JORDAN ALGEBRAS OF TYPE I

BY

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1. Introduction

Jordan, von Neumann, and Wigner [5] have classified all finite dimensional Jordan algebras over the reals. The present paper is an attempt to do the same in the infinite dimensional case. The following restriction will be imposed: we assume the Jordan algebras are weakly closed Jordan algebras of self-adjoint operators with minimal projections acting on a Hilbert space, i.e. are irreducible JW-algebras of type I.(1) The result is then quite analogous to that in [5], except we do not get hold of the Jordan algebra $\mathfrak{M}_3^8$ of that paper, as should be expected from the work of Albert [1]. We first classify all irreducible JW-algebras of type I_n, $n \geqslant 3$ (Theorem 3.9). These algebras are roughly all self-adjoint operators on a Hilbert space over either the reals, the complexes, or the quaternions. Then all JW-factors of type I_n, $n \geqslant 3$, will be classified (Theorem 5.2). In addition to those in the irreducible case we find an additional JW-factor, namely one which is the C^*-homomorphic image of all self-adjoint operators on a Hilbert space. JW-factors of type I_2 are studied separately (Theorem 7.1). They are the spin factors, and except when the dimensions are small, are exactly those JW-factors which are not reversible. Global results of this type are obtained in section 6. Finally we show that the von Neumann algebra generated by a reversible JW-algebra of type I is itself of type I (Theorem 8.2).

A *J-algebra* is a real linear space $\mathfrak{A}$ of self-adjoint operators on a (complex) Hilbert space $\mathfrak{H}$ closed under the product $A \circ B = \frac{1}{2}(AB + BA)$. Then $\mathfrak{A}$ is closed under products of the form ABA and $ABC + CBA$, $A, B, C \in \mathfrak{A}$ (see [4]). A *JC-algebra* (resp. *JW-algebra*) is a uniformly (resp. weakly) closed J-algebra. A *JW-factor* is a JW-algebra with center the scalars (with respect to operator multiplication). A projection E in a J-algebra $\mathfrak{A}$ is *abelian* if $E\mathfrak{A}E$ is an abelian family of operators. By a *symmetry* we shall mean a self-adjoint unitary operator. Two projections E and F in a JW-algebra $\mathfrak{A}$ are said to be *equivalent* if there

(1) In a forthcoming paper all irreducible JW-algebras will be shown to be of type I.

exists a symmetry S in $\mathfrak{A}$ such that $E = SFS$. The *central carrier* of a projection E in $\mathfrak{A}$ is the least central projection in $\mathfrak{A}$ greater than or equal to E. $\mathfrak{A}$ is of *type I* if there exists an abelian projection in $\mathfrak{A}$ with central carrier the identity. If n is a cardinal $\mathfrak{A}$ is of *type* I_n if there exists an abelian projection in $\mathfrak{A}$ n equivalent copies of which add up to the identity operator. We shall write I_∞ whenever n is infinite. A *Jordan ideal* in a J-algebra $\mathfrak{A}$ is a J-algebra $\mathfrak{J} \subset \mathfrak{A}$ such that $A \circ B \in \mathfrak{J}$ whenever $A \in \mathfrak{J}$, $B \in \mathfrak{A}$. Let $\mathfrak{R}(\mathfrak{A})$ denote the uniformly closed real algebra generated by $\mathfrak{A}$. If n is a positive integer then $\mathfrak{A}^n$ is the uniformly closed real linear space generated by products of the form $\prod_{i=1}^n A_i$, $A_i \in \mathfrak{A}$, and $(\mathfrak{A})$ denotes the C^*-algebra generated by $\mathfrak{A}$. $\mathfrak{A}$ is *reversible* if $\prod_{i=1}^n A_i + \prod_{i=n}^1 A_i \in \mathfrak{A}$ whenever $A_1, \ldots, A_n \in \mathfrak{A}$, $n = 1, 2, \ldots$. If $\mathfrak{A}$ is a JC-algebra then $\mathfrak{A}$ is reversible if and only if $\mathfrak{A} = \mathfrak{R}(\mathfrak{A})_{SA}$ [10]. We denote by $\mathfrak{M}_{SA}$ the set of self-adjoint operators in a family $\mathfrak{M}$ of operators. $\mathfrak{M}$ is said to be *self-adjoint* if $\mathfrak{M}$ contains the adjoint of each operator in $\mathfrak{M}$. $\mathfrak{M}^-$ is the weak closure and $\mathfrak{M}'$ the commutant of $\mathfrak{M}$. If $\mathfrak{B} \subset \mathfrak{H}$ then $[\mathfrak{M}\mathfrak{B}]$ is the subspace of $\mathfrak{H}$ generated by vectors of the form Ax, $A \in \mathfrak{M}$, $x \in \mathfrak{B}$. We identify subspaces of $\mathfrak{H}$ and their projections. We denote by $\mathfrak{B}(\mathfrak{H})$ the algebra of all bounded operators on $\mathfrak{H}$. Throughout this paper $\mathbf{R}$ denotes the real numbers, $\mathbf{C}$ the complex numbers, and $\mathbf{Q}$ the quaternions. We shall consider $\mathbf{Q}$ as a subalgebra of M_2—the complex 2×2 matrices.

We are indebted to L. Ingelstam for pointing out an error in an early version of Theorem 2.1.

2. Real algebras

As will be seen there is a close relationship between real self-adjoint algebras of operators and JW-algebras. Kaplansky [7] has classified (up to isomorphisms) the simplest real algebras. We shall need a more detailed description of them.

THEOREM 2.1. *Let $\mathfrak{R}$ be a real self-adjoint algebra of operators on a Hilbert space such that every self-adjoint operator in $\mathfrak{R}$ is a scalar multiple of the identity I. Then $\mathfrak{R}$ is characterized as follows:*

(1) $\mathfrak{R} = \mathbf{R}I$.

(2) $\mathfrak{R} = \mathbf{C}I$.

(3) *There exists a minimal projection $P' \in \mathfrak{R}'$ with central carrier I such that $P'\mathfrak{R} = \mathbf{Q}P'$.*

(4) *There exist two non zero projections P and Q with $P + Q = I$ such that*

$$\mathfrak{R} = \{\lambda P + \bar{\lambda} Q : \lambda \in \mathbf{C}\}.$$

Proof. Let A be a non zero operator in $\mathfrak{R}$ and suppose there exists a sequence (B_n) of operators in $\mathfrak{R}$ such that $B_n A \to 0$ or $AB_n \to 0$ uniformly, say $B_n A \to 0$. Then $A^* B_n^* B_n A =$

$(B_nA)^*B_nA \to 0$. By hypothesis $B_n^*B_n = \alpha_n I$ with $\alpha_n \in \mathbf{R}$. Thus $\alpha_n A^*A \to 0$, $\alpha_n \to 0$, and $B_n \to 0$. Similarly $AB_n \to 0$ implies $B_n \to 0$. Thus $\mathfrak{R}$ has no non zero topological divisors of 0. By [7, Theorem 3.1] $\mathfrak{R}$ is isomorphic to one of $\mathbf{R}$, $\mathbf{C}$ or $\mathbf{Q}$. This could also be shown by an application of [3]. If $\mathfrak{R} \cong \mathbf{R}$ we have case (1). Let $\mathfrak{B}$ denote the C^*-algebra generated by $\mathfrak{R}$. Since $\mathfrak{R}$ is finite dimensional being isomorphic to $\mathbf{C}$ or $\mathbf{Q}$, $\mathfrak{B} = \mathfrak{R} + i\mathfrak{R}$. Assume $\mathfrak{R} \cong \mathbf{C}$. Then the dimension of $\mathfrak{B}$ as a vector space over $\mathbf{C}$ is 1 or 2. In case dim $\mathfrak{B} = 1$ we have case (2). Assume the dimension is 2. Then there exist two orthogonal non zero projections P and Q in $\mathfrak{B}$ with $P + Q = I$ such that every operator in $\mathfrak{B}$ is of the form $\lambda P + \mu Q$ with $\lambda, \mu \in \mathbf{C}$. We may thus identify $\mathfrak{B}$ with $\mathbf{C} \times \mathbf{C}$. Let $(\lambda, \mu) \in \mathfrak{R}$. Then $(\bar{\lambda}, \bar{\mu}) \in \mathfrak{R}$, hence $(\lambda + \bar{\lambda}, \mu + \bar{\mu}) \in \mathfrak{R}$, hence is a scalar. Thus $\mathrm{Re}\lambda = \mathrm{Re}\mu$. Moreover, if $(\zeta, \xi) \in \mathfrak{R}$ then $(\lambda\zeta, \mu\xi) \in \mathfrak{R}$, hence $\mathrm{Re}\lambda\zeta = \mathrm{Re}\mu\xi$, and $\mathrm{Im}\lambda\ \mathrm{Im}\zeta = \mathrm{Im}\mu\ \mathrm{Im}\xi$. In particular, with $\lambda = \zeta$, $\mu = \xi$, $(\mathrm{Im}\lambda)^2 = (\mathrm{Im}\mu)^2$, If $\mathrm{Im}\lambda = \mathrm{Im}\mu \neq 0$ then by the above identity $\mathrm{Im}\zeta = \mathrm{Im}\xi$ for all (ζ, ξ) in $\mathfrak{R}$, so $\mathfrak{R} = \{(\lambda, \lambda) : \lambda \in \mathbf{C}\}$, and we have case (2). Otherwise $\mathrm{Im}\lambda = -\mathrm{Im}\mu$ for all $(\lambda, \mu) \in \mathfrak{R}$, $\mathfrak{R} = \{(\lambda, \bar{\lambda}) : \lambda \in \mathbf{C}\}$, and we have case (4).

It remains to consider the case when $\mathfrak{R} \cong \mathbf{Q}$. Then the dimension of $\mathfrak{R}$ as a real vector space is 4. Let $\mathfrak{H}$ be the underlying Hilbert space. Since $\mathfrak{R}$ is a vector subspace of $\mathfrak{B}(\mathfrak{H})$ the real dimension of $\mathfrak{B} = \mathfrak{R} + i\mathfrak{R}$ is less than or equal to 8, i.e. $\dim \mathfrak{B} \leqslant 4$ as a complex vector space. Since $\mathfrak{R}$ is non abelian so is $\mathfrak{B}$, hence $\mathfrak{B} \cong M_2$. In particular, $\mathfrak{B}$ being finite dimensional, is a factor of type I, hence $\mathfrak{B}'$ is a factor of type I. Let P' be a minimal projection in $\mathfrak{B}'$. Then its central carrier is I. Since P' is minimal, $(\mathfrak{B}P')' = P'\mathfrak{B}'P' = \{\lambda P'\}$, considered as a von Neumann algebra on the Hilbert space P', hence $\mathfrak{B}P' = M_2 P'$. Thus $\mathfrak{R}P' = \mathbf{Q}P'$. The proof is complete.

