CANONICAL POLYGONS FOR FINITELY GENERATED FUCHSIAN GROUPS

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1. Introduction

We consider finitely generated Fuchsian groups G. For such groups Fricke defined a class of fundamental polygons which he called canonical. The two most important distinguishing properties of these polygons are that they are strictly convex and have the smallest possible number of sides. A canonical polygon P in Fricke's sense depends on a choice of a certain "standard" system of generators S for G. Fricke proved that for a given G and S canonical polygons always exist. His proof is rather complicated. Also, Fricke's polygons are not canonical in the technical sense; there are infinitely many P for a given G and S.

In this paper, we shall construct a *uniquely determined* fundamental polygon P which satisfies all of Fricke's conditions for every given G of positive genus and for every given S. We call this P a canonical Fricke polygon.

For every given G of genus zero, and S, we shall define a *uniquely determined* fundamental polygon P which we call a canonical polygon without accidental vertices. From this P, one can obtain in infinitely many ways polygons satisfying Fricke's conditions.

Our canonical polygons are invariant under similarity transformations of the group G if G is of the first kind. If G is of the second kind, this statement remains true after a suitable modification which will be clear from the construction.

The proof involves elementary explicit constructions, and continuity arguments which use quasiconformal mappings, as developed by Ahlfors and Bers.

We give a geometric interpretation of the canonical polygons in the last section. This interpretation provided the heuristic idea for the formulation of the main theorem.

The author takes pride and pleasure in acknowledging the guidance of professor Lipman Bers in the preparation of this dissertation. His patient and penetrating counsel and his warm friendship will be remembered always.

2. Definitions

We say that S is a Riemann surface of *finite type* $(g; n; m)$ if it is conformally equivalent to $\hat{S} - \{(p_1, p_2, \dots, p_n) \cup (d_1, d_2, \dots, d_m)\}$ where $\hat{S}$ is a closed Riemann surface of genus g, the p_i are points and the d_j are closed conformal discs ($p_i \neq p_j$, $d_i \cap d_j = \emptyset$ for $i \neq j$, $p_i \notin d_j$, $n \geqslant 0$, $m \geqslant 0$). Suppose that to each "removed" point p_j, $j = 1, 2, \dots, n$, there is assigned an "integer" ν_j, $\nu_j = 2, 3, \dots, \infty$, $\nu_1 \leqslant \nu_2 \leqslant \dots \leqslant \nu_n$. Then we say that S has *signature* $(g; n; \nu_1, \dots, \nu_n; m)$. A surface's type is preserved under conformal equivalence, hence this definition is meaningful.

A surface S with signature $(g; n; \nu_1, \dots, \nu_n; m)$ is *represented* by a Fuchsian group G if:

1) G is a properly discontinous group of Möbius transformations leaving the unit disc U fixed.

2) If U_G denotes $U - \{\text{elliptic fixed points of } G\}$ then

2a) U/G is conformally equivalent to $\hat{S} - \{(p_r, \dots, p_n) \cup (d_1, \dots, d_m)\}$ where $\nu_r = \nu_{r+1} = \dots = \infty$. U_G/G is conformally equivalent to $\hat{S} - \{(p_1, \dots, p_n) \cup (d_1, \dots, d_m)\}$.

2b) The map $\pi_G: U \rightarrow U/G$ is locally 1 to 1 in the neighborhood of every point of U_G, and is ν_j to 1 at the pre-images of the points $p_1, \dots, p_{r-1}$.

3. Preliminaries

From now on we consider only Riemann surfaces of finite type.

The following three classical theorems are basic to the theory we are discussing.

THEOREM 1. *Given a Riemann surface with a signature, a Fuchsian group representing it is finitely generated and is determined up to conjugation by a Möbius transformation.*

The proof may be found in Appell–Goursat [4].

THEOREM 2. *All finitely generated Fuchsian groups represent surfaces of finite type.*

A direct proof will appear in a forthcoming paper by Bers [7], and can also be found as a special case of a theorem of Ahlfors [2].

THEOREM 3. *There exists a finitely generated Fuchsian group representing every surface of finite type with a given signature provided that* $3g-3+n+m>0$ *and if* $g=0$, $m=0$, $n=4$ *then* $\sum_{j=1}^{4}\nu_j>8$.

Note that we omit any discussion of the triangle groups in this paper.

Proof of this may be found in Ford [9] and Bers [5]. However, the statement also follows from the construction we will give later. We will sketch the proof at the appropriate place.

4. Fricke polygons

If $(g; n; \nu_1, \ldots, \nu_n; m)$ is the signature of S, and if G is the group representing S, we call $(g; n; \nu_1, \ldots, \nu_n; m)$ the *signature* of G. We remark that the signature of G is preserved under conjugation.

Let G be a finitely generated Fuchsian group with signature $(g; n; \nu_1, \ldots, \nu_n; m)$ and suppose R is a fundamental region for G. R is called a *standard fundamental region* for G if it satisfies the following conditions:

1) R is bounded by $4g+2n+2m$ Jordan arcs in U and m arcs on the boundary of U, forming a Jordan curve oriented so that the interior of R is on the left.

2) If the sides of R are suitably labelled in order:

$$a_1, b_1, a_1', b_1', a_2, b_2, a_2', b_2', \ldots, a_g' b_g', c_1, c_1', \ldots, c_n, c_n', d_1, e_1, d_1', \ldots, d_m, e_m, d_m',$$

there exist hyperbolic elements A_i, B_i and $D_j \in G$, $i=1,2,\ldots,g$, $j=1,2,\ldots,m$, such that $A_i(a_i)=-a_i'$, $B_i(b_i')=-b_i'$ and $D_j(d_j)=-d_j'$, and elliptic elements $C_k\in G$ of order ν_k (parabolic if $\nu_k=\infty$) $k=1,2,\ldots,n$, such that $C_k(c_k)=-c_k'$. These elements satisfy the relation:

$$D_n \ldots D_1 C_n \ldots C_1 B_g^{-1} A_g^{-1} B_g A_g \ldots B_1^{-1} A_1^{-1} B_1 A_1 = 1. \tag{1}$$

(Note: In this paper the notation AB means first apply B and then apply A.) The e_j, $j=1,2,\ldots,m$ are arcs on the unit circle.

