HOW A MINIMAL SURFACE LEAVES AN OBSTACLE

BY

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This paper is an investigation of the curve of separation determined by the solution to a variational inequality for minimal surfaces. A strictly convex domain Ω in the $z = x_1 + ix_2$ plane is given together with a smooth function ψ which assumes a positive maximum in Ω and is negative on $\partial\Omega$, the boundary of Ω. Let u denote the Lipschitz function which minimizes area among all Lipschitz functions in Ω constrained to lie above ψ in Ω and to vanish on $\partial\Omega$. For such u there is a coincidence set $I \subset \Omega$ consisting of those points z where $u(z) = \psi(z)$. Let us call $\Gamma = \{(x_1, x_2, x_3): x_3 = u(z) = \psi(z),\ z \in \partial I\}$ the "curve" of separation. The object of this paper is to show that Γ is analytic, as a function of its arc length parameter, provided that ψ is strictly concave and analytic.

The study of the coincidence set of the solution to a variational inequality and its curve of separation was originated, together with the study of the regularity of the solution, by H. Lewy and G. Stampacchia ([11]). They obtained, essentially, the result presented here for the variational inequality derived from the Dirichlet Integral. The topological conclusion that Γ is a Jordan curve was reached under the assumption that $\psi \in C^2(\overline{\Omega})$ be strictly concave, a conclusion valid for a wide variety of cases, in particular the problem treated in this paper ([6]).

Our demonstration relies on the resolution of a system of differential equations and the utilization of the solution to extend analytically a conformal representation of the minimal surface which is the graph of u in the subset of Ω where $u(z) > \psi(z)$. The idea of connecting an analytic function to its possible extension by means of the solution to a differential equation is due to Hans Lewy and was used by him to study the behavior of minimal surfaces with prescribed and with free boundaries ([9], [10]).

The problem at hand is distinguished from more well known problems in the calculus of variations because it has only a single boundary relation, impeding both the derivation

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and the identification of the solution to the differential equations to which we have alluded. The system of equations is found through differentiation of the single relation, which requires, predictably, information about the second derivatives of u. This was the subject of ([7]). The resulting system depends on the mean curvature of the obstacle. The identification of the solution is facilitated by the rectification of Γ.

Consequently it falls within the scope of this paper to determine conditions for which Γ has a smooth parameterization. We show here that if $\psi \in C^3(\overline{\Omega})$ and is strictly concave, then Γ has a smooth parameterization via the boundary values of a conformal representation. The interested reader will discover that the method applies when $\psi \in C^{2,\lambda}(\Omega)$, but in the interest of simplicity we do not assume this. Central portions in this development are the integrability of a suitable conformal mapping and the local behavior of functions satisfying a differential inequality much in the style of P. Hartman and A. Wintner ([5]). We rely on the results of ([6]). A different use of a differential inequality based on ([5]) was used in the theory of boundary regularity for minimal surfaces by J. C. C. Nitsche ([13]). Our method applies to the problem treated in ([11]) where it yields that the curve of separation has a C^1 parameterization when $\psi \in C^3(\overline{\Omega})$ is strictly concave.

The considerations in this work are local in character but we do not include a refined study of these aspects. We close with some remarks about obstacles which are not strictly concave.

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Let us introduce precise notations and review the pertinent literature.

§ 1

Let $\Omega \subset R^2$ be a strictly convex domain with smooth ($C^{2,\lambda}$) boundary $\partial\Omega$ in the $z = x_1 + ix_2$ plane and $\psi(z)$ in $C^3(\overline{\Omega})$ be strictly concave in Ω such that $\max_\Omega \psi > 0$ and $\psi < 0$ on $\partial\Omega$. Let K denote the convex set of functions v in $H_0^{1,\infty}(\Omega)$ (1) satisfying $v \geq \psi$ in Ω. We consider, in the language of variational inequalities, the problem

(1) We employ the usual notations for function spaces. $H^{m,q}(\Omega)$ denotes the completion of $C^\infty(\overline{\Omega})$ in the norm $\|u\|_{H^{m,q}(\Omega)} = \Sigma_1^m \|D^n u\|_{L^q(\Omega)} + \|u\|_{L^q(\Omega)}$, where $D^n u$ denotes the collection of derivatives of u of order n. $H^{m,\infty}(\Omega)$ is the set of functions whose $m-1^{\text{st}}$ derivatives satisfy a Lipschitz condition, and $C^{m,\lambda}(\Omega)$ is the set of functions whose m^{th} derivatives satisfy a Hölder condition with exponent λ. $H_0^{m,q}(\Omega)$ is the closed subspace of $H^{m,q}(\Omega)$ spanned by $C_0^\infty(\Omega)$. Also, $H^m(\Omega) = H^{m,2}(\Omega)$.

$$u\in K:\int_\Omega \frac{D_j u}{\sqrt{1+|Du|^2}}D_j(v-u)\,dx\geqslant 0 \quad v\in K, \tag{1.1}$$

$D_jf=\partial f/\partial x_j=f_{x_j}$ and $Df=f_x=(f_{x_1}, f_{x_2})$. Such u is shown to exist in the work of H. Lewy and G. Stampacchia ([12]) and also in the work of M. Giaquinta and L. Pepe ([4]). They prove, furthermore, that

$$u\in H^{2,q}(\Omega)\cap C^{1,\lambda}(\Omega) \quad \textit{for every} \quad q,\lambda,\ 1\leqslant q<\infty \quad \textit{and} \quad 0<\lambda<1.$$

The coincidence set of the solution is the closed set $I=\{z\in\Omega:\ u(z)=\psi(z)\}$ in whose complement $\Omega-I=\omega$, we observe

$$D_j\left(\frac{D_j u}{\sqrt{1+|Du|^2}}\right)=0 \quad \textit{in} \ \ \omega. \tag{1.2}$$

It is known that I is a closed Jordan domain (Theorem 3, [6]). The gradient of the solution, which for convenience we write in the complex form $f=u_{x_1}-iu_{x_2}$, is a homeomorphism of $\bar\omega$ onto a domain contained in $f^*(I)$, where $f^*=\psi_{x_1}-i\psi_{x_2}$ is a homeomorphism of Ω because ψ is strictly concave (Theorem 2, Corollary 1.3 [6]). Note that $f=f^*$ in I because $u\in C^1(\Omega)$. Consequently, the mapping $T_0(z)=(f^{*-1}\circ f)(z)$ is a homeomorphism of $\bar\omega$ onto its image in I such that $T_0(z)=z$ for $z\in\partial I$.

In addition, $u\in H^{2,\infty}(\Omega)$ and the Gauss curvature $K=(1+|Du|^2)^{-2}\{D_{11}uD_{22}u-(D_{12}u)^2\}$ satisfies $K\leqslant -c<0$ in ω for a positive c (Theorem 4 [7]). This fact about the Gauss curvature we shall not use. It is evident that T_0 is a Lipschitz mapping.

We restate this information in terms of the two dimensional surfaces in R^3 defined by the solution and the obstacle. We denote by

$$\Sigma=\{x\in R^3:\ x_3=u(z),\ z\in\Omega\},$$

the minimal surface by $\quad S=\{x\in R^3:\ x_3=u(z),\ z\in\omega\},$

and the concave surface determined by the obstacle

$$M=\{x\in R^3:\ x_3=\psi(z),\ z\in\Omega\}.$$

We refer to $M_I=M\cap\Sigma$ as the set of coincidence in space, or simply the set of coincidence. As a subset of M, M_I has a boundary $\Gamma=\{x\in R^3: x_3=u(z)=\psi(z), z\in\partial I\}$, a Jordan curve which we have agreed to call the curve of separation. We extend T_0 to a homeomorphism from $\bar S$ to its image in $M_I\subset M$ by the formula $T(x)=(T_0(z),\psi(T_0(z)))$. Observe that $T(x)=x$ for $x\in\Gamma$. That $f=u_{x_1}-iu_{x_2}$ is one to one implies that the normal mapping $z\to(1/W)(-D_1u,-D_2u,1)$, $W=\sqrt{1+|Du|^2}$, is also one to one, a statement independent of the choice of coordinates in R^3. A corresponding statement is valid for $f=\psi_{x_1}-i\psi_{x_2}$. Hence, independent of coordinates, it may always be said that the spherical image of S is contained in the spherical image of M_f.

