipschitz constant of $D\varphi$ on balls of H and $A(x_n)\underset{n}{\to} A(x)$ pointwise if $x_n \underset{n}{\to} x$ in H. Furthermore, the limits above are uniform on bounded sets of H.

Let us also remark that one can replace in (6) the condition on directional derivatives by the following conditions: $\partial_{ij}\varphi$ exists and is continuous on bounded sets of H for all $1\leq i,j<\infty$, or $\partial_i D\varphi(x)$ exists and is continuous on bounded sets of H for all $1\leq i<\infty$, where ∂_i denotes the partial derivation with respect to x_i and $x_1,x_2,x_3,\ldots$ are the coordinates of x with respect to an arbitrary orthonormal basis $(e_1,e_2,e_3,\ldots)$ of H. In all that follows, we will denote by $D^2\varphi(x)=A(x)$. Denoting by $\mathrm{BUC}_{\mathrm{loc}}(H)=\{u\in C(H)$, u is bounded uniformly continuous on balls of $H\}$, we may now give the

Definition II.1. Let $u\in \mathrm{BUC}_{\mathrm{loc}}(H)$. We will say that u is a viscosity subsolution (resp. supersolution) of (1) if the following holds for each $\varphi\in X$

$$\begin{gathered} \text{at each local maximum } x_0 \text{ of } u-\varphi, \text{ we have} \\ \liminf_{y\to x_0} F(D^2\varphi(y), D\varphi(x_0), u(x_0), x_0) \leq 0 \end{gathered} \tag{7}$$

(resp.

$$\text{at each local minimum } x_0 \text{ of } u-\varphi, \text{ we have} \quad \limsup_{y\to x_0} F(D^2\varphi(y), D\varphi(x_0), u(x_0), x_0) \geq 0). \tag{8}$$

And we will say that u is a viscosity solution of (1) if u is both a viscosity supersolution and subsolution of (1).

Remarks. (i) If H is finite dimensional, then $X=C^2(H)$ and the above definition is nothing but the usual one.

(ii) We may replace local by global, or local strict, or global strict where by strict we mean that $(u-\varphi)(x)\leq(u-\varphi)(x_0)-\omega(|x-x_0|)$ where $\omega(t)>0$ if $t>0$.

(iii) Let us remark that in view of (2), the definition of X and $\mathrm{BUC}_{\mathrm{loc}}(H)$ it is possible to replace in (7) (for instance)

$$\liminf_{y\to x_0} F(D^2\varphi(y), D\varphi(x_0), u(x_0), x_0) \quad \text{by} \quad \liminf_{y\to x_0} F(D^2\varphi(y), D\varphi(y), u(y), y).$$

(iv) Let us finally warm the expert reader that this definition is motivated by the optimal control problems treated here (and in Part II [30]) but might require some minor modification in the case of (very general) stochastic differential games in infinite dimensions (unless of course the above notion is equivalent in general to the classical one recalled below).

It will be useful to compare the above notion with more usual ones which involve either the class $X'=\{\varphi\in C^2(H,\mathbf{R}),\ \varphi,\ D\varphi,\ D^2\varphi\in \mathrm{BUC}_{\mathrm{loc}}(H)\}$ or subsuper differentials in the following sense

$$D_+^2 u(x_0)=\Big\{(A,p)\in L'(H)\times H;\ \limsup_{y\to x_0}\big[\{u(y)-u(x_0)-(p,u-x_0)-\tfrac{1}{2}(A(y-x_0),y-x_0)\}\cdot|x_0-y|^{-2}\big]\leq 0\Big\} \tag{9}$$

$$D_-^2 u(x_0)=\Big\{(A,p)\in L'(H)\times H;\ \liminf_{y\to x_0}\big[\{u(y)-u(x_0)-(p,y-x_0)-\tfrac{1}{2}(A(y-x_0),y-x_0)\}\cdot|x_0-y|^{-2}\big]\geq 0\Big\}. \tag{10}$$

To simplify notations, we will say that $u\in \mathrm{BUC}_{\mathrm{loc}}(H)$ is a classical viscosity subsolution (resp. supersolution) of (1) if (7) (resp. (8)) holds for all $\varphi\in X'$ or equivalently (see [27] for the proof of this assertion in finite dimensions which adapts trivially to our case) if the following holds

$$F(A, p, u(x_0), x_0) \leq 0, \quad \forall (A, p) \in D^2_+ u(x_0), \quad \forall x_0 \in H \tag{11}$$

(resp.

$$F(A, p, u(x_0), x_0) \geq 0, \quad \forall (A, p) \in D^2_- u(x_0), \quad \forall x_0 \in H). \tag{12}$$

The following result gives some condition on F under which both notions are equivalent—observe that clearly a viscosity (sub, super) solution is always a classical viscosity (sub, super) solution.

PROPOSITION II.1. *Let $u \in \mathrm{BUC}_{\mathrm{loc}}(H)$ be a classical viscosity subsolution (resp. supersolution) of* (1). *Then, u is a viscosity subsolution (resp. supersolution) of* (1) *if F satisfies the following condition: there exists an increasing sequence of finite dimensional subspaces H_N of H such that $\bigcup_N H_N$ is dense in H and*

$$\overline{\lim_{\delta \to 0+}}\ \overline{\lim_{N}} \left(F(A, p, t, x) - F\left(\tfrac{1}{2} A P_N + \tfrac{1}{2} P_N A + \delta P_N + \frac{C}{\delta} Q_N, p, t, x\right)\right)^+ = 0 \tag{13}$$

$$\overline{\lim_{\delta \to 0+}}\ \overline{\lim_{N}} \left(F(A, p, t, x) - F\left(\tfrac{1}{2} A P_N + \tfrac{1}{2} P_N A - \delta P_N - \frac{C}{\delta} Q_N, p, t, x\right)\right)^- = 0 \tag{14}$$

for all $x \in H$, $t \in \mathbf{R}$, $p \in H$, $A \in L'(H)$, $C \geq 0$, where P_N, Q_N denote respectively the orthogonal projections onto H_N, $H_N^{\perp}$.

Remarks. (1) The proof below shows that, in fact, (7) (resp. (8)) holds for all $\varphi \in C^1(H, \mathbf{R})$ such that $D\varphi$ is locally Lipschitz,

$$\lim_{t \to 0+} \frac{1}{t} (D\varphi(x + tk) - D\varphi(x), h)$$

exists and is continuous on H ($\forall h, k \in H$) whenever (13) (resp. (14)) holds.

(2) The assumptions (13) or (14) are not always satisfied for natural examples of F. For instance, if $F = \sup_{|\xi|=1} [-(A\xi, \xi)] + \tilde{F}(p, t, x)$, (13) holds while (14) does not hold (take $A = 0$ for instance ...).

(3) Actually, the proof below shows that (13) (resp. (14)) implies that (7) (resp. (8)) holds with $D^2\varphi(y)$ replaced by $D^2\varphi(x)$, that is a stronger property holds. It is therefore plausible that, in general, both notions coincide but we have been unable to prove it (even if it is possible to prove the equivalence between the classical notion and weaker

formulations than (7)–(8) involving similar "relaxation" ideas). At this point, it may be useful to give an example of a function φ belonging to X but not to X': take $H=l^2$, $(x=(x_n)_{n\geq 1})$ and

$$\Phi(x)=\sum_{n\geq 1}\frac{1}{n^2}\varphi(nx_n)$$

where (for instance) $\varphi\in C^2(\mathbf{R})$, $\varphi''>0$ on $\mathbf{R}$,

$$\lim_{t\to-\infty}\varphi''(t)<\varphi''(t)<\lim_{t\to+\infty}\varphi''(t)\quad\text{on }\mathbf{R}.$$

Then, Φ is convex, belongs to $X\cap C^{1,1}(H)$ and for all $x, h, k\in H$

$$(D^2\Phi(x)\,h,k)=\sum_{n\geq 1}\varphi''(nx_n)\,h_n k_n.$$

Clearly, $D^2\Phi(0)=\varphi''(0)I$

$$\left(D^2\Phi\left(\frac{1}{\sqrt{n}}e_n\right)e_n,e_n\right)=\varphi''(\sqrt{n})\underset{n}{\to}\varphi''(+\infty),$$

so $D^2\Phi(x)$ is not continuous at 0 (in the $L(H)$ topology).

In fact, this example provides a convex, $C^{1,1}$ function Φ (belonging to X) which has *nowhere* a second-order expansion (i.e. $\Phi(x+h)=\Phi(x)+(D\Phi(x),h)+\frac{1}{2}(Ah,h)+O(|h|^2)$ for some $A\in L'(H)$).

Proof of Proposition II.1. We will prove only the subsolution part since the supersolution part is proved by the same argument. We thus take $\varphi\in X$ such that $u-\varphi$ has a local maximum at x_0, hence there exists $\delta>0$ such that

$$u(x)\leq u(x_0)+\varphi(x)-\varphi(x_0),\quad\text{if}\quad |x-x_0|\leq\delta.$$

Therefore, we have for $|x-x_0|\leq\delta$

$$u(x)\leq u(x_0)+(D\varphi(x_0),x-x_0)+\frac{1}{2}\int_0^1(D^2\varphi(x_0+t(x-x_0))\,(x-x_0),x-x_0)\,dt$$

and we denote by $\bar{A}(x)=\int_0^1 D^2\varphi(x_0+t(x-x_0))\,dt$. We next observe that

$$(\bar{A}(x)\,(x-x_0),x-x_0)\leq(\bar{A}(x)\,P_N(x-x_0),x-x_0)+C|P_N(x-x_0)||Q_N(x-x_0)|+C|Q_N(x-x_0)|^2$$

where C denotes various constants independent of x and N. And we deduce from the properties of φ that for $|x-x_0|\leq\delta$

$$(\bar{A}(x)(x-x_0), x-x_0) \leq (D^2\varphi(x_0) P_N(x-x_0), x-x_0)+\varepsilon_N(|x-x_0|)|x-x_0| \times|P_N(x-x_0)|+\delta|P_N(x-x_0)|^2+\frac{C}{\delta}|Q_N(x-x_0|^2$$

where $\varepsilon_N(\sigma)\to 0$ as $\sigma\to 0+$. Next, we observe that (see [27] for more details) there exists $\psi_N\in C^2(\mathbf{R})$ such that $\psi_N(0)=\psi_N'(0)=\psi_N''(0)=0$ and

$$\varepsilon_N(|x-x_0|)\,|x-x_0||P_N(x-x_0)| \leq \varepsilon_N(|x-x_0|)|x-x_0|^2 \leq \psi_N(|x-x_0|).$$

Therefore, we have finally for $|x-x_0|\leq\delta$

$$u(x)\leq u(x_0)+(D\varphi(x_0), x-x_0)+\tfrac{1}{2}(D^2\varphi(x_0)P_N(x-x_0), x-x_0) +\tfrac{1}{2}\delta(P_N(x-x_0), x-x_0)+\frac{1}{2}\cdot\frac{C}{\delta}(Q_N(x-x_0), x-x_0)+\psi_N(|x-x_0|).$$

We may now apply (11) to deduce, denoting by $A=D^2\varphi(x_0)$, $p=D\varphi(x_0)$, $t=u(x_0)$

$$F\left(\tfrac{1}{2}AP_N+\tfrac{1}{2}P_N A+\delta P_N+\frac{C}{\delta}Q_N, p, t, x_0\right)\leq 0$$

and this yields (7), letting $N\to\infty$, $\delta\to 0$ and using (13). □

We conclude this section with a stability (or consistency) result that we state only for subsolutions and we leave to the reader the easy adaptation to supersolutions.