It should be remarked that in the quaternionian case there exist real algebras like those in case (4). However, if there exist projections P and Q with sum I such that $\mathfrak{R} = \{\lambda P + \bar{\lambda} Q : \lambda \in \mathbf{Q}\}$, then P and Q belong to $\mathfrak{R}'$ but not to $\mathfrak{B}$—the C^*-algebra generated by $\mathfrak{R}$. Moreover, the map $A \to AP$ is an isomorphism of $\mathfrak{B}$ onto $\mathfrak{B}P$. This should be kept in mind in order to get a full understanding of the classification theorems to follow.

COROLLARY 2.2. *Let $\mathfrak{A}$ be a reversible JW-factor of type I. Let E be an abelian projection in $\mathfrak{A}$. Let $\mathfrak{R} = E\mathfrak{R}(\mathfrak{A})E$. Then $\mathfrak{R}$ is a real self-adjoint algebra satisfying the conditions of Theorem 2.1, and $E(\mathfrak{A})E$ is characterized as follows:*

(1) $E(\mathfrak{A})E = \mathbf{C}E$.

(2) *There exists a minimal projection P' in $E(\mathfrak{A})E'$ such that $E(\mathfrak{A})EP' = M_2P'$.*

(3) *There exist two non zero projections P and Q with $P + Q = E$ such that*

$$E(\mathfrak{A})E = \mathbf{C}P \oplus \mathbf{C}Q.$$

Proof. Since E is abelian $E\mathfrak{A}E = \mathbf{R}E$ [11, Corollary 24]. Since $\mathfrak{A}$ is reversible it follows that $\mathfrak{R}$ satisfies the conditions of Theorem 2.1. The rest is clear from the theorem and its proof.

The next result will be used in section 8.

COROLLARY 2.3. *Let $\mathfrak{R}$ be an irreducible, uniformly closed, self-adjoint, real algebra with identity I acting on a Hilbert space $\mathfrak{H}$. Assume $\mathfrak{R}_{SA}$ is abelian. Then $\mathfrak{R}$ is one of the following algebras:*

(1) (*resp.* 2) $\mathfrak{R} = \mathbf{R}I$ (*resp.* $\mathbf{C}I$), *and* $\dim \mathfrak{H} = 1$.

(3) $\mathfrak{R} = \mathbf{Q}I$, *and* $\dim \mathfrak{H} = 2$.

Proof. Let $\mathfrak{M}$ be a maximal ideal in $\mathfrak{R}_{SA}$. Let f be a pure state of $\mathfrak{R}_{SA}$ with kernel $\mathfrak{M}$. Let $\bar{f}$ be a state extension of f to the C^*-algebra $\mathfrak{B}$ generated by $\mathfrak{R}$. Then $\bar{f} = \omega_x \varphi$ with φ a representation of $\mathfrak{B}$. Let $\overline{\mathfrak{M}} = \mathfrak{R} \cap \ker \varphi$. Then $\overline{\mathfrak{M}}$ is a uniformly closed self-adjoint ideal in $\mathfrak{R}$. If $A \in \overline{\mathfrak{M}}$ then $AA^* \in \overline{\mathfrak{M}}$, hence $0 = \omega_x \varphi(AA^*) = f(AA^*)$, and $AA^* \in \mathfrak{M}$. Let $\mathfrak{J}$ be a uniformly closed ideal in $\mathfrak{R}_{SA}$. Then $\mathfrak{J} = \cap\, \mathfrak{M}$, $\mathfrak{M}$ maximal ideals in $\mathfrak{R}_{SA}$. Let $\bar{\mathfrak{J}} = \cap\, \overline{\mathfrak{M}}$, where the $\overline{\mathfrak{M}}$ are constructed as above. Let $A \in \bar{\mathfrak{J}}$. Then $AA^* \in \bar{\mathfrak{J}}$, hence $AA^* \in \overline{\mathfrak{M}}$ for all $\mathfrak{M}$. By the above $AA^* \in \mathfrak{M}$ for all $\mathfrak{M}$, hence $AA^* \in \mathfrak{J}$.

We show $\mathfrak{R}_{SA} = \mathbf{R}I$. If not then there exist two non zero uniformly closed ideals $\mathfrak{J}$ and $\mathfrak{F}$ in $\mathfrak{R}_{SA}$ such that $\mathfrak{J}\mathfrak{F} = 0$. Let $\bar{\mathfrak{J}}$ and $\bar{\mathfrak{F}}$ be ideals in $\mathfrak{R}$ constructed as above. Let $A \in \bar{\mathfrak{J}}$ and $B \in \bar{\mathfrak{F}}$. Then $AB \in \bar{\mathfrak{J}} \cap \bar{\mathfrak{F}}$. Thus $AB(AB)^* \in \mathfrak{J} \cap \mathfrak{F} = \mathfrak{J}\mathfrak{F} = 0$, and $AB = 0$. Thus $\bar{\mathfrak{J}}\bar{\mathfrak{F}} = 0$. Since $\mathfrak{R}$ is irreducible $\mathfrak{R}$ has no non zero ideal divisors of zero [8, Lemma 2.5]. Thus $\bar{\mathfrak{J}}$ or $\bar{\mathfrak{F}} = 0$, contrary to assumption; $\mathfrak{R}_{SA} = \mathbf{R}I$. An application of Theorem 2.1 completes the proof.

3. Irreducible JW-algebras

We classify all irreducible JW-algebras of type I_n, $n \geq 3$. The key to this and later results lies in the following general lemma on the structure of JW-algebras.

LEMMA 3.1. *Let $\mathfrak{A}$ be a JW-algebra such that there exists a family $\{E_\sigma\}_{\sigma \in J}$ of orthogonal, non zero, equivalent projections in $\mathfrak{A}$ with $\sum_{\sigma \in J} E_\sigma = I$, and* $\operatorname{card} J \geq 3$. *For $\sigma, \varrho \in J$ let $\mathfrak{S}_{\sigma\varrho} = E_\sigma \mathfrak{A} E_\varrho$. Then the following relations hold:*

(1) $\mathfrak{S}_{\sigma\tau}\mathfrak{S}_{\eta\varrho}=\begin{cases}0 \text{ if } \tau\neq\eta\\ \mathfrak{S}_{\sigma\varrho} \text{ if } \tau=\eta,\ \sigma\neq\varrho\\ \mathfrak{S}_{\sigma\varphi}\mathfrak{S}_{\varphi\sigma} \text{ for all } \varphi\neq\sigma \text{ if } \sigma=\varrho\neq\tau=\eta.\end{cases}$

(2) $\mathfrak{S}_{\sigma\varrho}=\mathfrak{S}_{\sigma\varrho}\mathfrak{S}_{\varrho\sigma}\mathfrak{S}_{\sigma\varrho}$ if $\sigma\neq\varrho$.

(3) *If* $\sigma\neq\varrho$, *let* $\mathfrak{A}_\sigma$ *be the uniformly closed real linear space generated by* $\mathfrak{S}_{\sigma\varrho}\mathfrak{S}_{\varrho\sigma}$. *Then* $\mathfrak{A}_\sigma$ *is a real self-adjoint algebra with identity* E_σ, $\mathfrak{A}_\sigma$ *is independent of* ϱ, *and* $\mathfrak{A}_{\sigma SA}=\mathfrak{S}_{\sigma\sigma}$.

(4) $E_\sigma\mathfrak{R}(\mathfrak{A})E_\varrho=\mathfrak{S}_{\sigma\varrho}$ *if* $\sigma\neq\varrho$, $E_\sigma\mathfrak{R}(\mathfrak{A})E_\sigma\subset\mathfrak{A}_\sigma^-$.

(5) $\mathfrak{A}$ *is reversible, and* $\mathfrak{R}(\mathfrak{A})^-=\mathfrak{A}^{4-}$.

Proof. Clearly $\mathfrak{S}_{\sigma\tau}\mathfrak{S}_{\eta\varrho}=0$ if $\tau\neq\eta$. Now $\mathfrak{S}_{\sigma\tau}\mathfrak{S}_{\tau\varrho}\subset\mathfrak{S}_{\sigma\varrho}$ if $\sigma\neq\varrho$. In fact, take $B, C\in\mathfrak{A}$ and put $A=E_\sigma BE_\tau+E_\tau BE_\sigma+E_\tau CE_\varrho+E_\varrho CE_\tau$. Clearly $A\in\mathfrak{A}$, hence $A^2\in\mathfrak{A}$, so that $E_\sigma A^2E_\varrho\in\mathfrak{S}_{\sigma\varrho}$. There are three cases, namely, $\tau=\sigma$, $\tau=\varrho$, and $\tau\neq\sigma$ and $\tau\neq\varrho$. In the two first cases straightforward computations yield $E_\sigma BE_\tau CE_\varrho=\frac{1}{2}E_\sigma A^2E_\varrho\in\mathfrak{S}_{\sigma\varrho}$. In the third case a similar computation yields $E_\sigma BE_\tau CE_\varrho=E_\sigma A^2E_\varrho\in\mathfrak{S}_{\sigma\varrho}$, and $\mathfrak{S}_{\sigma\tau}\mathfrak{S}_{\tau\varrho}\subset\mathfrak{S}_{\sigma\varrho}$ for all τ whenever $\sigma\neq\varrho$, as asserted. The opposite inclusion is clear if $\tau=\sigma$ or $\tau=\varrho$. Assume therefore σ,ϱ,τ all distinct. Let S be a symmetry in $\mathfrak{A}$ such that $E_\varrho=SE_\tau S$. Let $V=E_\varrho SE_\tau$. Then V is a partial isometry in $\mathfrak{S}_{\varrho\tau}$ such that $V^*V=E_\tau$ and $VV^*=E_\varrho$. Let $A\in\mathfrak{S}_{\sigma\varrho}$. Then

$$A=AE_\varrho=(AV)V^*\in(\mathfrak{S}_{\sigma\varrho}\mathfrak{S}_{\varrho\tau})\mathfrak{S}_{\tau\varrho}\subset\mathfrak{S}_{\sigma\tau}\mathfrak{S}_{\tau\varrho},$$

by the above. Thus $\mathfrak{S}_{\sigma\varrho}\subset\mathfrak{S}_{\sigma\tau}\mathfrak{S}_{\tau\varrho}$, and they are equal.

Assume $\varrho=\sigma$. Let $\varphi\neq\tau$ in J, both different from σ. Then, by the above, $\mathfrak{S}_{\sigma\tau}\mathfrak{S}_{\tau\sigma}=\mathfrak{S}_{\sigma\tau}\mathfrak{S}_{\tau\varphi}\mathfrak{S}_{\varphi\sigma}=\mathfrak{S}_{\sigma\varphi}\mathfrak{S}_{\varphi\sigma}$, and (1) is proved.

If $\varrho\neq\sigma$ let $\tau\in J$ be distinct from both. Then by repeated applications of (1),

$$\mathfrak{S}_{\sigma\varrho}=\mathfrak{S}_{\sigma\tau}\mathfrak{S}_{\tau\varrho}=\mathfrak{S}_{\sigma\varrho}\mathfrak{S}_{\varrho\tau}\mathfrak{S}_{\tau\varrho}=\mathfrak{S}_{\sigma\varrho}\mathfrak{S}_{\varrho\sigma}\mathfrak{S}_{\sigma\tau}\mathfrak{S}_{\tau\varrho}=\mathfrak{S}_{\sigma\varrho}\mathfrak{S}_{\varrho\sigma}\mathfrak{S}_{\sigma\varrho},$$

and (2) is proved.

