On Leray's problem of steady Navier-Stokes flow past a body in the plane

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Contents

1. Introduction

In this paper, we consider the problem of steady Navier-Stokes flow past a prescribed body in the plane. Let K be a compact set in $\mathbf{R}^2$ with Γ, the boundary of K, consisting of n smooth components. Given the set K, a number $\nu>0$, and a constant vector

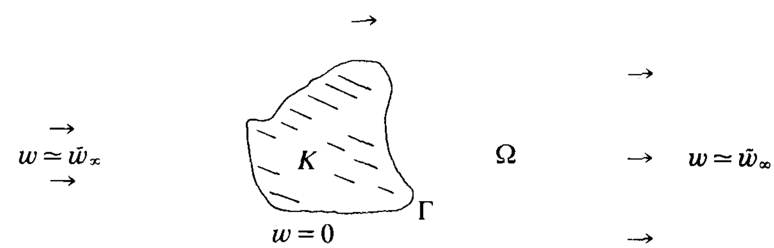

Figure 1. A picture of the physical problem represented by (1.1)–(1.4).

$\bar{w}_\infty \in \mathbf{R}^2 \setminus \{0\}$, the mathematical problem is to find a velocity $w: \bar{\Omega} \to \mathbf{R}^2$ and a pressure $p: \bar{\Omega} \to \mathbf{R}$, where $\Omega = \mathbf{R}^2 \setminus K$, satisfying

$$\left.\begin{aligned} -\nu\Delta w + (w\cdot\nabla)\, w &= -\nabla p \\ \nabla\cdot w &= 0 \end{aligned}\right\} \text{ in } \Omega, \qquad \begin{aligned}(1.1)\\(1.2)\end{aligned}$$

$$w = 0 \quad \text{on} \quad \Gamma, \tag{1.3}$$

and

$$w(x, y) \to \bar{w}_\infty \quad \text{as} \quad |(x, y)| \to \infty. \tag{1.4}$$

The function w is to have finite Dirichlet norm, that is,

$$\iint_\Omega |\nabla w|^2 \equiv \iint_\Omega \{u_x^2 + u_y^2 + v_x^2 + v_y^2\} < \infty, \tag{1.5}$$

where u and v denote the horizontal and vertical components of w, respectively. Such an estimate ensures that p and w are smooth in Ω; indeed, they are real-analytic. In the famous and still open problem of unsteady three dimensional Navier-Stokes flow in, say, a bounded domain, the problem is to show the presence or absence of singularities in the flow. Since any solution of (1.1)–(1.5) is steady and smooth in Ω, this problem of regularity does not arise. Rather, the problem is one of regularity at infinity, that is, the existence of a solution to (1.1)–(1.3) attaining a prescribed limit $\bar{w}_\infty \in \mathbf{R}^2$ at infinity. When $\bar{w}_\infty = 0$, we always have at least the trivial solution $w \equiv 0$.

The study of (1.1)–(1.5) began with Leray [18] who sought the solution as the limit of certain approximate solutions. For all sufficiently large R, let $\Omega_R = \Omega \cap \{(x, y): r < R\}$. Here r denotes distance from the origin which we may assume, without loss of generality, lies in the interior of K. Upon rescaling, we may also always assume that

$K\subset\{r<1\}$. Leray sought a solution of (1.1)–(1.2) in Ω_R with $w=0$ on Γ and $w(x,y)=\bar{w}_\infty$ on the circle of radius R. Leray was able to prove the existence of a solution (w_R, p_R) such that the Dirichlet norm has a uniform bound

$$\iint_{\Omega_R} |\nabla w_R|^2 \leq \text{const.}, \tag{1.6}$$

where the constant is independent of R. Upon taking a suitable subsequence as $R\to\infty$, a solution (w_L, p_L) of (1.1)–(1.3) was found. (Here the subscript 'L' denotes a Leray solution, that is, one constructed by the scheme given above.) The *a priori* bound in (1.6) is inherited by w_L, but the behaviour of w_L at infinity was not found. Indeed, it was not even apparent that w_L was non-trivial.

The problem (1.1)–(1.5) may also be posed in $\mathbf{R}^3$ with K compact and other obvious changes. The Leray construction gives rise to a solution satisfying the three-dimensional version of (1.6). Functions with finite Dirichlet norm in *three-dimensions* allow certain imbedding estimates:

$$\iiint_{\Omega_R} \left|\frac{w_R-\bar{w}_\infty}{r^2}\right|^2, \left\{\iiint_{\Omega_R} |w_R-\bar{w}_\infty|^6\right\}^{1/3} \leq \text{const.} \iiint_{\Omega_R} |\nabla w_R|^2. \tag{1.7}$$

In particular, the Leray solution satisfies

$$\iiint_{\Omega_R} \left|\frac{w_L-\bar{w}_\infty}{r^2}\right|^2, \iiint_{\Omega_R} |w_L-\bar{w}_\infty|^6 < \infty. \tag{1.8}$$

These estimates ensure that w_L is non-trivial, and were the basis for the proof, due to Finn [6] and Ladyzhenskaya [16], that (1.4) is satisfied. In addition, Babenko [2] has shown that any solution of (1.1)–(1.3) with $\nabla w \in L_2(\Omega)$ satisfies (1.4) for some $\bar{w}_\infty$ and the flow field has the expected wake properties. All of the results mentioned above were for the three-dimensional problem, and follow by fairly straightforward energy and potential-theoretic arguments.

The difference between the two-dimensional problem considered in this paper and the analogous three-dimensional one lies in the absence of estimates of the form (1.7). In two dimensions, we have

$$\iint_{\Omega_R} \frac{|w_R-\bar{w}_\infty|^2}{r^2(1+|\log r|)^2} \leq \text{const.} \left\{1+\iint_{\Omega_R} |\nabla w_R|^2\right\},$$

whence

$$\iint_{\Omega} \frac{|w_L - \tilde{w}_\infty|^2}{r^2(1+|\log r|)^2} \leqslant \text{const.} \left\{ 1 + \iint_{\Omega} |\nabla w_L|^2 \right\} < \infty. \tag{1.9}$$

Unfortunately, (1.9) gives little information about the behaviour of w_L at infinity. Indeed, if (1.9) holds, then it is also valid if $\tilde{w}_\infty$ is replaced by any other constant vector. Unlike (1.8), this estimate allows w_L to be trivial. If we had a better estimate on w_R, such as, say, $(\log r)\nabla w_R$ uniformly bounded in $L_2(\Omega_R)$, then (1.9) could be improved, and the problem solved. Unfortunately, the form of the nonlinear term in (1.1) appears only to allow (1.6), and so that is the information we must work with. The failure of energy methods for (1.1)–(1.5) forces us to use the more precise structure of the equations.

Before we discuss the results of Gilbarg and Weinberger [14], [15] in the next section, we note that (1.1)–(1.5) have been shown to have a solution by Finn and Smith [11] whenever $|\tilde{w}_\infty|/\nu$ is sufficiently small. The solution is not constructed by the Leray method, but begins with (1.1) replaced by

$$-\nu \Delta w + (\tilde{w}_\infty \cdot \nabla) w + \nabla p = (\tilde{w}_\infty - w) \cdot \nabla w.$$

The linear operator on the left is inverted with the addition of (1.2)–(1.4), and the resulting nonlinear equations solved with the aid of the contraction mapping principle. Additional results on the two and three-dimensional problem may be found in [1], [5], [7], [8], [9], [12], and [21].

1.1. Results of Gilbarg and Weinberger

A considerable step towards understanding (1.1)–(1.5) came in two important papers of Gilbarg and Weinberger, who exploited certain maximum principles for two physical quantities. If $w=(u,v)$, then the *vorticity* ω is defined by $\omega = u_y - v_x$. Equations (1.1)–(1.2) yield

$$-\nu \Delta \omega + w \cdot \nabla \omega = 0, \tag{1.10}$$

whence ω satisfies a two-sided maximum principle: The maximum and minimum values of ω on the closure on an open set U are taken on ∂U, the boundary of U. (If U is unbounded, we need to know something about ω in a neighborhood of infinity.) Equation (1.10) holds in both Ω_R and Ω.

A second quantity is the *total-head pressure* Φ defined by $\Phi=p+\frac{1}{2}|w|^2$. A calculation from (1.1)–(1.2) yields

$$-\nu\Delta\Phi+w\cdot\nabla\Phi=-\nu\omega^2, \tag{1.11}$$

whence Φ takes its maximum value on ∂U. Equation (1.11) also holds in Ω_R and Ω. In the first part of [14], Gilbarg and Weinberger considered the 'approximate' solutions (w_R, p_R) and used (1.11) in Ω_R and other arguments to show that

$$|w_R(x, y)|,\ |p_R(x, y)|\leq \text{const.} \quad \text{if} \quad |(x, y)|\leq R/2, \tag{1.12}$$

where the constant is independent of R. We hasten to add that any solution (w_R, p_R) of (1.1)–(1.5) in Ω_R has p_R determined only up to an additive constant. We assume that the pressures have been prescibed at some fixed point, say a point in $\Gamma=\partial\Omega$. (The arguments which prove (1.12) may be extended so that it holds on all of Ω_R, but there will be no need for them here.)

The important conclusion from (1.12) is that the Leray solution has $w_L\in L_\infty(\Omega)$. This is not surprising in view of our desire to show that (1.4) holds, but it is far from obvious. Indeed, one can easily construct solenoidal velocity fields in Ω with finite Dirichlet norm which become unbounded like $(\log r)^\alpha$, $0<\alpha<\frac{1}{2}$, as $r\to\infty$. In addition, it was shown that the pressure p_L has a limit at infinity, which may be taken to be zero after subtraction of a suitable constant. The main result of [14] was the use of (1.11) in Ω along with other estimates to show that the mean-value of w_L on circles attains a limit at infinity. Since we shall refer to these results and others, they are summarized in the following

THEOREM 1 (Gilbarg and Weinberger [14], [15]). *If (p_L, w_L) is a Leray solution (satisfying (1.1)–(1.3) and (1.5)), then*

(a) $w_L\in L_\infty(\Omega)$;

(b) *there exists a constant vector $w_\infty\in\mathbf{R}^2$ such that*

$$\int_0^{2\pi}|w_L(r,\theta)-w_\infty|\,d\theta\to 0 \quad as \quad r\to\infty; \tag{1.13}$$

(c) *if $w_\infty=0$, then $w_L\to 0$ at infinity;*

(d) *$p_L\to 0$ at infinity;*

(e) $$\iint_\Omega r|\nabla\omega|^2\leq \text{const.}\iint_\Omega \omega^2<\infty;$$

(f) $|\omega(r,\theta)|=o(r^{-3/4})$, $|\nabla w(r,\theta)|=o(r^{-3/4}\log r)$ *as $r\to\infty$, uniformly in θ.*

In (b), we have written $w(r,\theta)$ for what should correctly be termed $w(r\cos\theta, r\sin\theta)$; however, there is little chance of confusion. We note that $\nabla w \in L_2(\Omega)$ easily implies $\omega \in L_2(\Omega)$. Despite the very positive results represented in Theorem 1, a number of questions remain. Is w_L non-trivial, does w_∞ equal the desired $\bar{w}_\infty$, and can the convergence in (1.13) be improved to pointwise convergence? We shall return to these questions in the next section.

In [15], Gilbarg and Weinberger considered properties of any solution to (1.1)–(1.3) with $\nabla w \in L_2(\Omega)$. We know that the Leray solution has this property, and so their results were intended for 'solutions' constructed by some other method.

THEOREM 2 (Gilbarg and Weinberger [14], [15]). *Let* (w,p) *be a solution of* (1.1)–(1.3) *with* $\nabla w \in L_2(\Omega)$. *Then*

(a) $|w(r,\theta)|^2 = o(\log r)$ *as* $r\to\infty$, *uniformly in* θ; *and*

(b) p *has a limit at infinity, say* $p\to 0$ *at infinity.*

(c) *The quantities* ω *and* $|\nabla w|$ *satisfy estimates similar to those of Theorem* 1 (e) *and* (f).

(d) *If* $w \in L_\infty(\Omega)$, *then the estimates of Theorem* 1 (b)–(f) *hold.*

The difference between this theorem and the previous one lies in the absence of an estimate like (1.12) to ensure that $w \in L_\infty(\Omega)$. If one assumes this, then the methods for Theorem 1 give Theorem 2 (d). Since we shall show later that $w \in L_\infty(\Omega)$, we have not stated precisely what is meant in (c). A typical estimate would be

$$\int\!\!\int_\Omega \frac{r}{\sqrt{1+|\log r|}}\,|\nabla\omega|^2 \leq \text{const.} \int\!\!\int_\Omega \omega^2 < \infty$$

with similar small changes in the estimates for ω and $|\nabla w|$.

1.2. Methods and results

The results in this paper follow from the introduction of a new quantity γ which satisfies a two-sided maximum principle and is such that $\gamma + i\nu\omega$ is a pseudo-analytic function. The equations we shall display are derived solely from (1.1)–(1.2) and hold in any bounded domain, such as Ω_R, or in any unbounded domain, such as Ω. Fix any point (x_0,y_0) on $\Gamma=\partial\Omega$ and define a Stokes stream-function by

$$\psi(x,y) = \int_{(x_0,y_0)}^{(x,y)} u\,dy - v\,dx,$$

where we have integrated over any simple arc in Ω or Ω_R connecting the point (x_0, y_0) to (x, y). Since $w=(u, v)=0$ on Γ, the function ψ is well-defined. If we define $\gamma=\Phi-\psi\omega=p+\frac{1}{2}|w|^2-\psi\omega$, then a calculation from (1.1)–(1.2) yields

$$\gamma_x = \nu\omega_y - \psi\omega_x, \tag{1.14}$$

$$\gamma_y = -\nu\omega_x - \psi\omega_y \tag{1.15}$$

or, equivalently,

$$\omega_x = -(\nu\gamma_y + \psi\gamma_x)/(\nu^2+\psi^2). \tag{1.16}$$

$$\omega_y = (\nu\gamma_x - \psi\gamma_y)/(\nu^2+\psi^2). \tag{1.17}$$

Another calculation from (1.1)–(1.2) yields

$$-\nu\Delta\gamma + (w+q)\cdot\nabla\gamma = 0 \tag{1.18}$$

where

$$q = \{-2\nu^2 w + 2\nu\psi(-v, u)\}/(\nu^2+\psi^2).$$

Equation (1.18) ensures that γ satisfies a two-sided maximum principle, but it is the system (1.14)–(1.15) which is of the most interest. Standard theory [3] shows that (ω, γ) correspond to an elliptic system in the plane, and therefore should have many of the properties of conjugate harmonic functions. For example, let $(\bar{x}, \bar{y})$ be a point at which $\nabla\omega(\bar{x}, \bar{y})\neq 0$. Since ψ, γ, ω, and all of our quantities are real-analytic (in Ω_R or Ω), the level-set of $\omega=\omega(\bar{x}, \bar{y})$ is locally a real-analytic curve passing through $(\bar{x}, \bar{y})$. The maximum principle applied to (1.10) ensures that $\omega-\omega(\bar{x}, \bar{y})$ changes sign as the curve is crossed. The strong maximum principle ensures that the normal derivative of ω is one-signed on the curve. Equations (1.14)–(1.15) then show that the tangential derivative of γ is one-signed, whence γ is monotone as the curve is transversed.

The quantity γ is useful because it contains the speed $|w|$, but it is difficult to estimate, at first, because of the term $\psi\omega$. Since we expect the velocity w to have a limit at infinity in Ω, the stream function ψ should be of order r at certain points a distance r from Γ. The estimates of Theorems 1 and 2 only give us, roughly, that $\omega=o(r^{-3/4})$, so that the product $\psi\omega$ cannot be estimated. We shall get around this difficulty by using the monotonicity of γ along suitable level-sets of ω. Note that $\psi\omega$ vanishes on the level-set of $\omega=0$, and that $\gamma=\Phi$ there.

In section 2, we give results for any non-trivial solution of (1.1)–(1.3) with $\nabla w \in L_2(\Omega)$. Note that any non-trivial Leray solution (w_L, p_L) satisfies these conditions. In section 2.1, we give various properties of the vorticity and its level-sets. In particular, we show, roughly speaking, that there are at least two arcs connecting $\partial\Omega$ to infinity on which $\omega=0$ and on which $\Phi=p+\frac{1}{2}|w|^2$ is monotone increasing and decreasing, respectively, as the arcs are transversed from $\partial\Omega$ to infinity. We shall use this in Theorem 12 to show that $w \in L_\infty(\Omega)$. In particular, the results of Theorem 2(d) are applicable, and there exists a constant vector $w_\infty \in \mathbf{R}^2$ such that

$$\lim_{r\to\infty} \int_0^{2\pi} |w(r,\theta)-w_\infty|^2 d\theta = 0. \tag{1.19}$$

In Theorem 14, we use the monotonicity of γ along level-sets of ω to prove the important estimate

$$\gamma = \Phi - \psi\omega = p + \frac{1}{2}|w|^2 - \psi\omega \to \frac{1}{2}|w_\infty|^2 \tag{1.20}$$

at infinity, where the pressure is always normalized to vanish at infinity. The rest of section 2 is devoted to the behaviour of w and other quantities at infinity. If $w_\infty=0$, then Theorems 1(c) and 2(d) show that $w\to 0$ at infinity. For the moment, *we restrict attention to the case* $w_\infty \neq 0$, *and may assume that* $w_\infty=(1,0)$ after a suitable rotation and scaling. For each $\varepsilon \in (0, \pi/2)$, define the sectors

$$A_\varepsilon = \{(r,\theta): r \geq 1, |\theta| \in [\varepsilon, \pi-\varepsilon]\},$$

where $\theta \in (-\pi, \pi]$. The estimates (1.19)–(1.20) and others are used to show that $\psi \simeq y$ and $\omega = o(r^{-1})$ near infinity in A_ε. These estimates then give $w(x,y) \to w_\infty$ as (x,y) tends to infinity in the sectors A_ε. The pointwise convergence of w to w_∞ in the remaining sectors of angular width 2ε along the positive and negative x-axis is much more difficult. In these regions, we cannot expect the crucial estimate $\omega=o(r^{-1})$ proved in the A_ε. We get around this difficulty with the aid of (1.20). Since $|w|=|\nabla\psi|$, $\omega=u_y-v_x=\Delta\psi$, and $p\to 0$ at infinity, equation (1.20) gives

$$\frac{1}{2}|\nabla\psi|^2 - \psi\Delta\psi \simeq \frac{1}{2}|w_\infty|^2 = \frac{1}{2} \tag{1.21}$$

near infinity. On a set where $\psi>0$, this yields

$$\Delta(\sqrt{\psi}) \simeq -\frac{1}{4\psi^{3/2}}, \tag{1.22}$$

and similarly where $\psi<0$. Since ψ is defined in terms of a line integral involving w, and (1.19) gives some information about integrals of w, the right-hand side of (1.22) may be estimated in a suitable manner. We then invert the Laplacian to derive the desired information about $w=(\psi_y, -\psi_x)$ near infinity. This is done rigorously in section 4.1 for the case of symmetric flow (that is, for flow symmetric about the x-axis), but the method does not hold for asymmetric flow. Notwithstanding the difficulties of asymmetric flow, we show in section 2.4 that the speed $|w|$ always converges to $|w_\infty|$ at infinity, whence (1.20) yields $|\psi\omega|\to 0$ at infinity. We conclude section 2 by showing that $|\omega|$ and $|\gamma-\frac{1}{2}|w_\infty|^2|$ tend to zero *exponentially* at infinity away from the positive x-axis.

Section 3 is devoted to the 'approximate' solutions (w_R, p_R) due to Leray. We first show that the unique solution (w_R^s, p_R^s) of the Stokes problem in Ω_R (obtained by omitting $(w\cdot\nabla)w$ in (1.1)) satisfies

$$\iint_{\Omega_R} |\nabla w_R^s|^2 \leq \frac{\text{const.}}{\log R}|\bar{w}_\infty|^2. \tag{1.23}$$

This is a frightening result since one might expect the same to hold for w_R in which case Leray's solution w_L would be identically zero. However, we show in Theorem 23 that if there are level-sets of $\omega=0$ connecting (roughly) $\partial\Omega$ to $\{r=R\}$, then (1.23) does not hold, and

$$\liminf_{R\to\infty} \iint_{\Omega_R} |\nabla w_R|^2 > 0. \tag{1.24}$$

Finally, we show that if (1.24) holds for a general flow (that is, one that is not necessarily symmetrical), then Leray's solution is non-trivial.

In section 4, some of the results of sections 2 and 3 are improved for the case of symmetric flow. The equations (1.21)–(1.22) are analysed in section 4.1, and give the desired pointwise convergence $w(x,y)\to w_\infty$ at infinity in Ω. The inequality (1.24) is shown to hold, whence *Leray's solution is non-trivial.* Hence, for symmetric flow, Leray's construction gives rise to a non-trivial solution of (1.1)–(1.3), (1.5) which has a pointwise limit at infinity. The remaining great question is to show that the limiting value w_∞ equals the desired value $\bar{w}_\infty$ in (1.4).

A brief and preliminary version of this paper appeared in [1]. Many of the initial results were found during the author's visit to the Institute for Mathematics and its Applications in Minneapolis during the autumn of 1984, while this paper was written

during a visit to the Centre for Mathematical Analysis in Canberra. The author is grateful for the kind hospitality shown in both cities, and also to Professor Weinberger for his remarks on some of the preliminary results. The author was supported in part by the Sloan Foundation and the National Science Foundation.