PROPOSITION II.2. *Let* $u_n\in \mathrm{BUC}_{\mathrm{loc}}(H)$ *be a viscosity subsolution of*

$$F_n(D^2u_n, Du_n, u_n, x)=0 \quad \text{in } H, \quad n\geq 1 \tag{15}$$

for some F_n *bounded, uniformly continuous on bounded sets of* $L'(H)\times H\times\mathbf{R}\times H$. *We assume that there exist* $u\in\mathrm{BUC}_{\mathrm{loc}}(H)$ F *bounded, uniformly continuous on bounded sets of* $L'(H)\times H\times\mathbf{R}\times H$ *such that*

$$u_n(x)\underset{n}{\to} u(x) \quad \text{for all} \quad x\in H, \quad \overline{\lim_n}\, u^n(x_n)\leq u(x) \quad \text{if} \quad x_n\underset{n}{\to} x \quad \text{in } H \tag{16}$$

$$\underline{\lim}_n\, F_n(A_n, p_n, t_n, x_n)\geq \underline{\lim}_n\, F(A_n, p, t, x) \tag{17}$$

if A_n *is bounded in* $L'(H)$, $p_n\underset{n}{\to}p$ *in* H, $t_n\underset{n}{\to}t$ *in* $\mathbf{R}$, $x_n\underset{n}{\to}x$ *in* H.

Then, u is a viscosity subsolution of (1).

Proof. We just sketch the proof since it is a straightforward adaptation of the corresponding argument for first-order problems given in M. G. Crandall and P. L. Lions [5]. Indeed, let $\varphi \in X$, $x_0 \in H$ be such that $u-\varphi$ has a local maximum at x_0; replacing if necessary φ by $\varphi+|x-x_0|^4$ we may assume without loss of generality that there exists $\delta>0$ such that

$$(u-\varphi)(x) \leq (u-\varphi)(x_0)-|x-x_0|, \quad \textit{if} \quad |x-x_0| \leq \delta.$$

Then, exactly as in [5], we deduce the existence of $x_n \in B(x_0, \delta)$, $p_n \in H$ such that $u_n(x)-\varphi(x)+(p_n, x)$ has a local maximum at x_n for n large enough and $x_n \underset{n}{\to} x_0$, $u_n(x_n) \underset{n}{\to} u(x_0)$, $p_n \underset{n}{\to} 0$. (This is an easy consequence of the general perturbed optimisation results due to C. Stegall [34], I. Ekeland and G. Lebourg [11], J. Bourgain [3].) Therefore, applying (7), we see that there exists $y_n \underset{n}{\to} x_0$ such that

$$F_n(D^2\varphi(y_n), D\varphi(x_n)+p_n, u_n(x_n), x_n) \leq \frac{1}{n}.$$

And we conclude easily using (17). □

Let us make a few final comments on the arguments introduced in this section: first of all, everything we said extends trivially to the case of equations set in an *open set Q* of H instead of H itself. Next, as usual, we consider *"Cauchy" problems* of the form

$$\frac{\partial u}{\partial t}+F(D_x^2 u, D_x u, u, x, t)=0 \quad \text{in} \quad H\times(0,\infty)$$

as special cases of (1) where the equation takes place now in an open set $Q=H\times(0,\infty)$ of $\tilde{H}=H\times\mathbf{R}$ and where H is replaced by $\tilde{H}$, x by (x,t) ....

III. Optimal stochastic control in infinite dimensions

III.1. Notations and assumptions

We will be considering two examples of optimal control of "diffusion-type" processes in infinite dimensions: namely, discounted infinite horizon and finite horizon problems. Furthermore, to simplify the presentation and keep the ideas clear we will not try to make the most general assumptions on the coefficients and in the case of finite horizon problems we will assume that the coefficients are not time-dependent.

Let us now introduce the main notations and assumptions. Let $\mathscr{A}$ be a complete metric space, let V be a separable Hilbert space and let us denote by $(\bar{e}_1, \bar{e}_2, \bar{e}_3, \ldots)$ an orthonormal basis of V. An admissible (control) system $\mathscr{S}$ will be the collection of: (i) a probability space (Ω, F, F^t, P) with a right-continuous filtration of complete sub-σ fields F^t. (ii) a V-valued Brownian motion W_t, that is W_t is continuous, F^t-measurable, and $((W_t, \bar{e}_t))_n$ is a sequence of independent one-dimensional Brownian motions, (iii) a progressively measurable process α_t taking values in a compact subset of $\mathscr{A}$. Let us mention that we could as well fix the probability space and W_t. Then, for each $\mathscr{S}$ and for each $x \in H$, the state process X_t will be the continuous, F^t-adapted solution of the following stochastic differential equation in H (written in Itô's form)

$$dX_t = \sigma(X_t, \alpha_t) \cdot dW_t + b(X_t, \alpha_t)\, dt \quad \text{for} \quad t \geq 0, \quad X_0 = x, \tag{18}$$

where σ and b satisfy assumptions listed below which will insure in particular the existence and uniqueness of a solution of (18).

For each system $\mathscr{S}$, and for all $x \in H$, $t \geq 0$ we consider some cost functions and the associated minimal cost functions—the value functions. In the infinite horizon case, we consider

$$J(x, \mathscr{S}) = E \int_0^\infty f(X_t, \alpha_t) \exp\left(-\int_0^t c(X_s, \alpha_s)\, ds\right) dt \tag{19}$$

$$u(x) = \lim_{\mathscr{S}} J(x, \mathscr{S}), \quad \forall x \in H \tag{20}$$

while in the finite horizon case, we introduce

$$J(x, t, \mathscr{S}) = E \int_0^t f(X_s, \alpha_s) \exp\left(-\int_0^s c(X_\tau, \alpha_\tau)\, d\tau\right) ds + g(X_t) \exp\left(-\int_0^t c(X_s, \alpha_s)\, ds\right) \tag{21}$$

$$u(x, t) = \inf_{\mathscr{S}} J(x, t, \mathscr{S}), \quad \forall x \in H, \quad \forall t \geq 0 \tag{22}$$

where the infima are taken over all admissible systems $\mathscr{S}$. Here and below, f, g are given functions which satisfy conditions listed below that insure in particular that formula (19)–(22) are meaningful.

In all that follows (even if some of these assumptions are not necessary for most of the results presented in sections III and IV) we will assume that σ, b, f, c, g satisfy the assumptions that we detail now. First of all, for each $(x, \alpha) \in H \times \mathscr{A}$, $\sigma(x, \alpha) \in \mathscr{H}(V, H)$ that we define to be the Hilbert space contained in $L(V, H)$ (bounded linear operators from V into H) composed of those elements σ such that $\mathrm{Tr}(\sigma\sigma^*)<\infty$: $\mathscr{H}(V,H)$ is a Hilbert space for the scalar product $\mathrm{Tr}(\sigma_1\sigma_2^*)$, where Tr denotes the trace.

Then we assume

$$\begin{gathered}\sigma \in \mathrm{BUC}(H\times\mathscr{A};\mathscr{H}(V,H)),\\ D_x\sigma(\cdot,\alpha)\in C_b^{0,1}(H;L(H,\mathscr{H}(V,H))),\quad \text{for all } \alpha\in\mathscr{A};\quad \sup_{\alpha\in\mathscr{A}}\|D_x\sigma(\cdot,\alpha)\|_{C_b^{0,1}}<\infty.\end{gathered} \tag{23}$$

In less abstract words, (23) means that σ is differentiable with respect to x for all x,α, its differential (with respect to x) $D_x\sigma$ which is at each $(x,\alpha)\in H\times\mathscr{A}$ a bounded linear operator from H into $\mathscr{H}(V,H)$ is bounded (in operator norm) uniformly in $(x,\alpha)\in H\times\mathscr{A}$ and is Lipschitz in x with a uniform (in α) Lipschitz constant. We next turn to the assumptions we make on b

$$b\in\mathrm{BUC}(H\times\mathscr{A};H) \tag{24}$$

$$D_x b(\cdot,\alpha)\in C_b^{0,1}(H;L(H)),\quad \text{for all } \alpha\in\mathscr{A};\quad \sup_{\alpha\in\mathscr{A}}\|D_x b(\cdot,\alpha)\|_{C_b^{0,1}}<\infty. \tag{25}$$

Finally, we assume that f, c, g satisfy

$$f\in\mathrm{BUC}(H\times\mathscr{A};\mathbf{R}),\quad g\in\mathrm{BUC}(H),\quad c\in\mathrm{BUC}(H\times\mathscr{A};\mathbf{R}) \tag{26}$$

and in the case of the infinite horizon problem ((19)–(20)) we assume furthermore

$$\inf[c(x,\alpha);\, x\in H,\ \alpha\in\mathscr{A}]=c_0>0. \tag{27}$$

It is then easy to check that the assumptions made upon σ, b yield a unique solution X_t of (18) and that those made upon f, c, g give meaningful and finite expressions in (19)–(22).

Next, we denote by

$$a=\tfrac{1}{2}\sigma\sigma^*,\quad \forall(x,\alpha)\in H\times\mathscr{A}. \tag{28}$$

Observe that a is a nuclear operator on H ($\forall(x,\alpha)\in H\times\mathscr{A}$) and that in particular:

$$\sup[\mathrm{Tr}\, a(x,\alpha);\, x\in H,\alpha\in\mathscr{A}]<\infty.$$

In all that follows, we will denote indifferently a^α, σ^α, b^α, f^α, c^α or $a(\cdot,\alpha)$, $\sigma(\cdot,\alpha)$, $b(\cdot,\alpha)$, $f(\cdot,\alpha)$, $c(\cdot,\alpha)$.

From the classical dynamic programming considerations, one expects the value functions u ((20) or (22)) to solve respectively

(Infinite horizon problem)

$$F(D^2u, Du, u, x) = 0 \quad \text{in } H \tag{29}$$

(Finite horizon problem)

$$\frac{\partial u}{\partial t} + F(D^2u, Du, u, x) = 0 \quad \text{in } H \times (0, \infty) \tag{30}$$

with the initial condition

$$u(\cdot, 0) = g(\cdot) \quad \text{on } H. \tag{31}$$

Here and below, F is the HJB operator namely

$$F(A, p, t, x) = \sup_{\alpha \in \mathcal{A}} \{-\mathrm{Tr}(a^\alpha \cdot A) - (b^\alpha, p) + c^\alpha - f^\alpha\}. \tag{32}$$

Observe that F does satisfy (2) and (3).

III.2. Elementary regularity properties of the value functions

We will use the following conditions

$$\text{for all } \alpha \in \mathcal{A}, \quad f^\alpha, c^\alpha \in C_b^{0,1}(H; \mathbf{R}); \ \sup_{\alpha \in \mathcal{A}} \left\{ \|f^\alpha\|_{C_b^{0,1}} + \|c^\alpha\|_{C_b^{0,1}} \right\} < \infty \tag{33}$$

$$g \in C_b^{0,1}(H) \tag{34}$$

and

$$\text{for all } \alpha \in \mathcal{A}, \quad D_x f^\alpha, D_x c^\alpha \in C_b^{0,1}(H; H); \ \sup_{\alpha \in \mathcal{A}} \left\{ \|D_x f^\alpha\|_{C_b^{0,1}} + \|D_x c^\alpha\|_{C_b^{0,1}} \right\} < \infty \tag{35}$$

$$Dg \in C_b^{0,1}(H; H). \tag{36}$$

Then, exactly as in P. L. Lions [28], one can prove the following results.

THEOREM III.1. (Infinite horizon problem: (19)–(20).)

(i) *The value function* $u \in \mathrm{BUC}(H)$.

(ii) *There exists a constant* $\lambda_0 \geq 0$ *(bounded by a fixed multiple of the supremum over* $H \times \mathcal{A}$ *of* $\|D_x \sigma\| + \|D_x b\|$*) such that if* (33) *holds, then u satisfies*

$$|u(x) - u(y)| \leq C|x - y|^\alpha \quad \text{for all } x, y \in H, \quad \text{for some } C \geq 0, \tag{37}$$

where $\alpha=1$ *if* $c_0>\lambda_0$, α *is arbitrary in* $(0,1)$ *if* $c_0=\lambda_0$, $\alpha=c_0/\lambda_0$ *if* $c_0<\lambda_0$.

