SUBALGEBRAS OF C^*-ALGEBRAS II

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Contents

Introduction

This paper continues the study of non self-adjoint operator algebras on Hilbert space which began in [1]. Chapter 1 concerns dilation theory. The main results (1.2.2 and its corollary) imply that every commuting n-tuple of operators having a general compact set $X \subseteq \mathbf{C}^n$ as a "complete" spectral set has a (commuting) normal dilation whose joint spectrum is contained in ∂X, the Silov boundary of X relative to the rational functions which are continuous on X. This is a direct generalization of a known dilation theorem for single operators having for a spectral set a compact set $X \subseteq \mathbf{C}$ with connected complement, and it seems to clarify the relation between spectral sets and normal dilations. In section 1.3 we discuss non-normal dilations and present a result along these lines.

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Chapter 2 centers on boundary representations, the principal theme of [1]. Section 2.1 contains a general result that gives a concrete characterization of boundary representations for irreducible sets of operators whose generated C^*-algebras are not too pathological (i.e., are not NGCR). This "boundary theorem" provides some new information about the behavior of a broad class of irreducible Hilbert space operators. For example, in section 2.3 we show that many irreducible operators T are highly "deterministic" in roughly the sense that once one knows the norms of all low order polynomials in T then he knows not only the norms of all higher order polynomials but he in effect knows T to within unitary equivalence. In section 2.4 we show that the most "deterministic" operators are completely determined by an appropriate generalization of their numerical range.

Section 2.2 contains applications of the boundary theorem to operators T such that $T^*T - TT^*$ is compact, it contains the solution of a problem left open in [1] concerning parts of the backward shift, and also a unitary dilation theorem for certain commuting sets of contractions. In section 2.5 we give an application of the boundary theorem to model theory, which asserts that many classes of operators have a *unique* irreducible model.

Preliminaries

We want to recall one or two results from [1] which will be used freely throughout the sequel. Let S be a linear subspace of a C^*-algebra B, and let ϕ be a linear map of S into another C^*-algebra B_1. If M_k, $k=1, 2, \ldots$, denotes the C^*-algebra of all complex $k \times k$ matrices, then $M_k \otimes B$ is the C^*-algebra of all $k \times k$ matrices over B, and $M_k \otimes S$ is a linear subspace of this C^*-algebra. If id_k denotes the identity map of M_k, then $id_k \otimes \phi$ is a linear map of $M_k \otimes S$ into $M_k \otimes B_1$. We will say that ϕ is *completely positive, completely contractive*, or *completely isometric* according as every map in the sequence $id_1 \otimes \phi$, $id_2 \otimes \phi$, ... is positive, contractive, or isometric. We will use the notation $\mathfrak{L}(\mathfrak{H})$ (resp. $\mathfrak{C}(\mathfrak{H})$) to denote the algebra of all bounded (resp. compact) operators on the Hilbert space $\mathfrak{H}$.

THEOREM 0.1. (Extension theorem.) *Let S be a closed self-adjoint linear subspace of a C^*-algebra B, such that B contains an identity e and $e \in S$. Then every completely positive linear map ϕ: $S \to \mathfrak{L}(\mathfrak{H})$ has a completely positive linear extension ϕ_1: $B \to \mathfrak{L}(\mathfrak{H})$.*

COROLLARY 0.2. *Let S be a linear subspace of a C^*-algebra B with identity e, such that $e \in S$. Then every completely contractive linear map ϕ: $S \to \mathfrak{L}(\mathfrak{H})$ for which $\phi(e) = I$ has a completely positive linear extension to B.*

These are proved in 1.2.3 and 1.2.9 of [1].

Now let B be a C^*-algebra with identity e and let S be an arbitrary subset of B such that B is generated as a C^*-algebra by $S \cup \{e\}$ (this is expressed by the notation $B = C^*(S)$; similarly if T is an operator then $C^*(T)$ denotes the C^*-algebra generated by $\{I, T\}$). An irreducible representation π of B on a Hilbert space $\mathfrak{H}$ is called a *boundary representation* for S if the only completely positive linear map ϕ: $B \to \mathfrak{L}(\mathfrak{H})$ which agrees with π on S is $\phi = \pi$ itself. We will say that S has *sufficiently many boundary representations* if the intersection of the kernels of all boundary representations for S is the trivial ideal $\{0\}$ in B. As we pointed out in [1], in the commutative case $S \subseteq C(X)$ (here X is a compact Hausdorff space and S separates points and contains the constant 1) this condition asserts simply that X is the Silov boundary for the closed linear span of S. The following is a slight restatement of Theorem 2.1.2 of [1].

THEOREM 0.3. (Implementation theorem.) *Let S_i be a linear subspace of a C^*-algebra B_i, $i = 1, 2$, such that S_i contains the respective identity e_i of B_i and $B_i = C^*(S_i)$. If both S_1 and S_2 have sufficiently many boundary representations, then every completely isometric linear map ϕ of S_1 on S_2, which takes e_1 to e_2, is implemented by a *-isomorphism of B_1 onto B_2.*

We also recall once again a theorem of W. F. Stinespring [17] characterizing completely positive maps of C^*-algebras.

THEOREM 0.4. *Let B be a C^*-algebra and let ϕ be a completely positive linear map of B into $\mathfrak{L}(\mathfrak{H})$. Then there is a representation π of B on a Hilbert space $\mathfrak{K}$ and a linear map V: $\mathfrak{H} \to \mathfrak{K}$ such that $\phi(x) = V^*\pi(x)V$, $x \in B$.*

Finally, recall that if $\mathcal{S}$ is a multiplicative semigroup of operators on a Hilbert space $\mathfrak{H}$ which contains I, then a closed subspace $\mathfrak{M} \subseteq \mathfrak{H}$ is said to be *semi-invariant* under $\mathcal{S}$ if there are $\mathcal{S}$-invariant subspaces $\mathfrak{M}_1 \subseteq \mathfrak{M}_2$ such that $\mathfrak{M} = \mathfrak{M}_2 \ominus \mathfrak{M}_1$. Semi-invariant subspaces are characterized by the fact that the mapping $T \in \mathcal{S} \to P_{\mathfrak{M}} T|_{\mathfrak{M}}$ is multiplicative (cf. [20], Lemma 0).

The rest of our terminology is more or less standard, and conforms with [1]. For example, a set of operators $\mathcal{S} \subseteq \mathfrak{L}(\mathfrak{H})$ is called irreducible if $\mathcal{S}$ and $\mathcal{S}^*$ have no common closed invariant subspaces other than 0 and $\mathfrak{H}$; and $sp(T)$ denotes the spectrum of the operator T.

Chapter 1. Dilation theory

Let T be a Hilbert space operator and let X be a compact subset of the complex plane which contains the spectrum of T. X is called a *spectral set* for T if $\|f(T)\| \leq \|f\|_\infty = \sup \{|f(z)|: z \in X\}$ for every rational function f which is analytic on X [13]. We begin by recalling a dilation theorem which was proved independently by C. Foias [9], C. Berger [5], and A. Lebow [12].

THEOREM 1.0. *Let X be a compact subset of the complex plane whose complement is connected and let $T \in \mathcal{L}(\mathfrak{H})$ be an operator having X as a spectral set. Then there is a normal operator N on a Hilbert space $\mathfrak{K}$ and an isometric imbedding V of $\mathfrak{H}$ in $\mathfrak{K}$ such that $sp(N) \subseteq \partial X$ and $T^n = V^* N^n V$, $n = 0, 1, 2, \dots$.*

Note that if X is the closed unit disc $\{|z| \leqslant 1\}$ then the above operator N is unitary. So this result gives a generalization, more or less, of a familar theorem of Sz.-Nagy (appendix of [15]) which asserts that every contraction has a unitary (power) dilation. We remark that in most formulations of 1.0 $\mathfrak{K}$ appears as a space containing $\mathfrak{H}$ and V is the inclusion map of $\mathfrak{H}$ in $\mathfrak{K}$ (so that V^* is the orthogonal projection of $\mathfrak{K}$ on $\mathfrak{H}$). However, we shall find the above "invariant" formulation somewhat more convenient.

1.0 suggests generalizations of itself in a number of directions. For example, if X is a multiply connected spectral set for T then one might expect to find a normal operator $N \in \mathcal{L}(\mathfrak{K})$ and an isometry $V \in \mathcal{L}(\mathfrak{H}, \mathfrak{K})$ such that $sp(N) \subseteq \partial X$ and $f(T) = V^* f(N) V$ for every rational function f analytic on X (note that if one only requires $T^n = V^* N^n V$, $n \geqslant 0$, then the conclusion already follows from 1.0 by replacing X with its polynomially convex hull). In another direction, suppose $T_1, \dots, T_n \in \mathcal{L}(\mathfrak{H})$ are commuting operators such that

$$\|p(T_1, \dots, T_n)\| \leqslant \sup \{|p(z_1, \dots, z_n)| : |z_i| \leqslant 1\}$$

for every polynomial p in the n complex variables $z_1, \dots, z_n$ (i.e., the unit polydisc $\{|z| \leqslant 1\}^n$ in a "spectral set" for $\mathbf{T} = (T_1, \dots, T_n)$). Then one might expect to find n commuting unitary operators $U_1, \dots, U_n$ on a Hilbert space $\mathfrak{K}$ and an isometry $V \in \mathcal{L}(\mathfrak{H}, \mathfrak{K})$ such that $p(T_1, \dots, T_n) = V^* p(U_1, \dots, U_n) V$ for every polynomial p. Indeed a theorem of Ando implies somewhat more than this in the case $n = 2$ [19]. But suprisingly the answer for $n = 3$ is no, as shown by a recent example of S. Parrott [14], and it now appears that 1.0 may even be false in the one-dimensional case $X \subseteq \mathbf{C}$ when X is multiply connected (however, even the case where X is an annulus is to this day unresolved).

In spite of this negative evidence, there is an appropriate generalization of 1.0 which includes minor variations of all of the above conjectures. As we will see, what is required is a strengthening of the notion of spectral set, which reduces to the usual one in the context of 1.0.

1.1. The joint spectrum

In this section we collect for later use one or two facts about joint spectra which, while quite elementary, do not appear to be very widely known. Let E be a complex Banach space, and let $\mathbf{T}=(T_1, ..., T_n)$ be an n-tuple of commuting bounded operators on E, which will be fixed throughout this section. Define the *joint spectrum* $sp(\mathbf{T})$ to be the set of all complex n-tuples $\lambda=(\lambda_1, ..., \lambda_n)\in\mathbf{C}^n$ such that $p(\lambda)$ belongs to the spectrum of $p(\mathbf{T})$ for every multivariate polynomial $p=p(z_1, ..., z_n)$. This definition seems to have been first introduced by L. Waelbroeck, and the reader should consult [18] for additional information on the functional calculus in several variables. Let $\mathcal{A}$ be a subalgebra of the algebra $\mathcal{L}(E)$ of all bounded operators on E. $\mathcal{A}$ is *inverse-closed* if whenever an element S of $\mathcal{A}$ is an invertible operator on E (i.e., $S^{-1}\in\mathcal{L}(E)$) then $S^{-1}\in\mathcal{A}$. Note that every commutative subalgebra $\mathcal{A}\subseteq\mathcal{L}(E)$ is contained in a norm-closed inverse-closed commutative algebra (for example, the double commutant $\mathcal{A}''$ will do).

Proposition 1.1.1. *Let $\mathcal{A}$ be any inverse-closed commutative Banach subalgebra of $\mathcal{L}(E)$ which contains $T_1, ..., T_n$ and the identity. Let $\mathfrak{M}$ be the space of nontrivial complex homomorphisms of $\mathcal{A}$. Then $sp(\mathbf{T})$ contains $\{(\omega(T_1), ..., \omega(T_n))\colon \omega\in\mathfrak{M}\}$.*

Proof. Choose $\omega\in\mathfrak{M}$ and define $\lambda\in\mathbf{C}^n$ by $\lambda=(\omega(T_1), ..., \omega(T_n))$. Then for every n-variate polynomial p we have $\omega(p(\mathbf{T})-p(\lambda)I)=0$. Thus $p(\mathbf{T})-p(\lambda)I$ is not invertible in $\mathcal{A}$, and since $\mathcal{A}$ is inverse-closed it follows that $p(\lambda)\in sp(p(\mathbf{T}))$. That proves $\lambda\in sp(T_1, ..., T_n)$.

Corollary 1. *$sp(\mathbf{T})$ is not empty.*

Proof. Let $\mathcal{A}$ be the double commutant of $\{T_1, ..., T_n\}$. Then $\mathcal{A}$, being a commutative Banach algebra with identity, has at least one nonzero complex homomorphism. The conclusion follows from 1.1.1.

For each polynomial p, the set $\{\lambda\in\mathbf{C}^n\colon p(\lambda)\in sp(p(T))\}$ is closed, and so $sp(\mathbf{T})$ is an intersection of closed sets. Thus $sp(\mathbf{T})$ is closed. Note also that $sp(\mathbf{T})$ is contained in the Cartesian product $sp(T_1)\times sp(T_2)\times ...\times sp(T_n)$ (for if $(\lambda_1, ..., \lambda_n)\in sp(\mathbf{T})$ then choosing the polynomial $p_i(z_1, ..., z_n)=z_i$ we see that $\lambda_i\in sp(T_i)$). In particular $sp(\mathbf{T})$ is bounded, and is therefore compact.

Corollary 2. *Let p be an n-variate polynomial which has no zeros on $sp(\mathbf{T})$. Then $p(\mathbf{T})$ is invertible.*

Proof. Let $\mathcal{A}$ be the double commutant of $\{T_1, ..., T_n\}$, and let ω be a complex homomorphism of $\mathcal{A}$. By 1.1.1, $\omega(p(\mathbf{T}))\neq 0$. Thus $p(\mathbf{T})$ is contained in no maximal ideal of $\mathcal{A}$, hence $p(T)^{-1}\in\mathcal{A}$.

We can now make use of a rudimentary operational calculus. Let X be any compact set in $\mathbf{C}^n$ which contains $sp(\mathbf{T})$, and let $rat(X)$ denote the set of all rational functions on X, that is, all quotients p/q of polynomials p, q for which q has no zeros on X. The functions in $rat(X)$ form an algebra of continuous functions on X, and we can cause these functions to act on $\mathbf{T}$ as follows. If $f \in rat(X)$, say $f = p/q$ with p, q polynomials for which $0 \notin q(X)$, then by Corollary 2 $q(\mathbf{T})$ is invertible and we may define $f(\mathbf{T}) = p(\mathbf{T})q(\mathbf{T})^{-1}$. The map $f \to f(\mathbf{T})$ is clearly a homomorphism of $rat(X)$ into $\mathfrak{L}(E)$. Let us define $R(\mathbf{T})$ as the norm closure of $\{f(\mathbf{T}): f \in rat(sp(\mathbf{T}))\}$. Then $R(\mathbf{T})$ is a commutative Banach algebra containing the identity operator. Note that since X contains $sp(\mathbf{T})$, the range of the mapping $f \in rat(X) \to f(\mathbf{T})$ is contained in $R(\mathbf{T})$.

PROPOSITION 1.1.2 (Mapping theorem). *Let $X = sp(\mathbf{T})$. Then $sp(f(\mathbf{T})) = f(X)$ for every $f \in rat(X)$.*

Proof. Choose $f \in rat(X)$ such that $f \neq 0$ on X. Writing $f = g/h$ with g, h polynomials having no zeros on X, it follows from Corollary 2 that both $g(\mathbf{T})$ and $h(\mathbf{T})$ are invertible, and hence $f(\mathbf{T}) = g(\mathbf{T})h(\mathbf{T})^{-1}$ is invertible. Thus $0 \notin f(X)$ implies $0 \notin sp\ f(\mathbf{T})$. By translation, $z \notin f(X)$ implies $z \notin sp\ f(\mathbf{T})$ for every $z \in \mathbf{C}$, which proves $sp\ f(\mathbf{T}) \subseteq f(X)$.

For the opposite inclusion, let $\lambda \in X$. Then $f - f(\lambda)$ has the form g/h with g, h polynomials such that $h \neq 0$ on X. Then $g(\lambda) = 0$, so by definition of the joint spectrum we have $0 = g(\lambda) \in sp(g(\mathbf{T}))$. Thus $g(\mathbf{T})$ is singular. Since $h(\mathbf{T})$ is invertible (Corollary 2) if follows that $f(\mathbf{T}) - f(\lambda)I = g(\mathbf{T})h(\mathbf{T})^{-1}$ is singular. Thus $f(\lambda) \in sp\ (f(\mathbf{T}))$, as required.

COROLLARY 1. *$R(\mathbf{T})$ is inverse-closed.*

Proof. Suppose $S \in R(\mathbf{T})$ is invertible in $\mathfrak{L}(E)$. Choose a sequence $f_n \in rat(sp(\mathbf{T}))$ such that $\|S - f_n(\mathbf{T})\| \to 0$. Then $f_n(\mathbf{T})$ is eventually invertible and $f_n(\mathbf{T})^{-1}$ converges to S^{-1}. By 1.1.2 f_n has no zeros on X, hence $g_n = 1/f_n$ belongs to $rat(sp(\mathbf{T}))$ (for large n). Since $g_n(\mathbf{T}) = f_n(\mathbf{T})^{-1}$ we conclude that $S^{-1} = \lim_n g_n(\mathbf{T})$ belongs to $R(\mathbf{T})$.

Note that $R(\mathbf{T})$ is in fact the *smallest* inverse-closed Banach algebra of operators which contains $\{I, T_1, ..., T_n\}$. We can now identify $sp(\mathbf{T})$ with the joint spectrum of $\mathbf{T}$ relative to the commutative Banach algebra $R(\mathbf{T})$.