§ 2

In this section we discuss some properties of the uniformization of the solution surface Σ. For the existence and basic properties of the uniformization we refer to ([1]), [3]). At a fixed $0\in\Gamma$, we choose coordinates (X_1, X_2, X_3) in R^3 such that $0=(0, 0, 0)$ and $X_3=0$ is the tangent plane to Σ and M at 0. A uniformization (conformal representation) of the $C^{1,\lambda}$ surface Σ is a $1:1$ mapping

$$X: D\to\Sigma\subset R^3,\ D=\{\zeta=\xi_1+i\xi_2: |\zeta|<1\}$$
$$X(0)=0$$

with the properties, called isothermal relations, that

$$0<(X_{\xi_1})^2=(X_{\xi_2})^2 \text{ and } X_{\xi_1}X_{\xi_2}=0, \quad \zeta\in D \tag{2.1}$$

Also, $X\in C^{1,\lambda}(\bar{D})$. Let us write $X(\zeta)=(X_1(\zeta), X_2(\zeta), X_3(\zeta))$.

Our purposes require such a representation at each $P\in\Gamma$. We choose new coordinates $X_P=(x_1', x_2', x_3')$ with origin at $P\in\Gamma$ and $x_3'=0$ the tangent plane to M and Σ at P. We assume that the change of basis $A_P=(a_{kj}(P))$, defined so that

$$X_P=A_P(X-P), \tag{2.2}$$

depends continuously on $P\in\Gamma$. The conformal representation of Σ

$$X_P(\zeta)=(x_1'(\zeta), x_2'(\zeta), x_3'(\zeta))=A_P(X(\zeta)-P),\ P=X(\zeta_P),$$

has a $1:1$ projection onto $x_3'=0$ for $|\zeta-\zeta_P|$ sufficiently small, indeed, we may say in $B_\delta(\zeta_P)=\{\zeta: |\zeta-\zeta_P|<\delta\}$ with $\delta>0$ independent of P by the continuity of A_P. We denote this projection in complex form by

$$z'=\phi_P(\zeta)=x_1'(\zeta)+ix_2'(\zeta) \quad \zeta\in B_\delta(\zeta_P)$$

The function ϕ_P and its inverse ϕ_P^{-1} are $C^{1,\lambda}$ functions in $B_\delta(\zeta_P)$ and $\phi_P(B_\delta(\zeta_P))$ respectively. Near $z'=0$, a portion of Σ may be represented non parametrically by the $C^{1,\lambda}$ function

$$u_P(z')=x_3'(\phi_P^{-1}(z'))$$

which is a solution to the minimal surface equation (1.2) in $\{z'=x_1'+ix_2': (x_1', x_2', u_P(z'))\in S\}$.

That $X_P(\zeta)$ is a conformal representation of Σ implies that ϕ_P^{-1} satisfies the Beltrami equation

$$\frac{\partial}{\partial \bar{z}} \phi_P^{-1} = \mu_P \frac{\partial}{\partial z} \phi_P^{-1}, \quad \mu_P = \left(\frac{D_1 u_P + i D_2 u_P}{1+W}\right)^2, \quad W = \sqrt{1+|Du_P|^2}, \tag{2.3}$$

where $D_j = \partial/\partial x'_j$ and $|z'|$ is sufficiently small. By the chain rule, therefore,

$$\frac{\partial}{\partial \bar{\zeta}} \phi_P = \tau \mu_P \frac{\partial}{\partial \zeta} \phi_P \quad \text{in} \quad B_\delta(\zeta_P) \quad \text{where} \quad |\tau| = 1. \tag{2.4}$$

More precisely

$$\tau(\zeta) = -\frac{\partial}{\partial z} \phi_P^{-1} \Big/ \left(\overline{\frac{\partial}{\partial z} \phi_P^{-1}}\right)$$

evaluated at $\phi_P(\zeta)$.

In this study a useful role is fulfilled by the Gauss mapping of the surface

$$g_P(\zeta) = \frac{D_1 u_P - i D_2 u_P}{1 + \sqrt{1+|Du_P|^2}}\Bigg|_{z=\varphi_P(\zeta)} \qquad \zeta \in B_\delta(\zeta_P), \tag{2.5}$$

which has a geometric interpretation: it is the negative complex conjugate of the stereographic projection of the normal vector to Σ onto an equatorial plane from the south pole. Indeed, let us state as a lemma the well known properties of this and some related functions.

LEMMA 2.1. *Let $X: B \to R^3$, $B = \{|\zeta| < 1\}$ be a conformal representation of a minimal surface which admits a non parametric representation as a function of $z = x_1 + ix_2$*

$$x_3 = u(z), \ z \in G, \ G \text{ a domain},$$

the mapping $\zeta \to z(\zeta)$ assumed to be $1:1$. Let $p_j(\zeta) = D_j u(z)$, $j = 1, 2$, and $f'_k(\zeta) = x_{k\xi_1}(\zeta) - i x_{k\xi_2}(\zeta)$, $k = 1, 2, 3$. Then the functions

$$g_1(\zeta) = \frac{W}{(1+W)^2}\left\{1 + \frac{1+p_2^2}{W} + i\frac{p_1 p_2}{W}\right\}, \qquad W = \sqrt{1+|p|^2},$$

$$g_2(\zeta) = \frac{-W}{(1+W)^2}\left\{\frac{p_1 p_2}{W} + i\left(1 + \frac{1+p_1^2}{W}\right)\right\},$$

$$g_3(\zeta) = g(\zeta) = \frac{p_1 - i p_2}{1+W} \tag{2.6}$$

are holomorphic functions of ζ in B and satisfy the relations

$$g_1^2 + g_2^2 + g_3^2 = 0$$

$$g_1 = \tfrac{1}{2}(1 - g^2)$$

$$g_2 = \frac{1}{2i}(1 + g^2)$$

and

$$f'_1 : f'_2 : f'_3 = g_1 : g_2 : g_3.$$

We briefly indicate a proof (cf. Radò [15] or Nitsche [14]). In a neighborhood of any $z^\circ \in G$, define $y(z) = y_1 + iy_2$ by integration of the differentials

$$dy_1 = \left(1 + \frac{1+(D_1u)^2}{W}\right) dx_1 + \frac{D_1 u D_2 u}{W} dx_2,$$

$$dy_2 = \frac{D_1 u D_2 u}{W} dx_1 + \left(1 + \frac{1+(D_2u)^2}{W}\right) dx_2, \quad W(z) = \sqrt{1+|Du|^2},$$

which are path independent and 1: 1 in a neighborhood of z° since u is a solution of (1.2). We express $\tilde{X}(y) = X(\zeta) = (x_1(y), x_2(y), x_3(y))$ and compute that $(\partial x_j/\partial y_1) - i(\partial x_j/\partial y_2) = g_j(\zeta(y))$, i.e., are represented by the formulas (2.6). Since $\Sigma\,((\partial x_j/\partial y_1) - i(\partial x_j/\partial y_2))^2 = 0$, $\tilde{X}(y)$ is a conformal representation of the surface. Since the surface is minimal $\Delta\tilde{X}(y) = 0$, so that $(\partial x_j/\partial y_1) - i(\partial x_j/\partial y_2)$ are holomorphic functions of $y = y_1 + iy_2$. However, any two conformal representations are related by a conformal mapping, that is, there exists a holomorphic $h(\zeta)$ such that $y = h(\zeta)$. The proportion follows except at points where $f_3' = 0$. We interpret it to mean $f_1': f_2' = g_1: g_2$ at these points. Note that vanishing of any two f_j' at a point, and hence the vanishing of all three, violates the assumption that $z(\zeta)$ be 1: 1. The fact that $\partial y/\partial\bar{\zeta} = 0$ may also be verified directly using (2.3), (2.4).

LEMMA 2.2. *The family $(g_P: P \in \Gamma)$ defined by* (2.5) *is uniformly bounded and uniformly Lipschitz continuous in $\bar{D}$ and each g_P, $P \in \Gamma$, is holomorphic in $D_S = X^{-1}(S)$.*

Proof: This demonstration is a simple argument using that $u \in H^{2,\infty}(\Omega)$, [7], and that Σ admits a non-parametric representation. Given $\nu_P(\zeta) = (\nu_1', \nu_2', \nu_3')$ the normal vector to Σ at the point $X_P(\zeta)$ in the coordinates (2.2),

$$g_P(\zeta) = -\frac{\nu_1' - i\nu_2'}{1 + \nu_3'}. \tag{2.7}$$

In as much as $-\nu_3' = \langle A_P \nu, (0, 0, -1)\rangle$, where ν is the normal vector at $X_P(\zeta)$ in the (original) coordinates at 0 and $\langle,\rangle$ the scalar product, to show that $g_P(\zeta)$ is bounded it suffices to prove that

$$\sup \{\langle A_P \nu, (0, 0, -1)\rangle: \nu \text{ normal to } \Sigma, P \in \Gamma\} \leqslant \cos 2\tau,$$

for a τ, $0 < \tau < (\pi/4)$, independent of P. On the other hand, the spherical image of Σ is contained in that of M_I, so the supremum may be taken over ν normal to M. This would show also that the bound for g_P depends only on the obstacle. Now since M has a non-parametric representation $x_3 = \psi(z)$, cf. § 1, the angle between the normal at any point and (0, 0, 1) is less than $(\pi/2) - \tau$ for some $\tau > 0$. Therefore the maximum angle between any two normal vectors is $\pi-2\tau$. The orthogonal transformation A_P maps the normal to M at P into $(0, 0, 1)$ and hence the angle between $\nu_P(\zeta)=A_P\nu(\zeta)$, $\zeta\in D$, and $(0, 0, 1)$ is less than $\pi-2\tau$. Therefore $\nu_P(\zeta)$ makes an angle of at least 2τ with $(0, 0, -1)$. In other words,

$$\sup\{\langle A_P\nu, (0, 0, -1)\rangle: \nu \text{ normal to } M, P\in\Gamma\}\leqslant\cos 2\tau \tag{2.8}$$

In particular, $\|g_P\|_{L^\infty(D)}\leqslant(1-\cos 2\tau)^{-1}$.