(iii) *There exists a constant* $\lambda_1\geq 0$ *(bounded by a fixed multiple of the supremum over* $H\times\mathcal{A}$ *of* $\|D_x\sigma\|+\|D_x b\|$*) such that if* (33) *and* (35) *hold and* $c_0>\lambda_1$ *then* u *is semi-concave on* H *i.e. there exists a constant* $C\geq 0$ *such that*

$$u(x+h)+u(x-h)-2u(x)\leq C|h|^2, \quad \forall x, h\in H. \tag{38}$$

□

THEOREM III.2. (Finite horizon problem: (21)–(22).) *Let* $T\in(0,\infty)$.

(i) *The value function* $u\in \mathrm{BUC}(H\times[0,T])$ *and* $u(\cdot,0)=g$ *on* H.

(ii) *If* (33)–(34) *hold, then* u *satifies for some* $C\geq 0$

$$|u(x,t)-u(y,t)|\leq C|x-y| \quad \textit{for all } x,y\in H,\ t\in[0,T). \tag{39}$$

Furthermore, if (36) *holds, then* u *satisfies for some* $C\geq 0$

$$|u(x,t)-u(x,s)|\leq C|t-s| \quad \textit{for all } x\in H,\ t,s\in[0,T). \tag{40}$$

(iii) *If* (33)–(36) *hold, then* u *satisfies for some* $C\geq 0$

$$u(x+h,t)+u(x-h,t)-2u(x,t)\leq C|h|^2, \quad \forall x,h\in H,\ \forall t\in[0,T]. \tag{41}$$

□

Remark. If (35) holds and g is "very smooth" ($D^\alpha g\in C_b^{0,1}(H)$ for $0\leq\alpha\leq 3$) then similar arguments show that u is also semi-concave on $H\times[0,T]$ in (x,t) i.e.

$$u(x+h,t)+u(x-h,s)-2u\left(x,\frac{t+s}{2}\right)\leq C(|h|^2+(t-s)^2), \quad \forall x,h\in H,\ \forall t,s\in[0,T]. \tag{42}$$

□

III.3. Value functions are viscosity solutions of the HJB equation

In view of Theorems III.1 and III.2, we know that the value functions lie in BUC, so the following result makes sense.

THEOREM III.3. (i) (Infinite horizon problem.) *The value function* u *given by* (20) *is a viscosity solution of the* HJB *equation* (29).

(ii) (Finite horizon problem.) *The value function* u *given by* (22) *is a viscosity solution of the* HJB *equation* (30).

Proof. The strategy of the proof is basically the same as in P. L. Lions [27], [28], except that we have to pay some attention to difficulties associated with infinite dimensions namely that functions in X are not C^2 and that we have to be careful about Itô's formula.

Since the arguments are essentially the same, we will only show that, in the infinite horizon case, the value function is a viscosity supersolution. To this end, we take $\varphi \in X$ such that $u-\varphi$ has a global minimum at some point $x_0 \in H$. Without loss of generality (replacing if necessary φ by some modification of it), we may assume that $u(x_0)=\varphi(x_0)$ and that φ, $D\varphi$, $D^2\varphi$ are bounded over H, $D\varphi$ is Lipschitz over H, $(D^2\varphi(x)h, k)$ is uniformly continuous on H for all $h,k \in H$. Recall that we have to prove

$$\overline{\lim_{y\to x_0}} \sup_{\alpha}\{-\mathrm{Tr}(a^\alpha(x_0)\cdot D^2\varphi(y))-(b^\alpha(x_0), D\varphi(x_0))+c^\alpha(x_0)\,\varphi(x_0)-f^\alpha(x_0)\} \geq 0. \tag{43}$$

In order to do so, we will need several ingredients: the first of which is nothing but the usual optimality principle of the dynamic programming argument that we will not reprove here (see N. V. Krylov [22], M. Nisio [32], [33], K. Itô [18], N. El Karoui [10], W. H. Fleming [13] ...).

LEMMA III.1. *The value function satisfies for all* $h>0$, $x \in H$

$$u(x) = \inf_{\mathcal{S}}\left\{E\int_0^\theta f(X_t, \alpha_t)\exp\left(-\int_0^t c(X_s, \alpha_s)\,ds\right)dt + u(X_\theta)\exp\left(-\int_0^\theta c(X_t, \alpha_t)\,dt\right)\right\}. \tag{44}$$

Remark. In fact, u also satisfies the following identity: choose, for each $\mathcal{S}$, a stopping time θ, then for all $x \in H$

$$u(x) = \inf_{\mathcal{S}}\left\{E\int_0^\theta f(X_t, \alpha_t)\exp\left(-\int_0^t c(X_s, \alpha_s)\,ds\right)dt + u(X_\theta)\exp\left(-\int_0^\theta c(X_t, \alpha_t)\,dt\right)\right\}. \tag{45}$$

The other technical lemma is the justification of Itô's formula for $\varphi \in X$. We will prove this lemma after concluding the proof of Theorem III.3.

LEMMA III.2. *Let* $\varphi \in X$ *be such that* φ, $D\varphi$, $D^2\varphi$ *are bounded over* H, *then for each* $\mathcal{S}$ *and for each stopping time* θ *we have for all* $x \in H$

$$\begin{aligned}\varphi(x) = -E\int_0^\theta &\left\{\mathrm{Tr}(a^{\alpha_t}\cdot D^2\varphi)(X_t)+(b^{\alpha_t}, D\varphi)(X_t)-c^{\alpha_t}\varphi(X_t)\right\}\\ &\times\exp\left(-\int_0^t c(X_s, \alpha_s)\,ds\right)dt - \varphi(X_\theta)\exp\left(-\int_0^\theta c(X_t, \alpha_t)\,dt\right).\end{aligned} \tag{46}$$

We may now conclude the proof of Theorem III.3. In view of (44), (46) we have for all $h>0$

$$\varphi(x_0) \geqslant \inf_{\mathscr{S}} \left\{ E \int_0^h f(X_t, \alpha_t) \exp\left(-\int_0^t c(X_s, \alpha_s)\, ds\right) dt + \varphi(X_h) \exp\left(-\int_0^h c(X_s, \alpha_s)\, ds\right)\right.$$

$$\geqslant \inf_{\mathscr{S}} \left\{ E \int_0^h \{ f^{\alpha_t}(X_t) + \mathrm{Tr}(a^{\alpha_t} \cdot D^2\varphi)(X_t) + (b^{\alpha_t}, D\varphi)(X_t) - c^{\alpha_t}\varphi(X_t)\}\right.$$

$$\left.\times \exp\left(-\int_0^t c^{\alpha_s}(X_s)\, ds\right) dt \right\} + \varphi(x_0),$$

hence we deduce dividing by h

$$\sup_{\mathscr{S}} \left\{ \frac{1}{h} \int_0^h E\{-\mathrm{Tr}(a^{\alpha_t}(X_t) \cdot D^2\varphi(X_t)) - (b^{\alpha_t}(X_t), D\varphi(X_t)) + c^{\alpha_t}(X_t)\, \varphi(X_t)\right.$$

$$\left.\times \exp\left(-\int_0^t c^{\alpha_s}(X_s)\, ds\right) dt\right\} \geqslant 0.$$

Then, by standard arguments, one deduces easily

$$\sup_{\mathscr{S}} \left\{ \frac{1}{h} \int_0^h E\{-\mathrm{Tr}(a^{\alpha_t}(x_0) \cdot D^2\varphi(X_t)) - (b^{\alpha_t}(x_0), D\varphi(x_0)) + c^{\alpha_t}(x_0)\, \varphi(x_0) - f^{\alpha_t}(x_0)\}\, dt \right\}$$

$$\geqslant -\varepsilon(h) \to 0 \quad \text{as} \quad h \to 0_+.$$

Next, let $\delta>0$, the above inequality yields

$$\sup_{y \in B(x_0, \delta)} \sup_{\mathscr{S}} \left\{ \frac{1}{h} \int_0^h E\{-\mathrm{Tr}(a^{\alpha_t}(x_0) \cdot D^2\varphi(y)) - (b^{\alpha_t}(x_0), D\varphi(x_0)) + c^{\alpha_t}(x_0)\, \varphi(x_0) - f^{\alpha_t}(x_0)\}\, dt \right\}$$

$$\geqslant -\varepsilon(h) - C \sup_{\mathscr{S}} \left\{ \frac{1}{h} \int_0^h P(X_t \notin B(x_0, \delta))\, dt \right\}.$$

To conclude, we observe first that the $\sup_{\mathscr{S}}\{\ldots\}$ is nothing but

$$\sup_{\alpha \in \mathscr{A}} \{-\mathrm{Tr}(a^{\alpha}(x_0) \cdot D^2\varphi(y)) - (b^{\alpha}(x_0), D\varphi(x_0)) + c^{\alpha}(x_0)\, \varphi(x_0) - f^{\alpha}(x_0)\}$$

so that we deduce from the above inequality

$$\limsup_{y \to x_0} \left\{ \sup_{\alpha \in \mathscr{A}} \{-\mathrm{Tr}(a^{\alpha}(x_0) \cdot D^2\varphi(y)) - (b^{\alpha}(x_0), D\varphi(x_0)) + c^{\alpha}(x_0)\, \varphi(x_0) - f^{\alpha}(x_0)\} \right\}$$

$$\geqslant -\varepsilon(h) - C \lim_{\delta \to 0+} \sup_{\mathscr{S}} \left\{ \frac{1}{h} \int_0^h P(X_t \notin B(x_0, \delta))\, dt \right\}.$$

Therefore, (43) will be proved as soon as we show

$$\sup_{\mathscr{S}}\left\{\frac{1}{h}\int_0^h P(X_t\notin B(x_0,\delta))\,dt\right\}\to 0 \quad \text{as} \quad h\to 0+, \text{ for each } \delta>0. \tag{47}$$

The convergence for each $\mathscr{S}$ is obviously a trivial consequence of the continuity of X_t which implies that $P(X_t\notin B(x_0,\delta))\to 0$ as $t\to 0+$, for each $\delta>0$. To check that the convergence is uniform in $\mathscr{S}$, we just observe that by Itô's formula one obtains by routine arguments

$$E[|X_t-x_0|^2]\leqslant Ct, \quad \text{for all} \quad t\in[0,1]$$

where C is independent of $\mathscr{S}$. Hence,

$$\sup_{\mathscr{S}} P(X_t\notin B(x_0,\delta))\leqslant\frac{C}{\delta^2}t$$

and (47) is proved. □

Remarks. (1) We gave the proof only for the supersolution part. For the remaining part, the proof is actually a bit easier and yields a stronger result than (7) namely

$$\text{at each local maximum } x_0 \text{ of } u-\varphi, \text{ we have} \quad F(D^2\varphi(x_0),D\varphi(x_0),u(x_0),x_0)\leqslant 0 \tag{7'}$$

for all $\varphi\in X$.

(2) Observe also that the usual verification argument also yields that value functions are classical viscosity solutions, a fact that is also deduced from the above result since (section II) viscosity solutions are indeed classical viscosity solutions.

