BOUNDARY THEOREMS FOR A FUNCTION MEROMORPHIC IN THE UNIT CIRCLE.

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Introduction.

1. Let the function $w = f(z)$ be uniform and meromorphic in the unit circle $|z| < 1$. We adopt no general hypothesis regarding the unit circumference $|z| = 1$ of which every point may be a regular point or a pole of $f(z)$; or some points may be essential singularities; or every point may be an essential singularity of $f(z)$. For our purposes it is convenient to include the poles in the class of regular points of $f(z)$ for then the value ∞ is in no way exceptional.

In order to study and describe the behaviour of $f(z)$ near the circumference $|z| = 1$ we associate with $f(z)$ certain sets of values which are defined as follows.

(i) **The Cluster Set** $C(f)$. $a \in C(f)$ if there is a sequence $\{z_n\}$, $|z_n| < 1$, such that $\lim\limits_{n\to\infty} |z_n| = 1$ and $\lim\limits_{n\to\infty} f(z_n) = a$. An equivalent definition, which is applicable to a general domain, is that there is a point z_0 of the boundary $|z| = 1$ such that $\lim\limits_{n\to\infty} z_n = z_0$ and $\lim\limits_{n\to\infty} f(z_n) = a$. We call a a *cluster value of* $f(z)$. The complementary set of *non-cluster values* with respect to the closed complex plane is denoted by $\complement C(f)$. The frontier of $C(f)$ is denoted by $\mathfrak{F} C(f)$. We shall throughout use the notation $U(a, \varepsilon)$ for the ε-neighbourhood of a, i.e. the set of points w satisfying $|w - a| < \varepsilon$. Then $a \in \mathfrak{F} C(f)$ if and only if, for all $\varepsilon > 0$, $U(a, \varepsilon)$ contains at least one point of both the sets $C(f)$ and $\complement C(f)$.

Evidently $C(f)$ is closed so that $\complement C(f)$ is open and consists, if it is not void, of a finite or enumerable set of open domains.

(ii) **The Range of Values** $R(f)$. $a \in R(f)$ if there is a sequence $\{z_n\}$, $|z_n| < 1$, such that $\lim\limits_{n\to\infty} |z_n| = 1$ and $f(z_n) = a$ for all values of n. As for the cluster set, $\lim\limits_{n\to\infty} z_n = z_0$ for some z_0 on $|z| = 1$ and $f(z_n) = a$ is an equivalent condition. The complementary set with respect to the closed complex plane is denoted by $\complement R(f)$ and may be called the *Excluded Range*. A value $b \in \complement R(f)$ is an *excluded range value of* $f(z)$. The frontier of $R(f)$ is denoted by $\mathfrak{F} R(f)$.

(iii) **The Asymptotic Set** $\Gamma(f)$. $a \in \Gamma(f)$ if there is a continuous simple path $z = z(t)$, $\alpha < t < 1$, such that $|z(t)| < 1$; $\lim_{t \to 1} |z(t)| = 1$ and $\lim_{t \to 1} f(z(t)) = a$. We call a an *asymptotic value of* $f(z)$. The complementary set of *non-asymptotic values of* $f(z)$ is denoted by $\complement \Gamma(f)$.

(iv) **The Value Set** $X(f)$. $a \in X(f)$ if there is a point $z(a)$, where $|z(a)| < 1$, such that $f(z(a)) = a$. The frontier of $X(f)$ is denoted by $\mathfrak{F} X(f)$.

We see at once that $X(f)$ is open. For if $a \in X(f)$ we can find $\varepsilon > 0$ and $0 < \eta < 1 - |z(a)|$ such that every value in $U(a, \varepsilon)$ is taken by $f(z)$ in the circle $|z - z(a)| < \eta$ so that a is an internal point of $X(f)$. $\complement X(f)$ is therefore closed and $\mathfrak{F} X(f) \subseteq \complement X(f)$.

In the usual notation we denote closures by $\overline{C}(f)$, $\overline{\complement C}(f)$, $\overline{R}(f)$, $\overline{\complement R}(f)$, $\overline{\Gamma}(f)$ etc. and derived sets by $C'(f)$, etc.

Evidently

(1.1) $$R(f) \subseteq X(f),$$

(1.2) $$\overline{R}(f) \subseteq C(f)$$

and

(1.3) $$\overline{\Gamma}(f) \subseteq C(f).$$

Also

(1.4) $$C(f) \subseteq \overline{X}(f);$$

for if $a \in \complement \overline{X}(f)$ then, for some $\varepsilon > 0$, $|f(z) - a| > \varepsilon$ in $|z| < 1$ so that $a \in \complement C(f)$ and hence $\complement \overline{X}(f) \subseteq \complement C(f)$.

The foregoing definitions relate to the behaviour of $f(z)$ in the large either at the whole boundary $|z| = 1$ or in the whole domain $|z| < 1$ without specifying any particular point of the boundary or domain. In Part I of this paper we shall study these sets systematically and, in particular, we shall investigate the relations between the frontier sets $\mathfrak{F} C(f)$, $\mathfrak{F} R(f)$, $\mathfrak{F} X(f)$ and the excluded range $\complement R(f)$ on the one hand and the asymptotic set $\Gamma(f)$ on the other. By way of illustration we may recall corresponding theorems for the case of a function $F(z)$ meromorphic and non-rational in the finite plane $|z| < \infty$. For such a function the sets $C(F)$, $R(F)$ and

$\Gamma(F)$ are defined as above except that we put $\lim_{n\to\infty} z_n = \infty$ in the case of $C(F)$ and $R(f)$ and $\lim_{t\to\infty} z(t) = \infty$ in the case of $\Gamma(F)$. The three classical theorems of Weierstrass, Picard and Iversen can now be stated in terms of these sets as follows.

Theorem A (Weierstrass) $\complement C(F)$ *is void.*

Theorem B (Picard) $\complement R(F)$ *contains at most two values.*

From this theorem it follows at once that $\mathfrak{J} R(F) = \complement R(F)$.

Theorem C (Iversen) $\complement R(F) = \mathfrak{J} R(F) \subseteq \Gamma(f)$.

Theorem A is, of course, implied by the deeper and more difficult Theorem B. All three theorems apply also to the case of a function $F(z)$ having an isolated essential singularity, the sets C, R and Γ being defined in relation to the neigh bourhood of the singularity.

The corresponding system of theorems for a function $f(z)$ meromorphic in $|z| < 1$ is closely analogous. Theorem A holds for $f(z)$ under the condition that $T(r, f)$ is unbounded. We show that in the general case $C(f)$ is a continuum. Schottky's theorem and its variants are the analogues of Theorem B. The principal result to be proved in Part I is the analogue of Theorem C. The general form of this result is in fact very simple, namely

$$\mathfrak{J} R(f) \cup \mathfrak{J} C(f) = \overline{\complement R(f)} \cap C(f) \subseteq \overline{\Gamma}(f),$$

while if $\Gamma(f)$ is of linear measure zero $\complement C(f)$ is void and instead we have

$$\complement R(f) \subseteq \Gamma(f),$$

which is again of the form of Theorem C. These theorems lead to a number of results concerning the sets $\complement C(f)$, $\complement R(f)$, $\mathfrak{J} C(f)$, $\mathfrak{J} R(f)$, and $\Gamma(f)$ in special cases. The resulting system of theorems is related on the one hand to the order of ideas associated with Fatou's theorem on the boundary behaviour of a bounded function and its generalization by Nevanlinna to functions $f(z)$ of bounded characteristic $T(r, f)$ and on the other to a recent theorem of Collingwood[1] on deficient values of functions $f(z)$ of unbounded characteristic $T(r, f)$.

The theorems of Part I may be called boundary theorems in the large since they are concerned with the behaviour of $f(z)$ near the boundary of the unit circle

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[1] Collingwood (1) p. 336, Theorem IV and corollaries. The numbers against authors' names refer to the bibliography at the end of the paper.

and not near any particular point of the boundary and they belong to what may be called the boundary theory in the large.

Corresponding to the boundary theory in the large there is a boundary theory in the small. In this theory there is a selected point $z=e^{i\theta}$ of the boundary and we study the behaviour of $f(z)$ near this point. For the purposes of this theory we define sets relative to the function $f(z)$ and the point $z=e^{i\theta}$ analogous to $C(f)$, $R(f)$, and $\Gamma(f)$. It is also necessary to introduce a further conception, that of the uniform convergence of $f(z)$ to a value a on a sequence of arcs converging to a closed arc of $|z|=1$ which contains the point $z=e^{i\theta}$ and to define the set of values a for which $f(z)$ has this property. We postpone the formal definitions. In Part II of the paper we establish a system of boundary theorems in the small corresponding to the boundary theorems in the large proved in Part I.

In Part III we prove a group of theorems, of a type that originated with a well-known theorem of Plessner, concerning the distribution upon the circumference $|z|=1$ of certain classes of points, defined by the behaviour of $f(z)$ in their neighbourhoods.

The central idea of our method derives from Iversen's theory of the inverse function.[1] It consists in the continuation of an ordinary or algebraic element of the inverse function along an appropriate path free from non-algebraic singularities. The method appears to be one of considerable power in this field.

The first systematic work upon the sets C, R and Γ was that of Iversen (1, 2, 3) and Gross (1, 2) some thirty years ago and was concerned, in so far as it related to functions meromorphic in a domain having a contour, to the theory in the small. Subsequent developments in this theory are due notably to Seidel (1, 2), Doob (1—4), Beurling (1), Noshiro (1, 2, 3, 4) and, more recently, Caratheodory (1) and Weigand (1). But, as regards the theory in the large, while a number of individual theorems are known there has, so far as we are aware, been no systematic development of a general theory of the sets $C(f)$, $R(f)$, $\Gamma(f)$ and their mutual relations. It is the purpose of this paper to develop the main lines of such a theory for the unit circle both in the large and in the small. Our theorems can be extended by conformal mapping to Jordan domains, those of Parts I and II without restriction, and those of Part III subject to restrictions upon the boundary. These generalisations, which cannot be dismissed quite without discussion, are dealt with in the Appendix. We

[1] Iversen (1).

do not consider domains of a more general character for which interesting theorems have been proved by Gross (2), Besicovitch (1) and subsequent writers.[1]

It should be said that the present paper supersedes a paper by Cartwright (1) of 1935 in which the method of the inverse function was also used. Unfortunately Cartwright's investigation was vitiated by an oversight with the result that some of the theorems of the paper referred to, as well as the arguments, are incorrect.[2] Piecemeal correction would not be practicable and perhaps at this distance of time is hardly desirable. Recognition of the mistake has, however, led us to develop the theory afresh from a more general point of view which has enabled us, in particular, to elucidate in some detail the relations between the asymptotic set Γ, the excluded range CR and the frontier sets $\mathfrak{F}R$ and $\mathfrak{F}C$, both in the large and in the small.[3]

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[1] The terms *Cluster Set, Range of Values* and *Asymptotic Set* and the notation C, R and Γ have been adopted following Seidel since, so far as the terminology is concerned, this appears to be the established convention in English and it is clearly desirable to establish a conventional notation in a theory which is becoming elaborate. Practically the whole of the previous development of the theory has been in the small so that when we speak of previous conventions we refer to the conventions of that theory.

In the past there have been considerable variations both in terminology and notation. The Cluster Set was called *domaine d'indétermination* by Painlevé (1 and 2) with whom the concept originated and at first also by Iversen (1 and 2). Later Iversen (3) adopted the terminology and notation of Gross (2) who used H (*Haufungsbereich*) for C and W (*Wertbereich*) for R. Iversen (3) used K (*Konvergenzbereich*) for Γ. Noshiro and other Japanese writers have used Seidel's terminology but a different notation.

Doob in (3) wrote $F(z)$ for $C(f)$ which he called the cluster boundary function of $f(z)$. But he was only considering properties in the small. The functional notation and terminology are perhaps less well adapted to the theory in the large. In Doob (4) he used the term range for the value set.

In a recent paper Caratheodory (1), and Weigand (1) following him, used *Randwert* instead of *Haufungswert* for cluster value. Although this term does clearly relate the concept to boundary theory we have adhered to the term cluster value even though it is perhaps less suggestive. Moreover, the terms Randwert and boundary value are already used in other senses in the theory of functions.

[2] The sets which we subsequently call $\Gamma_A(f)$ and $\Phi(f)$ were overlooked in Cartwright (1). But the final section of that paper and two later papers, Cartwright (2 and 3), all of which are concerned with the boundary behaviour of functions at a boundary which is everywhere discontinuous, are unaffected by the error referred to.

[3] A short summary of some of our results, in particular Theorems 9 and 22 of the present paper, was communicated to the International Congress of Mathematicians at Harvard University, September 1950, see Collingwood and Cartwright (1).

PART I.

Boundary Theorems in the Large.

The analogue in the large of Theorem A.

2. We use the notation that is now standard.[1]

$$M(r, f) = \max_{|z|=r} |f(z)|$$

and

$$T(r, f) = m(r, f) + N(r, f),$$

where

$$m(r, f) = \frac{1}{2\pi} \int_0^{2\pi} \log^+ |f(re^{i\theta})| \, d\theta$$

and

$$N(r, f) = \int_0^r \frac{n(\varrho, \infty) - n(0, \infty)}{\varrho} d\varrho + n(0, \infty) \log r,$$

$n(\varrho, \infty)$ being the number of poles of $f(z)$ in the region $|z| \leq \varrho$, each being counted with its order of multiplicity. More generally, $n(\varrho, a)$ is the number of zeros of $f(z) - a$, each counted with its order of multiplicity, and

$$N(r, a) = \int_0^r \frac{n(\varrho, a) - n(0, a)}{\varrho} d\varrho + n(0, a) \log r.$$

Clearly, if $a \in \complement R(f)$ then $N(r, a) = O(1)$.

The functions $f(z)$ meromorphic in $|z| < 1$ fall into two classes, namely the class for which $T(r, f) = O(1)$, known as functions of bounded characteristic, and the complementary class for which $T(r, f)$ is unbounded, which may be called the class of functions of unbounded characteristic.

We recall that[2]

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[1] We refer generally to R. NEVANLINNA's two standard books: *Théorème de Picard-Borel et la théorie des fonctions méromorphes*, Paris 1929; and *Eindeutige Analytische Funktionen*. Berlin 1936, cited hereafter as E.A.F.

[2] If the values of $f(z)$ are transformed to the unit w-sphere $T(r, f)$ is defined in terms of the spherical metric and satisfies the equation $T\left(r, \dfrac{\alpha f + \beta}{\gamma f + \delta}\right) = T(r, f)$. Although the sets C, R, Γ etc. being in general unbounded are measured in the spherical metric, no inconvenience arises from retaining Nevanlinna's definition of $T(r, f)$. Projection onto the sphere may be made at any convenient stage.

$$T\left(\frac{\alpha f+\beta}{\gamma f+\delta}\right)=T(r,f)+O(1),$$

where $\dfrac{\alpha f+\beta}{\gamma f+\delta}$ is a linear transform of f.

All linear transforms of $f(z)$ thus have the same order defined by the limit

$$\overline{\lim_{r\to 1}}\ \frac{\log T(r,f)}{-\log(1-r)}.$$

A trivial argument shews that if $T(r,f)$ is unbounded $C(f)$ has the Weierstrass property of covering the closed plane, or its transform to the unit sphere. In fact we have

Theorem 1. *If $f(z)$ is meromorphic in $|z|<1$ and if $T(r,f)$ is unbounded, then $\complement C(f)$ is void.*

For suppose $a\in\complement C(f)$. Then there are numbers $K<\infty$ and $\varepsilon>0$ such that, for all r in $1-\varepsilon<r<1$, we have

$$\left|\frac{1}{f(re^{i\theta})-a}\right|<K$$

and hence $m(r,a)<K$ for $1-\varepsilon<r<1$. Since plainly $N(r,a)=O(1)$ we have

$$\begin{aligned}T(r,f)&=T\left(r,\frac{1}{f-a}\right)+O(1)\\&=m(r,a)+N(r,a)+O(1)=O(1)\end{aligned}$$

for $1-\varepsilon<r<1$; and since $T(r,f)$ is an increasing function of r this inequality holds for all $r<1$. This proves the theorem.

Evidently it is not true in general that $\complement C(f)$ is void. For functions $f(z)$ which are linear transforms of bounded functions and which constitute an important subclass of functions of bounded characteristic $\complement C(f)$ is not void. On the other hand we can find examples for which $\complement C(f)$ is void while $T(r,f)$ is bounded.

To complete the analogue of Theorem A we prove

Theorem 2. *If $f(z)$ is meromorphic and non-constant in $|z|<1$, then $C(f)$ is a continuum.*

If $C(f)$ contains only one point then $f(z)$ is a constant.[1] So we may assume

[1] It is easy to give a direct proof of this; but for economy we rely on the indirect proof given in § 4 below.

that $C(f)$ contains more than one point. To obviate special mention of the point at infinity we transform onto the unit w-sphere. Distances are accordingly to be understood as spherical distances.

First $C(f)$ is compact. For if $w_n \in C(f)$ and w is a limit of the sequence $\{w_n\}$ so that, for some sequence $\{n'\}$, $\lim_{n' \to \infty} w_{n'} = w$ it is easy to see that $w \in C(f)$. For given a sequence $\eta_{n'}$, $\lim_{n' \to \infty} \eta_{n'} = 0$, we can find a sequence $\{\varepsilon_{n'}\}$, $\lim_{n' \to \infty} \varepsilon_{n'} = 0$ such that the annulus $1 - \varepsilon_{n'} < |z| < 1$ contains a point $z_{n'}$ such that $|f(z_{n'}) - w_{n'}| < \eta_{n'}$. Hence $\lim_{n' \to \infty} |z_{n'}| = 1$, $\lim_{n' \to \infty} f(z_{n'}) = w$.

$C(f)$ is connected. To prove this we assume the contrary. There is then a partition of $C(f)$ into two compact subsets K_1 and K_2 which are at a positive distance 2δ apart. Let H_1 be the open set of points whose distance from K_1 is less than $\frac{1}{2}\delta$ and H_2 the similar set for K_2. The distance between H_1 and H_2 is δ, $\complement(H_1 \cup H_2)$ is closed and $\complement(H_1 \cup H_2) \subseteq \complement C(f)$.

Now choose $a \in K_1$, and $b \in K_2$. Given a sequence $\{\eta_n\}$, $\lim_{n \to \infty} \eta_n = 0$, we can find a sequence $\{\varepsilon_n\}$, $\lim \varepsilon_n = 0$, such that the annulus Δ_n defined by $1 - \varepsilon_n < |z| < 1$ contains a point $z_n(a)$ such that $|f(z_n(a)) - a| < \eta_n$ and a point $z_n(b)$ such that $|f(z_n(b)) - b| < \eta_n$. For $n > n_0$, say, $\eta_n < \frac{1}{2}\delta$ so that $U(a, \eta_n) \subseteq H_1$ and $U(b, \eta_n) \subseteq H_2$. We ignore values of $n \leq n_0$ and we join the points $z_n(a)$ and $z_n(b)$ in pairs by a standard curve. If $|z_n(a)| = |z_n(b)|$ then it is simply one of the arcs $z = |z_n(a)|$ joining these points. Otherwise if, say, $|z_n(a)| < |z_n(b)|$ it consists of the radial segment joining $z_n(a)$ to the circle $|z| = |z_n(b)|$ at $z_n(a)'$ and one of the arcs defined by $z_n(a)'$, $z_n(b)$. Call this curve C_n. The function $f(z)$ maps C_n, which lies wholly in Δ_n, on a continuous curve λ_n having its end points $w_n(a)$ and $w_n(b)$ in $U(a, \eta_n) \subseteq H_1$, and $U(b, \eta_n) \subseteq H_2$ respectively.[1] We can now find on λ_n a point $w_n \in \complement(H_1 \cup H_2)$. For λ_n is connected so that its end points are connected by a $\frac{1}{2}\delta$-chain $u_{n_1}, u_{n_2}, \ldots u_{n_m}$ and if all these m points of λ_n are contained in H_1, or H_2 it follows that the distance between H_1 and H_2 cannot exceed $\frac{1}{2}\delta$. But this distance is δ. We now choose a point $w_n \in \complement(H_1 \cup H_2)$ on each λ_n. The sequence $\{w_n\}$ has at least one limit point ω, which may be infinity, and there is a subsequence $\{w_{n'}\}$, $\lim_{n' \to \infty} w_{n'} = \omega$. Since $C_{n'}$ is contained in $\Delta_{n'}$ we have $\lim_{n' \to \infty} |z_{n'}| = 1$ where $w_{n'}$ and $z_{n'}$ are corresponding points of $\lambda_{n'}$ and $C_{n'}$. Multiple points of $\lambda_{n'}$ occasion no difficulty since the corresponding set of points of $C_{n'}$ must be finite. Therefore

[1] λ_n is not in general a simple curve.

$\omega \in C(f)$. But since $\omega = \lim_{n' \to \infty} w_{n'}$, where $w_{n'} \in \complement(H_1 \cup H_2)$, and $\complement(H_1 \cup H_2)$ is closed it follows that $\omega \in \complement(H_1 \cup H_2) \subset \complement(K_1 \cup K_2) = \complement C(f)$. We thus have a contradiction which proves that $C(f)$ is connected. This completes the proof of the theorem.

Analogues in the large of Theorem B.

3. The results here are well-known, but we set them down for completeness. The properties of $\complement R(f)$ with which we are concerned are closely related to the growth of $T(r, f)$. We have first the

Schottky-Nevanlinna Theorem.[1] *If $f(z)$ is meromorphic in $|z| < 1$ and if*

$$\overline{\lim_{r \to 1}} \frac{T(r, f)}{-\log(1-r)} = \infty$$

then $\complement R(f)$ contains at most two values.

In the Schottky-Nevanlinna Theorem the condition on the growth of $T(r, f)$ cannot be improved. For functions of unbounded characteristic we have

Frostman's Theorem.[2] *If $f(z)$ is meromorphic in $|z| < 1$ and if $T(r, f)$ is unbounded, then $\complement R(f)$ is of capacity zero.*

Although we do not use the theory of capacity[3] in any of our arguments we shall have occasion to state some comparative theorems in terms of this measure. The following metrical property is important.

We denote by an S-set[4] any set of points satisfying the following condition: suppose that $s(t)$ is a positive, continuous increasing function for $t > 0$ such that $s(0) = 0$ and

$$\int_0^k \frac{s(t)}{t} dt$$

is convergent for some $k > 0$. For any $\varepsilon > 0$ there is a sequence of circles with radii $\varrho_1, \varrho_2, \ldots$ covering the set such that

$$\sum_{\nu=1}^{\infty} s(\varrho_\nu) < \varepsilon.$$

[1] E.A.F. pp. 253–254.

[2] E.A.F. p. 260.

[3] *Zero capacity* and *zero harmonic measure* are equivalent and may be interchanged in our enunciations. For the theory of harmonic measure and capacity see E.A.F. pp. 29–41, 106–121 and 142–153. On zero capacity see also BEURLING (2).

[4] E.A.F. pp. 142–145.

An S-set is by definition a set of s-measure zero.[1]

A set of capacity zero is an S-set.

The definitions of s-measure and of capacity also apply when the set is projected on the unit sphere and the spherical metric is used. For unbounded sets they will be taken in this form.

Further, *an S-set is of α-dimensional measure zero for all* $\alpha > 0$.[2]

We now see that just as Theorem A is implied by the deeper Theorem B, so Theorem 1 is implied by the deeper Theorem of Frostman. For if $a \in \complement R(f)$ every neighbourhood $U(a, \varepsilon)$ contains points of $R(f)$, since $\complement R(f)$ is of linear measure zero. Hence $a \in R'(f)$; and so $R(f) \cup \complement R(f) \subseteq \bar{R}(f) \subseteq C(f)$, shewing that $\complement C(f)$ is void.