Writing the original non-parametric representation of Σ in terms of the conformal representation at 0,

$$(x_1, x_2, u(z)) = A'(X(\zeta)-X_0),\ X_0\in R^3,$$

A' an orthogonal matrix, we deduce that $x_1, x_2\in C^{1,\lambda}(D)$ for some $\lambda>0$. Since $u\in H^{2,\infty}(D)$, the components of $\nu=(1/W)(-D_1u, -D_2u, 1)$ are Lipschitz functions of $\zeta\in D$. Furthermore $\nu_P(\zeta)$ may be written $\nu_P(\zeta)=A_P{}^tA'\nu$ from which we deduce that $\nu_P(\zeta)$ is a triple of Lipschitz functions in ζ with derivatives bounded independently of P. Now by (2.7) and (2.8), we compute that

$$\frac{\partial g_P}{\partial\xi_j}=(1+\nu_3')^{-2}\left((1+\nu_3')\left(\frac{\partial\nu_1'}{\partial\xi_j}-i\,\frac{\partial\nu_2'}{\partial\xi_j}\right)-(\nu_1'-i\nu_2')\,\frac{\partial\nu_3'}{\partial\xi_j}\right)$$

$$\left|\frac{\partial g_P}{\partial\xi_j}\right|\leqslant 2(1-\cos 2\tau)^{-2}\left(\sum_1^3\left(\frac{\partial\nu_k'}{\partial\xi_j}\right)^2\right)^{\frac{1}{2}}\leqslant\text{const.},\qquad (j=1,2,\dots)$$

for a constant independent of $P\in\Gamma$. Q.E.D.

We combine the result of Lemma 2.2 with some of the earlier discussion.

LEMMA 2.3. *Given $\eta>0$, there exists an $\alpha>0$ independent of $P\in\Gamma$ such that ϕ_P is a 1:1 mapping of $B_{2\alpha}(\zeta_P)=\{\zeta:|\zeta-\zeta_P|<2\alpha\}$ onto a $z'=x_1'+ix_2'$ plane which satisfies*

$$\|\phi_P\|_{L^\infty(B_{2\alpha}(\zeta_P))}<\eta \ \text{ and } \ \|g_P\|_{L^\infty(B_{2\alpha}(\zeta_P))}<2\alpha\|Dg_P\|_{L^\infty(B_{2\alpha}(\zeta_P))}<\eta.$$

Furthermore ϕ_P satisfies (2.4) in $B_{2\alpha}(\zeta_P)$ and ϕ_P^{-1} satisfies (2.3) in $\phi_P(B_{2\alpha}(\zeta_P))$.

Evidently, $\mu_P(\zeta)=\overline{g_P(\zeta)^2}$ and $g_P(\zeta_P)=0$ which leads to the inequality

$$|\mu_P(\zeta)|\leqslant C|\zeta-\zeta_P|^2,\quad \zeta\in B_{2\alpha}(\zeta_P),\quad \textit{wite } C \textit{ independent of } P, \tag{2.9}$$

a fact of great importance in the sequel. Also it will be useful to recall that $|(\partial\phi_0/\partial\zeta)(\zeta)|>0$ for $\zeta\in D$ ([3]) and by virtue of this $|(\partial/\partial z')\phi_P^{-1}(z')|$ is bounded for $z'\in B_\varrho(0)$, some $\varrho>0$, independently of P for P near 0.

We wish to consider the behavior of the obstacle. In a neighborhood of each $P\in\Gamma$, M has a non-parametric representation in terms of orthogonal projection onto the tangent plane at P. As before, let us denote by $X_P = (x_1', x_2', x_3')$ a point written in terms of the coordinates at P with $x_3' = 0$ the tangent plane to M and Σ at P, that is,

$$X_P = A_P(X - P)$$

and consider the non-parametric representation

$$x_3' = \psi_P(z'),\ z' = x_1' + ix_2',\ |z'| < \delta,\ \delta \text{ small.}$$

In this neighborhood of $z' = 0$, the homeomorphism T induces a homeomorphism of $\{|z'| < \delta, X_P \in \bar{S}\}$ by $z' \to \operatorname{pr} T(X_P)$, with pr denoting projection onto the z' plane. This mapping may also be described by the composition

$$z' \to \frac{D_1 u_P - iD_2 u_P}{1 + \sqrt{1 + |Du_P|^2}} = t \to F_P^{-1}(t, \bar{t}),$$

$$F_P(z, \bar{z}) = \frac{D_1 \psi_P\left(\dfrac{z+\bar{z}}{2}, \dfrac{z-\bar{z}}{2i}\right) - iD_2 \psi_P\left(\dfrac{z+\bar{z}}{2}, \dfrac{z-\bar{z}}{2i}\right)}{1 + \sqrt{1 + |D\psi_P|^2}}. \tag{2.10}$$

The function F_P contains geometric information about M. To begin we note that the mean curvature $H(P)$ and the Gauss curvature $\varkappa(P)$ of M at P are given by half the trace and the determinant, respectively, of

$$b_P = \begin{pmatrix} D_{11}\psi_P & D_{12}\psi_P \\ D_{12}\psi_P & D_{22}\psi_P \end{pmatrix},\ D_{ij}\psi_P = D_{ij}\psi_P(0, 0).$$

The strict concavity of M implies the existence of constants $\varkappa_0$ and H_0 such that

$$\varkappa(P) \geqslant \varkappa_0 > 0 \quad\text{and}\quad H(P) \leqslant H_0 < 0,\ P \in M.$$

On the other hand,

$$\frac{\partial}{\partial \bar{z}'} F_P \,\Big|_{z'=\bar{z}'=0} = \tfrac{1}{4}\Delta\psi_P \,\Big|_{z'=\bar{z}'=0} = \tfrac{1}{2}H(P)$$

$$\frac{\partial}{\partial z'} F_P \,\Big|_{z'=\bar{z}'=0} = \tfrac{1}{4}(D_{11}\psi_P - D_{22}\psi_P) - \frac{i}{2} D_{12}\psi_P \,\Big|_{z'=\bar{z}'=0},$$

where
$$D_{ij} = \frac{\partial^2}{\partial x_i' \partial x_j'} \quad\text{and}\quad \Delta = \sum D_{jj}.$$

Consequently,
$$4\left|\frac{\partial}{\partial z'} F_P(0, 0)\right|^2 \leqslant H(P)^2 - \varkappa_0 = 4\left|\frac{\partial}{\partial \bar{z}'} F_P(0, 0)\right|^2 - \varkappa_0.$$

By the continuity of A_P and the C^3 character of ψ we conclude

LEMMA 2.4. *There exists an $\eta>0$ and $C>0$ independent of $P\in\Gamma$ such that F_P defined by* (2.10) *satisfies*

$$F_P(z,\bar z)=\frac{\partial}{\partial \bar z}F_P(0,0)\,\bar z+\frac{\partial}{\partial z}F_P(0,0)\,z+G_P(z,\bar z),\ |z|<\eta \tag{2.11}$$

where
$$\left|\frac{\partial}{\partial z}F_P(z,\bar z)\Big/\frac{\partial}{\partial \bar z}F_P(z,\bar z)\right|^2\leqslant 1-\frac{\varkappa_0}{8H(P)^2}<1,\ |z|<\eta,$$

$$\left|\frac{\partial}{\partial \bar z}F_P(z,\bar z)\right|\geqslant -\tfrac{1}{4}H_0,\ |z|<\eta,$$

and
$$\left|\frac{\partial}{\partial \bar z}F_P(0,0)\right|+\left|\frac{\partial}{\partial z}F_P(0,0)\right|+|z|^{-1}\left(\left|\frac{\partial}{\partial \bar z}G_P\right|+\left|\frac{\partial}{\partial z}G_P\right|\right)\leqslant C,\ |z|<\eta.$$

In the statement of Lemma 2.4, we have replaced the symbol z' by z.