It is clear from (1) that $\mathfrak{A}_\sigma$ is independent of ϱ. By (2),

$$(\mathfrak{S}_{\sigma\varrho}\mathfrak{S}_{\varrho\sigma})^2=\mathfrak{S}_{\sigma\varrho}\mathfrak{S}_{\varrho\sigma}\mathfrak{S}_{\sigma\varrho}\mathfrak{S}_{\varrho\sigma}=\mathfrak{S}_{\sigma\varrho}\mathfrak{S}_{\varrho\sigma},$$

so $\mathfrak{S}_{\sigma\varrho}\mathfrak{S}_{\varrho\sigma}$ is multiplicative. Since it is clearly self-adjoint $\mathfrak{A}_\sigma$ is a real self-adjoint algebra with identity E_σ. Let A be a self-adjoint operator in $\mathfrak{A}_\sigma$. Then A is a uniform limit of self-adjoint operators of the form $\sum_{i=1}^m B_i$ with

$$B_i=E_\sigma C_iE_\varrho D_iE_\sigma\in(\mathfrak{S}_{\sigma\varrho}\mathfrak{S}_{\varrho\sigma})_{SA},\quad C_i, D_i\in\mathfrak{A}.$$

Since $$B_i = \tfrac{1}{2}(B_i + B_i^*) = \tfrac{1}{2} E_\sigma (C_i E_\varrho D_i + D_i E_\varrho C_i) E_\sigma \in E_\sigma \mathfrak{A} E_\sigma = \mathfrak{S}_{\sigma\sigma},$$

$A \in \mathfrak{S}_{\sigma\sigma}$. Thus $\mathfrak{A}_{\sigma_{SA}} \subset \mathfrak{S}_{\sigma\sigma}$. The opposite inclusion follows as soon as we have shown $\mathfrak{S}_{\sigma\sigma} \subset \mathfrak{A}_\sigma$. For this let $P \leqslant E_\sigma$ be a non zero projection in $\mathfrak{A}$. Let S be a symmetry in $\mathfrak{A}$ such that $SE_\sigma S = E_\varrho$, $\varrho \neq \sigma$. Let $Q = SPS$. Then $Q \leqslant E_\varrho$. Let $R = P + Q$. Let $T = RSR$. Then $T \in \mathfrak{A}$, and
$$P = TQT = E_\sigma TQT E_\sigma = E_\sigma T E_\varrho T E_\sigma \in \mathfrak{S}_{\sigma\varrho} \mathfrak{S}_{\varrho\sigma} \subset \mathfrak{A}_\sigma.$$

Since real linear combinations of projections are uniformly dense in the JW-algebra $\mathfrak{S}_{\sigma\sigma}$, and since $\mathfrak{A}_\sigma$ is uniformly closed $\mathfrak{S}_{\sigma\sigma} \subset \mathfrak{A}_\sigma$, as asserted, (3) is proved. Notice that the same argument shows $(\mathfrak{A}_\sigma^-)_{SA} = \mathfrak{S}_{\sigma\sigma}$, a fact which will be used below.

We next show $\mathfrak{S}_{\sigma\varrho}$ is weakly closed whenever $\sigma \neq \varrho$. Let $A \in \mathfrak{S}_{\sigma\varrho}^-$. Then $A = E_\sigma A E_\varrho$. Let $\{A_\alpha\}$ be a net in $\mathfrak{A}$ such that $E_\sigma A_\alpha E_\varrho \to A$ weakly. Since the *-operation is weakly continuous, $E_\varrho A_\alpha E_\sigma \to A^*$ weakly, hence $A + A^*$ is the weak limit of the net $\{E_\sigma A_\alpha E_\varrho + E_\varrho A_\alpha E_\sigma\}$, the net consisting of operators in $\mathfrak{A}$. Therefore $A + A^* \in \mathfrak{A}$, as $\mathfrak{A}$ is weakly closed. Thus $A = E_\sigma (A + A^*) E_\varrho \in \mathfrak{S}_{\sigma\varrho}$, $\mathfrak{S}_{\sigma\varrho}$ is weakly closed.

In order to show (4) let $A \in \mathfrak{R}(\mathfrak{A})$. Then A is a uniform limit of operators of the form $\sum_{i=1}^n \prod_{j=1}^{m_i} A_{ij}$ with $A_{ij} \in \mathfrak{A}$. Therefore, in order to show $E_\sigma A E_\varrho \in \mathfrak{S}_{\sigma\varrho}$ if $\sigma \neq \varrho$ (resp. in $\mathfrak{A}_\sigma^-$ if $\sigma = \varrho$), it suffices by linearity and the fact that $\mathfrak{S}_{\sigma\varrho}$ and $\mathfrak{A}_\sigma^-$ are weakly closed so uniformly closed, to show that any operator of the form
$$E_\sigma \prod_{j=1}^n A_j E_\varrho \in \mathfrak{S}_{\sigma\varrho} \quad (\text{resp. } \mathfrak{A}_\sigma^-), \quad \text{where } A_j \in \mathfrak{A}.$$

For $n = 1$ this is trivial. Use induction and assume it holds for $n - 1$. Then
$$E_\sigma \prod_{j=1}^n A_j E_\varrho = E_\sigma \left(\prod_{j=1}^{n-1} A_j \right) A_n E_\varrho,$$
which is the strong limit of operators
$$\sum_{\tau \in J'} E_\sigma \prod_{j=1}^{n-1} A_j E_\tau A_n E_\varrho,$$
with J' a finite subset of J. By induction hypothesis
$$E_\sigma \prod_{j=1}^{n-1} A_j E_\tau \in \mathfrak{S}_{\sigma\tau} \quad \text{if } \sigma \neq \tau$$
and in $\mathfrak{A}_\sigma^-$ if $\sigma = \tau$. Hence, by (1)
$$E_\sigma \prod_{j=1}^{n-1} A_j E_\tau A_n E_\varrho \in \mathfrak{S}_{\sigma\varrho} \quad (\text{resp. } \mathfrak{A}_\sigma^-) \quad \text{if} \quad \tau \neq \sigma,$$
and if $\tau = \sigma$ then by (2) and the fact that $\mathfrak{S}_{\sigma\varrho}$ is weakly closed,

$$E_\sigma \prod_{j=1}^{n-1} A_j E_\sigma A_n E_\varrho \in \mathfrak{S}_{\sigma\varrho} \quad (\text{resp. } \mathfrak{A}_\sigma^-).$$

Thus
$$\sum_{\tau\in J'} E_\sigma \prod_{j=1}^{n-1} A_j E_\tau A_n E_\varrho \in \mathfrak{S}_{\sigma\varrho} \quad (\text{resp. } \mathfrak{A}_\sigma^-)$$

for all finite subsets J' of J. As $\mathfrak{S}_{\sigma\varrho}$ and $\mathfrak{A}_\sigma^-$ are weakly so strongly closed, it follows that

$$E_\sigma \left(\prod_{j=1}^{n-1} A_j\right) A_n E_\varrho \in \mathfrak{S}_{\sigma\varrho} \quad (\text{resp. } \mathfrak{A}_\sigma^-).$$

Thus $E_\sigma \mathfrak{R}(\mathfrak{A}) E_\varrho \subset \mathfrak{S}_{\sigma\varrho}$ (resp. $\mathfrak{A}_\sigma^-$). Since clearly $\mathfrak{S}_{\sigma\varrho} \subset E_\sigma \mathfrak{R}(\mathfrak{A}) E_\varrho$ (4) follows. Notice that $E_\sigma \mathfrak{R}(\mathfrak{A})^- E_\sigma \supset \mathfrak{A}_\sigma^-$, so they are equal.

Let $A \in \mathfrak{R}(\mathfrak{A})_{SA}$. Then $E_\sigma A E_\sigma \in \mathfrak{A}_{\sigma SA} = \mathfrak{S}_{\sigma\sigma}$. If $\sigma \neq \varrho$ then there exist $B, C \in \mathfrak{A}$ such that

$$\begin{aligned} E_\sigma A E_\varrho + E_\varrho A E_\sigma &= E_\sigma B E_\varrho + E_\varrho C E_\sigma \\ &= \tfrac{1}{2}(E_\sigma B E_\varrho + E_\varrho C E_\sigma) + \tfrac{1}{2}(E_\sigma B E_\varrho + E_\varrho C E_\sigma)^* \\ &= E_\sigma \tfrac{1}{2}(B+C) E_\varrho + E_\varrho \tfrac{1}{2}(B+C) E_\sigma \in \mathfrak{A}. \end{aligned}$$

Thus $A \in \mathfrak{A}$, $\mathfrak{A}$ is reversible.

It is clear that each $\mathfrak{S}_{\sigma\varrho} \subset \mathfrak{A}^2$. Thus $\mathfrak{A}_\sigma \subset \mathfrak{A}^4$. Hence by (4), $\mathfrak{R}(\mathfrak{A})^- \subset \mathfrak{A}^{4-}$, and they are equal. The proof is complete.

From now on the E_σ in Lemma 3.1 will be abelian projections in a JW-factor $\mathfrak{A}$ of type I. If $\mathfrak{A}$ is of type I_2 we define the $\mathfrak{S}_{\sigma\rho}$ as above. Whenever we write $\mathfrak{S}_{\sigma\rho}$ we shall assume $\sigma \neq \varrho$.

LEMMA 3.2. *Let $\mathfrak{A}$ be a JW-factor with orthogonal non zero abelian projections $\{E_\sigma\}_{\sigma\in J}$ such that $\sum_{\sigma\in J} E_\sigma = I$. Then every operator in $\mathfrak{S}_{\sigma\rho}$ is a scalar multiple of a partial isometry of E_ϱ onto E_σ. Moreover, if $S, T \in \mathfrak{S}_{\sigma\rho}$ then there exists a real number α such that*

$$S^*T + T^*S = \alpha E_\varrho, \qquad ST^* + TS^* = \alpha E_\sigma.$$

Proof. Let $S \in \mathfrak{S}_{\sigma\varrho}$. Then $S = E_\sigma A E_\varrho$, $A \in \mathfrak{A}$. Thus

$$S^*S = E_\varrho A E_\sigma A E_\varrho \in E_\varrho \mathfrak{A} E_\varrho = \mathbf{R} E_\varrho$$

[11, Corollary 24], hence $S^*S = \|S\|^2 E_\varrho$. Similarly $SS^* = \|S\|^2 E_\sigma$, and $S = \beta V$ with V a partial isometry of E_ϱ onto E_σ, $\beta \in \mathbf{C}$. Let T be another operator in $\mathfrak{S}_{\sigma\varrho}$. Since $\mathfrak{A}$ is linear so is $\mathfrak{S}_{\sigma\varrho}$. Thus $S + T = \gamma W$ with W a partial isometry of E_ϱ onto E_σ, $\gamma \in \mathbf{C}$. Therefore,

$$|\gamma|^2 E_\varrho = (S+T)^*(S+T) = S^*S + T^*T + (S^*T + T^*S) = |\beta|^2 E_\varrho + \delta E_\varrho + (S^*T + T^*S),$$

where $T^*T = \delta E_\varrho$. Thus $S^*T + T^*S = \alpha E_\varrho$. We may assume $T \neq 0$. Then

$$TS^* + ST^* = TS^*E_\sigma + E_\sigma ST^* = \delta^{-1}T(S^*T + T^*S)\,T^* = \delta^{-1}T\alpha E_\varrho T^* = \alpha E_\sigma.$$

The proof is complete.

