# EXTENSIONS IN VARIETIES OF GROUPS AND ALGEBRAS

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## Contents



## Introduction

The principal object of the following investigation is to study the *Schreier theory of extensions* and its analogue for any variety $V_S$ of groups or (not necessarily associative) linear algebras defined by a fixed (but arbitrary) set $S$ of identical relations, and then to show how the *Schreier theory* can be applied to give various qualitative results on extensions within such a variety.

Apart from dealing with groups and algebras separately, the discussion treats in a unified way extensions within such varieties as the varieties of (i) all groups, (ii) abelian groups, (iii) groups of fixed exponent $k$, (iv) groups of fixed nilpotency class $k$, (v) associative algebras, (vi) commutative and associative algebras, (vii) Lie algebras, and (viii) Jordan algebras, etc; although groups and algebras are treated separately, there are strong analogies between the results obtained for the two cases.

In establishing the Schreier theory of extensions lying within a fixed variety $V_S$ of groups (or algebras), an essential role is played by the free differential calculus of Fox [6], certain "free derivatives" which appeared in [13], and constructions with factor sets (2-cocycles) which are analogous to, or generalize, ones used at various times by Eilenberg and MacLane (cf. [3], [4] and [14]). This basic theory appears in Chapter I, § 2, and Chapter II, § 2.

Amongst the applications of the Schreier approach there is firstly a theorem stating that the equivalence classes of split extensions, within a given variety $V = V_S$, of an abelian group $M$ by $A \in V$ can be put into 1–1 correspondence with the set of $Z_S(A)$-module structures on $M$, where $Z_S(A)$ is a certain associative ring depending on $A$ and on $V$. This is Theorem 3.2 of Chapter I.

Consideration of the ideals of $Z_S(A)$, and of split "$V$-extensions" of abelian groups and endomorphism rings, leads naturally to the concept of a "$V$-envelope" of $A \in V$. This is a ring $Y$ together with a structure giving rise to a split $V$-extension of the abelian group of $Y$ by $A$. $Z_S(A)$ is an envelope which has the "universal" property that, given any $V$-envelope $Y$ of $A$, there exists a unique homomorphism $Z_S(A) \to Y$, which satisfies a certain naturality relation. Further, there is a 1–1 correspondence between the ideals of $Z_S(A)$ and the classes of $V$-envelopes of $A$, under a certain equivalence relation on envelopes. These, and other, properties of these concepts are discussed in Chapter I, § 4. $Z_S(A)$ and its properties are analogous to a functor $G_S(A)$ and its properties which were studied in [13], with regard to split extensions within varieties of linear algebras.

Next, given a variety $V$ of groups (or linear algebras), some results on general $V$-extensions yield, in particular, analogues of theorems of Eilenberg and MacLane [4, 5]: Let $A = F/R$ be a presentation of $A$ as a quotient of a "$V$-free" group (or algebra) $F$, and let $M$ be an abelian group (or zero algebra). Then the group of $V$-extensions of $M$ by $A$, realizing a given action of $A$ on $M$, is naturally isomorphic to the group of operator homomorphisms $R \to M$ modulo restrictions to $R$ of derivations $F \to M$, the isomorphism being obtainable by "cup products" with any chosen factor set of the extension $R \to F \to A$. (Cf. Chapter I, Theorem 5.2 and Chapter II, Theorem 3.2 — algebra extensions are taken to be *linearly split*. Some months after proving these theorems, I received the manuscript of [8] by M. Gerstenhaber, which includes an isomorphism of essentially these groups, for (not necessarily linearly split) algebra extensions; he sketches an approach different from that considered here.)

If $B \in V$, an analogue of another theorem of Eilenberg and MacLane [5] states that the set of $V$-extensions of $B$ by $A \in V$, under a certain class of actions of $A$ on $B$, can be put into 1–1 correspondence with the corresponding group of $V$-extensions of $N$ by $A$, where $N$

is the centre (or bicentre) of $B$; the correspondence is obtained by first choosing some $V$-extension of $B$ by $A$, and is not natural in general. (Cf. Chapter I, Theorem 6.1 and Chapter II, Theorem 4.1. Hochschild [9, 10] has given this result for associative and Lie algebras.)

Finally, in Chapter II, § 5, there is a result on the embedding of extensions of algebras in split extensions.

I would like to thank Professor J. F. Adams for some helpful comments, and Professor M. G. Barratt for an elegant proposition concerning $Z_S(A)$.

## I. Extensions in varieties of groups

### 1. Preliminary notation and definitions

Let $S$ be any subset of a free group $F$. A group $A$ is said to *satisfy the identical relations* $S$ if and only if

$$\phi(S) = \{1\}$$

for every homomorphism $\phi: F \to A$. If ${}^S F$ denotes the smallest fully invariant subgroup of $F$ containing $S$ then this condition is equivalent to the condition:

$$\phi({}^S F) = \{1\}$$

for every homomorphism $\phi: F \to A$. Any set $T$ such that ${}^T F = {}^S F$ will be called an *equivalent* set of identical relations to $S$. We shall denote by $V_S$ the variety of all groups satisfying the identical relations $S$. Thus, if $T$ is an equivalent set of identical relations to $S$, then $V_S = V_T$. The variety $V_S$ could equally well be defined in terms of any free group $F'$ with the following property: For some set $T$ of identical relations equivalent to $S$, there is a subset $\{x\}$ of a set of free generators of $F$ such that each element of $T$ is a word in the elements of $\{x\}$, and such that there exists an injection $\{x\} \to \{x'\}$ of $\{x\}$ into a set of free generators of $F'$.

For the sake of brevity, groups in $V_S$ will be called *S-groups*. An $S$-group $F_S$ is said to be *free* if and only if $F_S$ contains a subset $X$ such that any mapping of $X$ into an $S$-group $A$ can be extended uniquely to a homomorphism $F_S \to A$. If $F'$ is any free group, and ${}^S F'$ is the normal subgroup of $F'$ generated by

$$\bigcup_{\phi \in \operatorname{Hom}(F, F')} \phi(S),$$

then $F'_S = F' / {}^S F'$ is a free $S$-group. A discussion of identical relations in groups, and fully invariant subgroups of free groups, is given in [14].

Our basic reference on group extensions, and many of the results to be considered later, will be [14]. An *extension* of a group $B$ by a group $A$ is a group $E$ together with an epimorphism

$$\pi : E \to A$$

whose kernel is isomorphic to $B$, and gives rise to a short exact sequence

$$1 \to B \to E \xrightarrow{\pi} A \to 1.$$

The extension will be called an *$S$-extension* (*$V_S$-extension*) if and only if $E \in V_S$. Two extensions

$$1 \to B \to E \xrightarrow{\pi} A \to 1 \quad \text{and} \quad 1 \to B \to E' \xrightarrow{\pi'} A \to 1$$

are *equivalent* if and only if there exists a homomorphism $\tau : E \to E'$ and a commutative diagram:

$$\begin{tikzcd} & & E \arrow[dd, dashed, "\tau"'] \arrow[dr, "\pi"] & & \\ 1 \arrow[r] & B \arrow[ur] \arrow[dr] & & A \arrow[r] & 1; \\ & & E' \arrow[ur, "\pi'"'] & & \end{tikzcd}$$

$\tau$ is then necessarily an isomorphism: $E \cong E'$.

Let $\gamma : A \to E$ be a *cross section* mapping of an extension

$$1 \to B \to E \xrightarrow{\pi} A \to 1,$$

i.e. a mapping such that $\pi\gamma = \text{identity}$. Define a mapping

$$\eta : A \to \operatorname{Aut} B,$$

of $A$ into the group of automorphisms of $B$, by:

$$\eta(a)\,(m) = \gamma_a m \gamma_a^{-1} \quad [a \in A, m \in B],$$

and define the *factor set* mapping

$$\Gamma : A \times A \to B$$

of $\gamma$ (the deviation from multiplicativity of $\gamma$) by:

$$\gamma_a \gamma_b = \Gamma_{a,b}\, \gamma_{ab} \quad [a, b \in A].$$

Then, if $a \cdot m = \eta(a)\,(m)$ $[a \in A, m \in B]$ and $k \cdot m = kmk^{-1}$ $[k, m \in B]$,

$$a \cdot (b \cdot m) = \Gamma_{a,b} \cdot (ab \cdot m) \quad [a, b \in A;\ m \in B]. \tag{1}$$

Further, $\Gamma$ satisfies the "*cocycle*" condition:

$$a \cdot \Gamma_{b,c}\, \Gamma_{a,bc} = \Gamma_{a,b}\, \Gamma_{ab,c} \quad [a, b, c \in A]. \tag{2}$$

We also have:

$$(m\gamma_a)\ (n\gamma_b) = [m(a\cdot n)\ \Gamma_{a,b}]\gamma_{ab} \quad [m,n\in B;\ a,b\in A]. \tag{3}$$

Conversely, given mappings $\eta: A \to \operatorname{Aut} B$ and $\Gamma: A\times A\to B$ satisfying conditions (1) and (2), one can define an extension of $B$ by $A$ by letting $E$ be the set of all pairs $(m,a)$ $[m\in B,\ a\in A]$ with multiplication defined by:

$$(m,a)\ (n,b) = (m(a\cdot n)\Gamma_{a,b},\ ab).$$

It will always be assumed that the functions $\eta$ and $\Gamma$ are *normalized*, i.e. that $\eta(1)=$ identity, and

$$\Gamma_{1,a} = \Gamma_{a,1} = 1 \text{ for } a\in A.$$

(Similarly, cross sections $\gamma$ will be assumed to satisfy $\gamma_1 = 1$.) In this case, the unit element of the previous group $E$ will be $(1,1)$, and the inverse of a pair $(m,\ a)$ will be

$$(m,\ a)^{-1} = (\Gamma^{-1}_{a^{-1},a}\,(a^{-1}\cdot m)^{-1},\ a^{-1}). \tag{4}$$

PROPOSITION 1.1. *Two extensions $1\to B\to E\to A\to 1$ and $1\to B\to E'\to A\to 1$, given by (normalized) maps $\eta,\eta': A\to \operatorname{Aut} B$ and $\Gamma,\Gamma': A\times A\to B$ respectively, are equivalent if and only if there exists a (normalized) map $\psi: A\to B$ such that:*

(i) $\eta'(a) = \langle\psi_a\rangle\ \eta(a)\ [a\in A]$,

*where $\langle k\rangle$ denotes the inner automorphism $m\to k\cdot m$ induced by $k\in B$, and*

(ii) $\Gamma'_{a,b} = \psi_a(a\cdot\psi_b)\ \Gamma_{a,b}\ \psi^{-1}_{ab} \quad [a,b\in A]$.

The above facts are all standard. It will frequently be convenient to describe any extension group $E$ as a group of pairs in the above way; in this case, $\gamma: A\to E$ will always denote the cross section mapping $a\to(1,a)$.

Consider an extension

$$1\to B\to E\to A\to 1$$

determined by maps $\eta: A\to\operatorname{Aut} B$ and $\Gamma: A\times A\to B$. If $\phi: G\to A$ is any homomorphism of a group $G$ into $A$, consider the functions

$$\phi^*\eta = \eta\phi: G\to \operatorname{Aut} B$$

and

$$\phi^*\Gamma = \Gamma(\phi\times\phi): G\times G\to B.$$

A simple verification shows that $\phi^*\eta$ and $\phi^*\Gamma$ satisfy conditions (1) and (2) above, and so define an extension $\phi^*E$ of $B$ by $G$.