§ 3

In this section we shall demonstrate the differentiability of functions satisfying a certain differential inequality. The proof of the theorem will be familiar to the student of potential theory; we refer to P. Hartman and A. Wintner ([5]) or ([3]) for example.

THEOREM 1. *Let $w\in H^{1,q}(B_R)$, $B_R=\{|z|<R\}$, for a $q>2$ satisfy*

$$|z|^{-\sigma}w_{\bar z}(z)\in L^q(B_R)\quad \textit{for a}\quad \sigma>0.$$

If $q\sigma<2$, then

$$|w(z)-w(0)|\leqslant C(\|t^{-\sigma}w_{\bar t}\|_{L^q(B_R)}+\sup_{B_R}|w|)\,|z|^{1+\sigma-(2/q)}\quad z\in B_{R'},\ R'<R, \tag{3.1}$$

where $C=C(R,R',q,\sigma)$.

If $q\sigma>2$, then there is a c such that

$$\left|\frac{w(z)-w(0)}{z}-c\right|\leqslant C'(\|t^{-\sigma}w_{\bar t}\|_{L^q(B_R)}+\sup_{B_R}|w|)\,|z|^{\min(1,\sigma-(2/q))},\quad z\in B_{R'},\ R'<R' \tag{3.2}$$

where $|c|\leqslant R^{-2}\|w\|_{L^\infty(B_R)}+\dfrac{1}{\pi}\|z^{-\sigma}w_{\bar z}\|_{L^q(B_R)}R^{\sigma-(2/q)}$ *and* $C'=C'(R,R',q,\sigma)$.

Notice that $\sup|w|<\infty$ by Sobelof's Lemma.

Proof. We recall the formula of Green in complex form. Let $g=u+iv$ and $dz=dx_1+idx_2$. For any $E\subset\bar E\subset\overline{B_R}$ with ∂E smooth and $g\in H^{1,q}(E)$,

$$\int_E g_{\bar z}\,dx_1dx_2=-\frac{i}{2}\int_{\partial E}g\,dz.$$

In particular, if $\zeta(t)$ is analytic in a neighborhood of E and $w\in H^{1,q}(B_R)$ is given as above

$$\int_E w_{\bar t}\zeta\, dt_1 dt_2 = -\frac{i}{2}\int_{\partial E} w\zeta\, dt.$$

We choose $\zeta(t) = 1/t(t-z))$ for fixed z and the sets $E_\varepsilon = \{t: |t| > \varepsilon, |t-z| > \varepsilon, |t| < R\}$. Performing the integration above and passing to the limit as ε tends to zero, which exists since $w \in C^{0,\lambda}(\overline{B_R})$, $\lambda = 1-(2/q)$, yields the well known formula

$$\frac{w(z)-w(0)}{z} = \frac{1}{2\pi i}\int_{|t|=R} w\zeta\, dt - \frac{1}{\pi}\int_{B_R} w_{\bar t}\zeta\, dt_1 dt_2, \quad z \neq 0.$$

The first integral is a holomorphic function of $z \in B_R$. We write the second as

$$I(z) = -\frac{1}{\pi}\int_{B_R} w_{\bar t}\zeta\, dt_1 dt_2 = -\frac{1}{\pi}\int_{B_R} w_{\bar t}|t|^{-\sigma}|t|^{\sigma}\zeta\, dt_1 dt_2.$$

We first assume that $q\sigma < 2$. By Hölder's inequality

$$|I(z)| \leqslant \frac{1}{\pi}\|t^{-\sigma}w_{\bar t}\|_{L^q(B_R)}\left\{\int_{B_R}|t|^{p(\sigma-1)}|t-z|^{-p}dt_1 dt_2\right\}^{1/p}, \; z \in B_R, z \neq 0,$$

with $p^{-1}+q^{-1}=1$. The integral on the right may be estimated in a standard way which we indicate for the reader unfamiliar with this theory. Divide the set B_R into the three sets $A_1 = \{|t-z| < \frac{1}{2}|z|\} \cap B_R$, $A_2 = \{t: |t| < \frac{3}{2}|z|, \; |t-z| > \frac{1}{2}|z|\} \cap B_R$, and $A_3 = B_R - A_1 \cap A_2$, and set $I_j = \int_{A_j}|t|^{p(\sigma-1)}|t-z|^{-p}dt_1 dt_2$. It follows easily from the conditions $|t| > \frac{1}{2}|z|$ in A_1 and $|t-z| > \frac{1}{2}|z|$ in A_2 that

$$I_1 + I_2 \leqslant \text{const.}\,(q,\sigma)|z|^{2-(2-\sigma)p}.$$

For $t \in A_3$ we observe that

$$\left|\frac{t}{t-z}\right| \leqslant \frac{|t|}{|t|-|z|} \leqslant 3.$$

From this, one computes that

$$I_3 \leqslant 2\pi\int_{\frac{3}{2}|z|}^{R}\varrho^{p(\sigma-1)}\left|\frac{t}{t-z}\right|^p \varrho^{-p+1}d\varrho, \quad \varrho = |t|,$$

$$= 2\pi 3^p\int_{\frac{3}{2}|z|}^{R}\varrho^{1-(2-\sigma)p}d\varrho$$

$$\leqslant \text{const}\,(q,\sigma)\,|z|^{2-(2-\sigma)p} \qquad \text{since } 2-(2-\sigma)p < 0, |z| < R' < R.$$

Hence $(I_1+I_2+I_3)^{1/p} < \text{const.}\,(q,\sigma)|z|^{\sigma-2(1-1/p)} = \text{const.}\,(q,\sigma)|z|^{\sigma-(2/q)}$.

The proof when $q\sigma > 2$, nearly identical, will be omitted. Q.E.D.

§ 4

In this paragraph we show that $\Gamma=\{x\in R^3: x_3=u(z),\ z\in\partial I\}$ has a continuously differentiable parameterization. With reference to the notations of § 2, we shall actually show that a conformal mapping of $\{|t|<1,\ \operatorname{Im} t>0\}$ onto $\{\zeta: |\zeta|<\varepsilon,\ X(\zeta)\in S\}$ has boundary values of class $C^{1,\tau}$ for a $\tau>0$ on $\operatorname{Im} t=0$, $|t|<1$. Since the selection of the point 0 was arbitrary and Γ is compact, the result follows. On the other hand, the local character of the proof is clear. An important step in the demonstration is the lemma below for which we require concavity of the obstacle to effect a quasiconformal extension of this conformal mapping. The conclusion will follow from an important fact about such mappings, as we shall show.

LEMMA 4.1. *Let $\zeta=f_\varepsilon(t)$, for $\varepsilon>0$, be a conformal mapping of $G=\{|t|<1,\ \operatorname{Im} t>0\}$ onto a Jordan domain $f_\varepsilon(G)$ in D such that $\{|t|<1,\ \operatorname{Im} t=0\}$ is the preimage of $\gamma\cap \operatorname{cl} f_\varepsilon(G)$ (1), $\gamma=X^{-1}(\Gamma)$, $f_\varepsilon(0)=0$, and $D_S\cap B_\varepsilon(0)\subset f_\varepsilon(G)\subset D_S\cap B_{2\varepsilon}(0)$, $D_S=X^{-1}(S)$. Then there exists an $\varepsilon>0$ such that*

$$f_\varepsilon\in H^{1,q}(G\cap B_R),\ 0<R<1,\ \textit{for some}\ q>2.$$

Proof. Note first that f_ε is continuous in $\bar G$ since we have assumed $f_\varepsilon(G)$ to be a Jordan domain (Caratheodory [2]). We shall use the homeomorphism T to extend f_ε as a quasiconformal homeomorphism of B_1 onto a domain in the ζ plane. Given $f=f_\varepsilon$, we define

$$w(t)=\begin{cases} f(t) & t\in G \\ (X^{-1}\circ T\circ X\circ f)(\bar t) & t\in\{\operatorname{Im}\ t<0, |t|<1\}.\end{cases}$$

Recalling the formulas (2.10) and the discussion preceding them, the mapping w may be rewritten when $\operatorname{Im} t<0$ in terms of $F_0(p,\bar p)$, $p=(D_1u_0-iD_2u_0)/(1+W)=g_0(\zeta)$. The composite mapping has the expression

$$w(t)=(\phi_0^{-1}\circ F_0^{-1}\circ g_0\circ f)(\bar t)\qquad \operatorname{Im}\ t<0, |t|<1,\ \text{or}$$

$$w(t)=(F_0\circ\phi_0)^{-1}\circ(g_0\circ f)(\bar t)\qquad \operatorname{Im} t<0,\ |t|<1 \tag{4.1}$$

It is clear that w is continuous, because $T|\Gamma$ is the identity mapping, and that $w\in H^1(B_1)$. We shall prove the existence of an $\varepsilon>0$ and k, $0<k<1$, such that

$$\left|\frac{w_{\bar t}}{w_t}\right|<k\quad \text{a.e. in } B_1. \tag{4.2}$$

Delaying the proof of (4.2), we note it implies that $w\in H^{1,q}(B_R)$, for $R<1$ and for some $q>2$

(1) To avoid confusion, we shall use the notation $\operatorname{cl} f(G)$ for the closure of the image $f(G)$.

which is independent of R. In fact, any quasiconformal homeomorphism of the disc onto itself—that is any homeomorphism of the disc onto itself satisfying (4.2) with $k<1$ and in $H^1(B_1)$—is in $H^{1,q}(B_1)$ for a $q>2$ (cf., eg., [1] p. 276). Since the image of B_1 is not a disc, but some other domain, (4.2) implies only that $w \in H^{1,q}(B_R)$, $R<1$. This statement is verified easily by considering the composition $h \circ w$: $B_1 \to B_1$, where h: $w(B_1) \to B_1$ is conformal.