It is known that the elements of

$$S=\{A_1, B_1 \ldots, A_g, B_g, C_1, \ldots C_n, D_1, \ldots, D_m\}$$

with the single relation (1), generate G. We will call S a *standard sequence of generators*. R is said to belong to S.

Two standard fundamental regions are *equivalent* if they give rise to the same standard sequence of generators.

It follows easily from Theorem 2 that standard fundamental regions exist for all finitely generated Fuchsian groups. However, this will also follow from the main theorem of this paper; an explicit proof using this theorem will be given later (see Section 7).

If the boundary arcs of a standard fundamental region R belonging to S (except for the arcs on the unit circle) are non-Euclidean straight segments, R will be called a Fricke polygon belonging to S.

5. Fricke's theorem

Let S be a standard sequence of generators and let p_0 be a point in U. We will construct a closed "polygonal" curve $P(p_0; S)$ such that if this curve has no self-intersections, it is the boundary of a Fricke polygon belonging to S.

Let p_0 be given. Find the points:

$$p_0^1 = A_1^{-1}B_1^{-1}(p_0), \quad p_0^2 = A_1^{-1}(p_0), \quad p_0^3 = B_1^{-1}(p_0), \quad p_0^4 = p_1 = B_1^{-1}A_1^{-1}(p_0)$$

$$p_1^1 = A_2^{-1}B_2A_2(p_1), \quad p_1^2 = B_2A_2(p_1), \quad p_1^3 = A_2(p_1), \quad p_1^4 = p_2 = B_2^{-1}A_2^{-1}B_2A_2(p_1)$$

$$p_2^1 = A_3^{-1}B_3A_3(p_2), \quad p_2^2 = B_3A_3(p_2), \quad p_2^3 = A_3(p_2), \quad p_2^4 = p_3 = B_3^{-1}A_3^{-1}B_3A_3(p_2),$$

$$\vdots$$

$$p_{g-1}^1 = A_g^{-1}B_gA_g(p_{g-1}), p_{g-1}^2 = B_gA_g(p_{g-1}), p_{g-1}^3 = A_g(p_{g-1}), p_{g-1}^4 = p_g = B_g^{-1}A_g^{-1}B_gA_g(p_{g-1})$$

$$p_{g+1} = C_1(p_g), \quad p_{g+2} = C_2(p_{g+1}), \ldots, \quad p_{g+n} = C_n(p_{g+n-1}),$$

$$p_{g+n+1} = D_1(p_{g+n}), \quad p_{g+n+2} = D_2(p_{g+n+1}), \ldots, \quad p_{g+n+m} = p_0^1 = D_m(p_{g+n+m}),$$

and label the fixed points of $C_1, C_2, \ldots, C_n$ by $q_1, q_2, \ldots, q_n$.

If I_j is the isometric circle (see Ford [9]) of D_j, I_j intersects the axis of D_j. This axis has a direction in which it is moved by the translation D_j. Let r_j be the endpoint of I_j on the left side of the axis of D_j. Then $r_j' = D_j(r_j)$ is also to the left of the axis of D_j.

Join by non-Euclidean straight segments (see Figure 1):

$$p_0^1 \text{ to } p_0^2, \quad p_0^2 \text{ to } p_0, \quad p_0 \text{ to } p_0^3, \quad p_0^3 \text{ to } p_0^4 = p_1$$

$$p_1 \text{ to } p_1^1, \quad p_1^1 \text{ to } p_1^2, \quad p_1^2 \text{ to } p_1^3, \quad p_1^3 \text{ to } p_1^4 = p_2$$

$$p_2 \text{ to } p_2^1, \ldots$$

$$\ldots p_{g-1}^3 \text{ to } p_{g-1}^4 = p_g$$

$$p_g \text{ to } q_1, \quad q_1 \text{ to } p_{g+1}, \quad p_{g+1} \text{ to } q_2, \ldots, q_n \text{ to } p_{g+n}$$

$$p_{g+n} \text{ to } r_1, \quad r_1' \text{ to } p_{g+n+1}, p_{g+n+1} \text{ to } r_2, r_2' \text{ to } p_{g+n+2}, \ldots, r_m' \text{ to } p_{g+n+m} = p_0^1.$$

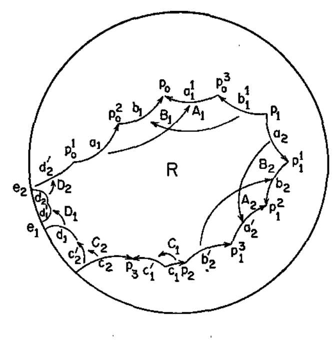

Fig. 1

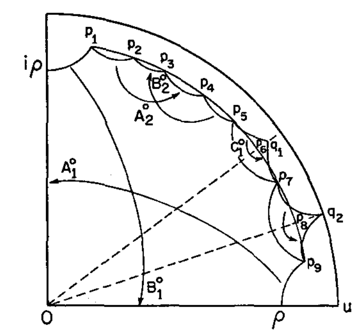

Fig. 2

Fig. 1. R is a standard fundamental region belonging to $S = \{A_1, B_1, A_2, B_2, C_1, C_2, D_1, D_2\}$ where S generates the group G with signature $(2; 2; k, \infty; 2)$. R is in fact a Fricke polygon.

Fig. 2. G has signature $(2; 2; k, \infty; 0)$.

We call the resulting polygonal curve $P(p_0; S)$. If it bounds a fundamental region for G (and hence is a Fricke polygon) we call p_0 *suitable*. If $P(p_0; S)$ is also strictly convex (i.e. all the interior angles are strictly less than π, except for those at vertices which are fixed points of elliptic transformations of order 2) we call p_0 *very suitable*.

