ON THE C^*-ALGEBRA OF A ONE-PARAMETER SEMIGROUP OF ISOMETRIES

(In memory of David M. Topping)

BY

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An isometry V on the complex Hilbert space $\mathcal{H}$ is a linear operator satisfying $\|Vf\| = \|f\|$ for f in $\mathcal{H}$. If Γ is a subgroup of the additive reals $\mathbf{R}$, then the semigroup Γ^+ of non-negative elements in Γ is a one-parameter semigroup and a one-parameter semigroup of isometries is a homomorphism $\gamma \to V_\gamma$ of Γ^+ into the collection of isometries on some $\mathcal{H}$, that is, a mapping such that $V_0 = I$ and $V_\gamma V_\delta = V_{\gamma+\delta}$ for γ and δ in Γ^+ (with no continuity assumption). In this note we are interested in the C^*-algebra $\mathcal{A}_\Gamma(V_\gamma)$ generated by $\{V_\gamma : \gamma \in \Gamma^+\}$ for a one-parameter semigroup of isometries.

If V_γ is onto for some nonzero γ in Γ^+, then all the V_γ are unitary and we have a unitary representation of the discrete topological abelian group Γ after setting $V_\gamma = V^*_{-\gamma}$ for negative γ in Γ. Although the complete description of such a representation would involve multiplicity theory, it is fairly routine to describe the C^*-algebra $\mathcal{A}_\Gamma(V_\gamma)$ in this case. If $\hat{\Gamma}$ denotes the compact dual group of Γ, then $\mathcal{A}_\Gamma(V_\gamma)$ is isometrically isomorphic to some quotient of the algebra $C(\hat{\Gamma})$ of continuous complex functions on Γ. Moreover, the isomorphism pairs V_γ with the coset containing the character γ on $\hat{\Gamma}$.

In this note we show that there is only one other possibility for $\mathcal{A}_\Gamma(V_\gamma)$, that is, any two nonunitary one-parameter semigroups of isometries for a fixed Γ generate canonically isomorphic algebras. Thus one can speak of the C^*-algebra $\mathcal{A}_\Gamma$. Although we are unable to give an explicit description of $\mathcal{A}_\Gamma$ we show $\mathcal{A}_\Gamma$ is type I [8] if and only if Γ is isomorphic to the group of integers $\mathbf{Z}$ and that $\mathcal{A}_{\Gamma_1}$ and $\mathcal{A}_{\Gamma_2}$ are isomorphic C^*-algebras if and only if the groups Γ_1 and Γ_2 are order isomorphic. Moreover, the commutator ideal $\mathcal{C}_\Gamma$ in $\mathcal{A}_\Gamma$ is shown to be a simple C^*-algebra with the quotient $\mathcal{A}_\Gamma/\mathcal{C}_\Gamma$ canonically isomorphic to $C(\hat{\Gamma})$. Lastly, certain applications of these results to some problems involving a real-valued analytical index are presented.

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If Γ is singly generated, then $\mathscr{A}_\Gamma(V_\gamma)$ is the C^*-algebra generated by an isometry and the preceding results were obtained in this case by Coburn in [4]. His analysis involved the von Neumann-Wold decomposition for isometries. Similar results were obtained by Berger and Coburn [2] for $\mathscr{A}_{\mathbf{R}}(V_\gamma)$ under the assumption that the mapping $\gamma \to V_\gamma$ is strongly continuous by analyzing the isometric cogenerator. The precursor of our methods was used in [6] to establish the isomorphism of the C^*-algebras generated by the one-parameter semigroups of translation operators on $L^2(\mathbf{R}^+)$ and $l^2(\mathbf{R}^+)$, respectively.

Our methods are closely related to certain analyses indigenous to prediction theory and the study of the Weyl commutation relations.