Now we prove (4.2). For any function h, set $\mu_h = h_{\bar z}/h_z$ and $m_h = 1/\mu_h$, and observe the relations about composite functions which follow from the chain rule.

$$m_{g\circ h} = \tau \frac{m_g + \overline{\tau \mu_h}}{1+\tau\mu_h m_g} \quad \text{where} \quad |\tau|=1, \quad \text{and} \quad |m_{h^{-1}}| = |m_h|.$$

Consider t in $\{\operatorname{Im} t<0, |t|<1\}$. By (4.1), since $(g_0 f)(\bar t)$ is an antianalytic function of t, a simple computation yields that

$$|\mu_w| = |m_{(F_0\circ\phi_0)^{-1}}|.$$

Hence from the formulas above

$$|\mu_w| = |m_{F_0\circ\phi_0}| = \left|\frac{m_{F_0} + \overline{\tau\mu_0}}{1+\tau\mu_0 m_{F_0}}\right|,$$

where $|\tau|=1$ and μ_0 is defined by (2.3). By Lemma 2.3, 2.4, we may choose $|\zeta|<\varepsilon$ with $\varepsilon>0$ so small that

$$|m_{F_0}| = \left|\frac{\partial F_0}{\partial z}\Big/\frac{\partial F_0}{\partial \bar z}\right| < k' < 1$$

and $|\mu_0|$, recalling (2.9), so small that $|\mu_w|<k<1$, for a $k \geq k'$, in $\{\operatorname{Im} t<0,\ |t|<1\}$. For $t \in G$, $\mu_w = 0$. Hence (4.2) is satisfied. Q.E.D.

As a remark we notice that only the continuity of the derivatives of F_0 was used in the proof. Hence, since quasiconformal mappings preserve null sets, the area of γ, and hence of Γ, is zero also when the obstacle $\psi \in C^2(\bar\Omega)$.

THEOREM 2. *Let $\zeta = f_\varepsilon(t)$ be a conformal mapping of $G=\{|t|<1, \operatorname{Im} t>0\}$ onto a Jordan domain $f_\varepsilon(G)$ in D such that $\{|t|<1, \operatorname{Im} t=0\}$ is the preimage of $\gamma \cap \operatorname{cl} f_\varepsilon(G)$, $\gamma = X^{-1}(\Gamma)$, $f_\varepsilon(0)=0$, and $D_S \cap B_\varepsilon(0) \subset f_\varepsilon(G) \subset D_S \cap B_{2\varepsilon}(0)$, $D_S = X^{-1}(S)$. Then there exists an $\varepsilon>0$ such that*

$$f_\varepsilon \in C^1(\bar G \cap B_R) \quad \textit{for each} \quad R<1.$$

To each point t_P, $-1<t_P<1$,(1) we shall associate a function $w=w_P \in H^{1,q}(B_R)$, $R<1$, which satisfies the condition

(1) The notation $-1<t<1$ means that the complex number $t \in (-1, 1)$.

$$|t-t_P|^{-\sigma} w_{\bar t}(t) \in L^q(B_R) \quad \textit{for a} \quad \sigma > 0. \tag{4.3}$$

To this w_P we shall apply Theorem 1. In the proof, we denote by const. any constant independent of P.

Proof. According to the previous lemma, there is an $\varepsilon > 0$ such that $f = f_\varepsilon \in H^{1,q}(G \cap B_R)$ for each $R < 1$ and a $q > 2$. It is no loss of generality to assume that $f \in H^{1,q}(G \cap B_1)$. We further restrict ε so that the conclusion of Lemma 2.3 is satisfied with the η of Lemma 2.4. For each $\zeta_P \in \gamma$, $|\zeta_P| < \varepsilon$ and $X(\zeta_P) = P \in \Gamma$, we define ϕ_P^* by

$$\begin{aligned}\phi_P^* &= \left(\frac{\partial F_P}{\partial \bar z}(0,0)\right)^{-1}\left\{g_P - \frac{\partial}{\partial z} F_P(0,0)\,\phi_P - G_P(\phi_P, \bar\phi_P)\right\}, \quad \zeta \in B_\varepsilon(0) \subset B_{2\varepsilon}(\zeta_P)\\ &= \left(\frac{\partial}{\partial \bar z} F_P(0,0)\right)^{-1}\left\{g_P + \frac{\partial}{\partial \bar z} F_P(0,0)\,\bar\phi_P - F_P(\phi_P, \bar\phi_P)\right\},\end{aligned}$$

according to (2.11). The ϕ_P^* so defined is continuous in $B_\varepsilon(0)$ and satisfies

$$\phi_P^* = \bar\phi_P \quad \text{on} \quad \gamma \cap B_\varepsilon(0)$$

in view of the relation $g_P = F_P(\phi_P, \bar\phi_P)$ on γ. In addition, for $\zeta \in D_S$

$$\frac{\partial}{\partial \bar\zeta}\phi_P^* = \left(\frac{\partial}{\partial \bar z} F_P(0,0)\right)^{-1}\left\{\left(\frac{\partial}{\partial \bar z} F_P(0,0) - \frac{\partial}{\partial \bar z} F_P(\phi_P, \bar\phi_P)\right)\overline{\frac{\partial \phi_P}{\partial \zeta}} - \frac{\partial F_P}{\partial z}(\phi_P, \bar\phi_P)\,\frac{\partial \phi_P}{\partial \bar\zeta}\right\},$$

Now $(\partial/\partial \bar z)\,F_P$ is a Lipschitz function of ζ since ψ_P is in C^3 and ϕ_P is in C^1, whence by (2.4), (2.9), and (2.11),

$$\left|\frac{\partial}{\partial \bar\zeta}\phi_P^*(\zeta)\right| \leqslant \text{const.}\,|\zeta - \zeta_P|, \quad \zeta \in B_\varepsilon(0) \cap D_S.$$

It is evident that ϕ_P^* is Lipschitz in ζ uniformly in P. We define

$$w(t) = w_P(t) = \begin{cases} \phi_P(f(t)) & t \in \bar G \\ \overline{\phi_P^*(f(\bar t))} & \operatorname{Im} t < 0, \quad |t| < 1, \end{cases}$$

a function continuous in B_1. Since $f \in H^{1,q}(B_1 \cap G)$ it follows that $w \in H^{1,q}(B_1)$. In view of the equation satisfied by ϕ_P, (2.4), and the inequality above derived for ϕ_P^*, we compute that

$$\begin{aligned} |w_{\bar t}(t)| &\leqslant \text{const.}\,|f'(t)|\,|f(t) - f(t_P)| \quad \operatorname{Im} t > 0,\ |t| < 1 \\ |w_{\bar t}(t)| &\leqslant \text{const.}\,|f'(\bar t)|\,|f(\bar t) - f(t_P)| \quad \operatorname{Im} t < 0,\ |t| < 1. \end{aligned} \tag{4.4}$$

Since $f \in C^{0,\lambda}(\overline{G \cap B_1}) \cap H^{1,q}(G \cap B_1)$, $\lambda = 1 - (2/q)$, and t_P is real, (4.3) is satisfied for $\sigma = \lambda$. Moreover,

$$\| |t-t_P|^{-\sigma} w_{\bar{t}} \|_{L^q(B_1)} \leqslant \text{const.}, |t_P| < R < 1. \tag{4.5}$$

Let R and R' be fixed, with $R' < R$. First we suppose that $q\sigma > 2$ to apply Theorem 1 (3.2). Then there exists a b_P, $|b_P| \leqslant$ const. (q, σ, R, R'), such that

$$\left| \frac{w(t) - w(t_P)}{t - t_P} - b_P \right| \leqslant \text{const.}\ (q, \sigma, R, R') |t - t_P|^\tau, \quad t \in B_R, t_P \in B_{R'}, \quad \tau = \min\left(1, \sigma - \frac{2}{q}\right) > 0.$$