If $G = F$ is a free group, such an extension $\phi^* E$ must *split*, i.e. it must have a cross section which is a homomorphism. In fact, any map $x_i \to \mu_{x_i} = (\psi_{x_i}, x_i)$ of a set of free generators $\{x_i\}$ of $F$ into $\phi^* E$ can be extended uniquely to a homomorphism $\mu: v \to (\psi_v, v)$ of $F$ into $\phi^* E$; since $\mu$ is a homomorphism one then has

$$1 = \psi_u(u \cdot \psi_v)\,\phi^* \Gamma(u, v)\,\psi_{uv}^{-1} \quad [u, v \in F].$$

It will be convenient to let

$$\Gamma_\phi : F \to B$$

denote the unique function such that

$$\Gamma_\phi(x_i) = 1,$$

$$\Gamma_\phi(uv) = \Gamma_\phi(u)\,[\phi u.\,\Gamma_\phi(v)]\,\Gamma_{\phi u, \phi v}.$$

Functions of this kind are considered in [15] in the case when $B$ is abelian; analogous functions are used in [3] and [4].

If $ZG$ denotes the integral group ring of a group $G$, and $F$ is a free group on free generators $\{x_i\}$, we shall need to consider the *free derivatives*

$$\partial_i = \frac{\partial}{\partial x_i} : ZF \to ZF$$

of Fox [6]. These are the linear maps given by:

$$\partial_i(x_j) = \delta_{ij},$$

$$\partial_i(uv) = \partial_i(u) + u\partial_i(v) \quad [u, v \in F].$$

The *existence* and *uniqueness* of the Fox derivatives can easily be established in the following way:

Let $M$ be an $F$-module, and consider the corresponding split extension

$$1 \to M \to E \to F \to 1.$$

Given any map $h: \{x_i\} \to M$, let $\psi: F \to E$ be the homomorphism such that $\psi(x_i) = (h(x_i), x_i)$. Then, if $u \in F$,

$$\psi(u) = (d(u), u)$$

for some uniquely determined element $d(u)$ of $M$. Since

$$\psi(uv) = \psi(u)\psi(v) = (d(u) + u \cdot d(v), uv),$$

$d$ is a derivation of $F$ into $M$.

Further, if $d'$ is any derivation of $F$ into $M$ such that $d'(x_i) = h(x_i)$ (all $i$), one can define a homomorphism $\psi' : F \to E$ by: $\psi'(u) = (d'(u), u)$. Since $\psi'$ agrees with $\psi$ on the free generators $x_i$, it must coincide with $\psi$, and hence $d' = d$.

This reproves the well-known result that *any map of a set of free generators of $F$ into an $F$-module $M$ can be uniquely extended to a derivation $F \to M$.*

The Fox derivative $\partial_i : F \to ZF$ is obtained by letting $M$ be the left $F$-module $ZF$, and by taking

$$h(x_j) = \delta_{ij}.$$

Given any set $S \subseteq F$, we shall write

$$\partial(S) = \bigcup_i \partial_i(S).$$

Finally, if, in the previous extension

$$1 \to B \to E \to A \to 1,$$

$B$ is abelian, then

$$\eta : A \to \operatorname{Aut} B$$

is a homomorphism, and we shall let

$$\hat{\eta} : ZA \to \operatorname{Hom}(B, B)$$

denote the linear extension of $\eta$. For any homomorphism $\phi : G \to A$,

$$\phi_* : ZG \to ZA$$

will denote the induced homomorphism of group rings.

## 2. Identities on factor sets and modules

Let $F$ be a free group on free generators $\{x_i\}$ having a subset $S$ defining the variety $V_S$ of $S$-groups.

THEOREM 2.1. *If $1 \to B \to E \to A \to 1$ is an $S$-extension with factor set $\Gamma : A \times A \to B$, then*

$$\Gamma_\phi(S) = \{1\}$$

*for every homomorphism $\phi : F \to A$.*

THEOREM 2.2. *Let $1 \to M \to E \to A \to 1$ be an extension of an abelian group $M$ by an $S$-group $A$, determined by a homomorphism $\eta : A \to \operatorname{Aut} M$ and a factor set cocycle $\Gamma : A \times A \to M$. Then $E \in V_S$ if and only if:*

(i) $\hat{\eta}\phi_*\partial(S') = \{0\}$, and

(ii) $\Gamma_\phi(S') = \{0\}$,

*for one set of identical relations $S'$ equivalent to $S$, and every homomorphism $\phi : F \to A$.*

The proofs depend on the following lemma:

LEMMA 2.3. *Let* $1 \to B \to E \xrightarrow{\pi} A \to 1$ *be an extension with factor set* $\Gamma: A \times A \to B$. *Let* $\bar{\phi}$: $F \to E$ *and* $\phi: F \to A$ *be homomorphisms such that* $\bar{\phi}(x_i) = (m_i, \phi x_i)$. *If* $u \in F$, *then:*

(i) $\bar{\phi}(u) = (X_{\bar{\phi}}(u)\, \Gamma_{\bar{\phi}}(u), \phi u)$,

*where* $X_{\bar{\phi}}(u)$ *is an element of* $B$ *which is unity if each* $m_i$ *is unity;*

(ii) *if* $\alpha: B \to M$ *is an* operator *homomorphism of* $B$ *into* $A$*-module* $M$, *i.e. a homomorphism such that* $\alpha(ebe^{-1}) = (\pi e) \cdot \alpha(b)$ $[e \in E, b \in B]$, *then*

$$\alpha(X_{\bar{\phi}}(u)) = \sum_j (\phi_* \partial_j u) \cdot \alpha(m_j).$$

*Proof.* The statements are true for $u = x_i$. Also, since $1 = \Gamma_\phi(1) = \Gamma_\phi(x_i^{-1} x_i)$, one obtains:

$$\Gamma_\phi(x_i^{-1}) = \Gamma^{-1}_{\phi x_i^{-1}, \phi x_i}.$$

Since $\pi\Gamma = \{1\}$, and $\partial_j(u^{-1}) = -u^{-1}\partial_j(u)$, the formula (4) for $(m_i, \phi x_i)^{-1}$ then shows that (i) and (ii) hold for $u = x_i^{-1}$. By considering products $w = uv$, the lemma is now easily proved by induction on the length $k$ of words $u = x_{i_1}^{\varepsilon_1} \dots x_{i_k}^{\varepsilon_k}$ $(\varepsilon_j = \pm 1)$.

Theorem 2.1 follows from this lemma on considering elements $u$ of $S$ and that homomorphism $\bar{\phi}: F \to E$ such that $\bar{\phi}(x_i) = (1, \phi x_i)$, when $\phi: F \to A$ is given.

If $B$ in the lemma is abelian, and $\alpha$ is taken to be the identity (operator) homomorphism $B \to B$, one obtains the formula:

$$\bar{\phi}(u) = (\sum_j (\phi_* \partial_j u) \cdot m_j + \Gamma_\phi(u), \phi u).$$

Since the elements $m_j$ of $B$ are arbitrary, it is now easy to deduce Theorem 2.2.

If, further, $\Gamma = \{0\}$ and $h: \{x_i\} \to B$ is given by $h(x_i) = m_i$, we obtain the formula:

$$\bar{\phi}(u) = (\sum_j (\phi_* \partial_j u) \cdot m_j, \phi u).$$

If one defines $d: F \to B$ by: $\bar{\phi}(u) = (d(u), \phi u)$, then $d$ is a derivation (via $\phi$). Thus the unique derivation $d: F \to B$ extending $h$ is given by:

$$d(u) = \sum_j (\phi_* \partial_j u) \cdot h(x_j).$$

If $A = F$, and $\phi$ is the identity map of $F$, we get

$$d(u) = \sum_j (\partial_j u) \cdot h(x_j),$$

and, when $B$ is the left $F$-module $ZF$, this is a result of Fox [6].

Again, if $\partial': A \to B$ is a derivation, and if $h(x_j) = \partial'(\phi x_j)$, then, since $\partial'\phi: F \to B$ is a derivation, we obtain a "*chain rule of differentiation*":

$$\partial'(\phi u) = \sum_j (\phi_* \partial_j u) \cdot \partial'(\phi x_j).$$

If $A = F'$ is a free group on free generators $\{x'_k\}$, and $B$ is $ZF'$ we obtain the "*chain rule*" of Fox [6]:

$$\frac{\partial}{\partial x'_k}(\phi u) = \sum_j (\phi_* \partial_j u) \frac{\partial}{\partial x'_k}(\phi x_j).$$

Now let $F'$ be any other free group, and let ${}^S F'$ denote the normal subgroup of $F'$ generated by $\bigcup_{\psi \in \mathrm{Hom}(F, F')} \psi(S)$. Then an $S$-group is also an ${}^S F'$-group, and we have:

COROLLARY 2.4. *Let* $1 \to B \to E \to A \to 1$ *be an* $S$*-extension given by maps* $\eta: A \to \mathrm{Aut}\, B$ *and* $\Gamma: A \times A \to B$. *If* $\phi \in \mathrm{Hom}(F', A)$, *then:*

(i) $\Gamma_\phi({}^S F') = \{1\}$;
(ii) *if* $B$ *is abelian, then* $\hat{\eta}\phi_* \partial({}^S F') = \{1\}$.

## 3. Classification of split $S$-extensions of abelian groups

The results of the next two sections are closely analogous to certain propositions concerning (not necessarily associative) linear algebras, which were considered in [13].

If $S$ is any subset of a free group $F$ on free generators $\{x_i\}$, and $G$ is any group, let $[\partial(S)]_G$ denote the (two-sided) ideal of $ZG$ generated by the set

$$\bigcup_{\phi \in \mathrm{Hom}(F, G)} \phi_* \partial(S).$$

If $A$ is any $S$-group, let

$$Z_S(A) = ZA / [\partial(S)]_A.$$

LEMMA 3.1. *If* $S'$ *is an equivalent set of identical relations to* $S$, *then*

$$Z_S(A) = Z_{S'}(A).$$

*Proof.* For any $u \in F$ and $\psi \in \mathrm{Hom}(F,F)$, the "chain rule of differentiation" of Fox [6], gives:

$$\partial_j(\psi u) = \sum_i (\psi_* \partial_i u) \partial_j(\psi x_i).$$

This shows that

$$\partial\Big(\bigcup_{\psi \in \mathrm{Hom}(F, F)} \psi(S)\Big) \subseteq [\partial(S)]_F.$$

The smallest normal subgroup of $F$ containing $\bigcup_{\psi \in \mathrm{Hom}(F,F)} \psi(S)$ is precisely the smallest fully invariant subgroup ${}^SF$ of $F$ containing $S$. We shall now show that $[\partial({}^SF)]_F$ coincides with the ideal in $ZF$ generated by the set

$$\Delta_S = (1 - {}^SF) \cup \bigcup_{\psi \in \mathrm{Hom}(F,F)} \psi_* \partial(S),$$

where $1 - {}^SF = \{1 - u : u \in {}^SF\}$.

The fact that $\partial({}^SF)$ is contained in the ideal generated by $\Delta_S$ follows from the formulae:

$$\partial_j(uv) = \partial_j(u) + u\partial_j(v), \quad \partial_j(u^{-1}) = -u^{-1}\partial_j(u),$$

and
$$\partial_j(uw^\varepsilon u^{-1}) = (1 - uw^\varepsilon u^{-1})\partial_j(u) + \varepsilon uw^{\varepsilon'}\partial_j(w) \quad [\varepsilon = \pm 1, \varepsilon' = (\varepsilon - 1)/2],$$

on regarding ${}^SF$ as the normal subgroup generated by $\bigcup_{\psi \in \mathrm{Hom}(F,F)} \psi(S)$, and making use of the first inclusion mentioned above. Therefore $\bigcup_{\zeta \in \mathrm{Hom}(F,F)} \zeta_* \partial({}^SF)$ is contained in the ideal generated by $\Delta_S$, because $\Delta_S$ is invariant under $\mathrm{Hom}(F,F)$.

Conversely, the "fundamental identity"

$$u - 1 = \sum_j (\partial_j u)\,(x_j - 1)$$

of Fox [6] shows that $\Delta_S$ is contained in the ideal generated by $\bigcup_{\zeta \in \mathrm{Hom}(F,F)} \zeta_* \partial({}^SF)$. Hence $[\partial({}^SF)]_F$ coincides with the ideal generated by $\Delta_S$.