1.3. Notation

We shall assume throughout this paper that Γ, the boundary $\partial\Omega$ of Ω, consists of n smooth closed curves. We denote points as (x, y) or $z=x+iy$, when convenient. The symbol $r=|z|$ will denote distance from the origin, which is assumed to lie in the interior of one of the components of Γ. We represent a function, say w, at a point in Ω by either $w(x, y)$ or $w(z)$ or $w(r, \theta)$ in polar coordinates. We use $w(r, \theta)$ as notation for $w(r\cos\theta, r\sin\theta)$. We assume that $\Gamma\subset\{|z|<1\}$.

Standard regularity theory ensures that all functions which occur in the problem are real-analytic in their open domain of definition, be it Ω or Ω_R. These functions are suitably smooth up to Γ and real-analytic up to $\{|z|=R\}$.

When Γ is symmetric about the x-axis and the desired boundary data is $\tilde{w}_\infty=(\alpha, 0)$ at infinity for the case Ω or at $\{|z|=R\}$ for Ω_R, one can seek a solution $(w, p)=((u, v), p)$ with p and u even in y and v odd in y. The functions Φ and γ will then be even, while ψ and ω will be odd functions of y. Such solutions will be referred to as 'symmetric' or 'symmetric flow'. If Γ and $\tilde{w}_\infty$ are symmetric (as described above), then the construction of Leray in Ω_R will refer to symmetrical solutions. (Since there need not be unique solutions in Ω_R, we cannot ensure that any solution is necessarily symmetric. However, the construction allows us to assume there is at least one.) Unless stated otherwise, the results in this paper hold for general flows, and results for symmetric flow will be carefully noted as such.

2. Flow in the unbounded domain Ω

Throughout this section, we assume that (w, p) satisfy (1.1)–(1.3) in Ω with $\nabla w\in L_2(\Omega)$. In addition, we assume that w is non-trivial. The results of Theorem 2 are then applicable, so that $|\omega(z)|$, $|p(z)|\to 0$ as $|z|\to\infty$. Section 2.1 is concerned with level-sets of ω and the open sets on which ω exceeds or is less than zero. Since we have assumed that w is non-trivial, the following lemma gives the same for the vorticity ω.

LEMMA 3. *If $\omega\equiv 0$ in Ω, then $w=(u, v)\equiv 0$ in Ω.*

Proof. The hypothesis and (1.2) together ensure that u and v are conjugate harmonic functions in Ω. Since $u=v=0$ on Γ by (1.3), it follows that $u=\partial u/\partial n=0$ on Γ, where n denotes a normal to Γ. Standard theory then proves that $u\equiv 0$ in Ω, and similarly for v. q.e.d.

2.1. Properties of the vorticity and its level-sets

The representation $\gamma=\Phi-\psi\omega$ in (1.14)–(1.15) yields

$$\Phi_x = \nu\omega_y+\psi_x\,\omega = \nu\omega_y-v\omega, \tag{2.1}$$

$$\Phi_y = -\nu\omega_x+\psi_y\,\omega = -\nu\omega_x+u\omega. \tag{2.2}$$

On a suitable level-set of $\omega=0$, we expect from (1.10) and the strong maximum principle that the normal derivative of ω to the curve will be one-signed. The use of this in (2.1)–(2.2) shows that the tangential derivative Φ is one-signed. Our aim is to find level sets of $\omega=0$ which go from infinity to points near to Γ, and on which Φ is monotone. Since we have estimates on Φ near Γ (indeed, on compact subsets of $\bar{\Omega}$), the monotonicity will provide some valuable information about $\Phi(z)$ for large $|z|$. We shall need the following technical result about certain level-sets of functions.

LEMMA 4. *Let τ be a real-analytic function in Ω. Let $z_1\in\Omega$ by such that $\nabla\tau(z_1)\neq 0$, and let L denote the component of $\{z\in\Omega: \tau(z)=\tau(z_1), \nabla\tau(z)\neq 0\}$ containing z_1. Then there exists an injective, real-analytic function φ defined on $I=(-M,N)$, where $M,N\in(0,\infty]$ with $\varphi(0)=z_1$, and $\varphi(I)\subset L$. Moreover, as $t\to N$, exactly one of the following hold*

(i) $|\varphi(t)|\to\infty$,

(ii) $\varphi(t)\to\Gamma=\partial\Omega$,

(iii) $\varphi(t)\to z_2\in\Omega$, $\nabla\tau(z_2)=0$,

or

(iv) $\varphi(t)\to z_2\in\Omega$, $\nabla\tau(z_2)\neq 0$.

If (i), (ii), *or* (iii) *occur, then $L=\varphi(I)$. If* (iv) *occurs, then $M,N<\infty$ and $\varphi(t)\to z_2$ as $t\to -M, N$, $L=\varphi(I)\cup\{z_2\}$, and L is a closed, simple real-analytic curve. As $t\to -M$, exactly one of* (i)–(iv) *hold.*

Proof. Since $\nabla\tau(z_1)\neq 0$, we know that in a neighborhood of z_1, L has a representation $(x(s),y(s))$, where $(x(0),y(0))=z_1$ and $s\in(-\varepsilon,\varepsilon)$. Here s denotes arc-length along L

measured from z_1 with the obvious meaning for negative values. The function $(x(\cdot), y(\cdot))$ is a real-analytic function of its arguments and is a solution of the equations

$$\tau_x(x(s),y(s))\,\dot{x}(s)+\tau_y(x(s),y(s))\,\dot{y}(s)=0,\quad \dot{x}(s)^2+\dot{y}(s)^2=1.$$

We may assume that the maximal interval of existence is $(-M, N)$, where $M, N \in (0, \infty]$ and that $\varphi(\cdot)\equiv(x(\cdot), y(\cdot))$ is injective there and that $\nabla\tau(x(\cdot), y(\cdot))\neq 0$. This yields the first part of the lemma.

Let $\{t_n\}$ denote a sequence increasing to N as $n\to\infty$. Assume that $|\varphi(t_n)|\to\infty$ as $n\to\infty$. We claim that $|\varphi(t)|\to\infty$ as $t\to N$. If not, there would exist a circle $\{|z|=S\}$ such that $\varphi(I)\cap\{|z|=S\}$ contains an infinite number of distinct points. Since τ is constant on $\varphi(I)$ and is real-analytic, it follows that $\tau(z)=\tau(z_1)$ for all z with $|z|=S$. It is immediate that $L=\{|z|=S\}$ and this contradicts $|\varphi(t_n)|\to\infty$ as $n\to\infty$. The claim is therefore proved.

If $\varphi(t_n)\to\partial\Omega$ as $n\to\infty$, then we claim that $\varphi(t)\to\partial\Omega$ as $t\to N$. To see this, one encloses the components of $\partial\Omega$ by suitable closed, real-analytic curves, and then argues as above.

Assume that $\varphi(t_n)\to z_2\in\Omega$ as $n\to\infty$. If $\nabla\tau(z_2)\neq 0$, then clearly $\varphi(t)\to z_2$ as $n\to\infty$. Assume that $\nabla\tau(z_2)=0$. As shown in Lemma 6, the level-set of $\tau=\tau(z_2)=\tau(z_1)$ in a punctured neighborhood of z_2 consists of a finite number of real-analytic arcs emanating from z_2. In addition, it is shown in Remark 1 following Lemma 6 that they each have finite arc-length as the point z_2 is approached. It follows that $\varphi(t)\to z_2$ as $t\to N$.

Assume that either (i), (ii), or (iii) occurs. It is then trivial to show that $\varphi(I)$ is both open and closed in the relative topology of L, whence $\varphi(I)=L$ by connectedness.

Assume that (iv) occurs, so that $\varphi(t)\to z_2\in\Omega$ as $t\to N$ where $\nabla\tau(z_2)\neq 0$. Clearly $N<\infty$, and we know that we cannot extend beyond N by the maximality of $(-M, N)$. Upon considering the possibility of (i)–(iv) as $t\to -M$, we are led to the conclusion that $M<\infty$ and $\varphi(t)\to z_2$ as $t\to -M, N$. It is immediate that $L=\varphi(I)\cup\{z_2\}$ is a simple, closed real-analytic curve. q.e.d.

THEOREM 5. *The function ω is not one-signed near infinity in Ω; that is, there exist sequences $\{z_n\}$, $\{\tilde{z}_n\}$ with $|z_n|$, $|\tilde{z}_n|\to\infty$ as $n\to\infty$ such that $\omega(z_n)>0$ and $\omega(\tilde{z}_n)<0$ for all n.*

Proof. Assume that contrary, so that for some $S>2$, $\omega\geq 0$, say, for all $|z|\geq S-1$. Since $\omega\to 0$ at infinity by Theorem 2, we may assume that $\omega>0$ for $|z|\geq S$. Set $m=\min_{|z|=S}\omega(z)>0$. Lemma 6 shows that $\nabla\omega$ vanishes at only isolated points in Ω, so there exists $d\in(0, m)$ such that $\nabla\omega(z)\neq 0$ if $|z|\geq S$ and $\omega(z)=d$. Let z_1 satisfy $|z_1|>S$ and be such that $\omega(z_1)=d$. If L denotes the level-set of ω passing through z_1, then Lemma 4 ensures that L is a closed, simple, real-analytic curve in $|z|\geq S$.

Let int L denote the open, bounded component of $\mathbf{R}^2\setminus L$. Since $\omega=\omega(z_1)$ on L, the strong maximum principle proves that the outward normal derivative of ω on L is one-signed. But then equations (1.14)–(1.15) show that the tangential derivative of γ is one-signed on L. Hence, as L is transversed from z_1 back to itself, the function γ is monotone increasing or decreasing. This is clearly impossible. q.e.d.

The following lemma gives some useful information about the local structure of the level-sets of ω.

LEMMA 6. (a) *The zeros of* $\nabla\omega$ *in* Ω *are isolated.*

(b) *For every* $\bar{z}\in\Omega$, *there exists* $\varepsilon>0$ *and a family of injective, real-analytic functions* $\{(x_i(\cdot),y_i(\cdot))\}_{i=1}^k$, *each defined on some interval* $(0,\varepsilon_i)$, *such that*

$$\{z\in\Omega: 0<|z-\bar{z}|<\varepsilon, \omega(z)=\omega(\bar{z})\} = \bigcup_{i=1}^{k} L_i,$$

where $L_i=\{(x_i(s),y_i(s)): s\in(0,\varepsilon_i)\}$. *In addition,* $L_i\cap L_j=\emptyset$ *if* $i\neq j$.

Proof. (a) It is immediate from (1.18) that the pair of functions $(-\gamma_x,\gamma_y)$ satisfy a linear elliptic system of two first-order equations:

$$(-\gamma_x)_x=(\gamma_y)_y-\frac{1}{\nu}(w+q)\cdot\nabla\gamma,$$

$$-(-\gamma_x)_y=(\gamma_y)_x.$$

Standard theory [3; pp. 255–261] then proves that the zeros of $\nabla\gamma$ are isolated in Ω. The equations (1.14)–(1.15) show that the same holds for $\nabla\omega$.

(b) The Weierstrass Preparation Theorem [4] applied to $\omega-\omega(\bar{z})$ allows us, after a suitable rotation of the axes, to assume that the zero set of this function coincides with the zero set of the function

$$P(x,y)=(y-\bar{y})^n+\sum_{i=0}^{n-1}a_i(x)(y-\bar{y})^i$$

in the set $\{|z-\bar{z}|<\varepsilon\}$. Here, a_i is real-analytic on $(\bar{x}-\varepsilon,\bar{x}+\varepsilon)$ and $a_0(\bar{x})=\ldots=a_{n-1}(\bar{x})=0$. We may take $n>1$ since the case $n=1$ is trivial. Standard theory [4; p. 40] together with (a) completes the proof. q.e.d.

Remark 1. (a) If $a_0(x)\equiv 0$ near $x=\bar{x}$, then $y=\bar{y}$ is a root of $P(x,y)=0$. Lemma 6(a) ensures in this case that $a_1(x)\not\equiv 0$ near $x=\bar{x}$. Hence, upon replacing P by $\tilde{P}=P/(y-\bar{y})$, we may always assume that $a_0(x)\not\equiv 0$ near $x=\bar{x}$. Note that $\tilde{P}(x,y)\neq 0$ on the axes $\{x=\bar{x}\}$ and $\{y=\bar{y}\}$ for small ε. Standard theory [4], [17], [19], [20] shows that each root in the 'first quadrant' (that is, where $x>\bar{x}$ and $y>\bar{y}$), if there are any, lies on a curve given by $y-\bar{y}=(x-\bar{x})^{\alpha}f((x-\bar{x})^{\beta})$. Here $\alpha>0$, $\beta>0$, and f is real-analytic in an open neighborhood of zero with $f(0)\neq 0$. Now α, β and f depend on the root, but there are only a finite number of distinct triples (α,β,f). A similar representation holds in the other quadrants. If follows immediately that each arc defined by $\{(x_i(s),y_i(s)): s\in(0,\varepsilon_i)\}$ has finite arc-length as the point $\bar{z}$ is approached.

(b) The maximum principle proves that $\omega-\omega(\bar{z})$ changes sign as each arc L_i is crossed, and so it is clear that k is an even positive integer.

As noted earlier, $\gamma=\Phi$ on level-sets of $\omega=0$, and it is natural to seek these sets as part of the boundary of the open sets where ω is one-signed. For $z\in\Omega$ with $\omega(z)>0$, let $U_+(z)\subset\Omega$ denote the maximal, open connected set containing z on which $\omega>0$. We define U_- similarly. The symbols $U_+(z)$ or $U_-(z)$ are only defined when $\omega(z)\neq 0$. Recall from section 1.3 that $\Gamma=\partial\Omega\subset\{|z|<1\}$.

LEMMA 7. *If* $|\bar{z}|>1$ *and* $\omega(\bar{z})>0$, *then* $\partial U_+(\bar{z})\cap\{|z|=1\}\neq\emptyset$, *and similarly for* $U_-(\bar{z})$ *if* $\omega(\bar{z})<0$.

Proof. Assume that the result is false so that $\partial U_+(\bar{z})\cap\{|z|=1\}=\emptyset$ for some $\bar{z}$ with $|\bar{z}|>1$ and $\omega(\bar{z})>0$. It follows that $U_+(\bar{z})\subset\{|z|>1\}$. Since ω then vanishes on $\partial U_+(\bar{z})$ and also at infinity by Theorem 2, the maximum principle for ω yields $\omega\equiv 0$ in $U_+(\bar{z})$. This yields $\omega\equiv 0$ in Ω which contradicts our assumption that the flow is non-trivial. q.e.d.

The following lemma is important and shows that $U_+(\bar{z})$ is unbounded for some $\bar{z}\in\Omega$. Since Lemma 7 ensures that $\partial U_+(\bar{z})$ must intersect $\{|z|=1\}$, this suggests there should be at least two unbounded arcs in $\partial U_+(\bar{z})$ on which $\omega=0$ and which connect infinity to points near Γ. The strong maximum principle and Lemma 6 suggest that the normal derivative of ω on these arcs is one-signed (except possibly at isolated points where it may not be defined). Equations (1.14)–(1.15) will then be used to show that $\Phi=p+\frac{1}{2}|w|^2$ is monotone decreasing and increasing, as these two arcs are transversed from near Γ to infinity. These statements will be made rigorous shortly.

THEOREM 8. *There exists* $\bar{z}\in\Omega$ *such that* $U_+(\bar{z})$ *is unbounded, and similarly for* U_-.

Proof. Assume the contrary so that every $U_+(z)$ is bounded. If we combine this with Theorem 5, then there exists a sequence $\{z_n\}$, with $|z_n|>2$, $|z_n|\to\infty$ as $n\to\infty$, such that $U_+(z_i) \cap U_+(z_j) = \emptyset$ if $i \neq j$. Lemma 6 gives the existence of a number $c \in (1, 2)$ such that $\nabla\omega(z) \neq 0$ it $|z| = c$. Lemma 7 shows that $\partial U_+(z_i) \cap \{|z| = c\} \neq \emptyset$ for all i. We claim that $\partial U_+(z_i) \cap \partial U_+(z_j) \cap \{|z| = c\} = \emptyset$ if $i \neq j$. If not, then for some i and j, there would exist a point $\hat{z}$ in this intersection. Now $\omega(\hat{z}) = 0$ and $\nabla\omega(\hat{z}) \neq 0$, so the level-set of $\omega = 0$ in a neighborhood of $\hat{z}$ is given by a real-analytic arc passing through $\hat{z}$. Since ω changes sign as the arc is crossed, we have $\omega > 0$ on one side, whence the contradiction that $U_+(z_i) \cap U_+(z_j) \neq \emptyset$.

The argument above gives the existence of disjoint points $\hat{z}_i \in \partial U_+(z_i) \cap \{|z| = c\}$ at which ω vanishes. Since ω is real-analytic, it follows that $\omega \equiv 0$ on $\{|z| = c\}$. Since ω vanishes at infinity, this gives the contradiction that $\omega \equiv 0$. q.e.d.

Recall the choice of $c \in (1, 2)$ such that $\nabla\omega \neq 0$ on $\{|z| = c\}$. For each point $\bar{z}$ with $|\bar{z}| > c$, $\omega(\bar{z}) > 0$, and for which the component of $U_+(\bar{z}) \cap \{|z| > c\}$ containing $\bar{z}$ is unbounded, let $V_+(\bar{z})$ denote this unbounded component. The arguments for Theorem 8 ensure that such sets exist. We define $V_-(\bar{z})$ similarly if $\omega(\bar{z}) < 0$. The proof of Theorem 8 suggests there are only a finite number of distinct V_+, and this is the case. The most important part of the following lemma is that V_+ is simply-connected, which we shall use in Theorem 10 to show that ∂V_+, with the point at infinity added, is a closed Jordan curve.

LEMMA 9. (a) *There exist at most a finite number of distinct sets V_+ and V_-.*
(b) *Each V_+ and V_- is simply-connected.*

Proof. (a) The proof of Theorem 8 yields (a).

(b) Let J be a closed Jordan curve in V_+. Either $\{|z| = c\} \subset \operatorname{int} J$ or $\operatorname{int} J \subset \{|z| > c\}$. The first case would imply that $\omega > 0$ in $\operatorname{ext} J$ which contradicts Theorem 5. The second case implies $\operatorname{int} J \subset \Omega$, whence $\omega > 0$ in $\operatorname{int} J$ by the maximum principle. Since J was arbitrary, we have shown that $\operatorname{int} J \subset V_+$ for all closed Jordan curves in V_+. This implies that V_+ is simply-connected. q.e.d.

Since V_+ is simply-connected, ∂V_+ is a connected set. The following theorem shows that it is homeomorphic to the interval $(0, 1)$.

THEOREM 10. *For each set V_+, there exists a continuous, injective, piecewise real-analytic map φ defined on $\mathbf{R}$, and such that $\partial V_+ = \varphi(\mathbf{R})$. Moreover, $|\varphi(s)| \to \infty$ as $|s| \to \infty$. Here s denotes arc-length along ∂V_+ measured from some point in ∂V_+. A similar result holds for V_-.*

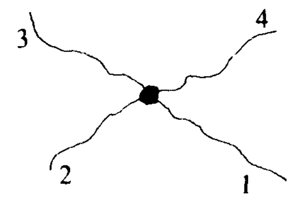

Figure 2. The level-set of $\omega=0$ in a neighborhood of z_2. The case $m=2$ has been shown, and the label i refers to L_i, $i=1,\ldots,4$.

Proof. Fix a set V_+ and choose $z_1\in\partial V_+$ such that $\nabla\omega(z_1)\neq 0$. As in the proof of Lemma 4, we can parametrize ∂V_+ in a neighborhood of z_1 by $\{(x(s),y(s)): s\in(-\varepsilon,\varepsilon)\}$, whence s denotes arc-length measured from z_1. Let $(-M,N)$, $M,N\in(0,\infty]$ denote the maximal interval in which $(x(\cdot),y(\cdot))$ has an extension as a subset of ∂V_+, and such that the map is injective, continuous, and piecewise real-analytic. The independent variable $s\in(-M,N)$ measures arc-length from z_1.

Assume that one of the numbers, say N, is finite. Since s measures arc-length, we may assume that $(x(s),y(s))\to z_2\in\partial V_+$ as $s\to N$. If $\nabla\omega(z_2)\neq 0$, then the maximality of $(-M,N)$ is contradicted unless $(x(s),y(s))\to z_2$ as $s\to -M$. In this case,

$$L=\{(x(s),y(s)): s\in(-M,N)\}\cup\{z_2\}$$

is a closed Jordan curve in ∂V_+. Clearly $\operatorname{int} L\subset V_+$ since V_+ is simply-connected, and we claim that $V_+\subset\operatorname{int} L$. If not, then there would exist a point $z_3\in V_+\setminus\operatorname{int} L$, and we can connect it by a Jordan arc in V_+ to some point $z_4\in\operatorname{int} L$. However, by connectedness, this arc must intersect $L\subset\partial V_+$. This contradiction gives $V_+=\operatorname{int} L$ whence $\partial V_+=L$. However, this is impossible since ∂V_+ is unbounded.