We also apply Theorem 1 (3.2) to $\phi_P^{-1}(z)$ which satisfies (3.3) in a neighborhood B_ϱ, ϱ independent of P. In this case, $q = \infty$ and $\sigma = 2$ so (3.2) applies. Hence for some b'_P, $|b'_P| \leqslant$ const.,

$$\left| \frac{\phi_P^{-1}(z)}{z} - b'_P \right| \leqslant \text{const.}\ |z|, |z| < \varrho.$$

Consequently for $c_P = b_P b'_P$, and further restricting ε if necessary so that $\phi_P(B_\varepsilon) \subset B_\varrho$,

$$\left| \frac{f(t) - f(t_P)}{t - t_P} - c_P \right| \leqslant \text{const.}\ |t - t_P|^\tau, \quad t \in B_R, t_P \in B_{R'}. \tag{4.6}$$

On the other hand, in the case where $q\sigma \leqslant 2$, or $q \leqslant 4$, we apply (3.1) to conclude that

$$|w(t) - w(t_P)| \leqslant \text{const.}\ (q, \sigma, R, R') |t - t_P|^{1+\tau'}, t_P \in B_{R'}, t \in B_R,$$

and $\tau' = \sigma - (2/q)$ if $q\sigma < 2$ and $\tau' = \sigma' - (2/q)$ for any $\sigma' < \sigma$ if $q\sigma = 2$. Since ϕ_P^{-1} is a Lipschitz function with constant independent of P, $t_P \in B_1$,

$$|f(t_P) - f(t)| \leqslant |\phi_P^{-1}(w(t)) - \phi_P^{-1}(w(t_P))| \leqslant \text{const.}\ |t - t_P|^{1+\tau'}, t \in \overline{G \cap B_R}, t_P \in B_{R'}.$$

Therefore by (4.4), w satisfies (4.3) and (4.5) with an improved $\sigma = \sigma_2 = \sigma_1 + 1 - (2/q)$ with $\sigma_1 = \lambda$ if $q\sigma_1 < 2$ and $\sigma_1 = \lambda'$ for any $\lambda' < \lambda$ if $\sigma_1 q = 2$. Repeating this process at most a finite number of times we obtain a σ_0 such that (4.3) is satisfied for $\sigma = \sigma_0 > 2/q$.

We return to the previous case to conclude (4.6). To complete the proof, we note the simple

LEMMA 4.2. *Let $f(t) \in C^\circ(\overline{G \cap B_R})$ be analytic in $G \cap B_R$ and satisfy*

$$|f(t) - f(t_P) - c_P(t - t_P)| \leqslant N_1 |t - t_P|^{\tau+1} \ \textit{for}\ t \in \overline{G \cap B_R} \ \textit{and} \ -R' < t_P < R', R' < R,$$

where $0 < \tau < 1$ and $|c_P| \leqslant N_2$, N_1 and N_2 positive constants. Then $f \in C^{1,\tau}(\overline{G \cap B_r})$ for each $r < R'$.

The proof follows upon writing Cauchy's formula for the derivative of $f(t) - f(t_P) - c_P(t - t_P)$.

§ 5

In this section we derive a system of differential equations to be used in the analytic extension of a conformal representation of S for which we assume that ψ is a real analytic strictly concave function. We recapitulate some notations. Let 0 be a fixed, but arbitrary, point of Γ, that is the point chosen in § 2, and $X: D \to \Sigma \subset R^3$, $D = \{|\zeta| < 1\}$, a conformal representation of Σ of class $C^{1,\lambda}(D)$ such that $X_3 = 0$ is the tangent plane to Σ and M at 0 and $X(0) = 0$. Let $\zeta = f(t)$ be a conformal mapping of $G = \{|t| < 1, \operatorname{Im} t > 0\}$ onto a Jordan domain $f(G)$ such that $\{\zeta: |\zeta| < \varepsilon, X(\zeta) \in S\} \subset f(G)$, for some $\varepsilon > 0$, and $-1 < t < 1$ is the preimage of a portion of $\gamma = X^{-1}(\Gamma)$, $f(0) = 0$. By Theorem 2, $f \in C^1(\bar{G})$ and hence $\tilde{X}(t) = X(f(t)) = (\tilde{X}_1(t), \tilde{X}_2(t), \tilde{X}_3(t))$ is a conformal representation of a portion of S which satisfies $\tilde{X}(t) \in \Gamma$ for $-1 < t < 1$, $\tilde{X}(0) = 0$, and $\tilde{X}_j(t) \in C^1(\bar{G})$, $1 \leqslant j \leqslant 3$. We shall use the notation $X_j(t)$ instead of $\tilde{X}_j(t)$. It is clear that $X_j(t)$ admit single valued harmonic conjugates $X_j^*(t) \in C^1(\bar{G})$, $X_j^*(0) = 0$, $1 \leqslant j \leqslant 3$. Let us set $f_j(t) = X_j(t) + iX_j^*(t)$ and write the isothermal relations in the form

$$f_1'(t)^2 + f_2'(t)^2 + f_3'(t)^2 = 0, \quad t \in \bar{G}. \tag{5.1}$$

Assume for the moment the existence of functions $\phi_j \in C^1(\bar{G})$ which satisfy

$$\phi_j(t) = \overline{f_j(t)}, \quad -1 < t < 1, \quad 1 \leqslant j \leqslant 3. \tag{5.2}$$

One such triple of functions is $\phi_j(t) = \overline{f_j(t)}$. The fact that $f_j \in C^1(\bar{G})$ allows us to deduce that

$$\phi_j'(t) = \overline{f_j'(t)}, \quad -1 < t < 1, \quad 1 \leqslant j \leqslant 3$$

and therefore by the isothermal relations (5.1),

$$\phi_1'(t)^2 + \phi_2'(t)^2 + \phi_3'(t)^2 = 0, \ -1 < t < 1. \tag{5.3}$$

Denote by $X_3 = \psi_0(X_1, X_2)$ a non parametric representation of a portion of M obtained by orthogonal projection onto the tangent plane of M at 0 and by $\psi_j = D_j \psi_0$, $\psi_{jk} = D_j D_k \psi_0$ (cf. § 2). The function ψ_0 and its derivatives with respect to X_j may be extended as holomorphic functions of two complex variables in the obvious manner. The fact that the surfaces M and Σ are tangent on Γ may be expressed by writing that the tangent plane to Σ is perpendicular to the normal direction of M, namely,

$$\psi_1 f_1' + \psi_2 f_2' - f_3' = 0, \qquad -1 < t < 1, \qquad \psi_j = \psi_j(X_1(t), X_2(t)). \tag{5.4}$$

Therefore

$$\psi_1 \phi_1' + \psi_2 \phi_2' - \phi_3' = 0, \qquad -1 < t < 1. \tag{5.5}$$

To obtain a third equation to complement (5.3) and (5.5), we would differentiate

(5.4), were that possible. Instead, the knowledge that the non-parametric solution u of (1.1) has bounded second derivatives, used in Lemma 2.2, provides us with three Lipschitz functions $g_j(t)$, $1\leqslant j\leqslant 3$, defined by (2.6) and satisfying f_1': f_2': $f_3'=g_1$: g_2: g_3 in G. This relation may be extended to any $t\in\bar{G}$ where not all of $f_j'(t)=0$, that is, almost everywhere. Hence

$$\psi_1 g_1+\psi_2 g_2-g_3=0,\qquad -1<t<1.$$

Writing $\psi_j=\psi_j(\frac{1}{2}(\phi_1+f_1),\ \frac{1}{2}(\phi_2+f_2))$, etc., we are led to the system of equations

$$\begin{aligned}\sum_{1\leqslant j,k\leqslant 2}\psi_{jk}\phi_j' g_k &= -\sum_{1\leqslant j,k\leqslant 2}\psi_{jk}f_j' g_k+2(g_3'-\psi_1 g_1'-\psi_2 g_2')\\ \psi_1\phi_1'+\psi_2\phi_2'-\phi_3' &=0\qquad t\in G.\\ (\phi_1')^2+(\phi_2')^2+(\phi_2')^2&=0\end{aligned}\tag{5.6}$$

with initial values $\qquad \phi_j(0)=\overline{f_j(0)}=0,\qquad 1\leqslant j\leqslant 3.$

The system (5.6) is satisfied by $\overline{f_j(t)}$ a.e., $-1<t<1$. In particular, since $\overline{f_j(t)}\in C^1(\bar{G})$, we may conclude that $g_j\in C^1((-1,1))$ by means of Lemma 2.1, but we shall not need this fact.