Therefore, since $A$ is an $S$-group, $[\partial({}^SF)]_A$ is generated by the set

$$\bigcup_{\phi \in \mathrm{Hom}(F,A)} \phi_* \bigcup_{\psi \in \mathrm{Hom}(F,F)} \psi_* \partial(S) = \bigcup_{\phi \in \mathrm{Hom}(F,A)} \phi_* \partial(S),$$

i.e. $[\partial({}^SF)]_A = [\partial(S)]_A$. The lemma follows.

We remark that $Z_S(-)$ is a covariant functor on the category of $S$-groups and homomorphisms.

Now let $M$ be an abelian group, and let

$$\mathrm{Splitext}_S\,(A,M)$$

denote the set of equivalence classes of split $S$-extensions of $M$ by $A$. With the aid of Theorem 2.2(i), it is easy to check that $\mathrm{Splitext}_S\,(-,M)$ is a contravariant functor on the category of $S$-groups and homomorphisms.

Given $M$, and a ring homomorphism $\alpha: Z_S(A) \to \mathrm{Hom}(M,M)$, let $\{E_\alpha\}$ denote the equivalence class of split extensions $E_\alpha$ of $M$ by $A$ corresponding to the homomorphism of groups:

$$\alpha\tau | A : A \to \operatorname{Aut} M,$$

where $\tau$ denotes the natural epimorphism $ZA \to Z_S(A)$.

THEOREM 3.2. *Let $M$ be an abelian group, and $A$ be an $S$-group. Then the mapping*

$$\alpha \to \{E_\alpha\}$$

*defines a 1–1 correspondence:*

$$\operatorname{Hom}(Z_S(A), \operatorname{Hom}(M, M)) \leftrightarrow \operatorname{Splitext}_S(A, M),$$

*which is natural with respect to homomorphisms $A' \to A$ of $S$-groups.*

*Proof*. Given a homomorphism $\alpha: Z_S(A) \to \operatorname{Hom}(M, M)$, let

$$\eta = \alpha\tau | A : A \to \operatorname{Aut} M.$$

Then $$\hat{\eta} = \alpha\tau : ZA \to \operatorname{Hom}(\mathrm{M}, M),$$

and hence $\hat{\eta}[\partial(S)]_A = \{0\}$. By Theorem 2.2, this shows that $\alpha \to \{E_\alpha\}$ is a mapping into $\operatorname{Splitext}_S(A, M)$.

If $\eta: A \to \operatorname{Aut} M$ is a homomorphism defining an element of $\operatorname{Splitext}_S(A, M)$, then, by Theorem 2.2,

$$\hat{\eta}[\partial(S)]_A = \{0\},$$

and so $\hat{\eta}: ZA \to \operatorname{Hom}(M, M)$ can be factorized into a composition

$$ZA \xrightarrow{\tau} Z_S(A) \xrightarrow{\alpha} \operatorname{Hom}(M, M).$$

Thus the mapping $\alpha \to \{E_\alpha\}$ is surjective.

Further, if $\alpha, \alpha': Z_S(A) \to \operatorname{Hom}(M, M)$ are homomorphisms such that

$$\alpha\tau | A = \alpha'\tau | A,$$

then $\alpha = \alpha'$, because $\tau$ is an epimorphism. Thus the given mapping is injective.

Finally, it is easy to check naturality.

*Example 3.3.* The split extension of the additive group of any ideal $K$ of $Z_S(A)$ by $A$, defined by left multiplication of $Z_S(A)$ on $K$, is an $S$-extension.

*Example 3.4.* If $V_S$ is the variety of *abelian groups*, then $Z_S(A)$ is isomorphic to the ring of integers $Z$: For, in this case $[\partial(S)]_A$ is generated by all elements of the form $1 - aba^{-1}$, $a - [a,b] = a - 1$ $(a, b \in A)$, i.e. $[\partial(S)]_A$ coincides with the "*augmentation ideal*" $IA$ of $ZA$.

*Example 3.5.* Let $V_S$ be the variety of *groups of exponent k*, i.e. groups $A$ such that $a^k = 1$ for all $a \in A$. Then $Z_S(A)$ is the quotient of $ZA$ by the ideal generated by all elements of the form

$$1 + a + a^2 + \dots a^{k-1} \quad (a \in A).$$

*Example 3.6.* The preceding two examples show that, for the variety of *abelian groups of exponent k*, there is an isomorphism of $Z_S(A)$ with the cyclic group $Z_k$ of order $k$.

*Example. 3.7.* Let $V_S$ be the variety of *nilpotent groups of class k*, i.e. groups $A$ satisfying

$$[a_1, \dots, a_{k+1}] \equiv [[a_1, \dots, a_k], a_{k+1}] = 1 \quad (a_i \in A).$$

By considering $\partial_{k+1}[x_1, \dots, x_{k+1}]$, one sees that here the relation

$$[a_1, \dots, a_k] = 1 \quad (a_i \in A)$$

is satisfied in $Z_S(A)$.

Further, M. G. Barratt has kindly shown me a proof that, for this variety,

$$Z_S(A) = ZA/(IA)^k.$$

## 4. Properties of the functor $Z_S$ (A)

Let $A$ be an $S$-group, and let $M$ be an abelian group with an $A$-module structure given by a homomorphism $\eta : A \to \operatorname{Aut} M$ such that the corresponding split extension of $M$ by $A$ is an $S$-extension; it will be convenient to call such a module $(M, \eta)$ an *S-module* for $A$.

Let $L = L_Y : Y \to \operatorname{Hom}_a(Y, Y)$ denote the homomorphism defined by left multiplication of a ring $Y$ on itself, where $\operatorname{Hom}_a(Y, Y)$ is the ring of endomorphisms of the abelian group of $Y$. Then Example 3.3 above states that $(Z_S(A), L\tau | A)$ is an $S$-module for $A$, with a similar property for any ideal $K$ of $Z_S(A)$.

If $(M, \eta)$ is any $S$-module for $A$, let $Y_M = \operatorname{Hom}(M, M)$. If $\sigma : A \to Y$ is any homomorphism of $A$ into the group of units of a ring $Y$ with identity, let $\hat{\sigma} : ZA \to Y$ denote the linear extension of $\sigma$. Then:

$$(L\eta)\hat{} = L\hat{\eta} : ZA \to \operatorname{Hom}_a(Y_M, Y_M).$$

Hence:

$$(L\eta)\hat{}\,[\partial(S)]_A = L\hat{\eta}[\partial(S)]_A = \{0\}.$$

Therefore, by Theorem 2.2,

$$(Y_M, L\eta) = (\operatorname{Hom}(M, M), L\eta)$$

is also an $S$-module for $A$.

Suppose that one now introduces the following definition: An *S-envelope* (*$V_S$-envelope*) of $A$ is a pair $(Y, \beta)$, consisting of a ring $Y$ with identity and a homomorphism $\beta : A \to Y$ of $A$ into the group of units of $Y$, such that $(Y, L\beta)$ is an $S$-module for $A$. Then $(Z_S(A), \tau | A)$ is an $S$-envelope of $A$, which has the following "universal" property:

THEOREM 4.1. *Given any S-envelope* $(Y, \beta)$ *of* $A$, *there exists a unique ring homomorphism* $\nu : Z_S(A) \to Y$ *such that* $\nu\tau | A = \beta$.

Since the propositions of this section are all closely analogous to ones concerning linear algebras [13], and since their proofs can be obtained by directly paraphrasing the corresponding proofs in [13], only Theorem 4.7 will be proved here.

The proof of Theorem 4.1 (which is analogous to Theorem 6.4 of [13]) is based on the following lemma (which is analogous to Theorem 5.2 of [13]):

LEMMA 4.2. *A pair* $(Y, \beta)$, *consisting of a ring* $Y$ *with identity and a homomorphism* $\beta: A \to Y$ of $A$ *into the group of units of* $Y$, *is an S-envelope of* $A$ *if and only if*

$$\beta[\partial(S)]_A = \{0\}.$$

This lemma is proved with the aid of Theorem 2.2(i) which is analogous to Theorem 4.5 of [13] (cf. Chapter II, Theorem 2.2(i)).

The $S$-envelope $(Z_S(A), \tau | A)$ has the property that $\tau(A)$ generates $Z_S(A)$. Further, every $S$-envelope $(Y, \beta)$ contains a *minimal* $S$-envelope $(Y_\beta, \beta)$, where $Y_\beta$ is the subring of $Y$ generated by $\beta(A)$. Suppose that one calls two $S$-envelopes $(Y, \beta)$ and $(Y', \beta')$ of $A$ *equivalent* if and only if their minimal subenvelopes $(Y_\beta, \beta)$ and $(Y'_{\beta'}, \beta')$ are isomorphic, under an isomorphism $\varepsilon: Y_\beta \to Y'_{\beta'}$, such that $\varepsilon\beta = \beta'$. Then we have:

PROPOSITION 4.3. *Every S-envelope of an S-group A is equivalent to an S-envelope of A of the form* $(\mathrm{Hom}(M, M), \mu)$.

(Cf. Theorem 5.5 of [13]—in fact, an $S$-envelope $(Y, \beta)$ is equivalent to

$$(\mathrm{Hom}_a(Y, Y),\ L\beta).)$$

The ring $Z_S(A)$ has the following further "classifying" property with respect to $S$-envelopes of $A$:

THEOREM 4.4. *If* $K$ *is a (two-sided) ideal of* $Z_S(A)$ *and* $\pi_K$ *is the natural epimorphism* $Z_S(A) \to Z_S(A)/K$, *then the mapping*

$$K \to (Z_S(A)/K,\ \pi_K\tau | A)$$

*induces a* 1–1 *correspondence between the ideals of* $Z_S(A)$ *and the equivalence classes of S-envelopes of A.*

(Cf. Theorem 6.6 of [13].)

Next we point out that the properties of $Z_S(A)$ given in Theorems 3.2 and 4.1 are equivalent and suffice to characterise the pair $(Z_S(A), \tau | A)$ axiomatically: For this purpose, call an $S$-envelope $(W, \mu)$ of $A$ *universal* if and only if it has the property that, for every $S$-envelope $(Y, \beta)$ of $A$, there exists a unique homomorphism $\nu: W \to Y$ such that $\nu\mu = \beta$. Thus, by Theorem 4.1, $(Z_S(A), \tau | A)$ is a universal $S$-envelope of $A$.

PROPOSITION 4.5. *Every universal $S$-envelope $(W, \mu)$ of $A$ is minimal, i.e. $W_\mu = W$. Any two universal $S$-envelopes $(W_1, \mu_1)$ and $(W_2, \mu_2)$ of $A$ are eqivalent under a unique isomorphism $\varepsilon: W_1 \to W_2$ such that $\varepsilon\mu_1 = \mu_2$.*

(Cf. Theorem 6.2 of [13].)

THEOREM 4.6. *Let $\mu: A \to W$ be a homomorphism of an $S$-group $A$ into the group of units of a ring $W$ with identity, such that $\mu(A)$ generates $W$. Then $(W, \mu)$ is a universal $S$-envelope of $A$ if and only if, for every abelian group $M$, if $\alpha: W \to \mathrm{Hom}(M, M)$ is a ring homomorphism, the mapping*

$$\alpha \to (M, \alpha\mu)$$

*induces a 1–1 correspondence:*

$$\mathrm{Hom}(W, \mathrm{Hom}(M, M)) \leftrightarrow \mathrm{Splitext}_S(A, M).$$

(Cf. Theorem 6.5 of [13].)

Finally, if $B$ is any group, let $B_S$ denote the largest quotient group of $B$ which is an $S$-group, i.e. $B_S = B/{}^S B$ where ${}^S B$ is the normal subgroup of $B$ generated by $\bigcup_{\phi \in \mathrm{Hom}(F, B)} \phi(S)$.