COROLLARY 2. *Let $\mathfrak{M}$ be the maximal ideal space of $R(\mathbf{T})$. Then*

$$sp(\mathbf{T}) = \{(\omega(T_1), ..., \omega(T_n)): \omega \in \mathfrak{M}\}.$$

Proof. The inclusion $\supseteq$ follows from the preceding corollary and 1.1.1. Conversely, choose $\lambda \in sp(\mathbf{T})$. Then for every $f \in rat(sp(\mathbf{T}))$ we have, by 1.1.2, $|f(\lambda)| \leqslant \sup |sp(f(\mathbf{T}))| \leqslant \|f(\mathbf{T})\|$. Thus $f(\mathbf{T}) \to f(\lambda)\ (f \in rat(sp(\mathbf{T})))$ is a bounded densely defined homomorphism of

$R(\mathbf{T})$, and so there is an $\omega \in \mathfrak{M}$ such that $\omega(f(\mathbf{T})) = f(\lambda)$, $f \in rat(sp(\mathbf{T}))$. The conclusion follows after evaluating this formula with the functions $f_i(z_1, ..., z_n) = z_i$, $1 \leqslant i \leqslant n$.

We remark that all of these results extend in a straightforward manner to the case of infinitely many commuting operators.

1.2. Spectral sets and normal dilations

Let $\mathbf{T} = (T_1, ..., T_n)$ be an n-tuple of commuting operators on a Hilbert space $\mathfrak{H}$. A compact set $X \subseteq \mathbf{C}^n$ is called a *spectral set* for $\mathbf{T}$ if X contains $sp(\mathbf{T})$ and $\|f(\mathbf{T})\| \leqslant \sup \{|f(\lambda)|: \lambda \in X\}$ for every $f \in rat(X)$. We shall require a somewhat stronger definition. For each $k \geqslant 1$ let M_k be the C^*-algebra of all $k \times k$ matrices over $\mathbf{C}$; the norm on M_k is realized by causing M_k to act on the Hilbert space $\mathbf{C}^k$ in the usual way. For each $k \geqslant 1$ let $rat_k(X)$ denote the algebra of all $k \times k$ matrices over $rat(X)$. Each element in $rat_k(X)$ is then a $k \times k$ matrix of rational functions $F = (f_{ij})$, and we may define a norm on $rat_k(X)$ in the obvious way $\|F\| = \sup \{\|F(\lambda)\|: \lambda \in X\}$, thereby making $rat_k(X)$ into a noncommutative normed algebra. For each element $F = (f_{ij})$ in $rat_k(X)$ we obtain a $k \times k$ operator matrix $F(\mathbf{T}) = (f_{ij}(\mathbf{T}))$, which can be regarded as an operator on the Hilbert space $\mathfrak{H} \oplus \mathfrak{H} \oplus ... \oplus \mathfrak{H}$, a direct sum of k copies of $\mathfrak{H}$. Note that the map $F \in rat_k(X) \to F(\mathbf{T})$ is an algebraic homomorphism, for each $k = 1, 2, ...$. X is called a *complete spectral set* for $\mathbf{T}$ if $sp(\mathbf{T}) \subseteq X$ and $\|F(\mathbf{T})\| \leqslant \sup \{\|F(\lambda)\|: \lambda \in X\}$ for every matrix-valued rational function F (more precisely, for every $F \in rat_k(X)$ and every $k \geqslant 1$). We first want to show that, in the setting of 1.0, spectral sets are complete spectral sets.

PROPOSITION 1.2.1. *Let X be a compact set in $\mathbf{C}$ having connected complement. If X is a spectral set for $T \in \mathfrak{L}(\mathfrak{H})$ then it is also a complete spectral set for T.*

Proof. Let ∂X denote the topological boundary of X, and let $A = rat(X)|_{\partial X}$, regarded as a subalgebra of $C(\partial X)$. By the maximum modulus principle we have $\sup \{|f(\lambda)|: \lambda \in X\} = \sup \{|f(\lambda)|: \lambda \in \partial X\}$ for every $f \in rat(X)$, and therefore $f \to f(T)$ can be regarded as a contractive homomorphism of A into $\mathfrak{L}(\mathfrak{H})$. Now a familiar theorem of Walsh [10] asserts that every real-valued continuous function on ∂X can be approximated in norm by real parts of polynomials. In particular, A is a Dirichlet algebra in $C(\partial X)$. By 3.6.1 of [1] the map $f \in A \to f(T)$ is completely contractive. In particular we have $\|F(T)\| \leqslant \sup \{\|F(\lambda)\|: \lambda \in \partial X\}$ for every matrix-valued rational function F, and the conclusion follows from this.

We remark that 1.2.1 is false in higher dimensions; there is a commuting triple $\mathbf{T} = (T_1, T_2, T_3)$ for which the polydisc $D^3 = \{(z_1, z_2, z_x): z_i \in \mathbf{C}, |z_i| \leqslant 1\}$ is a spectral set but not a complete spectral set (see the discussion following the corollary of 1.2.2).

Now let X be a compact Hausdorff space and let A be a subalgebra of $C(X)$ which contains the constant 1 and separates points. It will be convenient not to require A to be closed. A *representation* of A is a homomorphism ϕ of A into the algebra $\mathfrak{L}(\mathfrak{H})$ of all operators on some Hilbert space $\mathfrak{H}$ such that $\phi(1)=I$ and $\|\phi(f)\| \leq \|f\|$, $f \in A$. A *dilation* of ϕ is a pair (π, V) consisting of a representation π of $C(X)$ on some Hilbert space $\mathfrak{K}$ and an operator $V \in \mathfrak{L}(\mathfrak{H}, \mathfrak{K})$ such that $\phi(f) = V^*\pi(f)V$, $f \in A$. Since π is defined on all of $C(X)$ the conditions $\pi(1)=I$ and $\|\pi\| \leq 1$ imply that π is in fact a *-representation of $C(X)$ (cf. [1], Prop. 1.2.8). Note also that the condition $\phi(1)=I$ implies that V is an isometric imbedding of $\mathfrak{H}$ in $\mathfrak{K}$. Moreover, since ϕ is multiplicative on A the mapping $T \to VV^*TVV^*$ is multiplicative on the operator algebra $\pi(A)$; thus the range of V is a semi-invariant subspace for $\pi(A)$. We recall from [1] that ϕ is called *completely contractive* if for every $k=1, 2, \ldots,$ the induced homomorphism $id \otimes \phi$: $M_k \otimes A \to M_k \otimes \mathfrak{L}(\mathfrak{H})$ has norm 1. One thinks of $M_k \otimes A$ as the algebra of all $k \times k$ matrices $F=(f_{ij})$ whose entries f_{ij} belong to A, having the obvious norm $\|F\| = \sup \{\|F(x)\|: x \in X\}$; the map $id \otimes \phi$ takes a matrix of functions (f_{ij}) to the matrix of operators $(\phi(f_{ij}))$, the latter regarded as an operator on the direct sum of k copies of $\mathfrak{H}$. Finally, the *support* of a representation π of $C(X)$ is the smallest closed subset K of X such that π annihilates $\{f \in C(X): f(K)=0\}$.

THEOREM 1.2.2. (Dilation theorem.) *Every completely contractive representation of a function algebra $A \subseteq C(X)$ has a dilation (π, V) such that the support of π is contained in the Silov boundary of X relative to A.*

Proof. Let ϕ: $A \to \mathfrak{L}(\mathfrak{H})$ be a completely contractive representation of A and let ∂X be the Silov boundary of X relative to A. By definition of ∂X the restriction map $f \in A \to f|_{\partial X} \in C(\partial X)$ is an isometric isomorphism of A onto $A|_{\partial X}$. Since both $C(X)$ and $C(\partial X)$ are commutative C^*-algebras, 1.2.11 of [1] implies that $f \in A \to f|_{\partial X}$ is *completely* isometric. Thus we may regard ϕ as a completely contractive representation of $A|_{\partial X} \subseteq C(\partial X)$, and everything will follow if we simply show that ϕ has a dilation (π, V) where π is a representation of $C(\partial X)$. But by the corollary of the extension theorem (see 0.2) ϕ has a positive (in fact completely positive) extension $\tilde{\phi}$ to $C(\partial X)$, and by a theorem of Naimark (see [1], or Theorem 0.4) $\tilde{\phi}$ has the form $V^*\pi V$ where π is a representation of $C(\partial X)$ on a Hilbert space $\mathfrak{K}$ and $V \in \mathfrak{L}(\mathfrak{H}, \mathfrak{K})$. That completes the proof.

We remark that the converse of 1.2.2 is also true, since any map of $C(X)$ having the form $f \to V^*\pi(f)V$ (with π a representation and $\|V\| \leq 1$) is completely contractive ([1]. 1.2.10). Thus *a representation of A is dilatable if, and only if, it is completely contractive.*

Now let X be a compact subset of $\mathbf{C}^n$. We shall write ∂X for the Silov boundary of X relative to $rat(X)$. It follows easily from the maximum modulus principle that ∂X is always contained in the topological boundary of X, and in the one-dimensional case $n=1$ the two boundaries are identical. In higher dimensions, however, ∂X is usually much smaller. For example if X is the two-dimensional polydisc $D\times D=\{(z, w): |z|, |w|\leqslant 1\}$, then $\partial X=\partial D\times\partial D$ is the torus while the topological boundary of $D\times D$ is $\partial D\times D\cup D\times\partial D$. Now let $\mathbf{T}=(T_1, ..., T_n)$ be a commuting n-tuple of operators on a Hilbert space $\mathfrak{H}$ and let X be a compact set in $\mathbf{C}^n$ containing $sp(\mathbf{T})$. By a *normal dilation* of $\mathbf{T}$ we mean a pair $(\mathbf{N}, V)$, where $\mathbf{N}=(N_1, ..., N_n)$ is an n-tuple of commuting normal operators on a Hilbert space $\mathfrak{K}$ and V is an isometric imbedding of $\mathfrak{H}$ in $\mathfrak{K}$, such that $sp(\mathbf{N})\subseteq X$ and $f(\mathbf{T})=V^*f(\mathbf{N})V$ for every $f\in rat(X)$. Perhaps it would be better to call such an $(\mathbf{N}, V)$ a *normal X-dilation* for $\mathbf{T}$, but the shorter more ambiguous name does not usually cause problems. We can now extend 1.0 to the general case.

COROLLARY. *Let $\mathbf{T}=(T_1, ..., T_n)$ be a commuting n-tuple of operators which has the compact set $X\subseteq\mathbf{C}^n$ as a complete spectral set. Then $\mathbf{T}$ has a normal dilation $\mathbf{N}=(N_1, ..., N_n)$ such that* $\operatorname{sp}(\mathbf{N})\subseteq\partial X$.

Proof. By hypothesis, the map $f\in rat(X)\to f(\mathbf{T})$ defines a completely contractive representation of the function algebra $rat(X)$. By 1.2.2 there is a representation π of $C(X)$ on $\mathfrak{K}$, supported on ∂X, and an isometry V of the space on which $\mathbf{T}$ acts into $\mathfrak{K}$ such that $f(\mathbf{T})=V^*\pi(f)V$, $f\in rat(X)$. If we put $f_j(z_1, ..., z_n)=z_j$ and $N_j=\pi(f_j)$, $1\leqslant j\leqslant n$, then $N=(N_1, ..., N_n)$ is a commuting family of normal operators for which $\pi(f)=f(\mathbf{N})$, $f\in rat(X)$. It is easy to see that the spectrum of $\mathbf{N}$ is the support of π (this is well-known in the case $n=1$, and the general case has a similar proof), and so the conclusion follows.

As in the remark following 1.2.2, this sufficient condition for a normal dilation is also necessary. Note also that this result, together with 1.2.1, specializes to the dilation theorem 1.0 when $n=1$ and X has no holes.

The latter remark raises the question as to whether the conclusion of the corollary is generally valid if one deletes the term "complete" from the hypothesis. The answer is no. S. Parrott [14] has given an example of a commuting triple $\mathbf{T}=(T_1, T_2, T_3)$ such that $\|p(\mathbf{T})\|\leqslant\sup\{|p(z_1, z_2, z_3)|:|z_i|\leqslant 1\}$ for every polynomial p but which has no unitary dilation $(\mathbf{U}, V)$ (i.e., $\mathbf{U}=(U_1, U_2, U_3)$ is a commuting triple of unitary operators on a Hilbert space $\mathfrak{K}$ and V is an isometry such that $p(\mathbf{T})=V^*p(\mathbf{U})V$ for every polynomial $p(z_1, z_2, z_3)$). By a theorem of Oka [10] the first condition means that $\mathbf{T}$ has the unit polydic $D\times D\times D$ as a spectral set; while the second condition means that $\mathbf{T}$ has no normal dilation $\mathbf{N}$ with $sp(\mathbf{N})\subseteq\partial(D\times D\times D)$.

As a final note, the corollary implies in Parrott's example that $D\times D\times D$ is *not* a complete spectral set for $\mathbf{T}$. In particular, a *contractive representation of a function algebra* *need not be completely contractive*. This affirms a conjecture in ([1], p. 222) and gives another example such as in appendix A.3 of [1].

1.3. Nilpotent dilations

In the foregoing discussion we have been preoccupied with normal dilations. There are times, however, when one is led to seek *non-normal* dilations with special properties (we shall encounter such a situation in section 2.3). In this section we will illustrate this by answering the following question: given $T \in \mathcal{L}(\mathfrak{H})$ and an integer $n \geq 2$, under what conditions does there exist a contraction N on a larger Hilbert space $\mathfrak{K} \supseteq \mathfrak{H}$ such that $N^n = 0$ and $T^i = P_{\mathfrak{H}} N^i|_{\mathfrak{H}}$ for $i = 0, 1, ..., n-1$? Note that a necessary condition is that $\|T\| = \|P_{\mathfrak{H}} N|_{\mathfrak{H}}\| \leq 1$, and if in addition $T^n = 0$ then the answer is trivially yes for one can take $\mathfrak{K} = \mathfrak{H}$ and $N = T$. However if $T^n \neq 0$ then the question is nontrivial, and we will see in 1.2.1 below that the answer is yes iff T^n is "small" in an appropriate sense.

For each $n \geq 2$ let S_n be the "nilpotent shift" of index n; i.e., S_n is the operator on $\mathbf{C}^n$ whose matrix relative to the usual basis is

$$\begin{pmatrix} 0 & 1 & 0 & & \dots 0 \\ 0 & 0 & 1 & 0 & \dots 0 \\ & & & & \vdots \\ & & & & 1 \\ & & & & 0 \end{pmatrix}$$

For any operator T and any positive cardinal k, $k \cdot T$ will denote the direct sum of k copies of T. Any operator S which is unitarily equivalent to $k \cdot T$ will be called a *multiple* of T, and this relation is written $S \sim k \cdot T$.

THEOREM 1.3.1 *Let* $T \in \mathcal{L}(\mathfrak{H})$, $\|T\| \leq 1$, *and let* $n \geq 2$ *be an integer. Then the following are equivalent:*

(i) *there is a Hilbert space* $\mathfrak{K} \supseteq \mathfrak{H}$ *and a contraction* $N \in \mathcal{L}(\mathfrak{K})$ *such that* $N^n = 0$ *and* $T^i = P_{\mathfrak{H}} N^i|_{\mathfrak{H}}$ *for* $i = 0, 1, ..., n-1$.

(ii) *there is a Hilbert space* $\mathfrak{K} \supseteq \mathfrak{H}$ *and a multiple* $N \sim k \cdot S_n$ *of* S_n *which acts on* $\mathfrak{K}$ *such that* $T^i = P_{\mathfrak{H}} N_i|_{\mathfrak{H}}$ *for* $i = 0, 1, ..., n-1$.

(iii) $I + 2 \operatorname{Re} \sum_{i=1}^{n-1} \lambda^i T^i \geq 0$ *for each* $\lambda \in \mathbf{C}$, $|\lambda| = 1$.

(iv) $2 \operatorname{Re} (I - \lambda T)^* \lambda^n T^n \leq I - T^* T$ *for each* $\lambda \in \mathbf{C}$, $|\lambda| = 1$.

Proof. Since the implication (ii)$\Rightarrow$(i) is trivial, we will prove (i)$\Rightarrow$(iii)$\Rightarrow$(ii) and (iii)$\Leftrightarrow$(iv).

(i) *implies* (iii). Note that if $I+2\operatorname{Re}\sum_0^{n-1} z^i T^i \geqslant 0$ for all $|z|<1$, then (iii) follows by taking strong limits as $|z|\to 1$. Since the compression of a positive operator in positive, (i) implies (iii) follows from the fact that if N is as in (i) and $|z|<1$ then $I+2\operatorname{Re}\sum_1^{n-1} z^i N^i = I+2\operatorname{Re}\sum_1^{\infty} z^i N^i = \operatorname{Re}(I+zN)(I-zN)^{-1} = (I-\bar{z}N^*)^{-1}(I-|z|^2N^*N)(I-zN)^{-1}\geqslant 0$.