It is convenient to express (5.6) explicitly in terms of the unknowns φ_j' for which we shall employ that the mean curvature of M does not vanish. Let (a_1, a_2) be a pair of complex numbers and consider $\psi_j=\psi_j(\frac{1}{2}(a_1+f_1(t)),\ \frac{1}{2}(a_2+f_2(t)))$, $\psi_{jk}=\psi_{jk}(\frac{1}{2}(a_1+f_1(t)),\ \frac{1}{2}(a_2+f_2(t)))$. We abbreviate by $U_\delta=\{(a_1,a_2)\colon\ |a_1|^2+|a_2|^2<\delta^2,\ a_j \text{ complex}\}$ open in the space of two complex variables and $G_\varepsilon=\{\operatorname{Im} t>0,\ |t|<\varepsilon\}$. For the unknown complex numbers $\alpha=(\alpha_1,\alpha_2,\alpha_3)$ we consider the system of equations

$$\begin{aligned}(\psi_{11}g_1+\psi_{12}g_2)\alpha_1+(\psi_{12}g_1+\psi_{22}g_2)\alpha_2&=\beta\\ \psi_1\alpha_1+\psi_2\alpha_2-\alpha_3&=0\\ \alpha_1^2+\alpha_2^2+\alpha_3^2&=0.\end{aligned}\tag{5.7}$$

LEMMA 5.1. *There exist functions $Y_j(t,a_1,a_2)\in C(\bar{G}_\varepsilon)\times H^{1,\infty}(U_\delta)$, holomorphic in $G_\varepsilon\times U_\delta$, for ε, δ sufficiently small, such that* (5.7) *admits a unique solution*

$$\alpha_j=Y_j(t,\alpha_1,\alpha_2)\beta\qquad 1\leqslant j\leqslant 3\tag{5.8}$$

satisfying $\operatorname{Im}(\alpha_2/\alpha_1)>0$ *whenever* $\beta\neq 0$.

Proof. We present a straightforward proof. From the last two relations of (5.7), it follows that $(1+\psi_1^2)\alpha_1^2+2\psi_1\psi_2\alpha_1\alpha_2+(1+\psi_2^2)\alpha_2^2=0$ and therefore, assuming the moduli of t, a_1, a_2 to be small, either $\alpha_1=\alpha_2=0$ or

$$\frac{\alpha_2}{\alpha_1}=\frac{-\psi_1\psi_2 \pm i\sqrt{1+\psi_1^2+\psi_2^2}}{1+\psi_2^2},$$

where we have chosen a branch of the square root satisfying $\sqrt{t}>0$ for $t>0$. We consider this latter case. We choose the ratio

$$\frac{\alpha_2}{\alpha_1}=\frac{-\psi_1\psi_2 + i\sqrt{1+\psi_1^2+\psi_2^2}}{1+\psi_2^2}$$

implying, in particular, that $\operatorname{Im} \alpha_2/\alpha_1>0$ for (t, a_1, a_2) small. This inserted in the first equation of (5.7) yields that

$$\{(1+\psi_2^2)(\psi_{11}g_1+\psi_{12}g_2)+(\psi_{12}g_1+\psi_{22}g_2)(-\psi_1\psi_2+i\sqrt{1+\psi_1^2+\psi_2^2})\}\,\alpha_1=(1+\psi_2^2)\,\beta.$$

At the point $t=a_1=a_2=0$, $\psi_j=\psi_j(0,0)=0$ and $g_1(0)=\frac{1}{2}$, $g_2(0)=(1/2i)$, so that

$$(1+\psi_2^2)(\psi_{11}g_1+\psi_{12}g_2)+(\psi_{12}g_1+\psi_{22}g_2)(-\psi_1\psi_2+i\sqrt{1+\psi_1^2+\psi_2^2})=\tfrac{1}{2}\Delta\psi_0(0,0)=H_0,$$

the mean curvature of M at 0. Hence for ε, $\delta>0$ sufficiently small, there is a function $Y_1(t, a_1, a_2)$ satisfying the conclusions of the lemma. In the same way we find a function Y_2 and we define

$$Y_3=\psi_1 Y_1+\psi_2 Y_2$$

in view of (5.4). On the other hand, $\alpha_1=\alpha_2=\alpha_3=0$ is the unique solution of (5.7) when $\beta=0$, for ε, δ small, so that the equations (5.8) are valid for any β. The uniqueness of α under the hypothesis that $\operatorname{Im}(\alpha_2/\alpha_1)>0$ is clear from the derivation. Q.E.D.

From the functions Y_j in (5.8) we define

$$E_j(t, a_1, a_2)=Y_j\{-\sum_{1\le h,k\le 2}\psi_{hk}f_h' g_k+2(g_3'-\psi_1 g_1'-\psi_2 g_2')\}, \tag{5.9}$$

$\psi_{hk}=\psi_{hk}(\frac{1}{2}(a_1+f_1(t)), \frac{1}{2}(a_2+f_2(t)))$, $\psi_h=\psi_h(\frac{1}{2}(a_1+f_1), \frac{1}{2}(a_2+f_2))$.

Lemma 5.2. *The functions $\overline{f_j(t)}$ satisfy*

$$\overline{f_j(t)}=\int_0^t E_j(\tau, \overline{f_1(\tau)}, \overline{f_2(\tau)})\,d\tau,\ -\eta<t<\eta \textit{ for some } \eta>0.$$

Proof. Since the integrand on the right hand side is bounded, it suffices to verify that

$$\overline{f_j'(t)}=E_j(t, \overline{f_1(t)}, \overline{f_2(t)}) \text{ a.e. for } -\eta<t<\eta. \tag{5.10}$$

Since $\overline{f_j(t)}$ satisfy (5.3) and hence (5.4) with

$$\beta = -\sum \psi_{hk} f_h' g_k + 2(g_3' - \psi_1 g_1' - \psi_2 g_2'), \quad \psi_{hk} = \psi_{hk}(X_1(t), X_2(t)) \text{ etc},$$

except for a set of measure zero, it is only necessary to show that $\operatorname{Im}(\overline{f_2'(t)}/\overline{f_1'(t)}) > 0$ a.e. for $-\eta < t < \eta$. At any t where $f_1'(t) \neq 0$ and $f_2'(t) \neq 0$,

$$\overline{f_1'(t)} : \overline{f_2'(t)} = \overline{g_1(t)} : \overline{g_2(t)}.$$

The above is valid almost everywhere. Observing that $\lim_{t\to 0} (\overline{g_2(t)}/\overline{g_1(t)}) = i$, it follows that $\operatorname{Im}(\overline{f_2'(t)}/\overline{f_1'(t)}) > 0$ for $|t|$ sufficiently small. The lemma is proved. Q.E.D.

§ 6

In this paragraph we prove that the curve Γ is analytic.

THEOREM 3. *There exists a solution $\phi(t) = (\phi_1(t), \phi_2(t), \phi_3(t))$ continuous in $\bar{G}_\varepsilon$ and analytic in G_ε to the system*

$$\phi_j(t) = \int_0^t E_j(\tau, \phi_1(\tau), \phi_2(\tau))\, d\tau, \quad t \in \bar{G}_\varepsilon, \quad 1 \leq j \leq 3, \text{ for some } \varepsilon > 0. \tag{6.1}$$

$$\phi_j(0) = 0.$$

Furthermore, $\phi_j(t) = \overline{f_j(t)}$, $-\varepsilon < t < \varepsilon$.

Proof. The proof is by successive approximation. Choose a preliminary pair $\varepsilon_1, \delta > 0$ such that $E_j(t, a_1, a_2) \in C(\bar{G}_{\varepsilon_1}) \times H^{1,\infty}(U_\delta)$ and define the operator on triples of complex valued functions continuous in $\bar{G}_\varepsilon$ and analytic in G_ε

$$(A\phi)(t) = \int_0^t E(\tau, \phi_1(\tau), \phi_2(\tau))\, d\tau, \quad E = (E_1, E_2, E_3),$$

where the integral is extended over rectifiable arcs in $\bar{G}_\varepsilon$. For such triples ϕ, $A\phi$ is analytic in G_ε and continuous in $\bar{G}_\varepsilon$.