THEOREM 4.7. *The natural epimorphism $B \to B_S$ induces a natural isomorphism:*

$$ZB/[\partial({}^S F)]_B \cong Z_S(B_S).$$

This is analogous to Corollary 8.4 of [13]—a direct proof is as follows:

*Proof.* The natural epimorphism $\varrho: B \to B_S$ induces an epimorphism $\varrho_*: ZB \to ZB_S$, which sends $[\partial({}^S F)]_B$ onto $[\partial({}^S F)]_{B_S}$, since any homomorphism $\phi: F \to B_S$ can be lifted to a homomorphism $\psi: F \to B$ such that $\varrho\psi = \phi$. Further, $\varrho^*$ induces an epimorphism

$$\varrho_*': ZB/[\partial({}^S F)]_B \to ZB_S/[\partial({}^S F)]_{B_S} = Z_S(B_S).$$

To show that $\varrho_*'$ is also a monomorphism, recall that Ker $\varrho_*$ is generated by $\{u - 1 : u \in {}^S B\}$ (cf. [6], say). Further, the proof of Lemma 3.1 shows that $[\partial({}^S F)]_B$ is generated by the set

$$(1 - {}^S B) \cup \bigcup_{\psi \in \mathrm{Hom}(F, B)} \psi_* \partial(S);$$

hence $\operatorname{Ker} \varrho_* \subseteq [\partial({}^S F)]_B$. Then, since $\varrho_*$ sends $[\partial({}^S F)]_B$ onto $[\partial({}^S F)]_{BS}$, if $\varrho_* x \in [\partial({}^S F)]_{BS}$, then $\varrho_* x = \varrho_* y$, where $y \in [\partial({}^S F)]_B$.

Hence: $$x - y \in \operatorname{Ker} \varrho_* \subseteq [\partial({}^S F)]_B, \quad \text{i.e. } x \in [\partial({}^S F)]_B.$$

Therefore $\varrho'_*$ is a monomorphism.

## 5. General $S$-extensions

Any equivalence class of extensions

$$1 \to B \to E \to A \to 1$$

determines a homomorphism

$$\theta : A \to \overline{\operatorname{Aut}} B$$

of $A$ into the group of automorphisms of $B$ modulo inner automorphisms. Let

$$\operatorname{ext}_S^\theta (A, B)$$

denote the set of equivalence classes of $S$-extensions of $B$ by $A$ which determine the homomorphism $\theta$.

Suppose that $1 \to B \to E \to A \to 1$ is an $S$-extension of $B$ by $A$ determining $\theta$, which is given by maps $\eta : A \to \operatorname{Aut} B$ and $\Gamma : A \times A \to B$. Let $f : A' \to A$ be a homomorphism of $S$-groups. Consider the induced maps

$$f^*\eta = \eta f : A' \to \operatorname{Aut} B$$

and

$$f^*\Gamma = \Gamma(f \times f) : A' \times A' \to B.$$

With the aid of Section 2, and a somewhat tedious inductive proof, one has:

Proposition 5.1. *The mapping*

$$(\eta, \Gamma) \to (f^*\eta, f^*\Gamma)$$

*induces a mapping*

$$f^* : \operatorname{ext}_S^\theta (A, B) \to \operatorname{ext}_S^{f^*\theta} (A', B),$$

*which is a* 1–1 *corespondence if $f$ is an isomorphism.*

Now let $(M, \eta)$ be an $S$-module for $A$. Then, by Theorem 2.2(ii) or more directly, it follows that the set of equivalence classes of $S$-extensions of $M$ (with the $A$-module structure $\eta$) by $A$ forms an abelian group

$$\operatorname{Ext}_S^\eta (A, M) \quad \{\text{or } H_S^2 (A, M)\}.$$

If $f : A' \to A$ is a homomorphism of $S$-groups,then the mapping $\Gamma \to f^*\Gamma$ induces a homomorphism

$$f^*: \mathrm{Ext}_S^\eta(A, M) \to \mathrm{Ext}_S^{f^*\eta}(A', M),$$

which is an isomorphism if $f$ is one.

Now let $1 \to B \to E \xrightarrow{\pi} A \to 1$ be an arbitrary (but fixed) $S$-extension of a group $B$ by $A$, with factor set $\Gamma$. Let

$$\mathrm{Ophom}_E(B, M)$$

denote the abelian group of *operator homomorphisms* $B \to M$, i.e. homomorphisms $\alpha$ such that

$$\alpha(ebe^{-1}) = (\pi e) \cdot \alpha(b) \quad [e \in E, b \in B].$$

This group has a subgroup

$$\mathrm{Deriv}_E(B, M)$$

consisting of all restrictions $d|B$ of derivations $d: E \to M$ (via $\pi$).

THEOREM 5.2. *If $\alpha \in \mathrm{Ophom}_E(B, M)$, then the mapping $\alpha \to \{\alpha\Gamma: A \times A \to M\}$ induces a monomorphism*

$$T: \frac{\mathrm{Ophom}_E(B, M)}{\mathrm{Deriv}_E(B, M)} \to \mathrm{Ext}_S^\eta(A, M),$$

*which is independent of the factor set $\Gamma$ chosen for $E$.*

*If $E$ is a free $S$-group, then $T$ is an* isomorphism.

When $S = \{1\}$, this theorem reduces to one of Eilenberg and MacLane [5], [15].

If $M$ is an $S$-module under trivial action by $A$, one obtains the group

$$\mathrm{Ext}_S^c(A, M)$$

of equivalence classes of *central* $S$-extensions of $M$ by $A$. Then $E$ also acts trivially on $M$, and an operator homomorphism $B \to M$ is a homomorphism which sends every commutator $[e, b]$ $(e \in E, b \in B)$ to zero. Further, a derivation is now just a homomorphism. Hence, in this case, there is a monomorphism

$$\frac{\mathrm{Hom}(B, [E, B] \to M, 0)}{\mathrm{Hom}(E, M)|B} \to \mathrm{Ext}_S^c(A, M),$$

which is an isomorphism if $E$ is a free $S$-group. (When $S = \{1\}$, this reduces to a corollary of Eilenberg and MacLane [5].)

Finally, when $S$ defines the variety of abelian groups, and $E = F_a$ is a free abelian group, Theorem 5.2 reduces to another theorem of Eilenberg and MacLane [4], [5]:

$$\mathrm{Ext}_{\mathrm{Abel}}(A, M) \cong \frac{\mathrm{Hom}\,(B, M)}{\mathrm{Hom}\,(F_a, M)\,|\,B}.$$

Theorem 5.2 will be deduced from some further propositions:

Suppose that $A$ also acts on a (not necessarily abelian) group $N$ by means of a fixed map $\nu: A \to \mathrm{Aut}\,N$ arising from some $S$-extensions of $N$ by $A$.

Define an *extended operator homomorphism* $B \to N$ to be a map $\alpha: E \to N$ such that:

(i) $\alpha(ebe^{-1}) = \alpha(e) \cdot [(\pi e) \cdot \alpha(b)]$, and

(ii) $\alpha(be) = \alpha(b)\,\alpha(e) \quad [e \in E, b \in B]$.

Let $\mathrm{Ophom}_E^*(B,N)$ denote the set of extended operator homomorphisms $B \to N$. This set has the subset $\mathrm{Deriv}\,(E,N)$ of all *derivations* $E \to N$, i.e. maps $d$ satisfying $d(uv) = d(u)\,[\pi u \cdot d(v)]$; the set of restrictions to $B$ of derivations $E \to N$ will again be denoted by $\mathrm{Deriv}_E\,(B,N)$.

Now let $\theta: A \to \overline{\mathrm{Aut}}\,N$ be the homomorphism induced by $\nu: A \to \mathrm{Aut}\,N$. If $\alpha \in \mathrm{Ophom}_E^*\,(B,N)$, let $N_\alpha = \mathrm{Im}\,\alpha\,|\,B$.

THEOREM 5.3. (i) *If* $\alpha \in \mathrm{Ophom}_E^*(B,N)$, *then* $\alpha\Gamma: A \times A \to N$ *defines an element of* $\mathrm{ext}_S^\theta(A, N_\alpha)$.

*If $E$ is a free $S$-group, then every element of* $\mathrm{ext}_S^\theta(A,N)$ *can be defined by a factor set of the form* $\alpha\Gamma$ *as above.*

(ii) *Suppose that* $\nu: A \to \mathrm{Aut}\,N$ *is a homomorphism. Then, if* $d \in \mathrm{Deriv}\,(E,N)$, $d\Gamma$: $A \times A \to N$ *defines the split extension class in* $\mathrm{ext}_S^\theta\,(A, N_d)$. *Conversely, if* $\alpha \in \mathrm{Ophom}_E^*\,(B,N)$ *is such that* $\alpha\Gamma$ *defines the split extension class in* $\mathrm{ext}_S^\theta\,(A,N_\alpha)$, *then* $\alpha\,|\,B \in \mathrm{Deriv}_E\,(B,N)$.

*In each case, the class defined by* $\alpha\Gamma$ *is independent of the factor set* $\Gamma$ *chosen for* $E$.

The proof is analogous to an argument of Eilenberg and MacLane (cf. [14], for example):

*Proof.* (i) Suppose that the given factor set $\Gamma: A \times A \to B$ corresponds to a cross section function $\gamma: A \to E$. Let $\alpha \in \mathrm{Ophom}_E^*\,(B,N)$. Then it is easily verified that $\alpha\Gamma$ and the map $\nu_\alpha: A \to \mathrm{Aut}\,N_\alpha$, given by: $\nu_\alpha(a) = \langle \alpha(\gamma_a) \rangle\, \nu(a)\,|\,N_\alpha$, determine an element of $\mathrm{ext}^\theta\,(A, N_\alpha)$.

To show that this element lies in $\mathrm{ext}_S^\theta\,(A, N_\alpha)$, let $E_\alpha$ be the extension group of pairs $(n,a)$ $[n \in N_\alpha,\, a \in A]$ defined by $\nu_\alpha$ and $\alpha\Gamma$. Let $\bar{\phi}: F \to E$ be a homomorphism such that

$$\bar{\phi}(x_i) = (m_i, \phi x_i) \quad [m_i \in B,\ \phi x_i \in A],$$

and let $\alpha_*\bar{\phi}: F \to E_\alpha$ be the homomorphism such that $\alpha_*\bar{\phi}(x_i) = (\alpha m_i, \phi x_i)$. If $u \in F$, and $\bar{\phi}(u) = (W_{\bar{\phi}}(u), \phi u)$, it is easily shown by induction that $\alpha_*\bar{\phi}(u) = (\alpha W_{\bar{\phi}}(u),\ \phi u)$.

For $u \in S$, these formulae show that $E_\alpha$ is an $S$-extension of $N_\alpha$ by $A$.

Now suppose that $E$ is a free $S$-group of the form $F'_S = F'/{}^S F'$, where $F'$ is free on free generators $\{x'_i\}$. Let $\varrho: F' \to A$ be the composition of the natural epimorphism $F' \to F'_S$ with $\pi: F'_S \to A$. If $\Lambda: A \times A \to N$ is the factor set of an $S$-extension of $N$ by $A$ in which $A$ acts on $N$ by the map $\nu$, consider the function $\Lambda_\varrho: F' \to N$ such that:

$$\Lambda_\varrho(x'_i) = 1, \quad \Lambda_\varrho(uv) = \Lambda_\varrho(u)\ [\varrho u \cdot \Lambda_\varrho(v)]\ \Lambda_{\varrho u, \varrho v}.$$

Then, by Corollary 2.4, $\Lambda_\varrho({}^S F') = \{1\}$, and so $\Lambda_\varrho$ defines a function $\Lambda_\pi: F'_S \to N$ satisfying:

$$\Lambda_\pi(uv) = \Lambda_\pi(u)\ [\pi u \cdot \Lambda_\pi(v)]\ \Lambda_{\pi u, \pi v}.$$

Then $\alpha = \Lambda_\pi$ is an extended operator homomorphism $B \to N$, and, since

$$\Gamma_{a,b} = \gamma_a \gamma_b \gamma_{ab}{}^{-1} \quad (a, b \in A),$$

one has

$$\alpha\Gamma_{a,b} = \mu_a(a \cdot \mu_b)\ \Lambda_{a,b}\mu_{a,b}^{-1},$$

where $\mu = \alpha\gamma: A \to N$. This expression for $\alpha\Gamma$ leads to the relation

$$\nu_\alpha(a)\ \nu_\alpha(b) = \langle \alpha\Gamma_{a,b} \rangle\ \nu_\alpha(ab) \quad [a, b \in A]$$

in Aut $N$. Hence $\nu_\alpha$ and $\alpha\Gamma$ define an extension of the entire group $N$ by $A$ which is equivalent to that defined by $\nu$ and $\Lambda$.