From now on $\mathfrak{A}$ is a JW-factor of type I_n, $n \geqslant 3$, and the E_σ are as in Lemma 3.2. Then they are all equivalent [11, Corollary 26], and Lemma 3.1 is applicable. We keep the notation in Lemma 3.1.

LEMMA 3.3. *For each pair* $\sigma \neq \varrho$ *in* J *we can choose one partial isometry* $W_{\sigma\varrho} \in \mathfrak{S}_{\sigma\varrho}$ *such that whenever* ϱ, σ, τ *are three distinct elements in* J *then*

(1) $W_{\sigma\varrho} = W^*_{\varrho\sigma}$,

(2) $W_{\sigma\varrho} = W_{\sigma\tau} W_{\tau\varrho}$.

Proof. Let first ϱ, σ, τ be three distinct elements in J. Choose partial isometries $W_{\sigma\varrho}$ and $W_{\sigma\tau}$ in $\mathfrak{S}_{\sigma\varrho}$ and $\mathfrak{S}_{\sigma\tau}$ respectively (Lemma 3.2). Define $W_{\varrho\sigma}$ and $W_{\tau\sigma}$ by (1). Let

$$W_{\tau\varrho} = W_{\tau\sigma} W_{\sigma\varrho}, \qquad W_{\varrho\tau} = W_{\varrho\sigma} W_{\sigma\tau}.$$

By Lemma 3.1 $W_{\tau\varrho} \in \mathfrak{S}_{\tau\varrho}$ and $W_{\varrho\tau} \in \mathfrak{S}_{\varrho\tau}$, and $W_{\tau\varrho} = W^*_{\varrho\tau}$. It is straightforward to check (2) for the different rearrangements of ϱ, σ, and τ, e.g. $W_{\sigma\tau} = E_\sigma W_{\sigma\tau} = W_{\sigma\varrho} W_{\varrho\sigma} W_{\sigma\tau} = W_{\sigma\varrho} W_{\varrho\tau}$. Thus the lemma holds for the three elements ϱ, σ, and τ in J.

Let K be a maximal subset of J containing ϱ, σ, and τ, and for which the $W_{\eta\varphi}$ are chosen so that (1) and (2) hold for all elements η, φ in K. Then $K = J$. If not let $\eta \in J - K$. Choose $W_{\varrho\eta}$ and $W_{\eta\varrho}$ in $\mathfrak{S}_{\varrho\eta}$ and $\mathfrak{S}_{\eta\varrho}$ respectively such that $W_{\varrho\eta} = W^*_{\eta\varrho}$. Let

$$W_{\eta\varphi} = W_{\eta\varrho} W_{\varrho\varphi}, \qquad W_{\varphi\eta} = W_{\varphi\varrho} W_{\varrho\eta},$$

for all $\varphi \in K$, $\varphi \neq \varrho$. Then $W^*_{\eta\varphi} = W_{\varphi\eta}$, so (1) holds. Let $\mu \neq \varphi$ be in K and distinct form ϱ. This is possible by the preceding paragraph. As above we can show (2) holds for all rearrangements of η, ϱ, and φ. We show (2) holds for all rearrangements of η, μ, and φ. Indeed,

$$W_{\mu\eta} = W_{\mu\varrho} W_{\varrho\eta} = W_{\mu\varphi} W_{\varphi\varrho} W_{\varrho\varphi} W_{\varphi\eta} = W_{\mu\varphi} E_\varphi W_{\varphi\eta} = W_{\mu\varphi} W_{\varphi\eta},$$

$$W_{\eta\mu} = W^*_{\mu\eta} = W_{\eta\varphi} W_{\varphi\eta},$$

$$W_{\varphi\mu} = W_{\varphi\varrho} W_{\varrho\mu} = W_{\varphi\varrho} E_\varrho W_{\varrho\mu} = W_{\varphi\varrho} W_{\varrho\eta} W_{\eta\varrho} W_{\varrho\mu} = W_{\varphi\eta} W_{\eta\mu},$$

and (2) holds. Thus $K \cup \{\eta\}$ satisfies (1) and (2) for all its elements, contradicting the maximality of K. Thus $K = J$, the proof is complete.

LEMMA 3.4. *The* $\mathfrak{A}_\sigma$, $\sigma \in J$, *are all spatially isomorphic, and each* $\mathfrak{A}_\sigma$ *is one of the following algebras:*

(1) (*resp.* 2) $\mathfrak{A}_\sigma = \mathbf{R}E_\sigma$ (*resp.* $\mathbf{C}E_\sigma$).

(3) *There exists a projection* $P' \in \mathfrak{A}'$ *with central carrier* I *such that if* $\mathfrak{A}$ *is replaced by* $P'\mathfrak{A}$ *then* $\mathfrak{A}_\sigma = \mathbf{Q}E_\sigma$.

(4) *There exist two orthogonal non zero projections* P_σ *and* Q_σ *with sum* E_σ *such that* $\mathfrak{A}_\sigma = \{\lambda P_\sigma + \bar{\lambda} Q_\sigma : \lambda \in \mathbf{C}\}$.

Proof. Choose the $W_{\sigma\varrho} \in \mathfrak{S}_{\sigma\varrho}$ as in Lemma 3.3. By Lemma 3.1 $\mathfrak{S}_{\sigma\varrho} = \mathfrak{S}_{\sigma\varrho}\mathfrak{S}_{\varrho\sigma}\mathfrak{S}_{\sigma\varrho}$. If $V \in \mathfrak{S}_{\sigma\varrho}$ then $V = VE_\varrho = VW_{\varrho\sigma}W_{\sigma\varrho} = E_\sigma V = W_{\sigma\varrho}W_{\varrho\sigma}V$. Thus $\mathfrak{S}_{\sigma\varrho} = \mathfrak{S}_{\sigma\varrho}\mathfrak{S}_{\varrho\sigma}W_{\sigma\varrho} = W_{\sigma\varrho}\mathfrak{S}_{\varrho\sigma}\mathfrak{S}_{\sigma\varrho}$. Since $\mathfrak{S}_{\sigma\varrho}$ is linear $\mathfrak{S}_{\sigma\varrho} = \mathfrak{A}_\sigma W_{\sigma\varrho} = W_{\sigma\varrho}\mathfrak{A}_\varrho$. In particular $\mathfrak{A}_\varrho = W^*_{\sigma\varrho}\mathfrak{A}_\sigma W_{\sigma\varrho}$, and they are spatially isomorphic.

By Lemma 3.1 $\mathfrak{A}_{\sigma SA} = E_\sigma \mathfrak{A} E_\sigma = \mathbf{R}E_\sigma$. By Theorem 2.1 there are four cases. The cases (1), (2), and (4) of that theorem yield cases (1), (2), and (4) above. Assume $\mathfrak{A}_\sigma$ is given by (3) in Theorem 2.1 for all $\sigma \in J$. Then there exists a minimal projection $P'_\sigma \in \mathfrak{A}'_\sigma$, where $\mathfrak{A}_\sigma$ is considered as an algebra on the Hilbert space E_σ, such that $P'_\sigma \mathfrak{A}_\sigma = \mathbf{Q}P'_\sigma$. Let $\mathfrak{B}_\sigma$ denote the C^*-algebra generated by $\mathfrak{A}_\sigma$. Then $\mathfrak{B}_\sigma$ is a factor of type I. The map $A \to AP'_\sigma$ is an isomorphism of $\mathfrak{B}_\sigma$. Fix $\sigma \in J$. For each $\varrho \neq \sigma$ let $P'_\varrho = W_{\varrho\sigma}P'_\sigma W_{\sigma\varrho}$. Since the map $\mathfrak{A}_\sigma \to \mathfrak{A}_\varrho$ by $A \to W_{\varrho\sigma}AW_{\sigma\varrho}$ is an isomorphism, P'_ϱ is a minimal projection in $\mathfrak{A}'_\varrho$. Let $P' = \sum_{\varrho \in J} P'_\varrho$. Then $P' \in \mathfrak{A}'$. In fact, let $A \in \mathfrak{A}$. Then $P'E_\varrho A E_\varrho = P'_\varrho E_\varrho A E_\varrho = E_\varrho A E_\varrho P'_\varrho = E_\varrho A E_\varrho P'$, since $P'_\varrho \in \mathfrak{A}'_\varrho$. Since $\mathfrak{S}_{\tau\varrho} = \mathfrak{A}_\tau W_{\tau\varrho}$ there exists $A_\tau \in \mathfrak{A}_\tau$ such that $E_\tau A E_\varrho = A_\tau W_{\tau\varrho}$. Thus, with τ and ϱ distinct from σ,

$$\begin{aligned} P'E_\tau A E_\varrho &= P'_\tau A_\tau W_{\tau\varrho} = A_\tau P'_\tau W_{\tau\varrho} \\ &= A_\tau W_{\tau\sigma}P'_\sigma W_{\sigma\tau}W_{\tau\varrho} = A_\tau W_{\tau\sigma}P'_\sigma W_{\sigma\varrho} \\ &= A_\tau W_{\tau\sigma}W_{\sigma\varrho}P'_\varrho = A_\tau W_{\tau\varrho}P'_\varrho = E_\tau A E_\varrho P', \end{aligned}$$

and $P' \in \mathfrak{A}'$ as asserted. Let Q be a central projection in $\mathfrak{A}'$ such that $Q \geqslant P'$. Then $Q_\sigma = E_\sigma Q E_\sigma \geqslant P'_\sigma$. But Q_σ is central in $E_\sigma \mathfrak{A}'' E_\sigma = \mathfrak{B}_\sigma$, so $Q_\sigma = E_\sigma$. Thus $Q = \sum_\sigma E_\sigma = I$, and the central carrier of P' is I. The proof is complete.

As an immediate application of this lemma and its proof we have

LEMMA 3.5. *Let the* $W_{\sigma\varrho}$, $\sigma \neq \varrho \in J$, *be partial isometries in* $\mathfrak{S}_{\sigma\varrho}$ *chosen as in Lemma 3.3. Then* $\mathfrak{S}_{\sigma\varrho}$ *is characterized as follows:*

(1) $\mathfrak{S}_{\sigma\varrho} = \mathbf{R}W_{\sigma\varrho}$.

(2) $\mathfrak{S}_{\sigma\varrho} = \mathbf{C}W_{\sigma\varrho}$.

(3) *There exists a projection* $P' \in \mathfrak{A}'$ *with central carrier* I *such that if* $\mathfrak{A}$ *is replaced by* $P'\mathfrak{A}$ *then* $\mathfrak{S}_{\sigma\varrho} = \mathbf{Q}W_{\sigma\varrho}$.