If N is finite, then we have shown that $\nabla\omega(z_2)=0$; in particular, $|z_2|>c$. Lemma 6 and the Remark after it give the level-set of $\omega=0$ in a punctured neighbourhood of z_2 as the union of an even number of real-analytic arcs $L_1,\ldots,L_{2m}$ emanating from z_2, and each with finite arc-length as z_2 is approached (see Figure 2).

Although the figure shows the case of four arcs emanating from z_2, any (even) number can be handled by the arguments to follow. Without loss of generality, we may assume that $(x(s),y(s))\in L_1$ for all s sufficiently near to N. If there were only two arcs, say L_1 and L_2, instead of the four shown, then $\omega>0$ on one side of $L_1\cup L_2\cup\{z_2\}$, whence $L_2\subset\partial V_+$. The only way for the maximality of $(-M,N)$ to remain inviolate is that $(x(s),y(s))\to z_2$ along L_2 as $s\to -M$. As in the case $\nabla\omega(z_2)\neq 0$, this implies that ∂V_+ is bounded, which is impossible.

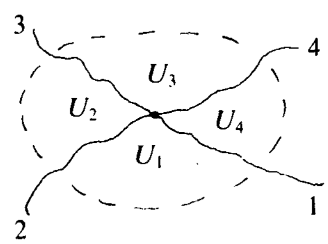

Figure 3.

It follows that there are at least four arcs emanating from z_2 for the case $N<\infty$ being considered. Let U_1 denote the open region 'between' L_1 and $L_2, \dots,$ and U_4 that between L_4 and L_1 as shown in Figure 3.

Since $L_1 \subset \partial V_+$, it follows by the maximum principle that either $U_1 \subset V_+$ or $U_4 \subset V_+$. We assume without loss of generality that the former holds. Then $L_2 \subset \partial V_+$. We claim that ∂V_+ in a neighborhood of z_2 is given by $L_1 \cup L_2 \cup \{z_2\}$. If not, then there would exist a point on L_3 or L_4 belonging to ∂V_+. Since $\omega>0$ in $U_1 \cup U_3$ and $\omega<0$ in $U_2 \cup U_4$, it follows that $U_3 \subset V_+$. Fix points $\bar{z}_1 \in U_1$ and $\bar{z}_3 \in U_3$. Since V_+ is connected, they may be joined by an arc $L_a \subset V_+$. The points $\bar{z}_1$ and $\bar{z}_3$ may be joined by an arc L_b, disjoint from L_a, lying in $U_1 \cup U_3 \cup \{z_2\}$ (see Figure 4).

Now $L=L_a \cup L_b$ is a closed Jordan curve on which $\omega \geqslant 0$, whence $\omega>0$ in $\operatorname{int} L$. It follows that either U_2 or U_4 lie in $\operatorname{int} L$, but this is impossible since $\omega<0$ on these sets. This contradiction implies that ∂V_+ equals $L_1 \cup L_2 \cup \{z_2\}$ in a neighborhood of z_2. Since $(-M, N)$ was maximal and $(x(s), y(s)) \in L_1$ for all s sufficiently near N, we arrive at a contradiction unless $(x(s), y(s)) \to z_2$ as $s \to -M$. But then ∂V_+ is bounded, and we have a contradiction. Hence, we have shown that $N=\infty$, and a similar argument gives $M=\infty$.

We claim that $|(x(s), y(s))| \to \infty$ as $s \to \infty$. Indeed, the proof of Lemma 4 ensures that we need only prove that $\{(x(s), y(s)) : s \geqslant 0\}$ is unbounded. If this were false, then for some unbounded sequence $\{s_n\}$, we would have $(x(s_n), y(s_n)) \to z_3 \in \partial V_+$ as $n \to \infty$. How-

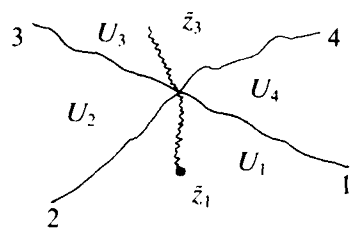

Figure 4. The arc $L_b \subset U_1 \cup U_3 \cup \{z_2\}$ connecting $\bar{z}_1$ to $\bar{z}_3$.

ever, ∂V_+ is given by a rectifiable Jordan arc in a neighborhood of z_3, and this is a contradiction. A similar result holds as $s \to -\infty$.

Define $\varphi(s)=(x(s),y(s)), s\in\mathbf{R}$. In the extended $\mathbf{R}^2$ plane ($\mathbf{R}^2$ plus the point $\{\infty\}$), $\varphi(\mathbf{R})\cup\{\infty\}$ in a closed Jordan curve on which $\omega\geqslant 0$. It follows that $\omega>0$ in the interior, which is easily seen to equal V_+. This implies that $\varphi(\mathbf{R})=\partial V_+$. An equivalent argument is to first apply the reflection $z\mapsto 1/z$ in $\{|z|\geqslant c\}$, and then add $\{0\}$ to the image of $\varphi(\mathbf{R})$. This yields a closed Jordan curve in the unit disc on which the function $\tilde{\omega}(z)=\omega(1/z)$ is non-negative. Since $\tilde{\omega}$ satisfies an equation of the form (1.10) in the interior of this curve, $\tilde{\omega}>0$ there. It again follows easily that $\varphi(\mathbf{R})=\partial V_+$. q.e.d.

Since V_+ is simply-connected and $\partial V_+\cup\{\infty\}$ is a closed Jordan curve in the extended plane, there is a conformal map f of the unit circle D onto V_+ such that f has a continuous and injective extension from $\bar{D}$ onto $V_+\cup(\partial V_+\cup\{\infty\})$. We may assume that $f(1)=\{\infty\}$. With $(r,\theta), \theta\in[0,2\pi)$, denoting polar coordinates in D, there exist numbers θ_1,θ_2 with $0<\theta_1<\theta_2<2\pi$ such that $f(C_1)\cup f(C_2)\subset\partial V_+\cap\{|z|>c\}$, where

$$C_1=\{e^{i\theta}: 0<\theta<\theta_1\},\quad C_2=\{e^{i\theta}: \theta_2<\theta<2\pi\}.$$

In addition, we may assume that $|f(e^{i\theta_j})|=c$, $j=1,2$. If we define $\tilde{\omega}$ in D by $\tilde{\omega}(\zeta)=\omega(f(\zeta))$, then

$$\Delta\tilde{\omega}+\beta\cdot\nabla\tilde{\omega}=0 \quad \text{in } D, \tag{2.3}$$

where β is real-analytic in D. Since ∂V_+ is real-analytic except possibly at discrete points converging to infinity (Lemma 6), it follows that $\tilde{\omega}$ is real-analytic across ∂D except possibly at discrete points which converge to $\zeta=1$. Since $\tilde{\omega}>0$ in D and $\tilde{\omega}=0$ on $C_1\cup C_2$, the strong maximum principle applied to (2.3) gives

$$\frac{\partial\tilde{\omega}}{\partial r}(e^{i\theta})<0,\quad e^{i\theta}\in C_1\cup C_2 \tag{2.4}$$

except possibly at a sequence tending to $\zeta=1$. At these points, the radial derivative need not exist.

If we define $\tilde{\gamma}(\zeta)=\gamma(f(\zeta))$ and $\zeta=\xi+i\eta$, $\xi,\eta\in\mathbf{R}$, then a calculation based on (1.14)–(1.15) shows that

$$\tilde{\gamma}_\xi=\nu\tilde{\omega}_\eta-\psi(f(\zeta))\tilde{\omega}_\xi, \tag{2.5}$$

$$\tilde{\gamma}_\eta=-\nu\tilde{\omega}_\xi-\psi(f(\zeta))\tilde{\omega}_\eta. \tag{2.6}$$

If we combine these equations with (2.4) and the fact that $\bar{\omega}=0$ on $C_1 \cup C_2$, then

$$\frac{d}{d\theta}\bar{\gamma}(e^{i\theta}) = -\nu\frac{\partial\bar{\omega}}{\partial r}(e^{i\theta}) > 0$$

for all $e^{i\theta} \in C_1 \cup C_2$ except possibly at a sequence tending to $\zeta=1$.

Let us restate what has been found; as one moves from $\{|z|=c\}$ to infinity along the Jordan arc $\{f(e^{i\theta}): \theta \in (\theta_2, 2\pi)\}$, the function $\gamma=\Phi-\psi\omega$ in monotone increasing. Since $\omega=0$ on this arc, it follows that Φ is *monotone increasing* as one goes from $\{|z|=c\}$ to infinity along this arc of ∂V_+. By considering γ along $f(C_1)$, we see that Φ is monotone decreasing as one goes from $\{|z|=c\}$ to infinity along it.

THEOREM 11. *Let V denote a set V_+ or V_-.*

(a) *The set $\partial V \cap \{|z|>c\}$ has precisely two unbounded components which may be parametrized as $\{(x_i(s), y_i(s): s \in (0,\infty)\}$, $i=1,2$. In addition, $(x_i(0), y_i(0)) \in \{|z|=c\}$, s denotes arc-length measured from these points, and the functions $x_i(\cdot)$ and $y_i(\cdot)$ are real-analytic on $(0,\infty)$ except possibly at isolated points which may accumulate at infinity. The function ω vanishes on these arcs, and $|(x_i(s), y_i(s))| \to \infty$.*

(b) *The maps $s \mapsto \Phi(x_i(s), y_i(s))$ are monotone decreasing and increasing on $(0,\infty)$, respectively, for $i=1,2$.*

2.2. Boundedness of the velocity, convergence of γ, and additional properties of the vorticity

We now have the necessary machinery to improve the results in [15] by showing that $w \in L_\infty(\Omega)$.

THEOREM 12. *Let (w,p) be a solution of (1.1)–(1.3) with $\nabla w \in L_2(\Omega)$. Then $w \in L_\infty(\Omega)$, and parts (b)–(f) of Theorem 1 hold for (w,p).*

Proof. Since $\nabla w \in L_2(\Omega)$, we know that

$$\int_{2^n}^{2^{n+1}} \frac{dr}{r} \int_0^{2\pi} \left|\frac{\partial}{\partial\theta} w(r,\theta)\right|^2 d\theta \to 0 \quad \text{as} \quad n \to \infty.$$

The integral mean-value theorem gives the existence of $R_n \in (2^n, 2^{n+1})$ such that

$$\int_0^{2\pi} \left|\frac{\partial}{\partial\theta} w(R_n,\theta)\right|^2 d\theta \to 0 \quad \text{as} \quad n \to \infty. \tag{2.7}$$

Consider any domain V which is a V_+ or V_-. Then $\Phi(x_1(s), y_1(s))$ is monotone decreasing on $(0, \infty)$ whence

$$\frac{1}{2}|w(x_1(s), y_1(s))|^2 \leqslant -p(x_1(s), y_1(s))+\Phi(x_1(0), y_1(0)).$$

Since the pressure vanishes at infinity it follows that $|w|$ is bounded on the curve

$$\{(x_1(s), y_1(s)) \colon s \in (0, \infty)\}.$$

By connectedness, there exists $\theta_n \in [0, 2\pi)$ such that $|w(R_n, \theta_n)| \leqslant$const., independently of n. Combining this with (2.7) yields

$$\max_{\theta \in [0, 2\pi]} |w(R_n, \theta)| \leqslant \text{const.},$$

independently of n. Hence, $\Phi = p + \frac{1}{2}|w|^2$ satisfies

$$\max_{z \in \partial A_n} \Phi(z) \leqslant \text{const.},$$

where $A_n = \{z \in \Omega \colon R_n < |z| < R_{n+1}\}$. The maximum principle for Φ from (1.11) yields $\Phi(z) \leqslant$const., $z \in \bar{A}_n$, where the constant is independent of n. Since p vanishes at infinity, the theorem is proved. q.e.d.

COROLLARY 13. *Let V denote a set V_+ or V_-, and $\{(x_i(\cdot), y_i(\cdot))\}$, $i=1,2$ be as in Theorem* 11. *Then*

$$\left.\begin{array}{r} |w(x_i(s), y_i(s))| \to |w_\infty|, \\ \Phi(x_i(s), y_i(s)), \gamma(x_i(s), y_i(s)) \to \dfrac{1}{2}|w_\infty|^2 \end{array}\right\} \quad as \quad s \to \infty,$$

where w_∞ is as in (1.13).

Proof. Let $L_i = \{(x_i(s), y_i(s)) \colon s \in (0, \infty)\}$, $i=1,2$, denote the unbounded components of $\partial V \cap \{|z| > c\}$ as in Theorem 11. Since $w \in L_\infty(\Omega)$ by Theorem 12, we may use Theorem 2(d) to conclude the existence of a constant vector $w_\infty \in \mathbf{R}^2$ such that

$$\int_0^{2\pi} |w(r, \theta) - w_\infty|^2 d\theta \to 0 \quad \text{as} \quad r \to \infty.$$

We may combine this with (2.7) to conclude that

$$\max_{\theta \in [0, 2\pi]} |w(R_n, \theta) - w_\infty| \to 0 \quad \text{as} \quad n \to \infty. \tag{2.8}$$

Let $s_n \in (0, \infty)$ denote the first value such that $(x_i(s), y_i(s)) \in \{|z|=R_n\}$. The sequence $\{s_n\}$ is monotone increasing. Since $\Phi(x_i(\cdot), y_i(\cdot))$ is monotone on $(0, \infty)$ and p vanishes at infinity, equation (2.8) implies that $\Phi(x_i(s), y_i(s)) \to \frac{1}{2}|w_\infty|^2$ as $s \to \infty$. Since ω vanishes on L_i, the result for γ is immediate. q.e.d.

We have shown that $\gamma = \Phi - \psi\omega$ tends to $\frac{1}{2}|w_\infty|^2$ if we approach infinity along suitable curves on which ω vanishes. We now prove that

$$\gamma(z) \to \frac{1}{2}|w_\infty|^2 \quad \text{as} \quad |z| \to \infty. \tag{2.9}$$

If we knew that $\nabla\gamma \in L_2(\Omega)$, then such a result would be immediate. Indeed, if we use (2.7) with w replaced by γ, then it is immediate that $|\gamma(r, \theta) - \frac{1}{2}|w_\infty|^2| \to 0$, uniformly in θ, for suitable $r \to \infty$. Since γ satisfies (1.18), the maximum principle would then give (2.9). Since $\nabla\omega \in L_2(\Omega)$ by Theorem 2, equations (1.14)–(1.15) show that $\nabla\gamma \in L_2(\Omega)$ if and only if $\psi\nabla\omega \in L_2(\Omega)$. However, we expect that $\psi(z) = O(|z|)$, and Theorem 2 only gives $\sqrt{|z|}\,\nabla\omega \in L_2(\Omega)$, not $|z|\nabla\omega \in L_2(\Omega)$. (One *can* show that $\psi\nabla\omega \in L_2(\Omega)$ with the aid of the identity $\nu\psi^2|\nabla\omega|^2 = \nu\psi^2\Delta(\omega^2/2) - \operatorname{div}(w\psi^2\omega^2/2)$ which follows from (1.10) and the identities $\operatorname{div} w = 0$, $w \cdot \nabla\psi = (\psi_y, -\psi_x) \cdot (\psi_x, \psi_y) = 0$. The idea of the proof is as follows: For suitably small $d > 0$, one integrates this equation over $B_d = \{z \in V_+ : \omega(z) > d\}$:

$$2\nu \int\int_{B_d} \psi^2|\nabla\omega|^2 = -4\nu \int\int_{B_d} \psi\omega\nabla\psi \cdot \nabla\omega - \int_{\partial B_d} \psi^2\omega^2(w \cdot n)\, ds + \nu \int_{\partial B_d} \psi^2 \frac{\partial}{\partial n}(\omega^2)\, ds.$$

On the subset P of ∂B_d where $\omega = d$, the term

$$-\int_P \psi^2\omega^2(w \cdot n)\, ds = -d^2 \int_P \psi^2(w \cdot n)\, ds = -d^2 \int_P \psi^2 \frac{\partial\psi}{\partial s}\, ds,$$

may be integrated exactly. The maximum principle ensures that the other boundary term

$$\nu \int_P \psi^2 \frac{\partial}{\partial n}(\omega^2)\, ds,$$

is negative. The remaining part of ∂B_d, that on which $\omega < d$, is necessarily a subset of $\{|z| = c\}$, and the boundary integrals on this set are estimated easily. One arrives at an inequality

$$2\nu \int\int_{B_d} \psi^2|\nabla\omega|^2 \leq -4\nu \int\int_{B_d} \psi\omega\nabla\psi \cdot \nabla\omega + \text{const.},$$

where the constant is independent of d. Since $\nabla\psi \in L_\infty(\Omega)$ and $\omega \in L_2(\Omega)$, the Schwarz inequality bounds $\psi\nabla\omega$ in $L_2(B_d)$, independently of d. Letting $d\to 0$ yields $\psi\nabla\omega \in L_2(V_+)$, and similarly $\psi\nabla\omega \in L_2(V_-)$. The proof of the following theorem could be simplified considerably if we used the fact (now known) that $\nabla\gamma \in L_2(\Omega)$, but we have not done so since certain of the techniques developed will be needed in later results.)

THEOREM 14. (a) *Let V denote a set V_+ or V_-. Then $\gamma(z)\to\frac{1}{2}|w_\infty|^2$ as $|z|\to\infty$, $z\in V$.*
(b) *$\gamma(z)\to\frac{1}{2}|w_\infty|^2$ and $\frac{1}{2}|w(z)|^2-\psi(z)\,\omega(z)\to\frac{1}{2}|w_\infty|^2$ as $|z|\to\infty$, $z\in\Omega$.*

Proof. (b) Assume that (a) holds. Lemma 9 ensures that there exists $R>2$ such that any point z with $|z|>R$ belongs to either a V_+, V_-, or their boundary. Corollary 13 shows that γ has the desired limit as infinity is approached along ∂V_+ or ∂V_-, and so (a) implies $\gamma(z)\to\frac{1}{2}|w_\infty|^2$ as $|z|\to\infty$. Since p vanishes at infinity, the other result is immediate.

(a) We shall assume that v is a V_+ since similar arguments hold for a V_-. We adopt some of the notation from the proof of Theorem 11. Let D denote the unit disc, and let f be a conformal map of D onto V_+ such that $f(1)=\{\infty\}$. The functions $\tilde\omega(\zeta)=\omega(f(\zeta))$ and $\tilde\gamma(\zeta)=\gamma(f(\zeta))$ satisfy (2.3) and (2.5)–(2.6), respectively. To prove (a), it suffices to show that $\tilde\gamma(\zeta)\to\frac{1}{2}|w_\infty|^2$ as $\zeta\to 1$, $\zeta\in D$.

Assume that this is false. Then there exists $\varepsilon>0$ and a sequence $\{\zeta_n\}$ with $\zeta_n\to 1$ as $n\to\infty$ and

$$\left|\tilde\gamma(\zeta_n)-\frac{1}{2}|w_\infty|^2\right|\geq 4\varepsilon \quad \text{for all } n. \tag{2.10}$$

Corollary 13 proves $\tilde\gamma(e^{i\theta})\to\frac{1}{2}|w_\infty|^2$ as $|\theta|\to 0$, where now $\theta\in(-\pi,\pi]$, and so there exists $\theta_3\in(0,\pi)$ such that

$$\left|\tilde\gamma(e^{i\theta})-\frac{1}{2}|w_\infty|^2\right|\leq\varepsilon \quad \text{if} \quad |\theta|\leq\theta_3. \tag{2.11}$$

We may assume without loss of generality that $\nabla\omega(f(e^{\pm i\theta_3}))\neq 0$, whence (2.4) shows that

$$\frac{\partial\tilde\omega}{\partial r}(e^{\pm i\theta_3})<0. \tag{2.12}$$

By continuity, there exists $r_1\in(0,1)$ such that

$$\left|\tilde\gamma(re^{\pm i\theta_3})-\frac{1}{2}|w_\infty|^2\right|\leq 2\varepsilon, \quad r\in[r_1,1], \tag{2.13}$$

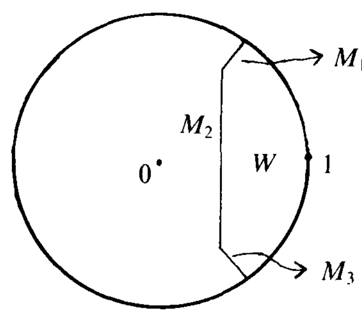

Figure 5. The arcs M_1, M_2, M_3 and the open set W.

and

$$\frac{\partial \tilde{\omega}}{\partial r}(re^{\pm i\theta_3})<0, \quad r\in[r_1,1]. \tag{2.14}$$

Define

$$M_1=\{re^{i\theta_3}: r\in(r_1,1)\}, \quad M_3=\{re^{-i\theta_3}: r\in(r_1,1)\},$$

and let $M_2\subset D$ denote the straight line containing and connecting $r_1e^{i\theta_3}$ to $r_1e^{-i\theta_3}$. Let $W\subset D$ denote the interior of $M_1\cup M_2\cup M_3\cup\{e^{i\theta}: |\theta|\leq\theta_3\}$, so that unity belongs to $\bar{W}$ (see Figure 5).

