UNIFORM APPROXIMATION ON SMOOTH CURVES

BY

GABRIEL STOLZENBERG

Brown University, Providence, R. I., U.S.A. (1)

Let $K_1, ..., K_n$ be compact subsets of complex N-space C^N, each the locus of a smooth (*continuously differentiable*) curve. Let $K = K_1 \cup ... \cup K_n$.

For any compact set Y in C^N define its polynomial convex hull $\hat{Y}$ as

$$\{p \in C^N : |f(p)| \leq \max_Y |f| \text{ for all polynomials } f\},$$

and say that Y is *polynomially convex* whenever $Y = \hat{Y}$.

Let X be a polynomially convex set in C^N.

THEOREM.

A. $\widehat{K \cup X} - (K \cup X)$ *is a (possibly empty) one-dimensional analytic subset of* $C^N - (K \cup X)$.

B. *Every continuous function on* $K \cup X$ *which is uniformly approximable on* X *by polynomials is uniformly approximable on* $K \cup X$ *by rational functions.*

C. *If* K *is simply-connected and disjoint from* X *or, more generally, if the map* $\check{H}^1(K \cup X; Z) \to \check{H}^1(X; Z)$ *induced by* $X \subset K \cup X$ *is injective then* $K \cup X$ *is polynomially convex.*

Comments (Technical)

1. N may be infinite, but n is finite.

2. A closed subset V of an open subset U of C^N is a *one-dimensional analytic subset of* U if and only if a neighborhood of each point in V can be covered by finitely many sets of the form $\Phi(\Delta)$ where Δ is an open disk in the plane and each $\Phi : \Delta \to V$ is a non-constant *analytic* mapping, i.e., for each complex coordinate z_j on C^N, $z_j \circ \Phi$ is analytic on Δ.

3. $\hat{Y}$ is the spectrum of the algebra of all uniform limits of polynomials on Y [18].

4. In part B, if $K \cup X$ is polynomially convex then the rational functions may be taken to be polynomials [18].

(1) This work was supported in part by N.S.F. grant GU–976.

AN APPLICATION: *If $\mathfrak{F}$ is a family of smooth complex-valued functions on a closed interval I such that for every pair $x \neq y$ in I there is an $f \in \mathfrak{F}$ with $f(x) \neq f(y)$ then every continuous function on I is a uniform limit of polynomial combinations of members of $\mathfrak{F}$.*

Proof. This follows directly from parts C, B and Comment 4 if we view the members of $\mathfrak{F}$ as the coordinates of a smooth injection $I \to C^{\mathfrak{F}}$, set K = the image of I and let X be empty.

Comments (Historical)

We are paving the path pioneered by John Wermer in [15], [16] and [17]. He proved the theorem for K a single non-singular real analytic arc or simple closed curve, X empty and N finite. He also constructed examples [14], [17] to show that without *some* smoothness restriction on K parts A, B and C can all be false, even for K an arc, X empty and N finite.

Next, Errett Bishop in [1] and Halsey Royden in [10], each emphasizing a different aspect of Wermer's approach, went further and settled the case of a general real analytic K and empty X.

Then, in [2], Bishop developed a completely new approach as part of an attack on the general problem of determining the extent of analytic structure on the spectrum of an algebra of analytic functions. In this way he redid the real analytic case (with X empty), but by methods which he knew could also be used to settle the general smooth case. He invented a theory of interpolating semi-norms with which he exposed and exploited the local nature of the problem.

We do not interpolate semi-norms; but the local character of our theorem is implied by the presence of the extra set X. Our information about $K \cup X$ is only local, smoothness on K. The smoothness is used, with Sard's Lemma [12] to get certain polynomials which project the part of $K \cup X$ lying over some sector in C^1 locally one-one onto a finite disjoint union of smooth non-singular arcs (Lemmas 1–4). Then, by our variation (Lemmas 5–9) on Bishop's argument from pp. 496–497 of [2], we produce *some* strategically located analytic disks near K in $\widehat{K \cup X} - (K \cup X)$. This uses

(i) *The Local Maximum Modulus Principle* (*L.M.M.P.*). *If $T \subset \hat{Y} \subset C^N$ and ∂ is the topological boundary of T in $\hat{Y}$ then $T \subset \widehat{\partial \cup (T \cap Y)}$, and*

(ii) (*The*) *Maximality Theorem. The uniform closure of the polynomials on a closed disk in C^1 is maximal among all uniformly closed algebras of continuous functions on that disk which satisfy the maximum modulus principle with respect to the boundary.*

(The L.M.M.P. was proved by Hugo Rossi [9]; there is a relatively short proof from first principles in [6]. Generalizations of this maximality theorem were proved by Walter

Rudin in [10] and by Wermer. There are *two* very short proofs of Wermer's result on pp. 93–94 of [8].)

Finally, to get from these isolated analytic disks to analyticity everywhere on $\widehat{K \cup X} - (K \cup X)$ we use ideas of Royden from his elegant and illuminating treatment of the real analytic case in [10]. In particular, we adapt to our local situation his way of using an analytic kernel to "cross over edges" [10, pp. 39–41], and with that plus his criterion (Lemma 10 and [10, p. 25]) for a linear functional on an algebra to be a linear combination of homomorphisms we get an explicit parameterization of the analytic structure on $\widehat{K \cup X} - (K \cup X)$. This is accomplished in Lemmas 10 and 11.

Note. Throughout this paper the elements of commutative Banach algebra theory are used freely, often without comment. For a general reference we have [18].

Proof of part B

(A special case was done in [7].)

By the theory of antisymmetric sets (see [4]) it suffices to prove that if $p \in K - X$ then for each $q \neq p$ in $K \cup X$ there is a *real*-valued f, with $f(q) \neq f(p)$, which is uniformly approximable by rational functions on $K \cup X$.

Since X is polynomially convex there is a polynomial g such that $g(p) = 1$ and $\operatorname{Re} g \leqslant 0$ on $X \cup \{q\}$. Let c be a real-valued continuous function on $g(K \cup X)$ which is identically 0 for $\operatorname{Re}\zeta \leqslant \frac{1}{2}$ and with $c(0) = 1$. The following argument of Wermer shows that c is a uniform limit of rational functions on $g(K \cup X)$.