(4) *There exist orthogonal projections* P_σ, Q_σ, P_ϱ, Q_ϱ *with* $P_\sigma + Q_\sigma = E_\sigma$, $P_\varrho + Q_\varrho = E_\varrho$ *such that*

$$\mathfrak{S}_{\sigma\varrho}=\{(\lambda P_\sigma+\bar{\lambda}Q_\sigma)W_{\sigma\varrho}:\lambda\in\mathbf{C}\}=\{W_{\sigma\varrho}(\mu P_\varrho+\bar{\mu}Q_\varrho):\mu\in\mathbf{C}\}.$$

The remaining part of the proof of Theorem 3.9 consists of eliminating case 4 in Lemma 3.5. We shall need a simple and probably well-known lemma of more general nature.

LEMMA 3.6. *Let E, F and G, H be two pairs of orthogonal non zero projections, let V be a partial isometry of $G+H$ onto $E+F$. Assume that for each $\lambda\in\mathbf{C}$ there exists $\mu\in\mathbf{C}$ such that*

(1) $(\lambda E+\bar{\lambda}F)V=V(\mu G+\bar{\mu}H)$.

Then either $V^*EV=G$ and $V^*FV=H$, or $V^*EV=H$ and $V^*FV=G$.

Proof. Multiply (1) on the right by G. Then

(2) $\lambda EVG+\bar{\lambda}FVG=\mu VG$.

Multiply (2) on the left by E. Then $\lambda EVG=\mu EVG$. Thus either $\lambda=\mu$ or $EVG=0$. Similarly, multiplication of (2) on the left by F yields $\bar{\lambda}=\mu$ or $FVG=0$. Since (1) holds for all $\lambda\in\mathbf{C}$ either $EVG=0$ or $FVG=0$, say $EVG=0$. Multiplication of (1) on the right by H yields $EVH=0$ or $FVH=0$. If $EVH=0$ then $0=EV(G+H)=EV=EVV^*=E$, contrary to assumption. Thus $FVH=0$. Now $(V^*EV)G=0$ hence $V^*EV\leqslant H$. Similarly $V^*FV\leqslant G$. Since $V^*EV+V^*FV=H+G$, $V^*EV=H$, $V^*FV=G$. The case $FVG=0$ is treated similarly.

LEMMA 3.7. *Assume the $\mathfrak{S}_{\sigma\varrho}$ are given by (4) in Lemma 3.5. Then the P_σ and Q_σ can be chosen so that $P_\sigma W_{\sigma\varrho}=W_{\sigma\varrho}P_\varrho$ and $Q_\sigma W_{\sigma\varrho}=W_{\sigma\varrho}Q_\varrho$, where the $W_{\sigma\varrho}$ are chosen as in Lemma 3.3.*

Proof. Fix $\tau\in J$ and P_τ. By relabelling P_ϱ if necessary whenever $\varrho\neq\tau$ Lemma 3.6 gives $P_\tau W_{\tau\varrho}=W_{\tau\varrho}P_\varrho$, hence $W_{\varrho\tau}P_\tau=(P_\tau W_{\tau\varrho})^*=(W_{\tau\varrho}P_\varrho)^*=P_\varrho W_{\varrho\tau}$ whenever $\varrho\neq\tau$. Let $\sigma\in J$. We may assume $\sigma\neq\tau$. If $\varrho\neq\sigma,\tau$, then $P_\sigma W_{\sigma\varrho}=P_\sigma W_{\sigma\tau}W_{\tau\varrho}=W_{\sigma\tau}P_\tau W_{\tau\varrho}=W_{\sigma\tau}W_{\tau\varrho}P_\varrho=W_{\sigma\varrho}P_\varrho$. The proof is complete.

Assume the $W_{\sigma\varrho}$ are chosen as in Lemma 3.3 and the P_σ as in Lemma 3.7.

LEMMA 3.8. *Assume the $\mathfrak{S}_{\sigma\varrho}$ are given by (4) in Lemma 3.5. Let P be the projection $P=\sum_{\sigma\in J}P_\sigma$. Then P belongs to the center of $\mathfrak{A}'$.*

Proof. Let $A\in\mathfrak{A}$. Then A is a strong limit of finite sums of the form

$$\sum_{\sigma\neq\varrho}(\lambda_{\sigma\varrho}P_\sigma+\bar{\lambda}_{\sigma\varrho}Q_\sigma)W_{\sigma\varrho}+\sum_\sigma\alpha_\sigma E_\sigma.$$

Hence $P\in\mathfrak{A}'$ if we can show

$$P_\sigma(\lambda P_\sigma+\bar{\lambda}Q_\sigma)W_{\sigma\varrho}=(\lambda P_\sigma+\bar{\lambda}Q_\sigma)W_{\sigma\varrho}P_\varrho$$

for all $\sigma, \varrho \in J$. But this is immediate from Lemma 3.7. Since the P_σ are all in $(\mathfrak{A})$ (see Corollary 2.2), $P \in \mathfrak{A}''$. Thus P is central in $\mathfrak{A}'$.

THEOREM 3.9. *Let $\mathfrak{A}$ be an irreducible JW-algebra of type I_n, $n \geq 3$. Let $\{E_\sigma\}_{\sigma \in J}$ be an orthogonal family of non zero abelian projections in $\mathfrak{A}$ with $\sum_{\sigma\in J} E_\sigma = I$. Let $\mathfrak{S}_{\sigma\varrho} = E_\sigma \mathfrak{A} E_\varrho$ for $\sigma \neq \varrho$. Then every operator in $\mathfrak{S}_{\sigma\varrho}$ is a scalar multiple of a partial isometry of E_ϱ onto E_σ. If $W_{\sigma\varrho}$ is a partial isometry in $\mathfrak{S}_{\sigma\varrho}$ then one of three cases occur.*

(1) $\mathfrak{S}_{\sigma\varrho} = \mathbf{R} W_{\sigma\varrho}$ *for all* $\sigma \neq \varrho$, *and* $\dim E_\sigma = 1$

(2) $\mathfrak{S}_{\sigma\varrho} = \mathbf{C} W_{\sigma\varrho}$ *for all* $\sigma \neq \varrho$, *and* $\dim E_\sigma = 1$

(3) $\mathfrak{S}_{\sigma\varrho} = \mathbf{Q} W_{\sigma\varrho}$ *for all* $\sigma \neq \varrho$, *and* $\dim E_\sigma = 2$.

Moreover, $\mathfrak{A}$ is reversible.

Proof. Case (4) in Lemma 3.5 cannot occur. Indeed, if it did, let $P = \sum_{\sigma\in J} P_\sigma$, $Q = \sum_{\sigma \in J} Q_\sigma$. By Lemma 3.8 both P and Q are non zero orthogonal projections in $\mathfrak{A}'$ contradicting the fact that $\mathfrak{A}$ is irreducible. By Lemma 3.2 every operator in $\mathfrak{S}_{\sigma\varrho}$ is a scalar multiple of a partial isometry of E_ϱ onto E_σ. By Lemma 3.5 $\mathfrak{S}_{\sigma\varrho}$ is one of the three sets described above. By Corollary 2.2 $\dim E_\sigma$ is also as described. Finally, by Lemma 3.1 $\mathfrak{A}$ is reversible. The proof is complete.

4. Abelian projections

One of the difficulties in the study of JW-algebras is due to the poor behaviour of cyclic projections. In this section and in section 8 we shall obtain useful results on such operators. Presently we shall find a formula connecting abelian and cyclic projections.

LEMMA 4.1. *Let $\mathfrak{A}$ be a JC-algebra with identity acting on a Hilbert space $\mathfrak{H}$. Let x be a vector in $\mathfrak{H}$ and assume $[(\mathfrak{A})x] = I$. Let F be a projection in $\mathfrak{A}$ such that $F \leq I - [\mathfrak{A}x]$. Then $F = 0$.*

Proof. By assumption the projections $[\mathfrak{A}^n x]$ converge strongly to I, $n = 1, 2, \ldots$. Also $F[\mathfrak{A}x] = 0$. Use induction and assume $F[\mathfrak{A}^{n-1}x] = 0$, $n \geq 2$. Let $n \geq 2$ and $A_1, \ldots, A_n \in \mathfrak{A}$. If $n = 2$ then

$$FA_1A_2x = F(FA_1A_2 + A_2A_1F)x \in F[\mathfrak{A}x] = 0.$$

If $n \geq 3$, then

$$F \prod_{i=1}^{n} A_i x = F(FA_1A_2 + A_2A_1F) \prod_{i=3}^{n} A_i x \in F[\mathfrak{A}^{n-1}x] = 0.$$

Thus $F[\mathfrak{A}^n x] = 0$, $n = 1, 2, \ldots$. Since $[\mathfrak{A}^n x] \to I$ strongly, $0 = F[\mathfrak{A}^n x] \to F$ strongly. $F = 0$.

For reversible JC-algebras similar techniques give an inequality in the opposite direction.

LEMMA 4.2. *Let $\mathfrak{A}$ be a reversible JC-algebra. Let E be a projection in $\mathfrak{A}$ and x a vector in E. Assume* $[(\mathfrak{A})x]=I$. Then $I-E\leq[\mathfrak{A}x]$.

Proof. Let $F=I-E$. Then $F[\mathfrak{A}^n x]=[\mathfrak{A}x]F$ for $n=1,2,\ldots$. Indeed, $Fx=0$, so with $A_1,\ldots,A_n\in\mathfrak{A}$,

$$F\prod_{i=1}^{n}A_i x=(F\prod_{i=1}^{n}A_i+\prod_{i=n}^{1}A_i F)\,x\in[\mathfrak{A}x]\leq[\mathfrak{A}^n x],$$

and the assertion follows. Since $[\mathfrak{A}^n x]\to I$ strongly, $[\mathfrak{A}x]F=F[\mathfrak{A}^n x]\to F$, strongly. Thus $[\mathfrak{A}x]F=F$, the proof is complete.