Since $\tilde{\omega}>0$ in D, we have

$$m=\min_{\zeta\in M_2}\tilde{\omega}(\zeta)>0, \quad M=\max_{\zeta\in M_2}\tilde{\omega}(\zeta)>0. \tag{2.15}$$

We may assume that $\zeta_n\in W$ for all n, and may use Lemma 6 to construct a new sequence $\{\xi_n\}$ contained in W and converging to unity such that

$$\left|\tilde{\gamma}(\xi_n)-\frac{1}{2}|w_\infty|^2\right|\geq 3\varepsilon \quad \text{for all } n, \tag{2.16}$$

and $\nabla\tilde{\omega}(\zeta)\neq 0$ if $\zeta\in D$ and $\tilde{\omega}(\zeta)=\tilde{\omega}(\xi_n)$. Fix a value of n, and write ξ for ξ_n. Since $\tilde{\omega}(\xi)$ may be made as small as we need, we may assume that $\tilde{\omega}(\xi)\in(0,m)$. Let L denote the component of $\{\zeta\in\bar{W}: \tilde{\omega}(\zeta)=\tilde{\omega}(\xi)\}$ containing ξ. The arguments for Lemma 4 ensure that L intersects ∂W. The intersection cannot be where $|\zeta|=1$ since $\tilde{\omega}=0$ there, nor on M_2 since $\tilde{\omega}\geq m$ there. It follows from (2.14) that L intersects M_1 at one point and similarly M_3. Hence, we may represent L as

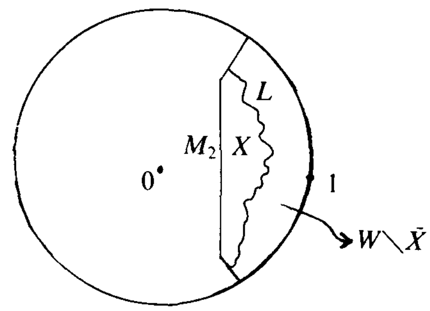

Figure 6. The regions X and $W\setminus\bar{X}$.

$$L=\{\varphi(s)\colon s\in[0,1]\},$$

where φ is real-analytic, $\varphi(s)\in W$, $s\in(0,1)$, and $\varphi(0)\in M_1$, $\varphi(1)\in M_3$.

Let $X\subset D$ denote the interior of the closed curve formed by L, M_2 and the straight lines connecting $r_1e^{i\theta_3}$ to $\varphi(0)$ and $r_1e^{-i\theta_3}$ to $\varphi(1)$ (see Figure 6).

By construction, $\tilde{\omega}\in[\tilde{\omega}(\xi),M]$ on ∂X, whence $\tilde{\omega}>\tilde{\omega}(\xi)$ in X. Since $\tilde{\omega}=\tilde{\omega}(\xi)$ on L, the strong maximum principle gives

$$\frac{\partial\tilde{\omega}}{\partial n}(\zeta)<0,\quad \zeta\in\mathring{L},\tag{2.17}$$

where $\mathring{L}$ denotes the interior $\varphi((0,1))$ of L and n is the outward normal to X. If we combine (2.17) with (2.5)–(2.6) and use the fact that $\tilde{\omega}$ is constant on L, then

$$\frac{d}{ds}\tilde{\gamma}(\varphi(s))<0,\quad s\in(0,1).$$

In particular, $\tilde{\gamma}(\varphi(0))>\tilde{\gamma}(\xi)>\tilde{\gamma}(\varphi(1))$. Since $\varphi(0)\in M_1$, and $\varphi(1)\in M_3$, we may use (2.13) and there results

$$\left|\tilde{\gamma}(\xi)-\frac{1}{2}|w_\infty|^2\right|\leqslant 2\varepsilon.$$

However, this contradicts (2.16), and so the theorem is proved. q.e.d.

Theorem 14 will have important applications in the sections to come. Before proceeding to them, we consider in more detail the relationship between ω and γ in Ω. Define $\sigma=\gamma+i\nu\omega$, and let

$$\partial_z=\frac{1}{2}(\partial_x-i\partial_y)\quad\text{and}\quad\partial_{\bar{z}}=\frac{1}{2}(\partial_x+i\partial_y).$$

Then a calculation based on (1.14)–(1.15) gives

$$\frac{\partial}{\partial \bar{z}}\sigma = N(z)\frac{\partial}{\partial \bar{z}}\bar{\sigma}, \tag{2.18}$$

where

$$N = (\psi^2 - 2i\nu\psi)/(4\nu^2 + \psi^2). \tag{2.19}$$

Although $|N(z)|<1$, $z \in \bar{\Omega}$, one cannot expect an upper bound strictly less than unity, since we expect ψ to be unbounded at infinity. This prevents us from using standard theory [3; p. 259] to examine (2.18) in a neighborhood of infinity. We now show that $\nabla\omega(z) \neq 0$ whenever $|z|$ is sufficiently large, say $|z| \geq S$. Although we have been unable to exploit this result in future sections, it does have two curious implications. Firstly, if $|z|>S$, the level-sets of ω are locally real-analytic arcs; in particular, the functions $((x_i(\cdot), y_i(\cdot))$ describing the unbounded components of $\partial V_\pm \cap \{|z|>c\}$ in Theorem 11 are real-analytic for all large values of their argument. Secondly, equation (2.18) may be rewritten as

$$\frac{\partial}{\partial \bar{z}}\sigma = \tilde{N}(z)\frac{\partial}{\partial z}\sigma \quad \text{in} \quad |z| \geq S, \tag{2.20}$$

where $\tilde{N} = N\bar{\sigma}_{\bar{z}}/\sigma_z$. Since $\nabla\omega \neq 0$ there, the same is true for $\nabla\gamma$ (cf. (1.14)–(1.15)), whence (2.20) is a Beltrami equation [3] with real-analytic coefficients in a neighborhood of infinity.

We begin the proof that $\nabla\omega(z) \neq 0$ near infinity with the case $\omega(z)=0$.

THEOREM 15. *There exists $S \geq 2$ such that $\nabla\omega(z) \neq 0$ if $|z| \geq S$ and $\omega(z)=0$.*

Proof. Assume the contrary and let $\{z_n\}$ denote an unbounded sequence for which $\omega(z_n)=|\nabla\omega(z_n)|=0$. By Lemma 9, we may assume that $z_n \in \partial V_1$ for all n, where V_1 is a specified V_+ or V_-. We shall assume that $V_1 = V_+$, and also, without loss of generality, that

$$z_n \in \{(x_1(s), y_1(s)) \colon s \in (0, \infty)\},$$

one of the two unbounded components of $\partial V_+ \cap \{|z|>c\}$. After taking a suitable subsequence, we may assume that $z_n = (x_1(s_n), y_1(s_n))$, where $\{s_n\}$ is monotonically increasing to infinity.

The proof consists of two natural steps: (i) we show that the level-set of $\omega=0$ in a neighborhood of z_n is a Jordan curve passing through z_n and (ii) that this implies $\nabla\omega(z_n) \neq 0$. This contradiction proves the theorem.

Fix n, set $\bar{z}=z_n$, and assume that the level-set of $\omega=0$ is not a Jordan curve through $\bar{z}$. It follows from Lemma 6 that there are at least four arcs emanating from $\bar{z}$ on which $\omega=0$. We shall consider the case of just four arcs since the general case is analogous. Referring to Figures 2 and 3 in the proof of Theorem 10, we shall assume that $L_1\subset\{x_1(s),y_1(s)): s>s_n\}$ while $L_2\subset\{(x_1(s),y_1(s)): 0<s<s_n\}$. Now $U_1\subset V_1$ so that $\omega>0$ in U_1, whence $\omega>0$ in U_3. Now U_3 lies in some V_+. It is not contained in V_1 since the proof of Theorem 10 showed that ∂V_+ is locally a Jordan curve. Hence, $U_3\subset V_3$, where V_3 is a V_+ and $V_1\cap V_3=\emptyset$. Now $\partial V_3\cap\{|z|>c\}$ has an unbounded component $\{(x_3(\bar{s}),y_3(\bar{s})): \bar{s}\in(0,\infty)\}$ where $\bar{z}=(x_3(\bar{s}_n),y_3(\bar{s}_n))$. Consider the unbounded arcs

$$\{(x_1(s),y_1(s)): s\geq s_n\} \quad\text{and}\quad \{(x_3(\bar{s}),y_3(\bar{s})): \bar{s}\geq\bar{s}_n\},$$

and recall that ω vanishes on these sets. If they do not intersect other than at $\bar{z}$, then $\omega=0$ on their union plus $\{\infty\}$, whence $\omega=0$ in the interior of this closed Jordan curve in the extended plane. If they do intersect at (first) points corresponding to $s=a>s_n$ and $\bar{s}=b>\bar{s}_n$, then

$$\{(x_1(s),y_1(s)): s_n\leq s\leq a\}\cup\{(x_3(\bar{s}),y_3(\bar{s})): \bar{s}_n<\bar{s}<b\}$$

is a closed Jordan curve on which $\omega=0$. This yields the contradiction $\omega\equiv0$.

We have shown that if $\omega(\bar{z})=0$, $\bar{z}=z_n$, then the level-set of $\omega=0$ through $\bar{z}$ is locally a Jordan curve. We claim that $\nabla\omega(\bar{z})\neq0$, and to prove this, we now assume that $\nabla\omega(\bar{z})=0$. After a suitable rotation of the coordinate axes, the Weierstrass Preparation Theorem [4] shows that

$$\omega(x,y)=g(x,y)\left\{(y-\bar{y})^n+\sum_{i=0}^{n-1}a_i(x)(y-\bar{y})^i\right\}$$

where g is real-analytic in a neighborhood of $\bar{z}$, $g(\bar{z})\neq0$, the a_i are real-analytic in a neighborhood of $x=\bar{x}$, and $a_i(\bar{x})=0$, $i=0,\ldots,n-1$. Since $\omega(\bar{z})=|\nabla\omega(\bar{z})|=0$, we have $n>1$. We also have $a_0(x)\not\equiv0$ else the line $\{y=\bar{y}\}$ would constitute the level-set of $\omega=0$ near $z=\bar{z}$. Since ω vanishes on this line and changes sign as it is crossed, the strong maximum principle would give $\partial\omega/\partial y(\bar{x},\bar{y})\neq0$. Hence, $a_0\not\equiv0$. The use of this fact gives the existence of $\varepsilon>0$ such that $\omega(x,y)\neq0$ for all points $(x,\bar{y})$, $0<|x-\bar{x}|<\varepsilon$, and points $(\bar{x},y)$, $0<|y-\bar{y}|<\varepsilon$. Since the level-set of $\omega=0$ is locally a Jordan curve passing through $\bar{z}$, there are precisely two arcs L_1 and L_2 emanating from $\bar{z}$. We assume without loss of generality that L_1 lies in the first quadrant:

$$L_1\subset\{z=(x,y): x>\bar{x}, y>\bar{y} \text{ and } |z-\bar{z}|<\varepsilon\}.$$

We claim that L_2 lies in the third quadrant, that is, where $x<\bar{x}$ and $y<\bar{y}$. Indeed, if L_2 was in the first quadrant, then ω would be one-signed below the line $\{y=\bar{y}\}$ and would vanish at $(\bar{x},\bar{y})$. The strong maximum principle then gives $\partial\omega/\partial y(\bar{x},\bar{y})\neq 0$. A similar argument holds if L_2 is in the second or fourth quadrant. If we use Remark 1 after Lemma 6, then we may assume (after choosing ε smaller if need be) that

$$L_1 = \{(x,y)\colon y=\bar{y}+(x-\bar{x})^{\alpha_1}f_1((x-\bar{x})^{\beta_1}), x\in(\bar{x},\bar{x}+\varepsilon)\},$$

$$L_2 = \{(x,y)\colon y=\bar{y}-(\bar{x}-x)^{\alpha_2}f_2((\bar{x}-x)^{\beta_2}), x\in(\bar{x}-\varepsilon,\bar{x})\}$$

where the f_i are real-analytic on $(\bar{x}-\varepsilon,\bar{x}+\varepsilon)$ and positive there. Here $\alpha_i,\beta_i>0$ for $i=1,2$.

If $\alpha_1>1$, then the function

$$\hat{y}(x)=\begin{cases}(x-\bar{x})^{\alpha_1}f_1((x-\bar{x})^{\beta_1}), & x\in(\bar{x},\bar{x}+\varepsilon),\\ 0, \quad x\in(\bar{x}-\varepsilon,\bar{x}]\end{cases}$$

is of class $C^{1+\sigma}$ for some $\sigma\in(0,1)$. Since ω is one-signed near $\bar{z}$ in the region $\{(x,y)\colon x\in(\bar{x}-\varepsilon,\bar{x}+\varepsilon), y>\bar{y}+\hat{y}(x)\}$ and vanishes at $\bar{z}$, the strong maximum principle for domains with $C^{1+\sigma}$ boundaries [13; p. 49] gives $\nabla\omega(\bar{x},\bar{y})\neq 0$. The case $\alpha_1\in(0,1)$ is similar: one first solves for x as a function of y, say $x=x_1(y)$, and then defines

$$\hat{x}(y)=\begin{cases}x_1(y), & y>\bar{y}\\ 0, & y\leq\bar{y}.\end{cases}$$

The function $\hat{x}\in C^{1+\sigma}$ for some $\sigma>0$, and the argument proceeds as before. Similar arguments hold if $\alpha_2\in(0,1)$ or if $\alpha_2>1$. Hence, we may assume that $\alpha_1=\alpha_2=1$. If $f_1(0)\neq f_2(0)$, then we can put a straightline through $\bar{z}$ such that ω is one-signed on one side of it. This yields $\nabla\omega(\bar{z})\neq 0$ again. If $f_1(0)=f_2(0)$, then it is straightforward to check that

$$\hat{y}(x)=\begin{cases}\bar{y}+(x-\bar{x})f_1((x-\bar{x})^{\beta_1}), & x\in(\bar{x},\bar{x}+\varepsilon)\\ \bar{y}+(x-\bar{x})f_2((\bar{x}-x)^{\beta_2}), & x\in(x-\varepsilon,\bar{x}]\end{cases}$$

is of class $C^{1+\sigma}$. The usual argument yields the contradiction that $\nabla\omega(\bar{z})\neq 0$. q.e.d.

THEOREM 16. *There exists $S\geq 2$ such that $\nabla\omega(z)\neq 0$ if $|z|\geq 2$.*

Proof. Because of Theorem 15, it suffices to show this for all points z with $|z|$ large and $z\in V$, where V is a V_+ or V_-. We assume that V is a V_+, and use the methods and notation from the proof of Theorem 14. It suffices to show that $\nabla\tilde{\omega}\neq 0$ in the region

$W\setminus\tilde{X}$ shown in Figure 6. Assume that $\nabla\tilde{\omega}$ vanishes at a point $\tilde{z}$ in that domain. Note that

$$\tilde{\omega}(\tilde{z}) < \max_{z\in\partial(W-\tilde{X})} \tilde{\omega}(z) = \tilde{\omega}(L).$$

If we apply the arguments in the proof of Theorem 15, then for some small $\delta>0$, the set

$$\{z\in W\setminus\tilde{X}: 0<|z-\tilde{z}|<\delta, \tilde{\omega}(z)=\tilde{\omega}(\tilde{z})\}$$

consists of at least four arcs $L_1, \ldots, L_4$ emanating from $\tilde{z}$.

Now L_1 may be parametrized locally by $L_1=\{\tau_1(s): s\in(0,\delta_1)\}$, where s denotes arc-length measured from $\tilde{z}$. It is easy to see that there is an extension τ_1 to a maximal interval $(0,N_1)$ such that τ_1 is injective there, $\tilde{\omega}(\tau_1(\cdot))=\tilde{\omega}(\tilde{z})$, and $\tau_1(\cdot)\in W\setminus X$. Indeed, if $\tau_1(s)\to z_1$, with $\nabla\tilde{\omega}(z_1)=0$, as $s\uparrow M<N_1$, then one can use the local structure of $\{z: \tilde{\omega}(z)=\tilde{\omega}(\tilde{z})=\tilde{\omega}(z_1)\}$ near z_1 to extend τ beyond M. We claim that $\tau_1(s)$ approaches a point on $\partial(W-X)$ as $s\to N_1$. If not, then $\tau_1(s)\to z_1\in W-X$, and the maximality of $(0,N_1)$ is contradicted unless that set $\tau_1((0,N_1))\cup\{z_1\}$ contains a closed Jordan curve on which $\tilde{\omega}=0$. This is impossible, and so we may assume that $\tau_1(N_1)\in\partial(W-X)$. Set $\tilde{L}_1=\tau_1((0,N_1])$.

In a similar manner, we can construct arcs $\tilde{L}_i=\tau_i((0,N_i])$, $i=2,3,4$. We may also assume that $\tilde{L}_i\cap\tilde{L}_j=\emptyset$ if $i\neq j$. Indeed, if $\tilde{L}_1\cap\tilde{L}_2\neq\emptyset$, then upon increasing s from zero, we would have a first point of intersection, say $z_2=\tau_1(s_1)=\tau_2(s_2)$. But then

$$\tau_1((0,s_1])\cup\tau_2((0,s_2))\cup\{\tilde{z}\}$$

is a closed Jordan curve on which $\tilde{\omega}$ vanishes. Since the $\tilde{L}_i$ are disjoint, we know that the points $\tau_i(N_i)$ are distinct. Since $\tilde{\omega}(e^{i\theta})=0$, $|\theta|\leq\theta_3$, and $\tilde{\omega}(L)>\tilde{\omega}(\tau_1(N_1))=\ldots=\tilde{\omega}(\tau_4(N_4))$, the four points $\tau_i(N_i)$ must lie on the straight lines connecting $\varphi(0)$ to $e^{i\theta_3}$ and $\varphi(1)$ to $e^{-i\theta_3}$. However, equation (2.14) shows that $\tilde{\omega}$ is monotone on these lines, and so there cannot be four distinct points at which $\tilde{\omega}$ is equal. q.e.d.

2.3. Convergence and decay rates in sectors

As noted earlier, Theorem 12 allows us to use the result of Gilbarg and Weinberger [14], [15] that

$$\int_0^{2\pi} |w(r,\theta)-w_\infty|^2 d\theta \to 0 \quad \text{as} \quad r\to\infty \tag{2.21}$$

for some constant vector w_∞. In addition, equation (2.8) gives

$$\max_{\theta\in[0,2\pi]} |w(R_n,\theta)-w_\infty|\to 0 \quad \text{as} \quad n\to\infty \tag{2.22}$$

for suitable $R_n\in(2^n,2^{n+1})$ (cf. (2.7)). Our wish is to strengthen (2.21)–(2.22) by showing that w tends to w_∞ pointwise at infinity. If $w_\infty=0$, then Gilbarg and Weinberger showed that (2.22) and the one-sided maximum principle for Φ give

$$|w(z)|\to 0 \quad \text{as} \quad |z|\to\infty. \tag{2.23}$$

We *conjecture* that if (2.23) occurs, then $w\equiv 0$ in Ω. This can be seen formally by taking the inner product of (1.1) with w, and then using (1.2) to show that

$$-\nu|\nabla w|^2=-\frac{\nu}{2}\Delta(|w|^2)+\operatorname{div}(w\Phi).$$

Integrate this over $\Omega\cap\{|z|<S\}$ and use (1.3), let $S\to\infty$, and conclude that $|\nabla w|\equiv 0$, whence $w\equiv 0$. The problem with this argument is the boundary term

$$S\int_0^{2\pi}\Phi(S,\theta)\{u(S,\theta)\cos\theta+v(S,\theta)\sin\theta\}\,d\theta.$$

Although $\Phi=p+\frac{1}{2}|w|^2$ tends to zero at infinity by (2.23) and the term in brackets has mean-value zero on $(0,2\pi)$ by (1.2)–(1.3), there are no estimates to show that the integral is $o(S^{-1})$ as $S\to\infty$. Notwithstanding this difficulty, we believe that the results and structure of the equations shown in the previous section may make the conjecture true. Assume for the moment that the conjecture is true, at least for the case of symmetric flow. Since we show in section 4.2 that Leray's solution is non-trivial when the flow is symmetric, it would then follow that $w_\infty\neq 0$.