Namely, it suffices to prove that any measure μ on $g(K \cup X)$ which annihilates all uniform limits of rational functions also annihilates c. This will be done if we can show that any such μ is supported on $\{\operatorname{Re}\zeta \leqslant \frac{1}{2}\}$. But K is a finite union of smooth curves and g is a polynomial, so $g(K)$ has zero planar measure and, hence, $\int (z-\zeta)^{-1} d\mu(z) = 0$ for almost all ζ with $\operatorname{Re}\zeta > \frac{1}{2}$. Therefore, by Fubini's Theorem, for almost all open disks $\Delta \subset \{\operatorname{Re}\zeta > \frac{1}{2}\}$, if $\partial =$ the boundary of Δ then

$$0 = \frac{-1}{2\pi i} \int_\partial d\zeta \int (z-\zeta)^{-1} d\mu(z) = \int d\mu(z) \cdot \frac{1}{2\pi i} \int_\partial (\zeta - z)^{-1} d\zeta = \int \chi_\Delta(z)\, d\mu(z),$$

where χ_Δ is the characteristic function of Δ. It follows that $\mu = 0$ on $\{\operatorname{Re}\zeta > \frac{1}{2}\}$.

Hence c is a uniform limit of rational functions on $g(K \cup X)$ and, hence, $f = c \circ g$ is a continuous real-valued function on $K \cup X$, with $f(q) \neq f(p)$, which is a uniform limit of rational functions.

That settles part B.

(I)

Lemma 1. *If* $p \notin K \cup X$ *there is a polynomial* f *such that* $f(p) = 0 \notin f(K \cup X)$ *and* $\operatorname{Re} f \leqslant -1$ *on* X.

Proof. By part B with X empty every continuous function on K is a uniform limit of rational functions. Hence ([17] [12]) for $p \notin K$ there is a polynomial g with $g(p) = 0 \notin g(K)$.

If also $p \notin X$ then, since $\hat{X} = X$, there is a polynomial h such that $h(p) = 0$ and $\operatorname{Re} h < -1$ on X. By compactness there is an $\varepsilon > 0$ such that $\operatorname{Re}(h - \lambda g) < -1$ on X for all $|\lambda| < \varepsilon$. Since h/g is smooth on K, $(h/g)(K)$ omits some complex number λ with $|\lambda| < \varepsilon$. Then, setting $f = h - \lambda g$, we have $f(p) = 0 \notin f(K \cup X)$ and $\operatorname{Re} f \leqslant -1$ on X.

Deduction of part C from part A and Lemma 1

Consider any $p \notin K \cup X$ and choose an f as in Lemma 1. Then f is a continuous invertible function on $K \cup X$ with a continuous logarithm on X. But, for any Y, $\check{H}^1(Y; Z)$ is isomorphic to the group of all continuous invertible complex-valued functions on Y modulo those with continuous logarithms. Therefore, since $\check{H}^1(K \cup X; Z) \to \check{H}^1(X; Z)$ is injective, there is a continuous branch of $\log(f)$ on all of $K \cup X$. However, *by part* A, $\widehat{K \cup X} - (K \cup X)$ is a one-dimensional analytic subset of $C^N - (K \cup X)$; so by the argument principle (see, for instance, [13, p. 271]) f has no zeros on $\widehat{K \cup X} - (K \cup X)$. Hence any such p is not in $\widehat{K \cup X}$; so $K \cup X$ is polynomially convex.

Proof of part A

Lemma 2. *Let* $p \notin K \cup X$ *and let* f *be a polynomial as in Lemma 1. Then there exist numbers* ε, r *and* s, *with* $-1 < \varepsilon < 0$ *and* $-\frac{1}{2}\pi < r < s < \frac{1}{2}\pi$ *such that if*

$$S = \{\zeta \in C^1 : r < \operatorname{Arg}(\zeta - \varepsilon) < s\}$$

and $J = f^{-1}(S) \cap K$ *then* $0 \in S$ *and* $J = J_1 \cup \ldots \cup J_k$ *where the* J_j *are disjoint arcs such that* $\operatorname{Arg}(f - \varepsilon)$ *maps the closure of each* J_j *in* K *one-one onto* $[r, s]$, *each* $f(J_j)$ *is a non-singular arc, and any two are either disjoint or identical.*

Proof. Let I be the closed unit interval. We shall repeatedly use the simple consequence of Sard's Lemma [12] that if E is a closed totally disconnected subset of I and φ is a smooth real-valued function on I then $\varphi(E)$ is also totally disconnected.

Let $\varphi_i : I \to C^N$ smoothly with $\varphi_i(I) = K_i$. Define

$$I_i = \{t \in I : \operatorname{Re} f(\varphi_i(t)) \geqslant 0\}$$

$$A_i : I_i \to [-\tfrac{1}{2}\pi, \tfrac{1}{2}\pi] \text{ by } A_i(t) = \operatorname{Arg}(f(\varphi_i(t)))$$

and $V = V_1 \cup \ldots \cup V_n$, where V_i is the set of critical values of A_i. Then V is compact and totally disconnected, so we can choose an interval $[a, b]$, with $a < b$, in $[-\frac{1}{2}\pi, \frac{1}{2}\pi] - V$. Then $\{t \in I_i : A_i(t) \in [a, b]\}$ is a finite union of disjoint closed intervals $I(i, 1), \ldots, I(i, k(i))$ on each of which A_i is non-singular and maps one-one onto $[a, b]$. By the Chain Rule φ_i and $f \circ \varphi_i$ are also non-singular and one-one on each $I(i, j)$.

Relabel the pairs $(I(1, 1), \varphi_1), \ldots, (I(1, k(1)), \varphi_1), \ldots (I(n, 1), \varphi_n), \ldots, (I(n, k(n)), \varphi_n)$ as $(I'_1, \varphi'_1), \ldots, (I'_m, \varphi'_m)$.

Define $K'_i = \varphi'_i(I'_i)$, $F_i = f(K'_i)$, and $K' = K'_1 \cup \ldots \cup K'_m$. Then

$$K' = \{x \in K : \operatorname{Arg} f(x) \in [a, b]\},$$

each K'_i is a compact arc in C^N, and each F_i is a compact arc in the plane.

If $\partial(i, j)$ is the boundary of $K'_i \cap K'_j$ in K'_i and $\delta(i, j)$ is the boundary of $(\varphi'_i)^{-1}(\partial(i, j))$ in I'_i then $\varphi'_i(\delta(i, j)) = \partial(i, j)$ and, setting $\partial = \bigcup_{i,j} \partial(i, j)$, $K' - \partial$ is a disjoint union of open *arcs*.

Similarly, if $\beta(i, j)$ is the boundary of $F_i \cap F_j$ in F_i and $b(i, j)$ is the boundary of $(f \circ \varphi'_i)^{-1}(\beta(i, j))$ in I'_i, then $f(\varphi'_i(b(i, j))) = \beta(i, j)$ and, setting $\beta = \bigcup_{i,j} \beta(i, j)$, $(F_1 \cup \ldots \cup F_m) - \beta$ is a disjoint union of open arcs.