LEMMA 4.3. *Let $\mathfrak{A}$ be a JC-algebra acting on a Hilbert space $\mathfrak{H}$. Let ω_x be a pure vector state on $\mathfrak{A}$. Let F denote the set of vectors $z\in\mathfrak{H}$ such that $\omega_z|\mathfrak{A}=\|z\|^2\omega_x|\mathfrak{A}$. Then F is a subspace of $\mathfrak{H}$, and $F\leq[x]+I-[\mathfrak{A}x]$.*

Proof. Clearly $z\in F$, $\lambda\in\mathbf{C}$ implies $\lambda z\in F$. Let w and z be unit vectors in F. Let $A\geq 0$ be in $\mathfrak{A}$. Then

$$\begin{aligned}\omega_{w+z}(A)&=\omega_w(A)+\omega_z(A)+2\,\mathrm{Re}(Aw,z)\\&=2\,(\omega_x(A)+\mathrm{Re}\,(A^{\frac12}w,A^{\frac12}z))\\&\leq 2\,(\omega_x(A)+\|A^{\frac12}w\|\,\|A^{\frac12}z\|)\\&=4\,\omega_x(A).\end{aligned}$$

Since ω_x is pure and $\omega_{w+z}\leq 4\omega_x$, $\omega_{w+z}=\|w+z\|^2\omega_x$ on $\mathfrak{A}$, $w+z\in F$. Thus F is a linear manifold. It is clear that F is closed, hence is a subspace. Now $x\in F$. Hence in order to show $F\leq[x]+I-[\mathfrak{A}x]$ it suffices to consider a unit vector $y\in F-[x]$. Let λ be a complex number of modulus 1. Then $x+\lambda y\in F$, so

$$\omega_{x+\lambda y}=\|x+\lambda y\|^2\omega_x=(\|x\|^2+\|y\|^2)\omega_x=2\omega_x\quad\text{on }\mathfrak{A}.$$

Thus for all $A\in\mathfrak{A}$,

$$2\omega_x(A)=\omega_{x+\lambda y}(A)=2\omega_x(A)+2\mathrm{Re}(A\lambda y,x),$$

so that $\mathrm{Re}(A\lambda y,x)=0$ for all complex numbers λ of modulus 1. Thus $0=(Ay,x)=(y,Ax)$ for all $A\in\mathfrak{A}$, i.e. $y\in I-[\mathfrak{A}x]$. The proof is complete.

THEOREM 4.4. *Let $\mathfrak{A}$ be a JW-factor acting on a Hilbert space $\mathfrak{H}$. Let E be a projection in $\mathfrak{A}$ and x a unit vector in E. Assume $[(\mathfrak{A})x]=I$. Then E is abelian if and only if*

(1) $E\leq[x]+I-[\mathfrak{A}x]$.

Moreover, if $\mathfrak{A}$ is reversible then the inequality (1) *is equality.*

Proof. Suppose E is abelian. Then $E\mathfrak{A}E = \mathbf{R}E$, hence ω_x is pure on $\mathfrak{A}$. Moreover, if y is a unit vector in E then $\omega_y | \mathfrak{A} = \omega_x | \mathfrak{A}$, so by Lemma 4.3, (1) follows.

Conversely, assume (1) holds. Let G be a projection in $\mathfrak{A}$ with $G \leqslant E$. If $x \in G$ then $E - G \leqslant I - [\mathfrak{A}x]$, hence $E - G = 0$ by Lemma 4.1. If $x \in E - G$ then similar arguments give $G = 0$. Assume $Gx \neq 0$. Since $G \leqslant E \leqslant [x] + I - [\mathfrak{A}x]$, $Gx = \lambda x + y$ with $y \in I - [\mathfrak{A}x]$, $\lambda \neq 0$. Then $y = (G - \lambda I)x \in [\mathfrak{A}x]$, hence $y = 0$. Thus $Gx = \lambda x$. Since $\lambda \neq 0$, $x \in G$, $G = E$ by the above. Thus E is a minimal projection in $\mathfrak{A}$, hence is abelian. If $\mathfrak{A}$ is reversible then Lemma 4.2 shows that (1) must be equality. The proof is complete.

5. *JW*-factors of type I_n, $n \geqslant 3$

In Lemma 3.5 we have practically classified all JW-factors of type I_n, $n \geqslant 3$. However, the description of case (4) is incomplete. The present section fills out this gap. For this we shall need an analogue of the result for von Neumann algebras, which states that a factor of type I has a faithful normal representation as all bounded operators on a Hilbert space.

THEOREM 5.1. *Let $\mathfrak{A}$ be a JW-factor of type I_n, $n \geqslant 3$. Then there exists a representation of $(\mathfrak{A})$ which, when restricted to $\mathfrak{A}$, is a faithful normal representation as an irreducible JW-algebra of type I_n.*

Proof. Let E be an abelian projection in $\mathfrak{A}$. Let x be a unit vector in E. Then ω_x is a pure state of $\mathfrak{A}$. Let f be a pure state extension of ω_x to $(\mathfrak{A})$. Then $f = \omega_y \varphi$ with φ an irreducible representation of $(\mathfrak{A})$. Thus $\varphi(\mathfrak{A})$ is an irreducible JC-algebra. Moreover, $y \in F = \varphi(E)$ with F an abelian projection in $\varphi(\mathfrak{A})$, hence in $\varphi(\mathfrak{A})^-$, which is thus of type I. By assumption $\mathfrak{A}$ is of type I_n, $n \geqslant 3$. Since all abelian projections in $\mathfrak{A}$ are equivalent [11, Corollary 26], φ is faithful on $\mathfrak{A}$, hence $\varphi(\mathfrak{A})^-$ is of type I_m, $m \geqslant n$. If n is finite clearly $m = n$. Otherwise $n = \infty$ in which case $m = \infty$, hence $\varphi(\mathfrak{A})^-$ is of type I_n. In particular $\varphi(\mathfrak{A})^-$ is reversible (Theorem 3.9). We show φ i ultra-weakly continuous on $\mathfrak{A}$. There are two cases.

Case 1. $\varphi(\mathfrak{A})^-$ is determined by (1) or (2) in Theorem 3.9. Then $\dim F = 1$, so by Theorem 4.4 $[\varphi(\mathfrak{A})y] = I$. Let $w = Ay + iBy$ with $A, B \in \varphi(\mathfrak{A})$. Let $S \geqslant 0$ in $\varphi(\mathfrak{A})$. Then

$$\begin{aligned} 0 &\leqslant (Sw, w) \\ &= (SAy, Ay) + (SBy, By) - 2\,\mathrm{Im}\,(SBy, Ay) \\ &\leqslant (ASAy, y) + (BSBy, y) + 2\,\|S^{\frac{1}{2}}By\|\,\|S^{\frac{1}{2}}Ay\| \\ &= ((ASAy, y)^{\frac{1}{2}} + (BSBy, y)^{\frac{1}{2}})^2. \end{aligned}$$

Thus, if $A = \varphi(A_1)$, $B = \varphi(B_1)$, $S = \varphi(S_1)$ then

$$0 \leqslant (\varphi(S_1)w, w) \leqslant (\omega_z(A_1 S_1 A_1)^{\frac{1}{2}} + \omega_z(B_1 S_1 B_1)^{\frac{1}{2}})^2.$$

Therefore, if $S_j \geqslant 0$ in the unit ball $\mathfrak{A}_1$ of $\mathfrak{A}$ and $S_j \to 0$ weakly, then $(\varphi(S_j)w, w) \to 0$. Let z be any unit vector in $\mathfrak{H}$. Suppose S_j are operators as above. Let $\varepsilon > 0$ be given. Choose a unit vector $w = Ay + iBy$ in $\mathfrak{H}$, A, $B \in \varphi(\mathfrak{A})$, such that $\|w - z\| < \varepsilon/4$. By the preceding we can choose j so large that $(\varphi(S_j)w, w) < \varepsilon/2$. Then

$$\begin{aligned} 0 &\leqslant (\varphi(S_j)z, z) \\ &\leqslant |(\varphi(S_j)z, z) - (\varphi(S_j)w, w)| + (\varphi(S_j)w, w) \\ &< |(\varphi(S_j)z, z-w) + (\varphi(S_j)(z-w), w)| + \varepsilon/2 \\ &\leqslant 2\,\|\varphi(S_j)\|\,\|z\|\,\|z-w\| + \varepsilon/2 \\ &< 2\varepsilon/4 + \varepsilon/2 = \varepsilon. \end{aligned}$$

Thus $S_j \to 0$ on the positive part of $\mathfrak{A}_1$ implies $(\varphi(S_j)z, z) \to 0$ for all unit vectors z in $\mathfrak{H}$. As in [6, Remark 2.2.3] it follows that φ is ultra-weakly continuous on $\mathfrak{A}$.

Case 2. $\varphi(\mathfrak{A})^-$ is determined by (3) in Theorem 3.9. Then $\dim F = 2$ so there exists a unit vector z orthogonal to y in F. By Theorem 4.4 $[y] + [z] = F = [y] + I - [\varphi(\mathfrak{A})y]$, hence $[\varphi(\mathfrak{A})y] + [z] = I$. Therefore, every vector w in $\mathfrak{H}$ is of the form $w = u + \lambda z$ with $u \in [\varphi(\mathfrak{A})y]$, $\lambda \in \mathbf{C}$. Exactly as in case 1 $\omega_u \varphi$ is weakly continuous at 0 in $\mathfrak{A}_1$. Let $S \geqslant 0$ be in $\varphi(\mathfrak{A})$. Then

$$\begin{aligned} 0 &\leqslant (Sw, w) \\ &= (Su, u) + |\lambda|^2 (Sz, z) + 2\,\mathrm{Re}\,(S\lambda z, u) \\ &= (Su, u) + |\lambda|^2 (Sy, y) + 2\,\mathrm{Re}\,(S\lambda z, u) \\ &\leqslant (Su, u) + |\lambda|^2 (Sy, y) + 2\,\|S^{\frac{1}{2}}\lambda z\|\,\|S^{\frac{1}{2}} u\| \\ &= ((Su, u)^{\frac{1}{2}} + |\lambda|(Sy, y)^{\frac{1}{2}})^2. \end{aligned}$$

As in case 1 we conclude that φ is ultra-weakly continuous on $\mathfrak{A}$, i.e. φ is weakly continuous on $\mathfrak{A}_1$, which is weakly compact. Thus the unit ball in $\varphi(\mathfrak{A})$ is weakly compact. As Topping has pointed out the Kaplansky density theorem holds for JC-algebras (see the proof in [2]). Since $\varphi(\mathfrak{A})$ is strongly dense in $\varphi(\mathfrak{A})^-$ and contains the unit ball in $\varphi(\mathfrak{A})^-$ it must be equal to $\varphi(\mathfrak{A})^-$, i.e. $\varphi(\mathfrak{A})$ is a JW-algebra. The proof is complete.

THEOREM 5.2. *Let $\mathfrak{A}$ be a JW-factor of type I_n, $n \geqslant 3$, acting on a Hilbert space $\mathfrak{H}$. Let $\{E_\sigma\}_{\sigma \in J}$ be an orthogonal family of non zero abelian projections in $\mathfrak{A}$ with $\sum_{\sigma \in J} E_\sigma = I$. For $\sigma \neq \varrho$ let $\mathfrak{S}_{\sigma\varrho} = E_\sigma \mathfrak{A} E_\varrho$. Let $W_{\sigma\varrho}$ be a partial isometry in $\mathfrak{S}_{\sigma\varrho}$. Then one of the following four cases occurs:*

(1) $\mathfrak{S}_{\sigma\varrho} = \mathbf{R} W_{\sigma\varrho}$ *for all* $\sigma \neq \varrho$.

(2) $\mathfrak{S}_{\sigma\varrho} = \mathbf{C} W_{\sigma\varrho}$ *for all* $\sigma \neq \varrho$.

(3) *There exists a projection* $P' \in \mathfrak{A}'$ *with central carrier* I *such that if* $\mathfrak{A}$ *is replaced by* $P'\mathfrak{A}$ *then* $\mathfrak{S}_{\sigma\varrho} = \mathbf{Q} W_{\sigma\varrho}$.