(iii) *implies* (ii). Choose T satisfying (iii) and define a linear map ϕ of *span* $\{S_n^i: 0\leqslant i\leqslant n-1\}$ onto *span* $\{T^i: 0\leqslant i\leqslant n-1\}$ by $\phi: \sum_{i=0}^{n-1} A_i S_n^i \to \sum_{i=0}^{n-1} A_i T^i$. Then ϕ is obviously linear and preserves identities. We claim that ϕ is completely contractive. For that, consider the sequence of operators $\{Z_i: -\infty<i<+\infty\}$ defined by $Z_i = T^i$ for $0\leqslant i\leqslant n-1$, $Z_i=0$ for $i\geqslant n$, and $Z_i = Z_{-i}^*$ for $i<0$. Then clearly $\sum_{-\infty}^{\infty}\|Z_i\|\leqslant 2n-1<\infty$, and the hypothesis on T means that the "Fourier transform" $\zeta(\lambda)=\sum_{-\infty}^{+\infty}\lambda^i Z_i$ is positive: $\zeta(\lambda)\geqslant 0$ for $|\lambda|=1$. It follows that the map ψ: $C(X)\to \mathfrak{L}(\mathfrak{H})$ defined by

$$\psi(f)=\frac{1}{2\pi}\int_0^{2\pi} f(e^{i\theta})\,\zeta(e^{-i\theta})\,d\theta$$

is a positive linear map, and note that $\psi(z^i)=Z_i$, $i=0, 1, 2, \ldots$, where $z\in C(X)$ is the function $z(\lambda)=\lambda$. Now a positive linear map of $C(X)$ is always completely positive [17], and a completely positive linear map which preserves identities is completely contractive ([1], 1.2.10). So if we let A be the disc algebra (the closed linear span in $C(X)$ of $1, z, z^2, \ldots$), then the restriction $\psi_0=\psi|_A$ is a completely contractive linear map such that $\psi_0(z^i)=T^i$ for $0\leqslant i\leqslant n-1$, and $\psi_0(z^i)=0$ for $i\geqslant n$. Now since $T^n\neq 0$ in general, ψ_0 is *not* multiplicative; however, it does vanish on the ideal z^nA, and therefore induces a completely contractive linear map $\dot{\psi}_0$ of the quotient A/z^nA into $\mathfrak{L}(\mathfrak{H})$ such that $\dot{\psi}_0(z^i+z^nA)=T^i$, $0\leqslant i\leqslant n-1$. On the other hand, it was proved in ([1], 3.6.6) that ω: $\sum_{i=0}^{n-1} a_i S_n^i \to \sum_{i=0}^{n-1} a_i(z^i+z^nA)$ is a completely isometric linear map of *span* $\{I, S_n, \ldots, S_n^{n-1}\}$ onto A/z^nA. Finally, since the original map ϕ has the decomposition $\phi=\dot{\psi}_0\circ\omega$, it follows that ϕ is completely contractive.

Now since $\phi(I)=I$, a corollary of the extension theorem shows that ϕ has a completely positive extension to $C^*(S_n)$ (see 0.2). By Stinespring's theorem the extension of ϕ has the form $V^*\pi V$, where π is a representation of $C^*(S_n)$ on a Hilbert space $\mathfrak{K}$ and $V\in\mathfrak{L}(\mathfrak{H},\mathfrak{K})$. Since $V^*V=\phi(I)=I$, V is an isometric imbedding of $\mathfrak{H}$ in $\mathfrak{K}$. Now $S_n\in\mathfrak{L}(\mathbf{C}^n)$ and is irreducible, so that $C^*(S_n)=\mathfrak{L}(\mathbf{C}^n)$. Thus, π must be unitarily equivalent to a multiple $k\cdot id$ of the identity representation. In particular, $\pi(S_n)$ has the form $N=k\cdot B$ where B is unitarily equivalent to S_n. Thus we see that $T^i=\phi(S_n^i)=V^*N^iV$ for $0\leqslant i\leqslant n-1$, and the conclusion (ii) follows after identifying $\mathfrak{H}$ with a subspace of $\mathfrak{K}$ in terms of the isometry V.

(iii) *implies* (iv). If we multiply the inequality (iii) on the left by $I-\bar{\lambda}T^*$ and on the right by $I-\lambda T$, making use of the identity $\sum_{k=1}^{n-1}\lambda^k T^k(I-\lambda T)=\lambda T-\lambda^n T^n$, we obtain $I-T^*T-2\operatorname{Re}(I-\lambda T)^*\lambda^n T^n\leqslant 0$, from which (iv) is evident.

(iv) *implies* (iii). Assume (iv). We claim first that T^* has no eigenvalues of modulus 1. Indeed, if μ is a complex number of modulus 1 and if, to the contrary, ξ is a unit vector such that $T^*\xi=\bar{\mu}\xi$, then $\|T\xi-\mu\xi\|^2=\|T\xi\|^2-2\operatorname{Re}(T\xi,\mu\xi)+1$. Since $\|T\xi\|\leq 1$ and $(T\xi,\mu\xi)=\bar{\mu}(\xi,T^*\xi)=|\mu|^2=1$, we have $\|T\xi-\mu\xi\|^2\leq 0$ and hence $T\xi=\mu\xi$. Applying the vector state $\varrho(X)=(X\xi,\xi)$ to the inequality $\operatorname{Re}\lambda^n(I-\lambda T)^*T^n\leq I-T^*T$ we obtain $\operatorname{Re}\lambda^n(1-\bar{\lambda}\bar{\mu})\mu^n\leq 0$ for all $|\lambda|=1$, hence $\operatorname{Re}\lambda^n(1-\bar{\lambda})\leq 0$ for all $|\lambda|=1$. But this inequality is absurd for $n\geq 2$ (because the continuous function $f(\lambda)=\lambda^n(1-\bar{\lambda})=\lambda^n-\lambda^{n-1}$ has nonzero real part and zero Haar integral, hence its real part could not be nonpositive), and the assertion follows.

Now we have already made use of the identity

$$(I-\lambda T)^*(I+2\operatorname{Re}\sum_1^{n-1}\lambda^k T^k)(I-\lambda T)=I-T^*T-2\operatorname{Re}(I-\lambda T)^*\lambda^n T^n.$$

So if $\lambda\in\mathbf{C}$ is of unit modulus, then condition (iv) implies that $((I+2\operatorname{Re}\sum_1^{n-1}\lambda^kT^k)\eta,\eta)\geq 0$ for every vector η of the form $(I-\lambda T)\xi$, $\xi\in\mathfrak{H}$. Since, by the preceding paragraph, the null-space of $(I-\lambda T)^*$ is trivial, these η's are dense in $\mathfrak{H}$, and now condition (iii) follows. That completes the proof.

Our main application of this result will be when $T^n\neq 0$. However, note that if $T^n=0$ then (iv) is satisfied, and we conclude that there is a multiple $N\sim\infty\cdot S_n$ acting on a larger space such that $T^k=P_{\mathfrak{H}}N^k|_{\mathfrak{H}}$, $0\leq k\leq n-1$. In this case the equation persists for $k\geq n$, so that N is a *power* dilation of T. We remark that this special case (but not 1.2.1 itself) could also have been deduced from the results of ([1], section 3.6), or by a direct argument sketched in section 2.5.

The following sufficient condition will be useful in chapter 2. It asserts, roughly, that T has a nilpotent dilation as above when T^n is "small". For an operator T, $|T|$ denotes the positive square root of T^*T.

PROPOSITION 1.3.2 *Let T be a contraction and let $n\geq 2$. Suppose there is a positive constant ϱ such that $T^nT^{*n}\leq\varrho T^{*n}T^n$ and $|T^n|\leq(8+8\varrho)^{-\frac{1}{2}}(I-T^*T)$. Then conditions* (i) *through* (iv) *of* 1.2.1 *are satisfied.*

Proof. It suffices to verify condition (iv) of 1.2.1; and for that we shall make use of the operator inequality $(\operatorname{Re}X)^2\leq\frac{1}{2}(X^*X+XX^*)$, which is easily proved by expanding the right side of the inequality in terms of the real and imaginary parts of X. Applying this to $X=(I-\lambda T)^*\lambda^nT^n$ (where λ is a complex number of modulus 1) we obtain

$$(\operatorname{Re}(I-\lambda T)^*\lambda^nT^n)^2\leq\tfrac{1}{2}(T^{*n}|I-\lambda T|^2T^n+(I-\lambda T)^*T^{*n}(I-\lambda T)).$$

Now since $\|T\|\leq 1$ we have $T^{*n}|I-T|^2T^n\leq\|I-\lambda T\|^2T^{*n}T^n\leq 4T^{*n}T^n$, and on the other hand $(I-\lambda T)^* T^n T^{*n}(I-\lambda T) \leqslant \varrho(I-\lambda T)^* T^{*n}T^n(I-\lambda T) = \varrho T^{*n}|I-\lambda T|^2 T^n \leqslant 4\varrho T^{*n}T^n$. Thus $(\text{Re}\,(I-\lambda T)\lambda^n T^n)^2 \leqslant (2+2\varrho)\,T^{*n}T^n$. Now the function $f(X)=X^{\frac{1}{2}}$ is operator-monotone on the set of all positive operators on $\mathfrak{H}$ [4], and we may conclude from the above inequality that $|\text{Re}\,(I-\lambda T)\lambda^n T^n| \leqslant (2+2\varrho)^{\frac{1}{2}}\,|T^n|$. By hypothesis, the right side is $\leqslant \frac{1}{2}(I-T^*T)$, and since $X \leqslant |X|$ is valid for every self-adjoint operator X we conclude that $\text{Re}\,(I-\lambda T)^*\lambda^n T^n \leqslant \frac{1}{2}(I-T^*T)$, for all $\lambda \in \mathbf{C}$, $|\lambda|=1$. Condition (iv) of 1.21 follows.

Chapter 2. More on boundary representations

2.1. The boundary theorem

Let S be a linear space of operators on a Hilbert space $\mathfrak{H}$, which contains the identity. The implementation theorem (0.3) asserts that certain isometric linear maps of S are implemented by *-isomorphisms of $C^*(S)$ provided that S has sufficiently many boundary representations. The special case of greatest interest is where S is an irreducible set of operators. Here the identity representation of $C^*(S)$ (abbreviated *id*) is irreducible and of course has kernel $\{0\}$, so the hypothesis of the boundary theorem will be satisfied if *id* is a boundary representation for S (significantly, this sufficient condition is often necessary as well, see 2.1.0 below). Unfortunately, *id* is frequently not a boundary representation for S, even in the "nice" situation where $C^*(S)$ is a GCR algebra; see 3.54 of [1] for a class of examples. In theorem 2.4.5 of [1] we gave a very general characterization of boundary representations which, while effective for dealing with certain "maximal" representations of the disc algebra, is apparently of no help in determining when *id* is a boundary representation for S in the general case where $C^*(S)$ is an irreducible GCR algebra. In this section we are going to take up this problem in a somewhat more general setting, namely that in which S is an irreducible set of operators such that $C^*(S)$ contains the algebra $\mathcal{C}(\mathfrak{H})$ of all compact operators (it is easy to see that the latter condition is equivalent to saying that $C^*(S)$ is *not* an NGCR algebra, see the discussion preceding 2.3.1). We will give a complete solution of this problem in terms of criteria that turn out to be very easy to check in special cases.

We begin with a simple result that provides a useful reduction.

PROPOSITION 2.1.0. *Let S be an irreducible subset of $\mathcal{L}(\mathfrak{H})$, such that S contains the identity and $C^*(S)$ contains $\mathcal{C}(\mathfrak{H})$. Then S has sufficiently many boundary representations if, and only if, the identity representation is a boundary representation for S.*

Proof. Sufficiency is trivial, so assume that S has sufficiently many boundary representations. Then there is a boundary representation π which does not annihilate $\mathcal{C}(\mathfrak{H})$. Since

$\mathcal{C}(\mathfrak{H})$ is an ideal and π is irreducible, the restriction of π to $\mathcal{C}(\mathfrak{H})$ is irreducible, and therefore is equivalent to the identity representation of $\mathcal{C}(\mathfrak{H})$. But this implies π is equivalent to the identity representation of $C^*(S)$ (again, because $\mathcal{C}(\mathfrak{H})$ is an ideal), proving that $id \sim \pi$ is a boundary representation.

THEOREM 2.1.1. (Boundary theorem.) *Let S be an irreducible set of operators on a Hilbert space $\mathfrak{H}$, such that S contains the identity and $C^*(S)$ contains the algebra $\mathcal{C}(\mathfrak{H})$ of all compact operators on $\mathfrak{H}$. Then the identity representation of $C^*(S)$ is a boundary representation for S if, and only if, the quotient map q: $\mathfrak{L}(\mathfrak{H}) \to \mathfrak{L}(\mathfrak{H})/\mathcal{C}(\mathfrak{H})$ is not completely isometric on the linear span of $S \cup S^*$.*

Remark. The sufficiency part of this theorem is of particular interest when S is a "small" subset of $C^*(S)$. For example, if T is an irreducible operator whose distance from the compact operators is less than $\|T\|$ (i.e., $\|T-K\| < \|T\|$ for some $K \in \mathcal{C}(\mathfrak{H})$ then it follows that $C^*(T) \supseteq \mathcal{C}(\mathfrak{H})$, and by the boundary theorem *id* is a boundary representation for $S = \{I, T\}$ (see the corollary below).

Proof. The necessity half is straightforward, and we dispose of that first. Contrapositively, assume that the quotient map q: $\mathfrak{L}(\mathfrak{H}) \to \mathfrak{L}(\mathfrak{H})/\mathcal{C}(\mathfrak{H})$ is completely isometric on *span* $(S \cup S^*)$. We will produce a completely positive map ϕ: $C^*(S) \to \mathfrak{L}(\mathfrak{H})$ such that $\phi \neq id$, but $\phi|_S = id|_S$.

Let S_1 be the norm closure of *span* $(S \cup S^*)$. Then q is completely isometric on S_1, and so is its inverse q^{-1}: $q(S_1) \to S_1$. Since q^{-1} preserves the identity it is completely positive on $q(S_1)$ ([1], 1.2.9) and by the extension theorem there is a completely positive map ψ: $\mathfrak{L}(\mathfrak{H})/\mathcal{C}(\mathfrak{H}) \to \mathfrak{L}(\mathfrak{H})$ which extends q^{-1} on $q(S_1)$. Define ϕ: $C^*(S) \to \mathfrak{L}(\mathfrak{H})$ by $\phi = \psi \circ q$. ϕ is completely positive (since both q and ψ are) and leaves each element of S fixed. On the other hand ϕ annihilates $\mathcal{C}(\mathfrak{H})$ (because q does), and since $C^*(S)$ contains $\mathcal{C}(\mathfrak{H})$, ϕ is not the identity map of $C^*(S)$.

Turning now to the other implication, we want to show that if q is not completely isometric on *span* $(S + S^*)$ then *id* is a boundary representation for S. Note first that it suffices to deduce the conclusion from the stronger hypothesis that q is not isometric on *span* $(S + S^*)$. For if, in the general case, we choose $k \geq 1$ so that $q \otimes id_k$: $C^*(S) \otimes M_k \to (C^*(S)/\mathcal{C}(\mathfrak{H})) \otimes M_k$ is not isometric on *span* $(S + S^*) \otimes M_k$ and realize $S \otimes M_k$ as operators on $\mathfrak{H} \otimes \mathbf{C}^k$ (i.e., all $k \times k$ operator matrices over S) and $q \otimes id_k$ as the canonical map of $C^*(S \otimes M_k)$ into $C^*(S \otimes M_k)/\mathcal{C}(\mathfrak{H} \otimes \mathbf{C}^k)$, then note that all hypotheses are preserved, so by the special case we conclude that the identity representation of $C^*(S \otimes M_k) = C^*(S) \otimes M_k$ is a boundary representation for $S \otimes M_k$. This, however, implies that the identity representation of

$C^*(\mathcal{S})$ is a boundary representation for $\mathcal{S}$; indeed if ϕ: $C^*(\mathcal{S}) \to \mathfrak{L}(\mathfrak{H})$ is a completely positive linear extension of $id|_{\mathcal{S}}$, then $\phi \otimes id_k$: $C^*(\mathcal{S}) \otimes M_k = C^*(\mathcal{S} \otimes M_k) \to \mathfrak{L}(\mathfrak{H} \otimes \mathbb{C}^k)$ is a completely positive extension of $id|_{\mathcal{S}} \otimes M_k$, hence $\phi \otimes id_k$ is the identity map, hence ϕ is the identity map.

Thus we may assume that there is an operator T in *span* $(\mathcal{S} + \mathcal{S}^*)$ and a compact operator K such that $\|T + K\| < \|T\|$. Let ϕ be a completely positive map of $\mathfrak{L}(\mathfrak{H})$ into itself, which will be fixed throughout the remainder of the proof, such that $\phi(S) = S$, $S \in \mathcal{S}$. We will show that ϕ leaves $C^*(\mathcal{S})$ elementwise fixed (note that this implies *id* is a boundary representation for $\mathcal{S}$, since by the extension theorem every completely positive map of $C^*(\mathcal{S})$ into $\mathfrak{L}(\mathfrak{H})$ extends to $\mathfrak{L}(\mathfrak{H})$). Let $\mathcal{J} \subseteq \mathfrak{L}(\mathfrak{H})$ be the set of *all* fixed points of ϕ. Then $\mathcal{J}$ contains $\mathcal{S}$, so the desired conclusion follows if we can prove that $\mathcal{J}$ is a C^*-algebra. Now $\mathcal{J}$ is a norm-closed self-adjoint linear space (since ϕ is bounded and self-adjoint) and we want to show that $x, y \in \mathcal{J}$ implies $y^*x \in \mathcal{J}$. From the polarization formula

$$y^*x = \tfrac{1}{4}[(x+y)^*(x+y) - (x-y)^*(x-y) + i(x+iy)^*(x+iy) - i(x-iy)^*(x-iy)]$$

it is evidently enough to establish the following assertion: *for every* $X \in \mathfrak{L}(\mathfrak{H})$, $\phi(X) = X$ *implies* $\phi(X^*X) = X^*X$.

In the proof of this claim, we will construct a normal idempotent map of a von Neumann algebra, which is suitably related to ϕ. The following lemma gives one of the key properties of such maps.

LEMMA 1. *Let ψ be a normal completely positive linear map of a von Neumann $\mathcal{R}$ into itself such that $\psi \circ \psi = \psi$ and $\|\psi\| \leq 1$. Let P be the support projection of ψ (i.e., $P^\perp$ is the largest projection in the kernel of ψ). Then P commutes with the fixed points of ψ.*

Proof of Lemma. We remark that the existence of P is established just as if ψ were a normal state, and moreover P satisfies $\psi(X) = \psi(PX) = \psi(XP)$, and $\psi(X^*X) = 0$ if and only if $PX^*XP = 0$, $X \in \mathcal{R}$ (see [7], p. 61).

Since ψ is self-adjoint its fixed points form a self-adjoint family of operators in $\mathcal{R}$. Thus it suffices to show that for every $X \in \mathcal{R}$, $\psi(X) = X$ implies $PXP = XP$; in turn, this follows if we prove $PX^*PXP = PX^*XP$, since for every vector ξ in the underlying Hilbert space we have $\|(I-P)XP\xi\|^2 = \|XP\xi\|^2 - \|PXP\xi\|^2 = (PX^*XP\xi, \xi) - (PX^*PXP\xi, \xi)$.