If $W = \bigcup_{i,j} \operatorname{Arg} f \circ \varphi'_i(\delta(i, j) \cup b(i, j)) = \operatorname{Arg} f(\partial) \cup \operatorname{Arg}(\beta)$ then W is again a compact totally disconnected subset of $[a, b]$; so we can choose another interval $[c, d]$ in $[a, b] - W$ (with $c < d$).

Let $K'' = \{x \in K : \operatorname{Arg} f(x) \in [c, d]\}$. Then K'' has finitely many components, each of which is a compact arc which $\operatorname{Arg} f$ maps one-one onto $[c, d]$, and the images under f of any two components are either disjoint or identical.

Since $f(K)$ is compact and disjoint from 0, if we choose $c < c' < d' < d$ then, for ε in $(-1, 0)$ close enough to 0, the set $\{x \in K : c' < \operatorname{Arg}(f - \varepsilon) < d'\}$ will be contained in K''. If we now choose $c' < r < s < d'$ so that $[r, s]$ contains no critical value of $\operatorname{Arg}(f - \varepsilon)$ then S and J (defined as in the statement of Lemma 2) will fulfill the requirements of Lemma 2.

LEMMA 3. *Let f, ε, r and s be as in Lemma 2. Then there are $r < t < u < s$ such that, if for each $j < k$ we define J^*_j to be $\{x \in J_j : t < \operatorname{Arg}(f(x) - \varepsilon) < u\}$, then each J^*_j which is not contained in $\widehat{(K \cup X) - J^*_j}$ can be described in the following way.*

There is a polynomial f_j and an interval $[r_j, s_j]$ in $(-\frac{1}{2}\pi, \frac{1}{2}\pi)$ (with $r_j < s_j$) such that $\operatorname{Re} f_j \leqslant -1$ on X, $0 \notin f_j(K \cup X)$, $\{x \in K : \operatorname{Arg} f_j(x) \in (r_j, s_j)\}$ is a finite union of disjoint arcs, the closure of each is mapped by $\operatorname{Arg} f_j$ one-one onto $[r_j, s_j]$, the images under f_j are non-singular arcs, any two are either identical or disjoint, AND the arc J^*_j is one of those components N of $\{x \in K : \operatorname{Arg} f_j(x) \in (r_j, s_j)\}$ for which the distance from $f_j(N_j)$ to 0 is maximal.

Proof. It will be enough to show that *if* the assertions of Lemma 3 hold for $r < t_1 < u_1 < s$ and for all $j \leqslant k_1 \leqslant k$ (with $J_j^*(1) = \{x \in J_j : t_1 < \operatorname{Arg}(f(x) - \varepsilon) < u_1\}$ for all $j \leqslant k$) then, for the first $j > k_1$ such that $J_j^*(1) \not\subset \widehat{(K \cup X) - J_j^*(1)}$, there are $t_1 < t_2 < u_2 < u_1$, an associated polynomial f_j and an interval $[r_j, s_j]$ in $(-\frac{1}{2}\pi, \frac{1}{2}\pi)$, with f_j, $[r_j, s_j]$ and

$$J_j^*(2) = \{x \in J_j : t_2 < \operatorname{Arg}(f(x) - \varepsilon) < u_2\}$$

related as in the statement of Lemma 3.

But that this is so follows directly from Lemmas 1 and 2 applied to K and $X_j = \widehat{(K \cup X) - J_j^*(1)}$ (and any point not in $K \cup X_j$). These lemmas supply f_j and $[r_j, s_j]$ with $\{x \in K : \operatorname{Arg} f_j(x) \in (r_j, s_j)\} \subset J_j^*(1)$. If we choose a component N_j of $\{x \in K : \operatorname{Arg} f_j(x) \in (r_j, s_j)\}$ for which the distance from $f_j(N_j)$ to 0 is maximal then N_j is a subarc of $J_j^*(1)$. Since f is one-one on $J_j^*(1)$ there must be $t_1 < t_2 < u_2 < u_1$ with

$$N_j = \{x \in J_j^*(1) : t_2 < \operatorname{Arg}(f(x) - \varepsilon) < u_2\}.$$

Then take $J_j^*(2) = N_j$.

LEMMA 4. *Let f, ε, t and u be as in Lemma 3. Let K_0 = the union of all J_j^* for which $J_j^* \subset \widehat{(K \cup X) - J_j^*}$ and set $L = K - K_0$. Then $\widehat{L \cup X} = \widehat{K \cup X}$.*

Proof. $L \cup X$ = the intersection of all $(K \cup X) - J_j^*$ for which $J_j^* \subset K_0$. But by assumption each such $(K \cup X) - J_j^*$ contains the Šilov boundary [18] for the polynomials on $K \cup X$, and, therefore, so does $L \cup X$.

(II)

LEMMA 5. *Let Y be compact in C^N and h a polynomial. If $\lambda \in \partial$, the boundary of the unbounded component of $C^1 - h(Y)$, and $M = \{m \in \hat{Y} : h(m) = \lambda\}$ then $M = \widehat{M \cap Y}$.*

Proof. It suffices to show that M is a maximum set in $\hat{Y}$ (see, for instance [13, p. 287]); hence, since $\partial \supset h(\hat{Y})$, that λ is a peak point for some uniform limit of polynomials on ∂. But for any such ∂ in C^1 the uniform closure of the polynomials is a Dirichlet algebra on ∂ [5]. Therefore, by the Bishop–de Leeuw characterization of peak points [3] every $\lambda \in \partial$ is a peak point.

Note. For our purpose we need Lemma 5 only when λ lies on a smooth non-singular arc which is open in ∂. Here is a more direct argument for that case.

There is a closed disk Δ_0 centered about a point $\lambda_0 \neq \lambda$ and a wedge

$$W_0 = \{\zeta \in C^1 : |\operatorname{Arg}(\zeta - \lambda) - \operatorname{Arg}(\lambda - \lambda_0)| < \varepsilon_0\}$$

such that $\partial \subset \Delta_0 - W_0$. Let $\mathcal{R}$ be the extended Riemann map of $\Delta_0 - W_0$ onto the closed unit disk, with $\mathcal{R}(\lambda) = 1$. Then, for $r \searrow 1$, the maps $\mathcal{R}_r$ defined by

$$\mathcal{R}_r(\zeta) = \mathcal{R}((\zeta - \lambda_0)/r + \lambda_0)$$

are each analytic on a neighborhood of $\Delta_0 - W_0$ and converge uniformly to $\mathcal{R}$ on $\Delta_0 - W_0$. Since every function analytic about the polynomially convex set $\Delta_0 - W_0$ is a uniform limit of polynomials [18], so is $1 + \mathcal{R}$ which peaks at λ.