The set $\Gamma(f)$. Preliminary Theorems and Lemmas.

4. We set out in this section a number of results, some of them classical, to which we shall frequently have to appeal.

We must begin by analysing rather more closely the conceptions of an asymptotic value and an asymptotic path. If $a \in \Gamma(f)$ there is a continuous curve $z = z(t)$, $0 < t < 1$, on which $\lim_{t \to 1} |z(t)| = 1$ and $\lim_{t \to 1} f(z(t)) = a$. The limiting set of the curve $z(t)$ on $|z| = 1$ is either a single point or a closed arc, which may be the whole circumference. This is expressed in the following lemma.

Lemma 1. *A continuous curve $z = z(t)$, $0 < t < 1$, such that $|z(t)| < 1$ and $\lim_{t \to 1} |z(t)| = 1$ has at least one limit point $z = e^{i\theta}$ on $|z| = 1$ and if there are two such points $e^{i\theta_1}$ and $e^{i\theta_2}$, then the set of limit points of $z = z(t)$ on $|z| = 1$ contains at least one of the two arcs defined by $e^{i\theta_1}$ and $e^{i\theta_2}$.*

The first assertion is trivial, for if we choose a sequence $\{t_n\}$, $t_n \to 1$ as $n \to \infty$, the sequence of points $z_n = z(t_n)$ has a limit point on $|z| = 1$.

To prove the second assertion suppose that both of the arcs contains a point

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[1] The s-measure of a set α is defined as follows: Given $\varepsilon > 0$, suppose the set α to be covered by an arbitrary sequence of circles c_ν with radii $\varrho_\nu < \varepsilon$, and denote by $m_\varepsilon(\alpha, s)$ the lower bound of the corresponding sums $\sum s(\varrho_\nu)$. This number increases with decreasing ε. Put

$$m(\alpha, s) = \lim_{\varepsilon \to 0} m_\varepsilon(\alpha, s).$$

Then $0 \leq m(\alpha, s) \leq +\infty$. By definition $m(\alpha, s)$ is the s-measure of α. For $s = \pi r^2$ the s-measure is the plane measure and for $s = 2r$ the linear measure of α.

[2] E.A.F. pp. 142–143.

which is not a limit point of $z(t)$. Denote these points by p and q. We can find $\varepsilon > 0$ such that neither of the circles of centres p and q and radius ε contains a point of $z(t)$; and we can find $t(\varepsilon)$ such that $|z(t)| > 1 - \varepsilon$ for $t > t(\varepsilon)$. If we cut the annulus $1 - \varepsilon < |z| < 1$ along the radii to p and q neither of these cross cuts is within a distance ε of the curve $z = z(t)$, $t(\varepsilon) < t < 1$, which is therefore not connected. This is contrary to hypothesis and the lemma is proved.

We call the limiting set on $|z| = 1$ of an asymptotic path its "end"; and we define $\Gamma_P(f) \subseteq \Gamma(f)$ and $\Gamma_A(f) \subseteq \Gamma(f)$ as follows: $a \in \Gamma_P(f)$ if there is an asymptotic path $z = z(t)$ whose end is a point of $|z| = 1$ and such that $\lim\limits_{t \to 1} f(z(t)) = a$, and $a \in \Gamma_A(f)$ if there is an asymptotic path $z = z(t)$ whose end is an arc of $|z| = 1$ (or the whole circumference) and such that $\lim\limits_{t \to 1} f(z(t)) = a$.

We also write $a \in \Gamma_P(f, e^{i\theta})$ if $e^{i\theta}$ is the end of $z(t)$ and $\Gamma_P(f, \theta_1 < \theta < \theta_2)$ for $\bigcup\limits_{\theta_1 < \theta < \theta_2} \Gamma_P(f, e^{i\theta})$ so that $\Gamma_P(f) = \bigcup\limits_{\theta} \Gamma_P(f, e^{i\theta})$; then $\Gamma(f) = \Gamma_P(f) \cup \Gamma_A(f)$. We must note that the intersection $\Gamma_P(f) \cap \Gamma_A(f)$ is not necessarily void.

An important class of asymptotic paths are spirals converging to the circumference $|z| = 1$. Functions tending to asymptotic values along such paths have been constructed and their properties discussed by Valiron (1 & 2). We may denote by $\Gamma_S(f)$ the set of values $\{a\}$ for which there is a path $z = z(t)$, $0 < t < 1$, such that $\lim\limits_{t \to 1} |z(t)| = 1$, $|\arg z(t)|$ is unbounded and $\lim\limits_{t \to 1} f(z(t)) = a$. Then $\Gamma_S(f)$ contains the set of asymptotic values of $f(z)$ for which there are spiral asymptotic paths. Further $\Gamma_S(f) \subseteq \Gamma_A(f) \subseteq \Gamma(f)$.

Evidently, when $\Gamma_S(f)$ contains more than one value $\Gamma_P(f)$ must be void. Functions constructed by Valiron (1) are examples. Equally, if $\Gamma_P(f)$ contains more than one value $\Gamma_S(f)$ is void. All bounded functions satisfy this condition; and it is also satisfied by the modular function $\mu(z)$ which is unbounded and has $\Gamma_P(f) = (0) \cup (1) \cup (\infty)$. It follows that if neither $\Gamma_P(f)$ nor $\Gamma_S(f)$ is void they must consist of one and the same value. A function with this property is Koenigs' function[1] $K(z)$ for which $\Gamma_P(f) = \Gamma_S(f) = (\infty)$.

Theorem 3. *If $f(z)$ is meromorphic in $|z| < 1$ and if $T(r, f) = O(1)$, then $\Gamma_A(f)$ is void.*

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[1] See Valiron (2).

This is an immediate consequence of the following two well-known theorems.

Fatou-Nevanlinna Theorem.[1] *If* $f(z)$ *is meromorphic in* $|z|<1$ *and if* $T(r,f)=O(1)$, *then for almost all* θ *in* $0\le\theta<2\pi$, $\lim\limits_{z\to e^{i\theta}} f(z)$ *exists uniformly in the angle* $|\arg(1-ze^{-i\theta})|\le\frac{\pi}{2}-\delta$ *for all* $\delta>0$.

A point $z=e^{i\theta}$ for which this property holds we call a *Fatou point* and the set of such points we denote by $F=F(f)$ and its complement by CF. Then CF *is of measure zero if* $T(r,f)=O(1)$. For $e^{i\theta}\in F$ we denote the unique limit of $f(z)$ as z tends to $e^{i\theta}$ in any angle $|\arg(1-ze^{-i\theta})|\le\frac{\pi}{2}-\delta$ by $f(e^{i\theta})$.

Riesz-Nevanlinna Theorem.[2] *If* $f(z)$ *is meromorphic in* $|z|<1$, $T(r,f)=O(1)$ *and there is a number* a *such that* $f(e^{i\theta})=a$ *for a set of values of* θ *of positive measure, then* $f(z)\equiv a$.

Denoting by F_a the subset of F for which $f(e^{i\theta})=a$, we note first that if $f(z)$ is not constant and $T(r,f)=O(1)$ then F_a is of measure zero for all values of a. It follows as an immediate corollary that *if* $C(f)$ *consists of a single value* a, *then* $f(z)\equiv a$. For we may assume $a\neq\infty$ so that $T(r,f)=O(1)$ and F_a is the whole circumference $|z|=1$. We note secondly that given any arc of $|z|=1$ there is an infinity of values of x for which this arc contains points of F_x.[3]

If now $T(r,f)=O(1)$ and $a\in\Gamma_A(f)$ there is an arc of $|z|=1$ which is the end of an asymptotic path $\gamma(a)$ on which $f(z)$ tends to a. But we can find a radius to this arc on which $\lim\limits_{r\to1} f(re^{i\theta})\neq a$. Since $\gamma(a)$ intersects this radius an infinity of times in every neighbourhood of $e^{i\theta}$ we have a contradiction. Hence $\Gamma_A(f)$ is void and Theorem 3 is proved.

For functions of finite order there is an interesting theorem of Valiron (2) who has proved that *if* $f(z)$ *is regular in* $|z|<1$ *and of finite order then there is no finite number* $a\in\Gamma_S(f)$.

5. There is another important condition under which $\Gamma_A(f)$ is void.

Theorem 4. *If* $f(z)$ *is meromorphic in* $|z|<1$ *and* $CR(f)$ *contains more than two values, then* (i) $\Gamma_A(f)$ *is void so that* $\Gamma(f)=\Gamma_P(f)$; *and* (ii) *if* $a\in\Gamma_P(f,e^{i\theta})$, *then* $\Gamma_P(f,e^{i\theta})$ *contains no other value and* $e^{i\theta}\in F$.

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[1] E.A.F. p. 197.

[2] E.A.F. p. 197.

[3] E.A.F. p. 198 Satz.

This is a special case of a more general theorem (Theorem 6) the proof of which depends upon the following lemma due to Koebe.[1]

Koebe's Lemma. *Let $\varphi(z)$ be regular and bounded in $|z|<1$ and let there be two sequences $\{z_n^{(1)}\}$ and $\{z_n^{(2)}\}$ such that $|z_n^{(1)}|<1$, $\lim_{n\to\infty} z_n^{(1)}=e^{i\theta_1}$; $|z_n^{(2)}|<1$, $\lim_{n\to\infty} z_n^{(2)}=e^{i\theta_2}$ where $\theta_1\neq\theta_2$. If there is a sequence of continuous curves γ_n joining $z_n^{(1)}$ to $z_n^{(2)}$ and contained in an annulus $1-\varepsilon_n<|z|<1$, where $\varepsilon_n>0$, $\lim_{n\to\infty}\varepsilon_n=0$, such that on γ_n we have $|\varphi(z)-x|<\eta_n$ where $\lim_{n\to\infty}\eta_n=0$, then $\varphi(z)\equiv x$.*

We now introduce a further definition relating to a function $f(z)$ meromorphic in $|z|<1$. Suppose that there is a closed arc $\theta_1\leq\theta\leq\theta_2$ of the circumference $|z|=1$ which is the limit of a set of curves γ_n satisfying the condition of Koebe's lemma and that $|f(z)-a|<\eta_n$, $\lim_{n\to\infty}\eta_n=0$, for all z on γ_n. By definition, $a\in\Phi(f)$ and, for any θ satisfying $\theta_1\leq\theta\leq\theta_2$, $a\in\Phi(f, e^{i\theta})$; so that $\Phi(f)=\bigcup_\theta \Phi(f, e^{i\theta})$.

We now prove

Theorem 5. *If $f(z)$ is meromorphic and non-constant in $|z|<1$ and $f(z)\neq a, b$ or c where a, b and c are distinct, then* (i) *$\Phi(f)$ is void;*[2] *and* (ii) *if for some θ here exists $x\in\Gamma_P(f, e^{i\theta})$, then $\Gamma_P(f, e^{i\theta})$ contains no other value and $e^{i\theta}\in F$.*

We may assume $f(z)$ has been transformed so as to make $c=\infty$ and we write

$$\psi(z)=\frac{f(z)-a}{b-a}.$$

$\psi(z)$ is then regular and does not take the values 0, 1 or ∞ in $|z|<1$. Now let g be any number which is not real and write

$$\varphi(z)=\frac{\nu(\psi(z))-\nu(g)}{\nu(\psi(z))-\overline{\nu(g)}}$$

where $\nu(w)$ is the inverse of the modular function for the half plane. Then $\varphi(z)$ is

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[1] Koebe (1 and 2). Proofs are also given in L. Bieberbach, *Lehrbuch der Funktiontheorie* Bd II, p. 19, 2'nd edition, Leipzig 1931; and in P. Montel, *Leçons sur les familles normales de fonctions analytiques*, p. 107, Paris 1927.

[2] This result is in fact due to Gross (1) pp. 35–36 who also derived it from Koebe's Lemma. He does not use the modular function but maps the universal covering surface of the w-plane punctured at a, b and c onto the unit circle $|\xi|<1$. We had overlooked this theorem of Gross until after our own paper had gone to the press; but we have allowed our proof to stand since Gross' paper is not now very accessible and the result is an essential piece of our apparatus.

regular and satisfies $|\varphi(z)| < 1$ in $|z| < 1$. (ii) now follows immediately from the Fatou-Nevanlinna theorem. The proof of (i) is carried out in two stages.

First suppose that $h \in \Phi(f)$, where $h \neq a$, b or ∞, so that $k = \dfrac{h-a}{b-a} \neq 0$, 1 or ∞ and $k \in \Phi(\psi)$. Hence $\varphi(k) \in \Phi(\varphi)$ and it follows at once from Koebe's lemma that $\varphi(z) \equiv \varphi(k)$, $\psi(z) \equiv k$ and so $f(z) \equiv h$.

Secondly, we assume that $h \in \Phi(f)$ is one of the omitted values a or b. We can exclude the case $h = \infty$ by transforming to $1/f(z)$. Clearly we may put $h = b$. We then have a sequence of curves γ_n converging to a closed arc of $|z| = 1$, such that $|\psi(z) - 1| < \eta_n$, $\lim\limits_{n\to\infty} \eta_n = 0$, for all z on γ_n. Now let the circle $|z| < 1$ be cut along a radius. Each of the two branches of $(\psi(z))^{\frac{1}{2}}$ is regular and omits the four values $0, -1, 1, \infty$ in the cut circle. We can choose a branch of $(\psi(z))^{\frac{1}{2}}$, which we will call $u(z)$, and which converges uniformly to -1 on a sub-sequence $\gamma_{n'}$ of the γ_n. Map the cut circle by a function $\xi = \xi(z)$ on the circle $|\xi| < 1$ and denote by $z(\xi)$ the inverse mapping function. Then $v(\xi) = u(z(\xi))$ is regular and not equal to 0, 1 or ∞ in $|\xi| < 1$; and $-1 \in \Phi(v)$ since the sequence of curves γ_n is mapped by $\xi(z)$ on a sequence in $|\xi| < 1$ satisfying the same condition of convergence to a closed arc of $|\xi| = 1$. It now follows from the previous argument with $v(\xi)$ in place of $\psi(z)$ that $u(z) \equiv -1$, $\psi(z) \equiv 1$ and $f(z) \equiv b = h$.

Since by hypothesis $f(z)$ is not a constant this proves (i).

From theorem 5 we can at once deduce the following more general theorem.

Theorem 6. *If $f(z)$ is meromorphic and non-constant in $|z| < 1$ and $C R(f)$ contains more than two values, then* (i) *$\Phi(f)$ is void; and* (ii) *if for some θ there exists $x \in \Gamma_P(f, e^{i\theta})$ then $\Gamma_P(f, e^{i\theta})$ contains no other value and $e^{i\theta} \in F$.*

Let a, b and c belong to $C R(f)$. Then we can find $\varepsilon > 0$ such that $f(z) \neq a, b$, or c in the annulus $1 - \varepsilon < |z| < 1$; and so also in the annulus cut along the segment $1 - \varepsilon \leq \Re z \leq 1$. We map the cut annulus conformally on $|\xi| < 1$ by a function $\xi(z)$, the inverse being $z(\xi)$. The function $\varkappa(\xi) = f(z(\xi))$ is meromorphic and not equal to a, b or c in $|\xi| < 1$ so that $\Phi(\varkappa)$ is void. But if there is a number $d \in \Phi(f)$, then evidently $d \in \Phi(\varkappa)$. For there is a sequence of curves γ_n converging to an arc of $|z| = 1$, which we may clearly assume does not contain the point $z = 1$, on which $|f(z) - d|$ tends uniformly to zero. This sequence is mapped upon a sequence of curves β_n in $|\xi| < 1$ converging to an arc of $|\xi| = 1$ on which $|\varkappa(\xi) - d|$ tends uniformly to zero. Therefore $\Phi(\varkappa)$ void implies $\Phi(f)$ void and the theorem is proved.

Theorem 4 is an immediate corollary of Theorem 6. For if $a \in \Gamma_A(f)$ we can find a sequence of segments of an asymptotic path, on which $f(z)$ tends to a, converging to an arc of $|z|=1$. Hence $\Gamma_A(f) \subseteq \Phi(f)$.

We know from the Schottky-Nevanlinna theorem that the condition of theorem 3 that $\complement R(f)$ contains more than two values also implies a restriction on the characteristic $T(r, f)$ namely $T(r, f) = O(-\log(1-r))$. The argument of paragraph 4 has shown that if $\Gamma_A(f)$ is not void, or indeed if $\Phi(f)$ is not void, then $T(r, f)$ is unbounded. We may ask whether unboundedness of some minorant of $T(r, f)$ is implied by either of the conditions $\Gamma_A(f)$ not void or $\Phi(f)$ not void. The theorem of Valiron on spiral paths quoted at the end of paragraph 4, suggests that some such relation may exist. We are, however, unable to answer the question.

The sets $\Gamma(f)$ and $\Phi(f)$ under conformal mapping.

6. Let D be a simply connected domain in $|z|<1$ whose frontier consists of an arc α of the circumference $|z|=1$ and a Jordan curve in $|z|<1$. Conformal mapping of domains of this type upon the unit circle is an essential feature of our technique.[1] The essential property of such mappings for our purposes follows from the symmetry principle. To avoid repetition we state the relevant case of this principle as

Lemma 2. *If D is mapped conformally upon the circle $|\xi|<1$ the mapping function $\xi(z)$ is regular upon the arc α.*

From this we derive the properties of $\Gamma(f)$ and $\Phi(f)$ under conformal mapping.

It follows at once from the lemma that a curve $z(t)$ having its end point on α and making an angle Θ with the radius at this point is mapped upon a curve $\xi(z(t))$ making the same angle Θ with the radius of $|\xi|=1$ at its end point. In particular, if a is a radial asymptotic value for $f(z)$ at a point $e^{i\theta_0}$ of α i.e. if $\lim\limits_{r\to 1} f(r e^{i\theta_0}) = a$, then a is a radial limit for $\varphi(\xi) = f(z(\xi))$ at the point $\xi(e^{i\theta_0})$, of the circumference $|\xi|=1$; while if $e^{i\theta_0}$ is a point of the set $\bar{F}$ for $f(z)$ then $\xi(e^{i\theta_0})$ is a point of the set F for $\varphi(\xi)$.

It also follows from Lemma 2 that if $a \in \Phi(f, e^{i\theta_0})$, then $a \in \Phi(\varphi, \xi(e^{i\theta_0}))$.

We have thus proved

Lemma 2 a. *Let $f(z)$ be meromorphic in $|z|<1$ and let the domain D defined above be mapped conformally by the function $\xi(z)$ upon the circle $|\xi|<1$. Then*

[1] More general mappings are discussed in the Appendix.

(i) *A point* $e^{i\theta}$ *in* α *which belongs to the set of points* F *for the function* $f(z)$ *transforms into a point* $e^{i\vartheta}$ *of the set* F *on the circumference* $|\xi|=1$ *for the function* $\varphi(\xi)=f(z(\xi))$; *and if* $a=f(e^{i\theta})$, *then* $a=\varphi(e^{i\vartheta})$.

(ii) *If* $a \in \Phi(f, e^{i\theta})$ *for* $e^{i\theta}$ *in* a *then* $a \in \Phi(\varphi, e^{i\vartheta})$.

The set $\Gamma(f)$ in relation to the inverse function.

7. We recall the elementary properties of the inverse function $z=z(w)$ of the function $w=f(z)$.[1] To every z in $|z|<1$ there corresponds an element $e_z=e_z(w', w)$, where $z=e_z(w, w)$, which is regular or algebraic according as w is a simple or multiple value of $f(z)$ and which has a positive radius of convergence. If $|z|=1$ is a natural boundary for $f(z)$ the inverse function $z(w)$ is the set of these elements $e_z(w', w)$ for $|z|<1$. Given any two elements e_{z_1} and e_{z_2} ($|z_1|<1$; $|z_2|<1$) of $z(w)$, there is an analytic continuation of e_{z_1} to e_{z_2} such that every element of the continuation is an element of $z(w)$.

The definition of the inverse function $z(w)$ is easily extended by analytic continuation to the case where $|z|=1$ is not a natural boundary. We shew first that to every continuation of $f(z)$ there corresponds a continuation of $z(w)$. In particular, suppose that there is a continuation from a point z' in $|z|<1$ to a point z'' in $|z|>1$. These points can be joined by a polygonal curve L at all points of which $f(z)$ is regular and $f'(z) \neq 0$ and this curve is mapped on a curve Λ in the w-plane. To every point z of L there corresponds an inverse element $e_z(w', w)$ which maps a circle $|w'-w|<\varepsilon(z)$, $\varepsilon(z)>0$ on a domain $d(z)$ containing z. Since for every z the domain $d(z)$ contains an arc of L of positive length it follows from the Borel-Lebesgue covering theorem that L is covered by a finite number of the domains $d(z)$. Since consecutive domains of this finite set overlap, the corresponding circles in the w-plane also overlap, the corresponding inverse elements are equal in the common parts of these circles and thus are immediate continuations of one another. Every other inverse element $e_z(w', w)$ corresponding to a point z on L is an immediate continuation of one of this finite set of elements and is thus an element of the continuation of $z(w)$ along Λ corresponding to the continuation of $f(z)$ along L.

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[1] The theory was developed systematically by IVERSEN (1) in the parabolic case i.e. for $w=F(z)$ meromorphic in the finite plane $|z|<\infty$. For the corresponding theory with $w=f(z)$ meromorphic in $|z|<1$, the circumference $|z|=1$ being a natural boundary, see E.A.F. pp. 269–275. For the general case see VALIRON (5) pp. 415–417. The theory has been developed in considerable detail for an arbitrary analytic function by NOSHIRO (4) pp. 43–73.

We have now only to reverse the argument to shew that to every continuation of an element $e_{z_1}(w', w_1)$ of $z(w)$ to another element $e_{z_2}(w', w_2)$ there corresponds a continuation of the element $e_{w_1}(z', z_1)$ of $f(z)$ $(w_1 = e_{w_1}(z_1, z_1) = f(z)_1)$ of which $e_{z_1}(w', w_1)$ is the inverse, to an element $e_{w_2}(z', z_2)$ $(w_2 = e_{w_2}(z_2, z_2) = f(z_2))$ of which $e_{z_2}(w', w_2)$ is the inverse.

It has thus been shewn (i) that given any two elements $e_{z_1}(w', w_1)$, and $e_{z_2}(w', w_2)$, being the inverses of $f(z)$ at $z = z_1$ and $z = z_2$ respectively, each is a continuation of the other; and (ii) that given any element $e_{z_2}(w', w_2)$ which is a continuation of an element $e_{z_1}(w', w_1)$ of $z(w)$ then $e_{z_2}(w', w_2)$ is the inverse of an element of $f(z)$. $z(w)$ is thus defined throughout its domain of existence by analytic continuation from any element inverse to an element of $f(z)$.

Now consider a path λ in the w-plane defined by a continuous function $w = w(t)$, $0 \leq t < 1$. Then, by definition, an analytic continuation along the path is a set of regular or algebraic elements $e_{z_t}(w', w(t))$, where $z_t = e_{z_t}(w(t), w(t))$, such that for any t in $0 \leq t < 1$, $e_{z_t}(w', w(t))$ exists and we can find $\varepsilon = \varepsilon(t) > 0$ such that for $|T - t| < \varepsilon$ all the elements $e_{z_T}(w', w(T))$ are immediate continuations of $e_{z_t}(w', w(t))$. Any two elements of the set can be joined by a finite chain of elements of the set which may be selected in an infinity of different ways. It follows at once from the foregoing argument that to every continuation of an element of $z(w)$ along λ there corresponds a continuation of an element of $f(z)$ along the corresponding path l in the z-plane. We note expressly that either or both of the paths λ and l may be closed curves described any number of times.

