# Regularity of variational maximal surfaces

by

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## Introduction

In [B1] we showed that the Dirichlet problem for the prescribed mean curvature (PMC) equation in a Lorentzian manifold is solvable, provided the boundary surface has bounded mean curvature and admits a strictly spacelike spanning hypersurface. For a more precise statement see [B1] Section 4; when the spacetime is conformal to a product, this result is due to Gerhardt [G]. However, in the special case of Minkowski space much more is true [BS]: the only condition on the boundary data is that it admit a weakly spacelike spanning hypersurface (which could be the graph over a domain with arbitrarily rough boundary), and then the solution of the associated variational problem is a (classical) solution of the Dirichlet problem, except for a singular set consisting of light rays within the solution surface and extending between boundary points.

In this paper we show that this situation holds in general; the Dirichlet problem is solvable for rough boundary data and the variational problem has a solution which is strictly spacelike away from a singular set consisting of null rays. Although the solutions cannot be unique in general, we do have uniqueness "in the small", or if some curvature conditions are satisfied. This latter situation is well-known ([BF], [CB], [MT]). Using these results and an idea of Klaus Ecker [E], we will show elsewhere [B2] an improvement of the Hawking singularity theorem ([HE] p. 272, see also [Ge], [Ga]), based on an existence result for constant mean curvature surfaces in cosmological spacetimes. In [B3] we give a fairly complete survey of the regularity theory and describe the major applications of these results.

In some early physics papers concerning maximal surfaces [A], [Ge], it was assumed that a variational extremal surface, if it existed, would be smooth. Our results show that this assumption is only half-way correct: as well as showing that the

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variational solution is *a priori* bounded (and thus exists), it is necessary to show that it does not contain *entire* null lines (i.e. the singular set (3.13) is empty). Verification of these conditions generally relies on suitable *a priori* conditions on the geometry of the problem, such as existence of barriers or causality conditions on cl($D(S)$). This amounts to controlling the nature of the singularities of the spacetime, since the existence of barriers says the singularity is crushing [ES], whilst the "standard data set" condition on $D(S)$ (see Section 2) implies $H(S)$ does not have bad causality structures such as closed null loops. This is essential for the treatment of the Dirichlet problem in Sections 3 and 4. However, the treatment of the variational problem in Section 6 already assumes the existence of a locally extremal hypersurface and thus sidesteps these singularity problems.

The approach taken is quite different from that of [BS], primarily because there is not the direct relationship between the Dirichlet and variational problems that holds in Minkowski space. Thus, we first solve the Dirichlet problem in general (Section 4) using an interior gradient bound based on [B1] Theorem 3.1 and appropriately constructed time functions (Section 3). This construction involves some delicate estimates on the Lorentz distance function $l(p, q)$ and leads naturally to the singular set $\Sigma$. The interior gradient estimate is new even for Minkowski space, although a simplified version can be derived from the estimates of Cheng and Yau [CY]. This is described in Section 3. When the linearisation of the PMC operator is invertible we can construct local foliations with prescribed curvature and a resulting integral identity can be used to relate solutions of the Dirichlet and variational problems when the boundary data is smooth. The main result of Section 5 is an eigenvalue estimate for the linearised PMC operator over small domains with arbitrary (smooth) boundary, which implies invertibility. A corollary is that classical solutions are locally unique and locally maximising. In the final section we use all the above results to show the regularity of a weakly spacelike hypersurface which is locally extremal for some variational problem. Because this surface may have rough boundary, the foliation results of Section 5 do not apply immediately; it is possible that the local foliation may develop a "gap" and most of the work is devoted to handling this case.

The final results for the variational problem require only that the metric be $C^2$ and the manifold be time-orientable, since we work only locally. However, the preliminary results on the Dirichlet problem require also that $D(S)$ be properly contained in a compact globally hyperbolic set. This is only mildly restrictive because of [HE] 6.6.3 (see also [O'N] 14.38): int$D(S)$ is globally hyperbolic for an achronal set $S$. Although the basic existence theorem for the Dirichlet problem, Theorem 4.1, requires the hypersurface be achronal and the spacetime metric $C^2$, we show that this can be weakened to (essentially) $C^{0,1}$ metric (Theorem 4.3), assuming a condition on the distributional components of the curvature, and to immersed hypersurfaces (Theorem 4.2). This last generalisation is based on the simple idea of "$T$-homotopy" which should be useful elsewhere. In particular, it allows us to generalise some results of Quien [Q]. The final results for the variational problem are completely local and thus should be widely applicable. As in [B1], we have to assume the mean curvature function is $C^1$, whereas the estimates of [BS] and [G] required only bounded mean curvature since they relied on integral methods. It may be that the maximum principle method extends to this case also.

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## 2. Notation and basic concepts

Let $\mathcal{V}$ be a smooth $(n+1)$-dimensional manifold with $C^2$ Lorentz metric $g$, connection $\nabla$ and curvature tensors Riem and Ric. We use the notations $ds^2$ and $\langle \cdot, \cdot \rangle$ for $g$, and the summation convention with ranges $0 \leq \alpha, \beta \leq n$, $1 \leq i, j \leq n$. Constants depending only on $n$ will be denoted $c$, and those depending on geometric quantities by $C$. We suppose that $\mathcal{V}$ is time-orientable and that $T$ is a $C^2$ unit timelike vector field on $\mathcal{V}$. Both $\mathcal{V}$ and $T$ will remain fixed throughout this paper.

From $T$ we construct a reference Euclidean metric

$$g_E = g + 2T \otimes T \tag{2.1}$$

(in local coordinates, $g_{E\alpha\beta} = g_{\alpha\beta} + 2T_\alpha T_\beta$), which we use to measure the size of tensors and their covariant derivatives. Thus, we define the supremum norms, for any tensor $\Phi$,

$$\begin{aligned} \|\Phi\|(x) &= (g_E(\Phi, \Phi)(x))^{1/2}, \quad x \in \mathcal{V} \\ \|\Phi\| &= \sup\{\|\Phi\|(x) : x \in \mathcal{V}\} \\ \|\Phi\|_k &= \sum_{j=0}^{k} \|\nabla^j \Phi\|, \end{aligned} \tag{2.2}$$

and use the notation $\|\cdot\|_{k;\mathcal{U}}$ to indicate the supremum taken over a subset $\mathcal{U} \subset \mathcal{V}$. The Riemannian geodesic distance function $d(x, y)$ of $g_E$ makes $\mathcal{V}$ a metric space; using $d(x, y)$ we can define the (Caratheodory) distance between two sets $A, B$ by

$$d(A, B) = \inf\{d(x, y) : x \in A,\ y \in B\}$$

and their Hausdorff distance,

$$d_H(A,B)=\max(\sup\{d(x,B):x\in A\},\sup\{d(A,y):y\in B\}).$$

We use $l(x,y)$ to denote the Lorentzian distance function. Convergence of sets will be taken in the sense of Hausdorff distance, unless specified otherwise. For $\varepsilon>0$ we define the $\varepsilon$-subset $A^{(\varepsilon)}$ of $A$ by

$$A^{(\varepsilon)}=\{x\in A,\ d(x,\mathcal{V}-A)>\varepsilon\}. \tag{2.3}$$

We use $\mathrm{cl}(A)$ and $\mathrm{int}(A)$ to denote the topological closure and interior of $A$ and then $b(A)=\mathrm{cl}(A)-\mathrm{int}(A)$ is the topological boundary. The closure, interior, boundary with respect to a subset $\mathcal{U}\subset\mathcal{V}$ will be denoted $\mathrm{cl}(A;\mathcal{U})$, etc. Recall $A$ is precompact if it has compact closure and $A\subset\subset B$ ($A$ strictly contained in $B$) means $A$ is precompact and $\mathrm{cl}(A)\subset\mathrm{int}(B)$.

A *time function* $t\in C^1(\mathcal{U})$, $\mathcal{U}\subset\mathcal{V}$ has everywhere past-timelike gradient $\nabla t$. Using the integral curves of $\nabla t$ to transport coordinates from a fixed level set of $t$, we obtain the zero-shift coordinates $(t,x)$ of $t$, in which the metric becomes

$$ds^2=-\alpha^2dt^2+g_{ij}\,dx^i\,dx^j, \tag{2.4}$$

where $\alpha=\alpha(x,t)$ is the lapse function of $t$. Unlike [B1], we do not need to assume $\mathcal{V}$ has a global time function.

We shall use the notations of Hawking and Ellis in describing causal relationships and refer there for terms not defined here. Recall a set $A$ is *achronal* (resp. *acausal*) if no pair of points $p,q\in A$, $p\neq q$ can be joined by a timelike (resp. nonspacelike) curve. The *future domain of influence* of $A$ is

$$I^+(A)=\{p\in\mathcal{V}:\exists q\in A \text{ such that } q\ll p\},$$

where $q\ll p$ if there is a future timelike curve from $q$ to $p$, and the *future domain of dependence* of $A$ is

$$D^+(A)=\{p\in\mathcal{V}:\text{ every past-inextendible nonspacelike curve }\gamma\text{ with } \gamma(0)=p\text{ intersects }A\}.$$

The past domains of influence and dependence, $I^-(A)$ and $D^-(A)$, are of course defined dually. We also set

$$D(A)=D^+(A)\cup D^-(A),\quad I(A)=I^+(A)\cup I^-(A).$$

Note that in general $A \not\subset I(A)$, but always $A \subset D(A)$. Recall ([HE],[O'N]) that if $A$ is globally hyperbolic, the time separation function

$$l(p, q) = \sup\{\text{length}(\gamma);\ \gamma \text{ is a nonspacelike curve from } p \text{ to } q\}$$

is defined and continuous for $p, q \in A$, and $l(p, q)$ is realised by a nonspacelike geodesic (timelike if $l(p, q) > 0$).

Since we will be talking a lot about hypersurfaces, some definitions will help: $S \subset \mathcal{V}$ is a *weakly spacelike hypersurface* (WSH) if for each $p \in S$ there is a neighbourhood $p \in \mathcal{U}$ such that

$$S \cap \mathcal{U} = b(I^+(S \cap \mathcal{U}; \mathcal{U}); \mathcal{U}). \qquad \text{(WSH)}$$

By [HE] 6.3.1. this is equivalent to

$$S \cap \mathcal{U} \text{ is an embedded, achronal, } C^{0,1} \text{ hypersurface which is closed in } \mathcal{U}. \qquad \text{(WSH}')$$

A useful consequence of this definition is that $S$ is *locally separating*: for each $p \in S$ there is a neighbourhood $p \in \mathcal{U}$ such that $\mathcal{U} = \mathcal{U}^+ \cup \mathcal{U}^- \cup (S \cap \mathcal{U})$ is a disjoint union where $\mathcal{U}^\pm = I^\pm(S \cap \mathcal{U}; \mathcal{U})$, and for any curve $\gamma: [0; 1] \to \mathcal{U}$ with $\gamma(0) \in \mathcal{U}^-$, $\gamma(1) \in \mathcal{U}^+$, there is $0 < s < 1$ such that $\gamma(s) \in S \cap \mathcal{U}$.

We define the boundary of a WSH $S$ by

$$\partial S = \text{cl}(S) - S; \qquad (2.5)$$

then since $S$ is locally separating it is easy to show that if $S$ is achronal then

$$\partial S = \text{edge}(S)$$

where edge($S$) is defined in [HE] p. 202. This boundary is clearly more general than a classical manifold-with-boundary. Two examples which are included are (a) $\partial S$ contains isolated points (so we do not want $S$ to be closed) and (b) $\partial S$ is a graph over the boundary of an arbitrary bounded set $\Omega \subset \mathbf{R}^n \subset \mathbf{R}^{n,1}$. The second example shows that this definition generalises the boundary data definitions of [BS].

A $C^{k,\alpha}$ *regular hypersurface* $M$ is a WSH which is locally $C^{k,\alpha}$ for some $k \geq 1$ and $0 < \alpha < 1$ and has everywhere timelike normal vector. (By *regular* we shall mean $C^{2,\alpha}$ regular.) We say $M$ is *uniformly regular* if $M \cup \partial M$ is a $C^3$ submanifold with boundary (in the classical sense) and has timelike normal vector on $M \cup \partial M$. For a $C^{2,\alpha}$ regular hypersurface $M$ we can define the following quantities:

- future unit normal vector $N$
- tilt factor $\nu=-\langle T,N\rangle$
- second fundamental form $A(X,Y)=\langle X,\nabla_Y N\rangle$, for $X$, $Y$ tangent to $M$
- mean curvature $H=\mathrm{tr}_M A$
- induced gradient $\nabla^M$
- induced Laplacian $\Delta_M$
- volume form $dv_M$.