So choose $X \in \mathcal{R}$ such that $\psi(X) = X$. Note first that $X^*X \leq \psi(X^*PX)$; for $X^*X = \psi(X)^*\psi(X) = \psi(PX)^*\psi(PX) \leq \psi(X^*PX)$, the last inequality by the Schwarz inequality for completely positive linear maps of norm 1 (which follows directly from the canonical representation $\psi = V^*\pi V$, see [1] 1.1.1). Thus $X^*PX \leq X^*X \leq \psi(X^*PX)$ and, multiplying on left and right by P, we obtain $PX^*PXP \leq PX^*XP \leq P\psi(X^*PX)P$. Thus it suffices to show that the two extreme members of this inequality are the same, i.e., $P(\psi(X^*PX)-X^*PX)P=0$. But by the preceding, $\psi(X^*PX)-X^*PX$ is positive, and it is annihilated by ψ because $\psi\circ\psi=\psi$. The conclusion therefore follows from the properties of support projections. The proof of the lemma is complete.

LEMMA 2. *Let $\mathcal{R}$ be a von Neumann algebra and let $\mathcal{R}_0$ be a weakly dense C^*-subalgebra of $\mathcal{R}$ such that every bounded linear functional on $\mathcal{R}_0$ has an ultraweakly continuous extension to $\mathcal{R}$. Then for every completely positive linear map ϱ: $\mathcal{R}_0\to\mathcal{R}_0$ for which $\|\varrho\|\leqslant 1$, there is a normal completely positive linear map ψ: $\mathcal{R}\to\mathcal{R}$ such that $\|\psi\|\leqslant 1$, $\psi\circ\psi=\psi$, $\psi\circ\varrho=\psi$ on $\mathcal{R}_0$, and $\varrho(T)=T$ implies $\psi(T)=T$ for all $T\in\mathcal{R}_0$.*

Proof. For each integer $n\geqslant 1$ define ϱ^n: $\mathcal{R}_0\to\mathcal{R}_0$ as the n-fold composition of ϱ with itself, and let Λ be a Banach limit on the additive semigroup of positive integers. Define ψ_0: $\mathcal{R}_0\to\mathcal{R}$ as follows; fix $T\in\mathcal{R}_0$ and define a bilinear form $[\cdot,\cdot]$ on $\mathfrak{H}\times\mathfrak{H}$ ($\mathfrak{H}$ being the underlying Hilbert space) by $[\xi,\eta]=\Lambda_n(\varrho^n(T)\xi,\eta)$, $\xi,\eta\in\mathfrak{H}$ ($\Lambda_n a_n$ denotes the value of Λ at the bounded sequence $\{a_n\}$). $[\cdot,\cdot]$ has norm at most $\sup_n\|\varrho^n(T)\|\leqslant\|T\|$, and by a familiar lemma of Riesz there is a bounded operator $\psi_0(T)$ on $\mathfrak{H}$ such that $(\psi_0(T)\xi,\eta)=\Lambda_n(\varrho^n(T)\xi,\eta)$. Clerarly $T\mapsto\psi_0(T)$ is a linear map of $\mathcal{R}_0$ into $\mathcal{L}(\mathfrak{H})$ of norm at most 1. Moreover, a standard separation theorem shows that $\psi_0(T)$ belongs to the weakly closed convex hull of $\{\varrho^n(T)$: $n\geqslant 1\}$. In particular $\psi_0(\mathcal{R}_0)\subseteq\mathcal{R}$. Since each ϱ^n is positive, it also follows that ψ_0 is positive. For each $k\geqslant 1$ we can apply a separation theorem to $\psi_0\otimes id_k$: $\mathcal{R}_0\otimes M_k\to\mathcal{R}\otimes M_k$ in a similar way to conclude that ψ_0 is in fact completely positive.

One can easily find a normal extension ψ of ψ_0 to $\mathcal{R}$. The details are, briefly, as follows. For each bounded linear functional f on $\mathcal{R}_0$ let $\tilde f$ denote its ultraweakly continuous extension to $\mathcal{R}$. Then $\|\tilde f\|=\|f\|$ (since by Kaplansky's density theorem the unit ball of $\mathcal{R}_0$ is ultraweakly dense in that of $\mathcal{R}$), so that $f\mapsto\tilde f$ is an isometric isomorphism of the dual of $\mathcal{R}_0$ onto the predual $\mathcal{R}_*$ of $\mathcal{R}$. For each $T\in\mathcal{R}$ define $\psi(T)$ as the unique element of $\mathcal{R}$ such that $f(\psi(T))=(f\circ\psi_0)^\sim(T)$, $f\in\mathcal{R}_*$. Clearly ψ is a linear extension of ψ_0, and it has the required continuity property because $f\circ\psi=(f\circ\psi_0)^\sim$ is an ultraweakly continuous functional for each $f\in\mathcal{R}_*$. The same formula shows ψ is positive, and in fact is completely positive since ψ_0 was.

That $\|\psi\|\leqslant 1$ is a trivial consequence of $\|f\|=\|\tilde f\|$ and $\|\psi_0\|\leqslant 1$. The condition $\psi\circ\varrho(T)=\psi(T)$, $T\in\mathcal{R}_0$, follows from the translation invariance of Λ. Now choose $T\in\mathcal{R}_0$ such that $\varrho(T)=T$. Then $\varrho^n(T)=T$ for every $n\geqslant 1$, thus $(\psi_0(T)\xi,\eta)=\Lambda_n(\varrho^n(T)\xi,\eta)=(T\xi,\eta)$, $\xi,\eta\in\mathfrak{H}$, and we have $\psi(T)=\psi_0(T)=T$. It remains to show that $\psi\circ\psi=\psi$. Fix $T\in\mathcal{R}_0$. Then $\psi\circ\varrho(T)=\psi(T)$ implies $\psi\circ\varrho^n(T)=\psi(T)$ for $n\geqslant 1$, so that $\psi(X)=\psi(T)$ for all X in the weakly closed convex hull of $\{\varrho^n(T)$: $n\geqslant 1\}$ and taking $X=\psi(T)$ we conclude $\psi(\psi(T))=$

$\psi(T)$. The condition $\psi\circ\psi=\psi$ on $\mathcal{R}$ now follows by continuity, completing the proof of Lemma 2.

Returning now to the proof of the boundary theorem, let π be the universal representation of $\mathcal{L}(\mathfrak{H})$ ([8], 2.7.6 and 12.1.3), and let $\mathcal{R}$ be the von Neumann algebra generated by $\pi(\mathcal{L}(\mathfrak{H}))$. π decomposes uniquely into a direct sum $\pi=\pi_1\oplus\pi_2$, where π_2 annihilates $\mathcal{C}(\mathfrak{H})$ and π_1 is a nonzero multiple of the identity representation. Let E be the projection on the range of π_1. Then E is a nonzero *central* projection in $\mathcal{R}$ (because π_1 and π_2 are disjoint [8], 5.2.4), and it is minimal central because $\mathcal{R}E=\pi_1(\mathcal{L}(\mathfrak{H}))$ is a factor (isomorphic to $\mathcal{L}(\mathfrak{H})$). Note, finally, that if $X\in\mathcal{L}(\mathfrak{H})$ and $\pi(X)E=0$ then $X=0$, because $\pi(X)E=\pi_1(X)$ and π_1 is a faithful representation of $\mathcal{L}(\mathfrak{H})$.

Now fix ϕ as in the discussion preceding Lemma 1. Taking $\mathcal{R}_0=\pi(\mathcal{L}(\mathfrak{H}))$, then every bounded linear functional on $\mathcal{R}_0$ has an ultraweakly continuous extension to $\mathcal{R}$ ([8], 12.1.3) and we may apply Lemma 2 to the map $\pi\circ\phi\circ\pi^{-1}$: $\mathcal{R}_0\to\mathcal{R}_0$ to obtain a normal completely positive idempotent linear map ψ: $\mathcal{R}\to\mathcal{R}$ such that $\|\psi\|\leqslant 1$, ψ leaves $\{\pi(T): \phi(T)=T\}$ fixed, and $\psi\circ\pi\circ\phi=\psi\circ\pi$ on $\mathcal{L}(\mathfrak{H})$. Let P be the support projection of ψ. Then we claim: $P\geqslant E$. Indeed P is in $\mathcal{R}$ and by Lemma 1 P commutes with $\pi(\mathcal{S})$. Thus PE commutes with $\pi(\mathcal{S})E=\pi_1(\mathcal{S})$. Since $\mathcal{S}$ is an irreducible set of operators it generates $\mathcal{L}(\mathfrak{H})$ as a von Neumann algebra, and since π_1 is a normal representation of $\mathcal{L}(\mathfrak{H})$, $\pi_1(\mathcal{S})$ generates $\pi_1(\mathcal{L}(\mathfrak{H}))=\mathcal{R}E$ as a von Neumann algebra. Thus PE commutes with $\mathcal{R}E$ and so PE is a central projection in $\mathcal{R}E$. By minimality of E we have $PE=0$ or E. Now we claim PE cannot be 0. For if it were then $P\leqslant E^\perp$ and so $\psi\circ\pi=\psi\circ P\pi P=\psi\circ P(0\oplus\pi_2)P=\psi\circ(0\oplus\pi_2)$. Thus if we choose $T\in span(\mathcal{S}+\mathcal{S}^*)$ and a compact operator K such that $\|T+K\|<\|T\|$, then $\phi(T)=T$ implies $\pi(T)=\psi\circ\pi(T)=\psi(0\oplus\pi_2(T))=\psi(0\oplus\pi_2(T+K))$ because $\pi_2=0$ on $\mathcal{C}(\mathfrak{H})$. But the left hand member has norm $\|\pi(T)\|=\|T\|$ while the norm of the right side is at most $\|\psi\|\cdot\|T+K\|\leqslant\|T+K\|<\|T\|$, a contradiction. This proves that $PE=E$, as asserted.

Now to complete the proof, choose $X\in\mathcal{L}(\mathfrak{H})$ such that $\phi(X)=X$, and let us prove $\phi(X^*X)=X^*X$. By the Schwarz inequality (cf. Lemma 1) we have $X^*X=\phi(X)^*\phi(X)\leqslant\phi(X^*X)$, and hence $\pi(\phi(X^*X)-X^*X)$ is a positive element of $\mathcal{R}$. This element is annihilated by ψ because $\psi\circ\pi\circ\phi=\psi\circ\pi$ on $\mathcal{L}(\mathfrak{H})$, and therefore $P\pi(\phi(X^*X)-X^*X)P=0$ because P is the support of ψ. Multiply on left and right by E and use $PE=E$ to obtain $\pi_1(\phi(X^*X)-X^*X)=E\pi(\phi(X^*X)-X^*X)E=0$. Since π_1 is a faithful representation of $\mathcal{R}$ we conclude that $\phi(X^*X)-X^*X=0$, and the proof is finished.

Remark 1. The compact operators on $\mathfrak{H}$ are "approximately" finite rank operators. To say that an operator $T\in\mathcal{L}(\mathfrak{H})$ satisfies $\|q(T)\|=\|T\|$ (q being the quotient map onto $\mathcal{L}(\mathfrak{H})/\mathcal{C}(\mathfrak{H})$) means that the norm of T cannot be decreased by a compact perturbation;

in other words, the norm of T is achieved at "infinity". Thus the condition of the boundary theorem is, roughly, that *span*$(\mathcal{S} \cup \mathcal{S}^*)$ contains some operators whose norms are *not* achieved at infinity.

Remark 2. The sufficiency proof can be easily adapted to establish the following slightly more general result. If ϕ: $\mathfrak{L}(\mathfrak{H}) \to \mathfrak{L}(\mathfrak{H})$ is a completely positive map of norm 1 whose set $\mathfrak{F}$ of fixed points is irreducible and is such that the quotient map q: $\mathfrak{L}(\mathfrak{H}) \to \mathfrak{L}(\mathfrak{H})/\mathfrak{C}(\mathfrak{H})$ is not completely isometric on $\mathfrak{F}$, then $\mathfrak{F}$ is a C^*-algebra. This result is of interest and appears to be nontrivial even in the finite-dimensional case: thus if ϕ is a norm 1 completely positive map of a matrix algebra into itself whose fixed points *algebraically* generate the full matrix algebra, then ϕ is the identity map. But the only proof we know in this special case is essentially the one given. One does not need the universal representation here, of course, but the main steps, Lemma 1 and the construction of the idempotent map ψ, seem essential. It would be desirable to have a simpler proof of the finite dimensional theorem. For example, one might conjecture that the fixed points of a completely positive map of a matrix algebra into itself always form an algebra (assuming, say, that the identity is left fixed). However, this conjecture is false: consider the completely positive map which takes a 3×3 matrix (a_{ij}) into

$$\begin{pmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & \frac{1}{2}(a_{11}+a_{22}) \end{pmatrix}$$

Note that this map is even idempotent, but not faithful.

Along these lines, we append the following observation. *If ϕ is a faithful completely positive idempotent linear map of a C^*-algebra into itself such that $\|\phi\| \leq 1$, then the fixed points of ϕ form a C^*-algebra.* For the proof, it suffices to show that $\phi(X)=X$ implies $\phi(X^*X)=X^*X$, as in the proof of the boundary theorem. Let $H=\phi(X^*X)$-X^*X. $\phi(X)=X$ and the Schwarz inequality imply that $H \geq 0$, and $\phi(H)=0$ follows from idempotence. Thus $H=0$ because ϕ is faithful.

Remark 3. Our original version of the sufficiency part of the boundary theorem assumed that $\mathcal{S}$ was an irreducible set of compact operators, see Theorem 1 of [3]. C. A. Akemann and the author then adapted the proof to include the case where $\mathcal{S}$ contains a single nonzero compact operator. The above is a third adaptation, which is evidently in final form.

Problem. Does there exist a subset S of a C^*-algebra (such that S contains the identity) such that $C^*(S)$ has *no* boundary representations for S? We remark that if such a set S exists then one can construct an algebra with the same feature. Indeed, let B_1 be the C^*-

algebra of all 3×3 matrices over $C^*(S)$ and take S_1 to be the subalgebra of B_1 consisting of all 3×3 matrices of the form

$$\begin{pmatrix} \lambda e & \mu e & s \\ 0 & \lambda e & \mu e \\ 0 & 0 & \lambda e \end{pmatrix}$$

where e is the identity of $C^*(S)$, λ, μ are complex numbers, and s runs over the linear span of S. One can verify that $B_1 = C^*(S_1)$, and S_1 is an example whenever S is.

COROLLARY. *Let $\mathcal{S}_1$ and $\mathcal{S}_2$ be irreducible linear spaces of operators on Hilbert space $\mathfrak{H}_1$ and $\mathfrak{H}_2$. Suppose $\mathcal{S}_i$ contains an operator T_i whose distance from the compact operators is less than $\|T_i\|$, $i=1, 2$. Then every completely isometric linear map of $\mathcal{S}_1$ onto $\mathcal{S}_2$ which takes I_1 to I_2 is implemented by a unitary operator from $\mathfrak{H}_1$ to $\mathfrak{H}_2$.*

Proof. First we note that $C^*(\mathcal{S}_1)$ contains $\mathcal{C}(\mathfrak{H}_1)$. For if not, then the quotient map q: $\mathcal{L}(\mathfrak{H}_1)\to\mathcal{L}(\mathfrak{H}_i)/\mathcal{C}(\mathfrak{H}_1)$ would be injective, therefore isometric, on $C^*(\mathcal{S}_1)$, and therefore isometric on the norm closure of $\mathcal{S}+\mathcal{S}^*$, a contradiction.

By the boundary theorem, *id* is a boundary representation for $\mathcal{S}_1$, and in particular the intersection of the kernels of all boundary representations of $C^*(\mathcal{S}_1)$ for $\mathcal{S}_1$ is $\{0\}$. The same is true of $\mathcal{S}_2$, so by the implementation theorem every completely isometric linear map ϕ: $\mathcal{S}_1\to\mathcal{S}_2$ such that $\phi(I_1)=I_2$ is implemented by a *-isomorphism π: $C^*(\mathcal{S}_1)\to C^*(\mathcal{S}_2)$. π is an irreducible representation of $C^*(\mathcal{S}_1)$, thus its restriction to the ideal $\mathcal{C}(\mathfrak{H}_1)\subseteq C^*(\mathcal{S}_1)$ is also irreducible and so is unitarily equivalent to the identity representation of $\mathcal{C}(\mathfrak{H})$ ([8], section 4.1). Since $\mathcal{C}(\mathfrak{H})$ is an ideal in $C^*(\mathcal{S}_1)$, π itself is unitarily implemented, and therefore so is $\phi=\pi|_{s_1}$.

2.2. Almost normal operators and a dilation theorem

The criterion of the boundary theorem becomes particularly easy to check in the presence of almost normal operators. An operator $T\in\mathcal{L}(\mathfrak{H})$ is *almost normal* if T^*T-TT^* is compact; i.e., T is normal modulo $\mathcal{C}(\mathfrak{H})$. It is an empirical fact that many of the most commonly studied operators are almost normal. For example, subnormal operators are often almost normal, a prototype being the unilateral shift S of finite multiplicity (here S^*S-SS^* is a finite rank projection); and the same is true of the compression of S to one of its semi-invariant subspaces (such a compression T is even "almost unitary" since both $I-T^*T$ and $I-TT^*$ have finite rank). As another type of example let T be a unilateral weighted shift with weights α_0, α_1, α_2, ..., that is, $0<|\alpha_n|\leqslant M<\infty$ for all n and T is defined on an orthonormal base e_0, e_1, e_2, ... by $Te_n=\alpha_n e_{n+1}$. A simple computation shows that T^*T-TT^*: $e_n\to(|\alpha_n|^2-|\alpha_{n-1}|^2)e_n$ for $n\geqslant 1$, so that *T is almost normal if and only if*

$|\alpha_{n+1}|-|\alpha_n|\to 0$ *as* $n\to\infty$. Thus the moduli $|\alpha_n|$ may oscillate so long as the period of oscillation becomes appropriately large at ∞; for example the weights $\alpha_n=1+\sin\sqrt{n}$ define an almost normal weighted shift. We also remark that *all* unilateral weighted shifts are irreducible, since a routine matrix calculation shows that the only self-adjoint matrices that commute with the matrix of a unilateral weighted shift (relative to the obvious basis) are scalars.