We begin by analyzing the C^*-algebra generated by a given nonunitary one-parameter semigroup of isometries. Fix Γ and let $\gamma \to V_\gamma$ be a one-parameter semigroup of isometries on $\mathscr{H}$. If we set $\mathscr{H}_\gamma = V_\gamma \mathscr{H}$, then $\{\mathscr{H}_\gamma\colon \gamma \in \Gamma^+\}$ is a decreasing family of closed subspaces of $\mathscr{H} = \mathscr{H}_0$. Moreover, standard arguments (cf. [12]) show that the subspace $\mathscr{H}_\infty = \bigcap_{\gamma\in\Gamma^+} \mathscr{H}_\gamma$ is a reducing subspace for the operators $\{V_\gamma\}_{\gamma\in\Gamma^+}$ such that $\{V_\gamma | \mathscr{H}_\infty\}_{\gamma\in\Gamma^+}$ is a one-parameter semigroup of unitary operators and $\{V_\gamma | \mathscr{H} \ominus \mathscr{H}_\infty\}$ is a one-parameter semigroup of isometries for which $\bigcap_{\gamma\in\Gamma^+} V_\gamma(\mathscr{H} \ominus \mathscr{H}_\infty) = (0)$. Thus we have an analogue of the von Neumann-Wold decomposition into unitary and "pure" one-parameter semigroups of isometries.

We consider first the pure case and thus assume that $\mathscr{H}_\infty = \bigcap_{\gamma\in\Gamma^+} \mathscr{H}_\gamma = (0)$. If we define $E_\gamma = V_\gamma V_\gamma^*$, then E_γ can be seen to be the orthogonal projection onto $\mathscr{H}_\gamma$. Let $\mathscr{Z}\ (V_\gamma)$ denote the C^*-subalgebra of $\mathscr{A}_\Gamma(V_\gamma)$ generated by $\{E_\gamma\colon \gamma \in \Gamma^+\}$. We show first that $\mathscr{Z}_\Gamma$ is unique and does not depend on the particular representation of Γ^+.

Let PC$(\mathbf{R}^+)$ denote the C^*-algebra of bounded complex functions on $\mathbf{R}^+$ given the supremum norm which are right continuous and which possess limits from the left at every point including infinity. Moreover, let B_Γ denote the closed subalgebra of PC $(\mathbf{R}^+)$ generated by the characteristic functions $\{I_{[\gamma,\infty)}\colon \gamma \in \Gamma^+\}$.

PROPOSITION 1. *The correspondence* $E_\gamma \leftrightarrow I_{[\gamma,\infty)}$ *extends to a *-isomorphism between the* C^*-*algebras* $\mathscr{Z}_\Gamma(V_\gamma)$ *and* B_Γ.

Proof. Clearly elements of the form $\sum_{j=0}^N \alpha_j (E_{\gamma_j} - E_{\gamma_{j+1}})$ and $\sum_{j=0}^N \alpha_j I_{[\gamma_j, \gamma_{j+1})}$, where $\{\alpha_j\}_{j=1}^N$ are complex numbers and $0 = \gamma_0 < \gamma_1 < \ldots < \gamma_N < \gamma_{N+1} = \infty$ is a partition of $[0, \infty)$ with elements of Γ^+, are dense in $\mathscr{Z}_\Gamma(V_\gamma)$ and B_Γ respectively. Moreover, the linear extension of the mapping $E_\gamma \leftrightarrow I_{[\gamma,\infty)}$ pairs these two elements and since

$$\left\| \sum_{j=0}^N \alpha_j (E_{\gamma_j} - E_{\gamma_{j+1}}) \right\| = \sup_{0 \leq j \leq N} |\alpha_j| = \left\| \sum_{j=0}^N \alpha_j I_{[\gamma_j, \gamma_{j+1})} \right\|_\infty,$$

the result follows.

Our problem now is to extend this uniqueness of the abelian subalgebra $\mathscr{Z}_\Gamma(V_\gamma)$ to all of $\mathscr{A}_\Gamma(V_\gamma)$.

If we extend the definition of E_γ to all γ in $\mathbf{R}^+$ by setting $E_\gamma = \inf\{E_\delta: \delta \geqslant \gamma, \delta \in \Gamma^+\}$, then we obtain a decreasing family $\{E_\gamma\}_{\gamma\in\Gamma^+}$ of orthogonal projections such that $E_0 = I$ and $\lim_{\gamma\to\infty} E_\gamma = 0$ in the strong operator topology. We can now define $U_t = \int_0^\infty e^{i\gamma t} dE_\gamma$ for t in $\mathbf{R}$ to obtain a strongly continuous one-parameter group $\{U_t\}_{t\in\mathbf{R}}$ of unitary operators on $\mathscr{H}$. Although these operators do not belong to $\mathscr{A}_\Gamma(V_\gamma)$, they shall be quite important in what follows.