Remark: If p_0 is suitable, the sum of the angles at the vertices composing the accidental cycle (i.e. the cycle containing all the vertices which are images of p_0) is 2π.

THEOREM 4. (*Fricke*) *Let G be a finitely generated Fuchsian group with signature* $(g; n; \nu_1, \ldots, \nu_n; m)$ *and assume* $3g-3+n+m>0$. *Then very suitable points exist for any standard sequence of generators* S.

Remark: Fricke calls polygons corresponding to very suitable points *canonical polygons*. This paper originated from an attempt to find a new proof of Fricke's theorem. However, we were able to prove a stronger theorem involving a polygon which is uniquely determined and hence "more canonical". Therefore we reserve the name "canonical" for this new polygon. The proof of Fricke's theorem is contained in the proofs of the main theorems below.

6. Main Theorem, Part I

THEOREM 5. (*Main Theorem, Part* I.) *Let G be a finitely generated group of the first kind with signature* $(g; n; \nu_1, \ldots, \nu_n)$ *and assume* $g>0$, $3g-3+n>0$. *Then, given any standard sequence of generators* $S = \{A_1, B_1, \ldots, A_g, B_g, C_1, \ldots, C_n\}$ *the axes of* A_1 *and* B_1 *intersect; their intersection point* p^* *is very suitable.*

We will call $P(p^*; S)$ a *canonical Fricke polygon* belonging to S.

Proof. We will first give the construction of a canonical Fricke polygon for a particular group G_0 which has the prescribed signature. The theorem will then follow by a continuity argument.

Consider the unit circle and mark off the non-Euclidean lengths ϱ on the positive real and positive imaginary axes. At these points erect non-Euclidean perpendiculars and mark off the non-Euclidean lengths ϱ on the segments in the first quadrant. These perpendiculars do not intersect. If they did, their angle of intersection would be greater than or equal to $\pi/2$. This is true because in a non-Euclidean triangle the bigger angle is opposite the bigger side. The sum of the interior angles of the resulting non-Euclidean quadrilateral would be at least 2π—an impossible situation.

Join these new points (the endpoints of the perpendiculars) by a circular arc concentric with the unit circle (see Figure 2). Partition this arc into $4(g-1)+2n$ equal subarcs and label the subdivision points successively $p_1, p_2, \ldots, p_{4(g-1)+2n+1}$. Join p_1 to p_2, p_2 to $p_3, \ldots, p_{4(g-1)}$ to $p_{4(g-1)+1}$ by non-Euclidean straight segments. Draw rays from the origin through $p_{4g-2}, p_{4g}, \ldots, p_{4(g-1)+2n}$.

If $n>0$, find the point q_1 on the ray through p_{4g-2} such that the angle formed by the non-Euclidean segment joining this point to $p_{4(g-1)+1}$ and the ray is π/ν_1. Join this point with $p_{4(g-1)+1}$; again the angle formed will be π/ν_1, and the total vertex angle $p_{4g-3}p_{4g-2}p_{4g-1}$ will be $2\pi/\nu_1$. Repeat the construction to find the points $q_2, \ldots q_n$. If $\nu_j=\infty$, q_j will lie on the unit circle.

Note that the sum of the angles at the accidental vertices (i.e. $p_j, j=1, \ldots, 4(g-1)+1, 4g-1, 4g+1, \ldots, 4g+2n+1$, and 0, ϱ and $i\varrho$) is a continuous function of ϱ. We will show that the limit of this sum as ϱ tends to 1 is $3\pi/2$. We break up the polygon in the following manner. Join p_{4g-3} to p_{4g-1}, p_{4g-1} to $p_{4g+1}, \ldots$, and $p_{4g+2n-1}$ to $p_{4g+2n+1}$ by non-Euclidean straight segments. The sum of the interior angles of this truncated polygon tends to $3\pi/2$ as ϱ tends to 1. The original polygon consists of this truncated one plus n triangles. As ϱ tends to 1, all the vertices of these triangles tend to the horizon of the non-Euclidean plane and therefore the sum of their interior angles tends to $2\pi/\nu_i$. When ϱ is close to zero, the truncated polygon is almost Euclidean. It is a convex figure with $4g+n$ sides. Since the sum of the interior angles of a Euclidean polygon is $(S-2)\pi$, where S is the number of sides, the sum of the angles of the original non-Euclidean polygon definitely exceeds 2π for small ϱ. We conclude that for some ϱ it is exactly 2π.

Performing our construction for *this* ϱ, we obtain a polygon R_0 satisfying all the conditions of Poincaré's theorem (see Appell–Goursat [4]). Therefore the group G_0

generated by the sequence S_0 of non-Euclidean motions indicated on Figure 2 is Fuchsian and of the desired signature. Moreover, R_0 is a Fricke polygon for G_0 belonging to the standard sequence of generators S_0. We note that A_1^0 and B_1^0 are hyperbolic motions whose axes are the real and imaginary axes respectively, so that R_0 also satisfies the conditions of Theorem 5.

7. A lemma

The following lemma is obvious.

LEMMA 1. *For any finitely generated Fuchsian group G, with signature, there exists a standard fundamental region R such that each bounding arc is C^∞ and such that the angles at the accidental vertices are never* 0 *or* π.

8. Continuity argument

We now return to the proof of the main theorem.

Let R be a given standard fundamental region for the given group G satisfying the conditions of Lemma 1. Since G_0 and G have the same signature, R_0 and R have the same number of sides. We can define a continuous mapping $w: R_0 \to R$ by defining it first on the boundaries of R_0 and R respecting the identifications, and then extending it quasiconformally inside (see Ahlfors–Bers [3]). Since the identifications are respected, we have that for every $A_0 \in S_0$ there is an $A \in S$ such that

$$w(A_0(z)) = A(w(z)) \tag{2}$$

on the boundary of R_0. The mapping $A_0 \to A$ gives us an isomorphism of G_0 onto G by which we may now extend w to the rest of the unit disk so that (2) holds. w is now a homeomorphism of U onto itself and, since it is quasiconformal inside R_0, it is quasiconformal everywhere. Since it is a quasiconformal map of the whole disk it can be extended to the closed disk.