Proof of Lemma III.2. We justify (46) by a finite dimensional approximation. Let H_N be an increasing sequence of finite dimensional subspaces of H such that $\bigcup_N H_N$ is dense in H and let us denote by P^N the orthogonal projection onto H_N. The system $\mathscr{S}$ and $x\in H$ being fixed, we denote by X_t^N the continuous F^t-adapted solution of

$$dX_t^N=P^N\sigma(P^NX_t^N,\alpha_t)\cdot dW_t+P^Nb(P^NX_t^N,\alpha_t)\,dt, \quad X_0^N=P^Nx. \tag{48}$$

Observe that $X_t^N\in H_N$ for all $t\geqslant 0$ and that $\varphi|_{H_N}$ is now C^2. Hence, (46) holds if we replace x by P^Nx, X_t by X_t^N, $b^\alpha(\cdot)$ by $P^Nb^\alpha(\cdot)$ and $a^\alpha(\cdot)$ by $P^Na^\alpha(\cdot)P^N$. Therefore, observing that $D^2\varphi(y)\to D^2\varphi(x)$ pointwise if $y\to x$, (46) is proved as soon as we show

$$E\left[\sup_{t\in[0,T]}|X_t^N-X_t|^2\right]\underset{N}{\to} 0, \quad \forall T<\infty. \tag{49}$$

In order to prove (49) we apply Itô's formula and we find

$$E[|X_s^N - X_s|^2] = |x - P^N x|^2 + E\int_0^s \mathrm{Tr}\,\{(\sigma^{\alpha_t}(X_t) - P^N\sigma^{\alpha_t}(X_t^N)) \times(\sigma^{\alpha_t}(X_t)^* - \sigma^{\alpha_t}(X_t^N)^* P^N)\} + 2(b^{\alpha_t}(X_t) - P^N b^{\alpha_t}(X_t^N), X_t - X_t^N)\,dt$$

hence,

$$E[|X_s^N - X_s|^2] \leq |x - P^N x|^2 + C\int_0^s E|X_t^N - X_t|^2\,dt + C\int_0^s E\{\mathrm{Tr}(\sigma^{\alpha_t}(X_t) - P^N\sigma^{\alpha_t}(X_t))\cdot(\sigma^{\alpha_t}(X_t)^* - \sigma^{\alpha_t}(X_t)^* P^N)\}\,dt + C\int_0^s E\{|b^{\alpha_t}(X_t) - P^N b^{\alpha_t}(X_t)|^2\}\,dt$$

for some constant $C \geq 0$ (independent of s, N). To conclude, we just observe that the last integrals converge to 0 as N goes to $+\infty$ by Lebesgue's lemma; therefore, by Grönvall's lemma we deduce

$$\sup_{0\leq t\leq T} E[|X_t^N - X_t|^2] \underset{N}{\rightarrow} 0.$$

And this yields (49) by standard arguments. □

III.4. Further regularity properties of the value functions

THEOREM III.4. *In the finite horizon case* ((21)–(22)) *we assume* (33)–(36) *while in the infinite horizon case* ((19)–(20)) *we assume* (33)–(35) *and* $c_0 > \lambda_1$. *Then, the following regularity properties of the value functions u hold* (*in the finite horizon problem, these properties hold uniformly for* $t \in [0, T]$ *for all* $T < \infty$).

(i) *There exists a constant* $C \geq 0$ (*independent of* $\alpha \in \mathcal{A}$) *such that u is a viscosity subsolution, respectively supersolution of*

$$-\mathrm{Tr}(a^\alpha \cdot D^2 u) \leq C \quad \text{in } H, \quad \text{resp.} \quad -\mathrm{Tr}(a^\alpha \cdot D^2 u) \geq -C \quad \text{in } H. \tag{50}$$

(ii) *Assume that there exist an open set* $\omega \subset H$, *a positive constant* $\nu > 0$, *and a closed subspace* H' *of* H *such that*

$$\sup_{\alpha\in\mathcal{A}} (a^\alpha(x)\,\xi, \xi) \geq \nu|\xi|^2, \quad \forall \xi \in H', \quad \forall x \in \omega. \tag{51}$$

Then, there exists a constant $C \geq 0$ (depending only on ν) such that for all $\xi \in H'$, $|\xi|=1$, u is a viscosity subsolution, respectively supersolution of

$$-(D^2u(x)\,\xi, \xi) \leq C \quad \text{in } \omega, \quad -(D^2u(x)\,\xi, \xi) \geq -C \quad \text{in } \omega. \tag{52}$$

And this is equivalent to say that if we write $x=(x', x'') \in H' \times H'^{\perp}$, then for each $x'' \in H'^{\perp}$, $u(\cdot, x'')$ is differentiable, $\nabla_{x'} u$ is Lipschitz on ω and

$$|\nabla_{x'} u(x_1', x'') - \nabla_{x'} u(x_2', x'')| \leq C|x_1' - x_2'|, \quad \forall x'' \in H'^{\perp} \tag{53}$$

for all $x_1', x_2' \in H'$ such that $\{\theta(x_1', x'') + (1-\theta)(x_2', x'');\ \theta \in [0,1]\} \subset \omega$.

Remarks. (1) (50) and (52) really mean that $\mathrm{Tr}(a^{\alpha} \cdot D^2u)$, $(D^2u\xi, \xi)$ are bounded (independently of $\alpha \in \mathscr{A}$, $\xi \in H'$ respectively) on H, ω respectively.

(2) The above regularity result are the exact infinite dimensional analogues of the regularity results obtained in P. L. Lions [28] for finite dimensional problems.

(3) In view of Proposition II.1, we see that inequalities (50), (52) in viscosity sense or in classical viscosity sense are equivalent.

Proof of Theorem III.4. To simplify notations, we will say that $F(D^2u, Du, u, x)$ is bounded in viscosity sense on an open set $\mathcal{O}$ of H if there exists $C \geq 0$ such that u is a viscosity subsolution, respectively viscosity supersolution of

$$F(D^2u, Du, u, x) \leq C \quad \text{in } \mathcal{O}, \quad F(D^2u, Du, u, x) \geq -C \quad \text{in } \mathcal{O}.$$

Next, we will make the proof of Theorem III.4 only in the case of the infinite horizon problem since the proof in the other case is very much the same. Recall also that by Theorem III.1 we know that u is Lipschitz and semi-concave on H (i.e. satisfies (37) with $\alpha=1$ and (38)). Observe finally that (38) immediately yields that u is a viscosity supersolution of

$$-(D^2u(x)\,\xi, \xi) \geq -C \quad \text{in } H, \quad \text{for all } \xi \in H,\ |\xi| = 1. \tag{38'}$$

We first prove (i). To this end, we denote by $S^{\alpha}(t)$ the Markov semigroup corresponding to a fixed control $\alpha_t \equiv \alpha \in \mathscr{A}$ i.e.

$$[S^{\alpha}(t)\,\varphi](x) = E\varphi(X_t), \quad \forall x \in H, \quad \forall \varphi \in \mathrm{BUC}(H)$$

where X_t is the solution of (18) corresponding to $\alpha_t \equiv \alpha$. Clearly, $S^{\alpha}(t)$ is order-preserving that is $S^{\alpha}(t)\,\varphi_1 \leq S^{\alpha}(t)\,\varphi_2$ if $\varphi_1 \leq \varphi_2$ on H. Therefore, for all $x \in H$

$$\frac{1}{t}\{u-S^\alpha(t)u\}(x) \geqslant \inf_{\mathscr{S}} \frac{1}{t}\{J(x,\mathscr{S})-S^\alpha(t)J(\cdot,\mathscr{S})(x)\}.$$

Then, the proof in P. L. Lions [28] adapts and shows that if $c_0 \geqslant \lambda_1$ then $J(\cdot,\mathscr{S}) \in C_b^{1,1}(H)$ i.e. $J(\cdot,\mathscr{S}) \in C^1(H)$, is bounded, and $\nabla_x J(\cdot,\mathscr{S})$ is bounded, Lipschitz on H. And using the same finite dimensional approximation procedure as in Lemma III.2, we deduce easily that

$$\left|\frac{1}{t}\{J(x,\mathscr{S})-S^\alpha(t)J(\cdot,\mathscr{S})(x)\}\right| \leqslant C \quad \text{for all} \quad t\in(0,1),\ \alpha\in\mathscr{A},\mathscr{S}.$$

Finally, we obtain

$$\frac{1}{t}\{u-S^\alpha(t)u\} \geqslant -C \quad \text{on } H. \tag{54}$$

And we deduce as in Theorem III.3 that u is a viscosity supersolution of

$$-\operatorname{Tr}(a^\alpha \cdot D^2u) \geqslant -C \quad \text{on } H.$$

To complete the proof of (i), we have to show the other inequality. But let us remark that, from the definition of viscosity solutions, u is by Theorem III.3 a viscosity subsolution of

$$-\operatorname{Tr}(a^\alpha \cdot D^2u)-(b^\alpha, Du) \leqslant C \quad \text{on } H.$$

And we conclude using the fact that u is Lipschitz on H: indeed, observe that if u is Lipschitz and $u-\varphi$ has a maximum at x_0 then

$$|D\varphi(x_0)| \leqslant \sup_{x\neq y}|u(x)-u(y)|\,|x-y|^{-1}$$

(this is proved and used in M. G. Crandall and P. L. Lions [4] for instance).

We next prove (52). Observe first that in view of (38′) we just have to show the first inequality of (52). Formally, this is rather easy since by (38′) there exists $C_0 \geqslant 0$ such that $(D^2u - C_0 I) \leqslant 0$, hence because of (50)

$$\sup_\alpha[\operatorname{Tr} a^\alpha \cdot (C_0 I - D^2u)] \leqslant C$$

and then (51) yields

$$\nu\{C_0|\xi|^2-(D^2u(x)\xi,\xi)\} \leqslant C|\xi|^2, \quad \forall \xi\in H', \quad \forall x\in\omega$$

and we may conclude. We only have to justify by viscosity considerations the above argument. To this end, let $\varphi \in X$ and let x_0 be a minimum point of $u-\varphi$ (global for instance). Using Proposition II.1, we have for each α using (50)

$$-\mathrm{Tr}(a^\alpha(x_0)\cdot D^2\varphi(x_0)) \geq -C.$$

Next, we claim that $D^2\varphi(x_0) \leq CI$ where C is in fact the constant appearing in (38'). Indeed, observe that $u-\frac{1}{2}C|x|^2$ is concave and thus x_0 is a minimum point of $(u-\frac{1}{2}C|x|^2)-(\varphi-\frac{1}{2}C|x|^2)$ which implies easily our claim. Hence, we have

$$-\mathrm{Tr}\ a^\alpha(x_0)\cdot(D^2\varphi(x_0)-CI) \geq -C, \quad \forall \alpha \in \mathscr{A}$$

or

$$\sup_{\alpha \in \mathscr{A}} \mathrm{Tr}\ a^\alpha(x_0)\cdot(CI-D^2\varphi(x_0)) \leq C, \quad \forall \alpha \in \mathscr{A}$$

and we deduce, using (51), that for all $\xi \in H'$, $|\xi|=1$

$$\nu((CI-D^2\varphi(x_0))\cdot\xi, \xi) \leq 0 \quad \text{if} \quad x_0 \in \omega$$

hence

$$-(D^2\varphi(x_0)\cdot\xi, \xi) \leq C \quad \text{if} \quad x_0 \in \omega.$$

And (52) is proved.

To conclude the proof of Theorem III.4, we have to show why (52) implies (and thus is equivalent to) (53). There are mainly two steps in the proof of this claim: first, we show that (52) still holds locally if we write $x=(x', x'') \in H' \times H'^{\perp}$ and if we fix x'' considering u as a function of x' only. Once this is done, it is not difficult to conclude observing that if we take any finite dimensional subspace of H' the above argument gives viscosity inequalities (52) in this finite dimensional subspace and we know (from P. L. Lions [27]) that (53) then holds with H' replaced by its subspace. Since all constants are independent of the chosen finite dimensional subspace, we then conclude easily.