We show first that $\{V_\gamma\}_{\gamma\in\Gamma^+}$ and $\{U_t\}_{t\in\mathbf{R}}$ satisfy the "Weyl commutation relations", Since $V_\gamma E_\delta V_\gamma^* = V_\gamma V_\delta V_\delta^* V_\gamma^* = V_{\gamma+\delta} V_{\gamma+\delta}^* = E_{\gamma+\delta}$ for γ and δ in Γ^+, it follows that $V_\gamma E_\delta V_\gamma^* = E_{\gamma+\delta}$ for γ in Γ^+ and δ in $\mathbf{R}^+$. Therefore, we have

$$V_\gamma U_t = V_\gamma \left(\int_0^\infty e^{i\delta t} dE_\delta\right) V_\gamma^* V_\gamma = \left\{\int_0^\infty e^{i\delta t} d(V_\gamma E_\delta V_\delta^*)\right\} V \,,$$

$$= e^{-i\gamma t}\left\{\int_\gamma^\infty e^{i\delta t} dE_\delta\right\} V_\gamma = e^{-i\gamma t}\left\{\int_0^\infty e^{i\delta t} dE_\delta\right\} V_\gamma = e^{-i\gamma t} U_t V_\gamma,$$

where the second from last equality follows from the fact that $\{\int_0^\gamma e^{i\delta t} dE_\delta\} V_\gamma = 0$. Thus we have $U_t V_\gamma = e^{i\gamma t} V_\gamma U_t$ for γ in Γ^+ and t in $\mathbf{R}$.

The importance of the $\{U_t\}_{t\in\mathbf{R}}$ lies in the one parameter group of automorphisms they define on $\mathscr{A}_\Gamma(V_\gamma)$. If for T in $\mathscr{L}(\mathscr{H})$ we define $\alpha_t(T) = U_t^* T U_t$, then α_t is an automorphism on $\mathscr{L}(\mathscr{H})$. Moreover, since $\alpha_t(V_\gamma) = U_t^* V_\gamma U_t = e^{-i\gamma t} U_t^* U_t V_\gamma = e^{-i\gamma t} V_\gamma$, it follows that $\alpha_t\{\mathscr{A}_\Gamma(V_\gamma)\}$ is contained in $\mathscr{A}_\Gamma(V_\gamma)$, and since $\alpha_{-t} = \alpha_t^{-1}$ we see that α_t is an automorphism on $\mathscr{A}_\Gamma(V_\gamma)$. Moreover, $\{\alpha_t\}_{t\in\mathbf{R}}$ is a one-parameter group of automorphisms on $\mathscr{A}_\Gamma(V_\gamma)$ such that $\alpha_t(A)$ is a norm-continuous almost periodic function on R for each A in $\mathscr{A}_\Gamma(V_\gamma)$. This follows by considering the action of α_t on the generators of $\mathscr{A}_\Gamma(V_\gamma)$.

The automorphisms $\{\alpha_t\}_{t\in\mathbf{R}}$ leave the elements of $\mathscr{Z}_\Gamma(V_\gamma)$ fixed since each U_t lies in the strong closure of $\mathscr{Z}_\Gamma(V_\gamma)$. Moreover, these are the only elements of $\mathscr{A}_\Gamma(V_\gamma)$ fixed by all of the α_t. To prove this we consider the mean ϱ defined on $\mathscr{A}_\Gamma(V_\gamma)$ by $\varrho(A) = \lim_{T\to\infty} (1/2T) \int_{-T}^T \alpha_t(A)\, dt$ for A in $\mathscr{A}_\Gamma(V_\gamma)$. The fact that the limit exists in the norm topology follows from the almost periodicity of the function $\alpha_t(A)$. Moreover, it is clear that the fixed point subalgebra of $\{\alpha_t\}_{t\in\mathbf{R}}$ and the range of ϱ coincide. Thus it is only necessary to show that the range of ϱ is $\mathscr{Z}_\Gamma(V_\gamma)$.