(4) *There exist two non zero projections* P_σ *and* Q_σ *with* $P_\sigma + Q_\sigma = E_\sigma$ *such that* $\mathfrak{S}_{\sigma\varrho} = \{(\lambda P_\sigma + \bar{\lambda} Q_\sigma) W_{\sigma\varrho} : \lambda \in \mathbf{C}\}$. *In this case there exist a Hilbert space* $\mathfrak{K}$, *a normal *-isomorphism* ψ_1, *and a normal *-anti-isomorphism* ψ_2 *of* $\mathfrak{B}(\mathfrak{K})$ *into* $\mathfrak{B}(\mathfrak{H})$ *such that* $\psi_1(I)\psi_2(I) = 0$, *and such that* $\mathfrak{A}$ *is the image of the* C^**-isomorphism* $\psi_1 + \psi_2$ *of* $\mathfrak{B}(\mathfrak{K})_{SA}$ *into* $\mathfrak{B}(\mathfrak{H})_{SA}$.

Proof. If the $\mathfrak{S}_{\sigma\varrho}$ are determined by (1), (2), (3) in Lemma 3.5 then we have cases (1), (2), (3) above. Assume the $\mathfrak{S}_{\sigma\varrho}$ are determined by (4) in Lemma 3.5. Let $P = \sum_{\sigma \in J} P_\sigma$, $Q = \sum_{\sigma \in J} Q_\sigma$, where the P_σ and Q_σ are as in Lemma 3.5. By Lemma 3.8 P and Q are central projections in $(\mathfrak{A})^-$ with $P + Q = I$. Let φ be the representation constructed in Theorem 5.1 of $(\mathfrak{A})$ into $\mathfrak{B}(\mathfrak{K})$. Then φ has an extension to an irreducible representation $\bar{\varphi}$ of $(\mathfrak{A})^-$, hence $\bar{\varphi}(P) = 0$ or $\bar{\varphi}(Q) = 0$, say $\bar{\varphi}(Q) = 0$. Then $\varphi(Q_\sigma) = 0$ for all $\sigma \in J$. Consequently $\varphi(\mathfrak{S}_{\sigma\varrho}) = \mathbf{C}\varphi(W_{\sigma\varrho}) = \varphi(E_\sigma)\varphi(\mathfrak{A})\varphi(E_\varrho)$. Thus $\varphi(\mathfrak{A}) = \mathfrak{B}(\mathfrak{K})_{SA}$. Let ψ be the map $\varphi^{-1} : \mathfrak{B}(\mathfrak{K})_{SA} \to \mathfrak{A}$. Then ψ is normal and has an extension to a normal C^*-isomorphism of $\mathfrak{B}(\mathfrak{K})$ onto $\mathfrak{A} + i\mathfrak{A}$. By [4, Corollary to Theorem 7] (or by [10, Theorem 3.3]) ψ is the sum of a normal *-isomorphism ψ_1, and a normal *-anti-isomorphism ψ_2 of $\mathfrak{B}(\mathfrak{K})$ into $\mathfrak{B}(\mathfrak{H})$. Since $\mathfrak{A} = \psi(\mathfrak{B}(\mathfrak{K})_{SA})$ the proof is complete.

6. Non reversible JW-algebras

It turns out that a JW-algebra can be decomposed along its center into three parts, one part being the self-adjoint part of a von Neumann algebra, one part more like the JW-algebras given by (1), (3), and (4) in Theorem 5.2, and a third part, which is practically a global form of a spin factor.

LEMMA 6.1. *Let* $\mathfrak{A}$ *be a reversible* JW*-algebra. Then there exist central projections* E *and* F *in* $\mathfrak{A}$ *with* $E + F = I$ *such that* $E\mathfrak{A}$ *is the self-adjoint part of a von Neumann algebra, and* $\mathfrak{R}(F\mathfrak{A}) \cap i\mathfrak{R}(F\mathfrak{A}) = \{0\}$.

Proof. Let $\mathfrak{K} = \mathfrak{R}(\mathfrak{A}) \cap i\mathfrak{R}(\mathfrak{A})$. Then $\mathfrak{K}$ is an ideal in $(\mathfrak{A})$ [10, Remark 2.2], hence its weak closure $\mathfrak{K}^-$ is an ideal in $(\mathfrak{A})^-$. Thus there exists a central projection E in $(\mathfrak{A})^-$ such that $\mathfrak{K}^- = E(\mathfrak{A})^-$ [2, p. 45], and $E \in \mathfrak{K}^-$. Now $\mathfrak{A}$ is reversible, hence $\mathfrak{K}_{SA} \subset \mathfrak{A}$, and $(\mathfrak{K}^-)_{SA} = (\mathfrak{K}_{SA})^- \subset \mathfrak{A}$. Thus $E \in \mathfrak{A}$, and $\mathfrak{K}^-{}_{SA} = E\mathfrak{A}$. Clearly $\mathfrak{K}^-$ is a von Neumann algebra. Let $F = I - E$. Then F is central in $\mathfrak{A}$, and

$$\mathfrak{R}(F\mathfrak{A}) \cap i\mathfrak{R}(F\mathfrak{A}) = F(\mathfrak{R}(\mathfrak{A}) \cap i\mathfrak{R}(\mathfrak{A})) = F\mathfrak{K} \subset FE(\mathfrak{A})^- = \{0\}.$$

The proof is complete.

LEMMA 6.2. *Let $\mathfrak{A}$ be a JC-algebra with identity I. Let $\mathfrak{J}$ denote the set of operators $A \in \mathfrak{A}$ such that $BAC + C^*AB^* \in \mathfrak{A}$ for all $B, C \in \mathfrak{R}(\mathfrak{A})$. Then $\mathfrak{J}$ is a uniformly closed Jordan ideal in $\mathfrak{A}$. Moreover, $\mathfrak{J}$ is a reversible JC-algebra.*

Proof. Let $A, B \in \mathfrak{J}$, $S, T \in \mathfrak{R}(\mathfrak{A})$. Then

$$S(A+B)T + T^*(A+B)S^* = (SAT + T^*AS^*) + (SBT + T^*BS^*) \in \mathfrak{A},$$

so $\mathfrak{J}$ is linear. Let $A \in \mathfrak{J}$, $B \in \mathfrak{A}$, $S, T \in \mathfrak{R}(\mathfrak{A})$. Then

$$S(AB+BA)T + T^*(AB+BA)S^* = (SA(BT) + (BT)^*AS^*) + ((SB)AT + T^*A(SB)^*) \in \mathfrak{A},$$

so $\mathfrak{J}$ is a Jordan ideal in $\mathfrak{A}$. Since multiplication is uniformly continuous $\mathfrak{J}$ is uniformly closed. Let $A_1 \in \mathfrak{J}$, $A_2, \ldots, A_n \in \mathfrak{A}$. Let $A = \prod_{i=2}^n A_i$. Then $A_1A + A^*A_1 \in \mathfrak{A}$ by definition of $\mathfrak{J}$. We show $A_1A + A^*A_1 \in \mathfrak{J}$, hence $\mathfrak{J}$ is in particular reversible (with $A_2, \ldots, A_n \in \mathfrak{J}$) Let $B, C \in \mathfrak{R}(\mathfrak{A})$. Then

$$\begin{aligned} &B(A_1A + A^*A_1)C + C^*(A_1A + A^*A_1)B^* \\ &\quad = (BA_1(AC) + (AC)^*A_1B^*) + ((BA^*)A_1C + C^*A_1(BA^*)^*) \in \mathfrak{A}. \end{aligned}$$

The proof is complete.

Definition 6.3. Let $\mathfrak{A}$ be a JC-algebra. We say $\mathfrak{A}$ is *totally non reversible* if the ideal $\mathfrak{J}$ in Lemma 6.2 is zero.

THEOREM 6.4. *Let $\mathfrak{A}$ be a JW-algebra. Then there exist three central projections E, F, G in $\mathfrak{A}$ with $E + F + G = I$ such that*

(1) *$E\mathfrak{A}$ is the self-adjoint part of a von Neumann algebra.*
(2) *$F\mathfrak{A}$ is reversible and $\mathfrak{R}(F\mathfrak{A}) \cap i\mathfrak{R}(F\mathfrak{A}) = \{0\}$.*
(3) *$G\mathfrak{A}$ is totally non reversible.*

Proof. Let $\mathfrak{J}$ be the ideal found in Lemma 6.2. $\mathfrak{J}$ is weakly closed. In fact, if $A_\alpha \in \mathfrak{J}$, $A_\alpha \to A$ weakly, then for all $S, T \in \mathfrak{R}(\mathfrak{A})$, $SA_\alpha T + T^*A_\alpha S^* \to SAT + T^*AS^*$ weakly. Since $\mathfrak{A}$ is weakly closed $SAT + T^*AS^* \in \mathfrak{A}$, $A \in \mathfrak{J}$. Let H be the central projection in $\mathfrak{A}$ such that $H\mathfrak{A} = \mathfrak{J}$ (see [11]). Then $H\mathfrak{A}$ is reversible, and the existence of E and F follows from Lemma 6.1. Let $G = I - H$. We must show $G\mathfrak{A}$ is totally non reversible. Let $A \in G\mathfrak{A}$. If for all B, C in $\mathfrak{R}(G\mathfrak{A}) = G\mathfrak{R}(\mathfrak{A})$, $BAC + C^*AB^* \in G\mathfrak{A}$, then, since $B = GS$, $C = GT$, $S, T \in \mathfrak{R}(\mathfrak{A})$, $BAC + C^*AB^* = G(SAT + T^*AS^*) \in G\mathfrak{A}$. But $A = GA$. Thus $SAT + T^*AS^* \in G\mathfrak{A} \subset \mathfrak{A}$ for all S, T in $\mathfrak{R}(\mathfrak{A})$. But then $A \in \mathfrak{J} = H\mathfrak{A}$, $A = 0$. Thus $G\mathfrak{A}$ is totally non reversible. The proof is complete.

COROLLARY 6.5. *A JW-factor is either reversible or totally non reversible.*

THEOREM 6.6. *A totally non reversible JW-algebra is of type I_2.*

Proof. From [11, Theorem 5] there exists a central projection E in $\mathfrak{A}$—the JW-algebra in question—such that $\mathfrak{A}E$ is of type I and $\mathfrak{A}(I-E)$ has no type I portion. If $\mathfrak{A}(I-E)\neq 0$ the "halving lemma" [11, Theorem 17] yields the existence of at least four orthogonal equivalent projections in $\mathfrak{A}(I-E)$ with sum I, hence $\mathfrak{A}(I-E)$ is reversible by Lemma 3.1, contrary to assumption. Thus $\mathfrak{A}$ is of type I. By [11, Theorems 15 and 16] there exists an orthogonal family $\{P_n\}$ of central projections in $\mathfrak{A}$ such that $P_n=0$ or $\mathfrak{A}P_n$ is of type I_n for all cardinals n, and $\sum_n P_n = I$. However, if $n\geqslant 3$ and $P_n\neq 0$ then $\mathfrak{A}P_n$ is reversible by Lemma 3.1, contrary to assumption. If $P_1\neq 0$ then $\mathfrak{A}P_1$ is abelian hence reversible. Thus $\mathfrak{A}$ is of type I_2, the proof is complete.