We now prove the above claim concerning the reduction of (52) to H'. This is basically the same proof as in M. G. Crandall and P. L. Lions [4]. Indeed, let x_0'' be fixed in $H'^{\perp}$ and let $(x_0', x_0'') \in \omega$ be a minimum point of $u(\cdot, x_0'')-\varphi(\cdot)$ where $\varphi \in X$ (space of functions over H'). We may assume without loss of generality that there exist $\delta>0$, $\gamma>0$ such that

$$\overline{B(x_0', \delta)} \times \overline{B(x_0'', \delta)} \subset \omega \quad \text{and} \quad u(x', x_0'') - \varphi(x') \leq u(x_0', x_0'') - \varphi(x_0') - \gamma$$

if $|x' - x_0'| = \delta$. Then, we consider on $Q = \overline{B(x_0', \delta)} \times \overline{B(x_0'', \delta)}$ the function

$$z = u(x', x'') - \varphi(x') - \frac{1}{2\varepsilon}|x'' - x_0''|^2.$$

We claim that, on ∂Q, $z \leq u(x_0', x_0'') - \varphi(x_0') - \gamma/2$ if ε is small enough. Indeed, since $|x'' - x_0''| = \delta$ if $x'' \in \partial B(x_0'', \delta)$, this inequality is obvious for ε small enough if $x'' \in \partial B(x_0'', \delta)$; while if $x' \in \partial B(x_0', \delta)$

$$\begin{aligned} z &\leq u(x', x_0'') - \varphi(x') + m(|x'' - x_0''|) - \frac{1}{2\varepsilon}|x'' - x_0''|^2 \\ &\leq u(x_0', x_0'') - \varphi(x_0') - \gamma + m(|x'' - x_0''|) - \frac{1}{2\varepsilon}|x'' - x_0''|^2 \end{aligned}$$

where $m(t) \to 0$ as $t \to 0+$, and our claim is proved.

Therefore, using Stegall's result [34] as in M. G. Crandall and P. L. Lions [5], we deduce that there exist $p_\varepsilon \in B(0, \varepsilon)$, $x_\varepsilon' \in (x_0', \delta)$, $x_\varepsilon'' \in B(x_0'', \delta)$ such that $z(\cdot) + (p_\varepsilon, \cdot)$ has a maximum over Q at $(x_\varepsilon', x_\varepsilon'')$. Furthermore, since we may assume without loss of generality that $u(\cdot, x_0'') - \varphi(\cdot)$ has a unique strict maximum at x_0', we deduce easily that

$$x_\varepsilon'' \underset{\varepsilon}{\to} x_0'', \quad x_\varepsilon' \underset{\varepsilon}{\to} x_0'.$$

Then, by the definition of viscosity solutions, we see that for all $\xi \in H'$, $|\xi| = 1$

$$-(D_{x'}^2 \varphi(x')\xi, \xi) \leq C$$

and we deduce

$$-(D_{x'}^2 \varphi(x_0')\xi, \xi) \leq C,$$

which concludes the proof of our claim. □

IV. Uniqueness results

THEOREM IV.1. (1) (Infinite horizon problem: (19)–(20).) *Let $v \in \mathrm{BUC}(H)$ be a viscosity subsolution (resp. supersolution) of the* HJB *equation* (29). *Then $v \leq u$ on H (resp. $v \geq u$ on H).*

(2) (Finite horizon problem: (21)–(22).) *Let $v \in \mathrm{BUC}(H \times [0, T])$ be a viscosity*

subsolution (*resp. supersolution*) *of the* HJB *equation* (30) *such that* $v(\cdot,0)\leq g(\cdot)$ *on* H (*resp.* $v(\cdot,0)\geq g(\cdot)$ *on* H). *Then* $v\leq u$ *on* $H\times[0,T]$ (*resp.* $v\geq u$ *on* $H\times[0,T]$).

Remarks. (1) Extensions of this result are given in section V.

(2) If the notion of viscosity supersolution we use differs from the classical one, we do not know in general if the above results are valid with classical viscosity supersolutions.

Once more, since the proofs of (1) and (2) are very similar, we will only prove (1). The proof will be divided into two steps: we first show that any viscosity subsolution lies below u, i.e. u is the maximum viscosity subsolution, next we prove that any viscosity supersoloution is above u.

IV.1. Maximum subsolution

In this section, we consider a viscosity subsolution of (29) that we denote by v and we assume (for instance) that $v\in \mathrm{BUC}(H)$. And we want to show that $v\leq u$ on H. In view of the method introduced in P. L. Lions [27]—which basically uses only the density of step controls—we only have to show that if α is fixed in $\mathscr{A}$ then for all $t>0$ and for all $x\in H$

$$v\leq E\int_0^t f^\alpha(X_s)\exp\left(-\int_0^s c^\alpha(X_\sigma)\,d\sigma\right)ds+v(X_t)\exp\left(-\int_0^t c^\alpha(X_s)\,ds\right) \tag{55}$$

where X_t is the Markov process corresponding to the constant control $\alpha_t\equiv\alpha$. We then denote by $w(x,t)$ the right-hand side which is of course a viscosity solution (by the results of section III) of

$$\frac{\partial w}{\partial t}-\mathrm{Tr}(a^\alpha\cdot D^2w)-(b^\alpha,Dw)+c^\alpha w-f^\alpha=0 \quad \text{in } H\times(0,\infty) \tag{56}$$

and $w(\cdot,0)=v(\cdot)$ on H.

In order to compare v and w, the strategy we shall adopt is to build a smooth (i.e. an element of X) approximation of w which will be close to w uniformly on H and which will solve (56) up to an arbitrary small constant. Once this is done, we will conclude easily by a simple application of the notion of viscosity solution. Let us finally mention that to simplify notations we will omit the superscript α in the rest of this section.

We begin by smoothing f, c, v: indeed, see for instance J. M. Lasry and P. L. Lions [24], there exist

$$f^n, c^n, v^n \in C_b^{1,1}(H) = \{\varphi \in C^1(H),\ \varphi \in C_b^{0,1}(H;\mathbf{R}),\ D\varphi \in C_b^{0,1}(H;H)\}$$

such that for $n \geq 1$

$$f \leq f^n \leq f+\frac{1}{n}, \quad c^n \leq c \leq c^n+\frac{1}{n}, \quad v \leq v^n \leq v+\frac{1}{n} \quad \text{on } H.$$

Then, we consider for all $x \in H$, $t \geq 0$

$$w^n(x,t) = E\int_0^t f^n(X_s)\exp\left(-\int_0^s c^n(X_\sigma)\,d\sigma\right)ds + v^n(X_t)\exp\left(-\int_0^t c^n(X_s)\,ds\right)$$

and we observe that $w \leq w^n \leq w + C/n$, for some $C \geq 0$, while w^n is now a viscosity solution of

$$\frac{\partial w^n}{\partial t} - \operatorname{Tr}(a\cdot D^2 w^n) - (b, Dw^n) + c^n w^n - f^n = 0 \quad \text{in } H\times(0,\infty) \tag{56'}$$

and $w^n(\cdot,0) = v^n(\cdot) \geq v(\cdot)$ on H.

But obviously $f^n - c^n w^n \geq f - cw^n - C_T/n$ on $H\times[0,T]$ for some $C_T \geq 0$, $(\forall T<\infty)$ therefore w^n is a viscosity supersolution of

$$\frac{\partial w^n}{\partial t} - \operatorname{Tr}(a\cdot D^2 w^n) - (b, Dw^n) + cw^n - f = -\frac{C_T}{n} \quad \text{in } H\times(0,T)\ (\forall T<\infty). \tag{57}$$

It is then easy to check that w^n is Lipschitz in $(x,t) \in H\times(0,T)$ $(\forall T<\infty)$, bounded on $H\times[0,T]$ $(\forall T<\infty)$ and $w^n(\cdot,t) \in C_b^{1,1}(H)$ $(\forall t \in [0,\infty))$ with Lipschitz bounds on $D_x w^n(\cdot,t)$ uniform in $t \in [0,T]$ $(\forall T<\infty)$: this is readily seen from the explicit formula defining w^n.

But we still need to regularize w^n in order to have a smooth function. This is done with the help of the following lemma that we will also need for "stationary equations" in the next section. In the result which follows,

$$\sigma \in \mathrm{BUC}(H;\mathcal{H}(V,H)), \quad b \in \mathrm{BUC}(H;H), \quad c \in \mathrm{BUC}(H;\mathbf{R}), \quad f \in \mathrm{BUC}(H;\mathbf{R})$$

and we denote by $a = \frac{1}{2}\sigma\sigma^*$.

LEMMA IV.1. (1) (Infinite horizon problem.) *Let $z \in C_b^{1,1}(H)$ be a viscosity subsolution (resp. supersolution) of*

$$-\operatorname{Tr}(a\cdot D^2z)-(b,Dz)+cz=\tilde{f} \quad in\ H. \tag{58}$$

Then, for each $\varepsilon>0$, *there exists* $z_\varepsilon\in X$ *such that* $|z_\varepsilon-z|\leqslant\varepsilon$ *on* H *and* z_ε *is a viscosity subsolution* (*resp. supersolution*) *of*

$$-\operatorname{Tr}(a\cdot D^2z_\varepsilon)-(b,Dz_\varepsilon)+cz_\varepsilon=\tilde{f}+C\varepsilon\ (resp.\ \tilde{f}-C\varepsilon) \quad in\ H \tag{58'}$$

for some $C\geqslant 0$ (*depending only on the bounds on* z *and its derivatives and the moduli of continuity of the coefficients* σ, b, c, $\tilde{f}$).

(2) (Finite horizon problem.) *Let* $T<\infty$, *let* $z\in C_b^{0,1}(H\times[0,T])$, $z(\cdot,t)\in C_b^{1,1}(H)$ *for all* $t\in[0,T]$ *with Lipschitz bounds on* $Dz(\cdot,t)$ *uniform in* $t\in[0,T]$, *be a viscosity subsolution* (*resp. supersolution*) *of*

$$\frac{\partial z}{\partial t}-\operatorname{Tr}(a\cdot D^2z)-(b,Dz)+cz=\tilde{f} \quad in\ H\times(0,T). \tag{59}$$

Then, for each $\varepsilon>0$, *there exists* $z_\varepsilon\in X$ *such that* $|z_\varepsilon-z|\leqslant\varepsilon$ *on* $H\times[\varepsilon,T-\varepsilon]$ *and* z_ε *is a viscosity subsolution* (*resp. supersolution*) *of*

$$\frac{\partial z_\varepsilon}{\partial t}-\operatorname{Tr}(a\cdot D^2z_\varepsilon)-(b,Dz_\varepsilon)+cz_\varepsilon=\tilde{f}+C\varepsilon\ (resp.\ \tilde{f}-C\varepsilon) \quad in\ H\times(\varepsilon,T-\varepsilon) \tag{59'}$$

for some $C\geqslant 0$ (*depending only on the bounds on* z *and its derivatives and the moduli of continuity of the coefficients* σ, b, c, $\tilde{f}$).

Remark. As we will see from the proof, this result can be "localized" in any open set of H or $H\times(0,T)$.

We postpone the proof of Lemma IV.1 until we conclude the proof of our claim concerning v and w. By the preceding lemma, we deduce the existence of $w_\varepsilon^n\in X$ which is a viscosity supersolution of

$$\frac{\partial w_\varepsilon^n}{\partial t}-\operatorname{Tr}(a\cdot D^2w_\varepsilon^n)-(b,Dw_\varepsilon^n)+cw_\varepsilon^n=f-\frac{C}{n}-C\varepsilon \quad \text{in } H\times(\varepsilon,T-\varepsilon)$$

and $|w_\varepsilon^n-w^n|\leqslant\varepsilon$ on $H\times[\varepsilon,T-\varepsilon]$. Observe that the definition of viscosity solution immediately implies that we have in fact at each $(x,t)\in H\times(\varepsilon,T-\varepsilon)$

$$\frac{\partial w_\varepsilon^n}{\partial t}(x,t)-\operatorname{Tr}(a(x)\cdot D^2w_\varepsilon^n(x,t))-(b(x),Dw_\varepsilon^n(x,t))+c(x)\,w_\varepsilon^n(x,t)\geqslant f(x)-\frac{C}{n}-C\varepsilon.$$

It is now easy to conclude by maximizing over $H\times[\varepsilon,T-\varepsilon]$

$$e^{-C_0 t}(v(x)-w_\varepsilon^n(x,t))-\delta$$

where C_0, δ are positive constants to be determined later on.