Define $\mu(z) = w_{\bar{z}}/w_z$ and note that $|\mu(z)| \leqslant k < 1$ in U and

$$\mu(A(z))\overline{A'(z)} = \mu(z)A'(z) \quad \text{for} \quad A \in G. \tag{3}$$

Without loss of generality we may assume that $w(0) = 0$, $w(1) = 1$, because if B is a Möbius transformation such that $B(w(0)) = 0$, $B(w(1)) = 1$, we can replace G by $B^{-1}GB$.

Consider the functions $t\mu(z)$, $0 \leqslant t \leqslant 1$. Define $w^{t\mu}$ as the solution of the Beltrami equation $w_{\bar{z}} = t\mu w_z$ which maps the closed unit disk onto itself leaving 0 and 1 fixed.

$w^{t\mu}$ is then quasiconformal by definition and is a continuous function of t. In fact, when $t=0$, $w^{t\mu}=$ identity, while when $t=1$, $w^{t\mu}=w$. Define

$$A^{t\mu}=(w^{t\mu})^{-1}\circ A\circ w^{t\mu} \quad \text{for} \quad A\in G.$$

By Bers [5], $A^{t\mu}$ is a Möbius transformation. Then $S^{t\mu}=(w^{t\mu})^{-1}S_0 w^{t\mu}$ is a standard sequence of generators of a group $G^{t\mu}$; $G^{t\mu}$ will be Fuchsian since G_0 is.

Let p_t^* be the intersection point of the axes of $A_1^{t\mu}$ and $B_1^{t\mu}$. We see that these axes always intersect, because the fixed points of $A_1^{t\mu}$ and $B_1^{t\mu}$ separate each other for $t=0$, and since $w^{t\mu}$ is a homeomorphism of the closed disk, they must always separate each other.

Set $P_t=P(p_t^*; S^{t\mu})$. We claim that the set of t's for which p_t^* is very suitable is non-empty, open and closed and hence consists of all t, $0\leqslant t\leqslant 1$. The set is non-empty since it contains $t=0$ and $p_0=p_0^*$ is very suitable by construction. Let $p_{t_0}^*$ be very suitable and $|t-t_0|$ be small. The curve P_t lies very close to the curve P_{t_0} and since P_{t_0} is simple and in fact strictly convex, P_t must also be simple and strictly convex. Since the elements of $S^{t\mu}$ satisfy (1), the sum of the angles at the accidental vertices is a multiple of 2π. Since P_t is close to P_{t_0}, the sum is close to 2π and therefore exactly 2π. By Poincaré's theorem, P_t is a fundamental region for $G^{t\mu}$ and therefore is a Fricke polygon belonging to $S^{t\mu}$.

Let $t_j\to t$ and let $p_{t_j}^*$ be very suitable for all j. Then for N large, $|p_{t_N}^*-p_t^*|$ is small. The limit of a sequence of strictly convex polygons is either a convex polygon, or is a straight line. In our case, if the limit were a straight line, the axes of $A_1^{t\mu}$ and $B_1^{t\mu}$ would coincide. This is impossible since the fixed points of $A_1^{t\mu}$ and $B_1^{t\mu}$ separate each other. Hence the limit polygon is a convex polygon and reasoning as we did before we see that it is a fundamental polygon for $G^{t\mu}$. Hence p_t^* is suitable.

To see that this P_t is strictly convex, suppose the angle at p_t^* is α, $(0<\alpha<\pi)$. The sides emanating from p_t^* lie along the axes of $A_1^{t\mu}$ and $B_1^{t\mu}$. Applying either of these transformations to P_t yields a polygon, $A_1^{t\mu}(P_t)$ or $B_1^{t\mu}(P_t)$, which is adjacent to the original polygon. Moreover, since the axes are invariant, the angle at p_t^* in either $A_1^{t\mu}(P_t)$ or $B_1^{t\mu}(P_t)$ is $\pi-\alpha$; hence both the angle at $A_1^{t\mu-1}(p_t^*)$ and the angle at $B_1^{t\mu-1}(p_t^*)$ is $\pi-\alpha$. Now the sum of the angles at the three vertices p_t^*, $A_1^{t\mu-1}(p_t^*)$ and $B_1^{t\mu-1}(p_t^*)$ is $2\pi-\alpha$. Since the sum of the angles at all the accidental vertices is just 2π, all the other angles must be strictly less than π. Therefore the set is closed and p_t^* is very suitable for all t, $0\leqslant t\leqslant 1$, q.e.d.

9. Main Theorem, Part II

THEOREM 6. (*Main Theorem, Part* II.) *Let G be a finitely generated Fuchsian group of the second kind with signature $(g; n; \nu_1, \ldots, \nu_n; m)$ and assume $g>0$, $3g-3+m+n>0$. Then given any standard sequence of generators, $S=\{A_1, B_1, \ldots, D_m\}$, the axes of A_1 and B_1 intersect. Their intersection point p^* is very suitable and leads to a unique strictly convex Fricke polygon $P(p^*, S)$. We again call this polygon a* canonical Fricke polygon.