(ii) Let $E$ be an arbitrary $S$-extension of $B$ by $A$ again, and suppose now that $\nu: A \to \operatorname{Aut} N$ is a homomorphism. If $d: E \to N$ is a derivation, then, since

$$\Gamma_{a,b} = \gamma_a \gamma_b \gamma_{ab}^{-1} \quad (a, b \in A),$$

one has

$$d\Gamma_{a,b} = d(\gamma_a)\ [a \cdot d(\gamma_b)]\ d(\gamma_{ab})^{-1}.$$

Therefore $\nu_d$ and $d\Gamma$ define an extension of $N_d$ by $A$ equivalent to the split extension defined by $\nu$. Conversely, suppose that $\nu_\alpha$ and $\alpha\Gamma$ define the split extension class in $\operatorname{ext}_S^\theta (A, N_\alpha)$. Then there exists a map $\psi: A \to N_\alpha$ such that

$$\nu_\alpha(a) = \langle \psi_a \rangle\ \nu(a) \mid N_\alpha$$

and

$$\alpha\Gamma_{a,b} = \psi_a(a \cdot \psi_b)\ \psi_{ab}^{-1} \quad [a, b \in A].$$

Since $\nu_\alpha(a) = \langle \alpha(\gamma_a) \rangle\ \nu(a) \mid N_\alpha$ also, $\psi_a = \alpha(\gamma_a) k_a$ where $k_a$ is an element of the centre of $N_\alpha$. Using this fact and the expression for $\alpha\Gamma$, one obtains a derivation $d: E \to N$ by setting $d(m, a) = \alpha(m)\psi_a$ $[m \in B, a \in A]$. Then $\alpha \mid B = d \mid B$.

Finally, a simple calculation involving Proposition 1.1 shows that, in all cases, the extension class defined by $\alpha\Gamma$ is independent of the factor set $\Gamma$ chosen for $E$.

In order to deduce Theorem 5.2, now suppose that $(N, \nu)$ is the $S$-module $(M, \eta)$ for $A$. Then there is a monomorphism of abelian groups

$$\tau : \mathrm{Ophom}_E\,(B, M) \to \mathrm{Ophom}_E^*\,(B, M)$$

given by $(\tau\alpha)\,(m, a) = \alpha(m)$ $[m \in B, a \in A]$; to see that $\tau$ is injective, one can consider the epimorphism

$$\mathrm{res}\colon \mathrm{Ophom}_E^*\,(B, M) \to \mathrm{Ophom}_E\,(B, M)$$

given by restriction, which satisfies

$$\mathrm{res} \circ \tau = \text{identity}.$$

If $\alpha \in \mathrm{Ophom}_E\,(B, M)$, then $\eta_{\tau\alpha}$ coincides with $\eta$, and the (additive) map $\alpha \to \{\alpha\Gamma\} = \{(\tau\alpha)\Gamma\}$ sends $\alpha$ to an $S$-extension of the entire group $M$ by $A$:

To show this, consider the function $\Gamma_\phi : F \to B$ $(\phi \in \mathrm{Hom}\,(F, A))$ of Section 1. In the general case when $N$ is not necessarily abelian, it is easily verified by induction that

$$\beta \circ \Gamma_\phi = (\beta\Gamma)_\phi : F \to N$$

for any $\beta \in \mathrm{Ophom}_E^*\,(B, N)$.

Therefore, by Theorem 2.1,

$$(\alpha\Gamma)_\phi(S) = \alpha\Gamma_\phi(S) = \{0\};$$

hence, by Theorem 2.2, $\alpha\Gamma$ defines an $S$-extension of $M$ by $A$.

Bearing these facts in mind, the proof of Theorem 5.3 (ii) now shows that the map $\alpha \to \{\alpha\Gamma\}$ induces a homomorphism

$$T' : \mathrm{Ophom}_E\,(B, M) \to \mathrm{Ext}_S^\eta\,(A, M),$$

whose kernel is $\mathrm{Deriv}_E\,(B, M)$. When $E$ is a free $S$-group, Theorem 5.3(i) implies that $T'$ is an epimorphism. This proves Theorem 5.2.

It can be deduced from Theorem 5.2 that $\mathrm{Ext}_S^\eta\,(A, M)$ is a covariant functor of the $S$-module $(M, \eta)$. Also, it is readily verified that the monomorphism $T$ of that theorem is a natural transformation of functors.

## 6. Reduction to abelian kernels

Consider an arbitrary (but fixed) extension

$$1 \to B \to G \xrightarrow{\sigma} A \to 1$$

determining a homomorphism $\theta : A \to \overline{\mathrm{Aut}}\ B$. If $Z_B$ denotes the centre of $B$, let

$$1 \to Z_B \to E \xrightarrow{\pi} A \to 1$$

be any extension of $Z_B$ by $A$ determining the homomorphism $\theta$: $A \to \operatorname{Aut} Z_B$ induced by $\theta$.

Let $G \otimes E$ denote the subgroup of all pairs $(g, e) \in G \times E$ such that $\sigma g = \pi e$, and let $N$ be the normal subgroup of all pairs $(z, z^{-1})$, $z \in Z_B$. If

$$G_E = G \otimes E/N,$$

Eilenberg and MacLane [5] show that one obtains an extension

$$1 \to B \to G_E \to A \to 1,$$

in which $B$ maps into $G_E$ by $b \to (b, 1)N$ and $G_E$ maps onto $A$ by the common projection, which determines the given homomorphism $\theta$. They prove that the map $E \to G_E$ induces a 1–1 correspondence:

$$\operatorname{Ext}^\theta (A, Z_B) \leftrightarrow \operatorname{ext}^\theta (A, B),$$

(which is not natural in general).

As a corollary to their theorem, we obtain:

THEOREM 6.1. *If there exists an S-extension*

$$1 \to B \to G \xrightarrow{\sigma} A \to 1$$

*determining the homomorphism*

$$\theta : A \to \overline{\operatorname{Aut}} B,$$

*then the mapping $E \to G_E$ induces a 1–1 correspondence:*

$$\operatorname{Ext}^\theta_S (A, Z_B) \leftrightarrow \operatorname{ext}^\theta_S (A, B).$$

By using the previous result of Eilenberg and MacLane, this theorem can be deduced with the aid of Section 2. Since an analogous theorem (Theorem 4.1) in Chapter II is proved in a similar way by a method analogous to that of Eilenberg and MacLane, this approach will be omitted here.

As an alternative method, one can use the analogue of an approach used by MacLane [16] for associative rings:

Following Baer [1], consider the *graph* $\Theta$ of $\theta : A \to \overline{\operatorname{Aut}} B$, defined to be the subgroup of all pairs $(x, \alpha) \in A \times \operatorname{Aut} B$ for which $\alpha \in \theta(x)$. Define an epimorphism

$$\psi : G \to \Theta$$

with kernel $Z_B$ by: $g \to (\sigma g, \langle g \rangle \mid B)$.

Given another $S$-extension $1 \to B \to G' \xrightarrow{\sigma'} A \to 1$ determining $\theta$, let $(G, G')$ be the subgroup of all pairs $(g, g') \in G \times G'$ such that $\psi g = \psi' g' \in \Theta$, and let $K$ be the normal subgroup of all pairs $(b, b)$ $[b \in B]$. If

$$E(G, G') = (G, G')/K,$$

then the monomorphism $b \to (1, b)K$ of $Z_B$ into $E(G, G')$ and the epimorphism $(g, g')K \to \sigma g = \sigma' g'$ of $E(G, G')$ onto $A$ define an $S$-extension

$$1 \to Z_B \to E(G, G') \to A \to 1.$$

Furthermore, the action of $A$ on $Z_B$ in this extension is that determined by $\theta$, since the extensions $G$ and $G'$ determine $\theta$.

We now prove that *the mappings $E \to G_E$ and $G' \to E(G, G')$ define inverse mappings:*

$$\mathrm{Ext}_S^\theta(A, Z_B) \leftrightarrow \mathrm{ext}_S^\theta(A, B).$$

In order to do this, observe that a typical element of $E(G, G_E)$ can be written in the form $(g, (g, e)N)K$, where $g \in G, e \in E$, and that this form specifies $e \in E$ uniquely. An equivalence homomorphism $E(G, G_E) \to E$ is then obtained by mapping $(g, (g, e)N)K$ to $e$.

Conversely, a typical element of $G_{E(G,G')}$ can be written in the form $(g, (g, h)K)N$, where $g \in G$, $h \in G'$, and this form specifies $h \in G'$ uniquely. The mapping $(g, (g, h)K)N \to h$ then defines an equivalence homomorphism from $G_{E(G,G')}$ into $G'$.

## II. Extensions in varieties of linear algebras

### 1. Preliminary definitions and discussion

All algebras considered will be, not necessarily associative, linear algebras over a *fixed* commutative ring $K$ with unit. Since most of the concepts of Chapter I, § 1, have obvious direct translations within the category of algebras and homomorphisms of algebras, we shall avoid repetition and take as understood such concepts as *$S$-algebra* (where $S$ is a subset of a free (non-associative) algebra), *extension, $S$-extension, the variety $V_S$ of $S$-algebras*, and so on; since the following sections will deal exclusively with algebras, no confusion should arise from the use of notations identical with those used in Chapter I.

In order to discuss $S$-extensions, we require the notion of an *$S$-bimultiplication* $\sigma$ of an $S$-algebra $A$:

This is a pair of linear maps $a \to \sigma a$, $a \to a\sigma$ of $A$ into itself with the following property: if $\hat{A}$ is the quotient of the "*free product*" of $A$ with the free algebra on $\{\sigma\}$ (i.e. the quotient of the free algebra on the set $A \cup \{\sigma\}$ given by the relations identifying the sub-algebra generated by the set $A$ with the algebra $A$) given by the relations identifying $\sigma a$ and $a\sigma$ $(a \in A)$ with the images of $a$ under the given maps, then $A$ is an $S$-algebra.

(The notion of a bimultiplication was introduced by Hochschild [9], and used by MacLane [16], in the case when $S$ defines the variety of associative algebras. When $S$ defines the variety of Lie algebras, an $S$-bimultiplication is just a self-derivation of the algebra considered.)

For each element $c$ in an $S$-algebra $A$, an *inner* $S$-bimultiplication $\mu_c$ is defined by the mappings $a \to ca$, $a \to ac$. The set of inner $S$-bimultiplications of $A$ forms a submodule $\mu(A)$ of the module $M(A)$ of all $S$-bimultiplications of $A$ (under the point-wise operations).

Let $\gamma: A \to E$ be a *cross section* mapping of an $S$-extension $0 \to B \to E \xrightarrow{\pi} A \to 0$. Since $B$ is an ideal in $E$, for each $e \in E$, $\mu_e \,|\, B \in M(B)$ and, if $\pi e = \pi e'$ then $\mu_e - \mu_{e'} \in \mu(B)$. Hence the mapping $a \to \mu_{\gamma_a}$ induces a mapping

$$\theta: A \to M(B)/\mu(B),$$

which is linear, since, for $a, b \in A, \lambda \in K$, $\gamma(a+b) - \gamma(a) - \gamma(b)$ and $\lambda\gamma(b) - \gamma(\lambda b) \in B$. If $B$ is a *zero* algebra, i.e. all products are zero, this gives a linear map $\theta: A \to M(B)$, which makes $B$ into an "$S$-bimodule" for $A$ (an idea introduced by Eilenberg [3], for the case of algebras characterized by multilinear identities, and considered further in [11], [12], [17] and [13]):

An *$S$-bimodule* for $A$ is a module $M$ together with linear maps $R, L: A \to \mathrm{Map}(M, M)$ such that the corresponding split extension of $M$ (as a zero algebra) by $A$ is an $S$-extension. (It will be convenient to let $\mathrm{Map}(X, Y)$ denote the set of linear maps $X \to Y$, while $\mathrm{Hom}(X, Y)$ will denote the set of (algebra) homomorphisms $X \to Y$.)