LEMMA 6. *Let Y be a compact set and q a point in C^N. Let σ and μ be finite complex Borel measures on Y such that $\int g\,d\sigma = g(q)$ and $\int g\,d\mu = 0$ for all polynomials g. If V is an open subset of Y such that $q \notin \widehat{Y - V}$ then $\sigma|_V \neq \mu|_V$.*

Proof. Let $W = Y - V$ and let h be a polynomial such that $h(q) = 1$ and $|h| < \frac{1}{2}$ on W. Then

$$\int_V h^n\,d\sigma = \int_Y h^n\,d\sigma - \int_W h^n\,d\sigma \to 1$$

and

$$\int_V h^n\,d\mu = \int_Y h^n\,d\mu - \int_W h^n\,d\mu \to 0.$$

Hence, $\sigma|_V \neq \mu|_V$.

LEMMA 7. *Let Y be a compact set in C^N and h a polynomial such that $h(Y)$ is a simple closed curve in C^1. If there is a non-empty open subarc Z of $h(Y)$ such that h is one-one on $V = h^{-1}(Z) \cap Y$ then for any $q_1, q_2 \in \hat{Y}$ with $h(q_1) = h(q_2) = \zeta_0 \notin h(Y)$ it must be that $q_1 = q_2$.*

Proof. $C^1 - (h(Y) - Z)$ is connected so $\zeta_0 \notin \widehat{h(Y) - Z}$ and, hence, $q_i \notin \widehat{Y - V}$. Let ν_i be a representing measure [18] for q_i on Y. That is, ν_i is a finite *positive* Borel measure on Y such that $\int g\,d\nu_i = g(q_i)$ for every polynomial g. Hence, if we define measures ν_i^* on $h(Y)$ by $\nu_i^*(E) = \nu_i(h^{-1}(E))$ then $\int j\,d\nu_i^* = j(\zeta_0)$ for every polynomial j on C^1. But such positive representing measures on a simple closed curve in the plane are, for a given ζ_0, unique [18]. Therefore, $\nu_1^* = \nu_2^*$, and since h is one-one over Z, it follows that $\nu_{1|_V} = \nu_{2|_V}$.

If $q_1 \neq q_2$ there is a polynomial f_0 with $f_0(q_1) = 1$ and $f_0(q_2) = 0$. Then, setting $\sigma = f_0 \cdot \nu_1$ and $\mu = f_0 \cdot \nu_2$ we arrive immediately at a contradiction to Lemma 6. So $q_1 = q_2$.

LEMMA 8. *Let Y, h and Z be as in Lemma 7 and let U be the bounded component of $C^1 - h(Y)$. If there is any $q \in \hat{Y}$ such that $h(q) \in U$ then h maps $h^{-1}(U) \cap \hat{Y}$ one-one onto U, and for every polynomial g, $g \circ h^{-1}$ is analytic on U.*

Proof. The boundary of $h(\hat{Y})$ in C^1 is contained in $h(Y)$, so either $h(\hat{Y}) = h(Y)$ or $h(\hat{Y}) - h(Y) = U$. In the latter case, choose for each $\zeta_0 \in U$ a closed disk $\Delta_0 \subset U$, centered at

ζ_0, and with boundary ∂_0. Then, by the L.M.M.P. (page 186) applied to $h^{-1}(\Delta_0) \cap \hat{Y}$ and its boundary in $\hat{Y} - Y$ (which is $h^{-1}(\partial_0) \cap \hat{Y}$) it follows that $\mathfrak{A}_0 = \{g \circ h^{-1}|_{\Delta_0} : g \text{ polynomial}\}$ is an algebra of continuous functions on Δ_0 whose Šilov boundary is the circle ∂_0. Also, $\mathfrak{A}_0$ contains the identity function $\zeta = h \circ h^{-1}$; so by the Maximality Theorem (page 186) every $g \circ h^{-1}$ in $\mathfrak{A}_0$ is analytic on the interior of Δ_0.

LEMMA 9. *Let f, ε, t, u and L be as in Lemmas 1–4. For each $\zeta \in C^1$ with $t < \operatorname{Arg}(\zeta - \varepsilon) < u$ there is a closed disk $\Delta(\zeta)$ centered at ζ such that if $\partial(\zeta)$ = the boundary of $\Delta(\zeta)$, $D(\zeta) = f^{-1}(\Delta(\zeta)) \cap (\widehat{L \cup X})$, $\delta(\zeta) = f^{-1}(\partial(\zeta)) \cap (\widehat{L \cup X})$, $D_1(\zeta)$ = the union of all components of $D(\zeta)$ which meet $L \cup X$, $D_2(\zeta) = D(\zeta) - D_1(\zeta)$, and $\delta_i(\zeta) = \delta(\zeta) \cap D_i(\zeta)$, $i = 1, 2$, then $D_1(\zeta)$ and $D_2(\zeta)$ are open and closed in $D(\zeta)$, $D_2(\zeta) = \widehat{\delta_2(\zeta)}$ and $D_1(\zeta) - (\delta_1(\zeta) \cup L)$ is a one-dimensional analytic subset of $f^{-1}(\Delta(\zeta)) - (f^{-1}(\partial(\zeta)) \cup L)$.*

Proof. By Lemma 2 there are at most finitely many $q \in L \cup X$ for which $f(q) = \zeta$ and they all lie in L. Thus, each such q lies on one of the $J_j^* = N(q)$ for which there is a polynomial f_j as in Lemma 3. Since, by the description of J_j^* in that lemma, $f_j(N(q))$ is a smooth non-singular arc which is an open subset of the boundary of the unbounded component of $C^1 - f_j(L \cup X)$, Lemma 5 applies, so that