Proof. The proof, as in Theorem 5, is by continuity. However, we must modify our construction as follows. Again draw the real and imaginary axes and mark the non-Euclidean lengths ϱ on the positive halves. Again erect perpendiculars and mark the length ϱ in the first quadrant on them. Again join these points by a circular arc concentric with the unit circle. But now partition it into $4(g-1)+2n+4m$ equal subarcs and label the subdivision points $p_1, \ldots, p_{4(g-1)+2n+4m+1}$. Find the points $q_1, \ldots, q_n$ as before and join by non-Euclidean straight segments:

$$p_1 \text{ to } p_2,\ p_2 \text{ to } p_3, \ldots, p_{4g-4} \text{ to } p_{4g-3},\ p_{4g-3} \text{ to } q_1,$$

$$q_1 \text{ to } p_{4g-1},\ p_{4g-1} \text{ to } q_2,\ q_2 \text{ to } p_{4g+1}, \ldots, q_n \text{ to } p_{4g-4+2n}.$$

Draw rays from the origin through $p_{4(g-1)+2n+2}, p_{4(g-1)+2n+4}, \ldots, p_{4(g-1)+2n+4m}$. Call the endpoints of these rays $\tilde{r}_1, \ldots, \tilde{r}_{2m}$. Draw non-Euclidean segments from $p_{4(g-1)+2n+1}$ to $\tilde{r}_1$, $\tilde{r}_2$ to $p_{4(g-1)+2n+5}$, $p_{4(g-1)+2n+5}$ to $\tilde{r}_3, \ldots, \tilde{r}_{2m}$ to $p_{4(g-1)+2n+4m+1}$. (See Figure 3.)

We use the same argument as we used before to fix the value of ϱ and to apply Poincaré's theorem. Hence we obtain a group G_0 generated by a standard sequence of generators $S_0=\{A_1^0, B_1^0, \ldots, C_1^0, \ldots, D_1^0, \ldots, D_1^m\}$ for which the constructed region R_0 is a Fricke polygon belonging to S_0.

Recall that in defining $P(p_0; S)$ we fixed the non-parabolic vertices lying on the boundary of U as endpoints of isometric circles. We can alter R_0 by replacing $\tilde{r}_1, \ldots, \tilde{r}_{2m}$ by any set of pairs of points related by $D_1, \ldots, D_m$ and again obtain a Fricke polygon. Therefore, in order to make R_0 a canonical Fricke polygon, we replace $\tilde{r}_1, \tilde{r}_3, \ldots, \tilde{r}_{2m-1}$ by the endpoints $r_1, \ldots, r_m$, of the isometric circles of $D_1, \ldots, D_m$ lying to the left of their axes. We then set

$$r_1^1 = D_1(r_1), \ldots, r_m^1 = D_m(r_m).$$

The resulting polygon is now the required canonical polygon for the group G_0.

We now prove Lemma 1 for the case $g>0$, $m>0$. All the arguments go through as they did before since the isometric circles change continuously, q.e.d.

10. Canonical polygons without accidental vertices

If $g=0$ we have no conjugate pairs of hyperbolic generators (A_1 and B_1) and hence can no longer take the intersection point of their axes as a starting point for our canonical polygon. In fact, we make use of a different kind of polygon, suggested by a construction of Fricke, to get an analogous result. We call the new polygon a *canonical polygon without accidental vertices* and obtain it as follows.

Let G be a Fuchsian group with signature $(0; n; \nu_1, \ldots, \nu_n; m)$ and let

$$S = \{C_1, \ldots, C_n, D_1, \ldots, D_m\}$$

be a standard sequence of generators for G. Label the fixed points of $C_1, \ldots, C_n$ by $q_1, \ldots, q_n$. Label the endpoints of the isometric circles of $D_1, \ldots, D_m$ lying to the left of their axes, $r_1, \ldots, r_m$ respectively.

Let:

$$q_2^1 = C_1^{-1}(q_2), \qquad q_3^2 = (C_2 C_1)^{-1}(q_3), \ldots, \qquad q_n^{n-1} = (C_{n-1} \ldots C_1)^{-1}(q_n),$$

$$\bar{r}_1 = D_1(r_1), \qquad \bar{r}_2 = D_2(r_2), \ldots, \qquad \bar{r}_m = D_m(r_m),$$

$$r_1^n = (C_n \ldots C_1)^{-1}(r_1) = (D_1 C_n \ldots C_1)^{-1}(\bar{r}_1) = \bar{r}_1^{n+1}$$

$$r_2^{n+1} = (D_1 C_n \ldots C_1)^{-1}(r_2) = (D_2 D_1 C_n \ldots C_1)^{-1}(\bar{r}_2) = \bar{r}_2^{n+2}$$

$$\vdots$$

$$r_{m-1}^{n+m-2} = (D_{m-2} \ldots D_1 C_n \ldots C_1)^{-1}(r_{m-1}) = (D_{m-1} \ldots D_1 C_n \ldots C_1)^{-1}(\bar{r}_{m-1}) = \bar{r}_{m-1}^{n+m-1}$$

$$r_m^{n+m-1} = (D_{m-1} \ldots D_1 C_n \ldots C_1)^{-1}(r_m) = (D_m \ldots D_1 C_n \ldots C_1)^{-1}(\bar{r}_m) = \bar{r}_m^{n+m} = \bar{r}_m$$

Join by non-Euclidean straight segments:

$$q_1 \text{ to } q_2, \quad q_2 \text{ to } q_3, \ldots, \quad q_n \text{ to } r_1, \quad \bar{r}_1 \text{ to } r_2, \quad \bar{r}_2 \text{ to } r_3, \ldots, \quad \bar{r}_{m-1} \text{ to } r_m,$$

$$\bar{r}_m \text{ to } r_{m-1}^{n+m-2} = \bar{r}_{m-1}^{n+m-1}, \quad r_{m-1}^{n+m-2} \text{ to } r_{m-2}^{n+m-3} = \bar{r}_{m-2}^{n+m-2}, \ldots,$$

$$r_1^n \text{ to } q_n^{n-1}, \; q_n^{n-1} \text{ to } q_{n-1}^{n-2}, \ldots, q_2^1 \text{ to } q_1.$$

Call the resulting polygonal curve $Q(S)$. A special case is shown on Figure 4.

LEMMA 2. *If $Q(S)$ is not self-intersecting, it is the boundary of a fundamental region for G.*

Proof. $Q(S)$ satisfies all the conditions of Poincaré's theorem.