To that end we consider the collection $\mathscr{F}$ of operators in $\mathscr{A}_\Gamma(V_\gamma)$ of the form $\sum_{j=1}^N Z_j V_{\gamma_j}$, where $Z_1, Z_2, \ldots, Z_N$ are in $\mathscr{Z}_\Gamma(V_\gamma)$ and $\gamma_1, \gamma_2, \ldots, \gamma_N$ lie in Γ with $V_\gamma = V_{-\gamma}^*$ for $\gamma < 0$. Using the fact that $V_\delta^* \mathscr{Z}_\Gamma(V_\gamma) V_\delta = \mathscr{Z}_\Gamma(V_\gamma)$ for δ in Γ, one sees that $\mathscr{F}$ is a selfadjoint subalgebra of $\mathscr{A}_\Gamma(V_\gamma)$ containing $\{V_\gamma\}_{\gamma\in\Gamma^+}$ and hence must be dense. Moreover, since

$$\varrho(ZV_\gamma)=\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\alpha_t(ZV_\gamma)\,dt=ZV_\gamma\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}e^{-i\gamma t}\,dt,$$

it follows that
$$\varrho(ZV_\gamma)=\begin{cases}0 & \text{if } \gamma\neq 0\\ Z & \text{if } \gamma=0.\end{cases}$$

Thus $\varrho(\mathscr{F})$ is contained in $\mathscr{Z}_\Gamma(V_\gamma)$ and since ϱ is continuous and $\mathscr{F}$ is dense, we see that the range of ϱ is $\mathscr{Z}_\Gamma(V_\gamma)$ which is the fixed point subalgebra of $\{\alpha_t\}_{t\in\mathbf{R}}$.

The mapping ϱ is a conditional expectation of $\mathscr{A}_\Gamma(V_\gamma)$ onto the subalgebra $\mathscr{Z}_\Gamma(V_\gamma)$. Although ϱ is not one-to-one, it is on the positive elements on $\mathscr{A}_\Gamma(V_\gamma)$ for the same reason that a nonnegative scalar almost periodic function on $\mathbf{R}$ having mean zero must vanish identically.

We now use the mapping ϱ to lift the uniqueness of the abelian subalgebra $\mathscr{Z}_\Gamma(V_\gamma)$ to that of $\mathscr{A}_\Gamma(V_\gamma)$.

Let $\{V_\gamma\}_{\gamma\in\Gamma^+}$ and $\{V'_\gamma\}_{\gamma\in\Gamma^+}$ be two pure isometric representations of Γ^+. Forming the direct sum we obtain a further pure isometric representation $\{V_\gamma\oplus V'_\gamma\}_{\gamma\in\Gamma^+}$. An easy argument shows that the mapping Φ defined $\Phi(V_\gamma\oplus V'_\gamma)=V_\gamma$ extends to a *-homomorphism from $\mathscr{A}_\Gamma(V_\gamma\oplus V'_\gamma)$ to $\mathscr{A}_\Gamma(V_\gamma)$. If $\{\alpha_t\}_{t\in\mathbf{R}}$ denotes the one-parameter group of automorphisms defined on $\mathscr{A}_\Gamma(V_\gamma)$ and $\{\alpha'_t\}_{t\in\mathbf{R}}$ the corresponding group on $\mathscr{A}_\Gamma(V_\gamma\oplus V'_\gamma)$, then $\Phi\circ\alpha'_t=\alpha_t\circ\Phi$ for t in $\mathbf{R}$. Thus we obtain $\varrho\circ\Phi=\Phi_0\circ\varrho'$, where ϱ' and ϱ are the means on $\mathscr{A}_\Gamma(V_\gamma\oplus V'_\gamma)$ and $\mathscr{A}_\Gamma(V_\gamma)$, respectively, and Φ_0 is the restriction of Φ to a mapping from $\mathscr{Z}_\Gamma(V_\gamma\oplus V'_\gamma)$ to $\mathscr{Z}_\Gamma(V_\gamma)$ which is an isomorphism by Proposition 1. Now if Φ were not an isomorphism, then there would exist T in $\mathscr{A}_\Gamma(V_\gamma\oplus V'_\gamma)$ such that $\Phi(T)=0$. Setting $Q=T^*T$ we obtain a nonnegative element of $\mathscr{A}_\Gamma(V_\gamma\oplus V'_\gamma)$ for which $(\Phi\circ\varrho')(Q)=(\varrho\circ\Phi)(Q)=0$. Since Φ_0 is an isomorphism, we have $\varrho'(Q)=0$ which must by our preceding remarks imply that $Q=0$. Thus Φ is an isomorphism. Since the mapping $\Phi'(V_\gamma\oplus V'_\gamma)=V'_\gamma$ must also extend to an isomorphism, it follows that the correspondence $V_\gamma\leftrightarrow V'_\gamma$ extends to a *-isomorphism of $\mathscr{A}_\Gamma(V_\gamma)$ onto $\mathscr{A}_\Gamma(V'_\gamma)$. Thus we have proved