To keep the ideas clear, let us assume that a maximum point $(\bar{x},\bar{t})$ exists. Then, we first observe that $\tilde{v}=e^{-C_0 t}v$ is a viscosity subsolution of

$$\frac{\partial \tilde{v}}{\partial t}-\operatorname{Tr}(a\cdot D^2\tilde{v})-(b,D\tilde{v})+(c+C_0)\,\tilde{v}=fe^{-C_0 t} \quad \text{in } H\times(0,\infty) \tag{60}$$

and that $\tilde{w}_\varepsilon^n=e^{-C_0 t}w_\varepsilon^n+\delta$, $\tilde{w}^n=e^{-C_0 t}w^n$ satisfy $|\tilde{w}_\varepsilon^n-\tilde{w}^n|\leq\varepsilon+\delta$ on $H\times[\varepsilon,T-\varepsilon]$ and at each point $(x,t)\in H\times(\varepsilon,T-\varepsilon)$

$$\frac{\partial \tilde{w}_\varepsilon^n}{\partial t}-\operatorname{Tr}(a\cdot D^2\tilde{w}_\varepsilon^n)-(b,D\tilde{w}_\varepsilon^n)+(c+C_0)\,\tilde{w}_\varepsilon^n\geq\left(f-\frac{C}{n}-C\varepsilon\right)e^{-C_0 t}+(c+C_0)\,\delta. \tag{61}$$

Then, we choose $C_0=\sup_H c^-+1$, $\delta=C/n+C\varepsilon+\varepsilon$, so that (61) yields

$$\frac{\partial \tilde{w}_\varepsilon^n}{\partial t}-\operatorname{Tr}(a\cdot D^2\tilde{w}_\varepsilon^n)-(b,D\tilde{w}_\varepsilon^n)+(c+C_0)\,\tilde{w}_\varepsilon^n\geq fe^{-C_0 t}+\varepsilon \quad \text{on } H\times[\varepsilon,T-\varepsilon]. \tag{62}$$

Next, if $\bar{t}=\varepsilon$, we just deduce that on $H\times[\varepsilon,T-\varepsilon]$

$$\begin{aligned}(v-w_\varepsilon^n)(x,t)&\leq\delta e^{C_0 T}+v(\bar{x},\varepsilon)-w_\varepsilon^n(\bar{x},\varepsilon)\\&\leq\delta e^{C_0 T}+C\varepsilon+v(\bar{x})-w^n(\bar{x},\varepsilon)+\varepsilon\\&\leq\delta e^{C_0 T}+C\varepsilon+\varepsilon+v(\bar{x})-w(\bar{x},\varepsilon)\end{aligned}$$

and since $w\in\mathrm{BUC}(H\times[0,T])$, we deduce from this

$$(v-w)(x,t)\leq\frac{C}{n}+C\varepsilon+m(\varepsilon)+v(\bar{x})-w(\bar{x},0)=\frac{C}{n}+C\varepsilon+m(\varepsilon) \quad \text{on } H\times[\varepsilon,T-\varepsilon]$$

where $m(\sigma)\to 0$ as $\sigma\to 0+$. And we conclude letting $\varepsilon\to 0$, $n\to\infty$.

On the other hand if $\bar{t}>\varepsilon$, we may apply the definition of viscosity subsolutions (we can even do that if $\bar{t}=T-\varepsilon$ by the usual argument for viscosity solutions of Cauchy type problems, see [4], [7] ...) and deduce

$$\frac{\partial \tilde{w}_\varepsilon^n}{\partial t}(\bar{x},\bar{t})-\operatorname{Tr}(a\cdot D^2\tilde{w}_\varepsilon^n)(\bar{x},\bar{t})-(b,D\tilde{w}_\varepsilon^n)(\bar{x},\bar{t})+(C_0+c)\,\tilde{v}(\bar{x},\bar{t})\leq f(\bar{x})\,e^{-C_0\bar{t}}.$$

And comparing with (62) yields the following inequality

$$\bar{v}(\bar{x},\bar{t})-\bar{w}_\varepsilon^n(\bar{x},\bar{t})\leq -\varepsilon\leq 0$$

hence $\bar{v}-\bar{w}_\varepsilon^n\leq 0$ on $H\times[\varepsilon, T-\varepsilon]$ and we conclude again letting ε go to 0, n go to ∞.

We now have to deal with the question of the existence of a maximum point of $\bar{v}-\bar{w}_\varepsilon^n$. This is easily solved by perturbation arguments (as in [5]): indeed, let us introduce $\hat{w}=\bar{w}_\varepsilon^n+\alpha(1+|x|^2)^{1/2}$ where $\alpha>0$. Obviously, $\bar{v}-\hat{w}\to-\infty$ as $|x|\to\infty$ uniformly for $t\in[\varepsilon, T-\varepsilon]$, while $\hat{w}$ satisfies

$$\frac{\partial\hat{w}}{\partial t}-\mathrm{Tr}(a\cdot D^2\hat{w})-(b,D\hat{w})+(c+C_0)\,\hat{w}\geq fe^{-C_0t}+\varepsilon-C\alpha \quad \text{in } H\times[\varepsilon, T-\varepsilon].$$

Now, by Stegall's result [34], we deduce the existence of $\varrho_\nu\in H$ for all $\nu>0$ such that $|\varrho_\nu|\leq\nu<\alpha$ and $\bar{v}(x,t)-\hat{w}(x,t)+(\varrho_\nu, x)$ has a (global) maximum over $H\times[\varepsilon, T-\varepsilon]$ at some $(\bar{x},\bar{t})$. If $\bar{t}=\varepsilon$, we argue as before, letting ν go to 0, then α go to 0 and then ε go to 0, n go to $+\infty$. If $\bar{t}>\varepsilon$, we deduce at $(\bar{x},\bar{t})$

$$\frac{\partial\hat{w}}{\partial t}-\mathrm{Tr}(a\cdot D^2\hat{w})-(b,D\hat{w})+(c+C_0)\,\bar{v}\leq fe^{-C_0t}+C\nu.$$

And this yields as before

$$\bar{v}(\bar{x},\bar{t})-\bar{w}(\bar{x},\bar{t})\leq C(\nu+\alpha)-\varepsilon$$

and we conclude easily letting $\nu\to 0$, then $\alpha\to 0$, then $\varepsilon\to 0$, $n\to\infty$.

This concludes the proof of our claim concerning the comparison of v and w. □

Proof of Lemma IV.1. As usual we will only make the proof in the infinite horizon case when z is a viscosity subsolution of (58), the other cases being treated similarly. Since the proof is rather technical, it might be worth explaining first the idea: let $\varrho\geq 0$, $\varrho\in\mathcal{D}(\mathbf{R})$, $\int_{\mathbf{R}}\varrho\,dx=1$, $\mathrm{Supp}\,\varrho\subset[-1,+1]$ and let $\varepsilon>0$, we introduce

$$z_\varepsilon(x)=\lim_k z_\varepsilon^k(x),\quad z_\varepsilon^k(x)=\int_{\mathbf{R}^k} z(y_1,\dots,y_k,x')\left(\prod_{i=1}^k \varrho_{\varepsilon_i}(x_i-y_i)\right)dy \tag{63}$$

where $\varepsilon_i=\varepsilon/2^{i+1}$, $\varrho_h(\cdot)=(1/h)\,\varrho(\cdot/h)$ for all $h>0$, $x=(x_1,\dots,x_k,x')=(x_1,x_2,\dots)$ corresponds to various decompositions or identifications of H: more precisely, we fix an orthonormal basis of H say $(e_1,e_2,e_3,\dots)$ and we indifferently identify H with l^2 or with $\mathbf{R}^k\times H_k^\perp$ where $H_k=\mathrm{vect}(e_1,\dots,e_k)$. In fact, we have to show that z_ε is defined by (63) i.e. we have to show that z_ε^k converges to z_ε.

We claim that $z_\varepsilon\in X$, z_ε satisfies (58′) and $|z_\varepsilon-z|\leq C\varepsilon$ on H (where C is in fact the

Lipschitz constant of z). We begin by proving that z_ε makes sense, belongs to X and is close to z. There will only remain to show that z_ε is a viscosity subsolution of (58′): a fact which would be an immediate exercise on convolution if H were finite dimensional since in that case z would be an "a.e." subsolution of (58) (see [27])! Now, we first observe that $z_\varepsilon^k \in C_b^{1,1}(H)$ for all $k \geq 1$, $\varepsilon > 0$ and that

$$\sup_H |Dz_\varepsilon^k| \leq C_0 = \sup_H |Dz|,$$
$$\sup_{x \neq y} |Dz_\varepsilon^k(x) - Dz_\varepsilon^k(y)| |x-y|^{-1} \leq C_1 = \sup_{x \neq y} |Dz(x) - Dz(y)| |x-y|^{-1}. \tag{64}$$

Next, we remark that we have

$$\sup_H |z - z_\varepsilon^1| \leq C_0 \varepsilon_1, \quad \sup_H |z_\varepsilon^{k+1} - z_\varepsilon^k| \leq C_0 \varepsilon_{k+1} \quad \text{for all} \quad k \geq 1 \tag{65}$$

$$\sup_H |Dz - Dz_\varepsilon^1| \leq C_1 \varepsilon_1, \quad \sup_H |Dz_\varepsilon^{k+1} - Dz_\varepsilon^k| \leq C_1 \varepsilon_{k+1} \quad \text{for all} \quad k \geq 1 \tag{66}$$

$$\sup_H |\partial_{ij} z_\varepsilon^{k+1} - \partial_{ij} z_\varepsilon^k| \leq \frac{C}{\varepsilon_i \varepsilon_j} \varepsilon_{k+1}, \quad \sup_H |\partial_{ij} z_\varepsilon^k| \leq \frac{C}{\varepsilon_i \varepsilon_j} \quad \text{if} \quad 1 \leq i, j \leq k \tag{67}$$

$$\sup_H |D\partial_{ij} z_\varepsilon^k| \leq \frac{C}{\varepsilon_i \varepsilon_j}, \quad \sup_H |D\partial_{ij} z_\varepsilon^{k+1} - D\partial_{ij} z_\varepsilon^k| \leq \frac{C}{\varepsilon_i \varepsilon_j} \varepsilon_{k+1} \quad \text{if} \quad 1 \leq i, j \leq k \tag{68}$$

for some $C \geq 0$.

From (64)–(68) and the fact that $\Sigma_k \varepsilon_k = \varepsilon$ we deduce easily that z_ε^k converges in $C_b^1(H)$ to some $z_\varepsilon \in C^{1,1}(H)$ such that $|z - z_\varepsilon| \leq C_0 \varepsilon$ on H and $\partial_{ij} z_\varepsilon \in C_b^{0,1}(H)$ (in fact $C_b^{1,1}(H)$ and this is also valid for any partial derivative of any order ...) for all $1 \leq i, j < \infty$. We only have to check that for each $h, k \in H$,

$$\lim_{t \to 0+} \frac{1}{t} (Dz(x+tk) - Dz(x), h)$$

exists and is uniformly continuous on H. To do so, we denote by $k^N = (k_1, \ldots, k_N, 0, \ldots)$, $h^N = (h_1, \ldots, h_N, 0, \ldots)$ and we observe that

$$\left| \frac{1}{t} (Dz(x+tk) - Dz(x), h) - \frac{1}{t} (Dz(x+tk^N) - Dz(x), h^N) \right| \leq C_1 |h - h^N| |k| + C_1 |k - k^N| |h|$$

while

$$\left| \frac{1}{t}(Dz(x+tk^N)-Dz(x), h^N) - \sum_{i,j=1}^{N} \partial_{ij} z(x)\, h_i k_j \right| \leq C_N |k||h| t;$$

and this concludes the proof of our claim concerning the regularity of z.