Define the *bicentre* $K_C$ of an algebra $C$ to be the set of all $b \in C$ such that $\mu_b = 0$, i.e. $bx = xb = 0$ for all $x \in C$. (This term has been used by MacLane [16] for associative rings.) Then, in the general case, the above mapping $\theta: A \to M(B)/\mu(B)$ defines operations of $A$ on $K_B$ which turn it into an $S$-bimodule for $A$.

Henceforth, unless otherwise stated, we shall consider *only* extensions which are *linearly split*, i.e. which admit a linear cross section mapping. (If one considers extensions by algebras which are free as modules, e.g. if $K$ is a field, then this is no restriction.) In this case, if $\gamma: A \to E$ is a linear cross section of an extension $0 \to B \to E \xrightarrow{\pi} A \to 0$, we obtain a pair of linear mappings

$$R, L: A \to \mathrm{Map}(B, B)$$

by setting

$$R_a(m) = m\gamma_a = m \cdot a, \quad L_a(m) = \gamma_a m = a \cdot m \quad [a \in A, m \in B].$$

Also we obtain a bilinear *factor set* mapping

$$\Gamma: A \times A \to B$$

as the deviation from multiplicativity of $\gamma$:

$$\Gamma_{a,b}=\gamma_a\gamma_b-\gamma_{ab} \quad (a,b\in A).$$

Since $\gamma$ is linear, $\Gamma$ is *normalized*, i.e. $\Gamma_{a,0}=\Gamma_{0,a}=0$ $(a\in A)$.

We have: $$(m+\gamma_a)\ (n+\gamma_b)=(mn+a\cdot n+m\cdot b+\Gamma_{a,b})+\gamma_{ab}.$$

Conversely, given linear mappings $R,L:A\rightarrow\mathrm{Map}(B,B)$ and a bilinear mapping $\Gamma:A\times A\rightarrow B$, one can define an extension of $B$ by $A$ by letting $E$ be the module of all pairs $(m,a)$ $[m\in B, a\in A]$ with multiplication defined by:

$$(m,a)\ (n,b)=(mn+a\cdot n+m\cdot b+\Gamma_{a,b},ab).$$

Furthermore, this extension is equivalent to any extension giving rise to the given mappings in the above manner. It will frequently be convenient to describe any extension algebra $E$ as an algebra of pairs in this way; then $\gamma:A\rightarrow E$ will always denote the cross section $a\rightarrow(0,a)$.

THEOREM 1.1. *Two extensions $E$ and $E'$ of $B$ by $A$, given by mappings $R, L, R', L'$: $A\rightarrow\mathrm{Map}(B,B)$ and $\Gamma,\Gamma':A\times A\rightarrow B$ respectively, are equivalent if and only if there exists a linear mapping $\psi:A\rightarrow B$ such that:*

(i) $R'_a=R_a+R_{\psi_a},\quad L'_a=L_a+L_{\psi_a}\quad (a\in A)$,

*where $R_m$ and $L_m$ denote the right and left multiplications defined by $m\in B$,*

(ii) $\Gamma'_{a,b}=\Gamma_{a,b}+(\delta\psi)\ (a,b)$,

*where* $(\delta\psi)\ (a,b)=L_a(\psi_b)-\psi_{ab}+R_b(\psi_a)+\psi_a\psi_b$ $[a,b\in A]$.

*Proof*. Suppose that $\gamma:A\rightarrow E$ and $\gamma':A\rightarrow E'$ are linear cross sections to which the given mappings correspond. If there exists an equivalence homomorphism $\tau:E\rightarrow E'$, then $\beta=\tau\gamma:A\rightarrow E'$ is a new cross section of $E'\rightarrow A$, and one can define a linear mapping $\psi:A\rightarrow B$ of the required kind by:

$$\psi_a=\gamma'_a-\beta_a \quad (a\in A).$$

Conversely, if there exists a linear mapping $\psi:A\rightarrow B$ satisfying the given relations, then the mapping $m+\gamma_a\rightarrow m+\gamma'_a-\psi_a$ defines an equivalence homomorphism $\tau:E\rightarrow E'$.

COROLLARY 1.2. (*Eilenberg* [3].) *The set of equivalence classes of (linearly split) extensions of a zero algebra $M$ by $A$, corresponding to given mappings $R,L:A\rightarrow\mathrm{Map}(M,M)$ is in 1–1 correspondence with the set of bilinear mappings $A\times A\rightarrow M$ modulo* "coboundaries" *i.e. mappings of the form $\delta\psi$ as above.*

Let $0 \to B \to E \to A \to 0$ be an extension with factor set $\Gamma$. Any algebra homomorphism $\phi: G \to A$ gives rise to an induced factor set $\phi^*\Gamma = \Gamma(\phi \times \phi): G \times G \to B$ which defines an extension $\phi^* E$ of $B$ by $G$ in conjunction with the mappings $\phi^* R = R\phi$, $\phi^* L = L\phi: G \to \mathrm{Map}(B, B)$. If $G = F$ is free, this extension must *split*, i.e. it must have a cross section which is a homomorphism. In fact, any mapping $x_i \to (\psi_{x_i}, x_i)$ of a set of free generators $\{x_i\}$ of $F$ into $\phi^* E$ can be extended uniquely to a homomorphism $v \to (\psi_v, v)$ of $F$ into $\phi^* E$, and one has $\phi^*\Gamma = -\delta\psi$. It will be convenient to let

$$\Gamma_\phi: F \to B$$

denote the unique linear mapping such that:

$$\Gamma_\phi(x_i) = 0, \qquad \Gamma_\phi(uv) = \phi u \cdot \Gamma_\phi(v) + \Gamma_{\phi u, \phi v} + \Gamma_\phi(u) \cdot \phi v + \Gamma_\phi(u)\Gamma_\phi(v).$$

(When $B$ is a zero algebra, such a function is defined in [3] for polynomials which are multilinear in the free generators.)

Finally we mention some further concepts to be used later. For any algebra $A$, $G(A)$ will denote the *tensor algebra* $K + \sum_n \otimes_K^n A^*$ on the module $A^* = A \oplus A'$, where $A'$ is the *opposite* algebra to $A$, i.e. the module $A$ with multiplication anti-isomorphic to that of $A$. If $F$ is the free algebra on a set $\{x_i\}$, we shall make use of the linear mappings

$$D_i: F \to G(F)$$

given by: $$D_i(x_j) = \delta_{ij}, \qquad D_i(uv) = D_i(u) \otimes v + D_i(v) \otimes u',$$

where, if $a$ is an element of an algebra $A$, $a$ and $a'$ will denote the elements $(a, 0)$ and $(0, a)$ respectively of $A^* = A \oplus A'$. If $T$ is any subset of $F$, let

$$D(T) = \bigcup_i D_i(T).$$

These maps, which are analogous to the Fox derivatives, arise conceptually in the following way:

By an argument similar to one used in Chapter I, § 1, one can prove that *any map of a set of free generators of $F$ into a bimodule $M$ for $F$ can be uniquely extended to a derivation $F \to M$.*

Now let $M$ be the bimodule $G(F)$ for $F$ defined by:

$$u \cdot m = m \otimes u', \quad m \cdot u = m \otimes u \quad [u \in F, m \in G(F)].$$

Letting $h(x_j) = \delta_{ij}$, we obtain a unique derivation $D_i: F \to G(F)$ such that $D_i(x_j) = \delta_{ij}$, and

$$D_i(uv) = u \cdot D_i(v) + D_i(u) \cdot v = D_i(v) \otimes u' + D_i(u) \otimes v.$$

(The mappings $D_i$ were introduced in [13].)

If $\phi: A \to B$ is a linear mapping, $\phi_*: G(A) \to G(B)$ will denote the homomorphism induced by $\phi$. Also, if $R, L: A \to \mathrm{Map}(B, B)$ are given linear maps,

$$\langle R | L \rangle : G(A) \to \mathrm{Map}(B, B)$$

will denote the (unique) homomorphism extending the map $R|L: A^* \to \mathrm{Map}(B, B)$ given by: $R|L(a, b) = R_a + L_b$. *Here* $\mathrm{Map}(B, B)$ is regarded as an associative algebra with respect to the *particular* multiplication $\circ$ given as follows: if $\alpha, \beta: B \to B$ are maps, $\alpha \circ \beta: B \to B$ is the composition $B \xrightarrow{\alpha} B \xrightarrow{\beta} B$.

Finally, we remark that we shall regard modules as zero algebras whenever this is convenient.

## 2. Identities on factor sets and bimodules

Let $S$ be a subset of a free algebra $F$ on a set $\{x_i\}$. Let $0 \to B \to E \xrightarrow{\pi} A \to 0$ be an extension of an algebra $B$ by an $S$-algebra $A$, given by mappings $R$, $L: A \to \mathrm{Map}(B, B)$ and a factor set $\Gamma: A \times A \to B$.

THEOREM 2.1. *If $E$ is an $S$-algebra, then*

$$\Gamma_\phi(S) = \{0\}$$

*for every homomorphism $\phi: F \to A$.*

THEOREM 2.2. *If $B$ is a zero algebra, then $E \in V_S$ if and only if:*

(i) $\langle R | L \rangle \phi_* D(S') = \{0\}$, *and*

(ii) $\Gamma_\phi(S') = \{0\}$,

*for one set $S'$ of identities equivalent to $S$, and every homomorphism $\phi: F \to A$.*

Part (ii) of Theorem 2.2 was given essentially by Eilenberg [3] in the case of multilinear identities; part (i) appeared essentially in [13]. These propositions can be deduced from a lemma:

LEMMA 2.3. *Let $\bar{\phi}: F \to E$ and $\phi: F \to A$ be homomorphisms such that $\bar{\phi}(x_i) = (m_i, \phi x_i)$. If $u \in F$, then:*

(i) $\bar{\phi}(u) = (X_{\bar{\phi}}(u) + \Gamma_\phi(u),\ \phi u)$,

*where $X_{\bar{\phi}}(u)$ is an element of $B$ which is zero if each $m_i$ is zero;*

(ii) *if $\alpha: B \to M$ is an* operator *(algebra) homomorphism of $B$ into a bimodule $M$ for $A$, i.e. a homomorphism such that*

$$\alpha(em) = \pi e \cdot \alpha(m), \quad \alpha(me) = \alpha(m) \cdot \pi e \quad [e \in E,\ m \in B],$$

*then*
$$\alpha(X_{\bar\phi}(u)) = \sum_i (\langle R' | L' \rangle \phi_* D_i u)\, \alpha(m_i),$$

*where $R', L' : A \to \mathrm{Map}(M, M)$ are the maps defining the bimodule structure of $M$.*

In view of the close analogy with Chapter I, proofs of most propositions in Chapter II will be omitted. For propositions, such as the previous lemma, concerning arbitrary elements $u$ of $F$, it is sufficient, by linearity, to consider monomials $u$. One can then use an inductive method as follows:

After verifying a statement for the free generators $x_i$, one supposes it true for any monomial $u = x_{i_1} \dots x_{i_k}$ (bracketed in some order) of degree $k < n$.

Then one proves the statement for any product $w = uv$ of degree $n$.

Let $M$ be a bimodule for an algebra $A$, and let $\phi \in \mathrm{Hom}(F, A)$. Given any map $h$: $\{x_i\} \to M$, a method similar to one of Chapter I shows that the unique derivation $d: F \to M$ (via $\phi$) extending $h$ is given by

$$d(u) = \sum_i (\langle R | L \rangle \phi_* D_i u)\, h(x_i).$$

If $M$ is the bimodule $G(A)$ on which $A$ acts by

$$a \cdot m = m \otimes a', \quad m \cdot a = m \otimes a \quad [a \in A, m \in G(A)],$$

we get
$$d(u) = \sum_i h\,(x_i) \otimes \phi_* D_i(u).$$

(For $m, w \in G(A)$, it is easily shown by induction (considering $w$) that $(\langle R | L \rangle w)m = m \otimes w$.)