We classify the elements $e_z(w', w)$ of the inverse function $z(w)$ of $f(z)$ as follows: $e_z(w', w)$ is an *internal element* of $z(w)$ if it is the inverse of an element $e_w(z', z)$ of $f(z)$ for which $|z| < 1$; it is a *boundary element* if $|z| = 1$; and an *external element* if $|z| > 1$. If $|z| = 1$ is a natural boundary for $f(z)$ then $z(w)$ has only internal elements.

It is readily seen that *any continuation* (as defined above) *of an element of* $z(w)$ *along a given continuous path* λ *in the* w*-plane which contains both internal and external elements contains at least one boundary element.* Since λ is arbitrary the continuation may contain algebraic elements in which case the particular continuation is determined by the choice of branches of these algebraic elements. Now the branch of $z(w)$ generated by any particular continuation maps λ upon a path l in the z-plane which joins a point in $|z| < 1$ corresponding to an internal element to a point in $|z| > 1$ corresponding to an external element. So l cuts the circumference $|z| = 1$ in a point z corresponding to the point $w(t_1)$, say, on λ and, since $f(z)$ is meromorphic or algebraic on l the continuation along λ contains an element $e_{z_1}(w', w(t_1))$ which is a boundary element. This proves our assertion. It follows as a corollary that *any continuation along a path* λ *in the* w*-plane of an element of* $z(w)$ *which contains no boundary element, contains only internal or only external elements.*

Suppose now that the circumference $|z|=1$ is a natural boundary for $f(z)$ and consider a path λ defined by a continuous function $w=w(t)$, $0 \leq t < 1$ such that $\lim_{t\to 1} w(t)=\omega$. Suppose further that there is a continuation of an element of $z(w)$ along λ towards the point ω such that the radius of convergence of the regular or algebraic elements $e_{z_t}(w', w(t))$ of the continuation tends to zero as $t \to 1$. Then ω is a transcendant singularity of the branch of $z(w)$ generated by the continuation. The path λ is mapped by this branch of $z(w)$ on a path l defined by the continuous function $z=\zeta(t)=z(w(t))$, $0 \leq t < 1$, where $|\zeta(t)| < 1$. We see at once that $\lim_{t\to 1} |\zeta(t)|=1$. For if not there is a number $r_0 < 1$ and a sequence $t_1 < t_2 < \cdots < t_n < \cdots$, $\lim_{n\to\infty} t_n = 1$, such that $|\zeta(t_n)|=r_0$ and hence there is a point $z_0=r_0 e^{i\theta_0}$ and a subsequence $\{t_\nu\}$ such that $\lim_{\nu\to\infty} \zeta(t_\nu)=z_0$. Now since $|z_0|<1$ there is a regular or algebraic element $e_{z_0}(w', w_0)$, $w_0=f(z_0)$, of $z(w)$ having a positive radius of convergence ϱ_0. We can find ν_0 such that $|\zeta(t_\nu)-z_0|<\varrho_0/2$ for all $\nu>\nu_0$ and it follows that each of the elements $e_z(w', w(t_\nu))$ is an immediate continuation of $e_{z_0}(w', w_0)$ and thus has a radius of convergence greater than $\varrho_0/2$. But this contradicts the hypothesis that the radius of convergence of $e_{z_t}(w', w(t))$ tends to zero as $t \to 1$ so that it follows that $\lim_{t\to 1} |\zeta(t)|=1$ and since $\lim_{t\to 1} f(\zeta(t))=\omega$ we see that $\omega \in \Gamma(f)$. Conversely, if $\omega \in \Gamma(f)$ then ω is a transcendant singularity of some branch of $z(w)$. For there is a path l in $|z|<1$ defined by $z=z(t)$, $0 \leq t < 1$, such that $|z(t)| \to 1$ and $f(z(t)) \to \omega$ as $t \to 1$. We may assume without loss of generality that l passes through no zero of $f'(z)$ so that l is mapped by $f(z)$ on a continuous curve λ, $w=w(z(t))$, $0 \leq t < 1$, without branch points such that $w(z(t)) \to \omega$ as $t \to 1$. Let $z_1=z(t_1) \in l$ and $e_{z_1}(w', w_1)$, $w_1=f(z_1) \in \lambda$, be the corresponding element of $z(w)$. If the radius of convergence of the elements $e_z(w', w)$ obtained by continuation of $e_{z_1}(w', w_1)$ along λ towards ω does not tend to zero this radius has a lower bound ϱ_0. We can find t_0 such that $|w(z(t))-\omega|<\varrho_0/2$ for $t>t_0$. The continuation contains the element $e_{z_0}(w', w_0)$, where $z_0=z(t_0)$, $w_0=w(z(t_0))$, whose circle of convergence, of radius not less than ϱ_0, contains the circle $|w-\omega|=\varrho_0/2$, which in turn contains the curve $w(z(t))$, $t_0<t<1$, which it therefore maps within a domain contained in $|z|<1$. But since the continuation maps λ upon l on which $|z| \to 1$ this gives a contradiction and it follows that ω must be a transcendant singularity for the branch generated by this continuation.

We have thus proved that *if* $|z|=1$ *is a natural boundary for* $f(z)$, *then* $\Gamma(f)=\Omega(f)$, *where* $\Omega(f)$ *is the set of transcendant singularities of* $z(w)$.[1]

Secondly, consider the case where $|z|=1$ is not a natural boundary for $f(z)$. There are then regular points, in which we include the poles, or algebraic points of $f(z)$ on $|z|=1$. These form an open set which thus consists of a finite or enumerable set of open intervals I_n on $|z|=1$. At each point $e^{i\theta}$ of $I=\bigcup\limits_n I_n$ $f(z)$ clearly has a unique asymptotic value $f(e^{i\theta})$, which we call a *regular boundary value*, and which corresponds to a boundary element of $z(w)$. The intervals I_n are mapped by $f(z)$ upon a set of analytic arcs every point of which is a regular asymptotic value; and it follows that *the set of regular asymptotic values of* $f(z)$ *is either void or of positive linear measure.*

Asymptotic values of $f(z)$ which are not regular we call *transcendant*. If there is a continuation along λ defined by $w=w(t)$, $0\leq t<1$, $\lim\limits_{t\to 1} w(t)=\omega$ consisting only of internal elements of $z(w)$ and such that the radius of convergence tends to zero as $t\to 1$ then ω is a transcendant singularity for the branch of $z(w)$ generated by the continuation and, by the argument used above, λ is mapped by this branch on a path l defined by $z=\zeta(t)$, $0\leq t<1$, which cannot have any point of $|z|<1$ or any regular point of $|z|=1$ in its limiting set as $t\to 1$. Hence $\lim\limits_{t\to 1}|\zeta(t)|=1$ and ω *is a transcendant asymptotic value.* Conversely, if $\omega\in\Gamma(f)$ and is not a regular asymptotic value then ω is a transcendant singularity for some *internal* branch of $z(w)$; i.e. for some branch consisting only of internal elements.

Denote by $\Omega(f)$ the set of transcendant singularities for the *internal* branch of $z(w)$ consisting of all *internal* elements[2] and by $\Pi(f)$ the set of regular boundary values. We have shewn that *if* $|z|=1$ *is not a natural boundary for* $f(z)$, *then*

$$\Gamma(f)=\Omega(f)\cup\Pi(f).$$

Combining these two results we have the following lemma.

Lemma 3. *If* $f(z)$ *is meromorphic in* $|z|<1$ *then*

$$\Gamma(f)=\Omega(f)\cup\Pi(f),$$

[1] This result is given by Nevanlinna, E.A.F. pp. 271–272.

[2] We should observe that if $|z|=1$ is not a natural boundary the complete function $f(z)$ generated by continuation across the circumference $|z|=1$ in both directions may not be uniform so that for any z in $|z|<1$ there may be more than one element $e_w(z',z)$ of the complete function. But as we are only concerned with a single branch of $f(z)$ which is, by hypothesis, uniform in $|z|<1$ no ambiguity arises.

where $\Omega(f)$ *is the set of transcendant singularities for the internal branch of* $z(w)$ *and* $\Pi(f)$ *is the set of regular boundary values.*

If $|z|=1$ is a natural boundary then $\Pi(f)$ is void and the internal branch is the complete function $z(w)$ and we have $\Gamma(f)=\Omega(f)$. We note as a corollary of lemma 3 that *if* $\Gamma(f)$ *is of linear measure zero,* $\Pi(f)$ *is void so that* $|z|=1$ *is a natural boundary for* $f(z)$ *and* $\Gamma(f)=\Omega(f)$.

The set $\Gamma_P(f)$ for bounded functions.

8. We wish to study $\Gamma(f)$ when $\complement C(f)$ is not void. If $a\in\complement C(f)$ there are positive numbers σ and ε such that $|f(z)-a|>\sigma$ (or $|f(z)|<1/\sigma$ if $a=\infty$) in the domain $1-\varepsilon<|z|<1$. If $a\neq\infty$ we make a linear transformation that puts a on ∞ so that the transform of $f(z)$ is bounded. It is therefore sufficient to consider functions regular and bounded in an annulus.

We write $\Gamma_P(f,\theta_1<\theta<\theta_2)=\bigcup_{\theta_1<\theta<\theta_2}\Gamma_P(f,e^{i\theta})$ and with this definition we prove the following lemma.

Lemma 4. *Suppose that* $f(z)$ *is meromorphic in* $|z|<1$ *and regular and bounded in a simply connected domain* D *in* $|z|<1$ *whose frontier consists of an arc* $\Theta_1\leq\theta\leq\Theta_2$, $z=e^{i\theta}$, *and a Jordan curve in* $|z|<1$. *Then, for any pair* θ_1, θ_2 *such that* $\Theta_1<\theta_1<\theta_2<\Theta_2$, $w_1=f(e^{i\theta_1})=\Gamma_P(f,e^{i\theta_1})$, $w_2=f(e^{i\theta_2})=\Gamma_P(f,e^{i\theta_2})$ *and* $w_1\neq w_2$, *the projection on the open straight line* L *between* w_1 *and* w_2 *of the set* $\Gamma_P(f,\theta_1<\theta<\theta_2)$ *of values of* $f(e^{i\theta})$ *in the open interval* $\theta_1<\theta<\theta_2$ *includes all points on* L, *and hence the set* $\Gamma_P(f,\theta_1<\theta<\theta_2)$ *is of positive linear measure.*

Let D be mapped conformally on the unit circle $|\xi|<1$. As we saw in paragraph 6, it follows from lemma 2 that the arc $\Theta_1\leq\theta\leq\Theta_2$, $z=e^{i\theta}$, denoted by α, transforms into an arc $\Psi_1\leq\psi\leq\Psi_2$, $\xi=e^{i\psi}$, denoted by β; the points of F for $f(z)$ in α transform to points of F for $\varphi(\xi)=f(z(\xi))$ in β; and

$$\Gamma_P(\varphi,\psi_1\leq\psi\leq\psi_2)=\Gamma_P(f,\theta_1\leq\theta\leq\theta_2)$$

where $e^{i\psi_1}$ and $e^{i\psi_2}$ are the transforms of $e^{i\theta_1}$ and $e^{i\theta_2}$ respectively. Further, since $\varphi(\xi)$ is bounded, almost all points of the arc β and hence almost all points of the arc α are points of F for the respective functions $\varphi(\xi)$ and $f(z)$.

Now we can find θ_1, θ_2 such that $\Theta_1<\theta_1<\theta_2<\Theta_2$ and $w_1=f(e^{i\theta_1})=\Gamma_P(f,e^{i\theta_1})$ is not equal to $w_2=f(e^{i\theta_2})=\Gamma_P(f,e^{i\theta_2})$. For if not $\varphi(\xi)$ which is bounded in $|\xi|<1$ and has

$$\Gamma_P(\varphi,\Psi_1\leq\psi\leq\Psi_2)=\Gamma_P(f,\Theta_1\leq\theta\leq\Theta_2)$$

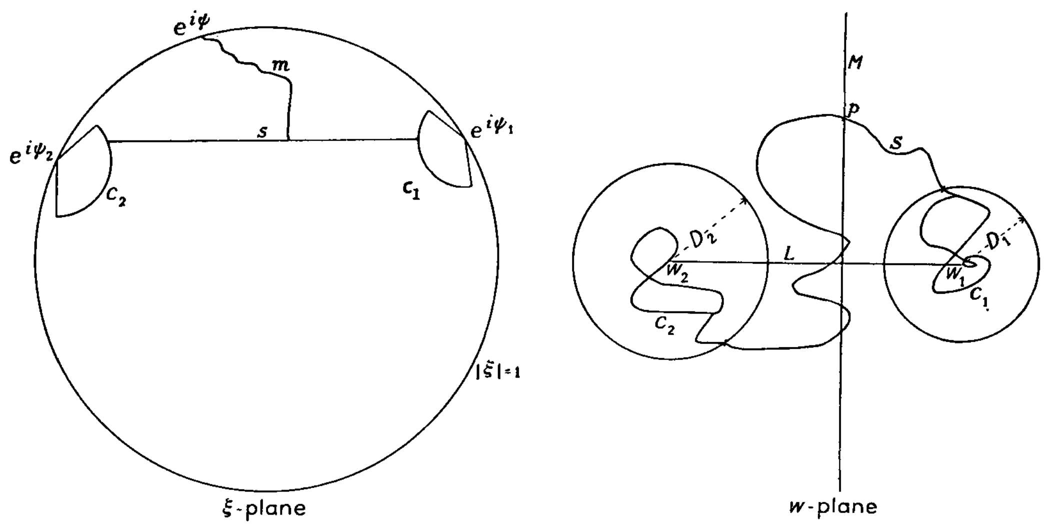

Fig. 1.

must be constant, by Riesz's theorem; and so $f(z)$ is constant in D and hence also in $|z| < 1$.

We now fix our attention on the function $\varphi(\xi)$. Draw the chord joining the points $e^{i\varphi_1}$, $e^{i\varphi_2}$ and consider the segment of $|\xi| < 1$ bounded by this chord and the arc of $|\xi| = 1$ contained in β. Let $\arg(1 - \xi e^{i\varphi_1}) = -\mu$ and $\arg(1 - \xi e^{-i\varphi_2}) = \mu$ be the lines bisecting the angle between the chord and tangent at $e^{i\varphi_1}$ and $e^{i\varphi_2}$ in the segment and, in the w-plane, let L be the straight line joining w_1 and w_2 and M any straight line perpendicular to L and intersecting it between w_1 and w_2. We now choose δ_1 so that in the domain $|\arg(1 - \xi e^{-i\varphi_1})| \leq \mu$, $|e^{i\varphi_1} - \xi| \leq \delta_1$ we have $|\varphi(\xi) - w_1| < \Delta_1$, where Δ_1 is the distance from w_1 to M.

Similarly, we choose δ_2 so that in $|\arg(1 - \xi e^{-i\varphi_2})| \leq \mu$, $|e^{i\varphi_2} - \xi| \leq \delta_2$ we have $|\varphi(\xi) - w_2| < \Delta_2$, where Δ_2 is the distance from w_2 to M. Let $\delta = \min(\delta_1, \delta_2)$ and denote by c_1 and c_2 the curves bounding the regions $|\arg(1 - \xi e^{-i\varphi_1})| \leq \mu$, $|e^{i\varphi_1} - \xi| \leq \delta$ and $|\arg(1 - \xi e^{-i\varphi_2})| \leq \mu$, $|e^{i\varphi_2} - \xi| \leq \delta$.

Since $\varphi(\xi)$ is regular for $|\xi| < 1$, the points at which $\varphi'(\xi) = 0$ are isolated; and so we can find a straight line s joining c_1 and c_2 inside $|\xi| < 1$ on which $\varphi'(\xi) \neq 0$. Hence s is mapped by $\varphi(\xi)$ on a simple curve S in the w-plane which joins C_1, the map of c_1, to C_2, the map of c_2. C_1 and C_2 may not be simple curves but they lie wholly within the circles $|\varphi - w_1| \leq D_1$ and $|\varphi - w_2| \leq D_2$ respectively. Therefore S certainly intersects M. Further, since $\varphi(\xi)$ is regular S is an analytic curve and it follows that *the number of intersections of S and M is finite.*

There is thus at least one point of intersection of S and M such that there is no other such point above (below) it on M. Let p be this uppermost (lowest) point of intersection. There is then an ordinary element $e_\xi(\varphi, p)$ of the inverse $\xi(\varphi)$ of the function $\varphi(\xi)$; and this element can be continued upwards (downwards) along M without encountering any other point of intersection. We choose p and the direction of continuation (if there is only one point of intersection of S and M so that a choice of direction remains) so that the curve m on which the path of continuation on M is mapped lies in the segment of $|\xi|<1$ bounded by the chord $e^{i\psi_1}$, $e^{i\psi_2}$ and the arc of $|\xi|=1$ contained in β. Now since $\varphi(\xi)$ is bounded, say $|\varphi(\xi)|<k$ for $|z|<1$, and since $e_\xi(\varphi, p)$ is an internal element of $\xi(\varphi)$ there are two possibilities: either (i) this continuation contains external elements in which case it contains a boundary element $e_\xi(\varphi, q)$, q being a point of M within the circle $|\varphi|<k$; or (ii) this continuation contains only internal elements in which case it must be brought to a stop by a transcendant singularity ω on M within the circle $|\varphi|<k$. In case (i) the path pq is mapped by the continuation on an asymptotic path in $|\xi|<1$ having its end point in the arc $\xi=e^{i\psi}$, $\psi_1<\psi<\psi_2$, contained in β, q being a regular asymptotic value at this point. In case (ii) it follows from Lemma 3 that the path $p\omega$ is mapped by the continuation on an asymptotic path in $|\xi|<1$ which, since $\varphi(\xi)|<k$ for $|\xi|<1$ also has its end point in the arc $\xi=e^{i\psi}$, $\psi_1<\psi<\psi_2$, contained in β and on which $\varphi(\xi)\to\omega$ as $|\xi|\to 1$. It follows that in case (i) $q\in\Gamma_P(f, \theta_1<\theta<\theta_2)$ and in case (ii) $\omega\in\Gamma_P(f, \theta_1<\theta<\theta_2)$, since $\varphi(\xi)=f(z(\xi))$. But M is any perpendicular to L so that there is at least one point q or one point ω in $|w|<k$ on every M. This proves the lemma.

Applying Lemma 4 to the case of a function $f(z)$ bounded in an annulus we may put $\Theta_1=0$ and $\Theta_2=2\pi$. We then have at once, by the remark at the head of this paragraph,

Theorem 7. *If $f(z)$ is meromorphic in $|z|<1$ and $CC(f)$ is not void, then $\Gamma(f)=\Gamma_P(f)$ is of positive linear measure.*

The Main Theorem in the Large.

9. The principal steps in the proof of our main theorem in the large can conveniently be isolated in separate lemmas. These lemmas and certain collateral results arising from them, are proved in this paragraph and in paragraph 10. The main theorem itself is proved in paragraph 11 with further developments in paragraphs 12–14.

Lemma 5. *If $f(z)$ is meromorphic in $|z|<1$ and if $a \in \complement\, \overline{\Gamma}(f)$, then either*

(i) $a \in \complement\, C(f)$; or

(ii) *a is an interior point of* $R(f)$.

For a given positive number σ we consider the set of domains $G(a, \sigma)$ in which $|f(z)-a|<\sigma$ $\left(\frac{1}{|f(z)|}<\sigma \text{ if } a=\infty\right)$ in $|z|<1$. The frontier of a $G(a, \sigma)$ consists only of level curves, which we shall call contours, on which

$$|f(z)-a|=\sigma \quad \left(\frac{1}{|f(z)|}=\sigma \text{ if } a=\infty\right)$$

in $|z|<1$ and points of the circumference $|z|=1$. Either of these elements may be absent; but if the frontier of $G(a, \sigma)$ contains a point of $|z|=1$ we say that it is unbounded.[1] Otherwise $G(a, \sigma)$ is bounded. The frontier of a bounded $G(a, \sigma)$ consists of a finite set of closed contours, while the frontier of an unbounded $G(a, \sigma)$ may contain either open or closed contours or both to any number finite or infinite.

By hypothesis we can find $\varepsilon>0$ such that $U(a, \varepsilon) \cap \Gamma(f)$ is void[2], and we have to consider the following possibilities:

(i) For some $\sigma<\varepsilon$ there is neither an unbounded $G(a, \sigma)$ nor an infinity of bounded $G(a, \sigma)$. Clearly in this case we can find $\eta>0$ such that $|f(z)-a| \geq \sigma$ in $1-\eta<|z|<1$ so that $a \in \complement\, C(f)$.

(ii) For all $\sigma<\varepsilon$ there is either an infinity of bounded $G(a, \sigma)$ or an unbounded $G(a, \sigma)$. In the former case each bounded $\overline{G}(a, \sigma)$[3] contains a zero of $f(z)-b$ for any b in $|b-a| \leq \sigma$ so that $U(a, \sigma) \subseteq R(f)$.

We treat the latter case in two stages. If the set of domains $G(a, \sigma)$ has only a finite number of closed contours and no open contour then there is only one unbounded $G(a, \sigma)$ which for a sufficiently small $\eta>0$ contains an annulus $1-\eta<|z|<1$ and in this annulus $|f(z)-a|<\sigma$. It follows that $\Gamma(f) \subseteq U(a, \varepsilon)$ and, by Theorem 7, that $\Gamma(f)=\Gamma(f) \cap U(a, \varepsilon)$ is of positive linear measure. Since this is contrary to hypothesis we conclude that if there is no unbounded $G(a, \sigma)$ having an open contour there is an unbounded $G(a, \sigma)$ having an infinity of closed contours.

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[1] We use the notation of Collingwood (1) p. 313.

[2] We may assume throughout that $a \neq \infty$; or alternatively $\Gamma(f)$ may be projected on the unit sphere so that $U(a, \varepsilon)$ is a neighbourhood on the sphere.

[3] We use the bar notation for closures. A bounded $\overline{G}(a, \sigma)$ is thus a connected region in which $|z|<1$, $|f(z)-a| \leq \sigma$.

We consider these two cases. First, if for a positive $\sigma < \varepsilon$ there is a $G(a, \sigma)$ having an unbounded contour we can choose a point z on it and continue the corresponding inverse element $e_z(w', w)$, which is an internal element of $z(w)$, indefinitely round the circumference $|w-a| = \sigma$. For, since $\Gamma(f) \cap U(a, \varepsilon)$ is void by hypothesis, the continuation contains no boundary element and there can be no transcendant point of the internal branch of $z(w)$ on the circumference $|w-a| = \sigma$. It follows that for any b having $|b-a| = \sigma$ there is an infinity of zeros of $f(z) - b$ on the open contour so that $b \in R(f)$. Secondly, if there is a $G(a, \sigma)$ having an infinity of closed contours, there is a zero of $f(z) - b$ on each of them so that again $b \in R(f)$. Since $\sigma < \varepsilon$ but is otherwise arbitrary we conclude that every point of $U(a, \varepsilon)$, except perhaps the point a, belongs to $R(f)$. We now shew that $a \in R(f)$ also. Let a' be any point in $U(a, \varepsilon/2)$ and put $\varrho = |a' - a| < \varepsilon/2$. Then $U(a', \varepsilon/2) \cap \Gamma(f)$ is void and we apply the foregoing argument to the domains $G(a', \varrho)$. Since $a' \in R(f) \subseteq C(f)$ case (i) is eliminated and we are in case (ii) so that $a \in R(f)$. Hence we have proved that $U(a, \varepsilon) \subseteq R(f)$. This proves the lemma.