We shall assume in the rest of this section and in sections 2.4–2.5, *that* $w_\infty\neq 0$, *and may assume after a suitable rotation and scaling that* $w_\infty=(1,0)$. Throughout this section, we shall let the polar coordinate θ vary over $[0,2\pi)$ or $[-\pi,\pi)$ as convenient. Since ψ is given as a line integral involving w and we have some such information in (2.21), it is plausible that $\psi(x,y)\simeq y$ in some sense near infinity. We now make this rigorous, and begin with some preliminaries. Let $\delta_1,\delta_2\in(0,\pi/2)$ with $\delta_1<\delta_2$. Now

$$\infty>\int_{\delta_1}^{\delta_2}d\theta\int_1^\infty r\left|\frac{\partial w}{\partial r}(r,\theta)\right|^2dr=(\delta_2-\delta_1)\int_1^\infty r\left|\frac{\partial w}{\partial r}(r,\tilde{\theta})\right|^2dr \tag{2.24}$$

for some $\tilde{\theta}=\tilde{\theta}(\delta_1,\delta_2)\in(\delta_1,\delta_2)$. If $r\in(R_n,R_{n+1})$, then

$$\begin{aligned}|w(r,\tilde{\theta})-w_\infty| &\le |w(R_n,\tilde{\theta})-w_\infty|+\int_{R_n}^{r}\left|\frac{\partial w}{\partial r}(r,\tilde{\theta})\right|dr\\ &\le |w(R_n,\tilde{\theta})-w_\infty|+\left\{2\log 2\int_{R_n}^{R_{n+1}} r\left|\frac{\partial w}{\partial r}(r,\tilde{\theta})\right|^2 dr\right\}^{1/2}.\end{aligned}$$

The use of (2.22) and (2.24) gives

$$|w(r,\tilde{\theta})-w_\infty|\to 0 \quad \text{as} \quad r\to\infty.$$

A small variation on these arguments ensures that for each $\delta_1,\delta_2\in(0,\pi/2)$ with $\delta_1<\delta_2$, there exists $\tilde{\theta}\in(\delta_1,\delta_2)$ such that

$$|w(r,\pm\tilde{\theta})-w_\infty|,\ |w(r,\pm(\pi-\tilde{\theta}))-w_\infty|\to 0 \quad \text{as} \quad r\to\infty. \tag{2.25}$$

Since $w=(u,v)=(\psi_y,-\psi_x)$, it is immediate from (2.25) that

$$\left|\frac{\psi(r,\theta)}{r\sin\theta}-1\right|\to 0 \quad \text{as} \quad r\to\infty, \tag{2.26}$$

if $\theta=\pm\tilde{\theta},\ \pm(\pi-\tilde{\theta})$. The use of (2.21) with (2.26) gives

$$\max_{|\theta|\in[\tilde{\theta},\pi-\tilde{\theta}]}\left|\frac{\psi(r,\theta)}{r\sin\theta}-1\right|\to 0 \quad \text{as} \quad r\to\infty, \tag{2.27}$$

where $\theta\in[-\pi,\pi)$. For any $\delta_1,\delta_2\in(0,\pi/2)$ with $\delta_1<\delta_2$, let

$$B_n=B_n(\tilde{\theta})=\{(r,\theta)\colon r\in(R_n,R_{n+1}),\ |\theta|\in(\tilde{\theta},\pi-\tilde{\theta})\}.$$

Note that B_n has two components—one in $\{y>0\}$ and the other in $\{y<0\}$. Now p vanishes at infinity, and if we combine this with (2.22) and (2.25), then

$$\max_{z\in\partial B_n}\left|\Phi(z)-\frac{1}{2}\right|\to 0 \quad \text{as} \quad n\to\infty.$$

Since $|\gamma(z)-\frac{1}{2}|w_\infty|^2|=|\gamma(z)-\frac{1}{2}|=|\Phi(z)-\frac{1}{2}-\psi(z)\,\omega(z)|\to 0$ as $|z|\to\infty$ by Theorem 14, we have

$$\max_{z\in\partial B_n}|\psi(z)\,\omega(z)|\to 0 \quad \text{as} \quad n\to\infty. \tag{2.28}$$

Note that $|\psi(z)| \geq |z||\sin\bar{\theta}||\psi(x,y)/y|$ on ∂B_n and that the final absolute value tends to unity as $n \to \infty$ by (2.26) or (2.27). The use of this in (2.28) yields

$$\max_{z \in \partial B_n} |z||\omega(z)| \to 0 \quad \text{as} \quad n \to \infty. \tag{2.29}$$

THEOREM 17. *Let* $\varepsilon \in (0, \pi/2)$. *Then*

$$\max_{|\theta| \in [\varepsilon, \pi-\varepsilon]} |\omega(r,\theta)| = o(1/r) \quad as \quad r \to \infty.$$

Proof. Given ε, choose $\delta_1 = \varepsilon/4$ and $\delta_2 = \varepsilon/2$. Any two points $z_1, z_2 \in \bar{B}_n$ have $|z_1|/|z_2| \leq 4$, and the theorem then follows from (2.29) and the maximum principle for ω. q.e.d.

COROLLARY 18. *Let* $\varepsilon \in (0, \pi/2)$. *Then*

$$\max_{|\theta| \in [\varepsilon, \pi-\varepsilon]} ||w(r,\theta)| - |w_\infty|| = \max_{|\theta| \in [\varepsilon, \pi-\varepsilon]} ||w(r,\theta)| - 1| \to 0 \quad as \quad r \to \infty.$$

Proof. Since $w = (\psi_y, -\psi_x) \in L_\infty(\Omega)$, we always have $|\psi(z)| \leq \text{const}\,|z|$, and so Theorem 17 gives $\psi(z)\,\omega(z) \to 0$ at infinity in these sectors. Since

$$|\gamma(z) - \tfrac{1}{2}| = |p(z) - \psi(z)\,\omega(z) + \tfrac{1}{2}|w(z)|^2 - \tfrac{1}{2}| \to 0 \quad \text{as} \quad |z| \to \infty$$

by Theorem 14, the result follows. q.e.d.

Remark 2. We shall show in Theorem 21 that $|w(z)| \to |w_\infty| = 1$ as $|z| \to \infty$.

Note that Corollary 18 does not show that $|w(z) - w_\infty|$ tends to zero at infinity in these sectors. This is done in the following theorem. The proof depends on a representation theorem for w in a domain in terms of its value on the boundary and the vorticity in the interior.

THEOREM 19. *Let* $\varepsilon \in (0, \pi/2)$. *Then*

$$\max_{|\theta| \in [\varepsilon, \pi-\varepsilon]} |w(r,\theta) - w_\infty| \to 0 \quad as \quad r \to \infty.$$

Proof. Given $\varepsilon > 0$, let $\delta_1 = \varepsilon/4$, $\delta_2 = \varepsilon/2$ and $\bar{\theta} = \bar{\theta}(\delta_1, \delta_2)$ be as before. Define

$$A_n = A_n(\varepsilon) = \{(r,\theta) : r \in (R_{n-2}, R_{n+3}), |\theta| \in (\bar{\theta}, \pi - \bar{\theta})\}.$$

For each $\hat{z}\in A_n$, we use the following representation theorem in [15; p. 388]:

$$u(\hat{z})-iv(\hat{z})-1=\frac{1}{2\pi i}\oint_{\partial A_n}\frac{u(z)-iv(z)-1}{z-\hat{z}}\,dz+\frac{1}{2\pi i}\iint_{A_n}\frac{\omega(z)}{z-\hat{z}}\,dx\,dy, \tag{2.30}$$

where $w=(u,v)$, $\hat{z}=\hat{x}+i\hat{y}$ and $z=x+iy$. Throughout the rest of this proof we shall restrict attention to points $\hat{z}\in C_n=\{z\colon |z|\in[R_n,R_{n+1}], |\theta|\in[\varepsilon,\pi-\varepsilon]\}$.

The first term on the right of (2.30) gives

$$\frac{1}{2\pi}\left|\oint_{\partial A_n}\frac{u(z)-iv(z)-1}{z-\hat{z}}\,dz\right|\le\frac{1}{2\pi}\max_{\partial A_n}|w(z)-w_\infty|\frac{|\partial A_n|}{\operatorname{dist}(\hat{z},\partial A_n)}, \tag{2.31}$$

where $w_\infty=(1,0)$, $|\partial A_n|$ denotes the length of ∂A_n, and the denominator is the distance from $\hat{z}$ to ∂A_n. The form of A_n and the assumptions on $\hat{z}$ ensure that the ratio in (2.31) is bounded by a const./ε, where the constant is independent of ε and n. The use of (2.22) and (2.25) in (2.31) yield

$$\max_{\hat{z}\in C_n}\left|\oint_{\partial A_n}\frac{u(z)-iv(z)-1}{z-\hat{z}}\,dz\right|\to 0\quad\text{as}\quad n\to\infty. \tag{2.32}$$

The final term on the right of (2.30) is estimated with the aid of Theorem 17:

$$\begin{aligned}\iint_{A_n}\frac{|\omega(z)|}{|z-\hat{z}|}\,dx\,dy&\le\max_{z\in\bar{A}_n}\{|z|\,|\omega(z)|\}\iint_{A_n}\frac{dx\,dy}{|z||z-\hat{z}|}\\&\le\text{const.}\max_{z\in\bar{A}_n}|z|\,|\omega(z)|\to 0\quad\text{as}\quad n\to\infty.\end{aligned} \tag{2.33}$$

The use of (2.32)–(2.33) in (2.30) gives the desired result

$$\max_{z\in C_n}|w(z)-w_\infty|\to 0\quad\text{as}\quad n\to\infty. \qquad\text{q.e.d.}$$

Theorem 19 shows that we only need to show the convergence at infinity of w to w_∞ in sectors about the x-axis of the form $\{(x,y)\colon |y|<\alpha|x|\}$ where $\alpha>0$ and may be taken as small as we need. (We shall do this in section 4.1 for symmetric flow.) One cannot expect that the estimate of Theorem 17 holds everywhere in Ω, that is, $|z|\,|\omega(z)|\to 0$ as $|z|\to\infty$. Indeed, one expects from [5] that $|\omega(z)|=O(1/|z|)$ and that $|z|\,|\omega(z)|$ tends to a positive limit at infinity within the wake along the positive x-axis. If we knew that $|\omega(z)|=O(1/|z|)$ in Ω, and set $A_n=\{(r,\theta)\colon r\in(R_{n-1},R_{n+2}), |\theta|<\alpha\}$, then the term in (2.33) could be shown to be $O(\alpha|\log\alpha|)$. This would give $w \to w_\infty$ in sectors along the positive x-axis, and a similar result would hold along the negative x-axis.

Remark 3. The convergence in small sectors of w to $w_\infty=(1,0)$ for general flow remains open. However, there are sufficient conditions which give this:

(i) $r^2\int_0^{2\pi} \omega^2(r,\theta)\,d\theta \to 0$ as $r\to\infty$;

or

(ii) $u(z)>0$ for all large $|z|$.

Since we cannot prove either of these conditions, we merely remark how they imply the desired convergence. Since

$$\int_1^\infty \frac{dr}{r}\int_0^{2\pi} r^2\omega^2(r,\theta)\,d\theta < \infty,$$

condition (i) is at least plausible. The use of (i) in the proof of Theorem 19 gives $w(z)\to w_\infty$ as $|z|\to\infty$. If (ii) holds, one sets $\tau(z)=\tan^{-1}[v(z)/u(z)]\in(-\pi/2,\pi/2)$, and derives an equation for $\nabla\tau$ from (1.1)–(1.2). This may be analyzed to give $\tau(z)\to 0$ as $|z|\to\infty$, whence $v(z)\to 0$. It is shown in the next section that $|w(z)|\to|w_\infty|$, whence $|u(z)|\to|u_\infty|$. Since u is one-signed at infinity by (ii), we are led to $u(z)\to u_\infty$.

2.4. Convergence of the speed

In this section, we shall prove that the speed

$$|w(z)| = |\nabla\psi(z)| \to |w_\infty| \quad \text{as} \quad |z|\to\infty. \tag{2.34}$$

(This is immediate for symmetric flow due to the stronger result $|w(z)-w_\infty|\to 0$ to be established in section 4.1.) We can easily derive some information about the speed from the one-sided maximum principle (cf. equation (1.11)) for the total-head pressure $\Phi=p+\frac{1}{2}|w|^2$. We know that

$$\max_{R_n\le|z|\le R_{n+1}} (p(z)+\tfrac{1}{2}|w(z)|^2) \le \max_{|z|=R_n,R_{n+1}} (p(z)+\tfrac{1}{2}|w(z)|^2),$$

and the right-hand side tends to $\frac{1}{2}|w_\infty|^2$ by (2.22) and since p vanishes at infinity. Hence, we have the upper bound

$$\limsup_{|z|\to\infty} |w(z)|^2 \leq |w_\infty|^2. \tag{2.35}$$

Since $\gamma(z)=\Phi(z)-\psi(z)\,w(z)\to\frac{1}{2}|w_\infty|^2$ as $|z|\to\infty$ (Theorem 14), it follows that

$$\limsup_{|z|\to\infty} \psi(z)\,\omega(z) \leq 0. \tag{2.36}$$

When we prove that $|w(z)|\to|w_\infty|$, we shall be able to strengthen (2.36) to

$$|\psi(z)\,\omega(z)|=|\psi(z)\,\Delta\psi(z)|\to 0 \quad \text{as} \quad |z|\to\infty.$$

From (2.22), (2.27), and Theorem 19, we have very good information about ψ and its gradient on the circles $\{|z|=R_n\}$ and in the regions $\{(x,y)\colon |y|>\alpha|x|, |z|\geq R_n\}$, where $\alpha>0$. If we combine this information with (2.34), then quite precise information can be found about the level-sets of ψ near to infinity. This is always useful since the representation $w=(u,v)=(\psi_y, -\psi_x)$ gives the velocity tangent to level-sets of ψ.

In the study of symmetric flow in section 4, it will be shown that $|\psi(x,y)/y-1|\to 0$ as $|z|\to\infty$, whence the level-set of $\psi=0$ near infinity, $\{z\colon \psi(z)=0, |z|\geq N\}$, consists of $\{(x,y)\colon y=0,\ \pm x\geq N\}$ for all large N. For general flow, we know that

$$\gamma(z)=p(z)+\tfrac{1}{2}|w(z)|^2-\psi(z)\,\omega(z)\to\tfrac{1}{2}|w_\infty|^2 \quad \text{as} \quad |z|\to\infty,$$

whence

$$|w(z)|\to|w_\infty| \quad \text{if} \quad |z|\to\infty \quad \text{and} \quad \psi(z)=0. \tag{2.37}$$

Our intention is to infer from (2.37) the desired result (2.34). First, we need some information about the level-set of $\psi=0$ near infinity.

LEMMA 20. *There exists $S>1$ such that $\{(x,y)\colon \psi(x,y)=0, |z|\geq S\}$ has precisely two components $C_\pm$, where*

$$C_\pm=\{(p_\pm(s), q_\pm(s))\colon s\in[0,\infty)\}.$$

Here $p_\pm$ and $q_\pm$ are real-analytic on $[0,\infty]$, $|p_\pm(\cdot)|\geq S$, $-p_-(s)$, $p_+(s)\to\infty$ as $s\to\infty$, and $|q_\pm(s)/p_\pm(s)|\to 0$ as $s\to\infty$. In addition, $|w(p_\pm(s), q_\pm(s))|\to|w_\infty|$ as $s\to\infty$.

Proof. Let $\alpha>0$ be small and choose N such that $\frac{1}{2}|\nabla\psi(z)|^2-\psi(z)\omega(z)\geq\frac{1}{4}$, whenever $|z|\geq R_N$, and $|\psi(x,y)/y-1|\leq\frac{1}{2}$ if $|y|>\alpha|x|$ and $|z|\geq R_N$. This is possible by Theorem 14 and equation (2.27), respectively. Define $F_n=\{(r,\theta): r\in(R_n,R_{n+1}), \theta\in(-\mu,\mu)\}$, where $\tan\mu=\alpha$. Since $\partial\psi/\partial\theta(r,\theta)=r\{\sin\theta\, v(r,\theta)+\cos\theta\, u(r,\theta)\}$, we may use (2.22) to choose N so large that $\partial\psi/\partial\theta(R_n,\theta)>0, \theta\in[-\mu,\mu], n\geq N$.

Since $\psi(r,\mu)>0$ and $\psi(r,-\mu)<0$, $r\geq R_N$, there are obviously points in F_n where ψ vanishes. Let $\bar{z}$ be such a point, and let L_n denote the component of $\{z\in\bar{F}_n: \psi(z)=0\}$ containing $\bar{z}$. Since $|\nabla\psi(z)|^2\geq\frac{1}{2}$, $z\in L_n$, it follows that L_n is locally a real-analytic curve. The arguments for Lemma 4 show that L_n is either an arc in $\bar{F}_n$ or a closed curve. The proof of Theorem 25 shows that the latter is impossible whenever n is sufficiently large. Since ψ is one-signed on $|y|=\alpha x$, L_n must intersect ∂F_n where $|z|=R_n, R_{n+1}$. Since $\partial\psi/\partial\theta(R_n,\theta)>0$, $\theta\in[-\mu,\mu]$, it follows that L_n intersects these circles in at most one point. Since L_n is maximal, it intersects $|z|=R_n$ and $|z|=R_{n+1}$. Hence, L_n is a real-analytic arc in $\bar{F}_n$ which intersects ∂F_n only at its endpoints. The monotonicity of ψ on $|z|=R_n, R_{n+1}$, ensures that $L_n=\{z\in\bar{F}_n: \psi(z)=0\}$. The set C_+ is just the union of the L_n for $n\geq N$, and may be parametrized by arc-length measured from $|z|=R_N$. Since $\alpha>0$ may be taken arbitrarily small, we have $|q_+(s)/p_+(s)|\to 0$ as $s\to\infty$. All of the arguments presented above are applicable to the case of x large and negative. q.e.d.

Our main result in this section is

THEOREM 21. (a) *The speed* $|w(z)|$ *satisfies*

$$|w(z)|=|\nabla\psi(z)|\to|w_\infty| \quad as \quad |z|\to\infty.$$

(b) *The quantity* $|\psi(z)\,\omega(z)|=|\psi(z)\,\Delta\psi(z)|\to 0$ *as* $|z|\to\infty$.
(c) *If* $|w(z)-w_\infty|\to 0$ *as* $|z|\to\infty$ *for* $z\in C_+\cup C_-$, *then* $|w(z)-w_\infty|\to 0$ *as* $|z|\to\infty$.

Note that (b) follows immediately from (a) and Theorem 14. The proof of (a) is quite technical, and appears in the Appendix. Part (c) follows immediately from the construction needed in (a), and shows that the pointwise behavior at infinity of w is determined by its behavior on the level-sets of $\psi=0$.

2.5. Some monotonicity results and decay rates for the vorticity

Recall the number $c\in(1,2)$ which appeared in Theorem 11. If $r\geq c$, equation (1.10) shows that the quantity $\max_{\theta\in[0,2\pi]}\omega(r,\theta)$ is either monotone decreasing on (c,∞), or is initially decreasing to some value of r, and thereafter is monotone increasing. Since ω vanishes at infinity, and is not one-signed in a neighborhood of infinity, only the first possibility occurs. If we apply the maximum principle to ω on all of Ω, we easily see that ω takes its maximum value on $\bar{\Omega}$ at $\partial\Omega=\Gamma$. A similar argument shows that $\min_{\theta\in[0,2\pi]}\omega(r,\theta)$ is monotone increasing on (c,∞), and ω takes its minimum value on $\bar{\Omega}$ at $\partial\Omega$.

Now $\gamma(z)=\Phi(z)-\psi(z)\,w(z)\to\frac{1}{2}|w_\infty|^2$ as $|z|\to\infty$, and Theorem 11 gives the existence of an arc connecting $\{|z|=c\}$ to infinity on which γ is monotone decreasing. It is immediate that $\max_{\theta\in[0,2\pi]}\gamma(r,\theta)$ is monotone decreasing on (c,∞). A similar argument shows that $\min_{\theta\in[0,2\pi]}\gamma(r,\theta)$ is monotone increasing on (c,∞). Finally, we note that γ takes its maximum and minimum values on $\bar{\Omega}$ at $\partial\Omega$.