$$f_j^{-1}(f_j(N(q))) \cap (\widehat{L \cup X}) \subset L.$$

Also, by Lemma 3, $N(q)$ is open in $f_j^{-1}(f_j(N(q))) \cap L$; so, if Δ_* is a small enough open disk about $f_j(q)$, then that component D_* of $f_j^{-1}(\Delta_*) \cap (\widehat{L \cup X})$ which contains q is open in $\widehat{L \cup X}$ and meets $L \cup X$ in an arc $N_*(q)$ of $N(q)$. If ∂_* is the boundary of Δ_* in C^1 and δ_* is the boundary of D_* in $\widehat{L \cup X}$ then $f_j(\delta_*) \subset \partial_*$ and, by the L.M.M.P., D_* is open in $\widehat{\delta_* \cup N_*(q)}$. Therefore, by Lemmas 7 and 8 with $Y = \delta_* \cup N_*(q)$, $h = f_j$ and $Z = f_j(N_*(q))$, either $D_* = N_*(q)$ or $D_* - N_*(q)$ is an analytic disk. In either case $\{q\}$ is a connected component of the set of zeros of $f - f(q)$ in $\widehat{L \cup X}$; so if Δ_q is a small enough open disk about $f(q) = \zeta$ then the component D_q of $f^{-1}(\Delta_q) \cap (\widehat{L \cup X})$ containing q is an open subset of D_*. Therefore D_q is open in $\widehat{L \cup X}$. If $D_* = N_*(q)$ then $D_q \subset L$; otherwise $D_q - L$ is an open subset of $D_* - N_*(q)$, and so is itself a one-dimensional analytic subset of $f^{-1}(\Delta_q) - L$.

Let $\Delta(\zeta)$ be a closed disk centered at ζ which is contained in the finite intersection of the Δ_q. Then (with the notation of the statement of Lemma 9) $D_1(\zeta)$ and $D_2(\zeta)$ are open and closed in $D(\zeta)$, and $D_1(\zeta)$ is the finite union of those components of $D(\zeta)$ which contain such q that $f(q) = \zeta$. Hence, $D_1(\zeta) - (\delta_1(\zeta) \cup L)$ is an open subset of the union of the $D_q - L$, each of which is a one-dimensional analytic subset of $f^{-1}(\Delta_q)-L$; so it itself is a (possibly empty) one-dimensional analytic subset of $f^{-1}(\Delta(\zeta))-(f^{-1}(\partial(\zeta))\cup L)$. Also, $D(\zeta)$ is polynomially convex, so by the L.M.M.P. $D_2(\zeta)=\widehat{\delta_2(\zeta)}$.

(III)

Definition. Let $\mathfrak{A}$ be a complex commutative algebra with unit, U an open subset of C^1, and $\mathfrak{L}:\mathfrak{A}\times U\to C^1$ linear on $\mathfrak{A}$ and analytic on U. Then $\mathfrak{L}$ is an *analytic linear functional*; and it is an *analytic character of order d* provided there is a discrete subset $E\subset U$ such that for each $\zeta\in U-E$ there are *d distinct* algebra homomorphisms $\pi_j(\zeta)$ and d non-zero complex numbers $c_j(\zeta)$ so that, for all $g\in\mathfrak{A}$,

$$\mathfrak{L}(g,\zeta)=\sum_{j=1}^{d}c_j(\zeta)\cdot\pi_j(\zeta)(g).$$

Note. This representation is unique.

LEMMA 10. (*Royden's Criterion*). *Let U be connected and let $\mathfrak{L}:\mathfrak{A}\times U\to C^1$ be an analytic linear functional.*

1. *$\mathfrak{L}$ is an analytic character of order d if and only if*

(i) *for all $e>d$ and all pairs of e-tuples $(\alpha_1,\dots,\alpha_e)$, $(\beta_1,\dots,\beta_e)$ of members of $\mathfrak{A}$, $\det(\mathfrak{L}(\alpha_i\beta_j,\zeta))=0$ on U, and*

(ii) *there exist $x_1,\dots,x_d$, $y_1,\dots,y_d$ and h in $\mathfrak{A}$ such that, if we let*

$$P_\zeta(\lambda)=\det(\mathfrak{L}(x_i(h-\lambda)y_j,\zeta)),$$

then for some $\zeta_0\in U$ the polynomial P_{ζ_0} has d distinct roots.

2. *If, for some non-empty open subset $U_0\subset U$, $\mathfrak{L}_{|\mathfrak{A}\times U_0}$ is an analytic character of order d, then also $\mathfrak{L}$ is an analytic character of order d on $\mathfrak{A}\times U$.*

3. *If $\mathfrak{L}$ is an analytic character of order d then there is a discrete subset $E_1\subset U$ such that about each point of $U-E_1$ there is a disk Δ, analytic functions $c_j:\Delta\to C^1-\{0\}$ and functions $\pi_j:\Delta\to$ the set of algebra homomorphisms of $\mathfrak{A}$, $j=1,\dots,d$ such that, for each ζ, $\pi_1(\zeta),\dots,\pi_d(\zeta)$ are distinct, for each $g\in\mathfrak{A}$, $\zeta\to\pi_j(\zeta)(g)$ is analytic, and $\mathfrak{L}(g,\zeta)=\sum_{j=1}^{d}c_j(\zeta)\cdot\pi_j(\zeta)(g)$.*

Proof. (Following Royden [10]). The function $\det(\mathfrak{L}(\alpha_i\beta_j,\zeta))$ is analytic on U, so if it vanishes on an open set it vanishes identically. Therefore, 2. is an immediate consequence of 1.

If $\mathfrak{L}$ is an analytic character of order d then (i) holds because $(\mathfrak{L}(\alpha_i\beta_j,\zeta))$ is a product of $(e\times d)$ and $(d\times e)$ matrices, and (ii) can be satisfied by selecting $\zeta_0\in U-E$ and $x_1,\dots,x_d\in\mathfrak{A}$ such that $\pi_i(\zeta_0)(x_j)=\delta_{ij}$, and then setting $y_j=x_j$ and $h=\sum_{j=1}^{d}j\cdot x_j$.

Now suppose $\mathcal{L}$ is an analytic linear functional which satisfies (i) and (ii). Let $\square(\zeta)=$ the discriminant of $P_\zeta(\lambda)$. Then $\square$ is analytic on U and $\square(\zeta_0)\neq 0$. Therefore $E_1=\{\zeta\in U:\square(\zeta)=0\}$ is a discrete subset of U. The leading coefficient of $P_\zeta(\lambda)$ which is $\pm\det(\mathcal{L}(x_i y_j,\zeta))$ has no zeros on $U-E_1$ so there is an inverse matrix $(a_{ij}(\zeta))$ whose entries are also analytic on $U-E_1$. Let

$$x_i(\zeta)=\sum_{j=1}^{d} a_{ij}(\zeta)\cdot x_j, \quad i=1,\dots,d.$$

Then $x_i(\zeta)$ is analytic on $U-E_1$ and the matrix $(\mathcal{L}(x_i(\zeta)hy_j,\zeta))$ is diagonalizable—its d distinct eigenvalues are $\lambda_1(\zeta),\dots,\lambda_d(\zeta)$, the d distinct roots of $P_\zeta(\lambda)$. Let $(\alpha_{ij}(\zeta))$ be a matrix with inverse $(\beta_{ij}(\zeta))$ such that $(\alpha_{ij}(\zeta))\cdot(\mathcal{L}(x_i(\zeta)hy_j,\zeta))(\beta_{ij}(\zeta))$ is diagonal. Then define functions

$$X_i(\zeta)=\sum_{j=1}^{d}\alpha_{ij}(\zeta)\cdot x_j(\zeta) \quad\text{and}\quad Y_i(\zeta)=\sum_{j=1}^{d}\beta_{ij}(\zeta)\cdot y_j.$$