Lemma 6. *Suppose that for some $\varepsilon > 0$ the set $U(a, \varepsilon) \cap \Gamma$ is of linear measure zero and that $a \in \complement \Gamma$. Then for all values of ϑ in $0 \leq \vartheta \leq 2\pi$ except perhaps for a set of measure zero there is no point of $U(a, \varepsilon) \cap \Gamma$ on the diameter of a circle $|w-a| \leq \sigma$ through the point $w = a + \sigma e^{i\vartheta}$ for any $\sigma < \varepsilon$.*

Put $\sigma < \varepsilon$ and consider the set of annular regions

$$\sigma \geq |w-a| \geq \frac{\sigma}{2}, \quad \frac{\sigma}{2} \geq |w-a| \geq \frac{\sigma}{2^2}, \cdots \frac{\sigma}{2^n} \geq |w-a| \geq \frac{\sigma}{2^{n+1}}, \cdots.$$

Call these regions $A_1, A_2, \ldots A_n, \ldots$. We say that a value ϑ is blocked in A_n if there is a point of Γ in A_n on the diameter through $w = a + \sigma e^{i\vartheta}$. Since $\Gamma \cap A_n$ is of linear measure zero the set B_n of blocked values ϑ is of measure zero. The set of values ϑ blocked in $|w-a| \leq \sigma$ is the union $\bigcup_n B_n$ of an enumerable set of sets B_n of measure zero and is therefore of measure zero. This proves the lemma.

This enables us to prove a generalisation of Lemma 5, namely

Lemma 7. *If $f(z)$ is meromorphic in $|z| < 1$, $a \in \complement \Gamma(f)$ and $U(a, \varepsilon) \cap \Gamma(f)$ is of linear measure zero for some $\varepsilon > 0$, then either*

(i) *$a \in \complement C(f)$; or*

(ii) *$U(a, \varepsilon) \subseteq \overline{R}(f)$ and $a \in R(f)$.*

Exactly as in the proof of Lemma 5, if $a \in C(f)$ then for all σ, $0 < \sigma < \varepsilon$, there is either an infinity of bounded domains $G(a, \sigma)$ or an unbounded $G(a, \sigma)$ having either an open contour or an infinity of closed contours.

Now since $\Gamma(f) \cap U(a, \varepsilon)$ is of linear measure zero the circumference $\gamma(\sigma_1)$ defined by $|w - a| = \sigma_1 < \varepsilon$ for almost all σ_1 in $0 < \sigma_1 < \varepsilon$ has no point of $\Gamma(f)$ upon it. If there is an infinity of closed contours of domains $G(a, \sigma_1)$, either bounded or unbounded, every value b on $\gamma(\sigma_1)$ belongs to $R(f)$. If there is not an infinity of closed contours there is at least one open contour of an unbounded $G(a, \sigma_1)$. Choose a point z upon such a contour and let $e_z(w', w)$, where w is on $\gamma(\sigma_1)$, be the corresponding element. We can continue $e_z(w', w)$ indefinitely round $\gamma(\sigma_1)$, the continuation containing only internal elements.[1] It follows that again every value b on $\gamma(\sigma_1)$ belongs to $R(f)$. Since every point of $U(a, \varepsilon)$ is arbitrarily near to a circumference $\gamma(\sigma_1)$ it follows that $U(a, \varepsilon) \subseteq \bar{R}(f)$. It will be observed that this does not require the condition $a \in \complement\, \Gamma(f)$.

It now remains to prove that $a \in R(f)$. Suppose on the contrary that $a \in \complement\, R(f)$. We can then find $\varepsilon(a) > 0$ such that for all $\sigma_1 < \varepsilon(a)$ all the zeros of $f(z) - a$ are contained in a finite set of bounded domains $G(a, \sigma_1)$. But since $a \in R'(f)$ we can find $b \in R(f)$ such that $|b - a| < \sigma_1$. Hence there is an unbounded domain $G_1(a, \sigma_1)$ which contains no zero of $f(z) - a$. Now by Lemma 6 we can find a diameter τ of the circle $\gamma(\sigma_1)$ on which there is no point of $\Gamma(f)$ and on which therefore every internal inverse element of $z(w)$ at an end point of τ on $\gamma(\sigma_1)$ can be continued through the point a to the antipodal point, the continuation again containing only internal elements. Choose a point z_1 on a contour of $G_1(a, \sigma_1)$ and continue the corresponding internal element $e_z(w, w_1)$, $|w_1 - a| = \sigma_1$ along the circumference to an end point of τ which is mapped on a point $z(\tau)$ of the contour. The corresponding element $e_{z(\tau)}$ can be continued along τ through the point a to the antipodal point and in this way τ is mapped on a cross cut of the domain $G_1(a, \sigma_1)$ on which there is a zero of $f(z) - a$ which therefore lies in $G_1(a, \sigma_1)$. But $G_1(a, \sigma_1)$ contains no zero of $f(z) - a$ so we have a contradiction and we conclude that $a \in R(f)$. This completes the proof of the lemma.

To complete the group of lemmas we have

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[1] In fact a continuation of an element $e_z(w', w)$ along any path contained in $U(a, \varepsilon)$ contains no boundary element since if there were such a boundary element then $U(a, \varepsilon) \cap \Gamma(f)$ would be of positive linear measure: see paragraph 7 above.

Lemma 8. *If $f(z)$ is meromorphic in $|z| < 1$ and*

(i) *if a is an isolated asymptotic value, i.e. if $a \in \Gamma(f) \cap \complement \Gamma'(f)$ then, for some $\varepsilon > 0$, $U(a, \varepsilon) - (a) \subseteq R(f)$; while*

(ii) *if $a \in \Gamma(f)$ and $U(a, \varepsilon) \cap \Gamma(f)$ is of linear measure zero for some $\varepsilon > 0$ then $U(a, \varepsilon) \subseteq \bar{R}(f)$.*

Choose $\sigma < \varepsilon$. Since $a \in \Gamma(f)$ there is at least one unbounded $G(a, \sigma)$.

(i) follows from the argument for the case of an unbounded $G(a, \sigma)$ in the proof of Lemma 5. This proves that $b \in R(f)$ for any

$$b = a + \sigma e^{i\vartheta}, \ 0 \leq \vartheta < 2\pi, \ 0 < \sigma < \varepsilon.$$

(ii) follows from the argument for the case of an unbounded $G(a, \sigma_1)$ in the proof of Lemma 7.

10. As a further preliminary to the proof of the main theorem in the large we prove

Theorem 8. *If $f(z)$ is meromorphic in $|z| < 1$, then the following relations are satisfied:*

(10.1) $$\text{Interior of } R(f) \subseteq \text{interior of } C(f)$$

(10.11) $$\subseteq \bar{R}(f)$$

(10.12) $$\subseteq C(f)$$

and from these, by taking complements,

(10.2) $$\complement C(f) \subseteq \text{interior of } \complement R(f)$$

(10.21) $$\subseteq \overline{\complement C}(f)$$

(10.22) $$\subseteq \overline{\complement R}(f).$$

Of these relations only (10.11) and its inverse (10.21) are not trivial. It is therefore sufficient to prove (10.21). This relation was first proved by Noshiro[1], but we give here a new and rather more direct proof based on dimension theory.[2]

Let a be an interior point of $\complement R(f)$ so that, for some $\varepsilon > 0$, $U(a, \varepsilon) \subseteq \complement R(f)$. Choose a sequence $\eta_n \to 0$ as $n \to \infty$ and define $X_n(f)$ as the set of values taken at

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[1] Noshiro (2) p. 230. The argument is reproduced in Noshiro (4) p. 67.

[2] We refer to Karl Menger, *Dimensionstheorie*, Leipzig 1928; or Witold Hurewicz and Henry Wallman, *Dimension Theory*, Princeton 1941.

least once in the ring $1-\eta_n<|z|<1$. Now $X_n(f)$ is an open set. For if $b \in X_n(f)$ there is a point $z(b)$ in $1-\eta_n<|z|<1$ such that $f(z(b))=b$ and a neighbourhood $U(z(b), \delta)$, $\delta>0$ in $1-\eta_n<|z|<1$ in which $f(z)$ takes all values in a neighbourhood $U(b, \varepsilon')$, $\varepsilon'>0$, in the w-plane so that $U(b, \varepsilon') \subseteq X_n(f)$ and b is an interior point of $X_n(f)$. Hence $\complement X_n(f)$ *is closed.*

Now not all of the sets $\complement X_n(f)$ can be of dimension ≤ 1. For if they were then, by Menger's summation theorem[1], $\complement R(f)=\bigcup_n \complement X_n(f)$ would be of dimension ≤ 1 which is impossible since $U(a, \varepsilon)$, which is contained in $\complement R(f)$, is of dimension 2. The same is true of the sets $\complement X_n(f) \cap U(a, \varepsilon)$.. We can therefore find n_0 and η_0 such that $\complement X_{n_0}(f) \cap U(a, \varepsilon)$ is of dimension 2, and hence, by a theorem of Menger and Urysohn[2], contains a domain. Let d be such a domain. Then $d \subseteq \complement C(f)$ and since $d \subseteq U(a, \varepsilon)$ and $\varepsilon>0$ is arbitrary it follows that $a \in \overline{\complement C}(f)$. This proves (10.21) and Theorem 8 follows.

The theorem has the following three corollaries.

Corollary 8.1. *If $f(z)$ is meromorphic in $|z|<1$ and if there is an interior point of $C(f)$, then $R(f)$ is dense in the interior of $C(f)$; and if $R(f)$ is nowhere dense, then $C(f)=\mathfrak{F}C(f)$ so that $C(f)$ is a Cantor curve.*

Immediate from (10.11).[3]

Corollary 8.2. *A necessary and sufficient condition that $\complement C(f)$ shall be void is that $\complement R(f)$ contains no interior point, i.e. $\complement R(f) \subseteq \mathfrak{F}R(f)$.*

The condition is necessary, for *if $\complement C(f)$ is void then $R(f)$ is everywhere dense* and it follows that there is no interior point of $\complement R(f)$. It is sufficient because if the interior of $\complement R(f)$ is void then $\complement C(f)$ is void by (10.2).

We require two further lemmas about the frontier sets $\mathfrak{F}C(f)$, $\mathfrak{F}R(f)$ and $\mathfrak{F}X(f)$.

Lemma 9. *If $f(z)$ is meromorphic in $|z|<1$, then*

$$\mathfrak{F}R(f) \cup \mathfrak{F}C(f)=\overline{\complement R}(f) \cap C(f). \tag{10.3}$$

We note that

$$\overline{\complement R}(f)=\mathfrak{F}R(f) \cup \text{ interior of } \complement R(f); \tag{10.31}$$

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[1] MENGER p. 91, or HUREWICS-WALLMAN p. 30.

[2] MENGER p. 242, or HUREWICS-WALLMAN p. 44,

[3] The weaker result: Interior of $C(f)$ not void implies $R(f)$ not void was proved, in quite a different way, by NOSHIRO (1) Theorems 5 and 6.

(10.32) $$\mathfrak{J} R(f) \cap C(f) = \mathfrak{J} R(f)$$

since $\mathfrak{J} R(f) \subseteq C(f)$; and by (10.21),

(10.33) $$\text{Interior of } \complement R(f) \subseteq \overline{\complement C}(f) = \complement C(f) \cup \mathfrak{J} C(f).$$

From (10.33),

$$C(f) \cap \text{ interior of } \complement R(f) \subseteq \mathfrak{J} C(f).$$

But also

$$\text{Interior of } \complement R(f) \subseteq \complement \mathfrak{J} R(f)$$

and so

(10.4) $$C(f) \cap \text{ interior of } \complement R(f) \subseteq \mathfrak{J} C(f) \cap \complement \mathfrak{J} R(f).$$

Now

$$\begin{aligned} \complement \mathfrak{J} R(f) &\subseteq \text{ interior of } R(f) \cup \text{ interior of } \complement R(f) \\ &\subseteq \text{ interior of } C(f) \cup \text{ interior of } \complement R(f) \end{aligned}$$

so that

$$\begin{aligned} \mathfrak{J} C(f) \cap \complement \mathfrak{J} R(f) &\subseteq \mathfrak{J} C(f) \cap \text{ interior of } \complement R(f) \\ &\subseteq C(f) \cap \text{ interior of } \complement R(f) \end{aligned} \quad (10.5)$$

since $\mathfrak{J} C(f) \subseteq C(f)$. Combining (10.4) and (10.5) we have

(10.6) $$C(f) \cap \text{ interior of } \complement R(f) = \mathfrak{J} C(f) \cap \complement \mathfrak{J} R(f).$$

Now from (10.31) and (10.32) we have

$$\begin{aligned} \overline{\complement R}(f) \cap C(f) &= \mathfrak{J} R(f) \cup (C(f) \cap \text{ interior of } \complement R(f)) \\ &= \mathfrak{J} R(f) \cup \mathfrak{J} C(f). \end{aligned}$$

by (10.6). This proves the lemma.

Lemma 10. *If $f(z)$ is meromorphic in $|z| < 1$, then*

(10.7) $$\mathfrak{J} X(f) \subseteq \mathfrak{J} R(f) \cup \mathfrak{J} C(f).$$

We see first that

(10.71) $$\mathfrak{J} X(f) \subseteq C(f).$$

For if $a \in \complement C(f)$ we can find $\varepsilon > 0$ and $\eta > 0$ such that $|f(z) - a| > \varepsilon$ for $1 - \eta < |z| < 1$ and hence the number N of a-points of $f(z)$ in $|z| < 1$ is finite. Now if $N > 0$ then $a \in$ interior of $X(f)$ and is therefore not a point of $\mathfrak{J} X(f)$; and if $N = 0$ then $\dfrac{1}{f(z) - a}$ is regular in $|z| < 1$ so that $|f(z) - a| > \varepsilon$ in $|z| < 1$ and $a \in$ interior of $\complement X(f)$ and is again not a point of $\mathfrak{J} X(f)$. Hence $\complement C(f) \subseteq \complement \mathfrak{J} X(f)$ and (10.71) follows.

Now suppose $a \in \mathfrak{J} X(f)$ is not a point of $\mathfrak{J} C(f)$. Then, by (10.71) and (10.11),

$$a \in \text{ Interior of } C(f) \subseteq \bar{R}(f).$$

Now a cannot be an interior point of $R(f)$ for it would then be an interior point of $X(f)$. Hence $a \in \mathfrak{F} R(f)$. This proves the lemma.

Lemma 10 implies a theorem of Persidskij[1] which in our notation may be stated as follows: *If $f(z)$ is meromorphic in $|z| < 1$, $\complement C(f)$ is not void and Δ is any component of $\complement C(f)$, then either $\Delta \subseteq X(f)$ or $\Delta \subseteq \complement X(f)$.* For $\mathfrak{F} X(f) \subseteq C(f)$.

Further, we observe that *if $C(f) = \mathfrak{F} C(f)$ then at least one component of $\complement C(f)$ is contained in $X(f)$.*

11. We are now in a position to prove our main theorem in the large, namely

Theorem 9. *If $f(z)$ is meromorphic in $|z| < 1$, then the following relations are satisfied:*

(i) *If $\Gamma(f)$ is unrestricted*

$$\mathfrak{F} R(f) \cup \mathfrak{F} C(f) = \overline{\complement R}(f) \cap C(f) \subseteq \overline{\Gamma}(f); \tag{11.1}$$

(ii) *if $\Gamma(f)$ is of linear measure zero*

$$\complement R(f) \subseteq \Gamma(f). \tag{11.2}$$

To prove (i) we use Lemma 5. By that lemma

$$\complement \overline{\Gamma}(f) \subseteq \complement C(f) \cup \text{ interior of } R(f) \tag{11.3}$$

and so, taking complements,

$$\overline{\complement R}(f) \cap C(f) \subseteq \overline{\Gamma}(f). \tag{11.4}$$

The complete relation (11.1) now follows from Lemma 9. Alternatively,

$$\mathfrak{F} R(f) \cup \mathfrak{F} C(f) \subseteq \overline{\Gamma}(f)$$

also follows immediately from (11.3). For, by (10.2) we have $\complement \overline{\Gamma}(f) \subseteq \complement \mathfrak{F} R(f)$ and, by (10.1), $\complement \overline{\Gamma}(f) \subseteq \complement \mathfrak{F} C(f)$ so that $\complement \overline{\Gamma}(f) \subseteq \complement \mathfrak{F} R(f) \cap \complement \mathfrak{F} C(f)$ and the result follows on taking complements.

To prove (ii) we use Lemma 7. It is convenient at this point to introduce a new notation. We define the set $\Gamma_+(f) \subseteq \overline{\Gamma}(f)$ as follows: *$a \in \Gamma_+(f)$ if, for all $\varepsilon > 0$, $U(a, \varepsilon) \cap \Gamma(f)$ is of positive linear measure.* A point of $\Gamma_+(f)$ is not necessarily a point of $\Gamma(f)$. Now from Lemma 7 we have

$$\complement \Gamma(f) \cap \complement \Gamma_+(f) \subseteq \complement C(f) \cup R(f)$$

and hence

$$\complement R(f) \subseteq \complement C(f) \cup \Gamma(f) \cup \Gamma_+(f). \tag{11.5}$$

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[1] Persidskij (1). The result is quoted by Doob (3) p. 450, who gives a proof.

But if $\Gamma(f)$ is of linear measure zero, $\Gamma_+(f)$ is void and, by Theorem 7, $\complement C(f)$ is also void. Therefore, under this condition $\complement R(f) \subseteq \Gamma(f)$ and (ii) is proved.

The corollary to Lemma 3 may be recalled at this point as supplementing (ii).

We shall now shew by an example that Theorem 9 (i) *is best possible* in the sense that in the general case $\overline{\Gamma}(f)$ cannot be replaced in (11.1) by $\Gamma(f)$. In fact we prove that *there exists a function* $w = g(z)$ *meromorphic in* $|z| < 1$ *such that* $\complement R(g) \cap C(g) \cap \complement \Gamma(g)$ *is not void.*

Using a well-known theorem of Koebe on the conformal mapping of symmetrical slit regions it is in principle a simple matter to construct an automorphic function $g(z)$ having the desired property.

Put $w = u + iv$ and denote by s_n the segment $u = \frac{1}{n}$, $-1 \leq v \leq 1$. We define the domain D, symmetrical about the real axis, by cutting the w-plane along all the segments s_n for $n = \pm 1, \pm 2, \ldots$ so that the frontier $\mathfrak{F} D$ of D consists of all the s_n and the segment s_∞ defined by $u = 0$, $-1 \leq v \leq 1$. All internal points of s_∞ are inaccessible points of $\mathfrak{F} D$. Let $t_n (n = 0, \pm 1, \pm 2 \ldots)$ be the segment of the real axis joining s_n and s_{n+1} with the convention that the segments $(-\infty, -1)$ $(+1, +\infty)$ are both designated t_0. Cut D along the real axis and let D_1 be the part above this axis and D_2 the part below. Denote by $s_n(1)$ the segments $u = \frac{1}{n}$, $(n = \pm 1, \pm 2, \ldots)$, $0 \leq v \leq 1$ and by $s_\infty(1)$ the segment $u = 0$, $0 \leq v \leq 1$. These together with the real axis form the frontier of D_1. We map D_1 conformally on the half-plane $\Im \zeta > 0$, the mapping function being denoted by $\zeta(w)$. The segments s_n and t_n are then mapped upon alternate segments $\sigma_n (n = \pm 1, \pm 2, \ldots)$ and $\tau_n (n = 0, \pm 1, \pm 2, \ldots)$ of $\Im \zeta = 0$ having a unique common limit point ϖ corresponding to the segment $s_\infty(1)$. We suppose $\zeta(w)$ to be normalised so that $\varpi = 0$ and $\zeta(1 + i) = \infty$. The τ_n then will lie in a finite segment $-k < \Re \zeta < k$.

Now consider the domain Δ of connectivity ∞ formed by cutting the ζ-plane along the segments $\tau_n (n = 0, \pm 1, \pm 2, \ldots)$. By a theorem of Koebe[1] there is a function $\xi = \xi(\zeta)$ which maps Δ conformally upon a symmetrical domain K (Kreisbereich) bounded by an infinity of distinct circles $\gamma_n (n = 0, \pm 1, \pm 2, \ldots)$ corresponding to the τ_n all lying outside of one another and all having their centres on the real axis $\Im \xi = 0$. We normalise the mapping so that $\xi(\infty) = \infty$ and $\xi = 0$ is the unique limit point of the γ_n. We now cut K along the real axis $\Im \xi = 0$ and denote by K_1 the upper part belonging to $\Im \xi > 0$ and by K_2 the lower part belonging

[1] Se Koebe (3) § 3 pp. 273 et seq.

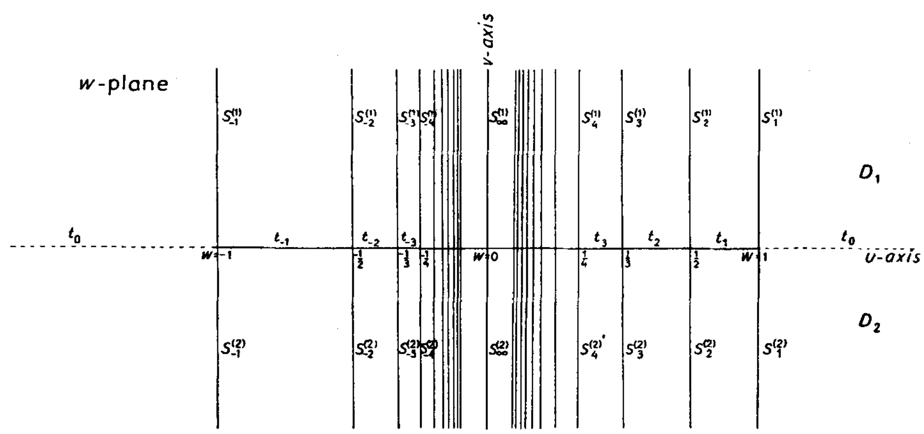

Fig. 2.

to $\Im \xi < 0$. We also denote by $s_n^{(2)}$ the segments $u = \frac{1}{n}$ $(n = \pm 1, \pm 2, \ldots)$, $-1 \leq v \leq 0$, in the w-plane. (See figure 2).