PROPOSITION 2. *If $\{V_\gamma\}_{\gamma\in\Gamma^+}$ and $\{V'_\gamma\}_{\gamma\in\Gamma^+}$ are two pure one-parameter semigroups of isometries, then the correspondence $V_\gamma\leftrightarrow V'_\gamma$ extends to a *-isomorphism between the C^*-algebras $\mathscr{A}_\Gamma(V_\gamma)$ and $\mathscr{A}_\Gamma(V'_\gamma)$.*

The bulk of the work has been done. It now remains to consider the case when there is a unitary part. For this we need to recall a result proved in [5] for a special pure one-parameter semigroup of isometries.

PROPOSITION 3. *If $\{V_\gamma\}_{\gamma\in\Gamma^+}$ is a pure one-parameter semigroup of isometries and $\mathscr{C}_\Gamma(V_\gamma)$ is the commutator ideal in $\mathscr{A}_\Gamma(V_\gamma)$, then the correspondence $\gamma\leftrightarrow V_\gamma+\mathscr{C}_\Gamma(V_\gamma)$ extends to a *-isomorphism between the C^*-algebras $C(\hat{\Gamma})$ and $\mathscr{A}_\Gamma(V_\gamma)/\mathscr{C}_\Gamma(V_\gamma)$.*

We can now prove our principal result.

THEOREM 1. *The C^*-algebra $\mathscr{A}_\Gamma$ generated by a nonunitary one-parameter semigroup of isometries is canonically unique.*

Proof. It is sufficient in view of Proposition 2 to show that the correspondence $V_\gamma \leftrightarrow V_\gamma \oplus U_\gamma$ extends to a *-isomorphism between the C^*-algebras $\mathscr{A}_\Gamma(V_\gamma)$ and $\mathscr{A}_\Gamma(V_\gamma \oplus U_\gamma)$, where $\{V_\gamma\}_{\gamma \in \Gamma^+}$ is a pure one-parameter semigroup of isometries and $\{U_\gamma\}_{\gamma \in \Gamma^+}$ is a one-parameter group of unitaries. To do this it is sufficient to show that the mapping $\psi(V_\gamma) = U_\gamma$ extends to a *-homomorphism from $\mathscr{A}_\Gamma(V_\gamma)$ to $\mathscr{A}_\Gamma(U_\gamma)$. Since the mapping ψ may be factored $\psi = \psi_1 \psi_2$, where ψ_2 is defined from $\mathscr{A}_\Gamma(V_\gamma)$ to $C(\hat{\Gamma})$ by $\psi_2(V_\gamma) = \gamma$ and ψ_1 is defined from $C(\hat{\Gamma})$ to $\mathscr{A}_\Gamma(U_\gamma)$ by $\psi_1(\gamma) = U_\gamma$ and both maps extend to *-homomorphisms by our previous remarks, the result follows.

COROLLARY. *If Γ is a one-parameter group, then the commutator ideal $\mathscr{C}_\Gamma$ in $\mathscr{A}_\Gamma$ is a simple C^*-algebra and the quotient $\mathscr{A}_\Gamma / \mathscr{C}_\Gamma$ is isomorphic to $C(\hat{\Gamma})$.*

Proof. The fact that $\mathscr{A}_\Gamma / \mathscr{C}_\Gamma$ is isomorphic to $C(\hat{\Gamma})$ follows from the special result in [5] and the theorem. To show that $\mathscr{C}_\Gamma$ is simple, let $\mathscr{I}$ be a closed two-sided ideal in $\mathscr{A}_\Gamma$ and let π be a faithful representation of the C^*-algebra $\mathscr{A}_\Gamma / \mathscr{I}$ into the bounded operators $\mathscr{L}(\mathscr{H})$ on some Hilbert space $\mathscr{H}$. Then $\{\pi(V_\gamma + \mathscr{I})\}_{\gamma \in \Gamma^+}$ is a one-parameter semigroup of isometries on $\mathscr{H}$ and there are two possibilities. Either they are non-unitary in which case $V_\gamma \leftrightarrow \pi(V_\gamma + \mathscr{I})$ extends to an isomorphism and $\mathscr{I} \equiv \{0\}$ or they are unitary in which case $\pi(V_\gamma + \mathscr{I})$ generates an abelian algebra and hence $\mathscr{I}$ contains $\mathscr{C}_\Gamma$. Thus $\mathscr{C}_\Gamma$ is a simple C^*-algebra.