By a direct computation, using (i), (see [10, p. 26]) the mapping $g\to(\mathcal{L}(X_i(\zeta)gY_j(\zeta),\zeta))$ is an algebra homomorphism of $\mathfrak{A}$ into the algebra of $d\times d$ matrices. Since $\mathfrak{A}$ is commutative and $(\mathcal{L}(X_i(\zeta)hY_j(\zeta),\zeta))$ is diagonal it follows that each $(\mathcal{L}(X_i(\zeta)gY_j(\zeta),\zeta))$ is also diagonal. By another computation (see [10, pp. 26–27]) if we define algebra homomorphisms $\pi_j(\zeta):\mathfrak{A}\to C^1$ by $\pi_j(\zeta)(g)=\mathcal{L}(X_j(\zeta)gY_j(\zeta),\zeta)$ and non-zero complex numbers $c_j(\zeta)=\mathcal{L}(X_j(\zeta),\zeta)\cdot\mathcal{L}(Y_j(\zeta),\zeta)$ then $\mathcal{L}(g,\zeta)=\sum_{j=1}^{d}c_j(\zeta)\cdot\pi_j(\zeta)(g)$. Also $\pi_1(\zeta),\dots,\pi_d(\zeta)$ are distinct, because $\{\pi_1(\zeta)(h),\dots,\pi_d(\zeta)(h)\}=\{\lambda_1(\zeta),\dots,\lambda_d(\zeta)\}$. This completes part 1. To complete part 3 we need only show that about each $\zeta_1\in U-E_1$ there is a disk Δ on which we can *choose* $\alpha_{ij}(\zeta)$ to be analytic.

This can be done as follows. Firstly (by the Cauchy formula for the inverse of an analytic function) on a disk Δ_1 about ζ_1 in $U-E_1$ the d distinct roots $\lambda_1(\zeta),\dots,\lambda_d(\zeta)$ of $P_\zeta(\lambda)$ can be parameterized as analytic functions. If, for each $\lambda_k(\zeta)$ we let $v_k(\zeta)$ be that associated eigenvector of $(\mathcal{L}(x_i(\zeta)hy_j,\zeta))$ whose first non-zero coordinate is 1 then, by Cramer's Rule, on a possibly smaller disk Δ about ζ_1 each coordinate of $v_k(\zeta)$ is analytic. Hence, the associated matrix $(\alpha_{ij}(\zeta))$ which diagonalizes $(\mathcal{L}(x_i(\zeta)hy_j,\zeta))$ will have its entries analytic on Δ.

That settles part 3.

LEMMA 11. *Let Y be compact in C^N and h a polynomial such that $h(\hat{Y})-h(Y)$ is connected, Let $Y_0=h^{-1}(h(Y))\cap\hat{Y}$. If there is an open disk $\Delta_*\subset h(\hat{Y})-h(Y)$ such that $h^{-1}(\Delta_*)\cap\hat{Y}$ is a one-dimensional analytic subset of $h^{-1}(\Delta_*)$ then $\hat{Y}-Y_0$ is a one-dimensional analytic subset of C^N-Y_0.*

Proof. There is an open disk $\Delta \subset \Delta_*$ for which $h^{-1}(\Delta) \cap \hat{Y}$ is a disjoint union of finitely many disks $D_1, \ldots, D_d$ on each of which h is one-one with analytic inverse.

Let h_j be the restriction of h to D_j. Let ζ_0 be the center of Δ and let $\Delta_2 \subset \Delta_1 \subset \Delta$ be two different concentric disks about ζ_0, with positively oriented boundaries ∂_i and interiors Δ_i^0, $i=1, 2$. Let A be the open annulus $\Delta_i^0 - \Delta_2$.

Let $\zeta_1 \in h(\hat{Y}) - (h(Y) \cup \Delta_1)$. We will show that there is an open disk $\Delta(\zeta_1)$ about ζ_1 such that $h^{-1}(\Delta(\zeta_1)) \cap \hat{Y}$ is a one-dimensional analytic subset of $h^{-1}(\Delta(\zeta_1))$.

For any such ζ_1 there is a simple closed curve Γ with $\zeta_1 \in \hat{\Gamma} - \Gamma \subset h(\hat{Y}) - h(Y)$ and such that $\Gamma \cap \Delta_1$ is a diameter. Let γ be a closed segment in $\Gamma \cap \Delta_2^0$ and set $\Delta_\gamma = \Delta_1^0 - \Gamma_*$ where Γ_* is the closure of $\Gamma - \gamma$ in C^1. Then Δ_γ is an open dense connected subset of Δ_1^0.

Define $B = h^{-1}(\Gamma) \cap \hat{Y}$, $B_* = h^{-1}(\Gamma_*) \cap \hat{Y}$ and $\beta_j = h_j^{-1}(\gamma)$, $j=1, \ldots, d$.

If $\zeta_* \in \hat{\Gamma} - \Gamma$ and $q_1, \ldots, q_e$ are distinct points of Y such that $h(q_i) = \zeta_*$ then, by the L.M.M.P., each $q_i \in \hat{B} - B$. Hence there are positive representing measures μ_i on B (vanishing on points) with $\int_B g\, d\mu_i = g(q_i)$ for all uniform limits of polynomials on $\hat{B}$.

Let $\mu = \sum_{i=1}^{e} \mu_i$, $\mu_* = \mu|_{B_*}$, and $\nu_j = \mu|_{\beta_j}$. Define measures ν_j^* on γ by $\nu_j^*(T) = \nu_j(h_j^{-1}(T))$ for $T \subset \gamma$; and then define analytic functions ψ_j on $C^1 - \gamma$ by

$$\psi_j(u) = \frac{1}{2\pi i} \int_\gamma \frac{1}{u - z}\, d\nu_j^*(z).$$

Let $\mathfrak{A}$ be the algebra of all uniform limits of polynomials on $\hat{Y}$.