The function $\xi(\zeta(w))$ maps D_1 conformally upon K_1. Mapping the half plane $\Im \xi > 0$ upon the unit circle $|z| < 1$ so that $\xi = 0$ is mapped on $z = 1$ and $\xi = \infty$ on $z = -1$ we obtain a function $z(\xi(\zeta(w))) = \varphi(w)$ which maps D_1 upon a domain J_1 in $|z| < 1$ bounded by an infinity of arcs $\varepsilon_n(1)$ $(n = \pm 1, \pm 2, \ldots)$ of the circumference $|z| = 1$ corresponding to the segments $s_n^{(1)}$ and an infinity of circular arcs $\delta_n(1)$ $(n = 0, \pm 1, \pm 2, \ldots)$ in $|z| < 1$ and orthogonal to the circumference $|z| = 1$. The point $z = 1$ is the unique limit point of the two sequences of arcs $\varepsilon_n(1)$ and $\delta_n(1)$. Denote by $w = g(z)$ the inverse of $z = \varphi(w)$. This function $g(z)$ is meromorphic in J_1 which it maps conformally upon D_1 and is real and continuous on all the arcs $\delta_n(1)$. By the symmetry principle $g(z)$ is therefore meromorphic on all the $\delta_n(1)$ and in the domains obtained by reflecting J_1 in these arcs. Denote by J_2 the reflection of J_1 in $\delta_0(1)$ and let I_0 be the domain consisting of J_1, J_2 and their common frontier $\delta_0(1)$. Then by the symmetry principle, $g(z)$ takes every value belonging to D once and once only in I_0 but takes no value belonging to $\mathfrak{J}D$ in I_0. The domain I_0 is bounded by the two sequences of arcs

$$\varepsilon_n(1) \ (n = \pm 1, \pm 2, \ldots), \ \delta_n(1) \ (n = \pm 1, \pm 2, \ldots)$$

and their transforms with respect to $\delta_0(1)$ which we may denote by

$$\varepsilon_n(2) \ (n = \pm 1, \pm 2, \ldots)$$

and

$$\delta_n(2) \ (n = \pm 1, \pm 2, \ldots).$$

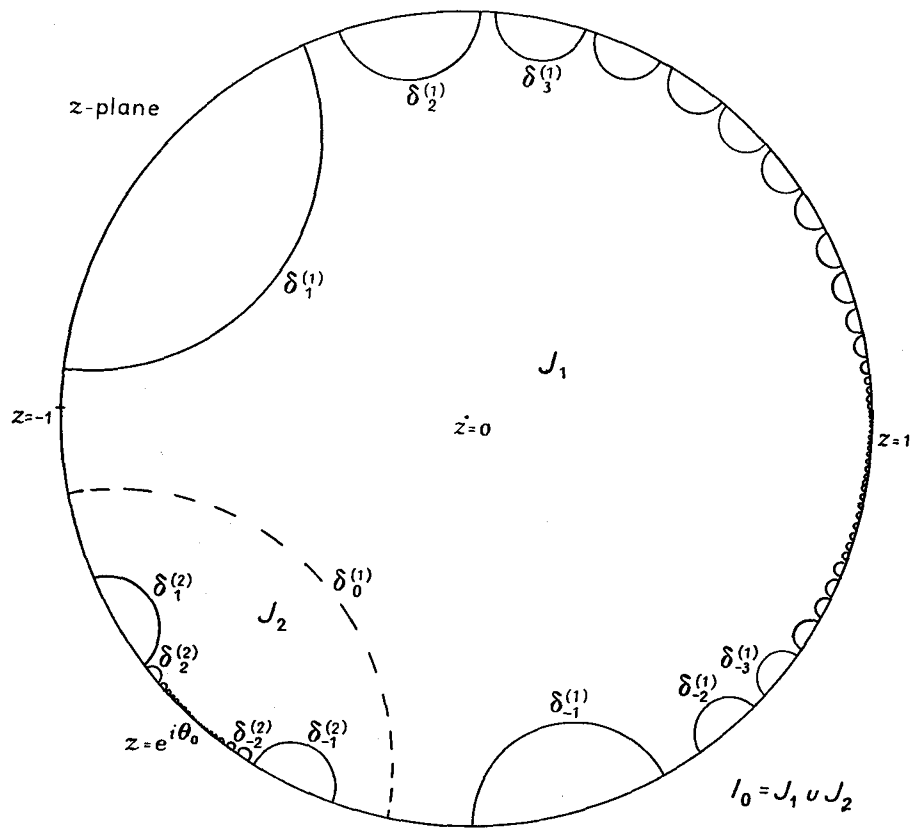

Fig. 3.

The boundary arcs of I_0 thus have the two limit points $z=1$ and its transform with respect to $\delta_0(1)$ which we may denote by $z=e^{i\theta_0}$. (See figure 3.)

Since $g(z)$ is real and continuous on the arcs $\delta_n(1)$ and $\delta_n(2)$ ($n=\pm 1, \pm 2, \ldots$) successive reflections with respect to these arcs and their transforms generate a group T of linear transformations of I_0 into a set of domains $I_1, I_2, \ldots I_m, \ldots$ which together with their common frontiers fill the unit circle $|z|<1$. The frontier of each such domain I_m consists of two sequences of arcs $\varepsilon_{mn}(1)$ and $\varepsilon_{mn}(2)$ of the circumference $|z|=1$, the transforms of $\varepsilon_n(1)$ and $\varepsilon_n(2)$, and two sequences of orthogonal arcs $\delta_{mn}(1)$ and $\delta_{mn}(2)$ in $|z|<1$, the transforms of $\delta_n(1)$ and $\delta_n(2)$, and the two limit points of these sequences, the transforms of $z=1$ and $z=e^{i\theta}$. The function $g(z)$ is thus automorphic with respect to the group T and the points $z=1$, $z=e^{i\theta_0}$ and their transforms are limit points of the group.

We have thus shewn that $g(z)$ is meromorphic in $|z|<1$ and that $R(g)=D$ and $\complement R(g)=\mathfrak{F}D$. Further, it follows from the structure of D that

$$\overline{\complement R}(g)=\complement R(g)=\mathfrak{F}D$$

and that

$$\complement C(g)=\complement \bar{R}(g)$$

is void. Now

$$\frac{i}{2}\subseteq \complement R(g)=\complement R(g)\cap C(g);$$

and it is easily shewn that $\frac{i}{2}\subseteq \complement \Gamma(g)$.

For consider any continuous path in $|z|<1$ tending to the circumference $|z|=1$, and let it be defined by a continuous function $z=p(t)$, $0<t<\infty$, where $|p(t)|<1$ and $\lim_{t\to\infty}|p(t)|=1$. Suppose first that for all $t>t_0$, say, $z=p(t)$ lies in one of the domains I_m which without loss of generality we may take to be I_0. Since $\frac{i}{2}$ lies outside D_2 and is an inaccessible frontier point of D_1 which is mapped conformally on J_1 it follows that $w=g(p(t))$ cannot tend to $\frac{i}{2}$ as t tends to ∞. Secondly, suppose that the path $z=p(t)$ has points in an infinity of the domains I_m. It must therefore cut an infinity of arcs $\delta_{mn}(1)$ or $\delta_{mn}(2)$. But at each point of intersection with one of these arcs $w=g(p(t))$ is real and it follows that $g(p(t))$ cannot tend to $\frac{i}{2}$ as $t\to\infty$. This proves our assertion. Finally, we note with regard to $g(z)$ that it is analytic or algebraic on the arcs $\varepsilon_{mn}(1)$ and $\varepsilon_{mn}(2)$ corresponding to $s_n^{(1)}$ and $s_n^{(2)}$ so that $\Gamma(g)$ is of positive linear measure. The function generated by continuation of $g(z)$ across the arcs $\varepsilon_{mn}(1)$ and $\varepsilon_{mn}(2)$, is a multiform function of which $g(z)$ is a uniform branch.

12. We proceed to deduce some of the consequences of Theorem 9 and the preceding lemmas. In the first place, if $\complement R(f)$ is restricted in some way so as to make $\complement R(f)\subseteq\mathfrak{F}R(f)$ results for $\complement R(f)$ follow at once from (11.1). For example, we have

Corollary 9.1. *A necessary and sufficient condition for*

$$\complement R(f)\subseteq\bar{\Gamma}(f) \tag{12.1}$$

is that $\complement R(f)$ *should contain no interior point; and if a value* $a\in\complement R(f)$ *is not in* $\bar{\Gamma}(f)$, *then* $a\in\complement C(f)\subseteq$ *interior of* $\complement R(f)$.

The condition is necessary since by (11.1)

$$\complement R(f) \cap C(f) \subseteq \overline{\complement R}(f) \cap C(f) \subseteq \overline{\Gamma}(f)$$

and so

$$\complement R(f) \cap \complement \overline{\Gamma}(f) \subseteq \complement C(f).$$

It is sufficient since it implies that

$$\complement R(f) \subseteq \mathfrak{J} R(f) \subseteq \overline{\Gamma}(f) \qquad \text{by (11.1)}.$$

An equivalent statement of the corollary follows from Corollary 8.2.

A necessary and sufficient condition for (12.1) *to be satisfied is that* $\complement C(f)$ *should be void.*

Since, by Theorem 1, $\complement C(f)$ is void if $T(r, f)$ is unbounded we have therefore

Corollary 9.2. *If* $f(z)$ *is meromorphic in* $|z| < 1$ *and* $T(r, f)$ *is unbounded, then* $\complement R(f) \subseteq \overline{\Gamma}(f)$.

Further corollaries follow if we impose a restriction upon the set $\Gamma(f)$.

Corollary 9.3. *If* $\Gamma(f)$ *is void then* $\complement R(f)$ *is void.*

This follows immediately from (11.2).

This result was originally proved by Noshiro[1], but in quite a different way. More generally, from (11.2) we have

Corollary 9.4. *If* $\Gamma(f)$ *is an isolated set then* $\complement R(f) \subseteq \Gamma(f)$ *is also isolated. In particular, if* $\Gamma(f)$ *is finite then* $\complement R(f) \subseteq \Gamma(f)$ *is also finite.*

By this corollary and Theorem 6 we have

Corollary 9.5. *If* $\complement R(f)$ *is infinite, then* $\Gamma(f)$ *is infinite and if* $\complement R(f)$ *is not an isolated set, then* $\Gamma(f)$ *is not an isolated set. In either case* $\Gamma(f) = \Gamma_P(f)$ *and* $\Phi(f)$ *is void.*

Corollaries 9.4 and 9.5 are illustrated by the modular function $\mu(z)$ regular in $|z| < 1$ for which $\complement R(f)$ and $\Gamma(f)$ both consist of $0, 1, \infty$ and by $Q(z) = \log \mu(z)$ for which $\complement R(f)$ and $\Gamma(f)$ consist of ∞, $\pm 2 n \pi i$ $(n = 0, 1, 2, \ldots)$.[2]

Two other special cases are of interest.

Corollary 9.6. (i) *If* $\complement R(f)$ *is of positive capacity, then* $\Gamma(f)$ *is of positive capacity; and* (ii) *if* $\complement R(f)$ *is of positive linear measure, then* $\Gamma(f)$ *is of positive linear measure.*

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[1] NOSHIRO (3) Theorem 4.

[2] J. E. LITTLEWOOD, *Theory of Functions*, Oxford 1944, p. 185; or LITTLEWOOD (1), p. 489. See also A. HURWITZ—R. COURANT, *Funktionentheorie*, Berlin 1929, p. 432.

Of these, (i) is a known result. For by Frostman's Theorem, if $\complement R(f)$ is of positive capacity $T(r, f) = O(1)$ and it follows from a theorem of Nevanlinna[1] that $\Gamma(f)$ is of positive capacity. As regards (ii), this is a stronger form of Theorem 7 which, in virtue of Corollary 8.3, may be stated in the form: *If $\complement R(f)$ is of positive plane measure, then $\Gamma(f)$ is of positive linear measure.*

Finally, combining Theorem 9 with Lemma 10 we have

Corollary 9.7.

$$\mathfrak{J}\, X(f) \subseteq \overline{\Gamma}(f);$$

and if $\Gamma(f)$ is of linear measure zero, then

$$\mathfrak{J}\, X(f) \subseteq \Gamma(f).$$

For since $X(f)$ is open we have $R(f) \subseteq X(f) \subseteq \complement\, \mathfrak{J}\, X(f)$ and so, if $\Gamma(f)$ is of linear measure zero,

$$\mathfrak{J}\, X(f) \subseteq \complement\, R(f) \subseteq \Gamma(f).$$

13. It follows from (11.1) that $\mathfrak{J}\, C(f) \subseteq \overline{\Gamma}(f)$, and since

$$\text{Interior of } \overline{\Gamma}(f) \subseteq \text{ interior of } C(f)$$

it follows that

(13.1) $$\mathfrak{J}\, C(f) \subseteq \mathfrak{J}\, \overline{\Gamma}(f).$$

Further, $\mathfrak{J}\, C(f)$ not void implies that $\Gamma(f) = \Gamma_P(f)$ by Theorem 7.

We can, however, prove a stronger result, namely

Theorem 10. *If $f(z)$ is meromorphic in $|z| < 1$, then*

(13.2) $$\mathfrak{J}\, C(f) \subseteq \mathfrak{J}\, \overline{\Gamma}(f) \cap \mathfrak{J}\, \Gamma_+(f).$$

From Lemma 7 we have

$$\complement\, \Gamma(f) \cap \complement\, \Gamma_+(f) \subseteq \complement\, C(f) \cup \text{ interior of } \overline{R}(f)$$

(13.3) $$\subseteq \complement\, C(f) \cup \text{ interior of } C(f);$$

and from Lemma 8 (ii)

$$\Gamma(f) \cap \complement\, \Gamma_+(f) \subseteq \text{ interior of } \overline{R}(f)$$

(13.4) $$\subseteq \text{ interior of } C(f).$$

Combining (13.3) and (13.4),

$$\complement\, \Gamma_+(f) \subseteq \complement\, C(f) \cup \text{ interior of } C(f)$$
$$\subseteq \complement\, \mathfrak{J}\, C(f)$$

[1] E.A.F. p. 198 Satz.

so that $\mathfrak{F}C(f) \subseteq \Gamma_+(f)$, and since

$$\text{Interior of } \Gamma_+(f) \subseteq \text{interior of } C(f)$$

it follows that

$$\mathfrak{F}C(f) \subseteq \mathfrak{F}\Gamma_+(f). \tag{13.5}$$

Combining (13.1) and (13.5) the theorem is proved.

Theorem 10 supplements Theorem 7. For, since $\Gamma_+(f)$ is closed, it associates with the frontier set $\mathfrak{F}C(f)$ a set of asymptotic values of positive linear measure while Theorem 7 merely asserts that if $\mathfrak{F}C(f)$ is not void then $\Gamma_+(f)$ is not void.

Corollary 10.1. *A component of one of the open sets $\complement\Gamma_+(f)$ or $\complement\overline{\Gamma}(f)$ is either a component of $\complement C(f)$ or is interior to $C(f)$.*

14. We have seen (Corollary 8.1) that if $R(f)$ is nowhere dence then $C(f)=\mathfrak{F}C(f)$. In particular this is true if $R(f)$ is an isolated set. We denote by $R_i(f)$ the set of isolated points of $R(f)$. With this definition we see more generally, that $R_i(f) \subseteq \mathfrak{F}C(f)$. For $R_i(f) \subseteq C(f)$; and if $a \in R_i(f)$, then for all $\varepsilon > 0$, $u(a, \varepsilon)$ contains interior points of $\complement C(f)$ and so also points of $\complement C(f)$. Therefore $a \in \mathfrak{F}C(f)$. In virtue of Theorem 10, we have thus proved

Theorem 11. *If $f(z)$ is meromorphic in $|z|<1$ and if $R(f)$ contains a set $R_i(f)$ of isolated values, then*

$$R_i(f) \subseteq \mathfrak{F}\overline{\Gamma}(f) \cap \mathfrak{F}\Gamma_+(f). \tag{14.1}$$

Also, if $R(f)$ is nowhere dense, then

$$R(f) \subseteq \mathfrak{F}\overline{\Gamma}(f) \cap \mathfrak{F}\Gamma_+(f). \tag{14.2}$$

Corollary 11.1. *If $R(f)$ is nowhere dense, then*

$$\overline{\Gamma}(f) = \Gamma_+(f) = \mathfrak{F}\overline{\Gamma}(f) = C(f).$$

For

$$\mathfrak{F}C(f) \subseteq \Gamma_+(f) \subseteq \overline{\Gamma}(f) \subseteq \mathfrak{F}\overline{\Gamma}(f) \subseteq C(f) = \mathfrak{F}C(f).$$

On comparing Theorem 11 with Lemma 8 (i) we see that, in a certain sense, the isolated points of $R(f)$ and the isolated points of $\Gamma(f)$ have a reciprocal property. For Lemma 8 (i) shews that, denoting by $\Gamma_i(f)$ the set of isolated points of $\Gamma(f)$ and by $(\complement R(f))_i$ the set of isolated points of $\complement R(f)$, we have

$$\Gamma_i(f) \subseteq (\complement R(f))_i \cup \text{interior of } R(f), \tag{14.3}$$

which we may regard as the counterpart to (14.1).

For $(\complement R(f))_i$ we also have the complementary relation

$$(14.4) \qquad (\complement R(f))_i \subseteq \Gamma(f) \cup \Gamma_+(f).$$

For plainly $(\complement R(f))_i \subseteq \complement R(f) \cap C(f)$ and (14.4) follows from (11.5). On the analogy with the case of $F(z)$ meromorphic in the plane $|z| < \infty$, where

$$\complement R(F) = (\complement R((F))_i \subseteq \Gamma(F),$$

it is natural to ask whether the set $\Gamma_+(f)$ on the right of (14.4) can be eliminated; but this question remains open.

Theorem 11 is illustrated by the function $h(z)$ defined as follows. Put

$$\varphi(\xi) = e^{-\xi^2} \cos \xi^{\,1}$$

and consider $\varphi(\xi)$ in the angle $|\arg \xi| \leq \frac{\pi}{4} - \delta$, when $0 < \delta < \frac{\pi}{4}$. $\varphi(\xi)$ tends uniformly to zero as ξ tends to infinity in this angle and takes the value 0 an infinity of times and every other value a finite number of times only. Let $z = z(\xi)$ be the function which maps the angle on the unit circle putting $\xi = 0$ and $\xi = \infty$ onto $z = -1$ and $z = 1$ respectively and let $\xi(z)$ be its inverse. Now put $h(z) = \varphi(\xi(z))$. Then plainly $R(h)$ consists of the single value 0; $h(z)$ is bounded; and every point of $|z| = 1$ with the exception of $z = 1$ is a regular point. So the set $F(h)$ consists of the whole circumference $|z| = 1$. Evidently $0 \in \Gamma_+(f)$.

PART II.

Boundary Theorems in the Small.

Preliminaries.

15. Let $e^{i\theta}$ be any point of the circumference $|z| = 1$. Although these are well established[2] we give here, for completeness and consistency of notation, the formal definitions of the cluster set and range of values of $f(z)$ at the point $e^{i\theta}$. The definitions of the sets $\Gamma_P(f, e^{i\theta})$ and $\Phi(f, e^{i\theta})$ have already been given in paragraph 4.

(i) **The Cluster Set** $C(f, e^{i\theta})$. $a \in C(f, e^{i\theta})$ if there is a sequence $\{z_n\}$, $|z_n| < 1$, such that $\lim_{n\to\infty} z_n = e^{i\theta}$ and $\lim_{n\to\infty} f(z_n) = a$. The complementary set is denoted by $\complement C(f, e^{i\theta})$ and the frontier by $\mathfrak{F} C(f, e^{i\theta})$.

[1] This example was given in a different connection by NOSHIRO (1) p. 29.

[2] These concepts were first formulated by PAINLEVÉ (1 and 2).

(ii) **The Range of Values** $R(f, e^{i\theta})$. $a \in R(f, e^{i\theta})$ if there is a sequence $\{z_n\}$, $|z_n| < 1$, such that $\lim_{n\to\infty} z_n = e^{i\theta}$ and $f(z_n) = a$ for all values of n. The complementary set is denoted by $\complement R(f, e^{i\theta})$ and the frontier by $\mathfrak{F} R(f, e^{i\theta})$.

If $b \in \complement R(f, e^{i\theta})$ we can find a positive number δ such that $f(z) \neq b$ in the domain $|z - e^{i\theta}| < \delta$, $|z| < 1$.

As before, we denote closures by $\bar{C}$, $\bar{R}$ etc. and derived sets by C', R' etc.

We observe trivially that $C(f, e^{i\theta})$ *is not void for any value of* θ. It is known[1], and can obviously be proved by the method of Theorem 2, that $C(f, e^{i\theta})$ *is either a single point or a continuum.*

Conversely, we have the following

Theorem of Gross.[2] *Given any continuum* C *and any* θ, *there is a function* $f(z)$ *meromorphic in* $|z| < 1$ *such that* $C = C(f, e^{i\theta})$.

Theorem 12. *If* $f(z)$ *is meromorphic in* $|z| < 1$ *and if for some value of* θ *the set* $C(f, e^{i\theta})$ *consists of a single value* a, *then* $\Gamma_P(f, e^{i\theta}) = a = C(f, e^{i\theta})$.

The proof is immediate. Let $z = z(t)$ $(0 < t \leq 1)$ define any continuous path such that $z(1) = e^{i\theta}$ and $|z(t)| < 1$ for $t < 1$. Then $\lim_{t\to 1} f(z(t)) = a$. For if not, we can find a sequence $t_1 < t_2 < \cdots < t_n < \cdots$, $t_n < 1$ and a number $\varepsilon > 0$ such that $|f(z(t_n)) - a| > \varepsilon$. We can therefore find a limit point b of the set $\{f(z(t_n))\}$ such that $|b - a| \geq \varepsilon$ and since $b \in C(f, e^{i\theta})$ the theorem is proved. The point $e^{i\theta}$ is a Fatou point for $f(z)$.

We need to be able to describe the behaviour of $f(z)$ at the boundary of $|z| < 1$ near a given point $z = e^{i\theta}$. For this purpose we adopt the following additional notations and definitions:

We write in general

$$C(f, \theta_1 \leq \theta \leq \theta_2) = \bigcup_{\theta_1 \leq \theta \leq \theta_2} C(f, e^{i\theta}) \tag{15.1}$$

with a similar definition for $C(f, \theta_1 < \theta < \theta_2)$; and, in particular,

$$C(f, 0 < |\theta - \theta_0| < \eta) = \bigcup_{0 < |\theta - \theta_0| < \eta} C(f, e^{i\theta})$$

and

$$C(f, |\theta - \theta_0| < \eta) = \bigcup_{|\theta - \theta_0| < \eta} C(f, e^{i\theta}).$$

[1] Gross (1) p. 20 § 6; or (2) p. 248 § 6: or Doob (1) p. 753.

[2] Gross (1) p. 20 § 7. More precise theorems were proved in Gross (2) pp. 248–253, see footnote 2, p. 123 below.

Then the *cluster set at $e^{i\theta_0}$ with respect to the boundary* is defined as the intersection of all the sets $\bar{C}(f, 0 < |\theta - \theta_0| < \eta)$, where η is arbitrarily small. This set[1] is denoted by

$$C_B(f, e^{i\theta_0}) = \bigcap_{\eta} \bar{C}(f, 0 < |\theta - \theta_0| < \eta). \tag{15.2}$$

The set $\Gamma_P(f, \theta_1 < \theta < \theta_2)$ was defined in paragraph 8. The definition is generalised as follows. We say that

$$a \in \Gamma(f, \theta_1 < \theta < \theta_2)$$

or

$$a \in \Gamma(f, \theta_1 \leq \theta \leq \theta_2)$$

if there is an asymptotic path on which $f(z)$ tends to a and whose end is contained in the open arc $z = e^{i\theta}$, $\theta_1 < \theta < \theta_2$ or the closed arc $z = e^{i\theta}$, $\theta_1 \leq \theta \leq \theta_2$, respectively. For brevity we write

$$\Gamma(f, \theta - \eta < \theta' < \theta + \eta) = \Gamma(f, |\theta' - \theta| < \eta)$$

and, in particular,

$$\Gamma(f, -\eta < \theta < \eta) = \Gamma(f, |\theta| < \eta),$$

and similarly for Γ_P and Γ_A.

The intersection of the sets $\Gamma(f, |\theta' - \theta| < \eta)$, which we denote by

$$\chi(f, e^{i\theta}) = \bigcap_{\eta} \Gamma(f, |\theta' - \theta| < \eta) \tag{15.3}$$

plays a similar role in the theory in the small to that of $\Gamma(f)$ in the theory in the large. It is convenient also to have the notation

$$\chi_P(f, e^{i\theta}) = \bigcap_{\eta} \Gamma_P(f, |\theta' - \theta| < \eta)$$

and

$$\chi_A(f, e^{i\theta}) = \bigcap_{\eta} \Gamma_A(f, |\theta' - \theta| < \eta).$$

Then

$$\chi(f, e^{i\theta}) = \chi_P(f, e^{i\theta}) \cup \chi_A(f, e^{i\theta}).$$

We note that

$$\chi'(f, e^{i\theta}) \subseteq \bigcap_{\eta} \Gamma'(f, |\theta' - \theta| < \eta)$$

and, writing

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[1] This definition, generalising the original concept of Painlevé, was introduced by GROSS (2) pp. 248–249.