We point out that the preceding corollary contains the well-known result that the algebra of compact operators is simple. Moreover, since the compact operators in $\mathscr{A}_\Gamma$ (in any representation) obviously form an ideal, it follows that the zero operator is the only one if Γ is not singly generated and hence $\mathscr{A}_\Gamma$ is not a type I C^*-algebra. Although we conjecture that two algebras $\mathscr{C}_{\Gamma_1}$ and $\mathscr{C}_{\Gamma_2}$ are isomorphic if and only if Γ_1 and Γ_2 are order isomorphic, we have been unable to prove it. Further, one form of the Bott Periodicity Theorem states that the homotopy groups π_k of the invertible elements in $I + \mathscr{C}_{\mathbf{Z}}$ are 0, $\mathbf{Z}$, 0, $\mathbf{Z}$, ... The analogous results would be of interest for $I + \mathscr{C}_\Gamma$. Atiyah and Singer [1] have shown that the homotopy groups of the invertibles in a certain algebra closely related to $I + \mathscr{C}_{\mathbf{R}}$ are 0, $\mathbf{R}$, 0, $\mathbf{R}$, ...

Although we are unable to prove that the commutator ideal determines the ordered group, we can establish that the algebra does. The simplification of this proof was obtained in conversation with Singer.

THEOREM 2. *If Γ_1 and Γ_2 are one-parameter groups, then $\mathscr{A}_{\Gamma_1}$ and $\mathscr{A}_{\Gamma_2}$ are isomorphic C^*-algebras if and only if Γ_1 and Γ_2 are order isomorphic.*

Proof. In one direction the result follows from Theorem 1. Thus assume that π is an isomorphism from $\mathcal{A}_{\Gamma_1}$ onto $\mathcal{A}_{\Gamma_2}$. Since π must respect commutator ideals, we have that π induces an isomorphism between $\mathcal{A}_{\Gamma_1}/\mathcal{C}_{\Gamma_1}$ and $\mathcal{A}_{\Gamma_2}/\mathcal{C}_{\Gamma_2}$ implying that $C(\hat{\Gamma}_1)$ is isomorphic to $C(\hat{\Gamma}_2)$. A result of Bohr and van Kampen (cf. [6]) implies that each component of the group of invertible elements in $C(\hat{\Gamma}_1)$ and $C(\hat{\Gamma}_2)$, respectively, contains a character and hence we obtain an isomorphism π_* between Γ_1 and Γ_2. Moreover, since an element γ in Γ_1 is positive if and only if some element in the coset $V_\gamma + \mathcal{C}_{\Gamma_1}$ has a left inverse, and this latter property is preserved by π, it follows that π_* is order preserving which completes the proof.

Let us consider briefly now some examples and applications. If α is a real number, then $\{(m, n) \in \mathbf{Z}^2 \colon m + n\alpha \geqslant 0\}$ determines a semigroup Σ_α of $\mathbf{Z}^2$. If $\mathcal{T}_\alpha$ denotes the C^*-algebra generated by translations in the direction of Σ_α on $l^2(\Sigma_\alpha)$, then for α irrational $\mathcal{T}_\alpha$ is one of the class of C^*-algebras studied in this note. In particular, $\mathcal{T}_\alpha$ is the C^*-algebra associated with $\Gamma_\alpha = \{m + n\alpha \colon (m, n) \in \mathbf{Z}^2\}$ and thus the problem of determining the isomorphism classes of these algebras for irrational α can be solved. After a few computations we find that $\mathcal{T}_\alpha$ and $\mathcal{T}_\beta$ are isomorphic for α and β irrational if and only if there exists integers i, j, k, l such that

$$\begin{pmatrix} i & j \\ k & l \end{pmatrix} \begin{pmatrix} 1 \\ \alpha \end{pmatrix} = \begin{pmatrix} 1 \\ \beta \end{pmatrix} \quad \text{and } il - jk = 1.$$

Among other things this implies that $\mathcal{T}_{\sqrt{2}}$ and $\mathcal{T}_{\sqrt{3}}$ are not isomorphic. For α rational the algebras $\mathcal{T}_\alpha$ have a center, and are, in fact, all isomorphic to the type I C^*-algebra $C(\mathbf{T}) \otimes \mathcal{A}_{\mathbf{Z}}$. There are obvious analogues of these examples in higher dimensions but the possible degeneracies would be quite complicated.