The function Φ satisfies a one-sided maximum principle, and $\Phi(z)\to\frac{1}{2}|w_\infty|^2$ as $|z|\to\infty$ by Theorem 21. With the aid of Theorem 11 we see that $\max_{\theta\in[0,2\pi]}\Phi(r,\theta)$ is monotone decreasing on (c,∞). It is immediate that Φ takes its maximum value on $\partial\Omega$, whence

$$\max_{\partial\Omega}\Phi=\max_{\partial\Omega}p>\tfrac{1}{2}|w_\infty|^2.$$

This can be improved slightly if $\partial\Omega$ has only one component so that $\psi=0$ on $\partial\Omega$. The fact that γ takes its minimum on $\partial\Omega$ yields

$$\tfrac{1}{2}|w_\infty|^2>\min_{\partial\Omega}\gamma=\min_{\partial\Omega}p.$$

We now give some more precise information about the vorticity ω in the case that $w(z)\to w_\infty\neq 0$ as $|z|\to\infty$. As usual, we may then take $w_\infty=(1,0)$. Let us begin with the case $x<0$. Equation (1.10) is easily seen to imply

$$\nu\Delta(\omega^2)=\operatorname{div}(w\omega^2)+2\nu|\Delta\omega|^2\quad\text{in }\Omega.$$

If we integrate this over $(-\infty,x)\times(-\infty,\infty)$, then

$$\frac{1}{2}\int_{-\infty}^{\infty}u(x,y)\,\omega^2(x,y)\,dy=\nu S'(x)-\nu\int_{-\infty}^{x}\int_{-\infty}^{\infty}|\nabla\omega|^2,$$

where

$$S(x)=\frac{1}{2}\int_{-\infty}^{\infty}\omega^2(x,y)\,dy.$$

For every small $\varepsilon>0$, let $M(\varepsilon)<0$ be such that $u(x,y)\geq 1-\varepsilon$ if $x\leq M(\varepsilon)$, $y\in\mathbf{R}$. It follows that

$$(1-\varepsilon)\,S(x)\leq \nu S'(x),\quad x\leq M(\varepsilon).$$

Integrating this equation yields

$$S(x)\leq \text{const.}\exp\left[-\frac{(1-\varepsilon)}{\nu}|x|\right],\quad x\leq M(\varepsilon). \tag{2.38}$$

It is then easy to derive the same bound for $|S'|$ and for

$$\int_{-\infty}^{\infty}|\nabla\omega(x,y)|^2\,dy.$$

The use of the equation (1.10) for ω and standard regularity estimates yield

$$|\omega(x,y)|,\ |\nabla\omega(x,y)|\leq \text{const.}\exp\left(-\frac{(1-\varepsilon)}{2\nu}|x|\right),\quad x\leq M(\varepsilon),\quad y\in\mathbf{R}. \tag{2.39}$$

Since $|\nabla\gamma(z)|$, $|\nabla\Phi(z)|\leq\text{const.}|z|(|\omega(z)|+|\nabla\omega(z)|)$, this yields

$$\begin{aligned}&|\gamma(x,y)-\tfrac{1}{2}|,\ |p(x,y)+\tfrac{1}{2}|w(x,y)|^2-\tfrac{1}{2}|,\ |\nabla\Phi(x,y)|,\ |\nabla\gamma(x,y)|\\&\quad\leq \text{const.}|x|\exp\left(-\frac{(1-\varepsilon)}{2\nu}|x|\right),\quad x\leq M(\varepsilon),\ y\in\mathbf{R}.\end{aligned} \tag{2.40}$$

Fix a small number $\alpha>0$, let $\varepsilon>0$ be small compared to α, and note that $|w(x,y)-(1,0)|\leq\varepsilon$ for all points (x,y) in the region $y\geq\alpha x+M(\alpha,\varepsilon)$, for some large number $M(\alpha,\varepsilon)$. The same arguments as before give (2.38)–(2.40) in this region, but with the exponential replaced by

$$\exp\left\{-\frac{1}{2\nu}(\alpha/\sqrt{1+\alpha^2}-\varepsilon)\,|y-\alpha x|/\sqrt{1+\alpha^2}\right\}.$$

An analogous result holds if $y\leq-\alpha x+M(\alpha,\varepsilon)$. These estimates on the vorticity greatly improve those of Theorem 17, but become less useful in a sector centred about the positive x-axis. For that case, let us take $\nu=1$. Equation (1.10) gives

$$\Delta\omega-\omega_x=(u-1)\,\omega_x+v\omega_y=\operatorname{div}(\omega(u-1,v))\quad\text{in}\quad\Omega. \tag{2.41}$$

The operator $\Delta\omega+\omega_x$ has a fundamental solution [5] in $\mathbf{R}^2$ given by

$$G(z;\bar z)=-e^{(\bar x-x)/2}K_0\left(\frac{|z-\bar z|}{2}\right),$$

where K_0 denotes the usual Bessel function. If this is applied to (2.41), then an equation for $\omega(\bar{z})$ may be given in terms of a boundary integral involving ω, $|\nabla\omega|$, and the fundamental solution, and an integral involving the right-hand side of (2.41). If $|u-1|$, $|v|$, and $|\nabla\omega|$ (or integrals involving them) have suitable decay at infinity, then the important term in $\omega(\bar{z})$ should be the boundary terms. If this is the case, then one expects

$$\omega(r,\theta)=\left(\left(\frac{|\theta|}{\sqrt{r}}+\frac{1}{r}\right)e^{-r\theta^2/4}\right)$$

when r is large and $|\theta|$ is small. This would give the expected estimate $\omega=O(1/r)$. The sets $\{(r,\theta)\colon \sqrt{r}\,|\theta|\leq\alpha\}$ are obviously of importance, and define the familiar 'parabolic wake'.

3. Flow in the bounded domain Ω_R

In this section, we return to the bounded domains $\Omega_R=\Omega\cap\{|z|<R\}$, and Leray's 'approximate' solutions (w_R, p_R) of (1.1)–(1.2) in Ω_R with $w_R=0$ on Γ and $w_R=\bar{w}_\infty$ on $\{|z|=R\}$. We know that the Dirichlet norm of w_R is bounded independently of R, and may assume that (w_R, p_R) and its derivatives converge as $R\to\infty$ on compact subsets of $\bar{\Omega}$ to a solution (w_L, p_L) of (1.1)–(1.3). Since the convergence is only on compact sets, the information provided by $w_R=\bar{w}_\infty$ on $\{|z|=R\}$ may be lost. We show how this information is transmitted to compact subsets with the aid of our maximum principles.

An important role is played by the level-sets of $\omega_R=0$, along which $\gamma_R=p_R+\frac{1}{2}|w_R|^2$. In Theorem 23, we assume there are two arcs connecting $\{|z|=R\}$ to points near to Γ along which Φ_R is monotone increasing and decreasing, respectively. We prove that

$$\iint_{\Omega_R}|\nabla w_R|^2\geq m>0, \tag{3.1}$$

where m is *independent* of R. We then prove in Theorem 24 that if (3.1) holds for a sequence $\{R_n\}$, with $R_n\to\infty$ as $n\to\infty$, *then Leray's solution* (w_L, p_L) *is non-trivial*. In section 4.2, we show that (3.1) holds for *symmetric* flow in Ω_R:

$$\inf_{R\geq 2}\iint_{\Omega_R}|\nabla w_R|^2>0. \tag{3.2}$$

3.1. Stokes flow

Before we consider the Navier-Stokes equations in Ω_R, let us make some remarks about those for Stokes flow:

$$\left.\begin{aligned} -\nu\Delta w &= -\nabla p \\ \nabla\cdot w &= 0 \end{aligned}\right\} \quad \text{in} \quad \Omega_R \tag{3.3, 3.4}$$

$$w=0 \quad \text{on} \quad \Gamma, \tag{3.5}$$

$$w=\tilde{w}_\infty \quad \text{on} \quad |z|=R. \tag{3.6}$$

Standard theory [16] ensures that this problem has a unique solution (w_R^s, p_R^s), where the superscript denotes Stokes flow.

THEOREM 22. *If* (w_R^s, p_R^s), *satisfies* (3.3)–(3.6), *then*

$$\iint_{\Omega_R} |\nabla w_R^s|^2 \leqslant \frac{\text{const.}|\tilde{w}_\infty|^2}{\log R}, \quad R \geqslant 2 \tag{3.7}$$

where the constant depends only on Γ, *and is independent of* R, $\tilde{w}_\infty$, *and* ν.

Proof. Let $\tau \in C^\infty(\mathbf{R})$ with $\tau(r)=0$ for $r\leqslant\frac{1}{2}$ and $\tau(r)=1$ for $r\geqslant 1$. Define $\mu(r)=\tau(\log r/\log R)$, so that $\mu(r)=0$ when $r\leqslant\sqrt{R}$. After a suitable rotation of the axes, we may assume without loss of generality that $\tilde{w}_\infty=(\tilde{u}_\infty,0)$. Define a solenoidal vector-field $A=(A_1,A_2)$ by

$$A_1=\frac{\partial}{\partial y}\{\tilde{u}_\infty y\mu(|z|)\}, \quad A_2=-\frac{\partial}{\partial x}\{\tilde{u}_\infty y\mu(|z|)\}.$$

Set $\tilde{w}=w_R^s-A$, and note that $\tilde{w}=0$ on $\partial\Omega_R$. Now $-\nu\Delta\tilde{w}-\nu\Delta A=\nabla p_R$, whence

$$\iint_{\Omega_R} |\nabla\tilde{w}|^2 = -\iint_{\Omega_R} \nabla\tilde{w}\cdot\nabla A,$$

and so

$$\iint_{\Omega_R} |\nabla\tilde{w}|^2 \leqslant \iint_{\Omega_R} |\nabla A|^2.$$

An explicit calculation gives

$$|\nabla A(z)|^2 \leqslant \frac{\text{const.}\,|\tilde{w}_\infty|^2}{|z|^2(\log R)^2}, \quad z\in\bar{\Omega}_R,$$

where the constant is independent of R and $\bar{w}_\infty$. It follows that

$$\iint_{\Omega_R} |\nabla\bar{w}|^2 \leqslant \iint_{\Omega_R} |\nabla A|^2 \leqslant \frac{\text{const.}\,|\bar{w}_\infty|^2}{\log R},$$

and the representation $w_R^s=\bar{w}+A$ completes the proof. q.e.d.

This result is not at all surprising in view of the result [10] that the only solution (w,p) of (3.3)–(3.4) in Ω with $w=0$ on Γ and $|w(z)|=o(\log|z|)$ at infinity is the trivial solution $w\equiv 0$. The result in Theorem 22 is frightening at first thought, because our solution (w_R,p_R) of the Navier-Stokes equations might satisfy a similar estimate. If this were the case, then the Leray solution $w_L\equiv 0$. (Such a result would not contradict the proof by Finn and Smith [11] of a non-trivial solution of (1.1)–(1.4) when $|\bar{w}_\infty|/\nu$ is small since the solution in [11] is not constructed as the limit of our (w_R,p_R).) However, we shall prove in section 4 that (3.1) is true for symmetric flow, so that (3.7) does not hold for Navier-Stokes flow. The difference between Stokes and Navier-Stokes flow is partly due to the different maximum principles present in the two cases. Equations (3.3)–(3.4) show that for Stokes flow $p+i\nu\omega$ is (complex) analytic in Ω_R or Ω, whence p and $\nu\omega$ are conjugate harmonic functions. However, for Navier-Stokes flow, it is $\gamma+i\nu\omega$, $\gamma=p+\frac{1}{2}|w|^2-\psi\omega$, which is a pseudo-analytic function.

3.2 Special level-sets of the vorticity

THEOREM 23. *Assume that there exist continuous, injective maps*

$$\{(x_i(s;R),y_i(s;R))\colon s\in[0,L_i]\}\subset\Omega_R\cup\{|z|=R\},\quad i=1,2,$$

such that the following hold

(a) $\operatorname{dist}(\Gamma,(x_i(0;R),y_i(0;R)))\leqslant 1$, $i=1,2$,

(b) $(x_i(L_i;R),y_i(L_i;R))\in\{|z|=R\}$, $i=1,2$,

and

(c)

$$\Phi_R((x_1(\cdot,R),y_1(\cdot,R)))$$

is monotone decreasing on $[0,L_1]$ *while*

$$\Phi_R((x_2(\cdot,R),y_2(\cdot R)))$$

is monotone increasing on $[0,L_2]$.

Then

$$\iint_{\Omega_R} |\nabla w_R|^2 \geqslant m > 0, \tag{3.8}$$

where the constant m is independent of $R \geqslant 2$.

Proof. We shall assume that (3.8) is false, and then derive a contradiction. Let $\{R_n\}$ be a sequence with $R_n \to Q \in [2, \infty]$ as $n \to \infty$, and such that

$$\iint_{\Omega_{R_n}} |\nabla w_{R_n}|^2 \to 0 \quad \text{as} \quad n \to \infty. \tag{3.9}$$

If $Q < \infty$, then standard theory ensures that w_{R_n} and its derivatives converge on $\bar{\Omega}_Q$ to w_Q. In particular, $w_Q = 0$ on Γ, $w_Q = \bar{w}_\infty$ on $\{|z| = Q\}$ and $|\nabla w_Q|_{L_2(\Omega_Q)} = 0$. Since we are always assuming that $\bar{w}_\infty \neq 0$, this is impossible, whence $R_n \to \infty$ as $n \to \infty$.

It follows that if (3.9) holds, then w_{R_n} and its derivatives converge on compact subsets of $\bar{\Omega}$ to $w_L \equiv 0$. After adding a suitable constant to p_{R_n}, we may assume that same convergence to zero of the pressure and its derivatives.

In the calculations to follow, we write w_n and p_n for w_{R_n} and p_{R_n}, respectively, and Ω_n for Ω_{R_n}. For a function $f(r, \theta)$, let

$$\bar{f}(r) = \frac{1}{2\pi} \int_0^{2\pi} f(r, \theta)\, d\theta.$$

Equations (1.1)–(1.2) give

$$\frac{\partial}{\partial r} p_n = \frac{\nu}{r} \frac{\partial}{\partial \theta} \omega_n + \frac{1}{r} \{u_n \frac{\partial}{\partial \theta} v_n - v_n \frac{\partial}{\partial \theta} u_n\}, \quad r \in (1, R_n),$$

where $w_n = (u_n, v_n)$. It follows that

$$\begin{aligned} \frac{d}{dr} \bar{p}_n &= \frac{1}{2\pi r} \int_0^{2\pi} \{u_n \frac{\partial}{\partial \theta} v_n - v_n \frac{\partial}{\partial \theta} u_n\}\, d\theta \\ &= \frac{1}{2\pi r} \int_0^{2\pi} \{(u_n - \bar{u}_n) \frac{\partial}{\partial \theta} v_n - (v_n - \bar{v}_n) \frac{\partial}{\partial \theta} u_n\}\, d\theta. \end{aligned}$$

The use of Wirtinger's inequality gives

$$\left|\frac{d}{dr}\bar{p}_n\right| \leq \frac{1}{2\pi r}\int_0^{2\pi}\left\{\left(\frac{\partial}{\partial\theta}u_n\right)^2+\left(\frac{\partial}{\partial\theta}v_n\right)^2\right\}d\theta$$
$$\leq \frac{1}{2\pi}\int_0^{2\pi}|\nabla w_n|^2\,d\theta, \quad r\in(1,R_n).$$

Integrating this yields

$$|\bar{p}_n(r)-\bar{p}_n(1)| \leq \int_1^{R_n}\left|\frac{d}{dr}\bar{p}_n\right|dr \leq \frac{1}{2\pi}\iint_{\Omega_n}|\nabla w_n|^2 \to 0 \tag{3.10}$$

as $n\to\infty$ by assumption.

As noted earlier, $p_n\to 0$ on compact subsets of $\bar{\Omega}$ as $n\to\infty$, and so

$$\max_{1\leq r\leq R_n}|\bar{p}_n(r)| \to 0 \quad \text{as} \quad n\to\infty. \tag{3.11}$$

We now show that

$$\iint_{U_n}|\nabla\omega_n|^2 \to 0 \quad \text{as} \quad n\to\infty, \tag{3.12}$$

where $U_n=\Omega\cap\{|z|<R_n/2\}$. Let $\tau\in C^\infty(\mathbf{R})$ be such $\tau(r)=1$ for $r\leq 0$ and $\tau(r)=0$ for $r\geq 1$. Define $\mu_n(r)=\tau(r-R_n/2)$. A calculation yields

$$\begin{aligned}-\nu\iint_{\Omega_n}\mu_n\omega_n\Delta\omega_n &= \nu\iint_{\Omega_n}\mu_n|\nabla\omega_n|^2-\frac{\nu}{2}\iint_{\Omega_n}\mu_n\Delta(\omega_n^2)\\ &= \nu\iint_{\Omega_n}\mu_n|\nabla\omega_n|^2-\frac{\nu}{2}\iint_{\Omega_n}\omega_n^2\Delta\mu_n+\{\text{B.T.}\},\end{aligned} \tag{3.13}$$

where {B.T.} denotes an integral around Γ of ω_n and its normal derivatives. As noted before, ω_n and its derivatives tend to zero on compact subsets of $\bar{\Omega}$, whence the boundary term {B.T.} tends to zero as $n\to\infty$. Since $\nu\Delta\omega_n=\operatorname{div}(w_n\omega_n)$ by (1.10), we have

$$-\nu\iint_{\Omega_n}\mu_n\omega_n\Delta\omega_n = -\frac{1}{2}\iint_{\Omega_n}\mu_n\operatorname{div}(w_n\omega_n^2) = \frac{1}{2}\iint_{\Omega_n}\omega_n^2 w_n\cdot\nabla\mu_n.$$

Equating this to (3.13) yields

$$\nu\iint_{U_n}|\nabla\omega_n|^2\leqslant\frac{\nu}{2}\iint_{\Omega_n}\omega_n^2\,\Delta\mu_n-\{\text{B.T.}\}+\frac{1}{2}\iint_{\Omega_n}\omega_n^2\,w_n\cdot\nabla\mu_n. \tag{3.14}$$

A result of Gilbarg and Weinberger [14] ensures that $|w_n|$ is bounded on $\Omega_n\cap\{|z|\leqslant 3R_n/4\}$, independently of n, while we also have

$$\iint_{\Omega_n}\omega_n^2\leqslant 2\iint_{\Omega_n}|\nabla w_n|^2\to 0 \quad\text{as}\quad n\to\infty$$

by assumption. The use of this with the vanishing of $\{\text{B.T.}\}$ as $n\to\infty$ proves (3.12).

Equations (1.1)–(1.2) give

$$\begin{aligned}\frac{\partial}{\partial x}p_n&=\nu\frac{\partial}{\partial y}\omega_n-u_n\frac{\partial}{\partial x}u_n-v_n\frac{\partial}{\partial y}u_n,\\ \frac{\partial}{\partial y}p_n&=-\nu\frac{\partial}{\partial x}\omega_n-u_n\frac{\partial}{\partial x}v_n-v_n\frac{\partial}{\partial y}v_n.\end{aligned}$$

The use of these equations with (3.9) and (3.12) yields

$$\iint_{U_n}|\nabla p_n|^2\to 0\quad\text{as}\quad n\to\infty. \tag{3.15a}$$

Equations (3.9) and (3.15a) prove that

$$\int_{R_n/4}^{R_n/2}\frac{dr}{r}\int_0^{2\pi}\left\{\left(\frac{\partial}{\partial\theta}p_n(r,\theta)\right)^2+\left(\frac{\partial}{\partial\theta}w_n(r,\theta)\right)^2\right\}d\theta\to 0\quad\text{as}\quad n\to\infty, \tag{3.15b}$$

and so there exists $\tilde R_n\in(R_n/4,R_n/2)$ such that

$$\int_0^{2\pi}\left\{\left(\frac{\partial}{\partial\theta}p_n(\tilde R_n,\theta)\right)^2+\left(\frac{\partial}{\partial\theta}w_n(\tilde R_n,\theta)\right)^2\right\}d\theta\to 0\quad\text{as}\quad n\to\infty. \tag{3.15c}$$

Say that the curves $\{x_i(\cdot\,;R_n,y_i(\cdot\,;R_n)\}$ intersect the circle $\{|z|=\tilde R_n\}$ at $z_i=z_i(n)$, $i=1,2$. Let $\bar z_i=\bar z_i(n)=(x_i(0;R_n),y_i(0;R_n))$, $i=1,2$. By (c) of the hypotheses, we know that

$$\Phi_n(z_1)=p_n(z_1)+\tfrac{1}{2}|w_n(z_1)|^2\leqslant p_n(\bar z_1)+\tfrac{1}{2}|w_n(\bar z_1)|^2, \tag{3.16}$$

$$\Phi_n(z_2)=p_n(z_2)+\tfrac{1}{2}|w_n(z_2)|^2\geqslant p_n(\bar z_2)+\tfrac{1}{2}|w_n(\bar z_2)|^2. \tag{3.17}$$

Since $\operatorname{dist}(\tilde{z}_i, \Gamma) \leq 1$ by (a) and $|p_n|, |w_n| \to 0$ on compact sets of $\bar{\Omega}$, the right-hand sides of (3.16)–(3.17) tend to zero as $n \to \infty$. Equations (3.11) and (3.15 c) give

$$\max_{|z|=\tilde{R}_n} |p_n(z)| \to 0 \quad \text{as} \quad n \to \infty,$$

and the use of this in (3.16) yields

$$|w_n(z_1(n))| \to 0 \quad \text{as} \quad n \to \infty.$$

If we combine this with (3.15 c) there results

$$\max_{|z|=\tilde{R}_n} |w_n(z)| \to 0 \quad \text{as} \quad n \to \infty. \tag{3.18}$$

Now

$$\begin{aligned}\int_0^{2\pi} |w_n(R_n, \theta)|\, d\theta - \int_0^{2\pi} |w_n(\tilde{R}_n, \theta)|\, d\theta &= \int_{\tilde{R}_n}^{R_n} dr \int_0^{2\pi} \frac{\partial}{\partial r} |w_n(r, \theta)|\, d\theta \\ &\leq \left(\int_{\tilde{R}_n}^{R_n} \frac{2\pi\, dr}{r} \right)^{1/2} \left(\iint_{\Omega_n} |\nabla w_n|^2 \right)^{1/2} \to 0 \quad \text{as} \quad n \to \infty\end{aligned}$$

by (3.9) and since $\tilde{R}_n \in (R_n/4, R_n/2)$. If we combine this estimate with (3.18) there results

$$\int_0^{2\pi} |w_n(R_n, \theta)|\, d\theta \to 0 \quad \text{as} \quad n \to \infty.$$

However, $w_n(z) = \bar{w}_\infty \neq 0$ on $\{|z| = R_n\}$, and so we have a contradiction. q.e.d.

3.3. A condition implying w_L is non-trivial

For an unbounded sequence $\{R_n\}$, we let $\{w_L, p_L\}$ denote the limit on compact subsets of $\bar{\Omega}$ of a suitable subsequence of (w_{R_n}, p_{R_n}). We know that (w_L, p_L) satisfies (1.1)–(1.3), (1.5) and has the various properties shown in section 2. The following theorem gives a necessary and sufficient condition for w_L to be non-trivial. We shall show in section 4.2 that the condition is satisfied for symmetric flow.

THEOREM 24. *Let $\{R_n\}$ denote an unbounded sequence such that $(w_n, p_n) \equiv (w_{R_n}, p_{R_n})$ and their derivatives converge on compact subsets of $\bar{\Omega}$ to a pair (w_L, p_L).*

Then w_L is non-trivial if and only if

$$\liminf_{n\to\infty} \iint_{\Omega_n} |\nabla w_n|^2 > 0,$$

where $\Omega_n = \Omega \cap \{|z| < R_n\}$.