(15.4) $$\chi^*(f, e^{i\theta}) = \bigcap_{\eta} \bar{\Gamma}(f, |\theta' - \theta| < \eta),$$

we have

(15.5) $$\chi(f, e^{i\theta}) \subseteq \bar{\chi}(f, e^{i\theta}) \subseteq \chi^*(f, e^{i\theta}).$$

The set $\chi(f e^{i\theta})$ may be void even though $\Gamma(f, |\theta' - \theta| < \eta)$ is not void for any $\eta > 0$; but $\chi^*(f, e^{i\theta})$ is not void unless $\Gamma(f, |\theta' - \theta| < \eta)$ is void for some $\eta > 0$.

Plainly

(15.6) $$\chi^*(f, e^{i\theta}) \subseteq C(f, e^{i\theta}).$$

The analogues in the small of Theorems B and C.

16. The characteristic Picard Theorem in the small is that of Gross and Iversen.[1] It is evident that

$$C_B(f, e^{i\theta}) \subseteq C(f, e^{i\theta}).$$

$C_B(f, e^{i\theta})$ is closed but not necessarily connected; but its two sub-sets

$$C_{Br}(f, e^{i\theta}) = \bigcap_{\eta} \bar{C}(f, -\eta < \theta' - \theta < 0)$$

and $C_{Bl}(f, e^{i\theta}) = \bigcap_{\eta} \bar{C}(f, 0 < \theta' - \theta < \eta)$ are both connected.[2] It is known[3] that

(16.1) $$\mathfrak{F}\, C(f, e^{i\theta}) \subseteq \mathfrak{F}\, C_B(f, e^{i\theta}).$$

Doob[4] has given a strikingly simple proof of the theorem of Gross-Iversen, which in our notation is stated as follows.

Theorem B′ (Gross-Iversen). *If $f(z)$ is meromorphic in $|z| < 1$, then for any value of θ, $f(z)$ takes every value belonging to $C(f, e^{i\theta})$ but not to $C_B(f, e^{i\theta})$, with two*

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[1] Gross (2) p. 291 § 6; Iversen (2) p. 13 § 12.

[2] We shall not be further concerned with the sets $C_{Br}(f, e^{i\theta})$ and $C_{Bl}(f, e^{i\theta})$. However, the relations between these sets and the sets $C(f, e^{i\theta})$ and $C_B(f, e^{i\theta})$ are significant and were studied in considerable detail by Gross and Iversen. See particularly Gross (2) pp. 248–253 and pp. 281–284 and Iversen (3) pp. 8–18.

Quite recently, interesting theorems on the structure of $C(f, e^{i\theta})$ of a rather different type from those proved by previous writers have been proved by Caratheodory (1) and Weigand (1). The principal theorem of Weigand does, however, contain Gross' theorem, quoted in § 15 above. But the methods of Gross and Weigand, which are similar in principle, do not apparently enable us to prove, in the abscence of any restriction on the continuum C, that $C = C(f)$ for some $f(z)$ meromorphic in $|z| < 1$.

[3] This was first proved by Beurling (1) p. 101. See also Noshiro (2 and 4).

[4] Doob (1).

possible exceptions, in every neighbourhood of $z = e^{i\theta}$ *contained in* $|z| < 1$; *i.e. the set*

$$\complement R(f, e^{i\theta}) \cap C(f, e^{i\theta}) \cap \complement C_B(f, e^{i\theta})$$

contains at most two values. Also, if this set contains two values, then $\complement R(f, e^{i\theta})$ *contains no other values and* $\complement C(f, e^{i\theta})$ *is accordingly void.*

This theorem follows easily from another theorem of Gross and Iversen which is an analogue in the small of Theorem C, namely

Theorem C′ (Gross-Iversen). *If* $f(z)$ *is meromorphic in* $|z| < 1$, *then for any value of* θ, *we have*

$$\complement R(f, e^{i\theta}) \cap C(f, e^{i\theta}) \cap \complement C_B(f, e^{i\theta}) \subseteq \Gamma_P(f, e^{i\theta}). \tag{16.2}$$

In other words, a Picard value at a point $e^{i\theta}$ which belongs to $C(f, e^{i\theta})$ but not to $C_B(f, e^{i\theta})$ is an asymptotic value. We observe that it follows at once from this theorem that if $R(f, e^{i\theta})$ is void then $C(f, e^{i\theta}) = C_B(f, e^{i\theta})$. For, in virtue of (16.1) and the fact that $\complement C_B(f, e^{i\theta})$ is an open set, $C(f, e^{i\theta}) \cap \complement C_B(f, e^{i\theta})$ is either void or an open set; but by Lindelöf's theorem $\Gamma_P(f, e^{i\theta})$ contains at most one value since $R(f, e^{i\theta})$ is void and hence $C(f, e^{i\theta}) \cap \complement C_B(f, e^{i\theta})$ is void.

The relation (16.2) has an obvious analogy with (11.5). However, the limitations of Theorem C′ are severe since it has no significance unless $C_B(f, e^{i\theta})$ differs from $C(f, e^{i\theta})$. For example, in the case of the modular function $\mu(z)$ the omitted values $0, 1, \infty$ belong to $\complement R(f, e^{i\theta})$ but both $\complement C(f, e^{i\theta})$ and $\complement C_B(f, e^{i\theta})$ are void for all values of θ and Theorem C′ tells us nothing about the set $\complement R(f, e^{i\theta})$. Our main theorem in the small (Theorem 16 below) leads to a generalization of Theorem C′ which is free from this limitation and leads to a generalization of Theorem B.

The Main Theorem in the Small.

17. Our method differs only in detail from that used to prove Theorem 9. We begin by proving the lemmas and collateral results analogous to those proved in paragraphs 9 and 10.

By analogy with the definition of the set $\Gamma_+(f)$ we say that $a \in \chi_*(f, e^{i\theta_0})$ *if for all* $\eta > 0$ *and* $\varepsilon > 0$ *the set* $U(a, \varepsilon) \cap \Gamma(f, |\theta - \theta_0| < \eta)$ *is of positive linear measure.*[1]

We now prove the analogue in the small of Lemma 5, namely

[1] This does not of course imply that $U(a, \varepsilon) \cap \chi(f, e^{i\theta})$ is of positive linear measure. But $a \in \complement \chi^*(f, e^{i\theta})$ does imply that for some ε_0, $U(a, \varepsilon_0) \cap \chi(f, e^{i\theta})$ is of linear measure zero.

Lemma 11. *If $f(z)$ is meromorphic in $|z|<1$ and if $a \in C\chi^*(f, e^{i\theta})$, then either*

(i) $a \in C\,C(f, e^{i\theta})$; *or*

(ii) *a is an interior point of $R(f, e^{i\theta})$; or*

(iii) *$a \in \Phi(f, e^{i\theta})$ and, for some $\varepsilon > 0$,*

$$U(a, \varepsilon) - (a) \subseteq R(f, e^{i\theta}).$$

We may clearly put $\theta = 0$. Denote by $E = E(1, \eta)$ the domain $|z-1| < 2\sin\eta/2$, $|z| < 1$, cut off from the unit circle by a circle of centre $z=1$ through the points $z = e^{\pm i\eta}$, and by $\Sigma = \Sigma(a, \sigma, \eta)$ the set of domains $G(a, \sigma) \cap E$.

By hypothesis we can find $\varepsilon > 0$ and $\eta > 0$ such that $U(a, \varepsilon) \cap \Gamma(f, |\theta| < \eta)$ is void and we have to consider the following possibilities.

(i) $a \in C\,C(f, 1)$; i.e. for some $\sigma \leq \varepsilon$ and some $\eta > 0$, $|f(z) - a| > \sigma$ in E so that Σ is void.

Otherwise, for all $\sigma > 0$ and $\eta > 0$ the set Σ is not void. In this case $z=1$ is a limit point of contours of Σ. For if not we can find η such that, for some σ, E is identical with Σ so that $|f(z) - a| < \sigma$ in E and so, by Lemma 4,

$$U(a, \varepsilon) \cap \Gamma(f, |\theta| < \eta)$$

is of positive linear measure, contrary to hypothesis. Case (i) being excluded we are now left with the two following alternatives.

(ii) For all $\sigma < \varepsilon_0 \leq \varepsilon$ the frontier of Σ contains either an infinity of closed contours or an open contour having an end in the arc $z = e^{i\theta}$, $|\theta| < \eta$. Then it follows by the method of Lemma 5 (ii) that, since there is no point of $\Gamma(f, |\theta| < \eta)$ on the circumference $\gamma(\sigma)$ defined by $|w - a| = \sigma$, there is an infinity of zeros of $f(z) - b$ in E for any b on $\gamma(\sigma)$ so that $b \in R(f, 1)$.

(iii) The frontier of Σ does not contain either an infinity of closed contours or an open contour having an end in the arc $z = e^{i\theta}$, $|\theta| < \eta$, but contains an infinity of open contours in E having no ends in this arc and therefore having their end points on the arc $|z-1| = 2\sin\eta/2$, $|z| < 1$. Since $f(z)$ is meromorphic in $|z| < 1$ only the points $e^{\pm i\eta}$ can be limit points of these end points. This clearly holds for all smaller η and hence such a set of contours converges to at least one of the arcs $z = e^{i\theta}$, $-\eta \leq \theta \leq 0$ or $0 \leq \theta \leq \eta$. If this condition is satisfied for any arbitrarily small $\sigma > 0$, then $a \in \Phi(f, 1)$; and so if $a \in C\,\Phi(f, 1)$ this condition is not satisfied for $\sigma < \varepsilon_0$, say.

Suppose now that $a \in \complement \Phi(f, 1)$. Then we can find ε_0 such that Σ satisfies condition (ii) for all $\sigma < \varepsilon_0$ and it follows by the argument above that $\gamma(\sigma) \subseteq R(f, 1)$ for all $0 < \sigma < \varepsilon_0$. We now shew that it also follows that $a \in R(f, 1)$. Choose a sequence $\sigma_1 > \sigma_2 > \cdots > \sigma_n \cdots$, $\lim_{n\to\infty} \sigma_n = 0$, $\sigma_n < \varepsilon$, and let b_1 by a point of $\gamma(\sigma_1)$. Since $b_1 \in R(f, 1)$ we can find a b_1-point z_1 of $f(z)$ i.e. a zero of $f(z) - b_1$, in $E(1, \eta/2)$. The corresponding element $e_{z_1}(w, b_1)$ of the inverse function $z(w)$ can be continued along the radius of $\gamma(\sigma_1)$ to the centre a and the continuation maps this radius on a curve $\varkappa_1$ in $|z| < 1$ one of whose end points is the point z_1 while the other is an a-point. Similarly, let b_2 be a point of $\gamma(\sigma_2)$ and choose a b_2-point z_2 of $f(z)$ in $E(1, \eta/3)$ and continue the corresponding element along the radius to a. This continuation maps the radius on $\varkappa_2$ joining z_2 to an a-point of $f(z)$. Repeat the process for b_3 on $\gamma(\sigma_3)$, z_3 in $E(1, \eta/4)$, and generally for b_n on $\gamma(\sigma_n)$, z_n in $E(1, \eta/n+1)$. In this way we obtain a sequence of curves $\varkappa_n$ for which $z = 1$ is a limit point and on which

$$|(f(z) - a| \leq \sigma_n \quad \left(\frac{1}{|f(z)|} \leq \sigma_n \text{ if } a = \infty\right).$$

Now suppose that $a \in \complement R(f, 1)$ and that η has been so chosen that E contains no a-point of $f(z)$. Then all the curves $\varkappa_n$ cross the circumference $|z - 1| = 2 \sin \eta/2$. Further, they have no limit point in E since at such a limit point z_0 we should have $f(z_0) = a$ contrary to hypothesis. Therefore the sequence of curves $\varkappa_n$ converges to at least one of the arcs $z = e^{i\theta}$, $-\eta \leq \theta \leq 0$, $0 \leq \theta \leq \eta$, so that $a \in \Phi(f, 1)$. It therefore follows that under the hypothesis of the lemma $U(a, \varepsilon) \subseteq R(f, 1)$ if

$$a \in C(f, 1) \cap \complement \Phi(f, 1).$$

Finally, $a \in \Phi(f, 1)$ implies $U(a, \varepsilon) - (a) \subseteq R(f, 1)$ since, by Theorem 6 and Lemma 2a, $\complement R(f, 1)$ contains at most two values. This completes the proof of the lemma.

As the analogue of Lemma 7 we prove

Lemma 12. *If $f(z)$ is meromorphic in $|z| < 1$ and if $a \in \complement \chi(f, e^{i\theta}) \cap \complement \chi_*(f, e^{i\theta})$ then either*

(i) $a \in \complement C(f, e^{i\theta})$; *or*

(ii) *a is an interior point of $\bar{R}(f, e^{i\theta})$ and*

$a \in R(f, e^{i\theta})$; *or*

(iii) $a \in \Phi(f, e^{i\theta})$ *and, for some* $\varepsilon > 0$,

$U(a, \varepsilon) - (a) \subseteq R(f, e^{i\theta})$.

Put $\theta=0$ and define E and Σ as in the proof of Lemma 11. Then, just as in that proof, if $a \in C(f,1) \cap \complement \Phi(f,1)$ we can find ε_0 such that for all $\sigma<\varepsilon_0$ the frontier of Σ contains either an infinity of closed contours or an open contour having an end in the arc $z=e^{i\theta}$, $|\theta|<\eta$. Now for almost all σ_1 in $0<\sigma_1<\varepsilon$ there is no point of $\Gamma(f, |\theta|<\eta)$ on $\gamma(\sigma_1)$; and it follows that $\gamma(\sigma_1) \subseteq R(f,1)$ and hence that $U(a,\varepsilon) \subseteq \bar{R}(f,1)$. To prove that $a \in R(f,1)$ we now choose a sequence

$$\sigma_{11} > \sigma_{12} > \cdots > \sigma_{1n} \ldots, \ \lim_{n\to\infty} \sigma_{1n}=0,$$

of numbers belonging to the set σ_1 and, as by Lemma 6 we may do, we choose a point b_{1n} on each $\gamma(\sigma_{1n})$ such that there is no point of $\Gamma(f, |\theta|<\eta)$ on the diameter of $\gamma(\sigma_{1n})$ through b_{1n}. We now repeat with the points b_{1n} the argument of Lemma 11 (iii) for the points b_n, which applies without modification. The lemma is therefore proved.

As the analogue in the small of Lemma 8 we prove

Lemma 13. *Suppose that $f(z)$ is meromorphic in $|z|<1$ and that $a \in \chi(f, e^{i\theta})$.*

(i) *If, for some $\eta<0$, a is an isolated point of $\Gamma(f, |\theta'-\theta|<\eta)$, then*

$U(a,\varepsilon)-(a) \subseteq R(f, e^{i\theta})$ for some $\varepsilon>0$.

(ii) *If $a \in \chi(f, e^{i\theta}) \cap \complement \chi^*(f, e^{i\theta})$, then a is an interior point of $\bar{R}(f, e^{i\theta})$.*

To prove this we use the methods of Lemmas 11 and 12. E and Σ are defined as before and we observe that $a \in C(f,1)$ so that Σ is not void. If $\Phi(f,1)$ is not void it follows, since $\complement R(f,1)$ then contains at most two values, that

$$U(a,\varepsilon)-(a) \subseteq R(f,1)$$

for some $\varepsilon>0$ independently of any condition on $\chi(f,1)$. We need then only consider the case when $\Phi(f,1)$ is void. Then, as we shewed in the proof of Lemma 11, we can find ε_0 such that the frontier of Σ contains either an infinity of closed contours or an open contour having an end in the arc $z=e^{i\theta}$, $|\theta|<\eta$. The argument of Lemma 11 (ii) then proves $U(a,\varepsilon)-(a) \subseteq R(f,1)$ if a is an isolated point of $\Gamma(f, |\theta|<\eta)$; and the argument of Lemma 12 (ii) proves that $U(a,\varepsilon) \subseteq \bar{R}(f,1)$ if $a \in \chi(f,1) \cap \complement \chi_*(f,1)$. The lemma is therefore proved.

18. The analogue in the small of Theorem 7 is the following theorem.

Theorem 14. *If $f(z)$ is meromorphic in $|z|<1$ and if for some θ the set $\complement C(f, e^{i\theta})$ is not void, then the set*

$$\Gamma(f, |\theta' - \theta| < \eta) = \Gamma_P(f, |\theta' - \theta| < \eta)$$

is of positive linear measure for all $\eta > 0$ *and thus the set* $\chi_*(f, e^{i\theta})$ *is not void.*

Put $\theta = 0$ and choose $a \in \complement C(f, 1)$. Then we can find η_0 such that the function $1/(f(z) - a)$ is regular and bounded in the domain $E(1, \eta_0)$. The theorem then follows immediately from Lemma 4.

To complete the preliminaries to the proof of our main theorem in the small, we observe that Theorem 8 holds for the sets $C(f, e^{i\theta})$ and $R(f, e^{i\theta})$. In fact we have

Theorem 15. *If* $f(z)$ *is meromorphic in* $|z| < 1$, *then for any* θ, $0 \leq \theta < 2\pi$,

(18.1) $$\textit{Interior of } R(f, e^{i\theta}) \subseteq \textit{interior of } C(f, e^{i\theta})$$

(18.11) $$\subseteq \bar{R}(f, e^{i\theta})$$

(18.12) $$\subseteq C(f, e^{i\theta})$$

and from these, by taking complements,

(18.2) $$\complement C(f, e^{i\theta}) \subseteq \textit{interior of } \complement R(f, e^{i\theta})$$

(18.21) $$\subseteq \overline{\complement C}(f, e^{i\theta})$$

(18.22) $$\subseteq \overline{\complement R}(f, e^{i\theta}).$$

Again, only (18.11) and (18.21) are not trivial. Putting $\theta = 0$ and defining $X_n(\mathrm{f}, 1)$ as the set of values taken at least once by $f(z)$ in the domain $E(1, \eta_n)$ when $\eta_n \to 0$ as $n \to \infty$ (18.11) is proved by precisely the same argument as (10.11) applied to the set $\complement R(f, 1) = \bigcup_n \complement X_n(f, 1)$.

Corollary 15.1. *If* $C(f, e^{i\theta})$ *has an interior point then* $R(f, e^{i\theta})$ *is dense in the interior of* $C(f, e^{i\theta})$; *and if* $R(f, e^{i\theta})$ *is nowhere dense then* $C(f, e^{i\theta}) = \mathfrak{F} C(f, e^{i\theta})$.[1]

Immediate from (18.11).

Corollary 15.2. *A necessary and sufficient condition that* $\complement C(f, e^{i\theta})$ *shall be void is that* $\complement R(f, e^{i\theta})$ *contains no interior point, i.e.* $\complement R(f, e^{i\theta}) \subseteq \mathfrak{F} R(f, e^{i\theta})$.

Just as for corollary 8.3, necessity follows from (18.21) and sufficiency from (18.2).

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[1] This latter result generalises a theorem of Gross (2), p. 260, § 10, who proved it for a function of bounded valency in $|z| < 1$.

Finally we have

Lemma 14. *If $f(z)$ is meromorphic in $|z| < 1$, then for any θ, $0 \leq \theta < 2\pi$,*

$$\mathfrak{F} R(f, e^{i\theta}) \cup \mathfrak{F} C(f, e^{i\theta}) = \overline{\complement R}(f, e^{i\theta}) \cap C(f, e^{i\theta}). \tag{18.3}$$

This lemma is deduced from (18.21) by precisely the same argument as Lemma 9 was deduced from (10.21). In view of this complete formal identity it is sufficient to refer to the proof of Lemma 9.

We now prove our main theorem in the small, namely

Theorem 16. *If $f(z)$ is meromorphic in $|z| < 1$, then for any θ, $0 \leq \theta < 2\pi$, the following relations are satisfied:*

(i) *If $\chi(f, e^{i\theta})$ is unrestricted*

$$\mathfrak{F} R(f, e^{i\theta}) \cup \mathfrak{F} C(f, e^{i\theta}) = \overline{\complement R}(f, e^{i\theta}) \cap C(f, e^{i\theta}) \subseteq \chi^*(f, e^{i\theta}) \cup \Phi(f, e^{i\theta}); \tag{19.1}$$

(ii) *If $\chi_*(f, e^{i\theta})$ is void i.e. if $\Gamma(f, |\theta' - \theta| < \eta)$ is of linear measure zero for some $\eta > 0$,*

$$\complement R(f, e^{i\theta}) \subseteq \chi(f, e^{i\theta}) \cup \Phi(f, e^{i\theta}) \tag{19.2}$$

and if, further, $\complement R(f, e^{i\theta})$ contains more than two values, $\Phi(f, e^{i\theta})$ and $\chi_A(f, e^{i\theta})$ are both void and

$$\complement R(f, e^{i\theta}) \subseteq \chi(f, e^{i\theta}) = \chi_P(f, e^{i\theta}). \tag{19.3}$$

To prove (i) we use Lemma 11. By that lemma

$\complement \chi^*(f, e^{i\theta}) \subseteq \complement C(f, e^{i\theta}) \cup$ interior of $R(f, e^{i\theta}) \cup \Phi(f, e^{i\theta})$ and hence

$$\complement \overline{R}(f, e^{i\theta}) \cap C(f, e^{i\theta}) \subseteq \chi^*(f, e^{i\theta}) \cup \Phi(f, e^{i\theta}). \tag{19.4}$$

The complete relation (19.1) now follows from Lemma 14.

To prove (ii) we use Lemma 12. This gives

$$\complement \chi(f, e^{i\theta}) \cap \complement \chi_*(f, e^{i\theta}) \subseteq \complement C(f, e^{i\theta}) \cup R(f, e^{i\theta}) \cup \Phi(f, e^{i\theta})$$

and hence

$$\complement R(f, e^{i\theta}) \subseteq \complement C(f, e^{i\theta}) \cup \chi(f, e^{i\theta}) \cup \chi_*(f, e^{i\theta}) \cup \Phi(f, e^{i\theta}). \tag{19.5}$$

Now, by Theorem 14, if $\chi_*(f, e^{i\theta})$ is void $\complement C(f, e^{i\theta})$ is also void and so we have (19.2) and if $\complement R(f, e^{i\theta})$ contains more than two values (19.3) follows by Theorem 6.

The function $g(z)$, which we constructed in paragraph 11 to shew that Theorem 9 (i) is best possible, also shews that Theorem 16 (i) is best possible in the sense that the set $\chi^*(f, e^{i\theta})$ cannot be replaced in (19.1) by $\chi(f, e^{i\theta})$. We have only to consider $g(z)$ in the neighbourhood of the point $z = 1$.

Theorem 16 is a concise statement of a somewhat complicated situation. We set out the various implications of the theorem in the general case, i.e. Theorem 16 (i), in order to shew that it is exhaustive.

For a given $a \in \overline{\complement R}(f, e^{i\theta}) \cap C(f, e^{i\theta})$ the alternative possibilities are tabulated as follows:

(i) $a \in \chi^*(f, e^{i\theta})$. In this case for all $\eta > 0$ we have $a \in \overline{\Gamma}(f, |\theta' - \theta| < \eta)$. This does not exclude the possibility that at the same time $a \in \Phi(f, e^{i\theta})$.

(ii) $a \in \complement \chi^*(f, e^{i\theta})$, but $a \in \Phi(f, e^{i\theta})$. In this case we can find η_0 and ε_0 such that $U(a, \varepsilon_0) \cap \Gamma(f, |\theta' - \theta| < \eta_0)$ is void. Also $\complement R(f, e^{i\theta})$ contains at most two values, one of which must be a since then $a \in \overline{\complement R}(f, e^{i\theta}) = \complement R(f, e^{i\theta})$. On further analysis we find three alternatives of this case.

(α) There is a set of curves C_n on which $f(z)$ tends uniformly to a as $n \to \infty$ and which converges to an arc $z = e^{i\theta'}$, $|\theta' - \theta| < \eta_1 \leq \eta_0$. Then, since

$$a \in \complement \Gamma(f, |\theta' - \theta| < \eta_1),$$

$\Gamma(f, |\theta' - \theta| < \eta_1)$ is void.