The results of this note can be used to simplify certain of the proofs in [6]. In particular, to obtain a faithful representation of $\mathcal{A}_\Gamma$, it is sufficient to map $\{V_\gamma\}_{\gamma \in \Gamma^+}$ to a one-parameter semigroup of isometries, and not to verify any further estimates.

Further, we can now throw a little light on an example mentioned in [6]. We shall adopt certain notation from [6] with a minimum of detail. Let H^2 denote the closed subspace of L^2 with respect to Lebesgue measure on $\mathbf{R}$ consisting of those functions possessing analytic extensions into the upper half plane and let P be the orthogonal projection of L^2 onto H^2. If AP denotes the algebra of almost periodic functions on $\mathbf{R}$ and C_0 the algebra of continuous functions on $\mathbf{R}$ which vanish at infinity, then $AP + C_0$ is the algebra of asymptotic almost periodic functions, C_0 is an ideal in $AP + C_0$ and we have the short exact sequence $(0) \to C_0 \xrightarrow{i} AP + C_0 \to AP \to (0)$, where i is the inclusion map. For φ in $AP + C_0$ let W_φ be the operator on H^2 defined by $W_\varphi f = P(\varphi f)$ for f in H^2. Then W_φ is a Wiener-Hopf operator and we are interested in the C^*-algebra $\mathcal{T}(AP + C_0)$ generated by

$\{W_\varphi\colon \varphi \in AP+C_0\}$. A moment's reflection will reveal that $\mathscr{T}(AP)$ is just $\mathscr{A}_{\mathbf{R}}$, while $\mathscr{T}(C_0)+I$ is $\mathscr{A}_{\mathbf{Z}}$.

Classical results state that an operator $I+W_\varphi$ for φ in C_0 is a Fredholm operator if and only if $1+\varphi \neq 0$ and the index of $I+W_\varphi$ is minus the winding number of the curve determined by $1+\varphi$. In [6] this result was extended to W_φ for φ in AP with the notion of Fredholm being replaced by that due to Breuer [3] in an appropriate II_∞-factor representation of $\mathscr{T}(AP)$. The corresponding result for φ in $AP+C_0$ is unknown and is of particular interest since the index would presumably lie in $\mathbf{R}\oplus\mathbf{Z}$.

If $\mathscr{C}(AP+C_0)$ denotes the commutator ideal in $\mathscr{T}(AP+C_0)$, then it is quite easy to show that the quotient $\mathscr{T}(AP+C_0)/\mathscr{C}(AP+C_0)$ is isomorphic to $AP+C_0$. One would like to be able to think of the function $\sigma(T)$ in $AP+C_0$ as the "symbol" of an operator T in $\mathscr{T}(AP+C_0)$. A minimum condition for this would be for the invertibility of the operator to imply that the symbol has index zero and this we prove and more.

The key to the proof is observing that $\mathscr{T}(C_0)$ is an ideal in $\mathscr{T}(AP+C_0)$ and identifying the quotient. Using the fact that $\mathscr{T}(C_0)$ contains the compact operators and that $W_\varphi W_\psi - W_{\varphi\psi}$ and $W_\psi W_\varphi - W_{\varphi\psi}$ are both compact for φ in C_0 and ψ in $AP+C_0$ (cf. [9], Chapter 7), we see that $\mathscr{T}(C_0)$ is a closed two-sided ideal in $\mathscr{T}(AP+C_0)$. Moreover, since the operators $\{W_{e^{i\lambda t}}\}_{\lambda\geq 0}$ and $\mathscr{T}(C_0)$ clearly generate $\mathscr{T}(AP+C_0)$, it follows that $\mathscr{T}(AP+C_0)/\mathscr{T}(C_0)$ is generated by the one-parameter semigroup of isometries $\{W_{e^{i\gamma t}}+\mathscr{T}(C_0)\}_{\gamma\geq 0}$ and hence the inclusion mapping of $\mathscr{T}(AP)$ into $\mathscr{T}(AP+C_0)/\mathscr{T}(C_0)$ is an isomorphism. Therefore we have the commutative diagram