Proof. If the limit infimum is zero, then clearly $w_L \equiv 0$. To prove the converse, assume that $w_L \equiv 0$. We may then assume that w_n and p_n converge on compact subsets of $\bar{\Omega}$ to zero. If we set $w_n = v_n + \tilde{w}_\infty$ then our equations become

$$\left.\begin{aligned} -\nu\Delta v_n + (w_n\cdot\nabla)\, v_n &= -\nabla p_n \\ \nabla\cdot v_n &= 0 \end{aligned}\right\} \quad \text{in } \Omega_n,$$

$$v_n = -\tilde{w}_\infty \quad \text{on} \quad \Gamma,$$

$$v_n = 0 \quad \text{on} \quad \{|z| = R_n\}.$$

If we take the inner product of the first equation with v_n, then

$$\begin{aligned} -\nu|\nabla v_n|^2 + \frac{\nu}{2}\Delta(|v_n|^2) &= \nu v_n\,\Delta v_n \\ &= \tfrac{1}{2}\operatorname{div}(w_n|v_n|^2) + \operatorname{div}(p_n v_n). \end{aligned}$$

Integrating this over Ω_n yields

$$-\nu \iint_{\Omega_n} |\nabla v_n|^2 + \frac{\nu}{2}\int_\Gamma \frac{\partial}{\partial \tilde{n}} |v_n|^2 = \int_\Gamma p_n v_n \cdot \tilde{n},$$

where $\tilde{n}$ denotes the outward unit normal to Ω_n on Γ. By hypothesis, $|\nabla v_n|$, $|p_n| \to 0$ on Γ as $n\to\infty$, whence

$$\iint_{\Omega_n} |\nabla w_n|^2 = \iint_{\Omega_n} |\nabla v_n|^2 \to 0 \quad \text{as} \quad n\to\infty. \qquad \text{q.e.d.}$$

4. Symmetric flow

4.1 Symmetric flow and convergence of the velocity

Throughout this section, we shall consider *symmetric flow* as defined in section 1.3. In particular, ψ, ω, and v are odd functions of y, while p, u, γ, and Φ are even. As before, we take $w_\infty = (1, 0)$. We shall prove in Theorem 27 that $w(z) \to w_\infty$ as $|z| \to \infty$. Because of

Theorem 19 and the symmetry, it suffices to show this in a region $\{(x,y): 0<y<\alpha|x|\}$ where α is as small as we wish. The main idea is to first show that $\psi(x,y)\simeq y$ near infinity and then use the result $\frac{1}{2}|\nabla\psi(z)|^2-\psi(z)\Delta\psi(z)\simeq\frac{1}{2}$ near infinity (Theorem 14(b)). On the set where $\psi>0$, this gives

$$\Delta(\sqrt{\psi})\simeq -\frac{1}{4\psi^{3/2}}\simeq -\frac{1}{4y^{3/2}},$$

and this equation will be analysed, and found to give the desired results.

THEOREM 25 (Symmetric flow). *The stream-function ψ satisfies $y\psi(x,y)>0$, $y\neq 0$, whenever $|z|=|(x,y)|$ is sufficiently large.*

THEOREM 26 (Symmetric flow). *With the convention that $\psi(x,y)/y$ equals $\psi_y(x,0)=u(x,0)$ when $y=0$, there holds*

$$\left|\frac{\psi(x,y)}{y}-1\right|\to 0 \quad as \quad |z|\to\infty.$$

THEOREM 27 (Symmetric flow). *The velocity w satisfies*

$$|w(z)-w_\infty|\to 0 \quad as \quad |z|\to\infty.$$

Proof of Theorem 25. We shall restrict attention to $y\geq 0$ and x large and positive. A similar argument holds for negative x. Now $\omega(x,0)=0$ and Theorem 15 or 16 ensures that $\omega_y(x,0)\neq 0$ for all large x. Equations (1.14)–(1.15) then show that $\gamma(x,0)=p(x,0)+\frac{1}{2}u(x,0)^2$ is monotone for all large x. If we combine this with (2.22) and the fact that p vanishes at infinity, then

$$\psi_y(x,0)=u(x,0)\to 1 \quad \text{as} \quad x\to\infty. \tag{4.1}$$

For each n, let $D_n=\{(x,y): 0<y<|x|, R_n<|z|<R_{n+1}\}$. Equation (2.26) shows that it is enough to prove that $\psi>0$ in D_n for all large n. Note that (2.22), (2.26), and (4.1) give

$$\max_{z\in\partial D_n}\left|\frac{\psi(x,y)}{y}-1\right|\to 0 \quad \text{as} \quad n\to\infty. \tag{4.2}$$

Assume that the theorem is false, so that $\psi(z_m)=0$ for some unbounded sequence $\{z_m\}$. We may assume that each $z_m\in D_n$ where $n=n(m)$. Since

$$\gamma(z)=p(z)+\tfrac{1}{2}|\nabla\psi(z)|^2-\psi(z)\,\omega(z)\to\tfrac{1}{2}$$

at infinity, we may restrict attention to large enough m and n such that

$$|\nabla\psi(z)|\geq\tfrac{1}{2}\quad\text{if}\quad z\in D_n \text{ and } \psi(z)=0. \tag{4.3}$$

Let L_m denote the component of $\{\psi=0\}$ in D_n, $n=n(m)$, containing z_m. Note that $L_m\subset D_n$ by (4.2), and that L_m is locally a real-analytic arc by (4.3). Using the arguments from Lemma 4, we conclude that L_m is a closed, real-analytic curve in D_n. Let U_m denote the (bounded) interior of L_m. Since the tangential derivative of ψ on $\partial U_m=L_m$ is zero, it follows from (4.3) that $|\partial\psi/\partial n|\geq\frac{1}{2}$ on L_m, where here n denotes the outward unit normal. Now

$$\left|\iint_{U_m}\omega\,dx\,dy\right|=\left|\iint_{U_m}\Delta\psi\,dx\,dy\right|=\left|\int_{L_m}\frac{\partial\psi}{\partial n}\,ds\right|\geq\tfrac{1}{2}|L_m|,$$

where $|L_m|$ denotes the length of L_m. On the other hand,

$$\left|\iint_{U_m}\omega\right|\leq\left\{\left(\iint_{U_m}\omega^2\right)|U_m|\right\}^{1/2},$$

where $|U_m|$ denotes the area of U_m. Combining these inequalities yields

$$\tfrac{1}{4}|L_m|^2\leq|U_m|\iint_{U_m}\omega^2.$$

However, the isoperimetric inequality gives $4\pi|U_m|\leq|L_m|^2$, whence

$$\pi\leq\iint_{U_m}\omega^2\leq\iint_{D_m}\omega^2.$$

Since $\omega\in L_2(\Omega)$, the right-hand side tends to zero as $n\to\infty$, and provides the desired contradiction. q.e.d.

Proof of Theorem 26. It suffies to show that

$$\max_{z\in \bar{D}_n}\left|\frac{\psi(x,y)}{y}-1\right|\to 0\quad\text{as}\quad n\to\infty.$$

We shall use the fact from Theorem 14 that

$$\tfrac{1}{2}|\nabla\psi(z)|^2-\psi(z)\Delta\psi(z)\to\tfrac{1}{2}\quad\text{as}\quad |z|\to\infty. \tag{4.4}$$

Set $\psi(x,y)=y(1+S(x,y))$ in D_n; note that $1+S>0$ by Theorem 25. If we substitute this in (4.4), there results

$$-y^2(1+S)\,\Delta S-y(1+S)\,S_y+\tfrac{1}{2}y^2|\nabla S|^2+\tfrac{1}{2}(1+S)^2\equiv M(z)\to\tfrac{1}{2}\quad\text{as}\quad |z|\to\infty. \tag{4.5}$$

Recall from equation (4.2) that

$$\max_{z\in\partial D_n}|S(z)|\to 0\quad\text{as}\quad n\to\infty.$$

If S has a local positive maximum at $\bar{z}\in D_n$, then (4.5) gives

$$\tfrac{1}{2}(1+S(\bar{z}))^2\leq M(\bar{z}),$$

whence

$$0\leq S(\bar{z})\leq\sqrt{2M(\bar{z})}-1\to 0\quad\text{as}\quad n\to\infty.$$

A similar argument holds at any local negative minima. q.e.d.

Proof of Theorem 27. Since $\gamma(z)\to\frac{1}{2}$ and $p(z)\to 0$ as $|z|\to\infty$, the following holds whenever $y>0$ and $|z|$ is sufficiently large, say $|z|\geq S$:

$$\Delta(\sqrt{\psi(z)}-\sqrt{y}\,)=\frac{N(z)}{4y^{3/2}}, \tag{4.6}$$

where $N(z)=1-2(\gamma(z)-p(z))\,(y/\psi(z))^{3/2}\to 0$ as $|z|\to\infty$. Fix a point $\bar{z}=(\bar{x},\bar{y})$ with $\bar{y}>0$ and such that the disc $E=E(\bar{z})=\{z\in\Omega\colon|z-\bar{z}|<\bar{y}/2\}\subset\{|z|\geq S\}$. Standard theory [13; p. 40] applied to (4.6) yields

$$|\nabla\tau(\bar{z})|\leq\text{const.}\left\{\frac{1}{\bar{y}}\max_{z\in\bar{E}}|\tau(z)|+\bar{y}\max_{z\in\bar{E}}\frac{|N(z)|}{y^{3/2}}\right\},$$

where $\tau=\sqrt{\psi}-\sqrt{y}$. In particular,

$$\begin{aligned}|\psi_y(\bar{z})-1|\leq{}&\left|\sqrt{\psi(\bar{z})/\bar{y}}-1\right|+\text{const.}\max_{z\in\bar{E}}\sqrt{y/\bar{y}}\,\left|\sqrt{\psi(z)/y}-1\right|\\&+\text{const.}\max_{z\in\bar{E}}(\bar{y}/y)^{3/2}\,|N(z)|,\end{aligned} \tag{4.7}$$

where $y \in [\bar{y}/2, 3\bar{y}/2]$ if $z=(x, y) \in \bar{E}$. The use of Theorem 26 and the fact that $N(z) \to 0$ as $|z| \to \infty$, yields

$$|u(\bar{z})-1| = |\psi_y(\bar{z})-1| \to 0 \quad \text{as} \quad |\bar{z}| \to \infty.$$

A similar argument gives $|v(\bar{z})| = |\psi_x(\bar{z})| \to 0$ as $|\bar{z}| \to \infty$. q.e.d.

4.2. Leray's solution w_L is non-trivial

In this section, we shall prove that the hypotheses of Theorem 23 are satisfied for symmetric flow in Ω_R. The use of this with Theorem 24 shows that the Leray solution w_L is non-trivial in this case.

In our arguments, $R \geq 2$ will be fixed, and it will be assumed as usual that $\Gamma = \partial\Omega \subset \{|z|<1\}$. We shall write (w, p) for the solution (w_R, p_R) in Ω_R, and similarly for other quantities. The sets on which ω is one-signed played an important role in section 2, and they will do so here also. Set

$$\tilde{\Omega}_R = \Omega_R \cap \{|z|>1\} = \{1<|z|<R\}.$$

Since $\omega(x, y)$ is an odd function of y, ω is not one-signed in $\tilde{\Omega}_R$.

For each $z \in \tilde{\Omega}_R$ at which $\omega(z)>0$, let $U_+(z)$ denote the maximal connected subset of $\tilde{\Omega}_R$ containing z and on which $\omega>0$. We define $U_-(z)$ similarly. Note that $U_+(z)$ or $U_-(z)$ are only defined when $z \in \tilde{\Omega}_R$ and $\omega(z) \neq 0$.

If V denotes a $U_+(z)$ or $U_-(z)$, then

$$V \text{ is simply-connected.} \tag{4.8}$$

Indeed, say V is a U_+ and let J be a closed Jordan curve in V. Since the flow is symmetric, $\omega=0$ on $\{y=0\}$, whence $J \subset \{y>0\}$ or $J \subset \{y<0\}$. It follows that $\operatorname{int} J \subset \tilde{\Omega}_R$, whence $\omega>0$ in $\operatorname{int} J$ by the maximum principle. This gives $\operatorname{int} J \subset V$ for any J so that V is simply-connected.

Remark 4. As we shall see later, (4.8) is the crucial result needed to satisfy the hypotheses of Theorem 23. If it holds for a problem involving general flow, then the following results of this section hold. For example, if $\Gamma = \partial\Omega$ has only one component and there is a level-set of $\omega=0$ connecting Γ to $\{|z|=R\}$, then (4.8) holds.

We wish to prove the existence of Jordan arcs $M_i \subset \bar{\tilde{\Omega}}_R$, $i=1, 2$, such that

$$M_i = \{(x_i(s), y_i(s)) : s \in [0, L_i]\} \text{ and } (x_i(0), y_i(0)) \in \{|z|=1\}$$

while $(x_i(L_i), y_i(L_i)) \in \{|z|=R\}$. In addition, $\omega(M_i)=0$ and the maps $s \mapsto \Phi(x_i(s), y_i(s))$ are monotone decreasing and increasing, respectively, on $[0, L_i]$ for $i=1,2$. We shall allow $M_1 \cap M_2 \neq \emptyset$. The proof of Theorem 10 ensures that ∂V, V a U_+ or U_-, is locally a Jordan arc, and since V is simply-connected, it follows that ∂V is a closed Jordan arc in $\bar{\bar{\Omega}}_R$. Assume that for some $\bar{z}$, $\partial V \cap \{|z|=R\} \neq \emptyset$ and $\partial V \cap \{|z|=1\} \neq \emptyset$, where V is either a $U_+(\bar{z})$ or $U_-(\bar{z})$. There are then two disjoint arcs of ∂V contained in $\bar{\Omega}_R$ which approach $\{|z|=1\}$ and $\{|z|=R\}$ as their endpoints are approached, and on which ω vanishes. The arguments in the proof of Theorem 11 then give the existence of the desired M_i, $i=1,2$. Hence if there exists V, a U_+ or U_-, such that

$$\partial V \cap \{|z|=R\} \neq \emptyset \quad \text{and} \quad \partial V \cap \{|z|=1\} \neq \emptyset \tag{4.9}$$

then the hypotheses of Theorem 23 are satisfied.

The remaining problem is to prove the existence of the arcs M_i when (4.9) is false for all $\bar{z} \in \bar{\Omega}_R$ with $\omega(\bar{z}) \neq 0$. *We shall assume this is the case unless stated otherwise.* We begin by noting that there are only a finite number of distinct sets U_+ or U_-. If not, then there would exist a sequence $\{z_n\}$, $z_n \in U_n$, with $\omega(z_n)>0$, say, and $U_n \cap U_k = \emptyset$ if $n \neq k$. Here $U_n = U_+(z_n)$. Without loss of generality, we may assume that the sequence converges to a point $\bar{z} \in \bar{\bar{\Omega}}_R$ at which ω vanishes. However, Lemma 6 shows that the level-set of $\omega=0$ in a neighborhood of $\bar{z}$ consists of a finite number of arcs emanating from $\bar{z}$, and this is a contradiction.

Let

$$A = \{z \in \bar{\Omega}_R : \partial U_+(z) \cap \{|z|=1\} \neq \emptyset \text{ or } \partial U_-(z) \cap \{|z|=1\} \neq \emptyset\},$$

and

$$B = \{z \in \bar{\Omega}_R : \partial U_+(z) \cap \{|z|=R\} \neq \emptyset \text{ or } \partial U_-(z) \cap \{|z|=R\} \neq \emptyset\}.$$

(If $z \in \bar{\Omega}_R$ and $\omega(z)>0$, say, then z belongs to either A or B. Indeed, if this were not the case, then $\omega=0$ on $\partial U_+(z)$, whence $\omega \equiv 0$ in $U_+(z)$ by the maximum principle.) Clearly $A \cap B = \emptyset$ by our assumption above. Since we know the structure of $\{\omega=0\}$ in a neighborhood of any point in $\partial\bar{\Omega}_R = \{|z|=1\} \cup \{|z|=R\}$, it is immediate that $\{|z|=1\} \subset \bar{A}$, $\{|z|=R\} \subset \bar{B}$, and that

$$\operatorname{dist}(\{|z|=1\}, \bar{B}),\ \operatorname{dist}(\{|z|=R\}, \bar{A}) > 0. \tag{4.10}$$

LEMMA 28. (a) $\bar{A} \cap \bar{B} \neq \emptyset$ *and* $\bar{A} \cap \bar{B} \subset \bar{\Omega}_R$.

(b) *If* $\bar{z} \in \bar{A} \cap \bar{B}$, *then* $\omega(\bar{z})=0$, *and there is a Jordan arc* J_1 *emanating from* $\bar{z}$ *such that* $J_1 \subset \bar{A} \cap \bar{B}$.

Proof. (a) Since $\bar{\bar{\Omega}}_R$ is connected and equals $\bar{A}\cup\bar{B}$, it follows that $\bar{A}\cap\bar{B}\neq\emptyset$. If $z_1\in\bar{A}\cap\bar{B}$ and $z_1\in\{|z|=1\}$, then $\bar{B}\cap\{|z|=1\}\neq\emptyset$, which contradicts (4.10). A similar argument holds if $z_1\in\{|z|=R\}$.

(b) If $\omega(\bar{z})\neq 0$, then either $\bar{z}\in A$ or $\bar{z}\in B$, whence $\bar{z}\notin\bar{A}\cap\bar{B}$. Hence, $\omega(\bar{z})=0$. We consider two cases; first, assume that the level-set of $\omega=0$ *through* $\bar{z}$ is locally a Jordan arc J_2. The function ω changes sign as J_2 is crossed, and since $\bar{z}\in\bar{A}\cap\bar{B}$ the points on one side belong to A and those on the other to B. Hence, $J_2\subset\bar{A}\cap\bar{B}$.

We now assume that the level-set of $\omega=0$ cannot be represented as a Jordan arc through $\bar{z}$ in a neighborhood of $\bar{z}$. We necessarily have $\nabla\omega(\bar{z})=0$. Lemma 6 ensures that in a punctured neighborhood N about $\bar{z}$, the set $\{\omega=0\}$ consists of $2m$, $m>1$, real-analytic arcs emanating from $\bar{z}$. We may assume that N is so small that $\{z\in N:\omega(z)\neq 0\}$ has $2m$ components. Each of these components is either in A or B, and the assumption that $\bar{z}\in\bar{A}\cap\bar{B}$ shows that at least one component is in A and at least one in B. Since ω changes sign across the arcs, there is at least one arc belonging to $\bar{A}\cap\bar{B}$. q.e.d.

Since the zeros of $|\nabla\omega|$ are isolated in Ω_R (Lemma 6), it follows from Lemma 28 that there is a point $\bar{z}\in\bar{A}\cap\bar{B}$ such that $\nabla\omega(\bar{z})\neq 0$. We shall take $\bar{y}\geq 0$ in the representation $\bar{z}=(\bar{x},\bar{y})$ since a similar argument holds if $\bar{y}\leq 0$. The level-set $\{\omega=0\}$ is locally a real-analytic arc J through $\bar{z}$ in some small neighborhood N centred at $\bar{z}$. The set $\{z\in N:\omega(z)\neq 0\}$ has precisely two components N_1 and N_2 in N, and we may assume that $\omega>0$ in N_1 and $\omega<0$ in N_2. Since $\bar{z}\in\bar{A}\cap\bar{B}$, one of the components is a subset of A while the other belongs to B. Assume that $N_1\subset A$ so that $N_2\subset B$. (The arguments to follow also hold if $N_1\subset B$ and $N_2\subset A$.) It follows that $N_1\subset U_+(z_1)$ and $N_2\subset U_-(z_2)$ for some points $z_1\in A$ and $z_2\in B$. Note that $J\subset\partial U_+(z_1)\cap\partial U_-(z_2)$.

Now $U_-(z_2)$ is simply-connected and $\partial U_-(z_2)$ is a closed Jordan curve in $\bar{\bar{\Omega}}_R$. The component J_3 of $\partial U_-(z_2)$ containing J and lying in $\bar{\Omega}_R$ has a representation

$$J_3=\{(x_3(s),y_3(s)):s\in(0,L_3)\}.$$

Note that $(x_3(s),y_3(s))\to\{|z|=R\}$ as $s\to 0,L_3$. We extend J_3 to be defined on the closed interval, set $p_1=(x_3(0),y_3(0))$, $p_2=(x_3(L_3),y_3(L_3))$, and may assume that the independent variable s measures arc-length along J_3 from p_1. As in the proof of Theorem 11, the strong maximum principle for ω on J_3 combined with (1.14)–(1.15) ensures that γ is monotone as the arc J_3 is transversed from p_1 to p_2. Upon replacing s by L_3-s if necessary, we may assume that γ is monotone increasing as J_3 is transversed from p_1 to

p_2. Note that $\gamma=\Phi$ on J_3 since $\omega=0$ on $\bar{\Omega}_R \cap \partial U_-(z_2)$. Choose a reference point $m \in J$ such that $\gamma(m)=\Phi(m)>\gamma(\bar{z})=\Phi(\bar{z})$.