(β) There are two sets of curves C_n^+ and C_n^- converging respectively to arcs $z = e^{i\theta'}$, $\theta \leq \theta' \leq \theta + \eta_1$, $\theta - \eta_1 \leq \theta' \leq \theta$ on which $f(z)$ tends uniformly to a as $n \to \infty$. Then $\Gamma(f, 0 < |\theta' - \theta| < \eta_1)$ and $U(a, \varepsilon_0) \cap \Gamma_P(f, e^{i\theta})$ are both void.

(γ) There is one set of curves C_n^+ or C_n^- with the property described in (β) above and the corresponding set $\Gamma(f, \theta < \theta' < \theta + \eta_1)$ or $\Gamma(f, \theta - \eta_1 < \theta' < \theta)$ is void. The opposite arc $e^{i\theta'}$, $\theta - \eta_1 < \theta' < \theta$ or $\theta < \theta' < \theta + \eta$, which we denote by δ, is then a Fatou arc almost all points of which belong to the set $F(f)$ and the corresponding set $\Gamma_P(f, \theta - \eta_1 < \theta' < \theta)$ or $\Gamma_P(f, \theta < \theta' < \theta + \eta)$ lies outside $U(a, \varepsilon_0)$. For consider a point $e^{i\theta'}$ of this opposite arc. Since $\eta_1 \leq \eta_0$ we have $a \in \complement \chi^*(f, e^{i\theta'})$ so that

$$a \in \complement C(f, e^{i\theta'}) \cup R(f, e^{i\theta'})$$

by Lemma 11. Hence, since $a \in \complement R(f, e^{i\theta})$ we can choose η_2 such that in $E(e^{i\theta'}, \eta_2)$ $1/(f(z) - a)$ is regular and bounded and it follows that the arc $z = e^{i\theta''}$, $|\theta'' - \theta'| < \eta_1$, is a Fatou arc by Lemma 2 a. Since $e^{i\theta'}$ is an arbitrary point of the arc δ our assertion is proved.

Evidently the three subsidiary cases (α), (β) and (γ) exhaust case (ii) and, by Theorem 16 (i), this together with case (i) exhausts all the possibilities when no restriction is imposed on $\chi(f, e^{i\theta})$.

Case (i) is illustrated by $g(z)$ and by the modular function $\mu(z)$ when a is one of the omitted values $0, 1, \infty$. But in this case $\Gamma(f, |\theta' - \theta| < \eta)$ is finite for all θ and $\eta > 0$ so that $a \in \chi(f, e^{i\theta})$. Other illustrations are Koenigs' function $K(z)$, for which $\infty \in \chi(f, e^{i\theta}) \cap \Phi(f, e^{i\theta})$ for all θ, and a function $f(z)$ constructed by Cartwright[1], for which $\infty \in \Gamma_P(f, e^{i\theta}) \cap \Phi(f, e^{i\theta})$ while $\Gamma(f, 0 < |\theta' - \theta| < \eta)$, $\eta < \eta_0$, is void.

Case (ii) (α) is illustrated by Valiron's regular function $f(z)$ tending to infinity on a spiral asymptotic path which was referred to in paragraph 4 above.[2]

20. The pattern of corollaries of Theorem 16 and collateral results is closely similar to that arising from Theorem 9. We have first

Corollary 16.1. *A necessary and sufficient condition for*

$$\complement R(f, e^{i\theta}) \subseteq \chi^*(f, e^{i\theta}) \cup \Phi(f, e^{i\theta}) \tag{20.1}$$

is that $\complement R(f, e^{i\theta})$ *should contain no interior point; and if a value* $a \in \complement R(f, e^{i\theta})$ *is not in* $\chi^*(f, e^{i\theta}) \cup \Phi(f, e^{i\theta})$, *then*

$$a \in \complement C(f, e^{i\theta}) \subseteq \text{interior of } \complement R(f, e^{i\theta}).$$

The condition is necessary since by (19.1)

$$\begin{aligned}\complement R(f, e^{i\theta}) \cap C(f, e^{i\theta}) &\subseteq \overline{\complement R}(f, e^{i\theta}) \cap C(f, e^{i\theta}) \\ &\subseteq \chi^*(f, e^{i\theta}) \cup \Phi(f, e^{i\theta})\end{aligned}$$

and so

$$\begin{aligned}\complement R(f, e^{i\theta}) \cap \complement \chi^*(f, e^{i\theta}) \cap \complement \Phi(f, e^{i\theta}) &\subseteq \complement C(f, e^{i\theta}) \\ &\subseteq \text{interior of } \complement R(f, e^{i\theta}).\end{aligned}$$

The condition is sufficient since it implies

$$\complement R(f, e^{i\theta}) \subseteq \mathfrak{F} R(f, e^{i\theta}).$$

An equivalent statement of this corollary follows from Corollary 15.2.

A necessary and sufficient condition for (20.1) *to be satisfied is that* $\complement C(f, e^{i\theta})$ *should be void.*

If now we impose a restriction upon the set $\chi(f, e^{i\theta})$ we obtain a further group of corollaries.

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[1] CARTWRIGHT (1) § 4.3 pp. 177—181. Cartwright's function is actually asymmetrical with respect to $e^{i\theta}$. The symmetrical function $f(z)$ is the sum of two such functions.

[2] VALIRON (2).

Corollary 16.2. *If $\chi^*(f, e^{i\theta})$ is void, then $C R(f, e^{i\theta})$ contains at most two values and*

$$C R(f, e^{i\theta}) \subseteq \Phi(f, e^{i\theta});$$

and if both $\chi^*(f, e^{i\theta})$ and $\Phi(f, e^{i\theta})$ are void then $C R(f, e^{i\theta})$ is void.

This follows immediately from Theorem 16 (ii). For $\chi(f, e^{i\theta}) \cup \chi_*(f, e^{i\theta}) \subseteq \chi^*(f, e^{i\theta})$.

More generally, from Theorem 16 (ii) we have

Corollary 16.3. *If, for some $\eta > 0$, $\Gamma(f, |\theta' - \theta| < \eta)$ is an isolated set, then $C R(f, e^{i\theta})$ is an isolated set. In particular, if $\Gamma(f, |\theta' - \theta| < \eta)$ is a finite set, then $C R(f, e^{i\theta})$ is also a finite set.*

We may assume that $C R(f, e^{i\theta})$ contains more than two values, so that $\Phi(f, e^{i\theta})$ is void, otherwise the assertion is trivial. So (19.3) is satisfied and the corollary is proved.

Conversely, we have at once

Corollary 16.4. *If $C R(f, e^{i\theta})$ is infinite, then for all $\eta > 0$, $\Gamma(f, |\theta' - \theta| < \eta)$ is infinite; and if $C R(f, e^{i\theta})$ is not isolated, then for all $\eta > 0$, $\Gamma(f, |\theta' - \theta| < \eta)$ is not isolated. In either case $\Phi(f, e^{i\theta})$ is void, and for all sufficiently small $\eta > 0$, $\Gamma(f, |\theta' - \theta| < \eta) = \Gamma_P(f, |\theta' - \theta| < \eta)$.*

We conclude this group with the analogue of Corollary 9.6.

Corollary 16.5. (i) *If $C R(f, e^{i\theta})$ is of positive capacity then $\Phi(f, e^{i\theta})$ is void and for all sufficiently small $\eta > 0$,*

$$\Gamma(f, |\theta' - \theta| < \eta) = \Gamma_P(f, |\theta' - \theta| < \eta)$$

is of positive capacity; and (ii) *if $C R(f, e^{i\theta})$ is of positive linear measure then $\Phi(f, e^{i\theta})$ is void and for all sufficiently small $\eta > 0$,*

$$\Gamma(f, |\theta' - \theta| < \eta) = \Gamma_P(f, |\theta' - \theta| < \eta)$$

is of positive linear measure and consequently $\chi_*(f, e^{i\theta})$ is not void.

This again follows immediately from Theorem 16 (ii), remembering that if $C R(f, e^{i\theta})$ contains more than two values $\Phi(f, e^{i\theta})$ is void and $\Gamma_A(f, |\theta' - \theta| < \eta)$ is void for sufficiently small $\eta > 0$.

The second part of this corollary is a stronger form of Theorem 14.

21. A further strengthening of Theorem 14 is given by the following theorem, which is the analogue in the small Theorem 10.

Theorem 17. *If $f(z)$ is meromorphic in $|z|<1$, then*

$$\mathfrak{F}\, C(f, e^{i\theta}) \subseteq \mathfrak{F}\, \chi^*(f, e^{i\theta}) \cap \mathfrak{F}\, \chi_*(f, e^{i\theta}). \tag{21.1}$$

If $\mathfrak{F}\, C(f, e^{i\theta})$ is not void, then $\Phi(f, e^{i\theta})$ is void and $\chi(f, e^{i\theta}) = \chi_P(f, e^{i\theta})$. It follows then from (19.1) that $\mathfrak{F}\, C(f, e^{i\theta}) \subseteq \chi^*(f, e^{i\theta})$. But

$$\text{Interior of } \chi^*(f, e^{i\theta}) \subseteq \text{interior of } C(f, e^{i\theta})$$

and therefore

$$\mathfrak{F}\, C(f, e^{r\theta}) \subseteq \mathfrak{F}\, \chi^*(f, e^{i\theta})^1. \tag{21.2}$$

From Lemma 12 we have

$$\begin{aligned} \complement\chi(f, e^{i\theta}) \cap \complement\chi_*(f, e^{i\theta}) &\subseteq \complement C(f, e^{i\theta}) \cup \text{interior of } \bar{R}(f, e^{i\theta}) \\ &\subseteq \complement\mathfrak{F}\, C(f, e^{i\theta}) \end{aligned} \tag{21.3}$$

and from Lemma 13 (ii)

$$\begin{aligned} \chi(f, e^{i\theta}) \cap \complement\chi_*(f, e^{i\theta}) &\subseteq \text{interior of } \bar{R}(f, e^{i\theta}) \\ &\subseteq \complement\mathfrak{F}\, C(f, e^{i\theta}). \end{aligned} \tag{21.4}$$

Combining (21.3) and (21.4) we have

$$\complement\chi_*(f, e^{i\theta}) \subseteq \complement F\, C(f, e^{i\theta})$$

so that

$$\mathfrak{F}\, C(f, e^{i\theta}) \subseteq \chi_*(f, e^{i\theta}).$$

Hence, since

$$\chi_*(f, e^{i\theta}) \subseteq \chi^*(f, e^{i\theta}),$$

$$\mathfrak{F}\, C(f, e^{i\theta}) \subseteq \mathfrak{F}\, \chi_*(f, e^{i\theta}). \tag{21.5}$$

Corollary 17.1. *A component of one of the open sets $\complement\chi_*(f, e^{i\theta})$ or $\complement\chi^*(f, e^{i\theta})$ is either a component of $\complement C(f, e^{i\theta})$ or is interior to $C(f, e^{i\theta})$.*

22. The considerations of paragraph 14 apply equally in the small. If $R(f, e^{i\theta})$ is an isolated set then $\complement\bar{R}(f, e^{i\theta})$ is everywhere dense so that, by (18.11), the interior of $C(f, e^{i\theta})$ is void and hence $C(f, e^{i\theta}) = \mathfrak{F}\, C(f, e^{i\theta})$. We denote by $R_i(f, e^{i\theta})$ the set of isolated points of $R(f, e^{i\theta})$. If $a \in R_i(f, e^{i\theta})$ then, for all $\varepsilon > 0$, $U(a, \varepsilon)$ contains interior points of $CR(f, e^{i\theta})$ and hence, by (18.21), points of $\overline{\complement C}(f, e^{i\theta})$ and so also points of $\complement C(f, e^{i\theta})$. But $a \in C(f)$ and it follows that

$$R_i(f, e^{i\theta}) \subseteq \mathfrak{F}\, C(f, e^{i\theta}).$$

In virtue of Theorem 17 we have thus proved

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[1] The relation (21.2) was recently proved, by a different method, by Ohtsuka (1).

Theorem 18. *If $f(z)$ is meromorphic in $|z|<1$, then*

$$R_i(f, e^{i\theta}) \subseteq \mathfrak{J}\, C(f, e^{i\theta}) \subseteq \mathfrak{J}\, \chi^*(f, e^{i\theta}) \cap \mathfrak{J}\chi_*(f, e^{i\theta}). \tag{22.1}$$

Also, if $R(f, e^{i\theta})$ is nowhere dense, then

$$R(f, e^{i\theta}) \subseteq \mathfrak{J}\, \chi^*(f, e^{i\theta}) \cap \mathfrak{J}\, \chi_*(f, e^{i\theta}). \tag{22.2}$$

Corollary 18.1. *If $R(f, e^{i\theta})$ is nowhere dense, then*

$$\chi^*(f, e^{i\theta}) = \chi_*(f, e^{i\theta}) = \mathfrak{J}\, \chi^*(f, e^{i\theta}).$$

For

$$\mathfrak{J}\, C(f, e^{i\theta}) \subseteq \chi_*(f, e^{i\theta}) \subseteq \chi^*(f, e^{i\theta}) \subseteq \mathfrak{J}\, C(f, e^{i\theta}) \subseteq \mathfrak{J}\, \chi^*(f, e^{i\theta}) \subseteq \mathfrak{J}\, C(f, e^{i\theta}).$$

A similar remark to that following Theorem 11 applies here also on comparing Theorem 18 with Lemma 13 (1). For by that lemma

$$\chi_i(f, e^{i\theta}) \subseteq \complement R(f, e^{i\theta})_i \cup \text{interior of } R(f, e^{i\theta}) \tag{22.3}$$

where we define $\chi_i(f, e^{i\theta})$ as the sets of points a such that, for some $\eta(a) > 0$, a is an isolated point of $\Gamma(f, |\theta' - \theta| < \eta(a))$ and $(\complement R(f, e^{i\theta}))_i$ is the set of isolated points of $\complement R(f, e^{i\theta})$.

We also have the complementary relation

$$(\complement R(f, e^{i\theta}))_i \subseteq \chi(f, e^{i\theta}) \cup \chi_*(f, e^{i\theta}) \cup \Phi(f, e^{i\theta}). \tag{22.4}$$

For $(\complement R(f, e^{i\theta}))_i \subseteq \complement R(f, e^{i\theta}) \cap C(f, e^{i\theta})$ and (22.4) follows from (19.5).

Theorem 18 is illustrated by the function $h(z)$ defined in paragraph 14 in the neighbourhood of $z=1$.

Generalisation of Theorems B′ and C′.

23. We consider first the relation between Theorem 16 and the known result Theorem C. We observe that, for a given $\eta > 0$,

$$\begin{aligned} \Gamma(f, |\theta' - \theta| < \eta) &= \Gamma_P(f, |\theta' - \theta| < \eta) \cup \Gamma_A(f, |\theta' - \theta| < \eta) \\ &= \Gamma_P(f, e^{i\theta}) \cup \Gamma_P(f, 0 < |\theta' - \theta| < \eta) \cup \Gamma_A(f, |\theta' - \theta| < \eta). \end{aligned} \tag{23.1}$$

Now if $a \in \Gamma_A(f, |\theta' - \theta| < \eta)$ and if there is an asymptotic path on which $f(z)$ tends to a and whose end contains the point $e^{i\theta}$ then $a \in \Phi(f, e^{i\theta})$. Hence

$$\Gamma_A(f, |\theta' - \theta| < \eta) \subseteq \Gamma_A(f, 0 < |\theta' - \theta| < \eta) \cup \Phi(f, e^{i\theta}) \tag{23.2}$$

and combining (23.1) and (23.2) we have

$$\Gamma(f, |\theta' - \theta| < \eta) \subseteq \Gamma_P(f, e^{i\theta}) \cup \Gamma(f, 0 < |\theta' - \theta| < \eta) \cup \Phi(f, e^{i\theta}). \tag{23.3}$$

Now write

$$\Psi(f, e^{i\theta}) = \bigcap_{\eta} \Gamma(f, 0 < |\theta' - \theta| < \eta)$$

and

$$\Psi^*(f, e^{i\theta}) = \bigcap_{\eta} \overline{\Gamma}(f, 0 < |\theta' - \theta| < \eta);$$

and we have from (23.3)

$$\chi^*(f, e^{i\theta}) \subseteq \overline{\Gamma}_P(f, e^{i\theta}) \cup \Psi^*(f, e^{i\theta}) \cup \Phi(f, e^{i\theta}). \tag{23.4}$$

Applying (23.4) to (19.1) we have

$$\begin{aligned} \complement R(f, e^{i\theta}) \cap C(f, e^{i\theta}) \cap \complement \Psi^*(f, e^{i\theta}) \\ \subseteq \overline{\Gamma}_P(f, e^{i\theta}) \cup \Phi(f, e^{i\theta}). \end{aligned} \tag{23.5}$$

Now

$$\Psi^*(f, e^{i\theta}) \subseteq C_B(f, e^{i\theta}) \quad \text{and} \quad \Phi(f, e^{i\theta}) \subseteq C_B(f, e^{i\theta}).$$

So (23.5) gives

$$\complement R(f, e^{i\theta}) \cap C(f, e^{i\theta}) \cap \complement C_B(f, e^{i\theta}) \subseteq \overline{\Gamma}_P(f, e^{i\theta}). \tag{23.6}$$

Now suppose that $a \in \complement R(f, e^{i\theta}) \cap C(f, e^{i\theta}) \cap \complement C_B(f, e^{i\theta})$ so that $a \in \overline{\Gamma}_P(f, e^{i\theta})$. If a is an isolated point of $\overline{\Gamma}_P(f, e^{i\theta})$ then $a \in \Gamma_P(f, e^{i\theta})$ and (16.2) follows from (23.6). Setting aside this trivial case, suppose that a is not an isolated point of $\overline{\Gamma}_P(f, e^{i\theta})$. Then $a \in \Gamma'_P(f, e^{i\theta})$ and there is a sequence $\{a_n\}$, $\lim\limits_{n\to\infty} a_n = a$ such that $a_n \in \Gamma_P(f, e^{i\theta})$. We can thus find distinct asymptotic paths γ_n such that $f(z) \to a_n$ as $z \to e^{i\theta}$ on γ_n. Without loss of generality we may take $a = \infty$ and a_n finite. We can choose $\varrho_0 < 1$ such that the circumference $|z - e^{i\theta}| = \varrho_0$ intersects two of the paths γ_n, say γ_1, and γ_2. Let D_0 be the domain having the point $e^{i\theta}$ as a frontier point and bounded by arcs of γ_1, γ_2 and the circumference $|z - e^{i\theta}| = \varrho_0$. Since $\infty \in \complement R(f, e^{i\theta})$ we can choose ϱ_0 such that the function $f(z)$ is regular in D_0 and since $f(z) \to a_1$ as $z \to e^{i\theta}$ on γ_1 and $f(z) \to a_2$ as $z \to e^{i\theta}$ on γ_2 it follows from Lindelöf's Theorem that $f(z)$ is unbounded in D_0 in the neighbourhood of the point $e^{i\theta}$. Let $\mu_0 = \max|f(z)|$ on the arcs of $|z - e^{i\theta}| = \varrho_0$ belonging to the frontier of D_0 and put $M_0 = \max(2|a_1|, 2|a_2|, \mu_0)$. Then we can find z_0 in D_0 such that $f(z_0) = w_0$, $|w_0| > 4M_0$. By choice of ϱ_0 we can ensure that $|f(z)| < \max(2|a_1|, 2|a_2|) \leq M_0$ on the arcs of γ_1 and γ_2 belonging to the frontier of D_0. There is thus a domain $G_0(\infty, 1/2M_0)$ containing z_0 and this domain is contained in D_0, since on its frontier in $|z| < 1$ we have $|f(z)| = 2M_0$, and it has $e^{i\theta}$ as a frontier point since D_0 contains no poles of $f(z)$. Now since, given $\varepsilon > 0$, $|f(z)| < 2M_0 + \varepsilon$ in some neighbourhood of every frontier point of $G_0(\infty, 1/2M_0)$ except perhaps $e^{i\theta}$ it follows that $f(z)$ is *unbounded* in

G_0 in the neighbourhood of $e^{i\theta}$. For otherwise we should have $|f(z)| \leq 2M_0$ in G_0 and in particular $|f(z_0)| \leq 2M_0$ contrary to hypothesis. We can therefore find a sequence $z_1, z_2, \dots z_n, \dots$ in G_0 such that $\lim_{n\to\infty} z_n = e^{i\theta}$ and $\lim_{n\to\infty} |f(z_n)| = \infty$. Take $\varrho_1 = \varrho_0/2$ and let D_1 be the domain common to G_0 and $|z - e^{i\theta}| < \varrho_1$ and having $e^{i\theta}$ as a frontier point, and denote by μ_1 the maximum of $|f(z)|$ on the arcs of $|z - e^{i\theta}| = \varrho_1$ contained in the frontier of D_1. We can choose z_1 in D_1 such that $|f(z_1)| > 4M_1$ where $M_1 = \max(2M_0, \mu_1)$. There is then a $G_1(\infty, 1/2M_1)$ containing z_1 and having $e^{i\theta}$ as a frontier point. Repeating the previous argument, we see that $f(z)$ is unbounded in G_1 in the neighbourhood of $e^{i\theta}$.

Proceeding in this way we obtaine a sequence of domains

$$G_0 \supset G_1 \supset G_2 \supset \cdots \supset G_n \supset \cdots$$

each having $e^{i\theta}$ as a frontier point and such that in $\bar{G}_n$ we have $|f(z)| \geq 2M_n$ where

$$M_0 < M_1 < M_2 < \cdots M_n < \cdots \lim_{n\to\infty} M_n = \infty.$$

We can therefore find a continuous path γ defined by $z = p(t)$, $0 < t < \infty$, such that $\lim_{t\to\infty} p(t) = e^{i\theta}$ and given any n we can find t_n such that $z = p(t) \in G_n$ for all $t > t_n$. γ is thus an asymptotic path on which $f(z) \to \infty$ so that $\infty \in \Gamma_P(f, e^{i\theta})$. We have thus shewn that in either case $a \in \Gamma_P(f, e^{i\theta})$ so that (16.2) again follows from (23.6). Theorem C′ is thus implied by Theorem 16.

By a straightforward adaptation of Doob's proof of Theorem B′ we now prove a generalisation of that theorem, namely

Theorem 19. *If $f(z)$ is meromorphic in $|z| < 1$, then for any value of θ, $f(z)$ takes every value belonging to $C(f, e^{i\theta})$ but not to $\Psi^*(f, e^{i\theta})$, with two possible exceptions, in every neighbourhood of $z = e^{i\theta}$ contained in $|z| < 1$; i.e. the set*

$$\complement R(f, e^{i\theta}) \cap C(f, e^{i\theta}) \cap \complement \Psi^*(f, e^{i\theta})$$

contains at most two values. Also, if this set contains two values, then $\complement R(f, e^{i\theta})$ contains no other values.

Suppose there is a value

$$a \in \complement R(f, e^{i\theta}) \cap C(f, e^{i\theta}) \cap \complement \Psi^*(f, e^{i\theta}).$$

Now if $a \in \complement \Gamma_P(f, e^{i\theta})$ it follows from (23.5) that $a \in \Gamma'_P(f, e^{i\theta}) \cup \Phi(f, e^{i\theta})$. If $a \in \Phi(f, e^{i\theta})$, $\Phi(f, e^{i\theta})$ is not void and it follows that $\complement R(f, e^{i\theta})$ can contain only one other value; if $a \in \Gamma_P'(f, e^{i\theta})$ there are two asymptotic paths ending in the point $e^{i\theta}$ on which $f(z)$ tends to different values and it follows from Lindelöf's well-known theorem that $\complement R(f, e^{i\theta})$ contains at most two values. If there are two values a and b both belonging to $\complement R(f, e^{i\theta}) \cap C(f, e^{i\theta}) \cap \complement \Psi^*(f, e^{i\theta})$ and both belonging to $\Gamma_P(f, e^{i\theta})$ then it follows from Lindelöf's Theorem that a and b are the only values belonging to $\complement R(f, e^{i\theta})$. This proves the theorem.