LEMMA 3. *If $Q(S)$ is not self-intersecting, it is strictly convex.*

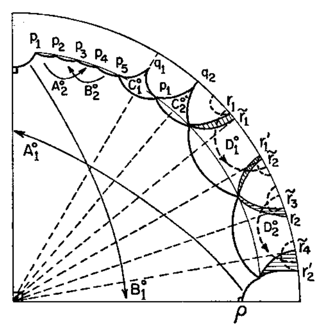

Fig. 3

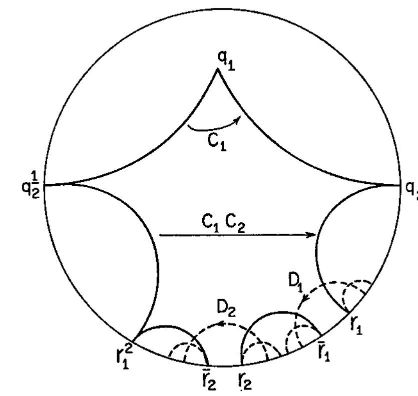

Fig. 4

Fig. 3. G has signature (2; 2; k, ∞; 2). Similar shading indicates congruency under D_i ($i = 1, 2$).

Fig. 4. G is generated by $S = \{C_1, C_2, D_1, D_2\}$, signature (0; 2; k, ∞; 2).

Proof. Each vertex is either part of an elliptic or parabolic cycle, or is the intersection of the unit circle with a non-Euclidean straight line, or is a cusp on the unit circle so that its angle is zero.

11. Main Theorem, Part III

THEOREM 7. (*Main Theorem, Part* III.) *$Q(S)$ is a strictly convex fundamental polygon for G. We call $Q(S)$ the* canonical polygon without accidental vertices *determined by S.*

Recall that if $m = 0$, $n = 4$, we require $\sum_1^4 \nu_i > 8$.

Proof. We must only show that $Q(S)$ it not self-intersecting. We need only construct a particular canonical polygon without accidental vertices since the continuity argument is completely analogous to the one given before.

Consider the regular non-Euclidean $n+m$-gon N with radius $\varrho < 1$ centered at the origin. Suppose that the point $(\varrho, 0)$ is a vertex. Then proceeding in a counterclockwise direction, label the sides $s_1, \ldots, s_{n+m}$. Using s_i, $i = 1, \ldots, n$, as a base, erect an isosceles triangle T_i outside N, with vertex angle $2\pi/\nu_i$. Call the vertex q_i. Trisect the angles subtending the sides s_j, $j = 1, \ldots, m$, by rays from the origin whose endpoints we call $\tilde{r}_{ij}$, $i = 1, 2$, $j = n+1, \ldots, n+m$. Then draw non-Euclidean straight segments $\tilde{l}_{ij}$, $i = 1, 2$, $j = 1, \ldots, m$, from the endpoints of the sides to the corresponding $\tilde{r}_{ij}$. Call the region bounded by $\tilde{l}_{1j}$, s_j, $\tilde{l}_{2j}$, and an arc of the unit circle, $\tilde{M}_j$. Now let

$$\hat{P} = N \cup \bigcup_{i=1}^{n} T_i \cup \bigcup_{j=1}^{m} \tilde{M}_j.$$

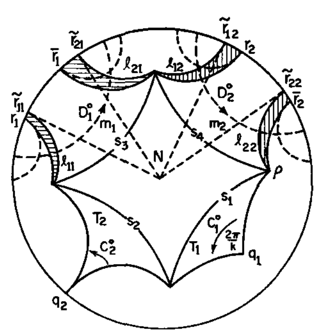

Fig. 5. First step of the construction of $\hat{P}$ for G with signature $(0; 2; k, \infty; 2)$.

We are now concerned with the sum of those interior angles of $\hat{P}$ lying at the vertices which are common to both $\hat{P}$ and N. (We call these the accidental vertices of $\hat{P}$.) We call this sum $\sum_{\hat{P}}$ and note that it depends continuously on ϱ. For ϱ close to 1 it is nearly equal to the sum $\sum_N$ of the interior angles of N, and $\sum_N$ is close to zero. For ϱ close to zero, $\sum_{\hat{P}} > \sum_N$ and $\sum_N$ is nearly $(N-2)\pi \geqslant 2\pi$ since $n \geqslant 4$, and N is nearly Euclidean. Hence for some ϱ, $\sum_{\hat{P}}$ is exactly equal to 2π.

By Poincaré's theorem, there exists for this ϱ, a Fuchsian group G_0, generated by the identifications indicated in the figure, with the relations

$$D_k^0 \dots D_1^0 C_n^0 \dots C_1^0 = 1, \quad C_j^{\nu_j} = 1, \quad j = 1, \dots, n. \tag{1'}$$

To make our constructed polygon the canonical polygon without accidental vertices for G_0, we proceed as follows. If I_j is the isometric circle of D_j, let r_j be the endpoint of I_j which lies to the left of the axis of D_j. Let $\tilde{r}_j = D_j(r_j)$. Draw the non-Euclidean straight segments l_{ij} from the endpoints of the sides s_j to the corresponding r_j and $\tilde{r}_j$. The region M_j bounded by l_{1j}, s_j, l_{2j} is congruent to $\tilde{M}_j$ so that the polygon

$$P_0(S_0) = N \cup \bigcup_{i=1}^{n} T_i \cup \bigcup_{j=1}^{m} M_j$$

is again a Fricke polygon for G_0.

For the sake of simplicity in drawing the figure, consider the transformation B which maps ϱ onto 0 and the fixed points of C_1 onto the positive half of the real axis. The image $B(P_0)$ we again call P_0. B will give us an isomorphism of $G_0 \to BG_0B^{-1}$ generated by BC_jB^{-1}, BD_kB^{-1}, $j = 1, \dots, n$, $k = 1, \dots, m$ which we again call G_0, C_j and D_k respectively.

Using this construction, we complete the proof of Lemma 1.