The *essential spectrum* of an operator $T\in\mathcal{L}(\mathfrak{H})$ is the spectrum of the image of T in the Calkin algebra $\mathcal{L}(\mathfrak{H})/\mathcal{C}(\mathfrak{H})$; this will be written $esp(T)$. It is clear that $esp(T)$ is a subset of $sp(T)$ which is invariant under compact perturbations of T, and thus $esp(T)\subseteq\bigcap sp(T+K)$, the intersection taken over all compact operators K. The larger set in this formula is usually called the *Weyl spectrum* of T, and it may contain $esp(T)$ properly, cf. [6]. $|esp(T)|$ will denote $\sup\{|\lambda|: \lambda\in esp(T)\}$, the essential spectral radius of T.

In the following theorem we assume that the underlying Hilbert space has dimension at least 2.

THEOREM 2.2.1. *Let S be an irreducible set of commuting almost normal operators which contains the identity. Assume that $|esp(T)|<\|T\|$ for some element $T\in S$. Then the identity representation of $C^*(S)$ is a boundary representation for S.*

Proof. Note first that $C^*(S)$ contains all compact operators. For if $T\in S$ is normal, then by Fuglede's Theorem $T\in C^*(S)'$ and hence T is a scalar. Thus for every non-scalar $T\in S$, T^*T-TT^* is a *nonzero* compact operator in $C^*(S)$; since $C^*(S)$ is irreducible a standard result (cf. [8]) implies that $C^*(S)$ contains the entire algebra $\mathcal{C}(\mathfrak{H})$ of compact operators.

Now let q be the canonical map of $\mathcal{L}(\mathfrak{H})$ onto the Calkin algebra $\mathcal{L}(\mathfrak{H})/\mathcal{C}(\mathfrak{H})$. Then $q(S)$ is a commuting set of normal elements in $\mathcal{L}(\mathfrak{H})/\mathcal{C}(\mathfrak{H})$, and in particular $\|q(T)\|$ is the spectral radius of $q(T)$ for every $T\in S$. The latter is $|esp(T)|$, so by hypothesis q is *not* isometric on S. The desired conclusion now follows from the boundary theorem.

When the set S of operators is an *algebra* one may obtain other criteria, of which the following is a sample.

THEOREM 2.2.2. *Let $\mathcal{A}$ be any non-commutative irreducible algebra of almost normal operators which contains the identity. Then the identity representation of $C^*(\mathcal{A})$ is a boundary representation for $\mathcal{A}$.*

Proof. As in the preceding result, $C^*(\mathcal{A})$ contains $\mathcal{C}(\mathfrak{H})$, and the canonical map q: $\mathcal{L}(\mathfrak{H})\to\mathcal{L}(\mathfrak{H})/\mathcal{C}(\mathfrak{H})$ maps $\mathcal{A}$ onto an algebra of normal elements in $\mathcal{L}(\mathfrak{H})/\mathcal{C}(\mathfrak{H})$. Now for

$X, Y \in q(\mathcal{A})$ define $[X, Y] = XY^* - Y^*X$. Then $[\cdot, \cdot]$ is a sesquilinear form on $q(\mathcal{A})$ such that $[X, X] = 0$, $X \in q(\mathcal{A})$, by normality of X. Thus the usual polarization identity shows that $[X, Y] = 0$ for all $X, Y \in q(\mathcal{A})$, hence $XY^* = Y^*X$. By Fuglede's Theorem we conclude that $q(\mathcal{A})$ is commutative. So choosing S, T to be any two elements of $\mathcal{A}$ which do not commute, then we see that $ST - TS \neq 0$ while $q(ST - TS) = 0$. In particular q is not isometric on $\mathcal{A}$, and the proof is completed by an application of the boundary theorem.

We remark that it is very easy to give examples of noncommutative algebras of almost normal operators. For instance, let T be any almost normal but non-normal operator, and consider any non-commutative subalgebra of $C^*(T)$.

We shall first apply 2.2.1 to settle a problem taken up in [1]. Let H^2 denote the usual Hardy space of all functions in L^2 of the unit circle $\mathbf{T}$ whose negative Fourier coefficients vanish. Let ϕ be an inner function, let $\mathfrak{H} = H^2 \ominus \phi H^2$, and let S_ϕ be the compression of the operator "multiplication by $e^{i\theta}$" to $\mathfrak{H}$. To avoid trivialities we will assume that the dimension of $\mathfrak{H}$ is greater than 1; equivalently, ϕ is not a constant and is not a trivial Blaschke product of degree 1. Then S_ϕ is an irreducible contraction (which is not normal since $\dim \mathfrak{H} > 1$), and we may ask if the identity representation of $C^*(S_\phi)$ is a boundary representation for $\{I, S_\phi, S_\phi^2, ...\}$. In section 3.5 of [1] a partial solution was given in terms of the "zero set" of ϕ, defined as the set Z_ϕ of all points $\lambda \in \mathbf{T}$ for which $1/\tilde{\phi}(z)$ is unbounded in every open subset of $\{|z| < 1\}$ which contains λ in its closure, where $\tilde{\phi}$ denotes the canonical analytic extension of ϕ to $\{|z| < 1\}$. The result of [1] was that if Z_ϕ has Lebesgue measure zero then *id* is a boundary representation, and if $Z_\phi = \mathbf{T}$ then *id* is not a boundary representation. The method of [1] gives no further information about the intermediate cases, Z_ϕ of positive measure but different from $\mathbf{T}$. The following result completes the discussion of this class of examples.

COROLLARY 1. *The identity representation of $C^*(S_\phi)$ is a boundary representation for $\{I, S_\phi, S_\phi, ...\}$ if, and only if, Z_ϕ is a proper subset of the unit circle.*

Proof. It remains to show that if $Z_\phi \neq \mathbf{T}$, then *id* is a boundary representation for $\{S_\phi^n \colon n \geqslant 0\}$. Since S_ϕ is irreducible and almost normal (recall that $I - S_\phi^* S_\phi$ and $I - S_\phi S_\phi^*$ are compact, cf. [1] Theorem 3.4.2), by 2.2.1 we need only show that if $Z_\phi \neq \mathbf{T}$ then there is a polynomial p such that $|esp(p(S_\phi))| = \sup\{|p(\lambda)| \colon \lambda \in esp(S_\phi)\}$ is less than $\|p(S_\phi)\|$. By ([1], 3.4.3 (ii)) we have $esp(S_\phi) = Z_\phi$, and thus it suffices to show that Z_ϕ is *not* a spectral set for S_ϕ.

But since the complement of Z_ϕ is connected and Z_ϕ has no interior, every operator having Z_ϕ as a spectral set must be normal (for example, see [15], p. 444). Since S_ϕ is not normal, the conclusion follows.

As a second application of Theorem 2.2.1, let us consider a weighted shift T with weights $\alpha_0, \alpha_1, \ldots$. As we have already pointed out, T is always irreducible, and it is almost normal if $|\alpha_{n+1}| - |\alpha_n| \to 0$ as $n \to \infty$. The following corollary implies in the almost normal case that *id* is a boundary representation for $\{I, T\}$ iff the sequence $|\alpha_0|, |\alpha_1|, \ldots$ does not achieve its maximum value at infinity. First, we require the following two lemmas.

LEMMA 1. *Let $\{P_\alpha\}$ be any net of projections in $\mathcal{L}(\mathfrak{H})$ such that $P_\alpha \mathfrak{H}$ has finite codimension and $\lim_\alpha P_\alpha = 0$ weakly. Let $q\colon \mathcal{L}(\mathfrak{H}) \to \mathcal{L}(\mathfrak{H})/\mathcal{C}(\mathfrak{H})$ be the natural map onto the Calkin algebra. Then $\lim_\alpha \|P_\alpha T P_\alpha\| = \|q(T)\|$, for every $T \in \mathcal{L}(\mathfrak{H})$.*

Sketch of Proof. A trivial computation (which we omit) shows that $\lim_\alpha \|P_\alpha K P_\alpha\| = 0$ for every operator K of rank 1, hence the same is true of every finite rank operator K. Since $\{K \in \mathcal{L}(\mathfrak{H})\colon \lim_\alpha \|P_\alpha K P_\alpha\| = 0\}$ is easily seen to be norm-closed, it follows that $\lim_\alpha \|P_\alpha K P_\alpha\| = 0$ for every $K \in \mathcal{C}(\mathfrak{H})$. Thus, if $T \in \mathcal{L}(\mathfrak{H})$ and $K \in \mathcal{C}(\mathfrak{H})$ then $\limsup_\alpha \|P_\alpha T P_\alpha\| = \limsup_\alpha \|P_\alpha (T+K) P_\alpha\| \leq \|T+K\|$; taking the inf over K we obtain $\limsup_\alpha \|P_\alpha T P_\alpha\| \leq \|q(T)\|$. On the other hand, since each $P_\alpha T P_\alpha = T + (P_\alpha^\perp T P_\alpha^\perp - P_\alpha^\perp T - T P_\alpha^\perp)$ is a finite rank perturbation of T, we see that $\|q(T)\| \leq \|P_\alpha T P_\alpha\|$, and hence $\|q(T)\| \leq \liminf_\alpha \|P_\alpha T P_\alpha\|$. The conclusion follows.

LEMMA 2. *Let T be an almost normal weighted shift with weights $\{\alpha_n\}$. Then $|esp(T)| = \limsup_n |\alpha_n|$.*

Proof. Choose an orthonormal base $e_0, e_1 \ldots$ so that $Te_n = \alpha_n e_{n+1}$. A familiar computation with weighted shifts (cf. [11], problem 77) shows that $\|T\| = \sup_n |\alpha_n|$. So if we let P_n be the projection onto the invariant subspace $[e_n, e_{n+1}, \ldots]$, $n \geq 1$, then the same calculation shows that $\|P_n T P_n\| = \sup_{k \geq n} |\alpha_k|$. By Lemma 1 we see that $\|q(T)\| = \lim_n \|P_n T P_n\| = \limsup_n |\alpha_n|$, and since $q(T)$ is normal the required formula $|esp(T)| = r(q(T)) = \|q(T)\| = \limsup_n |\alpha_n|$ follows.

COROLLARY 2. *Le T be an almost normal weighted shift with weights $\alpha_0, \alpha_1, \ldots$. If $\limsup_n |\alpha_n| < \sup_n |\alpha_n|$, then id is a boundary representation for $\{I, T\}$. If $\limsup_n |\alpha_n| = \sup_n |\alpha_n|$, then id is not even a boundary representation for $\{T^n\colon n = 0, 1, 2, \ldots\}$.*

Proof. If $\limsup_n |\alpha_n| < \sup_n |\alpha_n|$ then by Lemma 2 we see that $|esp(T)| < \|T\|$, and the first conclusion follows by applying 2.2.1 to $\mathcal{S} = \{I, T\}$.

Now assume $\limsup_n |\alpha_n| = \sup_n |\alpha_n|$. There is no essential loss if we assume $\sup_n |\alpha_n| = \|T\| = 1$. Then $\|q(T)\| \leq 1$ while by hypothesis (and Lemma 2) $r(q(T)) = 1$; hence $\|q(T)\| = r(q(T)) = 1$. We claim that the spectrum of $q(T)$ contains the unit circle $\mathbf{T}$. Indeed, $sp(q(T)) \cap \mathbf{T} \neq \varnothing$ by the preceding formula, so it suffices to show that $sp(q(T))$ is invariant under (complex) rotations. But T is unitarily equivalent to λT for $|\lambda|=1$, because λT is a weighted shift whose weights $\{\lambda\alpha_n\}$ have the same moduli as $\{\alpha_n\}$ ([11], problem 75). The claim obviously follows from this.

By ([1], Theorem 3.6.3) the map $p(z)\mapsto p(q(T))$ (where p ranges over all polynomials) defines a completely isometric representation of the disc algebra $P(\{|z|\leqslant 1\})$. The same argument shows that the same is true of the map $p(z)\mapsto p(T)$. This implies in particular that q is completely isometric on the linear span $\mathcal{A}$ of $\{T^k: k\geqslant 0\}$. From ([1], 1.2.9 and 1.2.10) we conclude that q is completely isometric on the norm closure of $\mathcal{A}+\mathcal{A}^*$, and now the conclusion follows from the boundary theorem. That completes the proof.

So for example, if we consider the three weight sequences $\alpha_n=n+2/n+1$, $\beta_n=n+1/n+2$, and $\gamma_n=1+\sin\sqrt{n}$, then $\alpha_{n+1}-\alpha_n$, $\beta_{n+1}-\beta_n$, and $\gamma_{n+1}-\gamma_n$ all tend to 0 so that each defines an almost normal weighted shift. Since $\lim\sup_n\alpha_n=1<\sup_n\alpha_n=2$, *id* is a boundary representation for $\{I, T_\alpha\}$. On the other hand, $\lim\sup_n\beta_n=\sup_n\beta_n=1$ and $\lim\sup_n\gamma_n=\sup_n\gamma_n=2$, so that *id* is not a boundary representation for $\{I, T_\beta, T_\beta^2, ...\}$ or for $\{I, T_\gamma, T_\gamma^2, ...\}$.

If one states the boundary theorem contrapositively, then it can be combined with the results of Chapter 1 to produce an unusual dilation theorem. To illustrate this, we will deduce a new unitary dilation theorem for certain sets of commuting contractions. We want to consider the following question: does every commuting n-tuple $T=(T_1, ..., T_n)$ of *almost normal* contractions on a Hilbert space have a unitary power dilation? The answer is no in general, for Parrott's class of examples [14] of commuting triples with no unitary dilation contains finite dimensional special cases, and clearly all finite rank operators are almost normal. Nevertheless, we will see that the answer is yes for a broad class of infinite-dimensional almost normal n-tuples.

First, we want to point out that the above question for general commuting n-tuples of almost normal contractions $\mathbf{T}=(T_1, ..., T_n)$ reduces to the case where $\{T_1, ..., T_n\}$ is irreducible. Indeed, we claim that the underlying Hilbert space $\mathfrak{H}$ decomposes into a direct sum $\mathfrak{H}_0\oplus\sum_\alpha^\oplus\mathfrak{H}_\alpha$ of reducing subspaces for $\{T_1, ..., T_n\}$ such that if $T_i=N_i\oplus\sum_\alpha^\oplus T_{i\alpha}$ is the corresponding decomposition of T_i, then $(N_1, ..., N_n)$ is a commuting n-tuple of normal contractions on $\mathfrak{H}_0$ and, for each α, $(T_{1\alpha}, ..., T_{n\alpha})$ is a commuting n-tuple of almost normal contractions on $\mathfrak{H}_\alpha$ such that $\{T_{1\alpha}, ..., T_{n\alpha}\}$ *is irreducible*. Granting that, the reduction follows from the obvious fact that an orthogonal direct sum of dilatable representations is dilatable and the known fact that a commuting n-tuple of *normal* contractions has a unitary power dilation. To obtain the indicated decomposition of $\mathfrak{H}$, let q be the canonical map of $\mathcal{L}(\mathfrak{H})$ into the Calkin algebra $\mathcal{L}(\mathfrak{H})/\mathcal{C}(\mathfrak{H})$. Then $\{q(T_1), ..., q(T_n)\}$ is a commuting set of normal elements in $\mathcal{L}(\mathfrak{H})/\mathcal{C}(\mathfrak{H})$ and hence generates a commutative C^*-algebra (by Fuglede's theorem). So if $com(\mathbf{T})$ is the commutator ideal in the C^*-algebra $C^*(T)$ generated by $\{I, T_1, ..., T_n\}$, this implies $q(C^*(\mathbf{T}))$ is commutative and hence $q(com(\mathbf{T}))=\{0\}$, i.e., $com(\mathbf{T})$ consists of compact operators. Define $\mathfrak{H}_0$ to be the nullspace of $com(\mathbf{T})$. Then the restriction of $com(\mathbf{T})$ to $\mathfrak{H}_0^\perp$ has trivial nullspace and so $\mathfrak{H}_0^\perp$ splits into a direct sum $\sum^\oplus \mathfrak{H}_\alpha$ such that $com(\mathbf{T})|_{\mathfrak{H}_\alpha} = \mathcal{C}(\mathfrak{H}_\alpha)$ (see 1.4.4. of [2] for a proof of this standard result). That this decomposition has the stated properties is a routine verification which we leave for the reader.

Thus we will only consider n-tuples $\mathbf{T}=(T_1, ..., T_n)$ for which $\{T_1, ..., T_n\}$ is irreducible. Now, let $\mathcal{S}$ be an irreducible subset of $\mathcal{L}(\mathfrak{H})$, which contains the identity and is closed under multiplication. Let $\infty\cdot\mathfrak{H}=\mathfrak{H}\oplus\mathfrak{H}\oplus...$ be the direct sum of $\aleph_0$ copies of $\mathfrak{H}$, and let $\infty\cdot\mathcal{S}\subseteq \mathcal{L}(\infty\cdot\mathfrak{H})$ be the semigroup of all operators of the form $\infty\cdot T=T\oplus T\oplus...$, $T\in\mathcal{S}$. Note that if $\mathfrak{M}$ is one of the "coordinate" subspaces of $\infty\cdot\mathfrak{H}$, then $\mathfrak{M}$ reduces $\infty\cdot\mathcal{S}$, and in fact the map $T\in\mathcal{S}\mapsto\infty\cdot T|_{\mathfrak{M}}$ is unitarily equivalent to the identity map of $\mathcal{S}$. By a *self-dilation* for $\mathcal{S}$ we mean an isometric imbedding V of $\mathfrak{H}$ in $\infty\cdot\mathfrak{H}$ such that $V^*\infty\cdot TV=T$, $T\in\mathcal{S}$; the self-dilation is called *trivial* if the range $V\mathfrak{H}$ of V reduces $\infty\cdot\mathcal{S}$. Thus, $\mathcal{S}$ has a nontrivial self-dilation if $\infty\cdot\mathcal{S}$ has a *non*-reducing semi-invariant subspace $\mathfrak{M}$ such that the map $T\in\mathcal{S}\mapsto P_{\mathfrak{M}}\infty\cdot T|_{\mathfrak{M}}$ is unitarily equivalent to the identity map of $\mathcal{S}$. Note that if there is a nonunitary isometry $V\in\mathcal{L}(\mathfrak{H})$ such that $V^*TV=T$, $T\in\mathcal{S}$ (i.e if, in the terminology of Section 3.2 of [1], $\mathcal{S}$ is infinite) then $\mathcal{S}$ has a nontrivial self-dilation. Thus we obtain a simple example of the latter by taking $\mathcal{S}$ to be any algebra of analytic Toeplitz operators on H^2 (here, one may take the isometry V to be the unilateral shift).