As in [B1], we say that a regular hypersurface $M$ satisfies the *mean curvature structure conditions* (MCSC) if there is a constant $\Lambda$ such that

$$|H_M|\leq\Lambda\nu \qquad \text{(MCSC1)}$$

$$|\nabla^M H_M|\leq\Lambda(\nu^2+\nu|A|) \qquad \text{(MCSC2)}$$

where $|\cdot|$ measures length on $M$. For example, if $\varphi\in C^1(\mathcal{V})$, $X\in C^1(T\mathcal{V})$ and $F\colon T\mathcal{V}\to\mathbf{R}$ is defined by

$$F(p,v)=\varphi(p)+\langle X,v\rangle \tag{2.6}$$

and the mean curvature $H_M$ of $M$ satisfies

$$H_M(p)=F(p,N(p))=\varphi(p)+\langle X,N\rangle \quad \text{for all} \quad p\in M,$$

then $M$ satisfies the MCSC with constant $\Lambda=\|\varphi\|_1+\|X\|_1$, independent of $M$. (If $M$ is uniformly regular and there is a suitable time function then by [B1] Theorem 3.1, there is a global bound on $\nu$, depending only on $\Lambda$, $\|H_{\partial M}\|$ and the time function.)

From the triangle inequality in the unit hyperboloid we have

LEMMA 2.1. *Suppose $T_1$, $T_2$, $T_3$ are unit future timelike vectors. Then*

$$\begin{gathered}\operatorname{arcosh}|\langle T_2,T_3\rangle|\leq\operatorname{arcosh}|\langle T_1,T_2\rangle|+\operatorname{arcosh}|\langle T_1,T_3\rangle|\\ 1\leq-\langle T_2,T_3\rangle\leq 2\langle T_1,T_2\rangle\langle T_1,T_3\rangle.\end{gathered} \tag{2.7}$$

Motivated by graphs, we make the definition:
the WSH's $S_0$, $S_1$ are *T-homotopic*,

$$S_0\approx S_1,$$

if there is a (continuous) map

$$h\colon S_0\times[0;1]\to\mathcal{V}$$

such that

$$S_t = h(S_0 \times \{t\}) \quad \text{is a WSH, for all } t \in [0;1],$$
$$h\colon S_0 \times \{t\} \to S_t \quad \text{is a homeomorphism,}$$
$$h(x,\cdot)\colon [0;1] \to \mathcal{V} \quad \text{has image in an integral curve of } T.$$

If in addition $\partial S_t = \partial S_0$ for all $t \in [0;1]$ then we say $S_0, S_1$ are *T-homotopic* rel $\partial S_0$;

$$S_0 \approx S_1 \ \operatorname{rel} \partial S_0.$$

The equivalence classes of this relation are natural spaces in which to consider the Dirichlet and variational problems. This will become clear in Section 4, especially with the generalisation to immersed WSH in Theorem 4.2. If the boundary is fixed and the surfaces are precompact, we see that the $T$-homotopy class does not depend on the choice of reference timelike vector $T$. The following useful lemma is a fairly straightforward consequence of the definitions of WSH and $T$-homotopy and the causal geometry results of [HE] (see also [O'N] Chapter 14):

LEMMA 2.2. *If $S$ is a* WSH *such that $S$ is achronal in $K$,*

$$D(S) \subset\subset K \subset \mathcal{V},$$

*where $K$ is compact and globally hyperbolic, and $M$ is a* WSH *with*

$$M \approx S \quad \operatorname{rel} \partial S$$

*then* $M \subset \operatorname{cl}(D(S))$.

*Proof.* Let $H^+, H^-$ denote the future, past horizons of $S$ in $K$, and let $H(S) = H^+ \cup H^-$ ([HE], [O'N]). Now $b(D(S))$ consists of null geodesics, with endpoints (past for $H^+$, future for $H^-$) on $\partial S$. Thus if $S_t \cap H(S) \neq \emptyset$, then $S_t$ must contain a null geodesic ending on $\partial S$, and since $S_t$ is weakly spacelike, the $T$-homotopy cannot push this ray out of $\operatorname{cl}(D(S))$. The proof follows by following $S_t$, $t \to 1$. □

*Remark.* The condition involving $K$ is fundamental to our work on the Dirichlet problem, saying roughly that $b(D(S))$ is bounded and does not meet any "singularities" (metric/curvature or causal, such as closed null loops) of $\mathcal{V}$. Any weakening of this condition will need to be balanced by additional information about the singular structure of $b(D(S))$ and/or a priori height bounds from barrier surfaces (e.g. the cosmological problem [G], [B1]) or pde height estimates (e.g. maximal surfaces in asymptotically flat spacetimes [B1]).

Motivated by this lemma we made the definition: $(S, K)$ is a *standard data set* if $S$ is a WSH, $K$ is a globally hyperbolic, compact set with

$$D(S) \subset\subset K$$

and $S$ is achronal with respect to $K$. Note that this definition implies that $\partial S \neq \varnothing$: the case where $S$ is a compact Cauchy surface (with $\partial S = \varnothing$) has been quite adequately treated in [G], [B1].

### 3. Interior gradient estimates

The basic estimate (3.1) follows from a maximum principle argument similar to that in [B1] Theorem 3.1 and the full interior estimates (Theorem 3.7) follow in turn from the basic estimate and the existence of "approximating time functions", which are constructed in Corollary 3.3. This construction is nearly optimal since the singular set $\Sigma$ ((3.13), see also [BS] Corollary 4.2) arises naturally. As straightforward consequences of the interior estimates we get the convergence Theorem 3.8 and the contained light ray Corollary 3.9 (compare [BS] Theorem 3.2).

THEOREM 3.1. *Let $M$ be a regular hypersurface satisfying the structure conditions* (MCSC) *and suppose $\tau \in C^2(\mathcal{V})$ is a time function in the region $\{\tau \geq 0\}$ such that*

$$M_{\tau \geq 0} \text{ is compact} \quad \text{and} \quad \partial M \cap \{\tau > 0\} = \varnothing,$$

*where $M_{\tau \geq a} = \{\tau \geq a\} \cap M$ for $a \in \mathbf{R}$. Further suppose there are constants $C_1$, $C_2$, $C_3$ such that*

$$\langle \nabla\tau, \nabla\tau \rangle \leq -C_1^{-2}$$
$$\|\tau\|_2 \leq C_1$$
$$\|T\|_2 \leq C_2$$
$$\|\mathrm{Ric}\| \leq C_3,$$

*where the norms $\|\cdot\|$ are taken over the region $\{\tau \geq 0\}$. Then there are constants $\tau_0 > 0$, $C$ such that in $M_{\tau > 0}$,*

$$\log \nu + f(\tau) \leq C(1 + \tau_{\max}) \tag{3.1}$$

*where*

$$\tau_{\max} = \max\{\tau(p), p \in M\}$$

*and* $f \in C^{1,1}(\mathbf{R}^+)$ *is defined by*

$$\begin{cases} n\log(\tau) & \text{for } 0<\tau<\tau_0 \\ n(\tau/\tau_0+\log(\tau_0)-1) & \text{for } \tau \geqslant \tau_0. \end{cases}$$

*In particular, for any* $\varepsilon>0$, *there is a constant* $C(\varepsilon^{-1}, \Lambda, C_1, C_2, C_3, \tau_{\max})$ *and the a priori estimate*

$$\nu(p) \leqslant C(\varepsilon^{-1}, \Lambda, C_1, C_2, C_3, \tau_{\max}) \quad \text{for all} \quad p \in M_{\tau \geqslant \varepsilon}. \tag{3.2}$$

*Proof.* Note that, in contrast to the situation in [B1], the vectors $\nabla\tau$ and $T$ are not linearly dependent. If we let $T_1$ be the future unit normal to the $\tau$-foliation and define

$$\nu_1 = -\langle T_1, T\rangle, \quad \nu_2 = -\langle T_2, N\rangle,$$

then we can estimate

$$\nu_1 \leqslant |\nabla\tau|^{-1}\|\tau\|_1 \leqslant C_1^2,$$
$$\nu_2 \leqslant 2\nu\nu_1,$$

using the triangle inequality. In the following calculations we use the fact that the components of $N$ and unit tangent vectors to $M$, with respect to a $T$-adapted orthonormal frame, are estimated by $\nu$.

We now apply the maximum principle argument of [B1] Theorem 3.1 to the function

$$\varphi(\nu, \tau) = \operatorname{arcosh}(\nu) + f(\tau).$$

Since $f(0) = -\infty$, $\varphi$ attains its maximum in $M_{\tau>0}$ and at the maximum point we have

$$\begin{aligned} \nabla^M \operatorname{arcosh}(\nu) + f'(\tau)\, \nabla^M \tau &= 0 \\ \Delta_M \operatorname{arcosh}(\nu) + f'(\tau)\, \Delta_M \tau + f''(\tau)\, |\nabla^M \tau|^2 &\leqslant 0. \end{aligned} \tag{3.3}$$

For the purposes of estimation, define $\varepsilon_k = k/4n^2$, $k=1,\ldots,4$, and let $\psi = \operatorname{arcosh}(\nu)$. Using [B1] Proposition 2.1 and estimating $\|\mathscr{L}_T g\|_1$ by $\|T\|_2$, we have

$$\Delta_M \psi \geqslant \operatorname{cotanh}(\psi)\left((1-\varepsilon_1/2n)\,|A|^2 - |\nabla^M \psi|^2 - C\nu^2\right), \tag{3.4}$$

where $C = C(C_1, C_2, C_3, \Lambda)$. The Schwarz inequality gives

$$|A|^2 \geqslant (1 + 1/(n-1) - \varepsilon_1)\,\lambda_1^2 - cH^2$$

where $\lambda_1$ is the eigenvalue of $A$ with greatest magnitude. Now $T^M$, the projection of $T$ tangent to $M$, has length $|T^M|^2=\nu^2-1$, so as in [B1],

$$|\nabla^M\nu|^2=-A(\nabla^M\nu, T^M)-\langle N, \nabla_{\nabla^M\nu}T\rangle$$
$$\leq|\nabla^M\nu|\sqrt{\nu^2-1}(|\lambda_1|+\nu\|T\|_1).$$

Thus

$$\lambda_1^2\geq(1-\varepsilon_1)|\nabla^M\psi|^2-C\nu^2,$$

and from the structure conditions we have

$$|A|^2\geq(1+1/(n-1)-\varepsilon_2)|\nabla^M\psi|^2-C\nu^2.$$

From [B1] 2.8, the structure conditions and $C_1$ we have

$$\Delta_M\tau\geq-C\nu^2.$$

Substituting everything into (3.3) we find at the maximum point of $\varphi$

$$\left(\left(\frac{1}{n-1}-\varepsilon_3\right)f'^2+\tanh(\psi)f''\right)|\nabla^M\tau|^2\leq C\nu^2(1+f'). \tag{3.5}$$

Now, the triangle inequality for hyperbolic angles (Lemma 2.1) gives

$$\nu\leq 2\nu_1\nu_2$$

and since $|\nabla^M\tau|^2=|\nabla\tau|^2(\nu_2^2-1)$ and $\nu_1$ is bounded we have

$$\nu^2\leq C(|\nabla^M\tau|^2+1).$$

Substituting this into (3.5) gives

$$\left(\left(\frac{1}{n-1}-\varepsilon_3\right)f'^2+\tanh(\psi)f''\right)|\nabla^M\tau|^2\leq C(1+f')(1+|\nabla^M\tau|^2)$$

and since $\varepsilon_3<n^{-2}$, the choice of $f$ shows that

$$|\nabla^M\tau|^2\leq C(\tau_0+\tau_0^2)(1+|\nabla^M\tau|^2).$$

Thus, choosing $\tau_0$ sufficiently small will bound $|\nabla^M\tau|$ and hence at the maximum point of $\varphi$ we have

$$\nu\leq C.$$

Since $f(\tau)\leq C\tau$, this means

$$\varphi(\nu,\tau)\leq\varphi_{\max}\leq C(1+\tau_{\max})$$

which gives the required estimates. □

*Remarks.* (1) By analysing the curvature terms more closely [B1] we see the conditions on Ric and $\nabla^2 T$ can be weakened to

$$\mathrm{Ric}(N,N)\geq -C_3\nu^2,\quad \|\mathcal{L}_T g\|_1\leq C_2,$$

where $\mathcal{L}_T$ is the Lie derivative.