Define analytic linear functionals:

$$\mathcal{N} : \mathfrak{A} \times (\Delta - \gamma) \to C^1 \quad \text{by} \quad \mathcal{N}(g, u) = (u - \zeta_*) \cdot \sum_{j=1}^{d} \psi_j(u) \cdot g(h_j^{-1}(u)),$$

$$O_i : \mathfrak{A} \times (C^1 - |\partial_i|) \to C^1 \quad \text{by} \quad O_i(g, \zeta) = \int_{\partial_i} \frac{\mathcal{N}(g, u)}{u - \zeta}\, du, \quad i = 1, 2,$$

$$\mathcal{P} : \mathfrak{A} \times (C^1 - \Gamma_*) \to C^1 \quad \text{by} \quad \mathcal{P}(g, \zeta) = \int_{B_*} \frac{h - \zeta_*}{h - \zeta}\, g\, d\mu_*$$

$$Q : \mathfrak{A} \times (C^1 - \Gamma) \to C^1 \quad \text{by} \quad Q(g, \zeta) = \int_B \frac{h - \zeta_*}{h - \zeta}\, g\, d\mu.$$

Then there are the following relations.

(R_1) $O_1 - O_2 = 2\pi i \mathcal{N}$ on $\mathfrak{A} \times A$

(R_2) $Q = \mathcal{P} + O_2$ on $\mathfrak{A} \times (A - \Gamma)$

(R_3) $Q = 0$ on $\mathfrak{A} \times (C^1 - \hat{\Gamma})$.

(R_1) and (R_2) are by the Cauchy Integral Formula. As for (R_3), if $\zeta \notin \hat{\Gamma}$, then $1/(h-\zeta)$ is a uniform limit of polynomials on $\hat{B}$, so

$$Q(g,\zeta)=\sum_{i=1}^{e}\frac{h(q_i)-\zeta_*}{h(q_i)-\zeta}\,g(q_i)=0.$$

Therefore, $\mathcal{D}+O_1$ is an analytic linear functional on $\mathfrak{A}\times\Delta_\gamma$ whose restriction to

$$\mathfrak{A}\times(\Delta_\gamma\cap(A-\hat{\Gamma}))$$

is $2\pi i\mathcal{N}$, which is evidently an analytic character of some order $d_1 \leqslant d$. Hence, by part 2. of Lemma 10, $\mathcal{D}+O_1$ is an analytic character of order d_1 on $\mathfrak{A}\times\Delta_\gamma$.

But, on $\mathfrak{A}\times(A\cap(\hat{\Gamma}-\Gamma))$, $Q=\mathcal{D}+O_1-2\pi i\mathcal{N}$ which is evidently an analytic character of some order $d_2\leqslant d_1$. Therefore, again by part 2 of Lemma 10, Q is an analytic character of order $d_2\leqslant d$ on $\mathfrak{A}\times(\hat{\Gamma}-\Gamma)$.

Now we shall show that $e\leqslant d$. For by (ii) of part 1, Lemma 10 applied to $\mathcal{C}(g,\zeta)=Q(g,\zeta_*)=\sum_{i=1}^{e}g(q_i)$ (where $q_1,\ldots,q_e$ are *distinct*) there exist $x_1,\ldots,x_e$, $y_1,\ldots,y_e$ and g_* in $\mathfrak{A}$ and λ_* in C^1 such that $\det(Q(x_i(g_*-\lambda_*)y_j,\zeta_*))\neq 0$. However, if $e>d$ then setting $\alpha_i=x_i(g_*-\lambda_*)$ and $\beta_i=y_i$ we would have, by (i) applied to Q on $\hat{\Gamma}-\Gamma$, that $\det(\alpha_i\beta_j,\zeta_*)=0$. Therefore $e\leqslant d$.

This means that for *any* $\zeta\in h(\hat{Y})-h(Y)$ there are *at most* d points $q(\zeta)$ in $\hat{Y}$ such that $h(q(\zeta))=\zeta$.

Next let the point ζ_* from the previous discussion be chosen to lie in $\Delta\cap\hat{\Gamma}-\Gamma$ and choose $q_j=h^{-1}(\zeta_*)$ for $j=1,\ldots,d$. In this case Q must be an analytic character on $\mathfrak{A}\times(\hat{\Gamma}-\Gamma)$ of order precisely d. Let E_1 be the discrete subset of $\hat{\Gamma}-\Gamma$ given by part 3 of Lemma 10. Then about any point of $\hat{\Gamma}-(\Gamma\cup E_1)$ there is a disk Δ on which Q has the local analytic representation $Q(g,\zeta)=\sum_{j=1}^{d}c_j(\zeta)\cdot\pi_j(\zeta)(g)$ as in part 3. Since $\hat{Y}$ is the spectrum of $\mathfrak{A}$ (see Technical Comment 3) each $\pi_j(\zeta)$ is a point of $\hat{Y}$ with $\pi_j(\zeta)(g)=g(\pi_j(\zeta))$. Moreover, by the formula for Q, we have $Q(h\cdot g,\zeta)=\zeta\cdot Q(g,\zeta)$ for all ζ and g. If we apply this for any g (depending on j and ζ) such that $g(\pi_j(\zeta))=\delta_{ij}/c_j(\zeta)$ we find that $h(\pi_j(\zeta))=\zeta$. But $\pi_1(\zeta),\ldots,\pi_d(\zeta)$ are distinct and there are *at most* d points in $\hat{Y}$ above ζ. Hence $h^{-1}(\Delta)\cap\hat{Y}=\pi_1(\Delta)\cup\ldots\cup\pi_d(\Delta)$ a disjoint union of d analytic disks.

Now if $\zeta_1\notin E_1$ we are done. Otherwise, let $\Delta(\zeta_1)$ be an open disk about ζ_1 containing no other point of $E_1\cup h(Y)$. If V is any connected component of $h^{-1}(\Delta(\zeta_1)-\{\zeta_1\})$ then $h: V\to\Delta(\zeta_1)-\{\zeta_1\}$ is, for some $d'<d$, a d'-sheeted regular analytic covering, and all the locally defined branches b of the inverse of this mapping are analytic continuations of one another. Therefore, if $\Delta(d',\zeta_1)$ is the disk about ζ_1 whose radius is the d'th root of the radius

of $\Delta(\zeta_1)$ then any locally defined branch $b((\zeta-\zeta_1)^{d'})$ on $\Delta(d', \zeta_1)-\{\zeta_1\}$ has a single-valued analytic continuation Φ mapping $\Delta(d', \zeta_1)-\{\zeta_1\}$ onto V. For each coordinate Z_j on C^N, the *bounded* analytic function $Z_j \circ \Phi$ extends analytically over $\Delta(d', \zeta_1)$ giving the coordinates of an analytic extension $\tilde{\Phi}$ of Φ where $\tilde{\Phi}(\Delta(d', \zeta_1)) = V \cup \tilde{\Phi}(\zeta_1)$ is the closure of V in $h^{-1}(\Delta(\zeta_1)) \cap \hat{Y}$. There are at most d such V, so the union $U(\zeta_1)$ of their closures in $h^{-1}(\Delta(\zeta_1)) \cap \hat{Y}$ is a one-dimensional analytic subset of $h^{-1}(\Delta(\zeta_1))$.