We now show that z_ε is a viscosity subsolution of (58″). Now, in view of the stability of subsolutions (Proposition II.2), we only have to show that z_ε^k is a viscosity subsolution of (58′) for each k. Therefore, we fix $k \geq 1$ and for all $N \geq k$ we consider $H_N = \text{vect}(e_1, \ldots, e_N)$ and we will write indifferently

$$x = (x_1, x_2, x_3, \ldots) = (x_1, \ldots, x_N, y) = (x^N, y) = x^N + y \quad \text{where} \quad y \in H_N^\perp.$$

Let $y_0 \in H_N^\perp$. We first want to show that $z(\cdot, y_0)$ is a viscosity subsolution in $H_N \simeq \mathbf{R}^N$ of a certain equation. To do so, let $\varphi \in C^2(H_N)$ and let x_0^N be a maximum point of $z(\cdot, y_0) - \varphi(\cdot)$. Since $z \in C_b^{1,1}(H)$ we have for all $x^N \in H_N$, $y \in H_N^\perp$

$$\begin{aligned} z(x^N, y) - \varphi(x^N) &\leq z(x^N, y) - z(x^N, y_0) - (D_y z(x^N, y_0), y - y_0) \\ &\quad + z(x^N, y_0) - \varphi(x^N) + (D_y z(x^N, y_0), y - y_0) \\ &\leq \frac{C_1}{2}|y-y_0|^2 + (D_y z(x_0^N, y_0), y - y_0) + C_1 |x^N - x_0^N||y - y_0| + z(x_0^N, y_0) - \varphi(x_0^N) \\ &\leq z(x_0^N, y_0) - \varphi(x_0^N) + (D_y z(x_0^N, y_0), y - y_0) + \frac{\delta}{2}|x^N - x_0^N|^2 \\ &\quad + \frac{C_1}{2}\left(1 + \frac{1}{\delta}\right)|y - y_0|^2 \end{aligned}$$

for all $\delta > 0$.

In particular, $z(x) - \varphi(x^N) - (D_y z(x_0^N, y_0), y - y_0) - (\delta/2)|x^N - x_0^N|^2 - \frac{1}{2}C_1(1 + 1/\delta)|y - y_0|^2$ has a maximum at $x_0 = (x_0^N, y_0)$ and we may apply the definition of viscosity solutions to find

$$\begin{aligned} &-\text{Tr}(a(x_0) \cdot D^2\varphi(x_0^N)) - (b(x_0), D\varphi(x_0^N)) + c(x_0)\, z(x_0) \\ &\leq f(x_0) + (b(x_0), D_y z(x_0)) + \delta\, \text{Tr}(a_N(x_0)) + C_1\left(1 + \frac{1}{\delta}\right) \text{Tr}(a'_N(x_0)) \end{aligned}$$

where $a_N = P_N a P_N$, $a'_N = Q_N a Q_N$ and P_N, Q_N are respectively the orthogonal projections onto H_N and $H_N^\perp$. But this means that, for each $y \in H_N^\perp$, $z_y = z(\cdot, y)$ is a viscosity subsolution of

$$
\begin{aligned}
&-\operatorname{Tr}(a(\cdot\,,y)\cdot D^2 z_y)-(b(\cdot\,,y),Dz_y)+c(\cdot\,,y)\,z_y \\
&\quad\leq \tilde{f}(\cdot\,,y)+C_0|Q_N b(\cdot\,,y)|+\delta\operatorname{Tr}(a_N(\cdot\,,y))+C_1\left(1+\frac{1}{\delta}\right)\operatorname{Tr}(a'_N(\cdot\,,y)).
\end{aligned}
\tag{69}
$$

And we observe that $z_y \in C^{1,1}(H_N)$. Hence, for each $y \in H_N^{\perp}$, (69) holds a.e. in H_N (see P. L. Lions [27]). We will denote by $\tilde{f}_N$ the right-hand side of (69).

Next, we fix $y \in H_N^{\perp}$ and consider $z_\varepsilon^k(x^N, y)$ as a function of x^N only. Obviously, we have, denoting by $z_{\varepsilon,y}^k$ this function

$$
\begin{aligned}
&-\operatorname{Tr}(a(x^N,y)\cdot D^2 z_{\varepsilon,y}^k(x^N))-(b(x^N,y)\cdot Dz_{\varepsilon,y}^k(x^N))+c(x^N,y)\,z_{\varepsilon,y}^k(x^N) \\
&\quad\leq \int_{H_k}\tilde{f}_N(\bar{x}^k,x_{k+1}^N,\ldots,x_N^N,y)\,\varrho_k(x^k-\bar{x}^k)\,d\bar{x}^k+m(\varepsilon),\quad \text{a.e. } x^N\in H_N,\quad \forall y\in H_N^{\perp}
\end{aligned}
\tag{70}
$$

where $m(h)\to 0$ as $h\to 0+$ (m depends only on the bounds on $D^2 z_\varepsilon^k$, Dz_ε^k, and the moduli of continuity of σ, b, c), and $\varrho_k(x^k)=\Pi_{i=1}^k \varrho_{\varepsilon_i}(x_i^N)$, $x^N=(x^k, x_{k+1}^N,\ldots,x_N^N)$.

To conclude, we have to pass to the limit as N goes to $+\infty$: observe first that (79) holds in viscosity sense since $z_{\varepsilon,y}^k \in C^{1,1}(H_N)$ (see [27]). Then, if $x^0=(x_0^N, y_0^N)\in H_N\times H_N^{\perp}$ is a maximum point of $z_{\varepsilon,y}^k-\Phi$ (over H) where $\Phi\in X$ then in particular x_0^N is a maximum point of $z_{\varepsilon,y_0^N}^k-\Phi(\cdot\,,y_0^N)$ and (70) implies

$$
\begin{aligned}
&-\operatorname{Tr}(a(x^0)\cdot D_{x^N}^2\Phi(x^0))-(b(x^0),D_{x^N}\Phi(x^0))+c(x^0)\,z_\varepsilon^k(x^0)\leq m(\varepsilon)+C\delta+\tilde{f}(x^0) \\
&\quad+\int_{H_k}\left\{C_0|Q_N b(\bar{x}^k,x_{k+1}^0,\ldots)|+C_1\left(1+\frac{1}{\delta}\right)\operatorname{Tr}\,a'_N(\bar{x}^k,x_{k+1}^0,\ldots)\right\}\varrho_k(x^k-\bar{x}^k)\,dx^k
\end{aligned}
$$

where $x^0=(x^k, x_{k+1}^0, x_{k+2}^0,\ldots)$. By Lebesgue's lemma, the integral goes to 0 as $N\to+\infty$, hence letting N go to ∞ we deduce

$$
-\operatorname{Tr}(a(x^0)\cdot D^2\Phi(x^0))-(b(x^0),D\Phi(x^0))+c(x^0)\,z_\varepsilon^k(x_0)\leq\tilde{f}(x^0)+m(\varepsilon)+C\delta
$$

and we may conclude letting δ go to 0. □

Remark. We were unable to show that the lemma is still valid if one replaces X by X' and this is the main reason why we weakened the class of test functions in our definition of viscosity solutions. If the lemma were true for X' then our uniqueness results would still be valid for classical viscosity solutions.

IV.2. Minimum supersolution

In this section, we consider a viscosity supersolution of (29) that we denote by v and we assume (for instance) that $v \in \mathrm{BUC}(H)$. And we want to show that $v \geq u$ on H. Exactly as in P. L. Lions [27], the method of proof relies on building a "smooth" subsolution of (29), close to u, for which the comparison with v will be a simple application of the definition of viscosity supersolutions. In fact, all the difficulty lies in the construction of the approximation since we cannot use any "elliptic" regularization as we did in [27] in infinite dimensions. Instead, we will use a highly nonlinear regularization.

But, first we observe that u is also the value function of the control problem where $f(\cdot,\alpha)$, $c(\cdot,\alpha)$ are replaced by $f(\cdot,\alpha)+\lambda u(\cdot)$, $c(\cdot,\alpha)+\lambda$ for all $\lambda>0$. This can be shown using Lemma III.1 as in N. V. Krylov [23], or by using the characterization of u we obtained in the preceding section in terms of maximum viscosity subsolution. Next, we choose λ so that $c_0+\lambda>\lambda_1$. Then, we regularize $f(\cdot,\alpha)$, $c(\cdot,\alpha)$, u as follows: by the results of [24], we see that there exist for all $n\geq 1, f^n(\cdot,\alpha)$, $c^n(\cdot,\alpha)$, $\tilde{u}^n \in C_b^{1,1}(H)$ (and all bounds are uniform in α for each n) such that

$$f^n(\cdot,\alpha)\leq f(\cdot,\alpha)\leq f^n(\cdot,\alpha)+\frac{1}{n},\quad c(\cdot,\alpha)\leq c^n(\cdot,\alpha)\leq c(\cdot,\alpha)+\frac{1}{n},$$
$$\tilde{u}_n\leq u\leq \tilde{u}_n+\frac{1}{n}\quad \text{on } H.$$

Next, we consider the value function u^n of the control problem where we replace

$$f(\cdot,\alpha)+\lambda u(\cdot),\quad c(\cdot,\alpha)+\lambda \quad \text{by} \quad f^n(\cdot,\alpha)+\lambda\tilde{u}^n(\cdot),\quad c^n(\cdot,\alpha)+\lambda.$$

One readily checks from the explicit formulas that we have

$$|u^n-u|\leq\frac{C}{n}\quad \text{on } H.$$

Furthermore, the regularity results Theorems III.1 and III.4 apply and we see that for each n, u^n is Lipschitz, semi-concave on H and $\mathrm{Tr}(a^\alpha\cdot D^2u^n)$ is bounded (in viscosity sense) on H uniformly in α. Finally, by Theorem III.3, u^n is a viscosity solution of

$$\sup_{\alpha\in\mathscr{A}}\{-\mathrm{Tr}(a^\alpha\cdot D^2u^n)-b^\alpha, Du^n)+c^n(\alpha)\,u^n-f^n(\alpha)\}+\lambda(u^n-\tilde{u}^n)=0\quad \text{in } H \tag{71}$$

and thus in particular u^n is a viscosity subsolution of

$$\sup_{\alpha\in\mathscr{A}}\{-\mathrm{Tr}(a^\alpha\cdot D^2u^n)-(b^\alpha, Du^n)+c^\alpha u^n-f^\alpha\}\leq\frac{C}{n}\quad \text{in}\quad H. \tag{72}$$