(2) From (3.1) and the definition of $f(\tau)$ we see $\nu=O(\tau^{-n})$ as $\tau\downarrow 0$ and it is clear from the proof that this can be improved to $O(\tau^{-(n-1+\varepsilon)})$ for any $\varepsilon>0$. This is nearly optimal, as can be seen from the spherically symmetric solutions in [BS].

The following approximation result constructs time functions adapted to a given hypersurface; we have in mind in particular the case where $S$ is a null surface. In that case the estimates are optimal, but if $S$ is a regular hypersurface then of course much better is possible.

PROPOSITION 3.2. *Let $(S,K)$ be a standard data set and let*

$$\mathcal{U}=K\cap I^+(S).$$

*Since $K$ is globally hyperbolic we can define*

$$l(x)=\sup\{l(y,x): y\in S\}\quad \textit{for}\quad x\in\mathcal{U}$$

*where $l(y,x)$ is the Lorentzian distance function* ([HE] 6.7). *Then $l(x)$ is Lipschitz and satisfies*

$$|l(x)-l(y)|\leq Cd(x,y)/\min(l(x),l(y)) \tag{3.6}$$

*for $x$, $y\in\mathcal{U}$ such that $d(x,y)\leq C^{-1}\min(l(x),l(y))^4$, and*

$$l(\gamma(s))-l(\gamma(0))\geq s\quad \textit{for}\quad s\geq 0, \tag{3.7}$$

*for any future directed unit speed geodesic $\gamma\subset\mathcal{U}$.*

*Proof.* Let $\beta=\beta_x(s)$ denote the geodesic from $\mathrm{cl}(S)$ to $x$ which realises $l(x)$. Since $K$ is globally hyperbolic it admits a time function, $t$ say, with lapse $\alpha$ and normalised gradient $T_1 = -\alpha \nabla t$. We can reparameterise $\beta$ by $t$, since along $\beta$,

$$\frac{dt}{ds} = \langle \beta', \nabla t \rangle = -\alpha^{-1} \langle \beta', T_1 \rangle \geq \min(\alpha^{-1}) > 0. \tag{3.8}$$

By the triangle inequality,

$$\tfrac{1}{2} \langle T, T_1 \rangle^{-1} \langle \beta', T \rangle \leq \langle \beta', T_1 \rangle \leq 2 \langle T, T_1 \rangle \langle \beta', T \rangle$$

so that (setting $\lambda = -\langle \beta', T \rangle \geq 1$),

$$C^{-1} \lambda^{-1} \leq \frac{ds}{dt} \leq C \lambda^{-1} \tag{3.9}$$

where $C = C(\alpha, |\langle T, T_1 \rangle|)$ depends only on the (fixed) time function $t$. Now letting $t_0 = t(\beta(0))$, $t_1 = t(x)$, we have

$$l(x) = \int_{t_0}^{t_1} \frac{ds}{dt} dt$$

and by (3.9) it remains to estimate $\langle \beta', T \rangle$ along $\beta$. Since $\nabla_{\beta'} \beta' = 0$,

$$\begin{aligned} \left| \frac{d\lambda}{dt} \right| &\leq \left| \frac{ds}{dt} \right| |\langle \beta', \nabla_{\beta'} T \rangle| \\ &\leq C |\lambda| \, \|\nabla T\|, \end{aligned}$$

so by Grönwalls inequality, for any $s_1, s_2 \in [0; l(x)]$,

$$\begin{aligned} \lambda(s_1) &\leq \exp(C \|\nabla T\| (t_1 - t_0)) \lambda(s_2) \\ &\leq C \lambda(s_2), \end{aligned}$$

where $C$ is independent of $x$ and $\beta$. From (3.9) and this estimate, there is a constant $C$ independent of $x \in \mathcal{U}$ such that

$$C^{-1} \leq l(x) |\langle \beta', T \rangle| (x) \leq C. \tag{3.10}$$

We now use this to show (3.6). We may assume $l(x) > l(y)$. Introduce $g$-geodesic normal coordinates $(z^\alpha)$ in a convex normal neighbourhood $\mathcal{W}$ of $x$ so that $\partial_{z^0} = T(x)$, and let $\bar{z} = \Lambda z$, $\Lambda \in SO(n, 1)$ be Lorentz-transformed $g$-geodesic coordinates such that

$\partial_{\bar{z}^0}|_x=\beta'(x)$, so that $\|\Lambda\|\leq c|\langle\beta', T\rangle|\leq c\lambda$. Letting $|z|^2=(\Sigma_0^n(z^\alpha)^2)^{1/2}$ and denoting the Minkowski metric by $\eta$, in $\mathscr{W}$ we have

$$\|g-\eta\|\leq\|\text{Riem}\|\,|z|^2\leq C\varepsilon^2$$

if $d(x,z)\leq\varepsilon$. Defining the distance function on $\mathscr{W}$,

$$\bar{d}(x,p)=\left(\sum_0^n(\bar{z}^\alpha)^2\right)^{1/2}\quad\text{where}\quad p=(\bar{z}^\alpha),$$

from (3.10) we have the estimates

$$C^{-1}l d(x,p)\leq\bar{d}(x,p)\leq C l^{-1}d(x,p). \tag{3.11}$$

Letting $\bar{g}$ be the Lorentz metric in the $(\bar{z}^\alpha)$ coordinates, we have

$$\begin{aligned}\|\bar{g}-\eta\|=\|\Lambda g'\Lambda-\eta\|&\leq\|\Lambda\|^2\|g-\eta\|\\&\leq C\lambda^2\varepsilon^2\leq Cl(x)^{-2}\varepsilon^2\end{aligned}$$

in $B_\varepsilon=\{p\in\mathscr{W}: d(x,p)\leq\varepsilon\}$. We suppose $\varepsilon$ chosen small enough that $\|\bar{g}-\eta\|\leq10^{-4}$ (say). Let $x_1=\beta(s_0)$, $s_0<l(x)$, be null-separated from $y$ so that

$$\bar{d}(x,x_1)\leq2\bar{d}(x,y)$$

for $x_1$, $y\in B_\varepsilon$. The existence of $x_1$ in the almost-Minkowski neighbourhood $B_\varepsilon$ is ensured if $d(x,y)\leq\varepsilon\lambda^{-2}$ (calculation) and we then have

$$\begin{aligned}l(x)&\leq l(x_1)+l(x_1,x)\\&\leq l(y)+\bar{d}(x,x_1)\\&\leq l(y)+2\bar{d}(x,y)\end{aligned}$$

which gives, using (3.11) and (3.10),

$$l(x)-l(y)\leq Cl(x)^{-1}d(x,y)$$

for $d(x,y)\leq\varepsilon\lambda^{-2}\leq C^{-1}l(x)^4$. This gives (3.6), and (3.7) is just the reverse triangle inequality. □

By smoothing we now have

COROLLARY 3.3. *In the setting of Proposition* 3.2, *let* $\varepsilon>0$ *be given. With* $K^{(\varepsilon)}$ *defined by* (2.3), *there is* $\tau\in C^{\infty}(K^{(\varepsilon)})$ *satisfying*

(1) $\tau$ *is a time function for* $\tau\geqslant 0$,
(2) $d_H(\{\tau=0\},S')\leqslant 2\varepsilon$, *where* $S'=b(I^+(S))\cap K^{(\varepsilon)}$,
(3) $l(x)/2\leqslant\tau(x)+\varepsilon\leqslant 2l(x)$ *for* $\tau(x)\geqslant 0$,
(4) $\|\nabla\tau\|(x)\leqslant C/(\tau(x)+\varepsilon)\leqslant C\varepsilon^{-1}$.

*Proof.* By (3.6), $l$ is differentiable almost everywhere in $I^+(S)$, with $-\nabla l(x)=\beta'(x)$ a uniformly timelike unit vector in $l(x)\geqslant\delta>0$ by (3.7) and (3.10). Standard causality results show that $l(x)=0$ exactly when $x\in b(I^+(S))$, so setting $l(x)=0$ for $x\in K-I^+(S)$ makes $l\in C^0(K)$. Mollifying $l$ with parameter sufficiently small (depending on $\varepsilon$ and $K$) produces $\tau+\varepsilon$, a function approximating $l$ and with uniformly timelike gradient for $\tau\geqslant 0$. Now (2) follows by noting that $d(x,S)\leqslant l(x)$ for all $x\in I^+(S)$. □

*Remarks.* (1) Since the interior gradient bound (3.1) depends on $\|\nabla^2\tau\|$, it would be helpful to estimate $\|\nabla^2 l\|$. This appears to be rather more difficult than the $\|\nabla l\|$ estimate, but fortunately such a bound is not essential to the arguments to follow—we just have to deal with non-explicit interior estimates.

(2) By adapting the construction of [B1] Proposition 3.2, we could combine a sequence of such time functions $\tau_\varepsilon$ to construct $\tau^*$, a time function for $\tau^*>0$ and such that $S=\{\tau^*=0\}$. We will not need this result.

(3) Similar constructions give time functions in $I^-(S)\cap K$.

(4) The level sets $\{\tau=0\}$ give $C^\infty$ regular (spacelike) hypersurfaces which approximate $S$, even when $S$ is a null surface.

From these time functions we derive

PROPOSITION 3.4. *Suppose* $(S,K)$ *is a standard data set. For any* $\varepsilon>0$ *there are functions* $\tau_\varepsilon^+$, $\tau_\varepsilon^-$, *in* $C^\infty(K^{(\varepsilon)})$, *time functions in* $\{\tau_\varepsilon^+(x)>0\}$, $\{\tau_\varepsilon^-(x)<0\}$ *respectively, such that the sets*

$$I_\varepsilon=\{x\in K^{(\varepsilon)}\colon \tau_\varepsilon^+(x)>0 \text{ or } \tau_\varepsilon^-(x)<0\} \tag{3.12}$$

*form an exhaustion (relative to K) of*

$$I=I^+(S)\cup I^-(S)\cup(S-\Sigma)$$

*(in the sense that there is a sequence* $\varepsilon_k\downarrow 0$ *such that* $I_{\varepsilon_j}\subset\subset I_{\varepsilon_k}$ *for* $j<k$ *and* $I=\bigcup_{\varepsilon>0}I_\varepsilon$*).*

*Here* $\Sigma=\Sigma(S)$ *is the* singular set *of* $S$, *defined by*

$$\Sigma = \{x \in S : x = \gamma(s_0) \text{ for some } 0<s_0<1, \text{ where } \gamma:[0;1]\to\mathcal{V} \text{ is a null geodesic such that } \gamma(s)\in S \text{ for all } s\in(0;1) \text{ and } \{\gamma(0),\gamma(1)\}\subset\partial S\}. \tag{3.13}$$

*Furthermore, there is a constant* $C$ *such that, for* $x\in\partial S\cup\Sigma$ *we have the estimates*

$$C^{-1}\varepsilon^2 \leqslant d(x, I_\varepsilon) \leqslant 2\varepsilon. \tag{3.14}$$

*Remarks.* (1) The singular set $\Sigma$ also appeared in [BS] Theorem 4.2 and is a natural construction, especially in view of the contained light ray result, Corollary 3.9. Note that the motive (and the method) for introducing $\Sigma$ here is quite different from that in [BS].

(2) The definition (3.13) makes sense even if $S$ is not achronal.