Now consider the polygons:

$$C_1^{-1}(P_0),\ (C_2C_1)^{-1}(P_0),\ \ldots,\ (D_{m-1}\ldots D_1C_n\ldots C_1)^{-1}(P_0).$$

They all have the origin as a vertex. Find the points $q_2^1, \ldots, q_n^{n-1}, r_1^n, \ldots, r_m^{n+m-1}$ de-defined by the relations on page 10. Join them in the order given there. Let:

$$C_1(0)=p_1,\ C_2(p_1)=p_2,\ \ldots,\ C_n(p_{n-1})=p_n,\ D_1(p_n)=p_{n+1},\ \ldots,\ D_m(p_{n+m-2})=p_{n+m-1}.$$

Then the following regions are congruent in pairs:

$$\begin{array}{ll}
T_1=(0,q_1,q_2^1) & \hat{T}_1=(p_1,q_1,q_2)\\
T_2=(0,q_2^1,q_3^2) & \hat{T}_2=(p_2,q_2,q_3)\\
\vdots & \vdots\\
T_{n-1}=(0,q_{n-1}^{n-2},q_n^{n-1}) & \hat{T}_{n-1}=(p_{n-1},q_{n-1},q_n)\\
T_n=(0,q_n^{n-1},r_1^n) & \hat{T}_n=(p_n,q_n,r_1)\\
T_{n+1}=(0,r_1^n,r_2^{n+1}) & \hat{T}_{n+1}=(p_{n+1},\bar{r}_1,r_2)\\
T_{n+2}=(0,r_2^{n+1},r_3^{n+2}) & \hat{T}_{n+2}=(p_{n+2},\bar{r}_2,r_3)\\
\vdots & \vdots\\
T_{n+m-1}=(0,r_{m-1}^{n+m-2},r_m^{n+m-1}) & \hat{T}_{n+m-1}=(p_{n+m-1},\bar{r}_{m-1},r_m)
\end{array}$$

Subtract $\bigcup_1^{n+m-1}\hat{T}_i$ from $P_0(S)$ and add $\bigcup_1^{n+m-1}T_i$. Call the resulting region $Q_0(S)$. This region is what we have called the canonical polygon without accidental vertices for this group.

The argument that $Q(S)$ is in general not self-intersecting is just the same continuity argument we used previously.

This completes the proof of the main theorem.

12. Fricke polygons for genus zero

From a canonical polygon without accidental vertices we may obtain many standard Fricke polygons. To find one, draw the non-Euclidean straight segment joining q_1 to $\bar{r}_m$. (If $m=0$ join q_1 to q_n.) Pick a point inside the convex polygon Θ whose vertices are $q_1, q_2^1, \ldots, \bar{r}_m$. Call it p_0. Join it to each of the vertices of Θ by non-Euclidean straight segments. The following pairs of triangles are congruent:

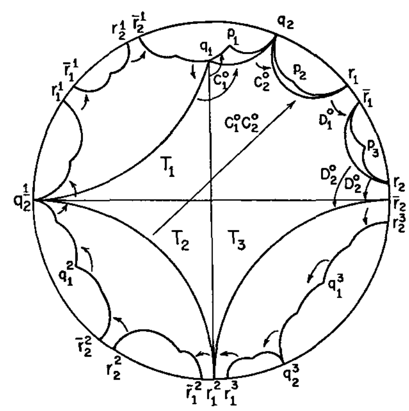

Fig. 6

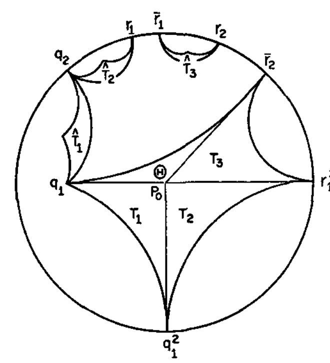

Fig. 7

Fig. 6. $Q(\mathcal{S}_0)$ is the inner polygon.

Fig. 7. Construction of a Fricke polygon from a canonical polygon without accidental vertices.

$$
\begin{array}{ll}
T_1 : (q_1, p_0, q_2^1) & \hat{T}_1 : (q_1, C_1(p_0), q_2) \\
T_2 : (q_2^1, p_0, q_3^2) & \hat{T}_2 : (q_2, C_2 C_1(p_0), q_3) \\
\vdots & \\
T_n : (q_n^{m-1}, p_0, r_1^n) & \hat{T}_n : (q_n, C_n \dots C_1(p_0), r_1) \\
T_{n+1} : (r_1^n, p_0, r_2^{n+1}) & \hat{T}_{n+1} : (\bar{r}_1, D_1 C_n \dots C_1(p_0), r_2) \\
\vdots & \\
T_{n+m-1} : (r_m^{n+m-1}, p_0, \bar{r}_m) & \hat{T}_{n+m-1} : (\bar{r}_{m-1}, D_{m-1} \dots C_1(p_0), r_m)
\end{array}
$$

Subtract $\bigcup_{i=1}^{n+m-1} T_i$ from $Q(\mathcal{S})$ and add $\bigcup_{i=1}^{n+m-1} \hat{T}_j$. Call the resulting polygon $P(p_0 : \mathcal{S})$.

Since the sum of the angles at p_0 is 2π, the sum of the angles at the accidental vertices of $P(p_0 : \mathcal{S})$ is 2π. Each angle is strictly less than π because it is an angle of a non-Euclidean triangle and the sum of the angles of a non-Euclidean triangle is strictly less than π. Hence $P(p_0 : \mathcal{S})$ is strictly convex and is therefore a Fricke polygon. Fricke's theorem is now completely proved.

If we chose our starting point p_0 in $Q(\mathcal{S}) - \Theta$, the above construction would not lead to a convex polygon. However, a similar construction works unless $m = 0$ and p_0 is on the line joining q_1 and q_n. In this case a convex polygon will result, but it will not be strictly convex.