The next step is to regularize u^n into $C_b^{1,1}(H)$ function which will still be (essentially) a viscosity subsolution of (72). In order to do so, we enlarge our original control problem: let $\mathcal{B}$ denote the closed unit ball of H, we replace $\mathcal{A}$ by $\mathcal{A}'=\mathcal{A}\times\mathcal{B}$ and we set $V'=V\times H$

$$\sigma(x,\alpha,\beta)=(\sigma(x,\alpha),\delta\beta),\quad b(x,\alpha,\beta)=(b(x,\alpha),0)$$
$$c(x,\alpha,\beta)=c^n(x,\alpha)+\lambda,\quad f(x,\alpha,\beta)=f^n(x,\alpha)+\lambda\bar{u}^n(x)$$
$$\forall x\in H,\quad \forall\alpha\in A,\quad \forall\beta\in\mathcal{B}$$

where $\delta>0$ is fixed. And we denote by u_δ^n the corresponding value function. One checks easily that u_δ^n satisfies

$$|u_\delta^n-u^n|\leq C_n\delta \quad \text{on } H. \tag{73}$$

Furthermore, by the regularity results Theorems III.1 and III.4, we see that the following holds

$$|u_\delta^n(x)-u_\delta^n(y)|\leq C_n|x-y| \tag{74}$$

$$\forall\xi\in H,\quad |\xi|=1,\quad (D^2u_\delta^n\cdot\xi,\xi)\geq -C_n \quad \text{on } H \tag{75}$$

$$-\mathrm{Tr}(a^{\alpha\beta}\cdot D^2u_\delta^n)\leq C_n \quad \text{on } H,\quad -\mathrm{Tr}(a^{\alpha\beta}\cdot D^2u_\delta^n)\geq -C_n \quad \text{on } H \tag{76}$$

where C_n denotes various constants independent of δ, α, β, where

$$a^{\alpha\beta}=a(\cdot,\alpha,\beta)+\tfrac{1}{2}\delta^2\beta\otimes\beta$$

and where (75), (76) hold in viscosity sense. Finally, observing that for all $\xi\in H$, we may choose $\beta=\xi|\xi|^{-1}$ so that

$$(a^{\alpha\beta}(x)\,\xi,\xi)\geq\tfrac{1}{2}\delta^2(\beta,\xi)^2=\tfrac{1}{2}\delta^2|\xi|^2$$

and thus (51) holds with $\nu=\frac{1}{2}\delta^2$, $H'=H=\omega$. Then, Theorem III.4 implies that $u_\delta^n\in C_b^{1,1}(H)$. And, by Theorem III.3, u_δ^n is a viscosity solution of

$$\frac{\delta^2}{2}\sup_{|\beta|\leq 1}-(D^2u_\delta^n\cdot\beta,\beta)+\sup_{\alpha\in\mathcal{A}}\{-\mathrm{Tr}(a^\alpha\cdot D^2u_\delta^n)-(b^\alpha,Du_\delta^n)+(c^n(\alpha)+\lambda)\,u_\delta^n-(f^n(\alpha)+\lambda\bar{u}^n)\}=0$$

in H. Since we may take $\beta=0$ in (77), we deduce immediately from (74) and (77) that u_δ^n is a viscosity subsolution of

$$\sup_{\alpha\in\mathscr{A}}\{-\mathrm{Tr}(a^\alpha\cdot D^2u^n_\delta)-(b^\alpha,Du^n_\delta)+c^\alpha u^n_\delta-f^\alpha\}\leq\frac{C}{n}+C_n\delta\quad\text{in } H. \tag{78}$$

We may now turn to our final approximation based upon Lemma IV.1 (and its proof). Indeed, (78) clearly shows that u^n_δ is a viscosity subsolution of

$$-\mathrm{Tr}(a^\alpha\cdot D^2u^n_\delta)-(b^\alpha,Du^n_\delta)+c^\alpha u^n_\delta-f^\alpha\leq\frac{C}{n}+C_n\delta\quad\text{in } H. \tag{78'}$$

for each $\alpha\in\mathscr{A}$ and $u^n_\delta\in C^{1,1}_b(H)$. Therefore, applying Lemma IV.1 for each $\alpha\in\mathscr{A}$ and observing that the construction of the regularization is independent of α, we deduce that there exists, for all $\varepsilon>0$, $u^n_{\delta,\varepsilon}\in X$ such that

$$|u^n_{\delta,\varepsilon}-u^n_\delta|\leq\varepsilon\quad\text{on } H \tag{79}$$

and $u^n_{\delta,\varepsilon}$ is a subsolution (in viscosity sense and thus pointwise since $u^n_{\delta,\varepsilon}\in X$) of

$$\sup_{\alpha\in\mathscr{A}}\{-\mathrm{Tr}(a^\alpha\cdot D^2u^n_{\delta,\varepsilon})-(b^\alpha,Du^n_{\delta,\varepsilon})+c^\alpha u^n_{\delta,\varepsilon}-f^\alpha\}\leq\frac{C}{n}+C_n\delta+C(n,\delta)\,\varepsilon\quad\text{in } H. \tag{80}$$

Finally, by (73) and (75), we deduce

$$|u-u^n_{\delta,\varepsilon}|\leq\frac{C}{n}+C_n\delta+\varepsilon\quad\text{on } H. \tag{81}$$

This tedious approximation being done, we may now conclude easily: indeed, assume first that $v-u^n_{\delta,\varepsilon}$ has a global minimum point $\bar{x}$ over H. Then, since $u^n_{\delta,\varepsilon}\in X$, we deduce from the definition of viscosity supersolution

$$\overline{\lim_{y\to x}}\sup_{\alpha\in\mathscr{A}}\{-\mathrm{Tr}(a^\alpha(\bar{x})\cdot D^2u^n_{\delta,\varepsilon}(y))-(b^\alpha(\bar{x}),Du^n_{\delta,\varepsilon}(\bar{x}))+c^\alpha(\bar{x})\,v(\bar{x})-f^\alpha(\bar{x})\}\geq 0$$

or equivalently

$$\lim_{\delta\downarrow 0+}\sup_{|y-x|\leq\delta}\sup_{\alpha\in\mathscr{A}}\{-\mathrm{Tr}(a^\alpha(y)\cdot D^2u^n_{\delta,\varepsilon}(y))-(b^\alpha(y),Du^n_{\delta,\varepsilon}(y))+c^\alpha(y)\,v(y)-f^\alpha(y)\}\geq 0.$$

And comparing this with (80), we deduce

$$c_0\inf_H(v-u^n_{\delta,\varepsilon})^-=c_0(v-u^n_{\delta,\varepsilon})^-(\bar{x})\leq\frac{C}{n}+C_n\delta+C(n,\delta)\,\varepsilon.$$

Therefore, by (81),

$$v \geq u - \frac{C}{n} - C_n \delta - C(n,\delta)\,\varepsilon \quad \text{on } H$$

and we conclude letting ε go to 0, then δ go to 0 and finally n go to 0. If $v - u^n_{\delta,\varepsilon}$ does not have a global minimum over H, we just modify the above argument using the same perturbation technique as in the preceding section. □

V. Extensions

As we already mentioned before, it is possible to extend the preceding results in various directions: we may weaken the regularity assumptions on σ, b and in particular we may require the continuity in α of the coefficients only on compact subsets of $\mathscr{A}$, next, we may treat finite horizon problems with time-dependent coefficients.

Let us also mention that everything we did easily adapts to various other control problems like for instance optimal switching, optimal stopping or optimal impulse control problems and combinations of the various possibilities .... We skip these easy variants.

We just want to mention a class of results which can be obtained using the method presented in the preceding sections (and combining it with the ideas used in P. L. Lions [27], [29]). To this end, we consider an open set $\mathcal{O}$ in H and *we now assume* that all the assumptions made in section III hold only for $x \in B_R$ (for all $R<\infty$). With these assumptions, we have now the

THEOREM V.1. (1) (Infinite horizon problem.) *Let* $v \in \mathrm{BUC}(\overline{\mathcal{O}_\delta \cap B_R})$ (*for all* $\delta>0$, $R<\infty$) *be a viscosity subsolution* (*resp. supersolution*) *of the* HJB *equation* (29) *in* $\mathcal{O}$, *where* $\mathcal{O}_\delta = \{x \in \mathcal{O}, \operatorname{dist}(x, \partial\mathcal{O}) > \delta\}$. *Then, for all* $\delta>0$, $R<\infty$ *and* $x \in \overline{\mathcal{O}_\delta \cap B_R}$, *denoting for each admissible system by* τ^R_δ *the first exit time of* X_t *from* $\overline{\mathcal{O}_\delta \cap B_R}$, *we have the following relation: for each admissible system* $\mathscr{S}$, *choosing a stopping time* θ, *then we have*

$$\begin{aligned} v(x) \leq \inf_{\mathscr{S}} \Bigg\{ & E \int_0^{\tau^R_\delta \wedge \theta} f^{\alpha_t}(X_t) \exp\left(-\int_0^t c^{\alpha_s}(X_s)\,ds\right) dt + v(X_{\tau^R_\delta \wedge \theta}) \\ & \times \exp\left(-\int_0^{\theta \wedge \tau^R_\delta} c^{\alpha_s}(X_s)\,ds\right)\Bigg\} \quad (\textit{resp.} \geq). \end{aligned} \tag{82}$$

(2) (Finite horizon problem.) *Let* $v \in \mathrm{BUC}(\overline{\mathcal{O}_\delta \cap B_R}) \times [0,T])$ (*for all* $\delta>0$, $R<\infty$) *be a viscosity subsolution* (*resp. supersolution*) *of the* HJB *equation* (30) *in* $\mathcal{O} \times (0,T)$. *Then,*

for all $\delta>0$, $R<\infty$, $x\in\overline{\mathcal{O}_\delta\cap B_R}$, $t\in[0,T]$ *and for each admissible system* $\mathcal{S}$, *let* θ *be a stopping time, we have*

$$v(x,t)\leq\inf_{\mathcal{S}}\left\{E\int_0^{t\wedge\tau_\delta^R\wedge\theta}f^{\alpha_s}(X_s)\exp\left(-\int_0^s c^{\alpha_\tau}(X_\tau)\,d\tau\right)ds\right.$$
$$\left.+v\left(X_{t\wedge\tau_\delta^R\wedge\theta},t-t\wedge\tau_\delta^R\wedge\theta\right)\exp\left(-\int_0^{t\wedge\tau_\delta^R\wedge\theta}c^{\alpha_s}(X_s)\,ds\right)\right\}\quad(\textit{resp.}\geq).\tag{83}$$

□

In particular, one can use the preceding result if $\mathcal{O}=H$, with $\theta=\tau^R$: this is useful when one wants to relax the boundedness assumptions made upon the coefficients. Let us give one example in that direction, concerning finite horizon problems (in fact all the general results along this line given in [27] adapt to the situation here). To the assumptions made above, we add that $c\equiv0$ (to simplify notations) and

$$\|\sigma(x,\alpha)\|^2\leq C+C|x|^2,\quad\forall x\in H,\quad\forall\alpha\in\mathcal{A}\tag{84}$$

$$2(x,b(x,\alpha))\leq C+C|x|^2,\quad\forall x\in H,\quad\forall\alpha\in\mathcal{A}\tag{85}$$

$$|f(x,\alpha)|\leq C+C|x|,\quad|g(x)|\leq C+C|x|,\quad\forall x\in H,\quad\forall\alpha\in\mathcal{A}\tag{86}$$

where $C\geq0$. Furthermore, we make the same assumptions on $D_x\sigma$, $D_x b$ as in section III and we assume (33)–(36). Then, the value function is Lipschitz in $(x,t)\in H\times[0,T]$ ($\forall T<\infty$), semi-concave in x uniformly in $t\in[0,T]$ ($\forall T<\infty$) and Theorem III.4 is still valid in this case. Now, if $v\in\mathrm{BUC}_{\mathrm{loc}}(H\times[0,T])$ is a viscosity subsolution (resp. supersolution) of (30) in $H\times(0,T)$ such that $v|_{t=0}\leq g$ on H (resp. $v|_{t=0}\geq g$) then we still have $v\leq u$ in $H\times[0,T]$ (resp. $v\geq u$) provided v satisfies the following growth condition (for instance)

$$v(x)\leq C(1+|x|^m)\quad(\text{resp.}\geq-C(1+|x|^m)),\quad\text{for some}\quad C\geq0,\ m\in(0,\infty).\tag{87}$$

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