*Proof.* All sets are defined relative to $K$. Let

$$\mathcal{U}_+ = I^+(D^-(S))$$
$$S_+ = b(\mathcal{U}_+)$$

so by [HE] 6.3.1, $\text{int}(K)\cap S_+$ is an achronal WSH with $\partial S\subset S_+$. Applying Corollary 3.3 to $S_+$ gives an approximating time function $\tau_\varepsilon^+$, with $\varepsilon$ normalised by condition (2) of Corollary 3.3. Now define

$$I_\varepsilon^+ = \{x\in K^{(\varepsilon)} : \tau_\varepsilon^+(x)>0\}$$

and note $d_H(S_+, \{\tau_\varepsilon^+=0\})\to 0$ by Corollary 3.3. Since $I_\varepsilon^+\cap S_+=\varnothing$ the $I_\varepsilon^+$ form an exhaustion of $\mathcal{U}_+$ in the sense described. We now have

Lemma 3.5. *If* $x\in S$ *and* $x\notin\mathcal{U}_+$, *then there is a future-directed null geodesic* $\gamma:[0;1]\to K$ *such that* $\gamma(0)=x$ *and* $\gamma(1)\in\partial S$.

*Proof.* Let $y_k\in I^-(x)$ be chosen so that $y_k\to x$. Since $y_k\notin D^-(S)$ (because $x\notin\mathcal{U}_+=I^+(D(S))$), there are future-inextendible nonspacelike curves $\gamma_k(s)$ such that $\gamma_k(0)=y_k$ and $\gamma_k\cap S=\varnothing$. Let $\gamma:[0;s_1]\to K$ be the affinely-parameterised future-inextendible limit curve, with $\gamma(0)=x$. Since $K$ is compact and globally hyperbolic, $0<s_1<\infty$. By reparameterising $\gamma_k$ we may assume $\gamma_k(s)\to\gamma(s)$ for $s\leqslant s_1$. Define $s_0\in[0;s_1]$ by

$$s_0=\inf\{s>0;\gamma(s)\notin S\},$$

so that $0\leqslant s_0<s_1$ because $S\subset\subset K$. If $y=\gamma(s_0)\notin S$ then $y\in\partial S$ by definition and we are done, since $\gamma$ is nonspacelike and $S$ is weakly spacelike. Thus, suppose $y=\gamma(s_0)\in S$. Since $S$ is locally separating, there is a neighbourhood $\mathcal{O}_1$ of $y$ separated by $S\cap\mathcal{O}_1$ into disjoint open sets $\mathcal{O}_1^{\pm}=\mathcal{O}_1\cap I^{\pm}(S)$. Since $S\cap\mathcal{O}_1$ is achronal and $\gamma$ is nonspacelike with $\gamma(s')\notin S\cap\mathcal{O}_1$ for $s'>s_0$, we must have $\gamma_k(s')\in\mathcal{O}_1^+$ for some $s'>s_0$ and $k\geqslant k_0$. Since $\gamma([0;s_0])\subset S$ is compact it can be covered by finitely many locally separating neighbourhoods $\mathcal{O}_j$ and then $\mathcal{O}=\bigcup_j\mathcal{O}_j$ is separated by $\mathcal{O}\cap S$ into $\mathcal{O}^{\pm}=\bigcup_j\mathcal{O}_j^{\pm}$. Then $\gamma_k(s')\in\mathcal{O}^+$ and $\gamma_k(0)\in\mathcal{O}^-$ so $\gamma_k(s)\in S$ for some $0<s<s'$, by the separating property. This contradicts the construction of $\gamma_k$ and finishes the proof. □

Now define

$$\Sigma_+=\{x\in S\colon \exists \text{ future-directed null geodesic } \gamma\colon[0;1]\to K \text{ such that } \gamma(0)=x,\ \gamma(1)\in\partial S \text{ and } \gamma[0;1)\subset S\} \tag{3.15}$$

and $\mathcal{U}_-$, $S_-$, $\tau_\varepsilon^-$, $I_\varepsilon^-$ and $\Sigma_-$ dually. Then $\Sigma=\Sigma_+\cap\Sigma_-$ and $I=\mathcal{U}_+\cup\mathcal{U}_-$, so the $I_\varepsilon$ form an exhaustion of $I$.

Let $d(x)$, $l(x)$, $x\in\mathcal{U}_+$ denote the Riemannian, Lorentzian distance to $S_+$ respectively. The estimates (3.14) follow by noting firstly that $d(x)\leqslant l(x)$ for $x\in\mathcal{U}_+$, by the Riemannian and Lorentzian triangle inequalities, and secondly that by (3.6), $\|\nabla l^2\|\leqslant C$ and thus

$$C^{-1}l^2(x)\leqslant d(x)\leqslant l(x).$$

Applying similar inequalities to $\mathcal{U}_-$ gives (3.14) for $x\in S_+\cap S_-=\partial S\cup\Sigma$. □

The following lemma summarises some basic properties of the singular set $\Sigma$.

LEMMA 3.6. *Let $S$, $K$, $\Sigma$, $I_\varepsilon$ be as in Proposition* 3.4.

(1) *Suppose $M$ is a* WSH *such that $M\approx S$* rel $\partial S$. *Then*

$$\Sigma(M)=\Sigma(S)\quad \textit{and}\quad M\subset \mathrm{cl}(D(S)).$$

(2) $\Sigma(S)$ *is a disjoint union of null geodesics which do not have conjugate points.*

(3) *Suppose $M$ is a* WSH *satisfying*

$$M \approx S' \quad \textit{for some} \quad S' \subset S \tag{3.16}$$

*where the T-homotopy satisfies $D(M_t) \subset\subset K$ and*

$$d_H(\partial M_t, \partial S) \leq C^{-1}\varepsilon^2, \tag{3.16'}$$

*where $C^{-1}\varepsilon^2$ is the constant of* (3.14). *Then*

$$\begin{gathered} \partial M \cap \mathrm{cl}(M \cap I_\varepsilon) = \emptyset, \\ M = (M \cap I_\varepsilon) \cup \{x \in M : d(x, \partial S \cup \Sigma) \leq 2\varepsilon\}. \end{gathered} \tag{3.17}$$

*Proof.* The arguments of Lemma 2.2 give (1) directly. To show (2) suppose $\gamma \subset \Sigma$ is a null geodesic with a pair of conjugate points, so there are points $p, q \in \gamma$ such that $p \ll q$. Then $\gamma$ may be perturbed to give a smooth future-timelike curve $\tilde\gamma$ with $p = \tilde\gamma(0)$, $q = \tilde\gamma(1)$. Since $S$ is locally separating we can find a neighbourhood $\mathcal{O}$ of $\gamma$ with $\tilde\gamma \subset \mathcal{O}$ and such that $S$ separates $\mathcal{O}$ into $\mathcal{O}^+$, $\mathcal{O}^-$. There is $\delta > 0$ such that $\tilde\gamma(s) \in \mathcal{O}^+$ for $0 < s \leq \delta$ and $\tilde\gamma(s) \in \mathcal{O}^-$ for $1-\delta \leq s < 1$, so there is $s_0 \in (\delta; 1-\delta)$ which is the first point with $\tilde\gamma(s_0) \in S$. Then $\tilde\gamma(s) \in \mathcal{O}^+$ for $0 < s < s_0$, contradicting $\tilde\gamma(s) \in I^-(\tilde\gamma(s_0))$ for $s < s_0$. If $\gamma_1, \gamma_2 \subset \Sigma$ are null geodesics with $p \in \gamma_1 \cap \gamma_2$, then in any neighbourhood of $p$ there is a broken null geodesic in $S$, which contradicts locally achronality. The first part of (3.17) follows from (3.14) and (3.16) and the second from the fact that $M$ lies in $I(S) \cup S \cup (\varepsilon$-neighbourhood of $\partial S)$. □

Combining these results gives the full interior gradient bound:

THEOREM 3.7. *Let $S$, $K$, $\Sigma$, $I_\varepsilon$ be as in Proposition* 3.4. *There is $\varepsilon_0 = \varepsilon_0(S, K)$ such that for any $0 < \varepsilon < \varepsilon_0$ and $\Lambda < \infty$ there is a constant $C = C(\varepsilon, \Lambda, K, S)$ such that if $M \subset\subset K$ is any regular hypersurface satisfying the structure conditions* (MCSC) *with constant $\Lambda$ and the conditions* (3.16), *so that in particular*

$$d_H(\partial M, \partial S) < C^{-1}\varepsilon^2,$$

*then $M$ satisfies the interior gradient estimate*

$$\nu(x) \leq C \exp\{-\max(f(\tau_\varepsilon^+(x)), f(\tau_\varepsilon^-(x)))\} \leq C(\varepsilon^{-1}) \quad \textit{for} \quad x \in M \cap I_\varepsilon, \tag{3.18}$$

*where $f(\tau)$ is defined in Theorem* 3.1.

*Proof.* Lemma 3.6 (3) shows that if $x \in M \cap I_\varepsilon$ then $x \notin \partial M$ and either $x \in \mathcal{U}_+$ or $x \in \mathcal{U}_-$, so the basic interior estimate applied with $\tau_\varepsilon^+$ or $\tau_\varepsilon^-$ (or both) gives the gradient estimate. □

Notice that this estimate does not depend on the precise form of $\partial M$, requiring only that $\partial M$ be close to $\partial S$. Lemma 3.6 also shows that if $M \approx S$ rel $\partial S$ then we get interior estimates on $M - \Sigma$, since $I_\varepsilon \cap M$ is an exhaustion of $M - \Sigma$. There is an alternative way of viewing the interior estimate (3.18), based on the "gap" at the boundary. We can illustrate this with an easy estimate derived from the gradient estimate of [CY]. If $M = \mathrm{graph}_\Omega u$ has constant mean curvature $\Lambda$ in $\mathbf{R}^{n+1}$, then the Cheng–Yau estimate gives ([CY], [E])

$$|\nabla^M l_y| \leq C_n(1+(L\Lambda)^2)$$

where (for this discussion)

$$l_y(x) = (|x-y|^2 - (u(x)-u(y))^2)^{1/2}$$

and $\{x : l_y(x) < L\} \subset\subset \Omega \subset \mathbf{R}^n$.

Now suppose $x \in \Omega$ and $B_{3R}(x) \subset \Omega$. Applying the $|\nabla^M l|$ estimate at $x \in B_{2R}(y) \subset \Omega$ gives

$$\begin{aligned} C_n(1+(R\Lambda)^2) &\geq |\nabla^M l_y(x)|^2 \\ &\geq 1 + l_y(x)^{-2} \langle (X-Y), N(x) \rangle^2 \end{aligned}$$

where $X, Y$ are the position vectors in $\mathbf{R}^{n,1}$ and $N(x)$ is the normal vector to $M$ at $x$. Defining the gap parameter $\delta$ of $x$ at $\partial B_R(x)$ by

$$1 - \delta = \sup_{y \in \partial B_R(x)} R^{-1}|u(x)-u(y)| \tag{3.19}$$

we see that

$$\begin{aligned} \langle X-Y, N(x) \rangle^2 &= \nu(x)^2((x-y) \cdot Du(x) - (u(x)-u(y)))^2 \\ &\geq \nu(x)^2 |x-y|^2 (|Du|(x) - (1-\delta))^2 \end{aligned}$$

if $y \in \partial B_R(x)$ is chosen so that $(x-y)$ and $Du(x)$ are parallel. Since $l_y(x)^2 \leq |x-y|^2$, we have either $|Du|(x) \leq 1 - \delta/2$ (and then $\nu(x) \leq \sqrt{2/\delta}$) or

$$|\nabla l_y(x)|^2 \geq 1 + \nu(x)^2 \delta^2/4$$

which gives the interior estimate

$$\nu(x) \leq C_n(1+(R\Lambda)^2)/\delta, \tag{3.20}$$

when $B_{3R}(x) \subset \Omega$. Thus the gradient is bounded in terms of the gap $\delta$, which measures the "distance" from the graph to the lightcone over $\partial B_R(x)$. The estimate (3.18) has similar qualitative behaviour, with the decay estimate of Corollary 3.4 for $\|\nabla\tau\|$ playing the role of the gap.

The first consequence of this estimate is a local convergence theorem for sequences of regular hypersurfaces. A corollary of this convergence theorem is a version of the "contained-light-ray" Theorem 3.2 of [BS] for Dirichlet problem solutions. In [BS] this was proved using comparison surfaces and applied to variational solutions: the proof here is quite different.