7. JW-factors of type I_2

Following [11] we define a *spin system* to be a set $\mathfrak{P}$ of symmetries $\neq \pm I$ such that $TS+ST=0$ for $S, T\in\mathfrak{P}$, $S\neq T$. If $\mathfrak{P}$ is a spin system let $\mathfrak{K}$ denote the weak closure of the real linear space spanned by $\mathfrak{P}$. If a JW-factor can be written in the form $\mathbf{R}I\oplus\mathfrak{K}$ with $\mathfrak{K}$ as above, it is said to be a *spin factor*.

THEOREM 7.1. *Let $\mathfrak{A}$ be a JW-factor. Then the following are equivalent.*

(1) $\mathfrak{A}$ *is of type I_2.*

(2) $\mathfrak{A}$ *is a spin factor.*

If $\dim \mathfrak{A}$ *as a vector space over* $\mathbf{R}$ *is greater than* 10(1) *then the above conditions are equivalent to*

(3) $\mathfrak{A}$ *is totally non reversible.*

Proof. (3) $\Rightarrow$ (1). This follows from Theorem 6.6.

(1) $\Rightarrow$ (3). Assume $\dim\mathfrak{A}>10$ and that (3) does not hold. Then $\mathfrak{A}$ is reversible (Corollary 6.5). Let E_1 and E_2 be non zero abelian projections in $\mathfrak{A}$ with $E_1+E_2=I$. Then $\dim\mathfrak{A}=1+1+\dim\mathfrak{S}_{12}$, as a vector space over $\mathbf{R}$. Since $\mathfrak{A}$ is reversible it follows from Corollary 2.2 that $E_j(\mathfrak{A})E_j$ is isomorphic to M_2, $\mathbf{C}$, or $\mathbf{C}\oplus\mathbf{C}$ hence $E_j(\mathfrak{A})E_j$ can be imbedded in M_2 $(j=1,2)$. Hence $\mathfrak{S}_{12}$ can be imbedded in M_2, and $\dim\mathfrak{A}\leqslant 1+1+\dim M_2=2+8=10$, contrary to assumption.

(2) $\Rightarrow$ (1). Let $\mathfrak{A}$ be a spin factor. Then $\mathfrak{A}=\mathbf{R}I\oplus\mathfrak{K}$, $\mathfrak{K}$ as above. By [11, Corollary 29] every non zero operator in $\mathfrak{K}$ is a positive multiple of a symmetry. Thus every operator in $\mathfrak{A}$ is of the form $T=\alpha I+\beta S$, S a symmetry in $\mathfrak{K}$, α, $\beta\in\mathbf{R}$. Since $S=E-F$ with E and F projections in $\mathfrak{A}$ such that $E+F=I$, T has at most two spectral projections. Thus $\mathfrak{A}$ is of type I_2.

(1) In fact it suffices to assume $\dim\mathfrak{A}>6$, see e.g. [5].

$(1) \Rightarrow (2)$. Let $\mathfrak{A}$ be of type I_2 and E and F orthogonal abelian projections in $\mathfrak{A}$ such that $E+F=I$. Then every operator in $\mathfrak{A}$ is of the form $A=\alpha E+\beta F+EAF+FAE$, $\alpha, \beta \in \mathbf{R}$, and where FAE (resp. EAF) is a scalar multiple of a partial isometry of E onto F (resp. F onto E) (Lemma 3.2). Let x and y be vectors of norm $2^{-\frac{1}{2}}$ in E and F respectively. Let Tr be the state $\omega_x+\omega_y$ on $\mathfrak{A}$. In view of Lemma 3.2 it is easy to show Tr is a faithful trace of $\mathfrak{A}$ in the sense of [11]. Define an inner product on $\mathfrak{A}$ by $(A, B)=\operatorname{Tr}(\frac{1}{2}(AB+BA))$. Let $\| \;\|_2$ denote the corresponding norm on $\mathfrak{A}$. Then $\mathfrak{A}$ is a real pre-Hilbert space. We show $\mathfrak{A}$ is closed. In fact, it is straightforward to show $\| \;\| \leqslant 2^{\frac{1}{2}} \| \;\|_2$. If A_n is a Cauchy sequence in $\mathfrak{A}$ with respect to $\| \;\|_2$ then $\|A_n-A_m\| \leqslant 2^{\frac{1}{2}} \|A_n-A_m\|_2 \to 0$. Hence there exists $A \in \mathfrak{A}$ such that $A_n \to A$ uniformly. Since Tr is uniformly continuous $\|A_n-A\|_2 \to 0$, $\mathfrak{A}$ is a real Hilbert space. Denote it by $\tilde{\mathfrak{A}}$. Now I and $E-F$ are orthogonal unit vectors in $\tilde{\mathfrak{A}}$. Extend them to an orthonormal base (S_α) for $\tilde{\mathfrak{A}}$. If $A \in \tilde{\mathfrak{A}}$ is orthogonal to I and $E-F$ then $EAE= FAF=0$. Thus, if S_α is in the base and $S_\alpha \neq I$ and $E-F$, then $S_\alpha=V_\alpha+V_\alpha^*$ with V_α a partial isometry of F onto E. Let S_α and S_β be distinct elements in the base different from I and $E-F$. Then

$$\begin{aligned} S_\alpha S_\beta+S_\beta S_\alpha &= (V_\alpha+V_\alpha^*)(V_\beta+V_\beta^*)+(V_\beta+V_\beta^*)(V_\alpha+V_\alpha^*) \\ &= (V_\alpha V_\beta^*+V_\beta V_\alpha^*)+(V_\alpha^* V_\beta+V_\beta^* V_\alpha) \\ &= \lambda E+\lambda F=\lambda I \end{aligned}$$

by Lemma 3.2. Since S_α and S_β are orthogonal, $0=\operatorname{Tr}(S_\alpha S_\beta+S_\beta S_\alpha)=\lambda$. Thus $S_\alpha S_\beta+ S_\beta S_\alpha=0$. Let $\mathfrak{P}$ be the set of S_α distinct from I. Then $\mathfrak{P}$ is a spin system. If $\mathfrak{K}$ denotes the weakly closed linear space generated by $\mathfrak{P}$ then $\mathfrak{A}=\mathbf{R}I \oplus \mathfrak{K}$, $\mathfrak{A}$ is a spin factor. The proof is complete.

8. Reversible *JW*-algebras

It would be easy by Theorem 5.2 to show that the von Neumann algebra generated by a JW-factor of type I_n, $n \geqslant 3$, is itself of type I. It is possible, however, to give a global version of this fact. For this some facts on central carriers will be needed. If $\mathfrak{A}$ is a JW-algebra or a von Neumann algebra the central carrier of a projection E in $\mathfrak{A}$ with respect to $\mathfrak{A}$ is the least central projection in $\mathfrak{A}$ greater than or equal to E. It will be denoted by $C_E(\mathfrak{A})$.

LEMMA 8.1. *Let $\mathfrak{A}$ be a JW-algebra and E a projection in $\mathfrak{A}$. Then*

$$C_E(\mathfrak{A})=[\mathfrak{A}E]=[\mathfrak{A}''E]=C_E(\mathfrak{A}'').$$

Proof. By [2, Corollaire 1, p. 7] $C_E(\mathfrak{A}'')=[\mathfrak{A}''E]$. Clearly $[\mathfrak{A}E]\leqslant[\mathfrak{A}''E]$. Now $[\mathfrak{A}E]\in\mathfrak{A}'$. In fact, if $x\in E$, A, $B\in\mathfrak{A}$ then

$$BAx=(BAE+EAB)x-EABx\in[\mathfrak{A}x]\vee E\leqslant[\mathfrak{A}E].$$

Thus B leaves $[\mathfrak{A}E]$ invariant, $[\mathfrak{A}E]\in\mathfrak{A}'$. Moreover, $[\mathfrak{A}E]\in\mathfrak{A}$. In fact, if $A\in\mathfrak{A}$, and $r(B)$ denotes the range projection of an operator B, then $r(AE)=r(AE(AE)^*)=r(AEA)\in\mathfrak{A}$, by spectral theory and the fact that $\mathfrak{A}$ is weakly closed. Thus $[\mathfrak{A}E]=\vee_{A\in\mathfrak{A}}r(AE)\in\mathfrak{A}$, as asserted. Thus $[\mathfrak{A}E]$ belongs to the center of $\mathfrak{A}$, which in turn is contained in the center of $\mathfrak{A}''$. Since $C_E(\mathfrak{A}'')=[\mathfrak{A}''E]\geqslant[\mathfrak{A}E]\geqslant E$, $[\mathfrak{A}E]=C_E(\mathfrak{A}'')$. Since clearly $C_E(\mathfrak{A})\geqslant C_E(\mathfrak{A}'')$ the proof is complete.

THEOREM 8.2. *If $\mathfrak{A}$ is a reversible JW-algebra of type I then $\mathfrak{A}''$ is a von Neumann algebra of type I.* (1)

Proof. There exists an abelian projection E in $\mathfrak{A}$ with $C_E(\mathfrak{A})=I$. Let φ be an irreducible representation of $E(\mathfrak{A})E$. Since $(\mathfrak{A})$ equals the uniform closure of $\mathfrak{R}(\mathfrak{A})+i\mathfrak{R}(\mathfrak{A})$, φ is an irreducible representation of $E\mathfrak{R}(\mathfrak{A})E$. Since $(E\mathfrak{R}(\mathfrak{A})E)_{SA}=E\mathfrak{A}E$ is abelian, $\varphi(E\mathfrak{R}(\mathfrak{A})E)$ is isomorphic to either $\mathbf{R}$, $\mathbf{C}$, or $\mathbf{Q}$, by Corollary 2.3. Thus $\varphi(E(\mathfrak{A})E)$ is isomorphic to either $\mathbf{C}$ or M_2, hence $E(\mathfrak{A})E$ is a CCR-algebra (see [8]). By [9, Theorem 6] $E(\mathfrak{A})^-E=(E(\mathfrak{A})E)^-$ is a von Neumann algebra of type I, hence $E\mathfrak{A}''E$ is of type I. Let F be an abelian projection in $E\mathfrak{A}''E$ with $C_F(E\mathfrak{A}''E)=E$ [2, Théorème 1, p. 123]. Then F is abelian in $\mathfrak{A}''$ since $F\mathfrak{A}''F=F(E\mathfrak{A}''E)F$. Let P be a central projection in $\mathfrak{A}''$ such that $P\geqslant F$. Then $PE\geqslant F$. But PE belongs to the center of $E\mathfrak{A}''E$, hence $PE=E$, and $P\geqslant E$. But by Lemma 8.1 $C_E(\mathfrak{A}'')=C_E(\mathfrak{A})=I$. Thus $P=I$, $C_F(\mathfrak{A}'')=I$, $\mathfrak{A}''$ is of type I [2, Théorème 1, p. 123]. The proof is complete.

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(1) In the paper referred to in footnote (1), p. 165, we shall show the converse of this theorem.

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