Finally, if $\mathbf{T}=(T_1, ..., T_n)$ is any n-tuple of operators on $\mathfrak{H}$ and $k=(k_1, ..., k_n)$ is any n-tuple of integers, then $k\geqslant 0$ will mean $k_1\geqslant 0, ..., k_n\geqslant 0$ and, for such a k, $\mathbf{T}^k$ denotes $T_1^{k_1}T_2^{k_2}\dots T_n^{k_n}$.

THEOREM 2.2.3. *Let $n=1, 2, ...$ and let $\mathbf{T}=(T_1, ..., T_n)$ be an n-tuple of commuting almost normal contractions such that $\{T_1, ..., T_n\}$ is irreducible. If $\{\mathbf{T}^k: k\geqslant 0\}$ has a nontrivial self-dilation, then $\mathbf{T}$ has a unitary power dilation.*

Proof. Let $\mathcal{S}=\{\mathbf{T}^k: k_1, ..., k_n\geqslant 0\}$ be the semigroup generated by $\{I, T_1, ..., T_n\}$. We claim first that id is not a boundary representation for $\mathcal{S}$. For that, since $\mathcal{S}$ has a non-trivial self-dilation, we can find an isometry $V\in\mathcal{L}(\mathfrak{H}, \infty\cdot\mathfrak{H})$ such that $V^*\infty\cdot TV=T$, $T\in\mathcal{S}$, and $V\mathfrak{H}$ does not reduce $\{\infty\cdot T: T\in\mathcal{S}\}$. Define a map ϕ: $C^*(\mathcal{S})\to\mathcal{L}(\mathfrak{H})$ by $\phi(X)=V^*\infty\cdot XV$. ϕ is clearly a completely positive linear map which fixes $\mathcal{S}$, and the claim will follow if we prove that ϕ is not multiplicative on $C^*(\mathcal{S})$ (for then $\phi\neq id$ on $C^*(\mathcal{S})$). But if ϕ were multiplicative on $C^*(\mathcal{S})$ then $V\mathfrak{H}$ would be a semi-invariant subspace for the *-algebra $\{\infty\cdot X: X\in C^*(\mathcal{S})\}$, and thus $V\mathfrak{H}$ reduces $\infty\cdot\mathcal{S}$, a contradiction. That proves the claim.

Next, note that if the dimension of $\mathfrak{H}$ is at least 2 (the only case of interest) then as in the proof of 2.2.1 we see that $C^*(\mathcal{S})$ contains the algebra $\mathcal{C}(\mathfrak{H})$ of all compact operators. The boundary theorem, together with the preceding paragraph, now shows that the Calkin map q: $\mathcal{L}(\mathfrak{H})\to\mathcal{L}(\mathfrak{H})/\mathcal{C}(\mathfrak{H})$ is completely isometric on *span* $\mathcal{S}$.

Thus, if we define $N_i=q(T_i)$, $i=1, 2, ..., n$, then $\mathbf{N}=(N_1, ..., N_n)$ is a commuting n-tuple of *normal* contractions in $\mathcal{L}(\mathfrak{H})/\mathcal{C}(\mathfrak{H})$ for which the map $p(\mathbf{N})\to p(\mathbf{T})$ (p ranging over all n-variate polynomials) is completely isometric. Letting $X=\{(z_1, ..., z_n): |z_i|\leqslant 1\}$ be the closed n-dimensional polydisc, we see that the joint spectrum of $\mathbf{N}$ is contained in $sp(N_1)\times ... \times sp(N_n)\subseteq X$. Since the joint spectrum of $\mathbf{N}$ is a complete spectral set for $\mathbf{N}$ (a fact which follows easily from the operational calculus for commutative C^*-algebras), we see in particular that X is a complete spectral set for $\mathbf{N}$. Thus X is also a complete spectral set for $\mathbf{T}$, and now the conclusion follows from the corollary of the dilation theorem (1.2.2).

2.3. The order of an irreducible operator

One of the basic (and hopelessly difficult) problems of operator theory is to classify separably-acting operators to unitary equivalence. One encounters a principal source of these difficulties immediately when he attempts to reduce the problem from general operators to irreducible operators by expressing the given operator T as a direct integral of irreducible operators. What happens is that if $C^*(T)$ is not a GCR algebra then this direct integral decomposition is badly non-unique, and it turns out to be all but useless (the numerous sources of this pathology are discussed at some length in [2]). If, on the other hand, $C^*(T)$ *is* a GCR algebra, then this procedure runs smoothly and allows a reduction to the case where T is an irreducible GCR operator (we will sketch this reduction presently). Thus one is led to seek unitary invariants for the class of all irreducible GCR operators. The self-adjoint theory takes us no farther, however, and in particular it gives virtually no insight into what kind of invariants one should look for in the latter class of operators.

In this section we will initiate the study of a somewhat broader class of irreducible operators. We are interested in the following admittedly vague question: what is the *minimum* knowledge of an irreducible operator T that one needs in order to know T to within unitary equivalence? We will find, for example, that with many operators T one can associate a numerical invariant $n(T)$ (the *order* of T) such that T is determined to unitary equivalence by the norms $\|p(T)\|$, where p ranges over all matrix valued polynomials *of degree at most* $n(T)$. Such operators are therefore highly "deterministic" in a sense analogous to the usage of the term in prediction theory, in that once one knows the norms of all low order polynomials in T then he knows not only the norms of all higher order polynomials, but he in effect knows every geometric property of T.

First, we want to sketch without details how the reduction of the classification problem to the irreducible case can be accomplished for GCR operators. It is known that every separable representation of a separable C^*-algebra is a direct integral of irreducible representations. This leads in a straightforward manner to the conclusion that every separably-acting operator T is a direct integral of irreducible operators. Thus one might hope that this device would reduce the general problem to the problem of classifying irreducible operators. However, profound difficulties appear when $C^*(T)$ is not a GCR algebra, not the least of which is that the above direct integral decomposition is highly non-unique. So one is lead to consider only those operators whose generated C^*-algebras are GCR algebras. Here a reduction is possible, and one may deduce from the self-adjoint theory that every such operator T has a direct integral decomposition of the form

$$T = \int_X^{\oplus} m(x) \cdot T_x d\mu(x),$$

where X is a standard Borel space (which can be taken as the spectrum of $C^*(T)$), μ is a finite Borel measure on X, $m(\cdot)$ is a "multiplicity function" (i.e., a Borel-measurable function from X into the set $\{1, 2, 3, ..., \aleph_0\}$ of all countable cardinals), $x \mapsto T_x$ is a Borel-measurable map of X into the Borel space of all separably-acting *irreducible* GCR operators such that $x \neq y$ implies T_x is not equivalent to T_y, and finally $m \cdot S$ denotes the direct sum of m copies of the operator S (see [2]). The key property of this decomposition is expressed as follows. Let S be any other separably-acting operator which is algebraically equivalent to T (see section 1.1 of [2]; when S and T are normal this means simply that they have the same spectrum, and this is also true in general provided one interprets the words appropriately). Then S has a decomposition

$$S = \int_X^{\oplus} n(x) \cdot S_x d\nu(x),$$

where ν is another Borel measure on X, $n(\cdot)$ is another multiplicity function defined on X, and $x \mapsto S_x$ is another operator-valued measurable function such that S_x is unitarily equivalent to T_x, for each $x \in X$. The key property is this: *T and S are unitarily equivalent iff μ and ν are mutually absolutely continuous and the multiplicity functions m and n agree almost everywhere.* Thus the self-adjoint theory has reduced the classification problem for arbitrary separably-acting GCR operators to the problem of classifying irreducible GCR operators.

Turning now to the current discussion, let T be a Hilbert space operator. We define the *order* of T (written $n(T)$) as follows. If a positive integer n exists such that $\{I, T,$

$T^2, \ldots, T^n\}$ has sufficiently many boundary representations, $n(T)$ will denote the smallest such n. If no such finite n exists but $\{T^k: k=0, 1, 2, \ldots\}$ has sufficiently many boundary representations, we define $n(T)=\omega$. If $\{T^k: k=0, 1, 2, \ldots\}$ does not have sufficiently many boundary representations then $n(T)$ is undefined. Now according to 2.1.0, if $C^*(T)$ happens to contain the algebra $\mathcal{C}(\mathfrak{H})$ of compact operators and $n(T)$ is defined, then $n(T)$ is the smallest positive integer such that *id* is a boundary representation for $\{I, T, T^2, \ldots, T^n\}$, provided such an integer exists, and is ω otherwise. Moreover, we will see momentarily that in this case $n(T)$ does not take on the infinite value ω, and in fact $n(T)=\omega$ only for the most pathological irreducible operators.

Before proceeding further, we want to recall one or two facts about irreducible C^*-algebras. Fist, recall that a C^*-algebra B is called NGCR if B has no nontrivial closed CCR ideals [8]. Now if $\mathcal{B}$ is an irreducible C^*-algebra acting on $\mathfrak{H}$, then the largest CCR ideal in $\mathcal{B}$ is $\mathcal{B} \cap \mathcal{C}(\mathfrak{H})$ (indeed the latter is clearly a CCR ideal in $\mathcal{B}$, and conversely if $T \in \mathcal{B}$ is such that $\pi(T)$ is compact for every irreducible representation π of $\mathcal{B}$ then take $\pi = id$ to see that $T \in \mathcal{B} \cap \mathcal{C}(\mathfrak{H})$). Since $\mathcal{B}$ is irreducible, $\mathcal{B} \cap \mathcal{C}(\mathfrak{H})$ mut be either $\{0\}$ or $\mathcal{C}(\mathfrak{H})$ ([2], Corollary 2 of 1.4.2), and thus we see that *an irreducible C^*-algebra acting on $\mathfrak{H}$ is not* NGCR *if it contains* $\mathcal{C}(\mathfrak{H})$. We now show that $n(T) \neq \omega$ for all but the "worst" irreducible operators T.

PROPOSITION 2.3.1. *Let T be an irreducible operator such that $C^*(T)$ is not an* NGCR *algebra. If $n(T)$ is defined at all then it is finite.*

Proof. By the preceding remarks we know that $C^*(T)$ contains $\mathcal{C}(\mathfrak{H})$. So assume that $n(T)$ is defined, that is, the identity representation of $C^*(T)$ is a boundary representation for $\{T^k: k=0, 1, 2, \ldots\}$. Let $\mathcal{S}$ be the linear span of the latter, and let $\mathcal{S}_n = span\ \{I, T, \ldots, T^n\}$ for $n=1, 2, \ldots$. By the boundary theorem 2.1.1 we see that the quotient map $q: C^*(T) \to C^*(T)/\mathcal{C}(\mathfrak{H})$ is not completely isometric on the closure of $\mathcal{S}+\mathcal{S}^*$. Now $\{\mathcal{S}_n + \mathcal{S}_n^*\}$ is an increasing sequence of subspaces of $\mathcal{S}+\mathcal{S}^*$ whose union is dense in $\mathcal{S}+\mathcal{S}^*$, and is such that $C^*(\mathcal{S}_n) = C^*(T)$ for each n. Now if q were isometric (resp. completely isometric) on each $\mathcal{S}_n$ it would follow that q is isometric (resp. completely isometric) on $(\mathcal{S}+\mathcal{S}^*)^-$. We conclude that there is a first n, $1 \leq n < \infty$, such that q fails to be completely isometric on $\mathcal{S}_n + \mathcal{S}_n^*$. By the boundary theorem again we see that n is the first positive integer such that *id* is a boundary representation for $\{I, T, T^2, \ldots, T^n\}$. Hence $n(T)=n$ is finite, and the proof is complete.

Before proceeding further, we want to point out that there exist irreducible GCR operators T for which $n(T)$ is undefined. Indeed, 3.6.3 of [1] implies that $n(T)$ is undefined for every GCR contraction T whose spectrum contains the unit circle $\{|z|=1\}$ (example: the unilateral shift of multiplicity one).

Before stating the basic result on operators of order n let us perturb slightly some terminology from Chapter 1. Let $A_0, A_1, \ldots, A_n$ be a sequence of $k \times k$ complex matrices with $A_n \neq 0$. Then $p(z)=A_0+A_1z+\ldots+A_nz^n$ defines an M_k-valued polynomial of degree n. If T is an operator on a Hilbert space $\mathfrak{H}$, then we define $p(T)$ to be the operator on $\mathbb{C}^k \otimes \mathfrak{H}$ given by $A_0 \otimes I + A_1 \otimes T + \ldots + A_n \otimes T^n$, where we have identified $k \times k$ matrices with operators on $\mathbb{C}^k$ in the obvious way. Thus for each k, $\|p(T)\| = \|A_0 \otimes I + A_1 \otimes T + \ldots + A_n \otimes T^n\|$ defines a seminorm on the algebra of all M_k-valued polynomials p. When we say that a statement is valid for all matrix-valued polynomials we mean that it is valid for every M_k-valued polynomial and for every $k=1, 2, \ldots$.

THEOREM 2.3.2. *Let S and T be irreducible operators acting on $\mathfrak{H}$ and $\mathfrak{K}$, respectively, such that $C^*(S)$ contains $\mathcal{C}(\mathfrak{H})$, and such that $n(S)$ and $n(T)$ are defined and equal. If $\|p(S)\| = \|p(T)\|$ for every matrix-valued polynomial p of degree $\leqslant n(T)$, then S and T are unitarily equivalent.*

Proof. Let us write n for $n(S)=n(T)$ (note that n is finite, by 3.3.1). The hypothesis asserts simply that the linear map

$$a_0 I + a_1 S + \ldots + a_n S^n \mapsto a_0 I + a_1 T + \ldots + a_n T^n$$

is a complete isometry of *span* $\{I, S, \ldots, S^n\}$ onto *span* $\{I, T, \ldots, T^n\}$. The implementation theorem (0.3) implies that this map extends to a *-isomorphism π of $C^*(S)$ onto $C^*(T)$. Note that $\pi(S)=T$. Since $C^*(S)$ contains $\mathcal{C}(\mathfrak{H})$ we may argue as in the proof of the corollary of the boundary theorem 2.1.1 to conclude that π is unitarily implemented. The conclusion is now immediate.

We conclude this section by describing a class of examples of operators of arbitrary finite order. The construction may be of independent interest, as it makes use of the "nilpotent dilation theorem" of section 1.3 as well as the boundary theorem.

THEOREM 2.3.3. *For every positive integer N there is a unilateral weighted shift T such that $C^*(T)$ contains the compact operators and $n(T)=N$.*

Proof. Note first that the case $N=1$ is simple. For any weighted shift T whose weights tend to 0 is compact and irreducible, hence by the boundary theorem *id* is a boundary representation for $\{I, T\}$, so that T has order 1.

So choose $N \geqslant 2$, and let $e_0, e_1, \ldots$ be an orthonormal base for a Hilbert space $\mathfrak{H}$. Let $a_1, a_2, \ldots$ be a sequence of positive numbers which increases to 1, and choose ε, r, $0<\varepsilon$,

$r<1$. Define a weight sequence $\{\alpha_n\}$ as follows; if $k\geqslant 0$ put $\alpha_{kN}=\alpha_{kN+1}=\ldots=\alpha_{kN+N-2}=a_{k+1}$, and put $\alpha_{kN+N-1}=\varepsilon r^{k+1}$. So if $N=3$, for example, then we have the weight sequence

$$a_1, a_1, \varepsilon r, a_2, a_2, \varepsilon r^2, a_3, a_3, \varepsilon r^3, \ldots.$$

Let T be the weighted shift on $\mathfrak{H}$ defined by $Te_n=\alpha_n e_{n+1}$, $n\geqslant 0$. We will show first that T^N is compact and nonzero (so that by the boundary theorem *id* is a boundary representation for $\{I, T, \ldots, T^N\}$; note that this also implies $C^*(T)$ contains $\mathcal{C}(\mathfrak{H})$). We will then prove that if $\{a_k\}$ increases not too rapidly to 1 and ε is sufficiently small then *id* is *not* a boundary representation for $\{I, T, \ldots, T^{N-1}\}$. Thus T will have order N.

Now T^N is defined on $\{e_n\}$ by $T^N e_n=\alpha_n\alpha_{n+1}\ldots\alpha_{n+N-1}e_{n+N}$, and so $|T^N|$ is the diagonal operator $|T^N|e_n=\alpha_n\alpha_{n+1}\ldots\alpha_{n+N-1}e_n$, $n\geqslant 0$. An inspection shows that the sequence $\{\alpha_n\alpha_{n+1}\ldots\alpha_{n+N-1}\}$ is term-by-term smaller than the sequence

$$\varepsilon r, \ldots, \varepsilon r, \varepsilon r^2, \ldots, \varepsilon r^2, \varepsilon r^3, \ldots, \varepsilon r^3, \ldots,$$

where the indicated blocks are of length N. Thus $|T^N|$ is a nonzero compact (in fact, trace class) operator, and the same is true of T^N. That proves the first assertion.