THEOREM 3.8. *Suppose $M_k$, $k=1,2,\ldots$ is a sequence of $C^3$ regular hypersurfaces with mean curvatures $H_k$ satisfying* (MSCS) *with constant $\Lambda$, and that $p$ is an accumulation point of the $M_k$ with neighbourhood $\mathcal{U}$, precompact, connected and simply connected, such that*

$$\partial M_k \cap \mathcal{U} = \emptyset,$$
$$M_k \cap \mathcal{U} \quad \text{is connected for} \quad k=1,2,\ldots.$$

*Then there is a subsequence, also denoted $M_k$, and a* WSH *$M \subset \mathcal{U}$ such that $p \in M$, $\partial M \cap \mathcal{U} = \emptyset$ and $d_H(M_k \cap \mathcal{U}, M) \to 0$ as $k \to \infty$. Furthermore, $M - \Sigma(M)$ is a $C^{2,\alpha}$ regular hypersurface with mean curvature $H \in C^{0,\alpha}(M)$ and $H_k \to H$ in $C^0(M)$, where the singular set $\Sigma(M)$ is defined by* (3.13).

*Proof.* The conditions on $M_k$ and $\mathcal{U}$ imply that $M_k$ separates $\mathcal{U}$ into two disjoint connected open sets, $\mathcal{U}_{k+}$, $\mathcal{U}_{k-}$ say. (This is an elementary homotopy argument.) Cover $\mathrm{cl}(\mathcal{U})$ by a finite number of coordinate neighbourhoods. Since each $M_k$ is locally a Lipschitz graph, we can apply Ascoli-Arzela to get a uniformly convergent subsequence $M_k \to M$. The limit surface $M$ also separates $\mathcal{U}$ (take the limits of $\mathcal{U}_{k+}$, $\mathcal{U}_{k-}$) so $\partial M \cap \mathcal{U} = \emptyset$ and hence $M$ is a weakly spacelike hypersurface. Let $x \in M - \Sigma$. From the interior gradient bound, Theorem 3.7, applied to $M \cap \mathcal{U}'$, where $\mathcal{U}'$ is a neighbourhood of $x$ such that $\mathcal{U}' \cap \Sigma = \emptyset$, there is a neighbourhood $x \in \mathcal{U}''$ such that $M_k \cap \mathcal{U}''$ is uniformly spacelike (since $x \in I_\varepsilon$ for some $\varepsilon > 0$, setting $\mathcal{U}'' \subset\subset I_\varepsilon$). Thus $M_k \cap \mathcal{U}''$ is the graph of a function satisfying a uniformly elliptic equation with $H_k \in C^1$, uniformly, so satisfying uniform $C^{2,\alpha}$ estimates by elliptic regularity ([GT] Chapter 8). The remaining conclusions follow immediately. □

*Remark.* The connectedness conditions on $\mathcal{U}$ are imposed for simplicity only: since we're only interested in applying this when $\mathcal{U}$ is a local coordinate neighbourhood, this causes no problem. Notice this result does not require that $M_k$ be achronal.

COROLLARY 3.9. ("Contained-light-ray", cf. [BS] Theorem 3.2.) *Suppose $M$ is a weakly spacelike hypersurface, relatively compact, such that there is a curve $\gamma: (0;1) \to M$ which is a null geodesic. If there is a sequence $M_k$, $k=1,2,\dots$ of $C^3$ regular hypersurfaces satisfying the mean curvature structure conditions with constant $\Lambda$ and such that $M_k \to M$, then there is a null geodesic extension*

$$\gamma^*: [s_0; s_1] \to M \cup \partial M, \quad s_0 \leq 0,\ s_1 \geq 1,$$

*of $\gamma$ such that $\{\gamma^*(s_0), \gamma^*(s_1)\} \subset \partial M$.*

*Proof.* By the convergence theorem, $\gamma((0;1)) \subset \Sigma$, and $\Sigma$ consists of null geodesics between points of $\partial M$. □

## 4. The Dirichlet problem

We are now ready to give general conditions under which the Dirichlet problem,

given a WSH $S$ with boundary set $\partial S$ and a mean curvature
function $F(x, v)$ satisfying (MCSC), find a regular (DP)
hypersurface $M$ with $\partial M = \partial S$ and $H_M = F(x, N)|_M$,

is solvable. Unlike [BS] which proceeded from the solution of the variational problem, we use the solvability for smooth data [B1] and the interior estimates of the previous section to obtain the solution, by approximation. In the following sections we will use these results to show regularity for local variational extremal surfaces, which in turn allows us to sharpen some DP results. As noted in [B1], the solution need not be unique.

We start with a basic existence theorem for achronal data, followed by two generalisations. The first generalisation deals with immersed WSH's and is based on a simple extension of the idea of $T$-homotopy equivalence. The construction of the auxiliary spacetime $\tilde{\mathcal{V}}$ there indicates that the $T$-homotopy approach is the natural one for this problem. It was recognised in [BS] and [B1] (see also [Q]) that the gradient estimates require only local achronality, and the result is the logical completion of this observation. Technical requirements prevent us from getting the strongest possible result here, but these can be overcome by invoking the variational regularity result of Section 6. Although this result will cover the case of boundary branch points (e.g. take an immersed surface spanning a double loop in $\mathbf{R}^{2,1}$), it does not allow for (moveable) interior branch points, and such solutions have been constructed using the Weierstrass representation ([K], see also [T]). It may be that an analogue of Osserman's theorem on branch points for minimal surfaces ([O]) holds here also.

The second generalisation concerns spacetimes with rough metric, $g\in C^{0,1}$. Such metrics have been considered in the literature (e.g. [DH], [Tb]) with distributional curvature representing some idealised matter distribution. Rather than take the metric to be $C^\infty$ except across some surface of discontinuity, as is usually done, it is more natural here to work with a sequence of $C^\infty$ approximating metrics with some additional control on components of the distributional curvature (RM2,3). As a consequence the statement of Theorem 4.3 is technical, although the hypotheses are physically rather natural.

THEOREM 4.1. *Suppose that $(S, K)$ is a standard data set and that $F(x, v)$ satisfies* MCSC. *Then there is an achronal regular hypersurface $M\subset K$ with singular set $\Sigma=\Sigma(S)$ (see* (3.13)*) such that*

(1) *$M\approx S$ rel $\partial S$ (so $\partial M=\partial S$)*
(2) *$M-\Sigma$ is a $C^{2,\alpha}$ regular hypersurface, for any $\alpha\in(0;1)$*
(3) *$H_M(x)=F(x, N(x))$ for all $x\in M-\Sigma$*

*where $N(x)$ is the future unit normal to $M$ at $x$.*

*Remarks.* (1) If $F\in C^{k,\alpha}$ then elliptic regularity shows that $M-\Sigma$ is $C^{k+2,\alpha}$ for $k\geq1$.

(2) There is no condition on the regularity of the boundary $\partial M$: compare [BS] Corollary 4.2.

(3) If there are barrier surfaces present, then the condition that $D(S)$ be precompact can be relaxed, along the lines of [B1] Theorem 4.3. Instead we need that the "accessible domain" $A(M)\subset D(M)$, bounded by the barrier surfaces and pieces of the Cauchy horizon $H(M)$, be compactly contained in a globally hyperbolic set.

(4) It is an interesting pde question to determine the regularity of $M$ across the singular set $\Sigma(M)$. If $p\in\gamma$, a null geodesic in $\Sigma$, then by comparing $M$ with light cones

based at points of $\gamma$ near $p$ we see that $M$ has a (null) tangent plane at $p$ and the second difference quotient of $M$ is bounded, so it is reasonable to conjecture that $M$ is $C^{1,\alpha}$ near $p$.

*Proof.* Proposition 3.4 constructs sets $I_j$ and comparison time functions $\tau_j^+$, $\tau_j^-$ where $\varepsilon=\varepsilon_j$ satisfies

$$\varepsilon_{j+1}<\tfrac{1}{2}C^{-1}\varepsilon_j^2, \quad j=0,1,\ldots$$

and the $I_j$ form an exhaustion of $I=\mathcal{U}_+\cup\mathcal{U}_-\cup(S-\Sigma)$, satisfying (3.17) of Lemma 3.6. Choose $S_j\subset\{\tau_{j+1}^+=0\}$ such that $\partial S_j$ is a smooth submanifold satisfying

$$d_H(\partial S_j, \partial S)<2\varepsilon_{j+1}$$

and $S_j$ satisfies (3.17) of Lemma 3.6, so that $D(S_j)\subset\subset K$. Since $K$ is globally hyperbolic and has a time function, we can apply [B1] Proposition 3.2 to construct a time function having $S_j$ as a level set. Then the argument of [B1] Theorem 4.2 gives a regular hypersurface $M_j\approx S_j \operatorname{rel} \partial S_j$ with mean curvature $H_j(x)=F(x, N_j(x))$ for $x\in M_j$. Since $M_j\subset\subset K$, compact, and $\partial M_j\cap I_k=\varnothing$ for $j\geq k$, we can apply the convergence Theorem 3.7 to construct a limiting regular hypersurface $M$ with boundary $\partial M=\lim\partial M_j=\partial S$, with smoothness determined by elliptic regularity. □

In order to extend this to immersed surfaces, we need a definition:

An *immersed* WSH is a pair $(f, S)$ where $S$ is a $C^\infty$ open $n$-manifold and $f: S\to\mathcal{V}$ is a $C^{0,1}$ map which is locally weakly spacelike. That is,

$$\forall x\in S,\ \exists \text{ neighbourhood } x\in\mathcal{U}\subset S \text{ such that } f(\mathcal{U}) \text{ is an achronal WSH.} \tag{4.1}$$

The boundary, denoted $\bar{\partial}S$, is defined in the usual way

$$\bar{\partial}S=\operatorname{cl}(f(S))-f(S). \tag{4.2}$$

For short we will say that $S$ is an immersed WSH, the map $f$ is understood, and the immersed WSH's $S_0, S_1$ are $T$-homotopic, $S_0\approx S_1$, if there is a homotopy

$$h: S\times[0,1]\to\mathcal{V}$$

where $h_t: S\times\{t\}\to\mathcal{V}$ is an immersed WSH for $0\leq t\leq 1$, such that $S_0=(S, h_0)$, $S_1=(S, h_1)$ and $h(x): [0,1]\to\mathcal{V}$ has image in an integral curve of $T$. Again, if $\bar{\partial}S_t=\bar{\partial}S$ for $0\leq t\leq 1$ then $S_0, S_1$ are $T$-homotopic relative to $\partial S$, $S_0\approx S_1 \operatorname{rel}\partial S$.

This clearly includes the previous definition of $T$-homotopy as a special case while preserving the idea of graphs. In fact this can be reduced to the previous (graphical) situation by the following standard construction.

Suppose $f(S)\subset\subset K$, a compact globally hyperbolic set, and let $\gamma_x(s)$ denote the unit parameterised integral curve of $T$ with $\gamma_x(0)=f(x)$, for $x\in K$. Define the auxiliary Lorentz manifold

$$\tilde{\mathscr{V}}=\{(x,t): x\in S, t^-(x)<t<t^+(x)\} \tag{4.3}$$

where $t^\pm\in C^0(S)$ satisfy

$$t^+(x)=\sup\{s: \gamma_{f(x)}(s)\in K\}$$

for $x\in S$, and $t^-$ is dual. $\tilde{\mathscr{V}}$ is equipped with the Lorentz metric

$$\tilde{g}=\tilde{f}^*(g)$$

where $\tilde{f}: \tilde{\mathscr{V}}\to\mathscr{V}$ is the smooth immersion

$$\tilde{f}(x,t)=\gamma_{f(x)}(t). \tag{4.4}$$

Note that, although $(\tilde{\mathscr{V}},\tilde{g})$ is naturally a $C^2$ Lorentz manifold, $(x,t)$ are not good coordinates in $\tilde{\mathscr{V}}$ (since they are only Lipschitz with respect to $\tilde{g}$) and $t$ need not be a time function. However, if $S_1$ is an immersed WSH and $S_1\approx S$ in $K$, then $S_1\approx S$ in $\tilde{\mathscr{V}}$ in the sense of the original definition (Section 2) since $S_1$ can be written as a graph over $S$ in $\tilde{\mathscr{V}}$.

The direct application of the existence Theorem 4.1 to the immersed WSH $S$, considered as a WSH in $\tilde{\mathscr{V}}$, meets with the difficulty that $\tilde{\mathscr{V}}$ may not have a (precompact, globally hyperbolic) neighbourhood $\mathscr{U}$ in some larger Lorentz manifold, since $b(\tilde{\mathscr{V}})$, defined via the metric space completion of $\tilde{\mathscr{V}}$, can be quite bizarre (e.g. if $S$ has a boundary branch point). The following result sidesteps this problem, at the cost of excluding such examples; the full result can be derived from the regularity for variational extrema in Section 6.