Now let $F_o$ be the subset of $F$ consisting of polynomials which, for each $x_i$, are homogeneous of the same degree $n_i$ in $x_i$ in each term. Then any map of $\{x_i\}$ into a bimodule $M$ for $F$ can be uniquely extended to a "derivation" $F_o \to M$, i.e. a mapping $d$ such that $d(uv) = u \cdot d(v) + d(u) \cdot v$. If we consider the "derivation" $u \to (\deg u)u$ of $F_o$ into $G(F)$, and the derivation of $F$ into $G(F)$ extending the map $x_i \to x_i$, we obtain "*Euler's relation*":

$$\sum_i x_i \otimes D_i(u) = (\deg u) u \quad (u \in F_o).$$

Again, if $D' : A \to M$ is a derivation, we obtain a "*chain rule of differentiation*":

$$D'(\phi u) = \sum_i (\langle R | L \rangle \phi_* D_i u) D'(\phi x_i).$$

When $A = F'$ is a free algebra on free generators $\{x'_k\}$, and $M$ is $G(F')$, this gives the "*chain rule*":

$$D_{x'_k}(\phi u) = \sum_i D_{x'_k}(\phi x_i) \otimes \phi_* D_i(u).$$

Next, if $F'$ is another free algebra, let ${}^SF'$ denote the ideal of $F'$ generated by $\bigcup_\psi \psi(S)$ $[\psi \in \mathrm{Hom}(F, F')]$.

COROLLARY 2.4. *Suppose $E \in V_S$, and $\phi \in \mathrm{Hom}(F', A)$. Then:*

(i) $\Gamma_\phi({}^SF') = \{0\}$;

(ii) *if $B$ is a zero algebra, then*

$$\langle R | L \rangle \phi_* D({}^SF') = \{0\}.$$

When $S$ is given, the maps $D_i$ provide a mechanical means for computing defining conditions for $S$-bimodules [13], while the inductive specification of the maps $\Gamma_\phi$ of Section 1 provides a mechanical means for computing defining conditions for "*$S$-2-cocycles*" (cf. Eilenberg [3]).

In connection with cohomology, we remark that an $S$-algebra $A$ has a *universal $S$-envelope* or "*enveloping algebra*" $G_S(A)$, defined to be the quotient of $G(A)$ by the ideal generated by the subset $\bigcup_\phi \phi_* D(S)$ $[\phi \in \mathrm{Hom}(F, A)]$, with properties directly analogous to those of the functor $Z_S(-)$ (cf. [13]). As remarked in [13], the enveloping algebras of associative and Lie algebras have been used to define *homology* and *cohomology* groups for such algebras. It was suggested by Jacobson in [12] that one might do the same for Jordan algebras. In general, the cohomology groups $H^n(G_S(A), M)$ of the "supplemented" algebra $G_S(A)$, in the sense of Cartan–Eilenberg [2], provide an exact connected sequence of covariant functors of the $S$-bimodule $M$ for $A$, which are zero for $n \geq 1$ when $M$ is $A$-injective. Similar remarks apply to the homology groups of $G_S(A)$. However, classically the second cohomology group provides a classification of extensions by $A$ realizing a given module structure. In general, the second cohomology groups of $G_S(A)$ would not do this. For example, if $V_S$ is the variety of zero algebras, $G_S(A) \cong K$ for all $A$.

In a similar way one might ask whether there exists a cohomology theory for $S$-groups, which is characteristic of the variety of $S$-groups. In this case, one type of answer would be obtained by considering ordinary cohomology groups of the rings $Z_S(A)$. However, again, the second cohomology groups would not always provide a classification of $S$-extensions: For example, for the variety of abelian groups, $Z_S(A) \cong Z$ for all $A$.

Thus this approach has certain limitations, perhaps.

However, Gerstenhaber [8] has sketched a cohomology theory for $S$-algebras which does include (not necessarily linearly split) $S$-extensions of algebras. With regard to both approaches, if is of interest to remark that the classification properties of $G_S(A)$, and standard module theory, show that *the category of $S$-bimodules for $A$ always contains "sufficiently many" injectives and projectives.* A similar remark applies to *categories of $S$-modules for $S$-groups.*

## 3. General $S$-extensions

By Section 1, any equivalence class of $S$-extensions $0 \to B \to E \to A \to 0$ determines a linear mapping $\theta: A \to M(B)/\mu(B)$. Let

$$\mathrm{ext}_S^\theta\,(A, B)$$

denote the set of equivalence classes of $S$-extensions of $B$ by $A$ which determine $\theta$.

Suppose that $0 \to B \to E \to A \to 0$ is an $S$-extension of $B$ by $A$ determining $\theta$, which is given by maps $R, L: A \to \mathrm{Map}\,(B, B)$ and a factor set $\Gamma: A \times A \to B$. Let $f: \bar{A} \to A$ be a homomorphism of $S$-algebras. Consider the maps $f^*R = Rf$, $f^*L = Lf: \bar{A} \to \mathrm{Map}(B, B)$ and $f^*\Gamma = \Gamma(f \times f): \bar{A} \times \bar{A} \to B$.

PROPOSITION 3.1. *The mapping* $(R, L, \Gamma) \to (f^*R, f^*L, f^*\Gamma)$ *induces a mapping*

$$f^*: \mathrm{ext}_S^\theta(A, B) \to \mathrm{ext}_S^{f^*\theta}(\bar{A}, B).$$

*If $f$ is an isomorphism, then $f^*$ is a* 1–1 *correspondence.*

Now let $(M, R, L)$ be an $S$-bimodule for $A$. The maps $R, L: A \to \mathrm{Map}\,(M, M)$ define a linear mapping $\eta$ of $A$ into the module of $S$-bimultiplications of $M$. By Theorem 2.2(ii), one sees that the set of equivalence classes of $S$-extensions of $M$ by $A$ determining $\eta$ forms a module

$$\mathrm{Ext}_S^\eta(A, M) \quad \{\text{or } H_S^2(A, M\}.$$

If $f: \bar{A} \to A$ is a homomorphism of $S$-algebras, then the mapping $\Gamma \to f^*\Gamma$ induces a homomorphism

$$f^*: \mathrm{Ext}_S^\eta(A, M) \to \mathrm{Ext}_S^{f^*\eta}(\bar{A}, M),$$

which is an isomorphism if $f$ is one.

Corresponding to any algebra $A$, there is a natural bimodule, the *regular bimodule* $A_0$ for $A$, defined to be the *module* $A$ with the bimodule structure given by the right and left multiplication maps $R, L: A \to \mathrm{Map}\,(A, A)$. Suppose, for a moment, that $K$ is a field and that the regular bimodule $A_0$ for an $S$-algebra $A$ is always an $S$-bimodule (e.g. this is always so if $K$ is infinite). Then we remark that $\mathrm{Ext}_S\,(A, A_0)$ has an interesting alternative interpretation as the module of "*infinitesimal deformations*" of $A$ in $V_S$, in the sense of Gerstenhaber's theory of "*deformations of algebras*" [7]. We shall not go into this statement further, since, for general $S$, it is at least alluded to in [7], while, for particular choices of $S$, it is explicitly proved by Gerstenhaber. (It can be proved with the aid of Theorem 2.2(ii).)

Now consider an arbitrary (but fixed) $S$-extension $0 \to B \to E \xrightarrow{\pi} A \to 0$ with factor set $\Gamma$. Let $M$ be an $S$-bimodule for $A$ defined by a fixed map $\eta$ as above. Consider the module

$$\mathrm{Ophom}_E(B, M)$$

of *operator* homomorphisms $B \to M$, i.e. algebra homomorphisms such that

$$\alpha(eb) = \pi e \cdot \alpha(b), \quad \alpha(be) = \alpha(b) \cdot \pi e \quad [e \in E, b \in B].$$

This module has a submodule

$$\mathrm{Deriv}_E(B, M)$$

consisting of all restrictions $d|B$ of *derivations* $d: E \to M$ (via $\pi$).

THEOREM 3.2. *If* $\alpha \in \mathrm{Ophom}_E(B, M)$, *then the mapping* $\alpha \to \{\alpha\Gamma: A \times A \to M\}$ *induces a monomorphism*

$$T: \frac{\mathrm{Ophom}_E(B, M)}{\mathrm{Deriv}_E\ (B, M)} \to \mathrm{Ext}_S^{\eta}(A, M),$$

*which is independent of the factor set* $\Gamma$ *chosen for* $E$.

*If* $E$ *is a free* $S$*-algebra, then* $T$ *is an* isomorphism.

(Some months after proving this theorem, I received the manuscript of [8] by M. Gerstenhaber, which includes an isomorphism of essentially these groups, for not necessarily linearly split extensions. He sketches an approach different from that considered here.)

If $A$ acts on $M$ by zero homomorphisms, then $M$ will lie in the bicentre of any extension of $M$ by $A$. Conversely, if $M$ lies in the bicentre of an extension of $M$ by $A$, then $a \cdot m = m \cdot a = 0$ for $a \in A$, $m \in M$.

Hence there is a fixed module

$$\mathrm{Ext}_S^c\ (A, M)$$

of *bicentral* $S$-extensions of $M$ by $A$, i.e. $S$-extensions whose bicentres contain $M$. In this case, an operator homomorphism $B \to M$ sends $EB \cup BE$ to zero, while a derivation $E \to M$ sends all products to zero. Hence, in this case, there is a monomorphism

$$\frac{\mathrm{Hom}\,(B, EB \cup BE \to M, 0)}{\mathrm{Hom}\,(E, M)\,|\,B} \to \mathrm{Ext}_S^c(A, M),$$

which is an isomorphism if $E$ is a free $S$-algebra.

Theorem 3.2 can be deduced from some further propositions:

Suppose that $A$ also acts on an algebra $N$ (not necessarily a zero algebra) by means of a fixed linear map $\nu: A \to M(N)$ arising from some $S$-extension of $N$ by $A$.

Define an *extended operator homomorphism* $B \to N$ to be a linear mapping $\alpha: E \to N$ such that

$$\alpha(be)=\alpha(b)\,\alpha(e)+\alpha(b)\cdot\pi e,\quad \alpha(eb)=\alpha(e)\,\alpha(b)+\pi e\cdot\alpha(b)\quad [e\in E, b\in B].$$

Let $\mathrm{Ophom}_E^*(B,N)$ denote the set of extended operator homomorphisms $B\to N$. Also, let $\mathrm{Deriv}^*(E,N)$ be the subset of $\mathrm{Ophom}_E^*(B,N)$ consisting of *extended derivations* $E\to N$, defined to be linear mappings $d:E\to N$ such that

$$d(uv)=d(u)\,d(v)+\pi u\cdot d(v)+d(u)\cdot\pi v;$$

the set of restrictions to $B$ of extended derivations will be denoted by $\mathrm{Deriv}_E^*(B,N)$. (If $N$ is a zero algebra, an extended derivation is just a derivation.)

Now let $\theta:A\to M(N)/\mu(N)$ be the linear mapping induced by $\nu:A\to M(N)$. If $\alpha\in\mathrm{Ophom}_E^*(B,N)$, let $N_\alpha=\mathrm{Im}\,\alpha\,|\,B$.

THEOREM 3.3. (i) *If* $\alpha\in\mathrm{Ophom}_E^*(B,N)$, *then* $\alpha\Gamma:A\times A\to N$ *defines an element of* $\mathrm{ext}_S^\theta(A,N_\alpha)$.