With the obvious notation $|\phi| = \sqrt{\sum |\phi_j|^2}$, we may estimate that

$$|A\phi(t)| \leq \sup_{|\tau| \leq \varepsilon_1} |E(\tau, \phi_1(\tau), \phi_2(\tau))|\,|t| \leq \sup\{|E(\tau, a_1, a_2)| : (\tau, a_1, a_2) \in G_{\varepsilon_1} \times U_\delta\}\,|t|$$

for the subset of functions ϕ satisfying $|\phi| < \delta$. Choose $\varepsilon_2 > 0$, $\varepsilon_2 \leq \varepsilon_1$, so that $\sup\{|E(\tau, a_1, a_2)| : (\tau, a_1, a_2) \in G_{\varepsilon_2} \times U_\delta\}\, \varepsilon_2 < \delta$. Hence $|\phi| < \delta$ implies that $|A\phi| < \delta$ and $A^2\phi$ is defined. In as much as E is a Lipschitz function of (a_1, a_2), there is a $C > 0$ such that

$$\begin{aligned}|A\phi(t)-A\phi^*(t)| &\leqslant C\sup_{[0,t]}|\phi(\tau)-\phi^*(\tau)|\,|t| \\ &\leqslant \tfrac{1}{2}\sup_{[0,t]}|\phi(\tau)-\phi^*(\tau)| \\ &\leqslant \tfrac{1}{2}\sup_{G_{\varepsilon_3}}|\phi(\tau)-\phi^*(\tau)|\end{aligned} \tag{6.2}$$

for $|t|\leqslant\varepsilon_3\leqslant\varepsilon_2$, where $[0, t]$ denotes the segment from 0 to t in the complex plane. We define $\phi^{(0)}(t)=(0, 0, 0)$ and $\phi^{(k+1)}(t)=A\phi^{(k)}(t)$. It is clear that $\phi^{(k)}\in C^0(\bar{G}_\varepsilon)$ is analytic in G_ε and $\phi^{(k)}(0)=(0, 0, 0)$, $\varepsilon=\varepsilon_3$. Moreover,

$$\sup_{G_{\varepsilon_3}}|\phi^{(k+1)}(t)-\phi^{(k)}(t)|\leqslant 2^{-k}\delta$$

It follows that $\{\phi^{(k)}(t)\}$ converges to a ϕ continuous in $\bar{G}_\varepsilon$ and analytic in G_ε. Moreover,

$$\phi(t)=\lim\phi^{(k)}(t)=A(\lim\phi^{(k-1)}(t))=(A\phi)(t),\quad t\in\overline{G_{\varepsilon_3}}.$$

That is, $\phi(t)$ satisfies (6.1). For $|t|<\varepsilon=\min(\varepsilon_3, \eta)$ and $\operatorname{Im} t=0$, it is also true that

$$\phi^*(t)=\int_0^t E(\tau, \overline{f_1(\tau)}, \overline{f_2(\tau)})\,d\tau,\ \phi^*(t)=(\overline{f_1(t)}, \overline{f_2(t)}, \overline{f_3(t)})$$

by Lemma 5.2. Hence by (6.2),

$$|\phi(t)-\phi^*(t)|\leqslant\tfrac{1}{2}\sup_{0\leqslant t\leqslant t}|\phi(\tau)-\phi^*(\tau)|,$$

which implies that $\phi(t)=\phi^*(t)$. Q.E.D.

COROLLARY 6.1. *The holomorphic functions $f_j(t)$ admit holomorphic extensions into a neighborhood $\{|t|<\varepsilon\}$ for some $\varepsilon>0$. In particular, the parameterization of a portion of Γ given by*

$$t\to\operatorname{Re}(f_1(t), f_2(t), f_3(t)),\ -\varepsilon<t<\varepsilon,$$

is analytic.

Proof. The proof is a direct application of the Schwarz Reflection Principle. With $\phi_j(t)$ the solution to (6.1) in $\bar{G}_\varepsilon$, we set

$$f_j(t)=\begin{cases} f_j(t) & \operatorname{Im} t\geqslant 0,\quad |t|<\varepsilon \\ \overline{\phi_j(\bar{t})} & \operatorname{Im} t<0,\quad |t|<\varepsilon.\end{cases}$$

By Theorem 3, f_j is continuous in $|t|<\varepsilon$ and hence holomorphic for $|t|<\varepsilon$.

THEOREM 4. *Let u be the solution to equation* (1.1) *and $I=\{z=x_1+ix_2\colon u(z)=\psi(z)\}$. Let $\Gamma=\{(x_1, x_2, x_3)\colon x_3=u(z)=\psi(z), z\in\partial I\}$. Then Γ is an analytic curve with an analytic tangent.*

Proof. The analyticity of Γ at each of its points was shown in Corollary 6.1. Hence Γ is analytic. In view of the analyticity of the conformal representation, the analyticity of the tangent to Γ is tantamount to showing that $\Sigma|f_j'(t)|^2>0$ for each t. Supposing the situation to be organized as in the preceding, we shall show that $\Sigma|f_j'(0)|^2>0$. The relations in Lemma 2.1 and the observation, from (2.6), that $g_1+ig_2=1$ imply that $g=f_3'/(f_1'+if_2')$. By Corollary 6.1, g is analytic in $0<|t|<\delta$, for some $\delta>0$, and furthermore, $g(0)=0$ since the normal to S at $0=X(0)$ is $(0, 0, 1)$. Hence g is analytic in B_δ. The expression for g in terms of the f_j' above implies, via the isothermal relations and Lemma 2.1, that the 1: 1 harmonic mapping from $G\cap B_\delta$ to the tangent plane of S at 0 has the form

$$z=z(t)=X_1(t)+iX_2(t)=at^\alpha(1+P(t,\bar{t})),\ a\neq 0,\ t\in G\cap B_\delta, \tag{6.3}$$

where $P(t,\bar{t})$ is a power series in $(t,\bar{t})$, $P(0,0)=0$. The integer $\alpha=1$ or 2, since $z(t)$ is 1: 1, with $\Sigma|f_j'(0)|^2>0$ only if $\alpha=1$. We suppose $\alpha=2$ for contradiction. Then the set

$$U=\{z=X_1+iX_2\in B_\varrho(0)\colon X(t)\in S\},$$

some $\varrho>0$, contains the outside of a cusp at $z=0$ by (6.3). By "the outside of a cusp at $z=0$" we intend "the frustum of a curvilinear sector of angle 2π with vertex at $z=0$."

Now $F_0(z,\bar{z})$ is a smooth homeomorphism of B_ϱ which preserves these properties, or the image of U under F_0, which we denote by an abuse of notation as $F_0(U)$, also contains the outside of a cusp with vertex at 0 and $F_0(B_\varrho-U)$ is bounded in part by the frustum of a curvilinear sector of angle 0.

As we have noted, the spherical image of S is contained in that of M_I, or $g(B_\delta\cap G)\subset F_0(B-U)$. The hypothesis that $\alpha=2$ implies that g maps $B_\delta\cap G$ onto a domain in $F_0(B-U)$ with the property that the variation of the argument of g on any semicircle $\{\operatorname{Im} t>0, |t|=\text{const.}\}$ tends to 0 as $|t|\to 0$. Since g is analytic in B_δ, this is untenable. Hence $\alpha=1$.

§ 7

The system of equations (5.6) depends only on the mean curvature of the obstacle M, but to solve it we were forced to use that the obstacle was strictly concave. Let us consider the special case of an obstacle which is in part a cylinder with negative mean curvature and show how a result analogous to Theorem 4 may be obtained. In this "degenerate" case, where one of the principle curvatures of M is always zero, the function g may be extended directly.

Consider an obstacle ψ such that $\psi(z)=h(x_1)$ for $z\in B\subset\Omega$, B a ball, where h is a real analytic function of x_1. Since the solution u to (1.1) is in $C^1(\Omega)$,

$$\frac{u_{x_1}-iu_{x_2}}{1+W}=\frac{h'}{1+\sqrt{1+h'^2}}\ \text{ in }\ B\cap I.$$

We assume also that ψ is concave and C^2 in $\bar{\Omega}$, from which it may be seen that the set of coincidence I is connected (a continuum) and it is well known that $\Omega - I$ is connected in this case. It follows that $\Omega - I$ is topologically an annulus and hence the surface $S=\{(x_1,x_2,x_3): x_3=u(z), z\in\Omega-I\}$ admits a conformal representation $X(t)=(X_1(t),X_2(t),X_3(t))$ in U, the upper half t plane from which a suitable circle is deleted. Let

$$g(t)=\frac{u_{x_1}-iu_{x_2}}{1+W}\bigg|_{z=z(t)}$$

analytic in U. We may suppose that as $t_k \to t$, $-1<t<1$, $t_k \in U$, $z(t_k)$ tends to a continuum of $\partial I \cap B$. Hence the bounded analytic g satisfies

$$\lim_{t_k\to t} g(t_k) \quad \text{is real a.e.,} \quad -1<t<1,$$

so that g may be extended analytically across $\operatorname{Im} t=0$, $|t|<1$. Solving for $X_1(t)$ in the relation

$$g(t)=\frac{h'(X_1(t))}{1+\sqrt{1+h'^2}},$$

possible because the mean curvature of M, and hence the second derivative of h, does not vanish, we conclude that $X_1(t)$ is analytic on $\operatorname{Im} t=0$, $|t|<1$. At this point we notice that it suffices to hypothesize only that $h\in C^2$ to obtain some sort of result. The other relations available imply that X_3 and X_2 are also analytic.

This idea of exploiting the relationships between several harmonics in a problem is of wide applicability.

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