Proposition 4.2. *Let $(f',S')$ be an immersed* WSH *and $\tilde{\mathscr{V}}$ the auxiliary Lorentz manifold constructed from $(f',S')$. Suppose that $S\subset S'$ is a* WSH *with $\partial S\subset S'$ (with respect to $\tilde{\mathscr{V}}$), and such that $D(f'(S))$ is precompact. Let $F(x,v)$ satisfy the* MCSC *in $\mathscr{V}$. Then there is an immersed hypersurface $(f,M)$ such that $M-\Sigma(S)$ is regular, $\tilde{\partial}M=\tilde{\partial}S$,*

*$M \approx S$ rel $\bar{\partial} S$, and with mean curvature $H_M(x)=F(f(x),\bar{f}_*(N(x)))$, $x \in M-\Sigma$, where $\bar{f}: \bar{\mathscr{V}} \to \mathscr{V}$ is the immersion defined by* (4.4) *from* $(f', S')$.

*Proof.* The hypotheses ensure that $D(S) \subset\subset \bar{\mathscr{V}}$, so the previous existence theorem gives $M$ as a regular hypersurface in $\bar{\mathscr{V}}$ with immersion $f: M \to \mathscr{V}$ defined by $f=\bar{f}|_M$. □

The second generalisation of the basic existence theorem is to the case of merely Lipschitz-continuous metric, $g \in C^{0,1}(\mathscr{U})$. Clearly some restrictions on $g$ are needed in order to carry through the previous arguments: rather than phrasing these in their weak (integral) form and then using mollifiers, we work directly with a sequence of approximating metrics. This is not an unnatural approach physically, since a metric with distributional curvature should be regarded as an idealisation of smooth metrics.

We say that $g \in C^{0,1}(\mathscr{U})$, $\mathscr{U}$ precompact, satisfies the *rough metric conditions* if there is a sequence of metrics $g_k \in C^2(\mathscr{U})$ and a constant $C$ such that

$$g_k \to g \quad \text{in} \quad C^0(\mathscr{U}) \tag{RM1}$$

$$\|\partial g_k\| \leq C, \tag{RM2}$$

for any future unit (w.r.t. $g_k$) vector $N$,

$$\mathrm{Ric}_k(N,N) \geq -C v_k^2 \tag{RM3.1}$$

where $v_k = -g_k(N,T)$ and $\mathrm{Ric}_k = \mathrm{Ricci}(g_k)$,

$$\|\mathscr{L}_T g_k\| + \|\nabla^{(k)} \mathscr{L}_T g_k\| \leq C \tag{RM3.2}$$

where $\nabla^{(k)}$ is the covariant derivative of $g_k$.

Roughly speaking, the second condition says that the delta-function components of $\mathrm{Ric}(g)$ satisfy the timelike convergence condition, and the third says that $T$ satisfies Killing's (isometry) equations up to non-distributional terms. Note that if $g \in C^{1,1}(\mathscr{U})$ then the sequence $g_k$ can be constructed by mollification.

THEOREM 4.3. *Suppose that $g \in C^{0,1}(\mathscr{V})$ satisfies* (RM), *$F(x,v)$ satisfies* (MCSC) *and that $(S,K)$ is a standard data set with $S$ a uniformly regular hypersurface.*

*Then there is a $C^{1,\alpha}$ regular hypersurface $M$ such that $M \approx S$ rel $\partial S$ with (weak) mean curvature*

$$H_M(x) = F(x, N(x)), \quad x \in M.$$

*That is, for all* $\varphi \in C_c^1(M)$,

$$\int_M \{\varphi(\nu^{-1}F(x,N) - \mathrm{div}_M T) - \langle \nabla^M \varphi, T\rangle\}\, dv_M(x) = 0, \tag{4.5}$$

*using* [B1] (2.7).

*Proof.* Since $S$ is uniformly strictly $g$-spacelike, by passing to a subsequence we can assume $S$ is also strictly $g_k$-spacelike, $k \geq 1$, and that there is an $\varepsilon > 0$ such that the metric

$$\tilde{g} = g_1 - \varepsilon T \otimes T$$

satisfies

$$\tilde{g}(X,X) > 0 \quad \Rightarrow \quad \begin{cases} g_k(X,X) > 0 & \text{for } k \geq 1, \quad \text{and} \\ g(X,X) > 0 & \text{any vector } X \end{cases} \tag{4.6}$$

and $S$ is strictly $\tilde{g}$-spacelike.

Using Corollary 3.3 we can thus construct $\tilde{g}$-approximating upper and lower time functions $\tau_j^{\pm}$ for $S$, which by (4.6) are also time functions for all the $g_k$. As in Theorem 4.1, we can solve the Dirichlet problem for mean curvature $F(x, N)$ in the metric $g_k$ with smooth boundary manifold in the level set $\{\tau_k^+ = 0\}$, giving a sequence $(M_k, \partial M_k)$ of $C^\infty$, $g_k$-spacelike hypersurfaces with $\partial M_k \to \partial M$ in Hausdorff distance. Since the $\tau_j^{\pm}$ are time functions with respect to all the $g_k$, Proposition 3.4 and the rough metric conditions (RM) give *uniform* interior gradient bounds in $M_k \cap I_j$. To see this we observe that (RM2) and (RM3) allow us the control the terms in the gradient estimate (3.1) which involve $\mathrm{Ric}_k$ and $T$ and its derivatives, while (RM1) controls the terms $\|\nabla^{(k)2}\tau_j^{\pm}\|$, since $\tau_j^{\pm}$ are already $C^2$ functions by construction.

Since $g_k \to g$ in $C^0(\mathcal{V})$, we have a subsequence $M_k$ converging to a $g$-spacelike hypersurface $M$ with $\partial M = \partial S$, satisfying an interior gradient estimate

$$\nu(x) \leq C_j \quad \text{for} \quad x \in M \cap I_j,$$

and taking the limit of the weak form of the mean curvature equations satisfied by the $M_k$ (notice the $M_k$ satisfy uniform interior $C^{1,\alpha}$ bounds), we see that $M$ satisfies (4.5), as required. □

## 5. Foliations and the eigenvalue condition

We intend to show regularity for variational extrema by comparing such surfaces with foliations by smooth surfaces of prescribed mean curvature; in this section we describe conditions under which such foliations can be constructed. This becomes an exercise using the implicit function theorem, once we can show that the linearised operator is invertible. Thus, the main result here is Theorem 5.2, which shows invertibility for surfaces given as graphs over sufficiently small domains. It is somewhat curious that this holds regardless of the boundary values, and that this is exactly the form in which the result will be required in the next section. Using the resulting foliation and its integral uniqueness identity, we can easily show (Corollary 6.3) that DP solutions are locally maximising for their associated variational problem. This integral identity is the key to showing the regularity of variational extrema in general, although the argument is more delicate than in the case of regular hypersurfaces.

If $X$ is a timelike vector field and $M$ is a regular spacelike hypersurface, then the variation of the mean curvature of $M$ when deformed by $X$ is given by ([CB], [B1])

$$\begin{aligned} X(H_X) &= \frac{\partial}{\partial s} H(s)\Big|_{s=0} \\ &= -\Delta_M\langle X, N\rangle + \langle X, N\rangle\,(|A|^2+\mathrm{Ric}(N,N)) + \langle X, \nabla^M H\rangle . \end{aligned}$$

Now suppose $M$ has prescribed mean curvature, $H_M(x)=F(x)$, $x\in M$ where $F\in C^1(\mathcal{V})$. Then the variation having mean curvature $F$ implies

$$\begin{aligned} \Delta_M\langle X, N\rangle &= \langle X, N\rangle\,(|A|^2+\mathrm{Ric}(N,N)) + \langle X, \nabla^M F\rangle - X(F) \\ &= \langle X, N\rangle\,(|A|^2+\mathrm{Ric}(N,N)+\langle N, \nabla F\rangle) \end{aligned} \tag{5.1}$$

so the linearised prescribed mean curvature operator is

$$L_M\varphi = -\Delta_M\varphi + (|A|^2+\mathrm{Ric}(N,N)+\langle N, \nabla F\rangle)\,\varphi$$

and we say the regular hypersurface $M$ satisfies the *eigenvalue condition* if $L_M>0$;

$$\lambda_1(L_M) = \inf_{\varphi\in C_c^1(M)} \int_M \varphi L_M\varphi \Big/ \int_M \varphi^2 > 0. \tag{5.2}$$

Note that the constant mean curvature foliation equation has a slightly different linearisation, determined by the condition $X(H_X)=-1$ rather than $X(H_X)=\langle X, \nabla F\rangle$. The standard situation satisfying the eigenvalue condition is where the timelike convergence condition, $\mathrm{Ric}(T', T') \geq 0$ for all timelike $T'$, holds and $F$ is nondecreasing to the future.

From the eigenvalue condition and the implicit function theorem we readily have

PROPOSITION 5.1. *Let $\{Q_\tau, -1<\tau<1\}$ be a $C^1$ family of uniformly regular hypersurfaces considered as graphs over $\Omega = Q_0$ with height defined by the lengths of T-integral curves through $\Omega$, of functions $\bar{\varphi}_\tau \in C^{3,\alpha}(\Omega)$ with boundary values $\varphi_\tau|_{\partial\Omega}$. (Thus $\tau \mapsto \bar{\varphi}_\tau$ is in $C^1((-1;1), C^{2,\alpha}(\Omega))$). Suppose that $F \in C^1(\Omega \times \mathbf{R})$ is such that $L_M$ satisfies the eigenvalue condition for all $M$ with boundary values $\varphi_\tau$ and mean curvature $F|_M$. Then there is a foliation with leaves $M_\tau = \mathrm{graph}_\Omega(u_\tau)$ such that $H(\tau) = H_{M_\tau} = F|_{M_\tau}$ and $\partial M_\tau = \partial Q_\tau = \mathrm{graph}_{\partial\Omega}\, \varphi_\tau$. Furthermore, the relation*

$$t = u_\tau(x) \tag{5.3}$$

*intrinsically defines a time function $\tau \in C^1(\Omega \times \mathbf{R})$.*

*Proof.* Let $\mathcal{Q}$ be the Banach manifold of uniformly regular $C^{2,\alpha}$ hypersurfaces with boundary $\partial Q_\tau$, $-1<\tau<1$, with local charts modelled on $\mathbf{R} \times \{w \in C^{2,\alpha}(\Omega), w|_{\partial\Omega} = 0\}$ about $u \in \mathcal{Q}$, $u|_{\partial\Omega} = \varphi_\tau$, by the map

$$(s, w) \mapsto \mathrm{graph}(u + \bar{\varphi}_{\tau+s} - \bar{\varphi}_\tau + w). \tag{5.4}$$