$$\begin{array}{ccccccccc}
 & & (0) & & (0) & & (0) & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
(0) & \longrightarrow & \mathscr{C}(C_0) & \xrightarrow{i} & \mathscr{C}(AP+C_0) & \xrightarrow{\pi} & \mathscr{C}(AP) & \longrightarrow & (0) \\
 & & \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle i} & & \\
(0) & \longrightarrow & \mathscr{T}(C_0) & \xrightarrow{i} & \mathscr{T}(AP+C_0) & \xrightarrow{\pi} & \mathscr{T}(AP) & \longrightarrow & (0) \\
 & & \downarrow{\scriptstyle \sigma} & & \downarrow{\scriptstyle \sigma} & & \downarrow{\scriptstyle \sigma} & & \\
(0) & \longrightarrow & C_0 & \xrightarrow{i} & AP+C_0 & \xrightarrow{\pi} & AP & \longrightarrow & (0) \\
 & & \vdots & & \downarrow & & \downarrow & & \\
 & & (0) & & (0) & & (0) & &
\end{array}$$

with short exact rows and columns.

Our problem is to show the Fredholmness of T in $\mathcal{T}(AP+C_0)$ implies the invertibility of $\sigma(T)$ in $AP+C_0$. Since $\mathcal{C}(C_0)$ is the compact operators and the sequence

$$(0) \longrightarrow \mathcal{C}(C_0) \xrightarrow{\ i\ } \mathcal{T}(AP+C_0) \xrightarrow{\pi\oplus\sigma} \mathcal{T}(AP) \oplus AP + C_0$$

is exact at $\mathcal{T}(AP+C_0)$ by ([10], Prop. 4), we see that T is a Fredholm operator if and only if $\pi(T)$ is invertible in $\mathcal{T}(AP)$ and $\sigma(T)$ is invertible in $AP+C_0$. Thus we have proved

THEOREM 3. *An operator T in $\mathcal{T}(AP+C_0)$ is a Fredholm operator if and only if $\sigma(T)$ is invertible in $AP+C_0$ and $\pi(T)$ is invertible in $\mathcal{T}(AP)$. Moreover, the index of such a T is equal to minus the topological index of $\sigma(T)$.*

The only thing needing comment is the moreover statement and since $\pi(T)$ is invertible it follows that the topological index of $\sigma(T)$ in $\mathbf{R}\oplus\mathbf{Z}$ must have first component zero and an elementary computation completes the proof.

Without going into detail we point out that the range of $\pi\oplus\sigma$ consists of the pairs $W\oplus\varphi$ for which $\sigma(W)=\pi(\varphi)$ by the exactness of the diagram using [7], and thus these pairs form the "symbol space" for $\mathcal{T}(AP+C_0)$. Hence one could say that a T in $\mathcal{T}(AP+C_0)$ is a "generalized Fredholm operator" if $\sigma(T)$ is invertible and $\pi(T)$ is a generalized Fredholm operator in $\mathcal{T}(AP)$ and using [6] this follows if $\sigma(T)$ is invertible. A reasonable analytical index can be defined which would agree with the negative of the topological index. We hope to present further details and justification in another paper.

We conclude with a couple of remarks. Firstly, the analysis of the W^*-algebras generated by a one-parameter semigroup of isometries is considered in a recent paper of Muhly [11] which as one might expect is considerably more complex. Secondly, the one-parameter group of automorphisms $\{\alpha_t\}_{t\in\mathbf{R}}$ crucial in the proof of Theorem 1 are what are called "weak-inner", that is, they are unitarily implemented by unitary operators in the weak closure of $\mathcal{C}_\Gamma$ in any of its representations. Since $\mathcal{C}_{\mathbf{Z}}$ contains the compact operators, all of its automorphisms are weak-inner. It is probably too much to expect the same to be true of $\mathcal{C}_\Gamma$.

Added in proof: Barry Johnson has given an example of an automorphism of $\mathcal{C}_\Gamma$ which is not weak-inner for $\Gamma \neq Z$.

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