For the second assertion, let S_N be the "nilpotent shift" of index N (cf. section 1.3). We will prove that for suitably chosen r, ε, and $\{a_k\}$, the linear map ϕ: $\sum_{k=0}^{N-1}\lambda_k T^k\mapsto \sum_{k=0}^{N-1}\lambda_k S_N^k$ is completely isometric. Granting that for a moment, note that *id* cannot be a boundary representation for $\{I, T, \ldots, T^{N-1}\}$. For if it were, then of course *id* is a boundary representation for $\{I, S^N, \ldots, S_N^{N-1}\}$ (S_N is compact and irreducible, cf. 2.1.1) and the implementation theorem would imply that ϕ is implemented by a *-isomorphism of $C^*(T)$ onto $C^*(S_N)$. But that is absurd, since for example $C^*(S_N)$ is finite-dimensional while $C^*(T)$ is not.

First, we claim that ϕ is completely contractive. For that, let q: $C^*(T)\to C^*(T)/\mathcal{C}(\mathfrak{H})$ be the canonical quotient map. We will produce an operator T_1 on $\mathfrak{H}$ such that $q(T)=q(T_1)$, and T_1 is unitarily equivalent to an infinite multiple $\infty\cdot S_N=S_N\oplus S_N\oplus\ldots$ of S_N. This leads to the claim because, on the one hand, the fact that $T_1\sim\infty\cdot S_N$ obviously implies that we can find a decreasing sequence P_n of reducing projections of finite codimension for T_1 such that $P_n\to 0$ weakly and T_1 is equivalent to each restriction $T_1|_{P_n\mathfrak{H}}$; and as in the proof of Lemma 1 of Section 2.2 we conclude that q is completely isometric on span $\{T_1^k\colon k\geqslant 0\}$. On the other hand, the map $\sum_{k=0}^{N-1}\lambda_k T_1^k\mapsto\sum_{k=0}^{N-1}\lambda_k S_N^k$ is clearly completely isometric (because T_1 is a multiple of S_N), and we conclude that the map ψ: $\sum_{k=0}^{N-1}\lambda_k q(T_1)^k\mapsto \sum_{k=0}^{N-1}\lambda_k S_N^k$ is completely isometric. Since $q(T)=q(T_1)$ we see that ϕ is the composition $\psi\circ q$, and since q is completely contractive, the claim will follow. Now define T_1 as follows. Let $\{c_n\}$ be the sequence of *nonnegative* weights defined by $c_n=0$ if $n=N-1$,

$2N-1, \ldots$, and $c_n = 1$ if $n \not\equiv N-1 \pmod N$. Define T_1 on $e_0, e_1, \ldots$ by $T_1 e_n = c_n e_{n+1}$. Then the matrix of T_1 appears in the $N \times N$ block form as

$$\begin{pmatrix} S_N & & 0 \\ & S_N & \\ 0 & & \ddots \end{pmatrix}$$

so that T_1 is equivalent to $\infty \cdot S_N$. Finally, we note that $q(T_1) = q(T)$, that is $T_1 - T$ is compact. For clearly $T_1 - T$ is a weighted shift with weight sequence

$$1-a_1, \ldots, 1-a_1, \ -\varepsilon r, 1-a_2, \ldots, 1-a_2, \ -\varepsilon r^2, \ldots;$$

since $a_k \to 1$ and $\varepsilon r^k \to 0$ as $k \to \infty$, $T_1 - T$ is compact.

Secondly, we claim that for suitable ε, r, $\{a_k\}$, ϕ^{-1}: $\sum_{k=0}^{N-1} \lambda_k S_N^k \mapsto \sum_{k=0}^{N-1} \lambda_k T^k$ is completely contractive. Of course that will follow if we can produce a Hilbert space $\mathfrak{K}$ containing $\mathfrak{H}$ and a multiple $A \sim \infty \cdot S_N$ acting on $\mathfrak{K}$ such that $T^k = P_{\mathfrak{H}} A^k |_{\mathfrak{H}}$ for $k = 0, 1, \ldots, N-1$. By 1.3.1 and 1.3.2, it suffices to prove the following assertion; if $0 < r \leqslant \frac{1}{2}$, $a_1, a_2, \ldots$ is any sequence which increases monotonically to 1 so that $0 < a_k \leqslant (1-r^k)^{\frac{1}{2}}$, and $0 < \varepsilon \leqslant (8+8\varrho)^{-1}$ where $\varrho = \max(r^{-1}, a_1^{-1})^{2N}$, then T satisfies $T^N T^{*N} \leqslant \varrho T^{*N} T^N$ and $|T^N| \leqslant (8+8\varrho)^{-1}(I - T^*T)$. We shall only sketch these routine calculations. Now $T^{*N}T^N$ and $T^N T^{*N}$ are, respectively, the diagonal operators whose weight sequences are $\{(\alpha_k \alpha_{k+1} \cdots \alpha_{k+N-1})^2\}_{k=0}^{\infty}$ and $\{(\alpha_{k-N}\alpha_{k-N+1} \cdots \alpha_{k-1})^2\}_{k=0}^{\infty}$ where we have made the convention $\alpha_j = 0$ if $j < 0$. Thus the condition $T^N T^{*N} \leqslant \varrho T^{*N} T^N$ is equivalent to the condition $(\alpha_k \alpha_{k+1} \cdots \alpha_{k+N-1})^2 \leqslant \varrho (\alpha_{k+N}\alpha_{k+N+1} \cdots \alpha_{k+2N-1})^2$ for $k \geqslant 0$. Now for each $k \geqslant 0$, $\alpha_k(\alpha_{k+N})^{-1}$ is either of the form $\varepsilon r^j(\varepsilon r^{j+1})^{-1} = r^{-1}$ or $a_j(a_{j+1})^{-1} \leqslant (a_{j+1})^{-1} \leqslant a_1^{-1}$. So in either case we have $\alpha_k \leqslant \max(r^{-1}, a_1^{-1})\alpha_{k+N}$, and the asserted condition follows. Secondly, note that $|T^N|$ and $I - T^*T$ are, respectively, the diagonal operators whose weight sequences are $\{\alpha_k \alpha_{k+1} \cdots \alpha_{k+N-1}\}$ and $\{1-\alpha_k^2\}$. Now we have already pointed out that $\{\alpha_k \alpha_{k+1} \cdots \alpha_{k+N-1}\}$ is term-by-term smaller than the sequence $\varepsilon r, \ldots, \varepsilon r, \varepsilon r^2 \ldots \varepsilon r^2, \varepsilon r^3, \ldots$, where each subsequence of terms of the form εr^j has length N. On the other hand, $\{1-\alpha_k^2\}$ is the sequence $1-a_1^2, \ldots, 1-a_1^2, 1-\varepsilon^2 r^2, \ 1-a_2^2, \ldots, 1-a_2^2, 1-\varepsilon^2 r^4, \ldots$. Since $\varepsilon \leqslant (8+8\varrho)^{-1}$, the inequality $|T^N| \leqslant (8+8\varrho)^{-1}(I-T^*T)$ will follow if we show that for $j \geqslant 1$, $r^j \leqslant \min(1-a_j^2, 1-\varepsilon^2 r^{2j})$. But $r^j \leqslant 1 - a_j^2$ follows from $a_j \leqslant (1-r^j)^{\frac{1}{2}}$ and $r^j \leqslant 1 - \varepsilon^2 r^{2j}$ follows from $r \leqslant \frac{1}{2}$. That completes the proof.

2.4. First order operators and the matrix range

We are now going to look at first order operators in more detail; we will introduce an invariant (the matrix range) which will turn out to be a complete unitary invariant for many of these operators.

Let T be a Hilbert space operator. Then as ϕ runs over the state space of $C^*(T)$, the complex numbers $\phi(T)$ fill out the closure of the numerical range of T. One may generalize this as follows. Let n be a positive integer and let M_n denote the C^*-algebra of all complex $n \times n$ matrices. $\mathcal{W}_n(T)$ is defined as the set of all $n \times n$ matrices of the form $\phi(T)$, where ϕ ranges over all completely positive linear maps of $C^*(T)$ into M_n which preserve the identity. Thus $\mathcal{W}_1(T)$ is the closure of the numerical range of T, and the sequence $\{\mathcal{W}_1(T), \mathcal{W}_2(T), \ldots\}$ will be called the *matrix range* of T.

Let us first collect some simple properties of $\mathcal{W}_n(T)$. Clearly $\mathcal{W}_n(T)$ is contained in the ball of radius $\|T\|$, and since the set of all completely positive maps ϕ: $C^*(T) \to M_n$ for which $\phi(I) = I$ is compact in the obvious topology, $\mathcal{W}_n(T)$ is also compact. Note also that $\mathcal{W}_n(T)$ has a very strong convexity property: If $X_1, X_2, \ldots$ is any sequence in $\mathcal{W}_n(T)$ and $Z_1, Z_2, \ldots$ is a sequence in M_n satisfying $\sum_k Z_k^* Z_k = I$, then $\sum_k Z_k^* X_k Z_k$ belongs to $\mathcal{W}_n(T)$. For we may find completely positive maps ϕ_k: $C^*(T) \to M_n$ with $\phi_k(I) = I$ such that $X_k = \phi_k(T)$, $k = 1, 2, \ldots$, and thus $\psi(S) = \sum_k Z_k^* \phi_k(S) Z_k$ is a completely positive map taking I to I and T to $\sum_k Z_k^* X_k Z_k$. Finally, the sequence $\{\mathcal{W}_n(T)\}$ is *coherent* in the sense that $\mathcal{W}_m(\mathcal{W}_n(T)) \subseteq \mathcal{W}_m(T)$ for all $m, n \geqslant 1$. More precisely, for every $X \in \mathcal{W}_n(T)$ one has $\mathcal{W}_m(X) \subseteq \mathcal{W}_m(T)$; this is a trivial consequence of the fact that the composition of two completely positive maps is completely positive.

It is not hard to see that these properties characterize the matrix range of an operator. Explicitly, suppose that for each $n \geqslant 1$, $\mathcal{V}_n$ is a closed subset of the ball of radius r in M_n having the above convexity property, for which $\{\mathcal{V}_n\}$ is a is a coherent sequence in the sense that $\mathcal{W}_m(\mathcal{V}_n) \subseteq \mathcal{V}_m$ for all $m, n \geqslant 1$. Then there is a separably-acting operator T such that $\|T\| \leqslant r$ and $\mathcal{W}_n(T) = \mathcal{V}_n$ for every $n \geqslant 1$. We omit the proof since this result does not bear on the sequel.

Experience has shown that while $\mathcal{W}_n(T)$ can be calculated for quite a variety of operators, it is not feasible to attempt to carry out the computations in general. Of course one would expect that to be so since by 2.4.3 below the matrix range is a complete unitary invariant for irreducible compact operators. Thus the purpose of this section is merely to point out the existence of this unusual invariant, its connection with dilation theory and first order operators (2.4.2 and 2.4.3), and the fact that for certain operators its structure is quite simple.

The following result shows that when T is normal, $\mathcal{W}_n(T)$ is the closed "matrix valued" convex hull of the spectrum of T.

PROPOSITION 2.4.1. *Let T be a normal operator and let n be a positive integer. Then $\mathcal{W}_n(T)$ is the closure in M_n of the set of operators of the form $\lambda_1 H_1 + \lambda_2 H_2 + \ldots + \lambda_r H_r$, where $r \geqslant 1$, $\lambda_i \in sp(T)$, and $\{H_i\}$ is a set of positive elements of M_n having sum I.*

Proof. Let K denote the set of all completely positive linear maps ϕ: $C^*(T) \to M_n$ satisfying $\phi(I) = I$. K is compact in the obvious "pointwise" topology ([1], Chapter 1) and it is a convex set of linear maps of $C^*(T)$ into M_n. Since $\phi \mapsto \phi(T)$ maps K continuously onto $\mathcal{W}_n(T)$, the proposition will follow from the Krein-Milman theorem provided we show that $\phi(T)$ has the asserted form for ϕ an *extreme point* of K. But by 1.4.10 of [1] such a ϕ has the form $\phi(X) = \sum_{k=1}^{r} \omega_k(X) H_k$, where $\omega_1, ..., \omega_r$ are complex homomorphisms of $C^*(T)$ and $H_1, ..., H_r$ are positive matrices in M_n having sum I. The conclusion follows by taking $X = T$ and noting that $\omega_k(T) \in sp(T)$.

So, for example, if T is a unitary operator whose spectrum fills out the unit circle, then $\mathcal{W}_n(T)$ is the closed unit ball in M_n, $n = 1, 2, ...$ (indeed 2.4.1 and the spectral theorem imply that $\mathcal{W}_n(T)$ contains every unitary operator in M_n, and since the closed convex hull of the unitary operators in M_n fills out the unit ball in M_n we see that $\mathcal{W}_n(T)$ contains ball M_n; the opposite inclusion is trivial). Similarly, if T is self-adjoint and $[a, b]$ is the smallest closed interval containing $sp(T)$, then $\mathcal{W}_n(T)$ is $\{X \in M_n: X = X^*, aI \leqslant X \leqslant bI\}$. Finally, as a different type of example, we remark that if T is a two-dimensional operator having a matrix representation

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

then $\mathcal{W}_n(T)$ consists of all $X \in M_n$ whose numerical radius is $\leqslant \frac{1}{2}$ (this follows easily from 2.4.2 below and 1.3.1).

We now examine the partial ordering of operators defined by the relation $\mathcal{W}_n(S) \subseteq \mathcal{W}_n(T)$, for every $n = 1, 2, ...$. We will say that an operator $S \in \mathfrak{L}(\mathfrak{H})$ is a *compression* of an operator $T \in \mathfrak{L}(\mathfrak{K})$ if there is a closed subspace $\mathfrak{M}$ of $\mathfrak{K}$ such that S is unitarily equivalent to $P_{\mathfrak{M}} T|_{\mathfrak{M}}$, $P_{\mathfrak{M}}$ denoting the projection of $\mathfrak{K}$ on $\mathfrak{M}$. Note that $\mathfrak{M}$ need not be invariant, or even semi-invariant, under T.

THEOREM 2.4.2. *Let S and T be Hilbert space operators (perhaps acting on different spaces). Then the following are equivalent.*

(i) $\mathcal{W}_n(S) \subseteq \mathcal{W}_n(T)$ *for every* $n \geqslant 1$.

(ii) $\|A \otimes I + B \otimes S\| \leqslant \|A \otimes I + B \otimes T\|$ *for every pair A, B of $n \times n$ matrices and every* $n \geqslant 1$.

(iii) *Every finite dimensional compression of S is a compression of $\pi(T)$, for some *-representation π of $C^*(T)$ (which may depend on the particular compression of S).*

(iv) *S is a compression of $\pi(T)$, for some *-representation π of $C^*(T)$.*

(v) *(for normal S) $sp(S)$ is contained in the closed numerical range of T.*

(vi) *(for T compact and irreducible) S is a compression of some multiple $k \cdot T$ of T.*

Proof. We first establish the equivalence of (i) through (iv). Since (iv) ⇒ (iii) is trivial, it suffices to prove (i) ⇒ (iv) and (iii) ⇒ (ii) ⇒ (i).

(i) *implies* (iv). We claim first that (i) implies that the linear map $\phi: aI+bT+cT^* \mapsto aI+bS+cS^*$ of *span* $\{I, T, T^*\}$ onto *span* $\{I, S, S^*\}$ is completely positive. Let $\{\mathfrak{M}_\alpha\}$ be an increasing directed set of finite dimensional subspaces of the space $\mathfrak{H}$ on which S acts such that $\bigcup_\alpha \mathfrak{M}_\alpha$ is dense in $\mathfrak{H}$. Let P_α be the projection of $\mathfrak{H}$ onto $\mathfrak{M}_\alpha$, and define ϕ_α: *span* $\{I, T, T^*\} \to \mathfrak{L}(\mathfrak{M}_\alpha)$ by $\phi_\alpha(aT+bT+cT^*) = P_\alpha(aI+bS+cS^*)|_{\mathfrak{M}_\alpha}$. We will show that each ϕ_α is completely positive. It will then follow from (1.2.10 of [1]) that $\|\phi_\alpha\| = \|\phi_\alpha(I)\| = 1$, while on the other hand $\lim_\alpha \phi_\alpha(X)P_\alpha = \phi(X)$ in the weak operator topology for each $X \in$ *span* $\{I, T, T^*\}$, and the complete positivity of ϕ will be proved. But if n_α is the dimension of $\mathfrak{M}_\alpha$, then we may regard $\mathfrak{L}(\mathfrak{M}_\alpha)$ as M_{n_α} (by making use of an orthonormal base in $\mathfrak{M}_\alpha$). Thus the map $X \in \mathfrak{L}(\mathfrak{H}) \to P_\alpha X|_{\mathfrak{M}_\alpha}$ becomes a completely positive map of $\mathfrak{L}(\mathfrak{H})$ into M_{n_α} carrying I to I and S to $P_\alpha S|_{\mathfrak{M}_\alpha}$. Thus $P_\alpha S|_{\mathfrak{M}_\alpha}$ belongs to $\mathcal{W}_{n_\alpha}(S)$, and by hypothesis it also belongs to $\mathcal{W}_{n_\alpha}(T)$. Thus there is a completely positive map ψ: $C^*(T) \to \mathfrak{L}(\mathfrak{M}_\alpha)$ such that $\psi(I) = I$ and $\psi(T) = P_\alpha S|_{\mathfrak{M}_\alpha}$. It follows that $\psi(T^*) = \psi(T)^* = P_\alpha S|^*_{\mathfrak{M}_\alpha} = P_\alpha S^*|_{\mathfrak{M}_\alpha}$, because ψ is self-adjoint, hence $\phi_\alpha = \psi$ is completely positive, and so is ϕ.