The existence results of [B1] show that $\mathcal{Q}$ contains hypersurfaces $\tilde{M}_\tau$ (not necessarily unique) with prescribed mean curvature $F|_{\tilde{M}_\tau}$, for each $-1<\tau<1$. Now define the $C^1$ map $\mathcal{P}: \mathcal{Q} \to C^{0,\alpha}(\Omega)$ by

$$\mathcal{P}(M) = H_M - F|_M.$$

The previous calculation shows that in the chart (5.4) about $M$ such that $\mathcal{P}(M) = 0$, $\mathcal{P}$ has linearisation

$$D_2 \mathcal{P}(M)\, w = -L_M(\nu w),$$

where the tilt function $\nu$ of $M$ is bounded by the gradient estimates of [B1], so that $D_2 \mathcal{P}(M)$ is invertible, by the eigenvalue condition. The implicit function theorem then gives a $C^1$ map $s \mapsto u_s$, $|s| \leq \varepsilon$, $\mathrm{graph}(u_0) = M$, such that

$$\mathcal{P}(\mathrm{graph}\, u_s) = 0.$$

Thus, starting with $M_0=\bar{M}_0$, this family of surfaces can be extended to $\tau\mapsto u_\tau$, $-1<\tau<1$, since the gradient estimates of [B1] and elliptic regularity ensure that $\bar{u}=\lim_{\tau\to\tau_0} u_\tau$ gives also a uniformly regular hypersurface with mean curvature $F$. Now $\mathcal{P}(u_\tau)=0$ implies that $\dot{u}_\tau=\partial u_\tau/\partial\tau$ satisfies

$$\begin{cases} L_{M_\tau}(\nu\dot{u}_\tau)=0 \\ \dot{u}_\tau|_{\partial\Omega}=\dfrac{\partial}{\partial\tau}\varphi_\tau>0, \end{cases}$$

so by the eigenvalue condition and the Hopf maximum principle, $\dot{u}_\tau>0$. Thus the level sets of $\tau$ define a *foliation* and by differentiating (5.3) we get

$$1=\dot{u}_\tau\frac{\partial\tau}{\partial t}$$

$$0=\frac{\partial u}{\partial x}+\dot{u}_\tau\frac{\partial\tau}{\partial x},$$

so $\tau\in C^1(\Omega\times\mathbf{R})$. □

The next result shows that the eigenvalue condition holds on sufficiently small regions. It will be somewhat simpler to describe this using the blowing up procedure to normalise the region of interest. Thus, consider geodesic normal coordinates $(x,t)$ about a fixed point $p\in\mathcal{V}$ and suppose for simplicity that $T(p)=\partial_t$. Given the parameter $\sigma\in(0;1]$, we define the blow-up metric $g_\sigma(x,t)$ on the cylinder $\mathcal{C}=B_1(0)\times(-2;2)$ in Minkowski space by

$$g_\sigma(x,t)=\sigma^{-2}g(\sigma x,\sigma t), \tag{5.5}$$

so that $g_\sigma$ is just a rescaling of $g$ in a $\sigma$-cylinder neighbourhood $\mathcal{C}_\sigma(p)$ of $p$. Thus, with $\partial$ representing derivatives with respect to the standard $(x,t)$ coordinates in $\mathcal{C}$, we have

$$\|g_\sigma-\eta\|+\|\partial g_\sigma\|+\|\partial^2 g_\sigma\|\leq C\sigma^2 \tag{5.6}$$

where $\eta$ is the standard Minkowski metric and $C$ is a geometric constant depending on curvature bounds near $p$.

THEOREM 5.2. *There is an $\varepsilon_0>0$ (computable and depending only on $n$) such that, if $M$ is any uniformly regular hypersurface defined as a graph in $\mathcal{C}$ equipped with metric $g$ satisfying*

$$\|g-\eta\|+\|\partial g\|+\|\partial^2 g\|\leq\varepsilon \tag{5.7}$$

*and with mean curvature $H_M = F|_M$, where $F \in C^1(\mathscr{C})$ satisfies*

$$\|F\|^2 + \|\partial F\| \leq \varepsilon, \tag{5.7'}$$

*such that $\varepsilon \leq \varepsilon_0$, then $L_M$ satisfies the eigenvalue condition. Furthermore, the first eigenvalue of $L_M$ defined by* (5.2) *satisfies*

$$\lambda_1(M) \geq \tfrac{1}{2}\lambda(n) > 0 \tag{5.8}$$

*where $\lambda(n)$ is the first Dirichlet eigenvalue of the flat Laplacian on the unit ball in $\mathbf{R}^n$.*

*Proof.* We refer to [B1] for any notation used here without explanation. Derivatives in spatial directions (with respect to $x$-coordinates) will be denoted by $D$, and $c$ denotes any constant depending only on $n$.

Suppose $\varphi \in C_c^1(M)$ and let $u$ be the height function of $M$, so $M = \text{graph}_{B_1(0)} u$. By extending $\varphi$ constant along the $t$-coordinate lines, we can also consider $\varphi \in C_c^1(B_1(0))$. An integration by parts followed by the Schwarz inequality shows that, for any $a \in \mathbf{R}$,

$$\int_M |\nabla^M u|^2 \varphi^2 \leq 4 \int_M (u-a)^2 |\nabla^M \varphi|^2 + 2 \int_M |u-a|\,|\Delta_M u|\,\varphi^2.$$

Since $u_{\max} - u_{\min} \leq 2(1+\varepsilon)$, by (5.6) and [B1] (2.8), we have

$$\int_M |\nabla^M u|^2 \varphi^2 \leq 4(1+\varepsilon)^2 \int_M |\nabla^M \varphi|^2 + c\varepsilon \int_M \varphi^2 \nu^2,$$

so the identity $\nu^2 = \alpha^2 |\nabla^M u|^2 + 1$ implies

$$\int_M \varphi^2 \nu^2 \leq (1+c\varepsilon) \int_M (4|\nabla^M \varphi|^2 + \varphi^2). \tag{5.9}$$

Now, by the definition of $\lambda(n)$,

$$\begin{aligned}
\int_M \varphi^2 &= \int_{B_1(0)} \varphi^2(x)\, \nu^{-1}(x) \sqrt{g(x, u(x))}\, dx \\
&\leq (1+\varepsilon)\, \lambda(n)^{-1} \int_{B_1} |D(\varphi \nu^{-1/2})|^2\, dx \\
&\leq \tfrac{5}{4}(1+\varepsilon)\, \lambda(n)^{-1} \int_{B_1} (|D\varphi|^2 + \varphi^2 \nu^{-2} |D\nu|^2)\, \nu^{-1}\, dx.
\end{aligned}$$

A standard computation (see [B1] Theorem 3.1) shows that

$$\begin{aligned}|A|^2 &\geq (1+1/n)\,\nu^{-2}|\nabla^M\nu|^2 - c\varepsilon\nu^2\\ &\geq (1+1/n)\,\nu^{-2}|D\nu|^2 - c\varepsilon\nu^2\end{aligned}$$

and thus

$$\int_M \varphi^2 \leq \tfrac{5}{4}(1+c\varepsilon)\,\lambda(n)^{-1}\int_M (|\nabla^M\varphi|^2 + |A|^2\varphi^2 + c\varepsilon\varphi^2\nu^2).$$

Using (5.9) to absorb the final term gives

$$\int_M \varphi^2 \leq \tfrac{3}{2}(1+c\varepsilon)\,\lambda(n)^{-1}\int_M (|\nabla^M\varphi|^2 + |A|^2\varphi^2), \tag{5.10}$$

and we can now estimate $\lambda_1(M)$:

$$\begin{aligned}\int_M \varphi L_M\varphi &\geq \int_M (|\nabla^M\varphi|^2 + |A|^2\varphi^2 - c\varepsilon\varphi^2\nu^2)\\ &\geq \int_M \{(1-c\varepsilon)(|\nabla^M\varphi|^2 + |A|^2\varphi^2) - c\varepsilon\varphi^2\}\\ &\geq (\tfrac{2}{3}(1-c\varepsilon)\,\lambda(n) - c\varepsilon)\int_M \varphi^2.\end{aligned}$$

The conclusion follows for $\varepsilon \leq \varepsilon_0$ where $\varepsilon_0$ is chosen so that the RHS coefficient is $\geq \lambda(n)/2$. □

Notice that this result does not require any a priori estimate on the tilt of $M$, but only that $M$ be smooth enough for the calculations to be sensible.

Now, if $M = \operatorname{graph} u$ is a regular hypersurface through $p$ with mean curvature $H_M = F|_M$ for $F \in C^1(\mathcal{V})$, the rescaling defined by

$$\begin{aligned}u_\sigma(x) &= u(\sigma x), \quad x \in B_1(0)\\ F_\sigma(x,t) &= \sigma F(\sigma x, \sigma t), \quad (x,t) \in \mathscr{C}\end{aligned}$$

puts us in the situation of Theorem 5.2, with $\varepsilon = C\sigma^2$. Thus, for all $\sigma$ sufficiently small, and any regular hypersurface $M$ with mean curvature $F|_M$ (and $\partial M \cap \mathscr{C}_\sigma(p) = \varnothing$) the eigenvalue condition is satisfied on $M \cap \mathscr{C}_\sigma(p)$.

## 6. The variational problem

Given a mean curvature function $F \in C^1(\mathcal{V})$, we define the variational functional $I_F(S)$ for $S$ a WSH by

$$I_F(S) = |S| - \int_{V(S_0, S)} F \, dv_{\mathcal{V}}, \tag{6.1}$$

where $|S|$ is the induced area of $S$ and $V(S_0, S)$ denotes the (signed) $(n+1)$-volume bounded by the reference surface $S_0$ and $S$, with $S \approx S_0$. If $\partial S \neq \partial S_0$ then the remaining component of $bV(S_0, S)$ is taken to be foliated by $T$-integral curves between $\partial S_0$ and $\partial S$. Given a connected $T$-homotopy class $\mathcal{F}$ with $S_0 \in \mathcal{F}$, we have the associated variational problem

$$(VP)_{\mathcal{F}}\text{: maximise } I_F(S) \text{ amongst } S \in \mathcal{F}.$$

It is well-known ([A], [AB], [Go]) that if $\mathcal{F}$ satisfies some boundedness condition, then the extremal of $(VP)_{\mathcal{F}}$ is attained by a WSH. (For completeness, we describe a basic existence result below.) In this section we will show (Theorem 6.4) that these extremals are in fact regular hypersurfaces. More generally, we show that if $M$ is a WSH which is *locally extremal* for $I_F$ in the sense that for every $p \in M$ there is a neighbourhood $p \in \mathcal{U} \subset \mathcal{V}$ such that $I_F(M) \geq I_F(M')$ for any $M'$ such that $V(M, M') \subset \mathcal{U}$, then $M$ is a regular hypersurface away from the singular set $\Sigma = \Sigma(M)$ defined by (3.13).

The existence of extremals for the variational problem can readily be deduced from the following basic result.

PROPOSITION 6.1. *Suppose $S_k$, $k=1,2,\ldots$ is a sequence of weakly spacelike hypersurfaces, $p$ is an accumulation point of the $S_k$ and $\mathcal{U}$ is a cylinder neighbourhood of $p$ such that $\partial S_k \cap \mathcal{U} = \emptyset$ and $S_k \cap \mathcal{U}$ is connected for $k=1,2,\ldots$. Then there is a subsequence, also denoted $S_k$, and a* WSH *$S \subset \mathcal{U}$ such that $p \in S$, $\partial S \cap \mathcal{U} = \emptyset$ and $d_H(S_k \cap \mathcal{U}, S) \to 0$ as $k \to \infty$. Furthermore, if we set*

$$I_{\mathcal{U}}(S) = |S| - \int_{\mathcal{U} \cap I^-(S)} F \, dv_{\mathcal{V}},$$

*then*

$$I_{\mathcal{U}}(S) \geq \limsup_{k \to \infty} I_{\mathcal{U}}(S_k \cap \mathcal{U}).$$

*Proof* (compare [BS]). The hypotheses ensure we can write $S_k$ as a Lipschitz graph in $\mathcal{U}$, and Ascoli-Arzela provides a subsequence converging strongly in $C^0$ and weakly in $H^1$. Now Serrins theorem ([M] 1.8.2) and concavity of the area functional gives the inequality on $I_{\mathcal{U}}$. □

Thus, if $\{S_k\}$ is a maximising sequence for $I_F$, this shows that if $\{S_k\}$ has a pointwise convergent subsequence, then the limit surface $S$ is at least locally maximising. The main regularity Theorem 6.4 will show that $S$ is a regular hypersurface, except for its singular set $\Sigma(S)$.