Let $p_1, ..., p_{d_1}$ $(d_1 \leq d)$ be those points in $U(\zeta_1)$ with $h(p_j)=\zeta_1$. Let $q \in \hat{Y}$ with $h(q)=\zeta_1$. If $q \neq$ any p_j there is a polynomial f_* with $f_*(q)=1$ and all $f_*(p_j)=0$. Then $\{m \in \hat{Y}: |f_*(m)| < \frac{1}{2}$ is a neighborhood of $\{p_1, ..., p_{d_1}\}$ in $\hat{Y}$ and for ∂ a small enough circle about ζ_1 in $\Delta(\zeta_1)$ it will contain $\delta = h^{-1}(\partial) \cap \hat{Y}$. But, by the L.M.M.P., $q \in \hat{\delta}$; while $f_*(q)=1>\max_\delta |f_*|$.

Hence every $q \in \hat{Y}$ with $h(q)=\zeta_1$ equals some p_j; so $h^{-1}(\Delta(\zeta_1) \cap \hat{Y} = U(\zeta_1)$ is a one-dimensional analytic subset of $h^{-1}(\Delta(\zeta_1))$, and we are done with Lemma 11.

Conclusion of the proof of part A

Let the point p of Lemma 2 be in $\widehat{K \cup X}-(K \cup X)$, let f, ε, t, u and L be as in Lemmas 1–4 and set $S_p=\{\zeta \in C^1: t<\operatorname{Arg}(\zeta-\varepsilon)<u\}$. Define $\mathcal{J}_p$ as the set of all $\zeta \in S_p$ for which there is an open disk Δ about ζ such that $f^{-1}(\Delta) \cap (\widehat{L \cup X}-(L \cup X))$ is a (possibly empty) one-dimensional analytic subset of $f^{-1}(\Delta)-(L \cup X)$.

We shall show that $\mathcal{J}_p=S_p$. Firstly, $\mathcal{J}_p$ is evidently open in S_p. Next, $\mathcal{J}_p$ is not empty, because any $\zeta \in S_p$ with $|\zeta|>\max_{L \cup X}|f|$ belongs to $\mathcal{J}_p$. So it remains to prove that $\mathcal{J}_p$ is closed in S_p.

If ζ is in the closure of $\mathcal{J}_p$ in S_p and $\Delta(\zeta)$ is a disk about ζ as in Lemma 9 then $\mathcal{J}_p \cap \Delta(\zeta)$ must contain an open disk Δ. By Lemma 9, $f^{-1}(\Delta) \cap D_2(\zeta)$ is open in

$$f^{-1}(\Delta) \cap (\widehat{L \cup X}-(L \cup X))$$

and so is a one-dimensional analytic subset of $f^{-1}(\Delta)$. Also, by Lemma 9, $D_2(\zeta)=\widehat{\delta_2(\zeta)}$; so by Lemma 11, $D_2(\zeta)-\delta_2(\zeta)$ is a one-dimensional analytic subset of $C^N-\delta_2(\zeta)$. This together with the description of $D_1(\zeta)-(\delta_1(\zeta) \cup L)$ in Lemma 9 implies that $\zeta \in \mathcal{J}_p$.

Hence $\mathcal{J}_p$ is closed in S_p; so $\mathcal{J}_p=S_p$.

Therefore, for each $p \in \widehat{K \cup X}-(K \cup X)$, $f^{-1}(S_p) \cap (\widehat{L \cup X})-(L \cup X)$ is a one-dimensional analytic subset of $f^{-1}(S_p)-(L \cup X)$ and is a neighborhood of p in $\widehat{K \cup X}-(K \cup X)$.

This completes the proof of part A.

References

[1]. Bishop, E., Analyticity in certain Banach algebras. *Trans. Amer. Math. Soc.*, 102 (1962), 507–544.

[2]. —— Holomorphic completions, analytic continuations and the interpolation of semi-norms. *Ann. of Math.*, 78 (1963), 468–500.

[3]. Bishop, E. & de Leeuw, K., The representation of linear functionals by measures on sets of extreme points. *Ann. Inst. Fourier* (*Grenoble*), 9 (1959), 305–331.

[4]. Glicksberg, I., Measures orthogonal to algebras and sets of antisymmetry. *Trans. Amer. Math. Soc.*, 105 (1962), 415–435.

[5]. Glicksberg, I. & Wermer, J., Measures orthogonal to a Dirichlet algebra. *Duke Math. J.*, 30 (1963), 661–666.

[6]. Gunning, R. & Rossi, H., *Analytic functions of several complex variables.* Prentice-Hall, 1965.

[7]. Helson, H. & Quigley, F., Existence of maximal ideals in algebras of continuous functions. *Proc. Amer. Math. Soc.*, 8 (1957), 115–119.

[8]. Hoffman, K., *Banach spaces of analytic functions.* Prentice-Hall, 1962.

[9]. Rossi, H., The local maximum modulus principle. *Ann. of Math.*, 72 (1960), 1–11.

[10]. Royden, H., Algebras of bounded analytic functions on Riemann surfaces. *Acta Math.*, 114 (1965), 113–142.

[11]. Rudin, W., Analyticity and the maximum modulus principle. *Duke Math. J.*, 20 (1953), 449–458.

[12]. Sternberg, S., *Lectures on differential geometry.* Prentice-Hall, 1964.

[13]. Stolzenberg, G., Polynomially and rationally convex sets. *Acta Math.*, 109 (1963), 259–289.

[14]. Wermer, J., Polynomial approximation on an arc in C^3. *Ann. of Math.*, 62 (1955), 269–270.

[15]. —— Function rings and Riemann surfaces. *Ann. of Math.*, 67 (1958), 45–71.

[16]. —— Rings of analytic functions. *Ann. of Math.*, 67 (1958), 497–516.

[17]. —— The hull of a curve in C^n. *Ann. of Math.*, 68 (1958), 550–561.

[18]. —— *Banach algebras and analytic functions.* Academic Press, 1961.

Received July 5, 1965