We may apply the arguments above to $U_+(z_1)$. We let the component of $\partial U_+(z_1) \cap \bar{\Omega}_R$ containing J have the representation

$$J_4 = \{(x_4(s), y_4(s)) : s \in (0, L_4)\},$$

and may assume that $(x_4(s), y_4(s))$ converges to $q_1, q_2 \in \{|z|=1\}$ as $s \to 0$, L_4, respectively. The function γ is monotone as J_4 is transversed from q_1 to q_2. Assume that as we go from q_1 to q_2, the point $\bar{z}$ is reached before m. Since $\gamma(m)>\gamma(\bar{z})$, it follows that $\gamma=\Phi$ is monotone increasing as we go from q_1 to q_2. On the other hand, if m is reached before $\bar{z}$, then we replace s by L_4-s to reverse the direction. Hence, we may assume Φ is monotone increasing as J_4 is transversed from q_1 to q_2. We set

$$\bar{z} = \begin{cases} (x_3(s_3), y_3(s_3)) & \text{for some } s_3 \in (0, L_3), \\ (x_4(s_4), y_4(s_4)) & \text{for some } s_4 \in (0, L_4). \end{cases}$$

Then the function

$$(x_2(s), y_2(s)) = \begin{cases} (x_4(s), y_4(s)) : s \in [0, s_4], \\ (x_3(s-s_4+s_3), y_3(s-s_4+s_3)) : s \in [s_4, L_3+s_4-s_3] \end{cases}$$

defines a Jordan arc going from $q_1 \in \{|z|=1\}$ to $p_2 \in \{|z|=R\}$ along which $\gamma=\Phi$ is monotone increasing.

The function

$$(x_1(s), y_1(s)) = \begin{cases} (x_4(L_4-s), y_4(L_4-s)) : s \in [0, L_4-s_4], \\ (x_3(s_3+L_4-s_4-s), y_3(s_3+L_4-s_4-s) : s \in [L_4-s_4, L_4-s_4+s_3] \end{cases}$$

defines a Jordan arc going from $q_2 \in \{|z|=1\}$ to $p_1 \in \{|z|=R\}$ along which Φ is monotone decreasing.

We have shown that if the flow is symmetric, then the hypotheses of Theorem 23 are satisfied whether or not (4.9) holds. We summarize this in the following

THEOREM 29 (Symmetric flow). (a) *For each $R \geq 2$, the hypotheses of Theorem* 23 *are satisfied.*

(b) $\inf_{R \geq 2} \iint |\nabla w_R|^2 > 2$.

(c) *If $\{R_n\}$ is an unbounded sequence such that (w_{R_n}, p_{R_n}) converges on compact subsets of $\bar{\Omega}$ to (w_L, p_L), then the Leray solution (w_L, p_L) to* (1.1)–(1.3) *is non-trivial, and there exists a constant vector w_∞ such that $|w(z)-w_\infty| \to 0$ as $|z| \to \infty$.*

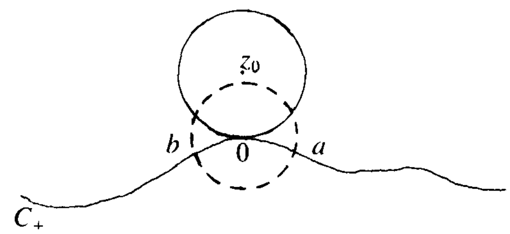

Figure A 1. The solid circle is that of radius R centred at $z_0=(0,R)$, and the dotted circle that of radius R_1, centred at 0. The points a and b have polar coordinates (R_1,θ_1) and (R_1,θ_2), respectively, in the (r',θ') system.

Proof. Part (a) was shown before the statement of the theorem, and (b) follows from Theorem 23. Theorem 24 proves that (w_L, p_L) is non-trivial, and the pointwise behavior of w_L at infinity was proved in Theorem 27. q.e.d.

Appendix

The proof of Theorem 21. Let S be as in Lemma 20 and for each $T>S$ set

$$G(T)=\left\{(x,y): x>2T, |y|<\frac{x}{4}, \psi(x,y)>0\right\}.$$

We shall prove that

$$\sup_{z\in G(T)} \big||w(z)|-|w_\infty|\big| \to 0 \quad \text{as} \quad T\to\infty, \tag{A 1}$$

where $w_\infty=(1,0)$. A similar argument holds if $\psi(x,y)<0$ or if $x<-2T$. Let $\delta>0$ be sufficiently small, but fixed. For any $z_0\in G(T)$, let $R=R(z_0)=\operatorname{dist}(z_0, \{z\in\Omega: \psi(z)=0\})$. Lemma 20 and our assumption that $x_0>2T$ ensures that $R=|z_0-\tilde{z}|$ for some $\tilde{z}\in C_+$, this set being defined in Lemma 20. Since $|w(\tilde{z})|-|w_\infty|\to 0$ as $|z_0|\to\infty$ by Lemma 20 and $|\nabla w(z)|=O(|z|^{-3/4}\log|z|)$ by Theorem 2, we may restrict attention to $R\geq 1$.

We take new coordinates $z'=(x',y')$ centred at $\tilde{z}$ and such that z_0 has the representation $(0,R)$ in this coordinate system. Let (r',θ') be polar coordinates, and let us write $\tilde{w}(x',y')$, $\tilde{w}(z')$ or $\tilde{w}(r',\theta')$ for w represented in this coordinate system (see Figure A 1). We do the same for other functions. We claim that there exists

$$R_1=R_1(z_0)\in(3R/4, (3/4+\delta)R)$$

such that

$$\int_0^{2\pi} \left|\frac{\partial}{\partial\theta'} \tilde{w}(R_1, \theta')\right|^2 d\theta' \leqslant \delta, \quad z_0 \in G(T). \tag{A 2}$$

Indeed, if we choose T so large that

$$\iint_{T \leqslant |z| < \infty} |\nabla w|^2 \leqslant \tfrac{2}{3}\delta^2,$$

then

$$\begin{aligned}
\tfrac{2}{3}\delta^2 &\geqslant \int_0^R r'\,dr' \int_0^{2\pi} \left|\frac{1}{r'}\frac{\partial}{\partial\theta'} \tilde{w}(r', \theta')\right|^2 d\theta' \\
&\geqslant \int_{3R/4}^{(3/4+\delta)R} \frac{dr'}{r'} \int_0^{2\pi} \left|\frac{\partial}{\partial\theta'} \tilde{w}(r', \theta')\right|^2 d\theta' \\
&= \log(1+\tfrac{4}{3}\delta) \int_0^{2\pi} \left|\frac{\partial}{\partial\theta'} \tilde{w}(R_1, \theta')\right|^2 d\theta'
\end{aligned}$$

for some $R_1 \in (3R/4, (3/4+\delta)R)$. This implies (A 2) when δ is sufficiently small.

We assume that θ' varies on the interval $(-3\pi/2, \pi/2]$, so that $\theta'=\pi/2$ corresponds to the positive y'-axis. Let $\theta_1, \theta_2 \in (-3\pi/2, \pi/2)$ be the largest and smallest numbers, respectively, such that $\tilde{\psi}(R_1, \cdot)$ vanishes. Set

$$\alpha = \alpha(z_0) = \tilde{u}(R_1, \theta_1), \quad \beta = \beta(z_0) = \tilde{v}(R_1, \theta_1),$$

and let $\tau \in (-3\pi/2, \pi/2]$ be such that

$$\cos\tau = \alpha/\sqrt{\alpha^2+\beta^2}, \qquad \sin\tau = -\beta/\sqrt{\alpha^2+\beta^2}. \tag{A 3}$$

Since $\tilde{\psi}(R_1, \theta_1)=0$ and $|\nabla\psi|^2-2\psi\Delta\psi \to |w_\infty|^2=1$ at infinity (Theorem 14), we may choose T so large that

$$|\sqrt{\alpha^2+\beta^2}-1| \leqslant \sqrt{\delta}, \quad z_0 \in G(T). \tag{A 4}$$

Next we must estimate the variation of $\tilde{\psi}(R_1, \theta')$ with respect to θ':

$$\begin{aligned}
\left|\frac{1}{R_1}\frac{\partial}{\partial\theta'} \tilde{\psi}(R_1, \theta') - \cos(\tau+\theta')\right| &= |(\sin\tau + \tilde{v}(R_1, \theta'))\sin\theta' + (\tilde{u}(R_1, \theta') - \cos\tau)\cos\theta'| \\
&\leqslant |\tilde{w}(R_1, \theta') - (\cos\tau, -\sin\tau)|
\end{aligned}$$

$$\leq |\tilde{w}(R_1,\theta')-\tilde{w}(R_1,\theta_1)|+|\tilde{w}(R_1,\theta_1)-(\cos\tau,-\sin\tau)|$$
$$\leq \int_{-3\pi/2}^{\pi/2}\left|\frac{\partial}{\partial\theta'}\tilde{w}(R_1,\theta')\right|d\theta'+\left|(\alpha,\beta)-(\alpha,\beta)/\sqrt{\alpha^2+\beta^2}\right|$$
$$\leq \sqrt{2\pi}\,\sqrt{\delta}+\left|\sqrt{\alpha^2+\beta^2}-1\right|\leq 4\sqrt{\delta} \tag{A 5}$$

by (A 2) and (A 4). Since $\tilde{\psi}(R_1,\theta_1)=0$, equation (A 5) yields

$$\left|\frac{1}{R_1}\tilde{\psi}(R_1,\theta')-\sin(\tau+\theta')+\sin(\tau+\theta_1)\right|\leq 2\pi\cdot 4\sqrt{\delta}\leq 26\sqrt{\delta},\quad \theta'\in\left(-\frac{3\pi}{2},\frac{\pi}{2}\right]. \tag{A 6}$$

We again choose new coordinates $z''=(x'',y'')$ centred at z_0 with $x''=x'$ and $y''=y'-R$. For ease of notation, we shall write $z=(x,y)$ for $z''=(x'',y'')$ and (r,θ) for (r'',θ''). We shall revert to the z'' coordinate system at the final important estimates. The type of estimate which gave (A 2) allows us to assume that

$$\int_{-3\pi/2}^{\pi/2}\left|\frac{\partial}{\partial\theta}w(R_2,\theta)\right|^2 d\theta\leq\delta \tag{A 7}$$

for some $R_2=R_2(z_0)\in(R/4,(1/4+\delta)R)$. The circle of radius R_2 centred at $z=z''=0$ intersects that of radius R_1 centred at $z'=0$ at some point $(R_2,\tilde{\theta})$ in the (r,θ) coordinate system. Equation (A 5) gives $|w(R_2,\tilde{\theta})-(\cos\tau,-\sin\tau)|\leq 4\sqrt{\delta}$, and combining this with (A 7) yields

$$|w(R_2,\theta)-(\cos\tau,-\sin\tau)|\leq\sqrt{\delta}\,(4+\sqrt{2\pi})\leq 7\sqrt{\delta} \tag{A 8}$$

for all $\theta\in(-3\pi/2,\pi/2]$. Hence, if we define the disc

$$D=D(z_0)=\{z:|z|<R_2\}, \tag{A 9}$$

then $||w|^2-1|\leq 49\delta+14\sqrt{\delta}\leq\text{const.}\sqrt{\delta}$ on ∂D.

Before proceeding, let us recall the size of certain terms:

$$R(z_0)=\operatorname{dist}(z_0,C_+),\quad R_1\in(3R/4,(3/4+\delta)R),\quad R_2\in(R/4,(1/4+\delta)R). \tag{A 10}$$

Since we have a good estimate for w on ∂D by (A 8), we can derive an estimate for ψ there after first estimating $\psi(R_2,\tilde{\theta})$.

Equation (A 6) gives

$$|\tilde{\psi}(x',y')-x'\sin\tau-y'\cos\tau+R_1\sin(\tau+\theta_1)|\leq 26R_1\sqrt{\delta}$$

and setting $x=x'$, $y=y'-R$ and restricting this estimate to the point of intersection yields

$$|\psi(R_2,\tilde{\theta})-R_2\sin(\tau+\tilde{\theta})-R\cos\tau+R_1\sin(\tau+\theta_1)|\leq 26\sqrt{\delta}\,R_1. \tag{A 11}$$

Equation (A 8) gives

$$\left|\frac{1}{R_2}\frac{\partial}{\partial\theta}\psi(R_2,\theta)-\cos(\tau+\theta)\right|\leq 7\sqrt{\delta},\quad \theta\in\left(-\frac{3\pi}{2},\frac{\pi}{2}\right], \tag{A 12}$$

and combining this with (A 11) proves the estimate

$$|\psi(R_2,\theta)-J(R_2,\theta)|\leq 70\sqrt{\delta}\,R,\quad \theta\in\left(-\frac{3\pi}{2},\frac{\pi}{2}\right], \tag{A 13}$$

where

$$J(r,\theta)=r\sin(\theta+\tau)+R\cos\tau-R_1\sin(\tau+\theta_1). \tag{A 14}$$

In the $z=z''=(x,y)$ coordinate system, we have

$$J(x,y)=x\sin\tau+y\cos\tau+R\cos\tau-R_1\sin(\tau+\theta_1).$$

The main technical problem facing us is the need for a lower bound on $\bar{D}$ for the complicated expression J. We shall show that $J(z)\geq R/16$, $z\in\bar{D}$ whenever δ is sufficiently small. The use of this in (A 13) yields $|\psi/J-1|\leq$ const. $\sqrt{\delta}$ on ∂D, and we shall then use the type of argument for Theorem 26 to show that $|\psi/J-1|\leq$ const. $\sqrt{\delta}$ on $\bar{D}$. The method of proving Theorem 27 will then be used with J playing the role of y. Equation (A 14) shows that

$$\begin{aligned}\min_{z\in\bar{D}}J(z)&=-R_2+R\cos\tau-R_1\sin(\tau+\theta_1)\\&=R\left\{\cos\tau-\frac{R_1}{R}\sin(\tau+\theta_1)-\frac{R_2}{R}\right\}\\&\geq R\left\{\cos\tau-\frac{3}{4}\sin(\tau+\theta_1)-\frac{1}{4}-2\delta\right\}\end{aligned} \tag{A 15}$$

by (A 10). In order to show that the term in brackets is bounded by 1/16, we shall need the following technical

LEMMA A 1. *For each sufficiently small* $\delta>0$, *let* $s\in(3/4,3/4+\delta)$ *and set* $\mu=\tan^{-1}(s/\sqrt{4-s^2})$. *Assume that*

$$0\leq\sin(\tau+\theta')-\sin(\tau+\theta_1)+28\sqrt{\delta} \tag{A 16}$$

for all $\theta' \in (\mu, \pi-\mu)$, *where* $\theta_1 \notin (\mu, \pi-\mu)$ *and* τ *are given. Then*

$$\cos\tau - \tfrac{3}{4}\sin(\tau+\theta_1) - \tfrac{1}{4} - 2\delta \geq \tfrac{1}{2} - \sin\mu + O(\sqrt{\delta}) \to \tfrac{1}{8} \quad \text{as} \quad \delta \to 0.$$

Proof. Set

$$B(\tau) = \min_{\theta' \in [\mu, \pi-\mu]} \sin(\tau+\theta').$$

so that $0 \leq B(\tau) - \sin(\tau+\theta_1) + 28\sqrt{\delta}$, whence

$$\begin{aligned} \cos\tau - \tfrac{3}{4}\sin(\tau+\theta_1) - \tfrac{1}{4} - 2\delta &\geq \cos\tau - \tfrac{1}{4} - 2\delta - \tfrac{3}{4}(B(\tau) + 28\sqrt{\delta}) \\ &\geq \cos\tau - \tfrac{3}{4}B(\tau) - \tfrac{1}{4} - 23\sqrt{\delta} \equiv \varphi(\tau). \end{aligned}$$

Note that

$$B(\tau) = \begin{cases} \sin(\mu-\tau), & \tau \in (0, \pi/2+\mu), \\ -1, & \tau \in [\pi/2+\mu, 3\pi/2-\mu], \\ \sin(\mu+\tau), & \tau \in (3\pi/2-\mu, 2\pi]. \end{cases}$$

For the case $\tau \in [\pi/2+\mu, 3\pi/2-\mu]$, equation (A 16) yields $\sin(\tau+\theta_1) \leq -1 + 28\sqrt{\delta}$, whence $\tau+\theta_1 \in [-\pi/2-\varrho, -\pi/2+\varrho]$ where $\varrho = \cos^{-1}(1-28\sqrt{\delta})$. Since $\theta_1 \notin (\mu, \pi-\mu)$, it follows that τ can only belong to $[\pi/2+\mu, \pi/2+\mu+\varrho]$ or $[3\pi/2-\mu-\varrho, 3\pi/2-\mu]$. Define I to be the union of $[0, \pi/2+\mu+\varrho]$ and $[3\pi/2-\mu-\varrho, 2\pi]$, and note that we may restrict attention to $\tau \in I$. An easy calculation gives

$$\min_{\tau \in I} \varphi(\tau) = \tfrac{1}{2} - \sin\mu + O(\sqrt{\delta})$$

and this proves the lemma. q.e.d.

In order to apply this lemma to (A 15), we must show that (A 16) is satisfied. In the (x', y') coordinate system, we know that $\tilde{\psi} > 0$ by construction in the disc $\tilde{D} = \{(x', y') : (x')^2 + (y'-R)^2 < R^2\}$. If we take a circle of radius $R_1 \in (3R/4, (3/4+\delta)R)$ centred at $z' = 0$, then points (R_1, θ') will belong to $\tilde{D}$ if $\theta' \in (\mu, \pi-\mu)$, where $\tan\mu = (4R^2/R_1^2 - 1)^{-1/2}$. Indeed, at the points of intersection of these two circles, we have

$$R^2 = (x')^2 + (y'-R)^2, \quad (x')^2 + (y')^2 = (R_1)^2,$$

whence $|y'/x'| = \tan\mu$. Hence, $\tilde{\psi}(R_1, \theta') > 0$, $\theta' \in (\mu, \pi-\mu)$, and the use of this in (A 6) yields

$$0 \leq \sin(\tau+\theta') - \sin(\tau+\theta_1) + 26\sqrt{\delta}, \quad \theta' \in (\mu, \pi-\mu).$$

This shows that (A 16) holds. Since $\tilde{\psi}(R_1, \theta_1)=0$, we know that $\theta_1 \notin (\mu, \pi-\mu)$, and so we have proved

LEMMA A2. *If* $\delta>0$ *is sufficiently small, then*

$$\min_{z\in\bar{D}} J(z) \geq R/16.$$

THEOREM A3. *If* $\delta>0$ *is sufficiently small, then*

$$\max_{z\in\bar{D}} \left|\frac{\psi(z)}{J(z)}-1\right| \leq \text{const.}\sqrt{\delta}.$$

Proof. If we set $\psi(x,y)=J(x,y)(1+S(x,y))$, then a calculation yields

$$\begin{aligned}|\nabla\psi|^2-2\psi\Delta\psi &= (1+S)^2+J^2|\nabla S|^2+2J(1+S)(S_x\sin\tau+S_y\cos\tau)\\ &\quad -2\psi\{2S_x\sin\tau+2S_y\cos\tau+J\Delta S\}.\end{aligned} \tag{A 17}$$

Upon choosing T sufficiently large in $G(T)$, we may ensure that $\big||\nabla\psi|^2-2\psi\Delta\psi-1\big|\leq\delta$ in $\bar{D}$. If S has a local positive maximum at $\hat{z}\in D$, then (A 17) gives $(1+S(\hat{z}))^2\leq 1+\delta$, whence $0<S(\hat{z})\leq\delta/2$. If S has a local negative minimum at $\hat{z}\in D$ then $0>S(\hat{z})\geq-\delta$. If we combine these estimates with (A 13) and Lemma A2, then the proof is complete. q.e.d.

We return to the equation $|\nabla\psi|^2-2\psi\Delta\psi=1+O(\delta)$ in $D(z_0)$ and recall that the left-hand side equals $-4\psi^{3/2}\Delta\sqrt{\psi}$. It follows that

$$\begin{aligned}\Delta(\sqrt{\psi}-\sqrt{J}) &= \frac{1+O(\delta)}{-4\psi^{3/2}}+\frac{1}{4J^{3/2}}\\ &= \frac{-1}{4\psi^{3/2}}\left\{1+O(\delta)-\left(\frac{\psi}{J}\right)^{3/2}\right\}\\ &= \frac{N(z)}{4R^{3/2}} \quad \text{in} \quad D(z_0).\end{aligned}$$

where

$$\max_{z\in\bar{D}}|N(z)|\leq\text{const.}\sqrt{\delta}.$$

The proof of Theorem 27 is now applicable, and we conclude that

$$|\psi_x(0,0)-\sin\tau|,\ |\psi_y(0,0)-\cos\tau|\leq\text{const.}\sqrt{\delta}. \tag{A 18}$$

If we write this in terms of the z' coordinate system, then

$$\left|\frac{\partial\tilde{\psi}}{\partial x'}(0,R)-\sin\tau\right|,\ \left|\frac{\partial\tilde{\psi}}{\partial y'}(0,R)-\cos\tau\right| \leq \text{const.}\sqrt{\delta}.$$

Returning finally to the physical coordinate system, we have

$$||w(z_0)|-1| \leq \text{const.}\sqrt{\delta},$$

whence,

$$\limsup_{T\to\infty}\sup_{z\in G(T)} ||w(z)|-|w_\infty|| \leq \text{const.}\sqrt{\delta}.$$

Since δ may be taken arbitrarily small, we have proved (A 1) and thereby Theorem 21 (a).

We now prove Theorem 21 (c). If $u(z)\to 1$, $v(z)\to 0$ as $|z|\to\infty$, $z\in C_+$, then the rotation necessary to go from the physical coordinates to the z' coordinates tends to zero as $|z_0|\to\infty$, $z_0\in G(T)$. It follows from (A 3) that $\cos\tau(z_0)\to 1$ and $\sin\tau(z_0)\to 0$ as $|z_0|\to\infty$, and the desired result follows immediately.

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Received May 13, 1987
Received in revised form November 4, 1987