Associated with any $C^1$ foliation with mean curvature $F$ there is an integral identity involving $I_F$ and based on Stokes theorem applied to $F=\mathrm{div}(N)$, where $N$ is the unit timelike normal to the leaves. Precisely, we have

LEMMA 6.2. *Suppose that $\tau \in C^1(\mathcal{U})$ is a time function with level sets $Q_\tau$ having mean curvature $H_{Q_\tau}=F|_{Q_\tau}$ for some $F \in L^1(\mathcal{U})$, and that $M$ is a* WSH *which is $T'$-homotopic to $Q'=Q_0$ where $T'=-\nabla\tau/|\nabla\tau|$. Then*

$$I_F(M) = I_F(Q') + \int_M (1-\nu_\tau)\, dv_M \tag{6.2}$$

*where $\nu_\tau = -\langle T', N\rangle$ and we interpret the term $\nu_\tau dv_M$ by* (6.3) *below if $M$ is not a regular hypersurface. Furthermore, the term $(1-\nu_\tau)dv_M$ is nonpositive and identically zero only if $M$ is contained in a level set of $\tau$.*

*Proof.* Using the flow lines of $T'$ we construct zero-shift coordinates $(x, \tau)$ with metric

$$ds^2 = -\alpha^2 d\tau^2 + g_{ij}\, dx^i dx^j,$$

and consider $M$ as a graph of $u \in C^{0,1}(Q')$ over $Q'$. Then a standard computation gives

$$\frac{\partial}{\partial\tau}\sqrt{g} = \alpha\sqrt{g}\, H_{Q_\tau} = \alpha\sqrt{g}\, F$$

so integrating over $(x, \tau)$ we have

$$\begin{aligned}\int_{V(Q',M)} F\alpha\sqrt{g}\, dx\, d\tau &= \int_{Q'} \sqrt{g(x,u(x))}\, dx - \int_{Q'} \sqrt{g(x,0)}\, dx \\ &= \int_M \nu_\tau\, dv_M - |Q'|.\end{aligned}$$

This last equality uses the identity

$$\nu_\tau \, dv_M = \sqrt{g(x, u(x))}\, dx, \tag{6.3}$$

which holds for regular hypersurfaces $M$, and will be valid for general weakly spacelike $M$ if we use (6.3) to interpret $\nu_\tau dv_M$. The definition of $I_F$ now gives (6.2), for all WSH $M$. The final assertion follows from the expression for the volume form ([B1] (5.16))

$$dv_M = \sqrt{1-\alpha^2|Du|^2}\,\sqrt{g(x, u(x))}\, dx. \qquad \square$$

As an immediate corollary we have that Dirichlet solutions are locally extremal:

COROLLARY 6.3. *Suppose $M$ is a regular hypersurface which has mean curvature $H_M=F|_M$ for $F\in C^1(\mathcal{V})$. Then $M$ is locally extremal for $I_F$ in the sense described above.*

*Proof.* Since $M$ is regular, the conditions of Theorem 5.2 can be met by blowing up a $\sigma$-cylinder neighbourhood $\mathcal{C}_\sigma(p)$ of any $p\in M$, for $\sigma$ sufficiently small, so that $M\cap\mathcal{C}_\sigma(p)$ satisfies the eigenvalue condition. Then Theorem 5.1 constructs a $C^1$ foliation with mean curvature $F$ in $\mathcal{C}_\sigma(p)$, with boundary data given by the level sets of any $C^1$ time function with $M$ as a level set. Such time functions can be constructed either by Proposition 3.2 or by [B1] Proposition 3.2. Let $\tau$ be the time function of the mean curvature $F$ foliation. If now $M'\approx(M\cap\mathcal{C}_\sigma(p))\ \mathrm{rel}\,\partial(M\cap\mathcal{C}_\sigma(p))$ then $M'$ and $M\cap\mathcal{C}_\sigma(p)$ are also $T'$-homotopic since they have common boundary and Lemma 6.2 shows that $I_F(M\cap\mathcal{C}_\sigma(p))\geq I_F(M')$, with equality only if $\nu_\tau\equiv 1$ on $M'$. This implies that equality holds only if $M'=M\cap\mathcal{C}_\sigma(p)$ and shows that $M$ is locally extremal. $\square$

Directly applying this argument to a variational extremal $M$ encounters two difficulties, both related to the fact that $M\cap\mathcal{C}_\sigma(p)$ can have rough boundary: the foliation may have a "gap" or "lens" spanning $\partial(M\cap\mathcal{C}_\sigma(p))$ because the conditions of Theorem 5.3 are not met and secondly, if the lens can be foliated, the normal vector to the foliation degenerates along $\partial(M\cap\mathcal{C}_\sigma(p))$ and Lemma 6.2 does not apply. Fortunately, the integral identity (6.2) is robust enough to deal with these problems.

THEOREM 6.4. *Suppose that $F\in C^1(\mathcal{V})$ and that $M$ is a* WSH *which is locally maximising for $(VP)_{\mathcal{F}}$. If $p\in M$ then either $p\in\Sigma(M)$, the singular set defined by* (3.13), *or $M$ is a regular hypersurface in a neighbourhood of $p$.*

*Remark.* Since this result is purely local it applies also if $M$ is an immersed WSH, thus providing an alternative approach to Theorem 4.2. More generally it will apply to any (constrained) variational problem which produces a WSH which is locally maximising on some subset, which subset will then be regular away from the singular set. Hence these results will apply to variational problems involving obstacles or barriers or free boundaries, for example. However, in such cases it may be necessary to slightly modify the definition of the singular set.

*Proof.* There is $\sigma_0>0$ such that $M$ is maximising for $(VP)_{\mathcal{F}}$ in the cylinder neighbourhoods $\mathscr{C}_\sigma(p)$ for $0<\sigma<\sigma_0$. Suppose first that $p\notin\Sigma(M\cap\mathscr{C}_\sigma(p))$; i.e. $p$ does not lie in a piece of null geodesic in $M$. By choosing $\sigma$ still smaller we may assume $\Sigma(M\cap\mathscr{C}_\sigma(p))=\emptyset$ and that $C\sigma^2<\varepsilon_0$, where $\varepsilon_0$ and $C$ are given in Theorem 5.2. By the blowing-up procedure described in Section 5 we may reduce our considerations to the case where $M$ is a WSH in the unit cylinder $\mathscr{C}=B_1(0)\times[-2;2]$ equipped with metric $g$ and mean curvature function $F$ satisfying the estimates (5.7), (5.7′) of Theorem 5.2, and with $p=(0,0)\in\mathscr{C}$.

Let $M=\text{graph}(u)$, where $u\in C^{0,1}(B_1)$ and set $\varphi=u|_{\partial B_1}$. By Corollary 3.3 we can construct a sequence $\varphi_k\in C^\infty(\partial B_1)$ of smooth, spacelike spanning boundary data such that $\varphi_k>\varphi$, $\varphi_k\to\varphi$ as $k\to\infty$. Let $M_k=\text{graph}(u_k)$, $u_k\in C^{2,\alpha}(B_1)$, be a solution of the Dirichlet problem with boundary $\varphi_k$ and mean curvature $F$, given by [B1] Theorem 4.3. The convergence Theorem 3.8 shows that

$$u_+=\liminf u_k$$

exists, $u_+\in C^{2,\alpha}_{\text{loc}}(B_1)$, and solves the Dirichlet problem with boundary data $\varphi$. We claim that $u\leq u_+$ in $B_1$. For if not, there is $k$ such that $u(x)>u_k(x)$ for $x\in\Omega_k\subset\subset B_1$, $u(x)=u_k(x)$ for $x\in\partial\Omega_k$ and $\Omega_k\neq\emptyset$. By Theorem 5.1 we can construct a $C^1$ foliation of $\mathscr{C}$ with mean curvature $F$ and including $M_k$ as a leaf, and now Lemma 6.1 shows that $I_F(u|_{\Omega_k})<I_F(u_k|_{\Omega_k})$. This contradicts the locally maximising property of $u$.

Let $M_+=\text{graph}(u_+)$. In a similar manner we construct the lower surface $M_-=\text{graph}(u_-)$, a regular hypersurface with mean curvature $F$ and boundary $\partial M_-=\partial M$, and we have

$$u_-(x)\leq u(x)\leq u_+(x)\quad\text{for all}\quad x\in B_1.$$

By the strong maximum principle, either $u_+\equiv u_-$ and $M$ is a regular hypersurface, or $u_-(x)<u_+(x)$ for all $x\in B_1$.

Thus we suppose $M_+ > M_-$ and proceed to foliate the lens-shaped region $\mathscr{L}$ between $M_+$ and $M_-$. (It is somewhat surprising that despite the uniqueness for smooth boundary data in $\mathscr{C}$, we cannot immediately exclude non-uniqueness for rough boundary data.) By [B1] Proposition 3.2 we have a time function in $B_{1-\delta}\times[-2;2]$, $\delta>0$, which includes $M_+$, $M_-$ as level sets. Over $B_{1-\delta}$, $M_-$ satisfies the eigenvalue condition so we can construct a $C^1$ time function $\tau_\delta$ in $\mathscr{L}$ over $B_{1-\delta}$ with level sets having mean curvature $F$ and such that $M_+, M_-$ are level sets of $\tau_\delta$. We can normalise $\tau_\delta$ by a $C^1$ change to have $\tau_\delta(0,t)=t$, so the level sets of $\tau_\delta$ are prescribed mean curvature hypersurfaces over $B_{1-\delta}$, parameterised by their intersection with the central axis $\{(0,t)\colon -2<t<2\}$. These level sets (and thus $\tau_\delta$) satisfy uniform interior estimates so we have a converging subsequence $\tau_{\delta_k}\to\tau$, where $\tau\in C^1_{\mathrm{loc}}(\mathscr{L})$ is a time function foliating the lens with prescribed mean curvature hypersurfaces $M_\tau=\mathrm{graph}(u_\tau)$, for $u_-(0)\leq\tau\leq u_+(0)$.

For $\delta>0$ define $M_0(\delta)=\{q\in M_0\colon d(q,\partial M_0)>\delta\}$ and the region

$$\mathscr{L}_\delta=\{q\in\mathscr{L};\ \text{the } \nabla\tau\text{-integral curve through } q \text{ intersects } M_0(\delta)\}.$$

Since $\mathscr{L}\subset D(M_0)$, the sets $\mathscr{L}_\delta$, $\delta\downarrow 0$ form an exhaustion of $\mathscr{L}$. Let $M(\delta)=M\cap\mathscr{L}_\delta$. Since $p\in M\cap M_0$ and $M\not\equiv M_0$, there is $\varepsilon>0$ and $\delta_0>0$ such that for all $0<\delta\leq\delta_0$,

$$\int_{M(\delta)}(1-\nu_\tau)\,dv_M\leq-\varepsilon$$

since $\nu_\tau\geq 1$ with equality only if $M$ is tangent to a level set of $\tau$. As noted previously, if $M$ is a WSH we interpret $\nu_\tau\,dv_M$ by (6.3).

Now by (6.2), since $M(\delta)\approx M_0(\delta)$ along $\nabla\tau$, for all $\delta\leq\delta_0$,

$$\begin{aligned}I_\delta(M(\delta))&=I_\delta(M_0(\delta))+\int_{M(\delta)}(1-\nu_\tau)\,dv_M\\&\leq I_F(M_0)-\varepsilon\end{aligned}$$

where

$$I_\delta(S)=|S|-\int_{V(M_0(\delta),S)}F\,dv_V$$

for $S \approx M_0(\delta)$ by $\nabla\tau$, and $I_F$ is measured from $M_0$. But

$$I_F(M) - I_\delta(M(\delta)) = |M - \mathscr{L}_\delta| - \int_{V(M_0, M) - \mathscr{L}_\delta} F\, dv_{\mathscr{V}}$$
$$= o(1) \quad \text{as} \quad \delta \downarrow 0$$

since $\mathscr{L}_\delta$ gives an exhaustion of $\mathscr{L}$. Thus choosing $\delta$ sufficiently small we have

$$I_F(M) \leq I_\delta(M(\delta)) + \varepsilon/2 \leq I_F(M_0) - \varepsilon/2$$

which contradicts the locally maximising property of $M$. Thus $M \equiv M_0$ and hence $M$ is regular about $p$.

If $p \in \Sigma(M \cap \mathscr{C}_\sigma(p))$ then $p$ lies on a piece of null geodesic $\gamma$ lying in $M$. We claim $\gamma \cap M$ does not have an endpoint $q \in M$, for if such a $q \in M$ existed we could apply the above argument to $M \cap \mathscr{C}_\sigma(q)$, which has empty singular set, $\Sigma(M \cap \mathscr{C}_\sigma(q)) = \varnothing$. (If $\Sigma(M \cap \mathscr{C}_\sigma(q)) \neq \varnothing$ for all $\sigma > 0$, then either $M$ is not locally achronal at $p$ or $p$ is not an endpoint of $\gamma \cap M$.) Thus $M \cap \mathscr{C}_\sigma(q)$ is regular, contradicting the fact that $\gamma \cap M \cap \mathscr{C}_\sigma(q) \neq \varnothing$. Thus $\gamma \cap M$ can have endpoints only on $\partial M$, so $p \in \Sigma(M)$. □

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