*If $E$ is a free S-algebra, then every element of* $\mathrm{ext}_S^\theta(A,N)$ *can be defined by a factor set of the form* $\alpha\Gamma$ *as above.*

(ii) *If* $d\in\mathrm{Deriv}^*(E,N)$, then $d\Gamma:A\times A\to N$ *defines the split extension class in* $\mathrm{ext}_S^\theta(A,N_d)$. *Conversely, if* $\alpha\in\mathrm{Ophom}_E^*(B,N)$ *is such that* $\alpha\Gamma$ *defines the split extension class in* $\mathrm{ext}_S^\theta(A,N_\alpha)$, *then* $\alpha\,|\,B\in\mathrm{Deriv}_E^*(B,N)$.

*In each case, the class defined by* $\alpha\Gamma$ *is independent of the factor set* $\Gamma$ *chosen for* $E$.

(If the factor set $\Gamma$ corresponds to a linear cross section $\gamma:A\to E$, and $\alpha\in\mathrm{Ophom}_E^*(B,N)$, one considers the extension of $N_\alpha$ by $A$ defined by $\alpha\Gamma$ and $\nu_\alpha:A\to M(N_\alpha)$, where $\nu_\alpha(a)$ is the bimultiplication $\mu_{\alpha(\gamma a)}+\nu(a)\,|\,N_\alpha$ $(a\in A)$.)

It can be deduced from Theorem 3.2 that $\mathrm{Ext}_S^\eta(A,M)$ is a covariant functor of the $S$-bimodule $(M,R,L)$. Also, the monomorphism $T$ of that theorem is a natural transformation of functors.

## 4. Reduction to zero algebra kernels

Let $\theta:A\to M(B)/\mu(B)$ be a linear map arising from some $S$-extension

$$0\to B\to G\xrightarrow{\sigma}A\to 0.$$

Given such an arbitrary (but fixed) extension, consider any $S$-extension

$$0\to K_B\to E\xrightarrow{\pi}A\to 0$$

of the bicentre $K_B$ of $B$ by $A$ determining the linear map $\theta:A\to M(K_B)$ induced by $\theta$.

Let $G \otimes E$ denote the subalgebra of all pairs $(g, e) \in G \times E$ such that $\sigma g = \pi e$, and let $N$ be the ideal of all pairs $(k, -k)$, $k \in K_B$. If

$$G_E = G \otimes E/N,$$

one obtains an $S$-extension

$$0 \to B \to G_E \to A \to 0,$$

in which $B$ maps into $G_E$ by $b \to (b, 0) + N$ and $G_E$ maps onto $A$ by the common projection.

THEOREM 4.1. *Assuming that there exists an $S$-extension $0 \to B \to G \xrightarrow{\sigma} A \to 0$ determining the linear map $\theta: A \to M(B)/\mu(B)$, the mapping $E \to G_E$ induces a 1–1 correspondence:*

$$\mathrm{Ext}_S^\theta\,(A, K_B) \leftrightarrow \mathrm{ext}_S^\theta\,(A, B).$$

(References to previously known cases of this theorem appear after the proof below.)

*Proof.* The argument below is analogous to one of Eilenberg and MacLane [5]: Let $\gamma: A \to G$ and $\omega: A \to E$ be linear cross sections defining factor sets $\Gamma: A \times A \to B$ and $\Omega$: $A \times A \to K_B$ respectively. Then $\bar{\gamma}: a \to (\gamma_a, \omega_a) + N$ is a linear cross section of $G_E$, with corresponding factor set

$$\bar{\Gamma}: (a, b) \to (\Gamma_{a,b}, \Omega_{a,b}) + N = (\Gamma_{a,b} + \Omega_{a,b}, 0) + N.$$

Further, if $b \in B$, then

$$\bar{\gamma}_a[(b, 0) + N] = (\gamma_a b, 0) + N, \quad [(b, 0) + N]\bar{\gamma}_a = (b\gamma_a, 0) + N,$$

and so the action of $A$ on $B$ in this extension gives rise to the map $\theta$.

Next we show that every $S$-extension $\tilde{G}$ of $B$ by $A$ determining $\theta$ is equivalent to one of the form $G_E$: Since $\tilde{G}$ determines $\theta$ there exists a linear cross section $\tilde{\gamma}: A \to \tilde{G}$ inducing the same mappings $A \to \mathrm{Map}(B, B)$ as $\gamma: A \to G$. If $\tilde{\Gamma}: A \times A \to B$ is the factor set of $\tilde{\gamma}$, it follows from the definition of factor sets that $\Gamma_{a,b}$ and $\tilde{\Gamma}_{a,b}$ $(a, b \in A)$ define the same right and left multiplication maps of $B$; hence they differ by an element of the bicentre of $B$. Thus there exists a bilinear mapping $\Omega: A \times A \to K_B$ such that

$$\tilde{\Gamma}_{a,b} = \Gamma_{a,b} + \Omega_{a,b} \quad (a, b \in A).$$

If $\phi \in \mathrm{Hom}(F, A)$, a simple inductive proof shows that then

$$\tilde{\Gamma}_\phi = \Gamma_\phi + \Omega_\phi: F \to B.$$

Therefore

$$\Omega_\phi(S) = [\tilde{\Gamma}_\phi - \Gamma_\phi](S) = \{0\},$$

by Theorem 2.1, and so, by Theorem 2.2, $\Omega$ is the factor set of an $S$-extension of $K_B$ by $A$. Further the equation $\tilde{\Gamma} = \Gamma + \Omega$ now shows that $E \to G_E$ induces a surjective mapping

$$\mathrm{Ext}_S^\theta (A, K_B) \to \mathrm{ext}_S^\theta (A, B).$$

Lastly it must be shown that the mapping is injective: Let $E$ and $\bar{E}$ be two $S$-extensions of $K_B$ by $A$ determining $\theta$ such that $G_E$ and $G_{\bar{E}}$ define equivalent extensions under an equivalence isomorphism $\tau: G_E \to G_{\bar{E}}$. Suppose that $\zeta, \bar{\zeta}: A \to G_E, G_{\bar{E}}$ are linear cross sections of $G_E$ and $G_{\bar{E}}$ respectively, such that $\zeta$ and $\bar{\zeta}$ induce the same mappings $A \to \mathrm{Map}(B, B)$. Then (for $a \in A$) $\tau\zeta_a$ and $\bar{\zeta}_a$ lie in the same coset of $B$, so that

$$\tau\zeta_a = \bar{\zeta}_a + \psi_a \quad (\psi_a \in B),$$

and also they induce the same linear maps $B \to B$. Hence the rule $a \to \psi_a$ defines a linear mapping $\psi: A \to K_B$.

Let $\Omega$ and $\bar{\Omega}$ be factor sets of $E$ and $\bar{E}$ respectively, corresponding to linear cross sections $\omega$ and $\bar{\omega}$, and suppose that

$$\zeta_a = (\gamma_a, \omega_a) + N, \quad \bar{\zeta}_a = (\gamma_a, \bar{\omega}_a) + N \quad (a \in A).$$

Then
$$\zeta_a \zeta_b = [(\Gamma_{a,b} + \Omega_{a,b}, 0) + N] + \zeta_{ab},$$

and, by applying $\tau$, one finds that $\Omega = \bar{\Omega} + \delta\psi$. Thus $E$ and $\bar{E}$ are equivalent extensions. This completes the proof.

Theorem 4.1 was proved by Hochschild [9] and [10] for associative and Lie algebras, by considering factor sets alone. For associative rings, such a result has also been established by MacLane [16], without requiring that the extensions be linearly split. MacLane's approach can be used to immediately provide a similar result in the case of any subvariety of the variety of associative algebras, without requiring that extensions be linearly split. His method can also be used to obtain an analogous reduction in the case of any subvariety of the variety of Lie algebras, without requiring extensions to be linearly split:

Suppose that $S$ defines a subvariety of the variety of Lie algebras. If $B$ is an $S$-algebra, then $M(B) \subseteq \mathrm{Deriv}(B, B)$, which is a Lie algebra under the bracket operation on derivations and has $\mu(B)$ as an ideal. In this case an $S$-extension $0 \to B \to G \xrightarrow{\sigma} A \to 0$ determines an algebra homomorphism

$$\theta: A \to \mathrm{Deriv}(B, B)/\mu(B),$$

and here $S$-extensions will *not* be required to be linearly split. Define the *graph* $\Theta$ of $\theta$ to be the subalgebra of all pairs $(x, \alpha) \in A \times \mathrm{Deriv}(B, B)$ for which $\alpha \in \theta(x)$. Then the mapping

$g \to (\sigma g, \mu_g | B)$ defines an epimorphism $\psi: G \to \Theta$ with kernel $K_B$. One can now proceed after the manner of MacLane [16] (cf. Chapter I, § 6).

As in the case considered by MacLane, one can observe that the mapping $(x, \alpha) \to x$ defines an epimorphism $\varrho: \Theta \to A$, whose kernel is isomorphic to $\mu(B)$ (given a homomorphism $\theta: A \to \mathrm{Deriv}(B, B)/\mu(B)$). If $K_B = \{0\}$, $B$ is isomorphic to $\mu(B)$ and one obtains a Lie algebra extension

$$0 \to B \to \Theta \xrightarrow{\varrho} A \to 0$$

determining $\theta$. By the above discussion, any Lie extensions of $B$ by $A$ determining $\theta$ will then be equivalent to $\Theta$. (By [5], analogous remarks apply to $S$-groups.)

In general, Theorem 4.1 has the corollary:

COROLLARY 4.2. *If $K_B = \{0\}$, and there exists an $S$-extension determining the linear map $\theta: A \to M(B)/\mu(B)$, then all such extensions of $B$ by $A$ are equivalent.*

## 5. Splitting algebras

Define a *splitting extension* of an extension $0 \to B \to E \to A \to 0$ with given linear cross section $\gamma: A \to E$ to be a split extension

$$0 \to \bar{B} \to \bar{E} \to A \to 0$$

such that $B$ and $E$ are subalgebras of $\bar{B}$ and $\bar{E}$ respectively and such that $\gamma: A \to E \subseteq \bar{E}$ is a linear cross section of $\bar{E} \to A$.

THEOREM 5.1. *Every extension $0 \to B \to E \to A \to 0$ has a splitting extension $0 \to \bar{B} \to \bar{E} \to A \to 0$ in which $\bar{B} = A_0 \oplus B$, where $A_0$ is a zero algebra isomorphic to the underlying module of $A$.*

*If $B$ is a zero algebra, $E$ is associative and $A \cdot B = \{0\}$ or $B \cdot A = \{0\}$, then $E$ has an associative splitting extension of this kind.*

*Proof.* The argument is analogous to Artin's proof of the existence of splitting groups for group extensions [18]:

*Let* $\bar{B} = A_0 \oplus B$, where $A_0$ is the underlying module of $A$ and is regarded as a zero algebra, and $0 \to B \to E \to A \to 0$ is some extension with linear cross section $\gamma: A \to E$ and corresponding factor set $\Gamma: A \times A \to B$.

Let $A$ act linearly on $B$ by using the action defined by $\gamma$ on $B$, and by letting

$$a \cdot b = ab - \Gamma_{a,b}, \quad b \cdot a = 0 \quad (a \in A, b \in A_0).$$

These operations and the mapping $\Gamma: A \times A \to B \subseteq \bar{B}$ define an extension $0 \to \bar{B} \to \bar{E} \to A \to 0$, such that $\bar{\bar{E}}$ contains $E$ monomorphically as a subalgebra and such that $\gamma : A \to E$ is a linear cross section of $\bar{\bar{E}} \to A$ under this inclusion. Furthermore, this extension splits, since $\Gamma = -\delta \bar{i}$, where $\bar{i} : A_o \to B$ is the inclusion mapping of $A_o$.

If $B$ is a zero algebra, $E$ is associative and $B \cdot A = \{0\}$, the fact that $\Gamma$ is an associative 2-cocycle implies that $\bar{\bar{E}}$ is associative. (If $A \cdot B = \{0\}$, an associative splitting extension is obtained by letting $a \cdot b = 0$, $b \cdot a = ba - \Gamma_{b,a}(a \in A, b \in A_o)$.)

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