By the extension theorem (0.1), ϕ may be extended to a completely positive map of $C^*(T)$ into $\mathfrak{L}(\mathfrak{H})$, and by Stinespring's theorem (0.4) the extension has the form $V^*\pi V$ where π is a *-representation of $C^*(T)$ on some Hilbert space $\mathfrak{K}$ and $V \in \mathfrak{L}(\mathfrak{H}, \mathfrak{K})$. Since $\phi(I) = I$ we see that $V^*V = I$, so that V is an isometry. The equation $S = \phi(T) = V^*\pi(T)V$ now shows that S is unitarily equivalent to the compression of $\pi(T)$ onto $VV^*\mathfrak{K}$.

(iii) *implies* (ii). Fix n, a positive integer, and let $\mathfrak{M}_\alpha$ be the net of finite dimensional subspaces of $\mathfrak{H}$ described above. Then for each α there is a *-representation π_α of $C^*(T)$ on $\mathfrak{K}_\alpha$ and an isometric imbedding V_α of $\mathfrak{M}_\alpha$ in $\mathfrak{K}_\alpha$ such that $P_\alpha S|_{\mathfrak{M}_\alpha} = V_\alpha^* \pi_\alpha(T) V_\alpha$. So if we put $\phi_\alpha = V_\alpha^* \pi_\alpha V_\alpha$, then ϕ_α is completely positive, takes T to $P_\alpha S|_{\mathfrak{M}_\alpha}$ and I to the identity in $\mathfrak{L}(\mathfrak{M}_\alpha)$. By (1.2.10 of [1]) ϕ_α is completely contractive. So if $A, B \in M_n$ then we have $\|A \otimes P_\alpha + B \otimes P_\alpha S P_\alpha\| = \|A \otimes \phi_\alpha(I) + B \otimes \phi_\alpha(T)\| \leq \|A \otimes I + B \otimes T\|$. Fixing A and B we can allow P_α to increase to the identity of $\mathfrak{H}$, and in the limit the left side is $\|A \otimes I + B \otimes S\|$, from which (ii) follows.

(ii) *implies* (i). Choose $X \in \mathcal{W}_n(S)$. We want to show that $X \in \mathcal{W}_n(T)$, assuming (ii). By (ii) the map $aI+bT \mapsto aI+bS$ is a completely contractive linear map of *span* $\{I, T\}$ on *span* $\{I, S\}$. By the corollary of the extension theorem, there is a completely positive linear map ψ: $C^*(T) \to \mathfrak{L}(\mathfrak{H})$ such that $\psi(I) = I$ and $\psi(T) = S$. On the other hand, by the definition of $\mathcal{W}_n(T)$ there is a completely positive linear map ϕ of $C^*(T)$ into M_n such that $\phi(T) = X$. By the extension theorem again we may extend ϕ to a map $\tilde{\phi}$: $\mathfrak{L}(\mathfrak{H}) \to M_n$. Thus

$\tilde{\phi}\circ\psi$: $C^*(T)\to M_n$ is completely positive, takes I to I and takes T to X. Thus $X\in \mathcal{W}_n(T)$, as required.

Now for (v), assume S is normal. Now if (i) is valid then in particular $\mathcal{W}_1(S)\subseteq\mathcal{W}_1(T)$ and it is well known that in this case $\mathcal{W}_1(S)$ is the closed convex hull of $sp(S)$ (see 2.4.1 for example). Thus $sp(S)\subseteq\mathcal{W}_1(T)$. Conversely, suppose $sp(S)\subseteq\mathcal{W}_1(T)$. By the preceding sentences $\mathcal{W}_1(S)\subseteq\mathcal{W}_1(T)$, and from this it is immediate that the linear map ϕ: $aI+bT+cT^*\mapsto aI+bS+cS^*$ is positive. Since $C^*(S)$ is commutative ϕ must be completely positive (1.2.2 of [1]), and therefore completely contractive (1.2.10 of [1]). Thus condition (ii) follows.

Finally, suppose T is compact and irreducible. Then $C^*(T)$ is the full algebra of all compact operators, and the equivalence of (iv) and (vi) is immediate from the fact that every representation of $C^*(T)$ is a multiple of the identity representation [8].

We now state a classification theorem for irreducible first order operators.

THEOREM 2.4.3. *Let S and T be irreducible first order operators such that neither $C^*(S)$ nor $C^*(T)$ is an* NGCR *algebra (i.e., both C^*-algebras contain nonzero compact operators). Then S and T are unitarily equivalent if, and only if, they have the same matrix range.*

Proof. The "only if" part is trivial, so assume $\mathcal{W}_n(S)=\mathcal{W}_n(T)$, $n\geqslant 1$. By 2.4.2 (ii) we see that the linear map $aI+bS\mapsto aI+bT$ is completely isometric and preserves identities. By hypothesis, *span* $\{I, S\}$ and *span* $\{I, T\}$ have sufficiently many boundary representations, so this map is implemented by a *-isomorphism π of $C^*(S)$ onto $C^*(T)$. Since $C^*(S)$ contains nonzero compact operators we may argue as in the proof of the corollary of the boundary theorem to conclude that π is unitarily implemented, and in particular $T=\pi(S)$ is equivalent to S.

Remarks. Since every irreducible operator T, with the property that some linear combination $aT+bT^*$ is at a distance less than $\|aT+bT^*\|$ from the compact operators, is of first order (boundary theorem) and its generated C^*-algebra contains the compact operators (cf. the proof of the corollary of the boundary theorem), these operators are classified by their matrix range. Of course, irreducible operators with compact imaginary part fall into this category, but as 2.2.1 and its corollary indicate, the latter form a rather small subclass.

It goes without saying that the matrix range is not a complete invariant for irreducible GCR operators which are not first order. As a rather extreme example, if S and T are any two contractions such that $sp(S)$ and $sp(T)$ both contain the unit circle, then $\|p(S)\|=$

$\|p(T)\|$ for every matrix valued polynomial p ([1], 3.6.3). In particular, 2.4.2 (ii) shows that S and T always have the same matrix range.

2.5. An application to model theory

Let $\mathcal{C}$ be a class of Hilbert space operators. For example, $\mathcal{C}$ might be the class of all contractions, or the class of all compact operators with nonnegative real part (thus we allow operators in $\mathcal{C}$ to act on different spaces, and we are deliberately ignoring set-theoretic anomalies). Broadening somewhat a term introduced by G.-C. Rota [16], we will say an operator T is a *model* for $\mathcal{C}$ if $T \in \mathcal{C}$ and each operator $S \in \mathcal{C}$ is unitarily equivalent to the compression of $\infty \cdot T = T \oplus T \oplus \dots$ to one of its semi-invariant subspaces: this relation between S and T will be written $S << T$. Thus, $S << T$ iff there is an isometric imbedding V of the space of S into the space of $\infty \cdot T$ such that $S^n = V^*(\infty \cdot T)^n V$, $n = 0, 1, 2, \dots$. It is easy to see that $S << T$ implies $\infty \cdot S << T$, and in turn this implies that the relation $<<$ is transitive. Thus $<<$ is a partial order in the class of all operators, and hence $S << T << S$ defines an equivalence relation (we omit these details). In the following discussion, we shall be primarily concerned with classes which have an *irreducible* model.

Note first that there are trivial examples of classes which do not have models; for example, if T is a model for $\mathcal{C}$ then every operator in $\mathcal{C}$ has norm at most $\|T\|$, so that a necessary condition for a model to exist is that $\mathcal{C}$ be bounded in norm. Similarly, it is easy to see that if $\mathcal{C}$ has an irreducible model then every operator in $\mathcal{C}$ must act on a separable space.

As a first example, let $\mathcal{C}$ be the class of all separably-acting operators T such that $\|T\| \leq 1$ and $\lim_{n\to\infty} T^{*n} = 0$ in the strong operator topology. It is not hard to see that the simple unilateral shift S (i.e., the weighted shift with weight sequence 1, 1, 1, ...) is an irreducible model for $\mathcal{C}$. While this is a restatement of a familiar result in dilation theory, we will briefly sketch the construction for completeness. Choose $T \in \mathcal{C}$, let D be the positive square root of $I - TT^*$, let $\mathfrak{H}$ be the space on which T acts, and let $\mathfrak{K}$ be the closed range of D. Let $\mathfrak{K}' = \mathfrak{K} \oplus \mathfrak{K} \oplus \dots$ be the Hilbert space of all square summable sequences in $\mathfrak{K}$, and let S' be the unilateral shift on $\mathfrak{K}'$; $S'(\xi_0, \xi_1, \dots) = (0, \xi_0, \xi_1, \dots)$. S' is unitarily equivalent to $n \cdot S$ where S is the simple unilateral shift and $n = \dim \mathfrak{K}$. Define a linear map V of $\mathfrak{H}$ into $\mathfrak{K}'$ by $V\xi = (D\xi, DT^*\xi, DT^{*2}\xi, \dots)$, $\xi \in \mathfrak{H}$. Because $\|T^{*n}\xi\| \to 0$ as $n \to \infty$, it follows easily that V is an isometry, and clearly $VT^* = S'^*V$. This implies that the range of V is invariant under S'^* and $VT^{*n} = S'^{*n}V$ for $n \geq 0$. Thus $T^{*N} = V^*S'^{*n}V$ and hence $T^n = V^*S'^nV$, $n \geq 0$. Since $n \leq \aleph_0$ and S' is equivalent to $n \cdot S$, this implies $T << S$. It is very easy to see that $S^{*n} \to 0$ stongly, so that S is a model for $\mathcal{C}$.

As another example, let $n \geq 2$ and let $\mathcal{C}_n$ be the class of all contractions T such that

$T^n=0$. Then the n-dimensional operator S_n whose matrix relative to some orthonormal basis has the form

$$\begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 \cdots & 0 \\ & & & \ddots & \\ & & & & 1 \\ & & & & 0 \end{pmatrix}$$

is an irreducible model for $\mathcal{C}_n$ (this can be proved by a simple variation of the preceding construction, or it follows as a somewhat trivial special case of 1.3.1).

Suppose now that $\mathcal{C}$ is a class which has an irreducible model. A natural question is, when does $\mathcal{C}$ have a *unique* irreducible model? More precisely, under what conditions are the irreducible models for $\mathcal{C}$ all unitarily equivalent? Now every irreducible operator T determines a largest class $\mathcal{C}(T)$ having T as a model (namely the class of all operators X such that $X \ll T$), and note that T is a model for $\mathcal{C}$ if $T \in \mathcal{C} \subseteq \mathcal{C}(T)$. Therefore T will be a unique model for $\mathcal{C}$ if it is a unique model for the larger class $\mathcal{C}(T)$. Note also that $\mathcal{C}(T)$ has a unique model if T satisfies the following condition: *for every irreducible operator* T_1, $T_1 \ll T \ll T_1$ *implies* T *and* T_1 *are unitarily equivalent.* Any irreducible operator T which has this property will be called a *unique model*, and we are led to ask which irreducible operators are unique models.

First, we want to point out that the simple unilateral shift S is *not* a unique model. To se this, let $\alpha_0, \alpha_1, \ldots$ be a sequence satisfying $0<\alpha_i\leqslant 1$, $\alpha_0<1$, and $\alpha_i=1$ for all $i\geqslant i_0\geqslant 1$. Let T_α be the weighted shift defined on an orthonormal base $e_0, e_1, \ldots$ by $T_\alpha e_n=\alpha_n e_{n+1}$, $n\geqslant 0$. Then $\|T_\alpha\|\leqslant 1$ and $T_\alpha^{*n}\to 0$ strongly, as $n\to\infty$, so by the preceding construction we have $T_\alpha \ll S$. On the other hand, the restriction of T_α to its invariant subspace $[e_{i_0}, e_{i_0+1}, \ldots]$ is clearly unitarily equivalent to S, and this implies $S \ll T_\alpha$. But T_α is *not* unitarily equivalent to S because two unilateral weighted shifts with different positive weight sequences cannot be equivalent (cf. [11], pp. 46–47).

In spite of that, the following result shows that a great variety of operators are unique models.

THEOREM 2.5.1. *Let T be an irreducible operator such that the Calkin map is not isometric on the ultraweakly closed linear span of $\{I, T, T^*, T^2, T^{*2}, \ldots\}$. Then T is a unique model.*

Proof. Let T_1 be any irreducible operator such that $T_1 \ll T \ll T_1$. Then as we have already pointed out, $\infty\cdot T_1 \ll T$. So letting $\mathfrak{H}$, $\infty\cdot\mathfrak{H}$, $\infty\cdot\mathfrak{H}_1$ be the Hilbert spaces on which T, $\infty\cdot T$, $\infty\cdot T_1$ respectively act, then there are isometries $V\in\mathcal{L}(\mathfrak{H}, \infty\cdot\mathfrak{H}_1)$ and

$W \in \mathfrak{L}(\infty\cdot\mathfrak{H}_1, \infty\cdot\mathfrak{H}_2)$ such that $T^n = V^*\infty\cdot T_1^n V$ and $\infty\cdot T_1^n = W^*\infty\cdot T^n W$, $n = 0, 1, 2, \ldots$. We claim first that the range of $WV \in \mathfrak{L}(\mathfrak{H}, \infty\cdot\mathfrak{H})$ reduces $\infty\cdot T$. Indeed, the preceding formulas imply $T^n = (WV)^*\infty\cdot T^n WV$ for $n \geq 0$, and hence the completely positive map $\phi(X) = (WV)^*\infty\cdot XWV$, $X \in \mathfrak{L}(\mathfrak{H})$, fixes $\mathcal{S} = \{T^n: n \geq 0\} \cup \{T^{*n}: n \geq 0\}$. Since ϕ is ultraweakly continuous it must fix the ultraweakly closed linearly span $\mathcal{S}_1$ of $\mathcal{S}$. The Boundary Theorem implies that ϕ fixes $C^*(\mathcal{S}_1)$, and in particular ϕ is multiplicative on $C^*(T)$. This means that the range of WV is semi-invariant under $\{\infty\cdot X: X \in C^*(T)\}$ and, since the latter is a *-algebra, the claim follows.

Next, we claim that the range of V reduces $\infty\cdot T_1$. Indeed, the preceding paragraph implies $(\infty\cdot X)WV = WVX$ for $X \in C^*(T)$, hence $\infty\cdot T_1 V = W^*\infty\cdot TWV = W^*WVT = VT$ and similarly $\infty\cdot T_1^* V = VT^*$. This shows that $\infty\cdot T_1$ and $\infty\cdot T_1^*$ leave the range of V invariant, as asserted.

Thus V implements a unitary equivalence between T and the restriction of $\infty\cdot T_1$ to one of its reducing subspaces. Now since $C^*(T_1)$ is irreducible, every irreducible subrepresentation of the representation $X \in C^*(T_1) \mapsto \infty\cdot X$ is equivalent to the identity representation of $C^*(T_1)$. It follows that the restriction of $\infty\cdot T_1$ to the range of V is unitarily equivalent to T_1, and it now follows that T is equivalent to T_1.

As an illustration of this theorem, let T be an irreducible operator, and suppose there is a sequence p_n of polynomials such that $\operatorname{Re} p_n(T)$ converges weakly to some nonzero compact operator K. Then T is a unique model (for the Banach–Steinhaus theorem implies that the sequence $X_n = \operatorname{Re} p_n(T)$ is bounded, and hence converges to K ultraweakly; since $K \neq 0$, 2.5.1 applies in a straightforward manner). Note that this also implies that for $2 \leq n < \infty$, the n-dimensional operator given by the matrix

$$\begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 \cdots & 0 \\ & & \ddots & \ddots & \\ & & & \ddots & 1 \\ & & & & 0 \end{pmatrix}$$

is a unique model for the "nilpotent" class $\mathcal{C}_n$ described above.

References

[1]. ARVESON, W., Subalgebras of C^*-algebras. *Acta Math.*, 123 (1969), 141–224.
[2]. —— Representations of C^*-algebras. To appear in the M.A.A. volume *Studies in Operator Theory*.
[3]. —— Unitary invariants for compact operators. *Bull. Amer. Math. Soc.*, 76 (1970), 88–91.

[4]. BENDAT, J. & SHERMAN, S., Monotone and convex operator functions. *Trans. Amer. Math. Soc.*, 79 (1955), 58–71.

[5]. BERGER, C., Ph.D. dissertation, Cornell University, 1963.

[6]. COBURN, L., Weyl's theorem for nonnormal operators. *Michigan Math. J.*, 13 (1966), 285–288.

[7]. DIXMIER, J., *Les algèbres d'operateurs dans l'espace Hilbertien.* Gauthier-Villars, Paris, 1957.

[8]. —— *Les C*-algèbres et leurs representations.* Gauthier-Villars, Paris, 1964.

[9]. FOIAS, C., Some applications of spectral sets I. (In roumainian.) *Stud. Cerc. Mat.*, 10 (1959), 365–401.

[10]. GAMELIN, T., *Uniform algebras.* Prentice Hall, Englewood Cliffs, 1969.

[11]. HALMOS, P., *A Hilbert space problem book.* Van Nostrand, Princeton, 1967.

[12]. LEBOW, A., On von Neumann's theory of spectral sets. *J. Math. Anal. Appl.*, 7 (1963), 64–90.

[13]. VON NEUMANN, J., Eine Spektraltheorie für allgemeine Operatoren eines unitären Raumes. *Math. Nachr.*, 4 (1951), 258–281.

[14]. PARROTT, S., Unitary dilations for commuting contractions. *Pacif. J. Math.*, 34 (1970), 481–490.

[15]. RIESZ, F. & SZ.-NAGY, B., *Functional analysis.* Frederick Ungar, New York, 1965.

[16]. ROTA, G.-C. On models for linear operators. *Comm. Pure Appl. Math.*, 13 (1960), 469–472.

[17]. STINESPRING, W., Positive functions on C^*-algebras. *Proc. Amer. Math. Soc.*, 6 (1955), 211–216.

[18]. WAELBROECK, L., Le calcule symbolique dans les algèbres commutatives. *Journal Math. Pures Appl.*, 33 (1954), 147–186.

[19]. ANDO, On a pair of commutative contractions. *Acta Sci. Math.* (*Szeged*), 24 (1963), 88–90.

[20]. SARASON, D., On spectral sets having connected complement. *Acta Sci. Math.* (*Szeged*), 26 (1965), 289–299.

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