13. Geometric interpretation

For the sake of simplicity we consider the case $g=2$, $n=0$, $m=0$. We can interpret the main theorem geometrically by looking at the Riemann surface S which G represents. We consider on S the Riemannian metric induced by the hyperbolic metric of the disk. On S we are given a set Γ of 4 curves, $\Gamma=\{\alpha_1, \beta_1, \alpha_2, \beta_2\}$ which intersect in only one point p and which are oriented as in figure 8. Cutting S along Γ will yield a simply connected surface. Γ is traditionally called a *canonical dissection* of S. There is an obvious natural correspondence between Γ and a standard sequence of generators $\mathcal{S}$ of G. The canonical Fricke polygon belonging to $\mathcal{S}$ corresponds to a particular set of curves Γ^* obtained from Γ as follows.

Consider all curves freely homotopic to α_1 and let the unique shortest be called a_1. Consider all curves freely homotopic to β_1 and let the unique shortest be called b_1. a_1 and b_1 intersect in a unique point p^*. In the disk a_1 and b_1 correspond to segments $\bar{a}_1$ and $\bar{b}_1$ of the axes of A_1 and B_1. This statement needs proof.

Consider z, $A(z)$ and $A^2(z)$ where A is an hyperbolic transformation. Then,

$$\min_{z \in U} \{\delta(z, A(z)) + \delta(A(z), A^2(z))\},$$

where $\delta(P, Q)$ is non-Euclidean distance will occur when the three points are co-linear; they will be co-linear only when z is on the axis (see Bers [8]).

If p is the intersection point of the original canonical dissection we need to define a curve σ_1 from p to p^* which has the properties

$$\sigma_1^{-1}\alpha_2\sigma_1 \sim a_2, \quad \sigma_1^{-1}\beta_2\sigma_2 \sim b_2$$

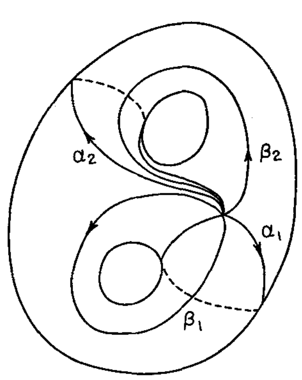

Fig. 8

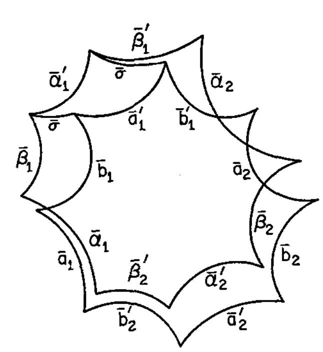

Fig. 9

Fig. 8. Canonical dissection of S of the type (2; 0; 0).

where $\sim$ denotes free homotopy and a_2 or b_2 is the shortest geodesic in the (bounded) homotopy class of α_1 or β_2. σ_1 (whose existence will be discussed below) is unique up to a homotopy. To see this, assume another such curve $\bar{\sigma}_1$ existed. Then $\bar{\sigma}_1^{-1}\alpha_2\bar{\sigma}_1 \sim a_2$, $\bar{\sigma}_1^{-1}\beta_2\bar{\sigma}_1 \sim b_2$ and consequently

$$\bar{\sigma}_1\sigma_1^{-1}\alpha_2\sigma_1\bar{\sigma}_1^{-1} \sim \alpha_2, \quad \bar{\sigma}_1\sigma_1^{-1}\beta_2\sigma_1\bar{\sigma}_1^{-1} \sim \beta_2.$$

Hence $\bar{\sigma}_1\sigma_1^{-1}$ commutes with both α_2 and β_2 and so is homotopic to the identity. We now take σ_1 as the shortest geodesic in its homotopy class.

The existence of σ_1 is equivalent to the existence of a deformation mapping f of Γ into $\Gamma^* = \{a_1, b_1, a_2, b_2\}$. We can prove the existence of f by the same kind of continuity argument used in the proof of the main theorem of this paper.

We must now verify that this new dissection has as one of its images in the disk, the canonical Fricke polygon $P(p^*)$ belonging to S. In the disk, draw $P(p^*)$ and draw a Fricke polygon P corresponding to Γ and containing p^* as an interior or boundary point. From figure 9, we then see that $\tilde{a}_2$ is homotopic to $\bar{\sigma}_1^{-1}\tilde{a}_2\bar{\sigma}$. Since $\tilde{a}_2$ is a geodesic and there is a unique geodesic in any homotopy class of curves through a given point on S, $\tilde{a}_2$ projects into a_2.

A similar interpretation can be given for all other surfaces of finite type and positive genus.

If S has type $(0; n; 0)$ join by geodesics the first distinguished point to the second, the second to third and so on until we reach the nth. This dissection will correspond to the canonical polygon without accidental vertices. This is the dissection originally used by Fricke in the case $g = 0$.

References

[1]. Ahlfors, L. V., On quasi-conformal mappings. *J. Analyse Math.*, 4 (1954), 1–58.
[2]. ——, Finitely generated Kleinian groups. *Amer. J. Math.*, 86 (1964), 413–429.
[3]. Ahlfors, L. V. & Bers, L., Riemann's mapping theorem for variable metrics. *Ann. of Math.*, 72 (1960), 385–408.
[4]. Appell, P. & Goursat, E., *Théorie des fonctions algébriques d'une variable*, Tome II, *Fonctions automorphes* (P. Fatou). Gautier-Villars, Paris, 1930.
[5]. Bers, L., Uniformization by Beltrami equations. *Comm. Pure Appl. Math.*, 14 (1961), 215–228.
[6]. ——, Spaces of Riemann surfaces. *Proc. Int. Congress*, 1958.
[7]. ——, Automorphic forms and Poincaré series for infinitely generated Fuchsian groups. *Amer. J. Math.*, 87 (1965), 196–214.
[8]. ——, Quasiconformal mappings and Teichmüller's Theorem. *Analytic Functions*, Princeton, 1960.
[9]. Ford, L., *Automorphic Functions*. Chelsea, 1929.
[10]. Fricke, R. & Klein, F., *Vorlesungen über die Theorie der automorphen Funktionen* 1. Pp. 285–315, Leipzig, 1926,

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