Since $\Psi^*(f, e^{i\theta}) \subseteq C_B(f, e^{i\theta})$ Theorem 19 contains Theorem B′.

Part III.

The classification and distribution of singularities of $f(z)$ on the unit circle.

24. The points of $|z|=1$ which are not regular points may be classified in terms of the excluded range $\complement R$ at those points. The appropriate definitions for the purpose of this classification are obvious enough. To begin with, *we define $W = W(f)$ as the set of points $e^{i\theta}$ for which $\complement C(f, e^{i\theta})$ is void.* Such a point we call a *Weierstrass point for $f(z)$.* By Corollary 15.1, *$R(f, e^{i\theta})$ is everywhere dense for $e^{i\theta} \in W(f)$.* In considering W we require a further definition, namely that of $F_1 = F_1(f)$. This is defined as the set of points $e^{i\theta}$ at which the set $F = F(f)$ of Fatou points is of density 1. The complements with respect to the circumference $|z|=1$ are denoted by $\complement W$ and $\complement F_1$. Evidently W is closed.

By a quite trivial argument we prove

Theorem 20. *If $f(z)$ is meromorphic in $|z|<1$, then every point of the circumference $|z|=1$ belongs either to W or to F_1.*

Suppose $e^{i\theta} \in \complement W$. Then there is a number $a \in \complement C(f, e^{i\theta})$ and so we can find $\eta > 0$ such that $1/(f(z)-a)$ is bounded in $E = E(e^{i\theta}, \eta)$ defined by $|z - e^{i\theta}| < 2 \sin \eta/2$ and $|z|<1$. We map E conformally on $|\xi|<1$ by the function $\xi(z)$ whose inverse is $z(\xi)$. The function

$$\varphi(\xi) = \frac{1}{f(z(\xi)) - a}$$

is then bounded in $|\xi|<1$ and it follows from Fatou's theorem that the set of points of $\complement F(\varphi)$ is of measure zero and hence, by Lemma 2 a, that the set $\complement F(f) \cap \alpha(\theta, \eta)$ is of measure zero, where $\alpha(\theta, \eta)$ denotes the arc $z = e^{i\theta'}$, $|\theta' - \theta| < \eta$, so that we have proved that $e^{i\theta} \in F_1$. This proves the theorem.

A more delicate argument gives a stronger theorem due to Littlewood. We define $H = H(f)$ *as the set of points* $e^{i\theta}$ *for which* $\complement R(f, e^{i\theta})$ *is of capacity zero.* Such a point we call a *Frostman point.* H is closed. For if for a sequence $\theta_n \to \theta$ as $n \to \infty$, $\complement R(f, e^{i\theta_n})$ is of capacity zero then $\complement R(f, e^{i\theta}) \subseteq \bigcup_n \complement R(f, e^{i\theta_n})$ is also of capacity zero. We begin by proving

Theorem 21 (Littlewood). *If* $f(z)$ *is meromorphic in* $|z| < 1$, *then every point of the circumference* $|z| = 1$ *belongs either to* H *or to* F_1.

For completeness we reproduce Littlewood's proof[1] in our own notation.

Suppose $e^{i\theta} \in \complement H$. Then we can find $\eta > 0$ such that $f(z)$ omits a set of values of positive capacity in the domain $E = E(e^{i\theta}, \eta)$. For if not, given any sequence $\eta_n \to 0$ as $n \to \infty$, the set

$$\complement R(f, e^{i\theta}) = \bigcup_n \complement X_n(f, e^{i\theta})$$

is of capacity zero contrary to hypothesis, where $X_n(f, e^{i\theta})$ is the set of values taken by $f(z)$ in $E(e^{i\theta}, \eta_n)$. We now map E on the unit circle $|\xi| < 1$, the mapping function being $\xi(z)$ and its inverse $z(\xi)$. The function $f(z)$ transforms into $\varphi(\xi) = f(z(\xi))$ for which $\complement R(\varphi)$ is of positive capacity. Hence, by Frostman's Theorem, $T(|\xi|, \varphi)$ is bounded and so $\complement F(\varphi)$ is of measure zero. It now follows from Lemma 2 a, as in the proof of Theorem 20, that $\complement F(f) \cap \alpha(\theta, \eta)$ is of measure zero so that $e^{i\theta} \in F_1$.

Since evidently $\complement W \subseteq \complement H$ this theorem contains Theorem 20.

Corollary 21.1. *If* $\complement F$ *is of positive measure then there is at least one point of* H *on* $|z| = 1$.

For if H is void then every point of $|z| = 1$ belongs to $\complement H \subseteq F_1$. But if $\complement F$ were of positive measure we could find a point at which $\complement F$ is of density 1. Therefore $\complement F$ is of measure zero if H is void.

As a further corollary of Theorem 21 we have

Theorem D (Littlewood).[2] *If* $f(z)$ *is meromorphic in* $|z| < 1$, *then almost all points of* $|z| = 1$ *belong either to* H *or to* F.

Suppose, on the contrary, that $\complement H \cap \complement F$ is of positive measure. Then there is a point $e^{i\theta}$ at which this set is of density 1. But, by Theorem 21, $e^{i\theta}$ belongs to

[1] This result is a stage in the proof of Theorem D, but deserves separate enunciation.

[2] The theorem was stated in a rather weaker form in Cartwright (1) p 181, where H was de ined as the set of $e^{i\theta}$ for which $\complement R(f, e^{i\theta})$ is an S-set. Our proof is the same except that Frostman's theorem is used in place of the earlier and weaker theorem of Ahlfors.

F_1 since it belongs to $\complement H$ so that $\complement F$ is of zero density at $e^{i\theta}$. Hence $\complement H \cap \complement F$ is of zero density at $e^{i\theta}$. We thus have a contradiction and the theorem is proved.

In this context it is interesting to recall Plessner's important generalisation, for functions of unbounded characteristic $T(r, f)$, of Fatou's theorem. To state the theorem we require a further definition. Given an angle $\triangle$ of vertex $e^{i\theta}$ and contained in $|z| < 1$, i.e. a Stolz angle at $e^{i\theta}$, the sets $C_\triangle(f, e^{i\theta})$ and $R_\triangle(f, e^{i\theta})$ are defined as in 15 (i) and (ii) except that the sequence $\{z_n\}$ is restricted to the angle $\triangle$. We now define $I = I(f)$ *as the set of points* $e^{i\theta}$ *for which* $\complement C_\triangle(f, e^{i\theta})$ *is void for every* $\triangle$. Such a point we call a *Plessner point*. Evidently $I \subseteq W$. But it is also clear that Theorem 15 holds for $C_\triangle(f, e^{i\theta})$ and $R_\triangle(f, e^{i\theta})$ so that for $e^{i\theta} \subseteq I(f)$ not only is $R(f, e^{i\theta})$ everywhere dense but, by Corollary 15.1, $R_\triangle(f, e^{i\theta})$ is everywhere dense for every Stoltz angle $\triangle$ at $e^{i\theta}$. We recall also that for $e^{i\theta} \subseteq F(f)$, $C_\triangle(f, e^{i\theta})$ consists of the single asymptotic value $f(e^{i\theta})$ for every Stolz angle $\triangle$. The theorem in question is

Theorem E (Plessner)[1]. *If* $f(z)$ *is meromorphic in* $|z| < 1$, *then almost all points of* $|z| = 1$ *belong either to* I *or to* F.

It will be noted that H does not contain I, nor does I contain H; and while F and H may have common points F and I cannot.

25. The results of the previous paragraph depend upon familiar theorems. However, Theorem 16 enables us to prove a new result of the type of Theorem 20. We define $P = P(f)$ *as the set of points* $e^{i\theta}$ *for which* $\complement R(f, e^{i\theta})$ *contains at most two values:* and we write $F' = F'(f)$ *for the derived set of* F. Then $P \subseteq H$ and $F_1 \subseteq F'$. P is the set of *Picard points* of $f(z)$.[2] P is closed since $\complement P$ is open. With these definitions we prove

Theorem 22. *If* $f(z)$ *is meromorphic in* $|z| < 1$, *then every point of the circumference* $|z| = 1$ *belongs either to* P *or to* F'.

Suppose $e^{i\theta} \in \complement P$. Then we can find three numbers a, b and c in the set $\complement R(f, e^{i\theta})$ so that $\Phi(f, e^{i\theta})$ is void. Now suppose that $e^{i\theta} \in \complement F'$. Since $\complement F' \subseteq \complement F_1$

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[1] Plessner (1).

[2] This definition of Picard points is that first formulated by Valiron (2 a) and (3) p 265 In a later paper, Valiron (4) p 13, he defined four categories of Picard points $P_1 \subseteq P_2 \subseteq P_3 \subseteq P_4$, of which $P_1 = P$ as we have defined it, while $H \subseteq P_4$. It is easily seen, however, that Valiron's method will allow P_4 to be replaced by H. It is only necessary to use Frostman's theorem in place of the theorem of Ahlfors referred to in footnote 2 on p. 138.

it follows from Theorem 20 that $C\,C(f, e^{i\theta})$ is void. Therefore, by Corollary 16.1 (equivalent form),

$$C\,R(f, e^{i\theta}) \subseteq \chi^*(f, e^{i\theta}) \subseteq \overline{\Gamma}(f, |\theta' - \theta| < \eta) \tag{25.1}$$

for all $\eta > 0$.

Now we can find η_0 such that $f(z)$ omits the values a, b and c in $E = E(e^{i\theta}, \eta_0)$ and it follows from Theorem 5 and Lemma 2a that, for $\eta < \eta_0$,

$$\Gamma(f, |\theta' - \theta| < \eta) = \Gamma_P(f, |\theta' - \theta| < \eta)$$

and that for any θ' in $|\theta' - \theta| < \eta$ for which $\Gamma_P(f, e^{i\theta'})$ is not void $e^{i\theta'} \in F$ and $\Gamma_P(f, e^{i\theta'}) = f(e^{i\theta'})$. In particular, $\Gamma_P(f, e^{i\theta})$ is either void or contains only one value which may be one of the values a, b or c.

Since $\overline{\Gamma}(f, |\theta' - \theta| < \eta) = \overline{\Gamma}(f, 0 < |\theta' - \theta| < \eta) \cup \Gamma_P(f, e^{i\theta})$ and since by (25.1) $\overline{\Gamma}(f, |\theta' - \theta| < \eta)$ contains all the values a, b and c it follows that if $f(e^{i\theta})$ is equal to one of these values, say a, then $\overline{\Gamma}(f, 0 < |\theta' - \theta| < \eta)$ contains the other two b and c; and if $f(e^{i\theta}) \neq a$, b or c then $\overline{\Gamma}(f, 0 < |\theta' - \theta| < \eta)$ contains a, b and c. Therefore, for any $\eta < \eta_0$, the arc $\alpha(\theta, \eta)$ contains points of F at which the asymptotic values are equal or arbitrarily near to a, b and c. This proves the theorem. A known result follows at once, namely

Corollary 22.1. *If $C\,R(f)$ contains more than two values then the points of F are everywhere dense on $|z| = 1$.*

For every point of $|z| = 1$ belongs to $C\,P \subseteq F'$. We may observe, for comparison with this corollary, that it follows from the theorem of Frostman and Fatou-Nevanlinna that if $C\,R(f)$ is of positive capacity then $C\,F$ is of measure zero. *The proof of Theorem 22 has in fact shewn that if, for a given θ, $C\,R(f, e^{i\theta})$ contains three or more values, of which a is any one, and if $C\,C(f, e^{i\theta})$ is void and $f(e^{i\theta}) \neq a$, then $e^{i\theta}$ is a limit point of points of F at which the asymptotic values are equal or arbitrarily near to a.* Familiar examples of functions with this property are the modular function $\mu(z)$ which omits the values $0, 1, \infty$ and $Q(z) = \log \mu(z)$ which omits the values ∞, $\pm 2n\pi i$, $n = 0, 1, 2, \ldots$. For both these functions every point of $|z| = 1$ is a limit point of points of F at vertices of the modular figure at which the asymptotic values are equal to the omitted values.

On the other hand, if $C\,C(f, e^{i\theta})$ is not void and $a \in C\,C(f, e^{i\theta})$ then a is at a positive distance from $\Gamma(f, |\theta' - \theta| < \eta)$ for all sufficiently small $\eta > 0$.

It should be mentioned in conclusion that while our methods do not give existence theorems for the sets W, H or P such theorems have been proved by Valiron.[1] In particular, he proved that *H is not void if $T(r, f)$ is unbounded*[2] *and that P is not void if*

$$\overline{\lim_{r\to 1}} \frac{T(r, f)}{-\log(1-r)} = \infty.$$

Appendix.

Application to Jordan domains.

26. The definition of the sets Γ and Φ can be generalised so as to apply to any Jordan domain D_j whose boundary $\mathfrak{F} D_j$ is a closed Jordan curve. It is easily shewn that the generalised sets Γ and Φ are invariant under conformal mapping of D_j onto any other Jordan domain. The definitions of the sets C, R and C_B already given are immediately applicable to any Jordan domain and these sets are also evidently invariant under conformal mapping. It follows that *the theorems we have proved in parts I and II remain valid for any bounded Jordan domain.* For the enunciations of these theorems involve only invariant sets.

Let $f(z)$ be meromorphic in D_j. For a simple continuous curve $z = z(t)$, $0 \leq t < 1$, we denote by $C(z(t))$ the cluster set of $z(t)$ as $t \to 1$, i.e. the set of points p such that $p = \lim_{n\to\infty} z(t_n)$ for some sequence $t_n \to 1$ as $n \to \infty$. We say that such a curve contained in a Jordan domain D_j converges to the boundary if $C(z(t)) \subseteq \mathfrak{F} D_j$, and that $C(z(t))$ is its end.

(i) *The Asymptotic Set* $\Gamma(f, D_j)$ is now defined as follows. $a \in \Gamma(f, D_j)$ if there is a continuous simple path $z = z(t)$ contained in D_j such that $C(z(t) \subseteq \mathfrak{F} D_j$ and $\lim_{t\to 1} f(z(t)) = a$.

If now D_j is mapped conformally upon the circle $|\xi| < 1$ by a function $\xi = \xi(z)$, $\mathfrak{F} D_j$ is mapped upon the circumference $|\xi| = 1$ and the path $z = z(t)$ upon a path $\xi = \xi(z(t))$ such that $|\xi(z(t))| < 1$ and $\lim_{t\to 1} |\xi(z(t))| = 1$. Hence $a \in \Gamma(\varphi)$, where $\varphi(\xi) = f(z(\xi))$ and $z(\xi)$ is the inverse of $\xi(z)$. The set $\Gamma(f, D_j)$ is therefore invariant under conformal mapping onto the unit circle and therefore onto any other Jordan domain.

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[1] Valiron (3 and 4).

[2] Valiron (4) pp. 28—30 actually proves that P_4 is not void if $T(r, f)$ is unbounded; but P_4 can be replaced by H; see footnote 2 p. 139 above.

By Lemma 1 the cluster set $C(\xi(z(t)))$ is either a point or an arc on the circumference $|z| = 1$ and it follows that $C(z(t)$ is either a point or an arc on $\mathfrak{F} D_j$. If the end $C(z(t))$ is an arc $a \in \Gamma_A(f, D_j)$ and if it is a point $a \in \Gamma_P(f, D_j)$; and these sets are invariant.

Let α be an arc and p a point of $\mathfrak{F} D_j$. Then we may write $a \in \Gamma(f, D_j, \alpha)$ or $a \in \Gamma_P(f, D_j, p)$ according as $C(z(t)) \subseteq \alpha$ or $C(z(t)) = p$; and $a \in \Gamma_P(f, D_j, \alpha)$ if $p \in \alpha$ and $a \in \Gamma_A(f, D_j, \alpha)$ if $C(z(t)) \subseteq \alpha$ is not a point. Then if D_j is mapped conformally upon another Jordan domain E_j we plainly have $\Gamma(g, E_j, \beta) = \Gamma(f, D_j, \alpha)$; $\Gamma_P(g, E_j, q) = \Gamma_P(f, D_j, p)$; $\Gamma_P(g, E_j, \beta) = \Gamma_P(f, D_j, \alpha)$ and $\Gamma_A(g, E_j, \beta) = \Gamma_A(f, D_j, \alpha)$ where $g(\zeta)$, q and β are the transforms of $f(z)$, p and α.

In order to generalise the definition of the sets $\Phi(f)$ and $\Phi(f, e^{i\theta})$ we require a definition of the convergence of a sequence of arcs in D_j such that the property is preserved by a conformal mapping. The required definition is that of *metrical convergence* of a sequence of bounded sets due to Hausdorff.[1]

Let M and N be two bounded sets and denote by $U(M, \varepsilon)$ and $U(N, \varepsilon)$ the ε-neighbourhoods of M and N. The Hausdorff distance $d_H(M, N)$ between M and N is defined as the lower bound of the numbers ε for which

$$N \subseteq U(M, \varepsilon) \text{ and } M \subseteq U(N, \varepsilon).$$

Then $d_H(M, N) = d_H(N, M)$ and $d_H(M, N) = 0$ if and only if $\bar{M} = \bar{N}$. Suppose that for a sequence of closed bounded sets c_n there is a closed set c such that $\lim\limits_{n\to\infty} d_H(c_n, c) = 0$. Then the sequence c_n is said to *converge metrically* to c.

For a function $f(z)$ meromorphic in a Jordan domain D_j we now define the set $\Phi(f, D_j)$ as follows.

(ii) $a \in \Phi(f, D_j)$ if there is a sequence of continuous arcs c_n (the end points being included) contained in D_j and converging metrically to an arc $c \subseteq \mathfrak{F} D_j$, and a sequence $\eta_n > 0$, $\lim\limits_{n\to\infty} \eta_n = 0$, such that, for all n, $|f(z) - a| < \eta_n$ for z on c_n. Also, by definition, $a \in \Phi(f, D_j, p)$ for any point $p \in c$. We see that for a circle this definition is equivalent to that given in § 5.

Now suppose that for $\varphi(\xi)$ meromorphic in $|\xi| < 1$ the set $\Phi(\varphi)$ is not void and that $a \in \Phi(\varphi)$. Then there is a sequence of arcs γ_n in $|\xi| < 1$ converging metrically to an arc γ of $|\xi| = 1$, and such that $|\varphi(\xi) - a| < \eta_n$, $\lim\limits_{n\to\infty} \eta_n = 0$, for all ξ

[1] F. Hausdorff, *Mengenlehre*, Berlin 1927, pp. 145—146; or P. Alexandroff and H. Hopf, *Topologie*, Berlin 1935, pp. 112—114.

on γ_n. The circle $|\xi| < 1$ is now mapped upon D_j and we denote by c the arc of $\mathfrak{F} D_j$ corresponding to γ and by c_n the arcs corresponding to γ_n. We have to shew that the sequence c_n converges matrically to c. Suppose the contrary. Then we can find $\varepsilon > 0$ and a subsequence c_m such that $d_H(c_m, c) > \varepsilon$. There are two cases: either there is a point $p_m \in c$ such that $U(p_m, \varepsilon)$ contains no point of c_m or there is a point $q_m \in c_m$ such that $U(q_m, \varepsilon)$ contains no point of c. Consider the former case first. We may assume the condition to be satisfied for the whole sequence p_m. There is then a subsequence p_s of the sequence p_m converging to a point $p_0 \in c$. The domain $U(p_0, \varepsilon/2)$ is contained in all the domains $U(p_s, \varepsilon)$ for $s > s(\varepsilon)$ and hence contains no point of any of the curves c_s for $s > s(\varepsilon)$. Now $U(p_0, \varepsilon/2) \cap D_j$ corresponds to a domain $\delta(e^{i\theta_0})$ in the unit circle whose boundary contains an arc of γ containing the point $e^{i\theta_0}$ corresponding to p_0. All the curves γ_s lie outside $\delta(e^{i\theta_0})$ and there is therefore an $\varepsilon' > 0$ such that γ is not contained in $U(\gamma_s, \varepsilon')$. But this is contrary to hypothesis and we conclude that $c \subseteq U(c_n, \varepsilon)$ for all $n > n(\varepsilon)$.

Now consider the second case. Assuming the condition to be satisfied for the whole sequence q_m, there is a subsequence q_s converging to a point q_0 and all contained in $U(q_0, \varepsilon/2)$ while $U(q_s, \varepsilon)$ contains no point of c. Now $U(q_0, \varepsilon/2)$ is contained in all the $U(q_s, \varepsilon)$ so that it contains no point of c and is therefore interior to D_j. The corresponding domain is interior to $|\xi| < 1$; it contains points of all the γ_s which therefore do not converge to any arc of $|\xi| = 1$ and in particular not to γ, contrary to hypothesis. We conclude that $c_n \subseteq U(c, \varepsilon)$ for all $n > n(\varepsilon)$.

We have thus proved that c_n converges metrically to c so that $a \in \Phi(f, D_j)$ and hence $\Phi(\varphi) \subseteq \Phi(f, D_j)$. We have only to reverse the argument, starting from the hypothesis that c_n converges metrically to c, to shew that γ_n converges metrically to γ and hence that $\Phi(f, D_j) \subseteq \Phi(\varphi)$. Therefore $\Phi(f, D_j) = \Phi(\varphi)$ and it follows that *the set* $\Phi(f, D_j)$ *is invariant under conformal mapping of* D_j *onto any bounded Jordan domain.* Further, $\Phi(g, E_j, q) = \Phi(f, D_j, p)$, where E_j, q and $g(\zeta)$ are the transforms of D_j, p and $f(z)$ under conformal mapping.

This establishes the validity of the results of Parts I and II for a general Jordan domain. The validity of the classical theorems of Gross and Iversen for such a domain is of course already well known.

27. On the other hand, the theorems of Part III cannot be generalised so widely since they relate to the measure of sets on the boundary of the domain

and to angular domains oriented with respect to the tangent to the boundary. It is therefore necessary to impose restrictions on the boundary. For the most part it is sufficient to assume that the boundary is a closed rectifiable curve. The measure of boundary sets is then determined in terms of the lengths of boundary arcs. Under conformal mapping boundary sets of measure zero are mapped upon boundary sets of measure zero. Further, a rectifiable curve has a unique tangent except perhaps at a set of measure zero. Leaving out of account the exceptional set, the sets F and I for a function meromorphic in such a domain are defined as for the circle. The definitions can indeed be enlarged to be applicable to corners at which there are distinct right and left tangents. The sets W, H and P are defined on the whole boundary as for the circle.

If $f(z)$ is meromorphic in D having $\mathfrak{F} D$ rectifiable and if D is mapped conformally upon $|\xi| < 1$ and $f(z)$ transforms to $\varphi(\xi)$, then $W(f)$, $H(f)$ and $P(f)$ are mapped on $W(\varphi)$, $H(\varphi)$ and $P(\varphi)$ respectively, while $F(f)$ and $I(f)$ are mapped on $F(\varphi)$ and $I(\varphi)$ excluding points corresponding to the exceptional set on $\mathfrak{F} D$. This set being of measure zero, $F_1(f)$ is mapped on $F_1(\varphi)$. It follows from this that the theorems of § 24 can be extended by conformal mapping to any Jordan domain with a rectifiable boundary.

Finally, in order to extend the results of § 25 to a Jordan domain it is necessary that F' should be invariant under conformal mapping. F must therefore be defined except for a finite set of boundary points. This is secured if the boundary is sectionally smooth, i.e. consists of a finite set of arcs with a continuously turning tangent at every point.

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